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AN ANALOGUE OF AN IDENTITY OF JACOBI

HENG HUAT CHAN, SONG HENG CHAN, AND PATRICK SOLÉ

ABSTRACT. In 2002, H.H. Chan, K.S. Chua and P. Solé found that for each positive integer d , there are theta series A_d, B_d and C_d of weight one that satisfy the Pythagoras-like relation $A_d^2 = B_d^2 + C_d^2$. In this article, we show that there are two collections of theta series $A_{b,d}, B_{b,d}$ and $C_{b,d}$ of weight one which satisfy $A_{b,d}^2 = B_{b,d}^2 + C_{b,d}^2$, where b and d are certain integers.

1. INTRODUCTION

One of the most famous identity of Jacobi states that

$$(1.1) \quad \left(\sum_{m,n=-\infty}^{\infty} q^{m^2+n^2} \right)^2 = \left(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{m^2+n^2} \right)^2 + \left(\sum_{m,n=-\infty}^{\infty} q^{(m+1/2)^2+(n+1/2)^2} \right)^2.$$

One can view (1.1) as a solution to

$$(1.2) \quad A^2 = B^2 + C^2$$

where A, B and C are theta series of weight 1. This identity is instrumental in the parametrization of Gauss' AGM by modular forms [8, 2].

In [6], Chan, Chua and Solé, motivated by the study of codes and lattices, found that for any positive integer d ,

$$(1.3) \quad \left(\sum_{m,n=-\infty}^{\infty} q^{2(m^2+mn+dn^2)} \right)^2 = \left(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{m^2+mn+dn^2} \right)^2 + \left(\sum_{m,n=-\infty}^{\infty} q^{2((m+1/2)^2+(m+1/2)n+dn^2)} \right)^2.$$

Identity (1.3) provides an infinite number of solutions in theta functions of weight 1 to (1.2). For more information on this generalized Jacobi identity, we refer the reader to [5, 7].

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Recently, while studying theta series associated with binary quadratic forms of discriminant -15 , we discovered the identity

$$(1.4) \quad \left(\sum_{m,n=-\infty}^{\infty} q^{2m^2+mn+2n^2} \right)^2 = \left(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{2m^2+mn+2n^2} \right)^2 + \left(2 \sum_{m,n=-\infty}^{\infty} q^{2(2(m+1/2)^2+(m+1/2)n+2n^2)} \right)^2.$$

In this article, we will establish an analogue of (1.3) for which (1.4) is a special case.

Theorem 1. *Let d be any positive integer and let $1 \leq b \leq d - 1$. Then*

$$(1.5) \quad \left(\sum_{m,n=-\infty}^{\infty} q^{dm^2+bm+dn^2} \right)^2 = \left(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{dm^2+bm+dn^2} \right)^2 + \left(2 \sum_{m,n=-\infty}^{\infty} q^{2(d(m+1/2)^2+b(m+1/2)n+dn^2)} \right)^2.$$

We note that when $d = 2$ and $b = 1$, we recover (1.4) from (1.5). The proof of (1.5) will be given in Section 2.

Our discovery of (1.5) provides a motivation for deriving the following two-variable version of (1.3):

$$(1.6) \quad \left(\sum_{m,n=-\infty}^{\infty} q^{2(bm^2+bm+dn^2)} \right)^2 = \left(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{bm^2+bm+dn^2} \right)^2 + \left(\sum_{m,n=-\infty}^{\infty} q^{2(b(m+1/2)^2+b(m+1/2)n+dn^2)} \right)^2.$$

Observe that when $b = 1$, (1.6) implies (1.3). We will give a proof of (1.6) in Section 3.

2. PROOF OF (1.5)

The Jacobi one-variable theta functions are defined by

$$\vartheta_2(q) = \sum_{j=-\infty}^{\infty} q^{(j+1/2)^2},$$

$$\vartheta_3(q) = \sum_{j=-\infty}^{\infty} q^{j^2}$$

and

$$\vartheta_4(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2}.$$

We first express the theta functions in (1.5) in terms of $\vartheta_j(q)$, $j = 2, 3, 4$.

Lemma 2. For $|q| < 1$,

(2.7)

$$\mathcal{A}_{b,d} = \sum_{m,n=-\infty}^{\infty} q^{dm^2+bmndn^2} = \vartheta_3(q^{2d+b})\vartheta_3(q^{2d-b}) + \vartheta_2(q^{2d+b})\vartheta_2(q^{2d-b}),$$

(2.8)

$$\mathcal{B}_{b,d} = \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{dm^2+bmndn^2} = \vartheta_3(q^{2d+b})\vartheta_3(q^{2d-b}) - \vartheta_2(q^{2d+b})\vartheta_2(q^{2d-b})$$

and

(2.9)

$$\mathcal{C}_{b,d} = 2 \sum_{m,n=-\infty}^{\infty} q^{2(d(m+1/2)^2+b(m+1/2)n+dn^2)} = \vartheta_2(q^{d+b/2})\vartheta_2(q^{d-b/2}).$$

Proof. We observe that

$$dm^2 + bmn + dn^2 = \begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} d & b/2 \\ b/2 & d \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}.$$

Next, since

$$\begin{pmatrix} d & b/2 \\ b/2 & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} d+b/2 & 0 \\ 0 & d-b/2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

we find that

$$dm^2 + bmn + dn^2 = \frac{2d+b}{4}(m+n)^2 + \frac{2d-b}{4}(m-n)^2.$$

Therefore,

$$\begin{aligned} \sum_{m,n=-\infty}^{\infty} q^{dm^2+bmndn^2} &= \sum_{m,n=-\infty}^{\infty} q^{(2d+b)(m+n)^2/4+(2d-b)(m-n)^2/4} \\ &= \sum_{\substack{m,n=-\infty \\ m+n \text{ even}}}^{\infty} q^{(2d+b)(m+n)^2/4+(2d-b)(m-n)^2/4} + \sum_{\substack{m,n=-\infty \\ m+n \text{ odd}}}^{\infty} q^{(2d+b)(m+n)^2/4+(2d-b)(m-n)^2/4} \\ &= \vartheta_3(q^{2d+b})\vartheta_3(q^{2d-b}) + \vartheta_2(q^{2d+b})\vartheta_2(q^{2d-b}) \end{aligned}$$

and the proof of (2.7) is complete. The proof of (2.8) is similar to the proof of (2.7).

To prove (2.9), we need the identity

$$(2.10) \quad \sum_{m=-\infty}^{\infty} (-1)^m q^{(m+1/2)^2} = 0.$$

Identity (2.10) is true since

$$\sum_{m=-\infty}^{\infty} (-1)^m q^{(m+1/2)^2} = \sum_{s=-\infty}^{\infty} (-1)^s q^{(s-1/2)^2} = \sum_{t=-\infty}^{\infty} (-1)^{t+1} q^{(t+1/2)^2}.$$

From (2.10), we deduce that for any integer ℓ ,

$$(2.11) \quad \sum_{m=-\infty}^{\infty} (-1)^m q^{(m+\ell+1/2)^2} = 0.$$

A consequence of (2.11) is that

$$(2.12) \quad \sum_{n=-\infty}^{\infty} q^{(2n+\ell+1/2)^2} = \sum_{n=-\infty}^{\infty} q^{(2n+1+\ell+1/2)^2}.$$

We are now ready to prove (2.9). Write

$$\mathcal{C}_{b,d} = 2 \sum_{m,n=-\infty}^{\infty} q^{(2d+b)(m+1/2+n)^2/2+(2d-b)(m+1/2-n)^2/2}.$$

Let $k = m - n$. Then

$$\begin{aligned} \mathcal{C}_{b,d} &= 2 \sum_{k=-\infty}^{\infty} q^{(2d-b)(k+1/2)^2/2} \sum_{n=-\infty}^{\infty} q^{(2d+b)(2n+k+1/2)^2/2} \\ &= \sum_{k=-\infty}^{\infty} q^{(2d-b)(k+1/2)^2/2} \sum_{s=-\infty}^{\infty} q^{(2d+b)(s+1/2)^2/2} = \vartheta_2(q^{(2d-b)/2})\vartheta_2(q^{(2d+b)/2}), \end{aligned}$$

which is (2.9). The last equality follows by writing

$$\begin{aligned} 2 \sum_{n=-\infty}^{\infty} q^{(2d+b)(2n+k+1/2)^2/2} &= \sum_{n=-\infty}^{\infty} q^{(2d+b)(2n+k+1/2)^2/2} + \sum_{n=-\infty}^{\infty} q^{(2d+b)(2n+k+1+1/2)^2/2} \\ &= \sum_{s=-\infty}^{\infty} q^{(2d+b)(s+1/2)^2/2}, \end{aligned}$$

where we have used (2.12) in the first equality. □

Using (2.7) and (2.8), we deduce that

$$\mathcal{A}_{b,d}^2 - \mathcal{B}_{b,d}^2 = 4\vartheta_2(q^{2d+b})\vartheta_2(q^{2d-b})\vartheta_3(q^{2d+b})\vartheta_3(q^{2d-b}).$$

Next, it is known from Jacobi's triple product identity that

$$\vartheta_2(q) = 2q^{1/4} \prod_{j=1}^{\infty} (1 - q^{2j})(1 + q^{2j})^2$$

and

$$\vartheta_3(q) = \prod_{j=1}^{\infty} (1 - q^{2j})(1 + q^{2j-1})^2.$$

Therefore,

$$(2.13) \quad 2\vartheta_2(q^2)\vartheta_3(q^2) = \vartheta_2^2(q).$$

Replacing q^2 by q and using (2.9), we deduce that

$$\mathcal{A}_{b,d}^2 - \mathcal{B}_{b,d}^2 = \mathcal{C}_{b,d}^2$$

and the proof of (1.5) is complete.

It is possible to derive (2.13) without using Jacobi's triple product identity. For more details, see [4, p. 58].

We end this note with the observation that when $d = 1$ and $b = 0$, (1.5) is given by

$$(2.14) \quad \left(\sum_{m,n=-\infty}^{\infty} q^{m^2+n^2} \right)^2 = \left(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{m^2+n^2} \right)^2 + \left(2 \sum_{m,n=-\infty}^{\infty} q^{2((m+1/2)^2+n^2)} \right)^2.$$

Note that this identity reduces to

$$(2.15) \quad \vartheta_3^4(q) = \vartheta_4^4(q) + 4\vartheta_2^2(q^2)\vartheta_3^2(q^2).$$

Note that by (2.13), we arrive at (1.1). Next, (2.15) can then be written as

$$(2.16) \quad \vartheta_3^4(q) + \vartheta_2^4(q) = \vartheta_3^4(q) - \vartheta_2^4(q) + 8\vartheta_2^2(q^2)\vartheta_3^2(q^2).$$

Identity (2.16) appeared in [1, p. 140] and the functions

$$\vartheta_3^4(q) + \vartheta_2^4(q), \quad \vartheta_3^4(q) - \vartheta_2^4(q) = \vartheta_4^4(q) \quad \text{and} \quad 2\vartheta_2^2(q)\vartheta_3^2(q)$$

play important roles in Ramanujan's theory of elliptic functions to the quartic base (see [3, Theorem 2.6(b)] and [1, (1.10),(1.11)]).

3. PROOF OF (1.6)

The proof of (1.6) is similar to the proof of (1.3). First, we need a lemma.

Lemma 3. *Let $0 < b < 4d$. Then*

$$(3.17) \quad A_{b,d} = \sum_{m,n=-\infty}^{\infty} q^{2(bm^2+bmndn^2)} = \vartheta_3(q^{2b})\vartheta_3(q^{2(4d-b)}) + \vartheta_2(q^{2b})\vartheta_2(q^{2(4d-b)}),$$

$$(3.18) \quad B_{b,d} = \sum_{m,n=-\infty}^{\infty} (-1)^{m-n} q^{bm^2+bmndn^2} = \vartheta_4(q^b)\vartheta_4(q^{4d-b})$$

and

$$(3.19) \quad C_{b,d} = \sum_{m,n=-\infty}^{\infty} q^{2(b(m+1/2)^2+b(m+1/2)n+dn^2)} = \vartheta_2(q^{2b})\vartheta_3(q^{2(4d-b)}) + \vartheta_3(q^{2b})\vartheta_2(q^{2(4d-b)}).$$

Proof. The proof of (3.17) follows by writing $A_{b,d}$ as

$$A_{b,d} = \sum_{m,n=-\infty}^{\infty} q^{2b(m+n/2)^2+n^2(4d-b)/2}.$$

Splitting the sum into two sums with one summing over even integers $n = 2\ell$ and the other summing over odd integers $n = 2\ell + 1$, we find that

$$\begin{aligned} A_{b,d} &= \sum_{m,\ell=-\infty}^{\infty} q^{2b(m+\ell)^2+2\ell^2(4d-b)} + \sum_{m,\ell=-\infty}^{\infty} q^{2b(m+\ell+1/2)^2+2(\ell+1/2)^2(4d-b)} \\ &= \vartheta_3(q^{2b})\vartheta_3(q^{2(4d-b)}) + \vartheta_2(q^{2b})\vartheta_2(q^{2(4d-b)}) \end{aligned}$$

and this completes the proof of (3.17). Next, write $B_{b,d}$ as

$$B_{b,d} = \sum_{m,n=-\infty}^{\infty} (-1)^{m-n} q^{b(m+n/2)^2+n^2(4d-b)/4}.$$

Splitting the sum into two sums with one summing over even integers $n = 2\ell$ and the other summing over odd integers $n = 2\ell + 1$ and using (2.11), we find that

$$\begin{aligned} B_{b,d} &= \sum_{m,\ell=-\infty}^{\infty} (-1)^m q^{2b(m+\ell)^2+2\ell^2(4d-b)} + \sum_{m,\ell=-\infty}^{\infty} q^{2b(m+\ell+1/2)^2+2(\ell+1/2)^2(4d-b)} \\ &= \sum_{m,\ell=-\infty}^{\infty} (-1)^\ell q^{(4d-b)\ell^2} \sum_{m=-\infty}^{\infty} (-1)^{m+\ell} q^{b(m+\ell)^2} \\ &= \vartheta_4(q^{4d-b})\vartheta_4(q^b) \end{aligned}$$

and (3.18) follows. Finally, to prove (3.19), write

$$C_{b,d} = \sum_{m,n=-\infty}^{\infty} q^{2b(m+1/2+n/2)^2+2n^2(4d-b)/4}.$$

Splitting the sum into two sums with one summing over even integers $n = 2\ell$ and the other summing over odd integers $n = 2\ell + 1$, we deduce that

$$\begin{aligned} C_{b,d} &= \sum_{m,\ell=-\infty}^{\infty} q^{2b(m+\ell+1/2)^2+2(2\ell)^2(4d-b)/4} + \sum_{m,\ell=-\infty}^{\infty} q^{2b(m+\ell+1)^2+2(2\ell+1)^2(4d-b)/4} \\ &= \vartheta_2(q^{2b})\vartheta_3(q^{8d-2b}) + \vartheta_3(q^{2b})\vartheta_2(q^{8d-2b}) \end{aligned}$$

and the proof of (3.19) is complete. \square

To complete the proof of (1.6), we note that

$$A_{b,d} - C_{b,d} = (\vartheta_3(q^{2b}) - \vartheta_2(q^{2b})) (\vartheta_3(q^{8d-2b}) - \vartheta_2(q^{8d-2b}))$$

and

$$A_{b,d} + C_{b,d} = (\vartheta_3(q^{2b}) + \vartheta_2(q^{2b})) (\vartheta_3(q^{8d-2b}) + \vartheta_2(q^{8d-2b})).$$

But it is immediate that

$$\vartheta_3(q^4) - \vartheta_2(q^4) = \vartheta_4(q)$$

and

$$\vartheta_3(q^4) + \vartheta_2(q^4) = \vartheta_3(q).$$

Therefore,

$$(\vartheta_3(q^4) - \vartheta_2(q^4))(\vartheta_3(q^4) + \vartheta_2(q^4)) = \vartheta_4(q)\vartheta_3(q) = \vartheta_4^2(q^2),$$

where the last equality follows from [2, p. 34]. Therefore,

$$\begin{aligned} A_{b,d}^2 - C_{b,d}^2 &= (\vartheta_3(q^{2b}) - \vartheta_2(q^{2b})) (\vartheta_3(q^{8d-2b}) - \vartheta_2(q^{8d-2b})) \\ &\quad \times (\vartheta_3(q^{2b}) + \vartheta_2(q^{2b})) (\vartheta_3(q^{8d-2b}) + \vartheta_2(q^{8d-2b})) \\ &= \vartheta_4^2(q^b)\vartheta_4^2(q^{4d-b}) = B_{b,d}^2 \end{aligned}$$

and the proof of (1.6) is complete.

4. CONCLUDING REMARKS

We have found infinitely many solutions to $X^2 + Y^2 = Z^2$ where X, Y and Z are theta series of weight 1. The Borweins' identity states that

$$(4.20) \quad \left(\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \right)^3 = \left(\sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2} \right)^3 + \left(\sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2} \right)^3$$

where $\omega = e^{2\pi i/3}$. This is the only example of a solution to $X^3 + Y^3 = Z^3$ with X, Y and Z being theta series of weight 1. Are there infinitely many solutions to $X^3 + Y^3 = Z^3$ where X, Y and Z are theta series of weight 1 besides (4.20)? This appears to be an interesting question.

REFERENCES

- [1] B.C. Berndt, H.H. Chan and W.-C. Liaw, *On Ramanujan's quartic theory of elliptic functions*, J. Number Theory **88** (2001), no. 1, 129–156.
- [2] J.M. Borwein and P.B. Borwein, *π and the AGM, a study in analytic number theory and computational complexity*, Wiley, Chichester (1987).
- [3] J.M. Borwein and P.B. Borwein, *A cubic counterpart of Jacobi's identity and the AGM*, Trans. Amer. Math. Soc. **323** (1991), 691–701.
- [4] H.H. Chan, *Theta Functions, Elliptic Functions and π* , De Gruyter, Berlin/Boston, 2020.
- [5] H.H. Chan, K.S. Chua and P. Solé, *Seven modular lattices and a septic base Jacobi identity*, J. Number Theory **99**, (2003), 361–372.
- [6] H.H. Chan, K.S. Chua and P. Solé, *Quadratic iterations to π associated to elliptic functions to the cubic and septic base*, Trans. Amer. Math. Soc. **355**, (4),(2002), 1505–1520.
- [7] K.S. Chua and P. Solé, *Jacobi identities, modular lattices, and modular towers*, European J. Combin. **25**, (4), (2004), 495–503.
- [8] P. Solé and P. Loyer, *U_n lattices, Construction B, and AGM iterations*, European J. Combin. **19**, (1998), 227–236.

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