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Euler Discriminant of Complements of Hyperplanes

Claudia Fevola and Saiei-Jaeyeong Matsubara-Heo

Abstract

The Euler discriminant of a family of very affine varieties is defined as the locus where the Euler characteristic drops. In this work, we study the Euler discriminant of families of complements of hyperplanes. We prove that the Euler discriminant is a hypersurface in the space of coefficients, and provide its defining equation in two cases: (1) when the coefficients are generic, and (2) when they are constrained to a proper subspace. In the generic case, we show that the multiplicities of the components can be recovered combinatorially. This analysis also recovers the singularities of an Euler integral. In the appendix, we discuss a relation to cosmological correlators.

1 Introduction

The study of Euler integrals has a long tradition dating back to works by Aomoto and Gelfand, among others. For an extensive overview, we refer the readers to [3, 4] and references therein. In this work, we are mainly interested in families of Euler integrals of the form

$$I_{\Gamma}(z) = \int_{\Gamma} h_{k+1}(\alpha; z)^{s_1} \cdots h_n(\alpha; z)^{s_n} \alpha_1^{\nu_1} \cdots \alpha_k^{\nu_k} \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_k}{\alpha_k}, \quad (1.1)$$

where $\alpha = (\alpha_1, \dots, \alpha_k)$ are coordinates on $(\mathbb{C}^*)^k$ and $h_j(\alpha; z) = z_{0j} + z_{1j}\alpha_1 + \dots + z_{kj}\alpha_k$ for $j = k+1, \dots, n$ are linear forms in the variables α and coefficients $z_{ij} \in \mathbb{C}$. The exponents s_j, ν_i take on complex values, making the integrand multivalued. Twisted de Rham (co)-homology provides a rigorous framework for defining Euler integrals. More precisely, the integration contour Γ is a twisted k -cycle on the complement of hyperplanes

$$X_z = (\mathbb{C}^*)^k \setminus (\{h_{k+1}(\alpha; z) = 0\} \cup \cdots \cup \{h_n(\alpha; z) = 0\}), \quad (1.2)$$

for fixed coefficients $z_{ij} \in \mathbb{C}$ in each linear form. The term *twisted* essentially means that Γ also records the choice of which branch of the integrand to integrate. We refer to [31, Section 3] for an introduction to twisted cycles and co-cycles. When the coefficients are generic, the signed Euler characteristic $(-1)^k \cdot \chi(X_z)$ of the hyperplanes complement counts the local solutions of an *A-hypergeometric system* and determines the dimension of the top-dimensional twisted de Rham cohomology group. For a discussion of these equalities in a broader framework, we refer to [1, Theorem 1.1]. Furthermore, for X_z as in (1.2), the signed Euler characteristic also coincides with the *beta invariant* of X_z , and with the number of bounded regions if the arrangement is real. The latter is known as Varchenko's conjecture

[39], proved by Orlik–Terao [34]. The value of $(-1)^k \cdot \chi(X_z)$ for arrangements in general position was computed in [4] via an explicit computation of a twisted cohomology basis of logarithmic forms.

In [15, 16], it is proposed to explore the variation of the Euler characteristic to determine the singularities of Feynman integrals, expressed in the form of Euler integrals with coefficients varying in a subspace of the space of physical parameters. This approach led to the re-discovery of the *Euler discriminant*, which was originally introduced by Esterov [14] in the context of studying singular solutions to polynomial systems. Here, we present a version of this definition adapted to our framework, with the general definition provided in Section 2.1. Let z denote the matrix of size $(k+1) \times (n-k)$ whose columns are the coefficients of the hyperplanes $h_j(\alpha; z)$. We denote $Z \subset \mathbb{C}^{(k+1)(n-k)}$ the subspace where these coefficients vary. The Euler discriminant $\nabla_\chi(Z)$ is the locus of points $z \in Z$ for which the Euler characteristic $(-1)^k \cdot \chi(X_z)$ is smaller than its generic value. When Z is the entire coefficients space, the Euler discriminant coincides with the zero locus of the *principal A -determinant*. This is a polynomial in the variables z_{ij} from the theory of A -hypergeometric systems [19]. We refer to Section 2.2 for its definition. In this article, we investigate the Euler discriminant associated to the complement of hyperplanes X_z for two distinct choices of coefficients spaces:

- (1). Z is $\mathbb{C}^{(k+1) \times (n-k)} \cap \{z_{ij} = 0 \text{ for some } i = 0, \dots, k, j = k+1, \dots, n\}$: in this case, the Euler discriminant coincides with the vanishing locus of a principal A -determinant. Theorem 3.9 describes the defining polynomial, expressed as a product of A -discriminants corresponding to various faces of a polytope. Importantly, we address the combinatorial problem of detecting the faces of the polytope that do not contribute to a discriminant. We characterize them using a condition on an associated bipartite graph.
- (2). Z is a general subvariety of $\mathbb{C}^{(k+1) \times (n-k)}$: in applications, coefficients are often constrained to a subspace of the full coefficients space. This setup is more general than the previous case in the sense that the principal A -determinant is not defined for general subvarieties. Using the Orlik-Solomon algebra associated to the family X_z , Theorem 4.1 describes the defining polynomial of the Euler discriminant once again through a determinantal formula.

As for Theorem 3.9, we note that [17, Proposition 4.4] determines the normal fan of the principal A -determinant, whereas Theorem 3.9 further provides an explicit description of its Newton polytope. In both cases (1) and (2) the matroid associated to the hyperplane arrangement is connected to the Euler discriminant. A clear consequence of Theorems 3.9 and 4.1 is that the Euler discriminant corresponds to locus where the matroid changes. This result aligns with [9, Theorem 1.3], where, motivated by the study of maximum likelihood (ML) degree in algebraic statistics, it was shown that the signed Euler characteristic is a matroid invariant. The paper also studies the resulting matroid stratification in several small-dimensional cases. An interesting problem would be to relate this stratification to the Euler stratification, defined in [41].

A key question raised in [16] concerns the relationship between the Euler discriminant and the singular locus of the D -module annihilating the Feynman integral. This question is particularly relevant in scattering amplitudes, where studying the differential equations

that annihilate Feynman integrals has been a deeply explored method to be able to evaluate these integrals. In Section 5, we formalize this question in the case of Euler integrals by defining an appropriate D -module. Theorem 5.3 shows that the singular locus of such a D -module and the Euler discriminant coincide in their codimension one part. Furthermore, these loci are purely one-codimensional in the case of complements of hyperplanes, yielding the expected equality in this case, as stated in Theorem 5.4.

Notably, examples of Euler integrals have appeared in theoretical cosmology [6, 8], where their integrands often takes the form of products of linear forms as in 1.1. The Appendix to this work offers a self-contained introduction to cosmological integrals as mathematical objects. We will use these integrals as concrete examples of Euler integrals of linear forms with coefficients parametrized by physical parameters and compute their Euler discriminant. **Outline.** The structure of the paper is as follows. In Section 2.1, we introduce the general notions of Euler discriminants and the principal A -determinant for Euler integrals. We explore their relationship with the Cayley configuration (Theorem 2.2). Section 3 focuses on the principal A -determinant for the choice of Z as in (1). We recall the definition of edge polytope and associated bipartite graph, which are central to proving Theorem 3.9. An implementation to compute the principal A -determinant from Theorem 3.9 is available at

$$\text{https://mathrepo.mis.mpg.de/EulerDiscriminantHyperplanes} \quad (1.3)$$

Subsection 3.4 presents a method for computing the multiplicities of the components. Section 4 examines the Euler discriminant for coefficients as in (2). In Section 5, we relate the Euler discriminant to the singular locus of a D -module, proving in Theorem 5.1 that these loci coincide in codimension one. Finally, the Appendix provides an introduction to cosmological integrals, illustrating their connection to Euler integrals of linear forms. This includes computational examples of Euler discriminants for cosmological integrals.

2 Euler discriminant and principal A -determinant

2.1 Euler discriminant

Let f_0, \dots, f_ℓ be Laurent polynomials, where each f_i is given as

$$f_i(\alpha; z) = \sum_{u \in A_i} z_{u,i} \alpha^u,$$

with variables $\alpha \in (\mathbb{C}^*)^n$ and fixed monomial support represented by a set $A_i \subset \mathbb{Z}^n$. The coefficients $z_i = (z_{u,i})_{u \in A_i}$ take values in $\mathbb{C}^{A_i} := \mathbb{C}^{|A_i|}$, and we assume that $\bigcup_{i=0}^{\ell} A_i$ spans the ambient lattice \mathbb{Z}^n . Once coefficients $z_i \in \mathbb{C}^{A_i}$ are fixed, the Laurent polynomials $f_i(\alpha; z)$ define a hypersurface in the algebraic torus $(\mathbb{C}^*)^n$:

$$V_{A,z} = V_{(\mathbb{C}^*)^n} \left(\prod_{i=0}^{\ell} f_i(\cdot; z) \right) = \left\{ \alpha \in (\mathbb{C}^*)^n \mid \prod_{i=0}^{\ell} f_i(\alpha; z) = 0 \right\} \quad (2.1)$$

Here, $f(\cdot; z)$ stands for the Laurent polynomial $f(\alpha; z)$ viewed as a regular function on $\alpha \in (\mathbb{C}^*)^n$. Consider $Z \subset \mathbb{C}^A$, a smooth subvariety of the full parameter space $\mathbb{C}^A := \mathbb{C}^{A_0} \times \cdots \times \mathbb{C}^{A_\ell}$. For each $z \in Z$, we set

$$X_z := (\mathbb{C}^*)^n \setminus V_{A,z}. \quad (2.2)$$

We denote χ_z the signed Euler characteristic $(-1)^n \cdot \chi(X_z)$, and χ^* the maximal value $\max\{\chi_z \mid z \in Z\}$. The *Euler discriminant* $\nabla_\chi(Z)$ is the locus

$$\nabla_\chi(Z) := \{z \in Z \mid \chi_z < \chi^*\}. \quad (2.3)$$

It follows from [16, Theorem 3.1] that $\nabla_\chi(Z)$ is a closed subvariety of Z .

Remark 2.1. Esterov first defined the Euler discriminant in [14], and it was later applied to Feynman integrals in [16]. Esterov's definition, in particular, also takes into account the multiplicities of each component: the Euler discriminant in [14, Definition 3.1] is the divisor given by the formal sum of the closure of the codimension-one strata P_i in Z where the Euler characteristic χ_z for $z \in P_i$ is equal to a fixed value χ_i . The multiplicity of each component is defined as the drop in Euler characteristic $\chi^* - \chi_i$. This definition is motivated by the notion of multiplicity, which was established for the generic setting where the subvariety Z is the entire parameter space, c.f. Theorem 2.5. \diamond

The *Cayley configuration* of $\{A_i\}_{0 \leq i \leq \ell}$ is represented by the columns of the $(n + \ell) \times \sum_{i=0}^{\ell} |A_i|$ matrix given by

$$\left(\begin{array}{ccc|ccc| \cdots |ccc} 0 & \cdots & 0 & 1 & \cdots & 1 & & & 0 & \cdots & 0 \\ & & \vdots & & & \vdots & & \cdots & & \vdots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & & 1 & \cdots & 1 \\ & & A_0 & & & A_1 & & & & & A_\ell \end{array} \right). \quad (2.4)$$

The Cayley configuration also has a natural interpretation as the monomial support of a multivariate Laurent polynomial in the variables $\beta = (\beta_1, \dots, \beta_\ell)$ and α as above, given by

$$f(\alpha, \beta) := f_0 + \sum_{i=1}^{\ell} \beta_i f_i(\alpha). \quad (2.5)$$

The following result relates the Euler characteristic of the variety defined by the Laurent polynomials f_0, \dots, f_ℓ and the one given by f .

Theorem 2.2. *Let $f_0, \dots, f_\ell \in \mathbb{C}[\alpha_1^{\pm 1}, \dots, \alpha_n^{\pm 1}]$ and set $T := \text{Spec } \mathbb{C}[\beta_1^{\pm 1}, \dots, \beta_\ell^{\pm 1}]$. Then*

$$\chi(V_{(\mathbb{C}^*)^n}(f_1 \cdots f_n)) = (-1)^\ell \cdot \chi(V_{(\mathbb{C}^*)^n \times T}(f)).$$

Proof. By an induction on ℓ , the proof is reduced to the case that $\ell = 1$. Hence, let $\ell = 1$ and denote $\mathcal{T} := (\mathbb{C}^*)^n \times T$. We consider the fiber bundle

$$\pi : \mathcal{T} \setminus (V_{\mathcal{T}}(f_0 + \beta_1 f_1) \cup V_{\mathcal{T}}(f_0 f_1)) \rightarrow (\mathbb{C}^*)^n \setminus V_{(\mathbb{C}^*)^n}(f_0 f_1)$$

defined by $\pi(\alpha, \beta_1) = \alpha$. Any fiber $\pi^{-1}(\alpha)$ is a complex plane with two points removed. By the multiplicative property of Euler characteristic, we obtain

$$\chi((\mathbb{C}^*)^n \setminus V_{(\mathbb{C}^*)^n}(f_0 f_1)) = -\chi(\mathcal{T} \setminus (V_{\mathcal{T}}(f_0 + \beta_1 f_1) \cup V_{\mathcal{T}}(f_0 f_1))). \quad (2.6)$$

The excision property of the Euler characteristic implies the following equalities:

$$\begin{aligned} \chi(\mathcal{T} \setminus V_{\mathcal{T}}(f_0 + \beta_1 f_1)) &= \chi(\mathcal{T} \setminus (V_{\mathcal{T}}(f_0 + \beta_1 f_1) \cup V_{\mathcal{T}}(f_0 f_1))) \\ &\quad + \chi(V_{\mathcal{T}}(f_0 f_1) \setminus V_{\mathcal{T}}(f_0 + \beta_1 f_1)); \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} \chi(V_{\mathcal{T}}(f_0 f_1) \setminus V_{\mathcal{T}}(f_0 + \beta_1 f_1)) &= \chi(V_{\mathcal{T}}(f_0) \setminus V_{\mathcal{T}}(f_1)) \\ &\quad + \chi(V_{\mathcal{T}}(f_1) \setminus V_{\mathcal{T}}(f_0)) = 0. \end{aligned} \quad (2.8)$$

Here, the last equality follows from the equality $V_{\mathcal{T}}(f_1) \setminus V_{\mathcal{T}}(f_0) = V_{(\mathbb{C}^*)^n}(f_1) \setminus V_{(\mathbb{C}^*)^n}(f_0) \times T$ and $\chi(T) = 0$. In view of equations (2.6), (2.7) and (2.8), the theorem for $\ell = 1$ follows. \square

In general, for a morphism of algebraic varieties $\pi : F \rightarrow Z$, we define the Euler discriminant $\nabla_{\chi}^{\pi}(Z)$ by the formula (2.3) where we replace χ_z by $|\chi(\pi^{-1}(z))|$. Using this notation, we can state the following corollary.

Corollary 2.3. *Let Z be a complex algebraic variety, and $f_0, \dots, f_{\ell} : (\mathbb{C}^*)^n \times Z \rightarrow \mathbb{C}$ regular functions. Consider the projections $\pi_0 : (\mathbb{C}^*)^n \times Z \setminus V(f_0 \cdots f_{\ell}) \rightarrow Z$ and $\pi : (\mathbb{C}^*)^{n+\ell} \times Z \setminus V(f_0 + \beta_1 f_1 + \cdots + \beta_{\ell} f_{\ell}) \rightarrow Z$. Then, one has the equality $\nabla_{\chi}^{\pi_0}(Z) = \nabla_{\chi}^{\pi}(Z)$.*

2.2 Principal A-determinant

We begin by recalling a definition of principal A -determinant [19, Chapter 10]. We follow the notation of Section 2.1, except that we restrict to the case of a unique Laurent polynomial $f = f_0$ with monomial support $A = A_0 \subset \mathbb{Z}^n$ such that the first entry of any $u \in A$ is 1. The A -discriminant variety ∇_A is defined as the closure of the image of the incidence variety

$$\left\{ (\alpha, z) \in (\mathbb{C}^*)^n \times \mathbb{C}^A \mid \frac{\partial f}{\partial \alpha_i}(\alpha; z) = 0, i = 1, \dots, n \right\}$$

through the natural projection $\mathbb{C}^A \times (\mathbb{C}^*)^n \rightarrow \mathbb{C}^A$. The A -discriminant variety is a proper homogeneous subvariety of \mathbb{C}^A . When ∇_A is a hypersurface, we say A is *non-defective*, and the reduced defining polynomial of ∇_A is called the A -discriminant, denoted by Δ_A . If ∇_A is not a hypersurface, we set $\Delta_A = 1$.

Let $\text{Conv}(A)$ be the polytope obtained as the convex hull of the columns of A in $\mathbb{R}^n = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}^n$. For a face Q of $\text{Conv}(A)$, we denote

$$f_Q(\alpha; z) := \sum_{u \in A \cap Q} z_u \alpha^u, \quad (2.9)$$

where $A \cap Q$ denotes the elements u of A such that $u \in Q$.

Next, we recall the notion of multiplicity necessary to define the principal A -determinant. Let $S \subset \mathbb{Z}^n$ be an abelian semigroup admissible in the sense of [19, Chapter 5, Definition 3.4]. We write $\Xi(S)$ for its group completion and $\Xi_{\mathbb{R}}$ for $\Xi(S) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $K(S)$ (resp. $K_+(S)$) denote the convex hull of S (resp. $S \setminus \{0\}$) in $\Xi_{\mathbb{R}}$. The subdiagram volume $u(S)$ is the normalized volume of the closure of the set $K(S) \setminus K_+(S)$, denoted $K_-(S)$, with respect to the volume form induced by $\Xi(S)$, see [19, Chapter 3, §3.D]:

$$u(S) := \text{vol}_{\Xi(S)}(K_-(S)). \quad (2.10)$$

For the trivial semigroup $S = \{0\}$, we set $u(S) := 1$. We write $S(A)$ for the subsemigroup of \mathbb{Z}^n generated by A and 0. In our setup, a face Q of $\text{Conv}(A)$ is identified with $S(Q)$ which is the subsemigroup of $S(A)$ generated by $A \cap Q$ and 0.

Given a face Q of $\text{Conv}(A)$, we define the number $i(Q, A)$ as the index

$$i(Q, A) := [\mathbb{Z}^n \cap \text{Lin}_{\mathbb{R}}(Q) : \text{Lin}_{\mathbb{Z}}(A \cap Q)]. \quad (2.11)$$

Here, for a subset $B \subset \mathbb{Z}^n$, $\text{Lin}_{\mathbb{R}}(B)$ (resp. $\text{Lin}_{\mathbb{Z}}(B)$) denotes the vector subspace spanned by B in $\mathbb{R}^n = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}^n$ (resp. \mathbb{Z}^n). We define a semigroup $S(A)/Q$ as the subsemigroup of $\mathbb{Z}^n / \mathbb{Z}^n \cap \text{Lin}_{\mathbb{R}}(Q)$ generated by A and 0. The principal A -determinant is defined as the product over all faces $Q < \text{Conv}(A)$:

$$E_A := \prod_{\substack{Q < \text{Conv}(A) \\ Q: \text{non-defective}}} \Delta_{A \cap Q}^{m_Q}, \quad (2.12)$$

where the multiplicity m_Q is given by $m_Q = i(Q, A) \cdot u(S(A)/Q)$.

Proposition 2.4. *The degree of the principal A -determinant equals $n \cdot \text{vol}_{\mathbb{Z}}(\text{Conv}(A))$.*

Proof. Let us recall the definition of the principal A -determinant of f as the A -resultant R_A ([19, Chapter 8, §2]):

$$E_A = R_A \left(f, \alpha_1 \frac{\partial f}{\partial \alpha_1}, \dots, \alpha_n \frac{\partial f}{\partial \alpha_n} \right) \in \mathbb{C}[z_u; u \in A]. \quad (2.13)$$

The equivalence with (2.12) is proved in [19, Chapter 10, Theorem 1.2]. For a non-zero complex number a , the principal A -determinant E_A is multiplied by $a^{n \cdot \text{vol}_{\mathbb{Z}}(\text{Conv}(A))}$ by replacing f by $a \cdot f$ ([19, Corollary 2.2, Chapter 8]). This proves the proposition. \square

The multiplicity m_Q can be interpreted in terms of the difference of Euler characteristics of very affine varieties [14, Theorem 2.36]. Although the following theorem is essentially known in the literature, we present a proof that uses Kashiwara's local index theorem [25].

Theorem 2.5. *For Q a non-defective face of $\text{Conv}(A)$, the multiplicity m_Q is given by*

$$\chi_{z^*} - \chi_{z_Q},$$

where $z^* \in \mathbb{C}^A \setminus \{E_A = 0\}$ and $z_Q \in \{z \in \mathbb{C}^A \mid \Delta_{A \cap Q}(z) = 0\}$ are generic points.

Proof. Firstly, note that the characteristic cycle for the GKZ system $M_A(c)$ with generic parameter c is given by

$$m_0 T_{\mathbb{C}^A}^* \mathbb{C}^A + \sum_{Q < \text{Conv}(A)} m_Q T_{\nabla_{A \cap Q}}^* \mathbb{C}^A,$$

where T^* denotes the conormal bundle of $\nabla_{A \cap Q}$. This result is stated in [30, 6.3] and its proof can be read off from the proof of [18, Theorem 4.6]. In view of Kashiwara's local index formula [25] (see also [11, Theorem 4.3.25]), it follows that $\chi_{z_Q} = m_0 - m_Q$. This concludes the proof since $m_0 = \chi_{z^*}$. \square

Let $\text{vol}(A)$ denote the lattice volume of the polytope $\text{Conv}(A)$. The proof of the following theorem is available in [2, Theorem 13], establishing a connection between the vanishing locus of the principal A -determinant and the decrease in the Euler characteristic of X_z .

Theorem 2.6. *The signed Euler characteristic $(-1)^n \cdot \chi(X_z)$ is equal to $\text{vol}(A)$ if and only if $z \in \mathbb{C}^A \setminus V_{\mathbb{C}^A}(E_A)$. Furthermore, when $E_A(z) = 0$, we have $(-1)^n \cdot \chi(X_z) < \text{vol}(A)$.*

Note that, when the subvariety Z as in (2.3) is the entire parameter space, we have $\nabla_\chi(\mathbb{C}^A) = V_{\mathbb{C}^A}(E_A)$. The following sections will concentrate on the case where the very affine variety X_z is the complement of hyperplanes. We will present formulas for the Euler discriminant and the principal A -determinant.

3 Principal A -determinant of a sparse arrangement

3.1 Sparse hyperplane arrangements

Any hyperplane arrangement in $\mathbb{P}^k = \text{Proj } \mathbb{C}[\alpha_0, \dots, \alpha_k]$ with $n + 1$ hyperplanes is specified by a point in the Grassmannian $\text{Gr}(k + 1, n + 1)$ which is realized as a quotient of the space of $(k + 1) \times (n + 1)$ matrices of rank $k + 1$ by the left multiplication by $\text{GL}(k + 1, \mathbb{C})$. Namely, a point $z \in \text{GL}(k + 1, \mathbb{C})$ represented by a $(k + 1) \times (n + 1)$ matrix $(z_{ij})_{\substack{i=0, \dots, k \\ j=0, \dots, n}}$ defines a collection of linear forms $\{h_j(\alpha; z) := z_{0j}\alpha_0 + \dots + z_{kj}\alpha_k\}_{j=0}^n$, whose vanishing locus defines a hyperplane arrangement

$$\mathcal{A}_z := \bigcup_{j=0}^n H_j, \tag{3.1}$$

where $H_j := \{\alpha \in \mathbb{P}^k \mid h_j(\alpha; z) = 0\}$. Note that H_j can be the whole projective space \mathbb{P}^k which we do not exclude. In this paper, we work on an open affine chart U of $\text{Gr}(k + 1, n + 1)$ where the matrix consisting of the first $k + 1$ columns of the matrix is invertible. By $\text{GL}(k + 1, \mathbb{C})$ action, any point in U is given by a $(k + 1) \times (n + 1)$ matrix

$$[E_{k+1} \mid z] \text{ where } z = \begin{bmatrix} z_{0k+1} & \cdots & z_{0n} \\ \vdots & \ddots & \vdots \\ z_{kk+1} & \cdots & z_{kn} \end{bmatrix}, \tag{3.2}$$

and E_{k+1} stands for the unit matrix of size $k + 1$. We identify U with the Euclidean space $\mathbb{C}^{(k+1)(n-k)}$. Consistently with Section 2.1, for any matrix $z \in \mathbb{C}^{(k+1) \times (n-k)}$, we set

$$X_z := \mathbb{P}^k \setminus \mathcal{A}_z. \quad (3.3)$$

We are interested in the study of its Euler discriminant $\nabla_x(Z)$. By [35, Theorem 3.3.7], the signed Euler characteristic of X_z is given by a combinatorial invariant called the *beta invariant*. The beta invariant is also equal to the number of bounded regions when the hyperplane arrangement is real ([35, Theorem 3.3.9]). Therefore, the Euler discriminant in such a case describes the locus for which some of the bounded chambers shrink.

When $Z = \mathbb{C}^{(k+1)(n-k)}$, the Euler discriminant coincides with the vanishing locus of a principal A -determinant which was described in [19, Chapter 9, §1]:

$$E_A(z) = \prod_{I, J: |I|=|J|} \det(z_{I, J}), \quad (3.4)$$

where the product runs over all subsets I , and J , of the set of rows, respectively columns, of the matrix z from (3.2).

We will focus on the case of a general subvariety Z in Section 4. In this section, we explore the case when Z is an intersection of coordinate hyperplanes. Also in this scenario, the Euler discriminant is given by the vanishing locus of a principal A -determinant.

Given a subspace Z as above, we define a bipartite graph G with vertex set $V(G) = \{0, 1, \dots, n\}$ and edge set $E(G) = \{ij \mid z_{ij} \neq 0\}$. In what follows, we write z_G for a point $z \in Z$ to stress that the non-zero entries are specified by the edge set of the graph G . In the context of graph theory, z_G is called the Edmonds matrix of G [38, 12.8.3]. We assume that any vertex of G is an endpoint of an edge. For a subset T of the vertex set $V(G)$, the symbol G_T denotes the subgraph of G induced by T . More precisely, G_T denotes a subgraph obtained by deleting all the vertices in $V(G) \setminus T$ and the edges connected to them. The *graph neighborhood* $N(G; T)$ is the set of vertices in $V(G)$ which are connected to an element of T by an edge of G . Let us set $V_1 := \{0, 1, \dots, k\}$ (the left vertex set) and $V_2 := \{k + 1, \dots, n\}$ (the right vertex set).

By assigning a variable to each vertex $i \in V(G)$, we consider the vector space \mathbb{R}^{n+1} and write e_i for its i -th standard unit vector. Here by convention, we note that $e_0 = (1, 0, \dots, 0)$. Consider the set

$$A_G := \{a_{ij} := e_i + e_j \mid ij \in E(G)\}. \quad (3.5)$$

Then, the convex hull of A_G is called an *edge polytope* of G and denoted by P_G .

We assume that G is a connected graph. Otherwise, G is decomposed into a disjoint union $G_1 \sqcup \dots \sqcup G_\ell$ with each G_i connected. The associated decomposition of vertices $V(G) = V(G_1) \sqcup \dots \sqcup V(G_\ell)$ induces a direct product decomposition $\mathbb{R}^{n+1} = \mathbb{R}^{V(G_1)} \times \dots \times \mathbb{R}^{V(G_\ell)}$. In this case, the edge polytope of G is the direct product of those of G_1, \dots, G_ℓ :

$$P_G = P_{G_1} \times \dots \times P_{G_\ell}. \quad (3.6)$$

For a comprehensive introduction to polytopes, we refer to [42]. For a polytope $P \subset \mathbb{R}^{n+1}$, a face $F \subset P$ is the set of points in the polytope P that maximizes a linear functional $\phi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$. The faces of dimension $\dim(P) - 1$ are called *facets*, where the dimension of a face is the dimension of the affine space spanned by its points.

Remark 3.1. It is clear that the edge polytope P_G is the convex hull of the monomial support of the polynomial arising from the Cayley configuration of the monomial supports of the linear forms h_j , see (2.5). More explicitly, consider the polynomial ring $\mathbb{C}[\alpha]$ where $\alpha = (\alpha_0, \dots, \alpha_k, \alpha_{k+1}, \dots, \alpha_n)$, then the polynomial associated to A_G is

$$f_G(\alpha; z) := \sum_{ij \in E(G)} z_{ij} \alpha_i \alpha_j. \quad (3.7)$$

When setting $\alpha_{k+1} = 1$ we get precisely the polynomial (2.5) in the context of this section. Corollary 2.3 then guarantees that to study the Euler discriminant of X_z we can look at the principal A -determinant for the matrix A_G . \diamond

3.2 Face structure of edge polytope of a bipartite graph

Ohsugi and Hibi [33] gave an explicit description of the facet structure of the edge polytope P_G associated to a bipartite graph G , and also investigated its dimension. The following result resumes the facet structure of P_G , as stated in [33, Theorem 1.7]:

Theorem 3.2 ([33]). *Let Q be a facet of P_G . Then, Q is one of the following types:*

- (a) *Q is associated to an ordinary vertex i_0 , i.e., a vertex such that $G_{V(G) \setminus \{i_0\}}$ is connected. In this case, F is given by the convex hull of a set $\{a_{ij} \mid i, j \neq i_0\}$.*
- (b) *F is associated to an acceptable subset $\emptyset \neq T \subset V_1$, i.e., a subset such that both $G_{T \cup N(G;T)}$ and $G_{V(G) \setminus (T \cup N(G;T))}$ are connected and the latter one is non-trivial in the sense that it has at least one edge. In this case, F is given by the convex hull of a set $\{a_{ij} \mid ij \in E(G_{T \cup N(G;T)}) \cup E(G_{V(G) \setminus (T \cup N(G;T))})\}$.*

Thus, any facet of P_G is of the form $P_{G_{I \cup J}}$ for some $I \subset V_1$ and $J \subset V_2$, or $P_{G_{I_1 \cup J_1}} \times P_{G_{I_2 \cup J_2}}$ for some $I_1, I_2 \subset V_1$ and $J_1, J_2 \subset V_2$. In each case, the graphs $G_{I \cup J}$, $G_{I_1 \cup J_1}$ and $G_{I_2 \cup J_2}$ are all connected. Note that a face of the product $P_{G_{I_1 \cup J_1}} \times P_{G_{I_2 \cup J_2}}$ is a product of faces of $P_{G_{I_1 \cup J_1}}$ and $P_{G_{I_2 \cup J_2}}$. In the next two results, we characterize the facet structure of an edge polytope. In particular, the following lemma is an immediate consequence of consecutively considering facets of a facet of P_G and applying Theorem 3.2.

Lemma 3.3. *For any face Q of P_G , there exist disjoint subsets $I_1, \dots, I_\ell \subset V_1$ and $J_1, \dots, J_\ell \subset V_2$ such that*

1. $G_{I_i \cup J_i}$ is connected for any $i = 1, \dots, \ell$
2. $Q = P_{G_{I_1 \cup J_1}} \times \dots \times P_{G_{I_\ell \cup J_\ell}}$

Conversely, the next lemma shows that any pair $(I, J) \in V_1 \times V_2$ gives rise to a face of P_G . We denote $Q_{I,J}$ the convex hull of the set $\{a_{ij} \mid ij \in E(G_{I \cup J})\}$.

Lemma 3.4. *Let $\emptyset \neq I \subset V_1$ and $\emptyset \neq J \subset V_2$. Then, $Q_{I,J}$ is a face of the polytope P_G .*

Proof. We may assume that $\{a_{ij} \mid ij \in E(G_{I \cup J})\}$ is non-empty. Let $\phi_i : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be the i -th coordinate projection. Then, it is easily seen that the linear functional that maximize the face $Q_{I,J}$ is given by $\phi_{I,J} := \sum_{i \in I} \phi_i + \sum_{j \in J} \phi_j$. \square

Example 3.5 ($n = 6, k = 2$). Let G be the bipartite graph with $V(G) = \{0, 1, \dots, 6\}$, and $E(G) = \{03, 04, 13, 15, 16, 24, 25, 26\}$. This graph will serve as running example for the results in the rest of the paper.

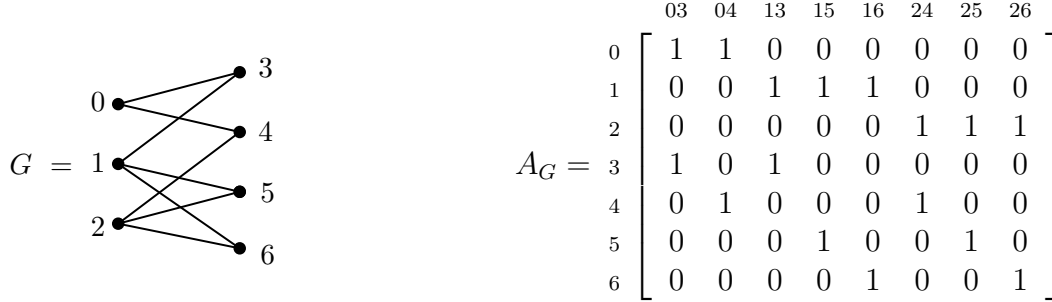


Figure 1: The bipartite graph G and the matrix whose columns give the set A_G from (3.5).

The edge polytope $P_G \subset \mathbb{R}^6$ is the 5-dimensional polytope obtained as the convex hull of the columns of the matrix A_G in Figure 1. We computed using `Oscar.jl` [36] that the f -vector is $(8, 26, 41, 31, 10)$ and the normalized volume $\text{vol}(P_G)$ equals 5. One can check that a facet of such a polytope is given by $P_{G_{I_1 \cup J_1}}$ for $I_1 = \{0, 1, 2\}$ and $J_1 = \{3, 4, 6\}$, while one of the codimension-2 faces is $P_{G_{I_2 \cup J_2}}$ for $I_2 = \{0, 1, 2\}$ and $J_2 = \{3, 4\}$. \diamond

3.3 Principal A-determinant of a sparse arrangement

In what follows, we will use the previous results on the face structure of an edge polytope P_G to derive an explicit formula for the corresponding principal A -determinant, namely the one for the polynomial $f_G(\alpha; z)$ defined in Remark 3.1.

A subset m of $E(G)$ is called a *matching* if no pair of edges in m is adjacent, i.e., they share a common vertex. We say that m is a *perfect matching*, if it covers the whole set $V(G)$. While, we say that m *saturates* V_1 if it covers the whole V_1 . The following characterization of the existence of a perfect matching is due to Hall ([32, Theorem 5.2]):

Theorem 3.6 ([32]). *Suppose that $|V_1| \leq |V_2|$. Then, the graph G has a matching which saturates V_1 if and only if, for any subset $W \subset V_1$, it holds that $|W| \leq |N(G; W)|$.*

In view of Theorem 3.6, let us assume $|V_1| = |V_2|$. We define the following condition on G :

- (*) $|V_1| = 1$ and G is connected, or $|V_1| > 1$ and for any non-empty, proper subset W of V_1 , the inequality $|W| < |N(G; W)|$ holds.

Note that, there is a clear relation between the hypothesis on the matching on the graph and the determinant of the Edmonds matrix z_G . In fact, it is readily seen that G has a perfect matching if and only if the determinant $\det(z_G)$ is not identically zero. Furthermore, condition (*) corresponds to the irreducibility of the polynomial $\det(z_G)$:

Proposition 3.7. *Let G be a bipartite graph with $|V_1| = |V_2|$. Then, $\det(z_G)$ is a non-zero irreducible polynomial in the variables z_{ij} over any field if and only if G satisfies condition $(*)$.*

Proof. Suppose that G does not satisfy $(*)$. When $|V_1| = 1$, it is clear that $\det(z_G)$ is zero since G is disconnected, hence there is no edge. Suppose that $|V_1| \geq 2$. By our assumption, we can find a non-empty proper subset W of V_1 such that $|W| = |N(G; W)|$. By the definition of graph neighborhood, the matrix z_G takes the following block-triangular form:

$$z_G = \begin{array}{c} \\ \\ \\ \\ \end{array} \begin{array}{cc} & M & V_2 \setminus M \\ W & \begin{array}{|c|c|} \hline * & 0 \\ \hline \end{array} \\ V_1 \setminus W & \begin{array}{|c|c|} \hline * & * \\ \hline \end{array} \end{array} \quad (3.8)$$

where $M := N(G; W)$. It follows that $\det(z_G)$ is either reducible or zero. The argument above shows the necessary condition.

Let us prove that the condition $(*)$ implies that $\det(z_G)$ is non-zero and irreducible. First of all, let $e = ij \in E(G)$ and write G' for the subgraph of G obtained by removing the vertices on e . Let i be the vertex on e which lies on V_1 . For any subset W of $V_1 \setminus \{i\}$, we have $|N(G'; W)| \geq |N(G; W)| - 1$. Therefore, condition $(*)$ implies that the graph G' verifies the hypothesis of Theorem 3.6. By cofactor expansion, we obtain that

$$\det(z_G) = \pm z_e \cdot \det(z_{G'}) + (\text{terms that do not contain } z_{ij}). \quad (3.9)$$

Since $\det(z_{G'}) \neq 0$, we proved that z_{ij} appears in $\det(z_G)$ for any edge $e = ij$ of G . Now, G having a perfect matching is equivalent to each connected component of G having a perfect matching. If G has a connected component H that is strictly smaller than G , we obtain $|N(G; V_1 \cap V(H))| = |V_1 \cap V(H)|$. This contradicts the condition $(*)$. Hence, G is connected. Let $\det(z_G) = p(z) \cdot q(z)$ be a decomposition. Fixing an edge $e = ij$ of G we may assume that z_{ij} appears in $p(z)$. By the definition of determinant, any element $z_{i'j}$ ($i'j \in E(G)$) or $z_{ij'}$ ($ij' \in E(G)$) must appear in $p(z)$. The connectedness of G proves that any variable z_{ij} ($ij \in E(G)$) belongs to $p(z)$, hence q is a scalar. \square

Before presenting the main theorem of this section, we must first introduce the following lemma, which is established using linear algebra.

Lemma 3.8. *For a connected bipartite graph G , one has an identity*

$$\mathbb{Z}^{n+1} \cap \text{Ker}_{\mathbb{R}} \phi = \sum_{ij \in E(G)} \mathbb{Z} \cdot a_{ij}, \quad (3.10)$$

where $\phi := \phi_0 + \dots + \phi_k - \phi_{k+1} - \dots - \phi_n$ and a_{ij} as in (3.5).

Proof. By definition, the right-hand side of (3.10) is contained in the left-hand side. On the other hand, for any $0 \leq i \leq k$ and $k+1 \leq j \leq n$, a vector $e_i + e_j$ belongs to the right-hand side of (3.10). In fact, there is a sequence $i_1 j_1, i_2 j_1, i_2 j_2, \dots, i_\ell j_\ell \in E(G)$ with $i_1 = i$ and $j_\ell = j$ since G is connected. It follows that $e_i + e_j = a_{i_1 j_1} - a_{i_2 j_1} + a_{i_2 j_2} - \dots + a_{i_\ell j_\ell}$. Since any element of the right-hand side of (3.10) can be written as a \mathbb{Z} -linear combination of vectors $e_i + e_j$, the lemma follows. \square

We can finally state the main result of this section.

Theorem 3.9. *Let G be a connected bipartite graph. Then, one has a formula*

$$E_{A_G}(z) = \prod_{\substack{I, J: |I|=|J|, \\ G_{I \cup J} \text{ is connected and } (*)}} \det(z_{I, J})^{u(S(A_G)/Q_{I, J})}, \quad (3.11)$$

where the product runs over all non-empty subsets $I \subset V_1$ and $J \subset V_2$, and $z_{I, J}$ denotes the submatrix of z_G obtained by selecting the rows in I and the columns in J .

Proof. To simplify the notation, we write A for A_G below. Let Q be a face of P_G and let H be the corresponding subgraph of G , i.e., $P_H = Q$ as in Lemma 3.3. We denote $f = f_G(\alpha; z)$ as in (3.7). We divide the proof into three main cases.

- (i) The subgraph H has several connected components: let $H_1 = G_{I_1 \cup J_1}, \dots, H_\ell = G_{I_\ell \cup J_\ell}$ be its connected components ($\ell \geq 2$). The face polynomial, defined in (2.9), is given by $f_Q = \sum_{a=1}^{\ell} \left(\sum_{i \in I_a, j \in J_a} z_{ij} \alpha_i \alpha_j \right)$. Let us denote $\alpha_{I_a} = (\alpha_i)_{i \in I_a}$ and $\alpha_{J_a} = (\alpha_i)_{i \in J_a}$ for $a = 1, \dots, \ell$. Then, the vanishing locus of $(A \cap Q)$ -discriminant is given by the Zariski closure of the set:

$$\nabla_{A \cap Q} = \bigcap_{a=1}^{\ell} \left\{ z \mid \exists (\alpha_{I_a}, \alpha_{J_a}) \in (\mathbb{C}^*)^{I_a} \times (\mathbb{C}^*)^{J_a} : \alpha_{I_a} \cdot z_{I_a, J_a} = z_{I_a, J_a} \cdot \alpha_{J_a}^T = 0 \right\}.$$

Therefore, by definition, $\nabla_{A \cap Q}$ is the union of proper varieties. It follows that the variety $\overline{\nabla_{A \cap Q}}$ has codimension higher than one. Thus, the $(A \cap Q)$ -discriminant is 1.

- (ii) The subgraph H is connected and of the form $H = G_{I \cup J}$ with $|I| < |J|$: we distinguish two sub-cases.

(ii-a) We assume there exists a matching m which saturates I : let us take a generic point z from the $(A \cap Q)$ -discriminant variety. By definition, there exists a row vector $\alpha_I \in (\mathbb{C}^*)^I$ such that $\alpha_I \cdot z_{I, J} = 0$, which means that the set of row vectors of $z_{I, J}$ is linearly dependent. Therefore, for any subset $J_1 \subset J$ with $|J_1| = |I|$, the minor $\det(z_{I, J_1})$ must vanish at z . Let $J_1 \subset J$ be the subset consisting of $j \in J$ that appear in m , then $\det(z_{I, J_1})$ is not a zero polynomial. On the other hand, there must exist a vertex $i_1 \in I$ which is connected to $j_2 \in J \setminus J_1$ since $G_{I \cup J}$ is connected. We write $j_1 \in J_1$ for the vertex paired with i_1 in m . We define the new matching $m' := (m \setminus \{(i_1, j_1)\}) \cup \{(i_1, j_2)\}$ and set $J_2 := (J_1 \setminus \{j_1\}) \cup \{j_2\}$. We obtain that $\det(z_{I, J_2})$ is not a zero polynomial. Summing up the argument above, the $(A \cap Q)$ -discriminant variety is contained in the intersection $\{\det(z_{I, J_1}) = 0\} \cap \{\det(z_{I, J_2}) = 0\}$. We conclude that the $(A \cap Q)$ -discriminant variety has codimension higher than one.

(ii-b) We assume that there is no matching m that saturates I : by definition of the $(A \cap Q)$ -discriminant, there exists a vector $\alpha_I \in (\mathbb{C}^*)^I$ such that $\alpha_I \cdot z_{I, J} = 0$. Theorem 3.6 and the connectedness of $G_{I \cup J}$ imply that there exists a non-empty proper subset $\emptyset \neq W \subsetneq I$ such that $|N(G_{I \cup J}; W)| < |W|$. It follows that

$$\alpha_{(I \setminus W)} \cdot z_{I \setminus W, J \setminus N(G_{I \cup J}; W)} = 0.$$

Since $|I \setminus W| < |J \setminus N(G_{I \cup J}; W)|$, there exist subsets $I' \subset I \setminus W$ and $J' \subset J \setminus N(G_{I \cup J}; W)$ such that $|I'| < |J'|$ and $G_{I' \cup J'}$ is a connected component of $G_{(I \setminus W) \cup (J \setminus N(G_{I \cup J}; W))}$. It follows by construction that $\alpha_{(I')} \cdot z_{I', J'} = 0$. If there exists a matching m of $G_{I' \cup J'}$ which saturates I' , the same argument as (ii-a) shows that the $(A \cap Q)$ -discriminant has codimension higher than one. If there is no such matching m , we repeat the argument above to find smaller subsets I'', J'' such that $|I''| < |J''|$ and $G_{I'' \cup J''}$ is connected. Repeating this argument eventually yields the case (ii-a).

(iii) The subgraph H is connected and of the form $H = G_{I \cup J}$ with $|I| = |J|$:
we distinguish two sub-cases.

(iii-a) The graph $H = G_{I \cup J}$ does not satisfy (*): in view of the proof of Proposition 3.7, $z_{I, J}$ is decomposed into a block triangular form as in (3.8). If z is a generic point of the $(A \cap Q)$ -discriminant variety, there exists $\alpha_I \in (\mathbb{C}^*)^{|I|}$ and $\alpha_J \in (\mathbb{C}^*)^{|J|}$ such that $\alpha_I \cdot z_{I, J} = 0$ and $z_{I, J} \cdot \alpha_J^T = 0$. Let z_{I_1, J_1} be its upper-left block and let $\det(z_{I_2, J_2})$ be its lower-right block. Then, we obtain $z_{I_1, J_1} \cdot \alpha_{J_1}^T = 0$ and $\alpha_{I_2} \cdot z_{I_2, J_2} = 0$. This means that the F -discriminant variety is contained in $\{\det(z_{I_1, J_1}) = \det(z_{I_2, J_2}) = 0\}$. Thus, the $A \cap Q$ -discriminant is 1.

(iii-b) The graph $H = G_{I \cup J}$ satisfies (*): it is readily seen that the $A \cap Q$ -discriminant variety is contained in $\{\det(z_{I, J}) = 0\}$. Let us prove the other inclusion. For any $i \in I$ and $j \in J$, condition (*), together with Theorem 3.6, implies that $G_{I \cup J \setminus \{i, j\}}$ has a perfect matching. This means that the (i, j) -cofactor Δ_{ij} of the matrix $z_{I, J}$ is a non-zero polynomial. It follows from Laplace expansion of matrices that $\{\det(z_{I, J}) = 0\} \setminus \cup_{i, j} \{\Delta_{ij} = 0\}$ is contained in the $A \cap Q$ -discriminant variety. Taking the Zariski closure, we conclude that the $A \cap Q$ -discriminant is precisely given by $\det(z_{I, J})$.

Formula (3.11) follows since $i(Q_{I, J}, A) = 1$ by Lemma 3.8. Note that $G_{I \cup J}$ is connected when the condition (*) is satisfied. \square

Note that, in [17, Proposition 4.4], the secondary fan, i.e., the normal fan of the principal A -determinant, is determined as the normal fan of a polynomial f which is different from (3.11). The polynomial f is defined as a product of the form (3.11) where the indices I, J run over pairs that satisfy the condition of Theorem 3.6. The reduced polynomial of f is identical to that of (3.11).

Example 3.10 ($n = 6, k = 2$). Let G be as in Example 3.5. Figure 2 shows the matrix z_G and the hyperplane arrangement associated to it. Using Theorem 4.1 we compute the factors of the principal A -determinant $E_{A_G}(z_G)$:

$$z_{03}^3 z_{13}^3 z_{04}^3 z_{24}^3 z_{15}^2 z_{25}^2 z_{16}^2 z_{26}^2 (z_{15} z_{26} - z_{16} z_{25})^2 (z_{03} z_{24} z_{15} + z_{13} z_{04} z_{25}) (z_{03} z_{24} z_{16} + z_{13} z_{04} z_{26}). \quad (3.12)$$

This is a polynomial of degree $30 = 6 \cdot 5$, consistently with Proposition 2.4. Exponents are computed by the method as in Example 3.12 in §3.4. Clearly, the factors of the form z_{ij} correspond to vertices of the polytope and therefore to minors determined by subsets $I = \{i\}$ and $J = \{j\}$. The three non-linear factors correspond respectively to the subgraphs displayed in Figure 3. Choosing vertices, for instance, in the sets $I = \{0, 1, 2\}$ and $J = \{4, 5, 6\}$ leads to

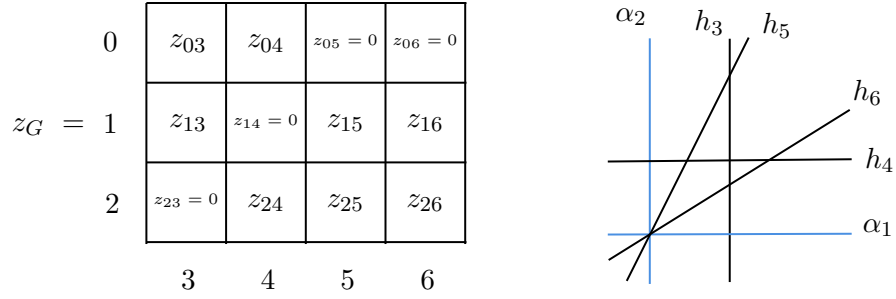


Figure 2: The matrix z_G with G as in Example 3.5 and the induced hyperplane arrangement.

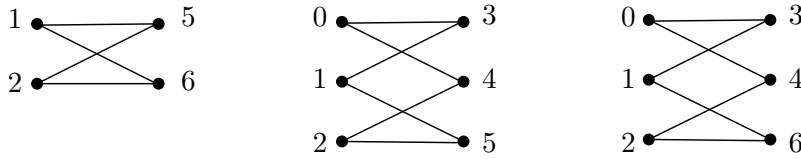


Figure 3: Subgraphs $Q_{12,56}$, $Q_{012,345}$, $Q_{012,346}$.

a non-connected graph and therefore to a reducible determinant of the associated Edmonds matrix, e.g., $z_{I,J} = z_{04}(z_{15}z_{26} - z_{16}z_{25})$. The choice $I = \{0, 1\}$ and $J = \{5, 6\}$ corresponds to a non-connected diagram which induces an identically vanishing determinant. Finally, the subgraph $G_{I \cup J}$ with $I = \{0, 1\}$ and $J = \{5, 6\}$ is an example of a connected graph which does not verify (*), leading again to an identically zero determinant.

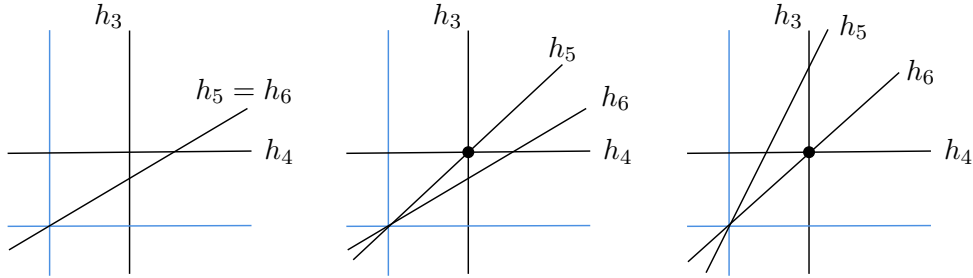


Figure 4: The degeneration of the generic hyperplane arrangement in Figure 2 to coefficient choices on the subspaces defined by the last three factors of $E_{A_G}(z_G)$ in (3.12).

Figure 2 (right) shows the hyperplane arrangement induced by a choice of coefficients z_{ij} in the matrix z_G that do not belong to the locus $\{E_{A_G}(z_G)\} = 0$. In Figure 4 instead we show the hyperplane arrangements associated to choices of coefficients which lie in the vanishing locus of the principal A -determinant. \diamond

3.4 Subdiagram volume from bipartite graphs

Computing the subdiagram volume based on its definition, c.f. (2.10), is often nontrivial and may pose computational challenges. We present a method tailored to the case where

the matrix A arises from a bipartite graph G , exploiting the graph's combinatorial structure. This approach uncovers intriguing connections with non-homogeneous toric ideals.

Let G be a bipartite graph. By abuse of notation, we write $A_G : \mathbb{Z}^{E(G)} \rightarrow \text{Ker}\phi \subset \mathbb{Z}^{n+1} = \mathbb{Z}^{V(G)}$ for the linear map which sends $e(ij)$ to a_{ij} defined by (3.5), where $e(ij)$ denotes the unit vector corresponding to the edge $ij \in E(G)$, and ϕ as in Lemma 3.8. The kernel of the map A_G is naturally identified with the first homology group of the graph G :

$$0 \longrightarrow H_1(G, \mathbb{Z}) \longrightarrow \mathbb{Z}^{E(G)} \xrightarrow{A_G} \text{Ker}\phi \longrightarrow 0. \quad (3.13)$$

Indeed, a cycle $C := \{i_1j_1, i_2j_1, i_2j_2, \dots, i_\ell j_\ell, i_1j_\ell\} \subset E(G)$ defines an element of the kernel of A_G by taking the linear combination:

$$e(i_1j_1) - e(i_2j_1) + e(i_2j_2) - \dots + e(i_\ell j_\ell) - e(i_1j_\ell). \quad (3.14)$$

The homology space $H_1(G, \mathbb{Z})$ is furthermore generated by the independent cycles of G . Let H be a connected subgraph of G defined by a non-defective face $Q < P_G$. Therefore, the subgraph H also admits a short exact sequence as in (3.13). The quotient of these short exact sequences determines another short exact sequence:

$$0 \longrightarrow H_1(G, H; \mathbb{Z}) \longrightarrow \mathbb{Z}^{E(G) \setminus E(H)} \xrightarrow{A_{G/H}} \mathbb{Z}^{n+1} / \text{Lin}_{\mathbb{Z}}(Q) \longrightarrow 0. \quad (3.15)$$

The image of $\mathbb{Z}_{\geq 0}^{E(G) \setminus E(H)}$ defines the semigroup $S(A)/Q$. Our goal is to identify this short exact sequence with the one representing the homology of a specific graph. To achieve this, we denote G/H the graph obtained from G by contracting the edges in $E(H)$. The kernel of the map $A_{G/H}$ can be identified with the homology group $H_1(G/H; \mathbb{Z})$ via the identification of homology groups $H_1(G, H; \mathbb{Z}) \simeq H_1(G/H; \mathbb{Z})$. Note that an even cycle of G/H is represented by a set $\{i_1j_1, i_2j_1, i_2j_2, \dots, i_\ell j_\ell, i_{\ell+1}j_\ell\} \subset E(G) \setminus E(H)$ where i_1 and $i_{\ell+1}$ are identified in the quotient G/H . Likewise, an odd cycle of G/H is represented by $\{i_1j_1, i_2j_1, i_2j_2, \dots, i_\ell j_\ell\} \subset E(G) \setminus E(H)$ where i_1 and j_ℓ are identified in the quotient G/H . Analogously to (3.14), every even cycle defines an element of $\text{Ker}A_{G/H}$, whereas an odd cycle, represented as above, gives rise to a sum

$$e(i_1j_1) - e(i_2j_1) + e(i_2j_2) - \dots + e(i_\ell j_\ell).$$

On the other hand, there is an identification $\mathbb{Z}^{V(G) \setminus V(H)} \simeq \mathbb{Z}^{n+1} / \text{Lin}_{\mathbb{Z}}(Q)$ which is induced by the natural embedding $\mathbb{Z}^{V(G) \setminus V(H)} \hookrightarrow \mathbb{Z}^{n+1} = \mathbb{Z}^{V(G)}$.

Summing up the arguments above, the short exact sequence in (3.15) can be identified with the canonical short exact sequence associated to the homology group of the graph G/H :

$$0 \longrightarrow H_1(G/H; \mathbb{Z}) \longrightarrow \mathbb{Z}^{E(G/H)} \xrightarrow{A_{G/H}} \mathbb{Z}^{V(G) \setminus V(H)} \longrightarrow 0. \quad (3.16)$$

By construction, the matrix $A_{G/H}$ is obtained from A_G by removing columns labeled by $E(H)$ and rows labeled by $V(H)$. We have proved the following

Proposition 3.11. *Let G be a bipartite graph, and let P_G denote its edge polytope. Let A be the matrix whose columns are the vertices of P_G . Then for any non-defective face Q of P_G , the following holds:*

$$u(S(A)/Q) = \text{vol}_{\mathbb{Z}}(\text{Conv}(\{0\} \cup A_{G/H})) - \text{vol}_{\mathbb{Z}}(\text{Conv}(A_{G/H})). \quad (3.17)$$

Example 3.12. Let G be as in Example 3.5. Given a set $S \subset V(G)$, we denote G_S the subgraph of G which is obtained from G by selecting the vertices which appear in S and the edges attached to such vertices. Figure 5 illustrates the quotients graphs G/H for the choices of H in $\{G_{03}, G_{15}, G_{1256}, G_{012345}, G_{012346}\}$. Note that any other choice of subdiagram H corresponding to a single edge leads to one of the two topologies of graph in the first line of Figure 5. We provide some of the matrices $A_{G/H}$, which can be derived from the matrix

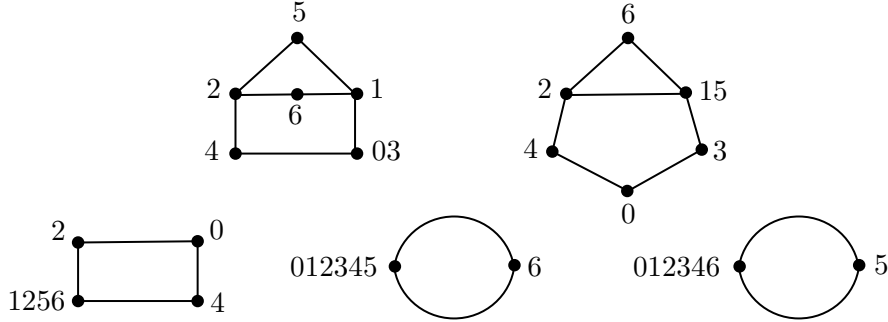


Figure 5: Quotients graphs G/H for $H = G_{03}, G_{15}, G_{1256}, G_{012345}, G_{012346}$.

A in Figure 1 by deleting specific rows and edges, as described above:

$$A_{G/G_{1256}} = \begin{matrix} & 03 & 04 & 13 & 24 \\ \begin{matrix} 0 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{matrix}, \quad A_{G/G_{012345}} = \begin{matrix} & 16 & 26 \\ 6 & \begin{bmatrix} 1 & 1 \end{bmatrix} \end{matrix}, \quad A_{G/G_{012346}} = \begin{matrix} & 15 & 25 \\ 5 & \begin{bmatrix} 1 & 1 \end{bmatrix} \end{matrix}.$$

The matrices $A_{G/H}$ for $H = G_{03}, G_{15}$ are obtained in the same manner. \diamond

In addition to the description given above, the subdiagram volume has also an interpretation as the multiplicity at the point 0 of a toric variety, see [19, Theorem 3.14]. We conclude this subsection by exploiting this perspective in our case of interest.

Consider $Y_{G/H}$ to be the affine toric variety associated to the linear map $A_{G/H}$. In general, $Y_{G/H}$ is not homogeneous. We write $I_{G/H} \subset \mathbb{C}[z_{ij}; ij \in E(G) \setminus E(H)]$ for its defining toric ideal. The following lemma is a direct consequence of the fact that $H_1(G/H; \mathbb{Z})$, as defined in (3.16), is generated by cycles of G/H . A practical consequence of this lemma is that $I_{G/H}$ is obtained from the toric ideal $I_G \subset \mathbb{C}[z_{ij}; ij \in E(G)]$, associated to G , by substituting $z_{ij} = 1$ for $ij \in E(Q)$.

Lemma 3.13. *Given a cycle C of G/H , we set f_C to be*

$$\begin{cases} z_{i_1 j_1} \cdots z_{i_\ell j_\ell} - z_{i_2 j_1} \cdots z_{i_\ell j_{\ell-1}} z_{i_{\ell+1} j_\ell} & (C = \{i_1 j_1, i_2 j_1, i_2 j_2, \dots, i_\ell j_\ell, i_{\ell+1} j_\ell\} \subset E(G) \setminus E(Q) : \text{even}) \\ z_{i_1 j_1} \cdots z_{i_\ell j_\ell} - z_{i_2 j_1} \cdots z_{i_\ell j_{\ell-1}} & (C = \{i_1 j_1, i_2 j_1, i_2 j_2, \dots, i_\ell j_\ell\} \subset E(G) \setminus E(Q) : \text{odd}). \end{cases}$$

Then, the set $\{f_C \mid C : \text{cycle of } G\}$ generates the ideal $I_{G/H}$.

Therefore, the generators obtained in the lemma can be used to compute the sub-diagram volume $u(S(A)/Q)$ as the local multiplicity $\text{mult}_0(Y_{G/H})$. Let $R_{G/H} := \mathbb{C}[z_{ij}; ij \in E(G) \setminus E(H)]/I_{G/H}$ and write $\mathfrak{m} \subset R_{G/H}$ for the maximal ideal corresponding to the origin. Then, $\text{mult}_0(Y_{G/H})$ can be read off from the leading coefficient of the Hilbert polynomial of the associated graded ring $\text{gr}_{\mathfrak{m}}(R_{G/H})$ with respect to the \mathfrak{m} -adic filtration of $R_{G/H}$. To compute the associated graded ring, a standard technique of Gröbner basis can be adapted. For these claims, see, e.g., [12, §12.1 and Proposition 15.28].

Example 3.14. Let G be as in Example 3.5. For the subgraphs $H = G_{03}, G_{15}$, the Hilbert polynomials of $\text{gr}_{\mathfrak{m}}(R_{G/H})$ are given by

$$\frac{1}{8}t^4 + \frac{11}{12}t^3 + \frac{19}{8}t^2 + \frac{31}{12}t + 1, \quad \frac{1}{12}t^4 + \frac{2}{3}t^3 + \frac{23}{12}t^2 + \frac{7}{3}t + 1.$$

The local multiplicity $\text{mult}_0(Y_{G/H})$ is thus given by $\frac{1}{8} \times 4! = 3$, and $\frac{1}{12} \times 4! = 2$, respectively. On the other hand, for subgraphs $H = G_{1256}, G_{012345}, G_{012346}$, the Hilbert polynomials are given by $t^2 + 2t + 1$, $t + 1$, and $t + 1$.

For the subgraph $H = G_{03}$, the toric ideal $I_{G/H}$ is generated by the following elements corresponding to three cycles in G/H :

$$z_{16} z_{25} - z_{15} z_{26}, \quad z_{04} z_{13} z_{26} - z_{16} z_{24}, \quad z_{04} z_{13} z_{25} - z_{15} z_{24}.$$

The lowest homogeneous parts of these polynomials $z_{16} z_{25} - z_{15} z_{26}, z_{16} z_{24}, z_{15} z_{24}$ generates an ideal $I' \subset \mathbb{C}[z_{04}, z_{13}, z_{15}, z_{16}, z_{24}, z_{25}, z_{26}]$. Using the algorithm [12, Proposition 15.28], we find that $\text{gr}_{\mathfrak{m}}(R_{G/H}) \simeq \mathbb{C}[z_{04}, z_{13}, z_{15}, z_{16}, z_{24}, z_{25}, z_{26}]/I'$. In several examples, we confirmed that the lowest homogeneous parts of binomials in Lemma (3.13) generate an ideal I' such that $\text{gr}_{\mathfrak{m}}(R_{G/H}) \simeq \mathbb{C}[z_{ij}; ij \in E(G) \setminus E(H)]/I'$, which seems to hold in general. In (1.3), we provide the code in Macaulay2 [20] to perform such a computation. \diamond

4 Determinants and Euler discriminants

In this section, we focus on the case when Z is a subvariety of $\mathbb{C}^{(k+1)(n-k)}$. Namely, we consider a matrix z of size $(k+1) \times (n-k)$, as in (3.2), except that here z varies in Z . Let us set $S := \{(I, J) \in \{0, \dots, k\} \times \{k+1, \dots, n\} : |I| = |J|, \det(z_{I,J}) \neq 0\}$. We define the reduced discriminant of Z as

$$E_{\chi}^{\text{red}}(z) := \prod_{(I,J) \in S} \det(z_{I,J}). \quad (4.1)$$

To emphasize the choice of Z , we may write $E_{\chi}^{\text{red}}(Z)$ for $E_{\chi}^{\text{red}}(z)$. Analogously to Section 3, for each $z \in Z$, the column vectors in (3.2) define a hyperplane arrangement \mathcal{A}_z in $\mathbb{P}^k(\mathbb{C}) = \text{Proj } \mathbb{C}[\alpha_0, \dots, \alpha_k]$, and its complement X_z , see (3.1) and (3.3), respectively.

Theorem 4.1. *Suppose that $\chi_z > 0$ for some $z \in Z$. Then, the following identity is satisfied:*

$$\nabla_{\chi}(Z) = \{z \in Z \mid E_{\chi}^{\text{red}}(z) = 0\}. \quad (4.2)$$

Proof. By abuse of notation, We write \mathcal{A}_z for the dehomogenized hyperplane arrangement that (3.2) determines by choosing $\{\alpha_0 = 0\}$ as the hyperplane at infinity. Suppose that $z_1, z_2 \in Z \setminus (\text{R.H.S.})$. Then, the intersection posets are equivalent: $L(\mathcal{A}_{z_1}) = L(\mathcal{A}_{z_2})$. This is because the matroids corresponding to the points z_1 and z_2 have the same basis set. Therefore, $\chi_z = \chi^*$ for any $z \notin (\text{R.H.S.})$ since the complement of the (R.H.S.) has non-empty intersection with the locus where the Euler characteristics takes its maximum. This proves the inclusion (L.H.S.) \subset (R.H.S.).

Let us show the other inclusion. We write M_z for the matroid of arrangements defined by the matrix (3.2). As long as $E_\chi^{\text{red}}(z) \neq 0$, the matroid M_z does not depend on the choice of z . We denote this matroid by M^* . Let $z \in Z$ such that $E_\chi^{\text{red}}(z) = 0$. We denote $A_z = A(M_z)$ the Orlik-Solomon algebra of M_z . To describe this algebra, we introduce the following notation. For an ordered subset $T = \{t_1, \dots, t_p\} \subset \{1, \dots, n\}$, we define

$$e_T := e_{t_1} \cdots e_{t_p} \quad \text{and} \quad \partial e_T := \sum_{k=1}^p (-1)^{k-1} e_{t_1} \cdots \widehat{e_{t_k}} \cdots e_{t_p},$$

where $\widehat{e_{t_k}}$ means that the element e_{t_k} is omitted. The Orlik-Solomon algebra A_z is then given as the quotient of the exterior algebra $\wedge^\bullet \mathbb{C}^{\{1, \dots, n\}}$ by the ideal $I(\mathcal{A}_z)$ generated by the set

$$\left\{ e_T \mid \bigcap_{t \in T} H_t = \emptyset \right\} \cup \left\{ \partial e_T \mid \bigcap_{t \in T} H_t \neq \emptyset \text{ and } T \text{ is dependent} \right\}.$$

The i -th graded part of A_z is denoted by A_z^i . Since any independent subset of M_z is that of M^* , there is a natural surjection

$$A(M^*) \rightarrow A_z \tag{4.3}$$

which respects the grading. We consider generic parameters $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and set

$$\omega_\lambda := \lambda_1 e_1 + \cdots + \lambda_n e_n,$$

where e_i is the element of the Orlik-Solomon algebra (either $A(M^*)$ or A_z) which corresponds to i -th hyperplane. The surjection (4.3) induces a surjection

$$H^k(M^*) := A^k(M^*) / \omega_\lambda A^{k-1}(M^*) \xrightarrow{\varphi} A_z^k / \omega_\lambda A_z^{k-1} =: H^k(M_z). \tag{4.4}$$

By the results of [13] (see also [37]), we have $\dim_{\mathbb{C}} H^k(M^*) = \chi^*$ and $\dim_{\mathbb{C}} H^k(M_z) = \chi_z$.

It remains to prove that the (4.4) has a non-trivial kernel. Let us consider a set $\tilde{I} = \{i_0, i_1, \dots, i_k\} \subset \{0, 1, \dots, n\}$ of cardinality $k+1$ with $i_0 < \cdots < i_k$. We set

$$\omega_{\tilde{I}} := \begin{cases} e_{i_1} \cdots e_{i_k} & i_0 = 0 \\ (e_{i_1} - e_{i_0}) \cdots (e_{i_k} - e_{i_0}) & i_0 \neq 0. \end{cases}$$

Since $E_\chi^{\text{red}}(z) = 0$, there exists a pair $(I, J) \in \{0, \dots, k\} \times \{k+1, \dots, n\}$ such that $|I| = |J|$ and $\det(z_{I,J}) = 0$ at z . We set $\tilde{I} := (\{0, \dots, k\} \setminus I) \cup J$. Then, $\omega_{\tilde{I}} \in H^k(M^*)$ is in the kernel of the morphism (4.4). Indeed, if $i_0 = 0$, then $\{i_1, \dots, i_k\}$ is a dependent subset of M_z and

$$\varphi(\omega_{\tilde{I}}) = e_{i_1 \cdots i_k} = e_{i_1} \partial(e_{i_1 \cdots i_k}) \in I(\mathcal{A}_z).$$

If $i_0 \neq 0$, this means that $\cap_{j=0}^k H_{i_j} \neq \emptyset$ and

$$\varphi(\omega_{\bar{I}}) = e_{i_1 \dots i_k} - e_{i_0 i_2 \dots i_k} + \dots = \partial e_{i_0 \dots i_k} \in I(\mathcal{A}_Z).$$

Hence, it is enough to show that $\omega_{\bar{I}}$ is a non-zero element of $H^k(M^*)$. Given $z^* \in Z$ generic, we write \mathcal{O}_* for the structure ring of the affine variety $\mathbb{C}^k \setminus L(\mathcal{A}_{z^*})$. By abuse of notation we write h_j for the dehomogenization of $h_j(\alpha; z)$ introduced in §3.1 with $\alpha_0 = 1$. We consider the ideal $\mathcal{I} \subset \mathcal{O}_*$ generated by a_1, \dots, a_k where a_i is the coefficient of $d\alpha_i$ of a one-form

$$\lambda_1 d \log(h_1) + \dots + \lambda_n d \log(h_n) \quad (4.5)$$

on $\mathbb{C}^k \setminus L(\mathcal{A}_{z^*})$. Then, there is an isomorphism $H^k(M^*) \simeq \mathcal{O}_*/\mathcal{I}$. Indeed, given an element $e_{i_1} \dots e_{i_k} \in A^k(M^*)$, we assign the coefficient of $d\alpha_1 \wedge \dots \wedge d\alpha_k$ in $d \log(h_{i_1}) \wedge \dots \wedge d \log(h_{i_k})$. Let V be the zero set of the one form (4.5). Note that the cardinality of V is $\chi^* > 0$ by [23, Theorem 1]. Through this correspondence, $\omega_{\bar{I}}$ corresponds to $\pm \det(z_{I,J}^*) \frac{d\alpha_1 \wedge \dots \wedge d\alpha_k}{h_{i_1} \dots h_{i_k}}$ when $i_0 = 0$ or to $\pm \det(z_{I,J}^*) \frac{d\alpha_1 \wedge \dots \wedge d\alpha_k}{h_{i_0} \dots h_{i_k}}$ when $i_0 \neq 0$, where $z^* \in Z$ is a generic point. Furthermore, there is an isomorphism $\mathcal{O}_*/\mathcal{I} \ni [f(x)] \mapsto (f(a))_{a \in V} \in \mathbb{C}^V$. Clearly, $\omega_{\bar{I}}$ corresponds to a non-zero vector in \mathbb{C}^V . \square

Note that the hypothesis that $\chi_z > 0$ is fundamental. Central arrangements, for instance, provide examples with $\chi_z = 0$ for all $z \in Z$ and non-trivial polynomial $E_\chi^{\text{red}}(z)$. A simple example is given for $k = 2, n = 1$, and $z = [0 \ z_{11} \ 0]^T$. Let

$$E_\chi^{\text{red}}(z) = \prod_{\Delta} \Delta(z),$$

be the irreducible decomposition of $E_\chi^{\text{red}}(z)$. On a generic point $z \in V_Z(\Delta)$, the number χ_z is independent of z . We set $m_\Delta := \chi^* - \chi_z$. It is convenient to set

$$E_\chi(z) := \prod_{\Delta} \Delta(z)^{m_\Delta}$$

to keep track of the drop of Euler characteristics.

Example 4.2. Consider the hyperplane arrangement introduced in Example 3.5. We compute the Euler discriminant for two choices of subspaces $Z_1, Z_2 \subset \mathbb{C}^{12}$. In particular, let Z_1 be parametrized by the matrix

$$C_1 = \begin{bmatrix} w_1 + w_2 & 1 & 0 & 0 \\ 1 & 0 & 1 & w_2 + w_3 \\ 0 & w_1 - w_3 & w_1 + w_2 + w_3 & 1 \end{bmatrix}, \quad (4.6)$$

and $Z_2 = Z_1 \cap \{w_1 + w_2 + w_3 = 0\}$. Then, since Z_1 intersects transversally the subspace defined by the matrix z_G in Example 3.5, we have

$$\begin{aligned} E_\chi(Z_1) &= (w_1 + w_2)^3 (w_1 - w_3)^3 (w_2 + w_3)^2 (w_1 + w_2 + w_3)^2 \\ &\quad (w_1 w_2 + w_1 w_3 + w_2^2 + 2w_2 w_3 + w_3^2 - 1)^2 \\ &\quad (w_1^2 + w_1 w_2 - w_1 w_3 + w_1 - w_2 w_3 + w_2 + w_3) \\ &\quad (w_1^2 w_2 + w_1^2 w_3 + w_1 w_2^2 - w_1 w_3^2 - w_2^2 w_3 - w_2 w_3^2 + 1). \end{aligned}$$

Note that the multiplicities of the components remain unchanged from those in (3.12), consistently with [14, Theorem 3.3]. To compute these exponents using [14, Theorem 3.3], we employed the function `EulerDiscriminantQ` from the `julia` package `PLD.jl`, [16]. This function returns the value χ_{z_Q} for each component defined by $\{\Delta_{A \cap Q}(z) = 0\}$ of the principal A -determinant. For the subspace Z_2 , some of the minors in (3.12) identically vanish when plugging in the parametrization. In fact the value of the generic Euler characteristic drops to $\chi^*(Z_2) = 3$, and

$$E_\chi(Z_2) = (w_1 + w_2)^2(w_1 - w_3)^2(w_2 + w_3)^2 \\ (w_1^2w_2 + w_1^2w_3 + w_1w_2^2 - w_1w_3^2 - w_2^2w_3 - w_2w_3^2 + 1).$$

Note that, in this case, the exponents are not preserved in contrast to those in (3.12). \diamond

5 Euler discriminant and singular locus

In this section, we discuss a relation between Euler discriminants and singular locus of a \mathcal{D} -module which an Euler integral underlies. Note that f_i below do not have to be linear polynomials. We use a standard notation of D -modules as in [22]. In this section, we assume Z is a smooth algebraic subvariety of $\mathbb{C}^{(k+1)(n-k)}$.

5.1 D -modules and Euler discriminant

We first prove some general propositions on D -modules and then relate it to Gauss-Manin connection.

Let D_Z be the sheaf of differential operators on Z . Given a coherent D_Z -module M , its characteristic variety $\text{Ch}(M)$ is canonically defined as a closed conic subvariety of the cotangent bundle T^*Z . We write $\varpi : T^*Z \rightarrow Z$ for the natural projection. The singular locus $\text{Sing}(M)$ of M is defined by

$$\text{Sing}(M) = \varpi(\text{Ch}(M) \setminus T_Z^*Z),$$

where T_Z^*Z is the zero section of T^*Z . For a given point $z \in Z$, we write $\iota_z : \{z\} \hookrightarrow Z$ for the canonical embedding. For a holonomic D_Z -module M , we set

$$\chi_z(M) := \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{C}} \text{Ext}_{D_Z}^i(M, \mathcal{O}_z),$$

where \mathcal{O}_z is the ring of convergent series at z . We write $\chi^*(M)$ for the holonomic rank of M . Note that $\chi^*(M)$ is $\chi_z(M)$ for generic z . Thanks to the formula (5.6), the notations $\chi_z(M)$ and $\chi^*(M)$ are compatible with χ_z and χ^* . This observation motivates the following definition

$$\nabla_\chi(Z, M) := \{z \in Z \mid \chi_z(M) < \chi^*(M)\}.$$

Note that $\nabla_\chi(Z, M)$ may not be a closed subvariety of Z for general M . A simple counter example is given by local cohomology. Nonetheless, we can prove the following theorem.

Theorem 5.1. *Let M be a holonomic \mathcal{D}_Z -module and let $\text{Sing}^1(M)$ be the union of the codimension 1 components of the singular locus of M . Then, one has the inclusion*

$$\text{Sing}^1(M) \subset \overline{\nabla_\chi(Z, M)}. \quad (5.1)$$

Proof. Let z be a smooth point of $\text{Sing}^1(M)$. The multiplicity of the characteristic cycle at z is at least one. By Kashiwara's index formula, it follows that $\chi_z(M) < \chi^*(M)$, which implies $z \in \nabla_\chi(Z, M)$. The inclusion (5.1) follows by taking the Zariski closure. \square

5.2 Gauss-Manin connection

In this section, we will use the notation and concepts introduced in Section 2.1. Let $(s, \nu) = (s_0, \dots, s_\ell, \nu_1, \dots, \nu_n) \in \mathbb{C}^{\ell+1+n}$ be complex parameters. The total space of variables and coefficients $X := \{(\alpha, z) \in (\mathbb{C}^*)^n \times Z \mid \prod_{i=0}^\ell f_i(\alpha; z) \neq 0\}$ is fibered over Z via the natural projection $\pi : X \rightarrow Z$. For any integer k , the space of relative differential k -forms $\Omega_{X/Z}^k(X)$ is given by

$$\Omega_{X/Z}^k(X) := \sum_{1 \leq i_1 < \dots < i_k \leq n} \mathcal{O}(X) d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_k}.$$

We set

$$\omega := d_\alpha \log(f(\alpha; z)^{-s} \alpha^\nu) = - \sum_{i=0}^\ell s_i \frac{d_\alpha f_i(\alpha; z)}{f_i(\alpha; z)} + \sum_{i=1}^n \nu_i \frac{d\alpha_i}{\alpha_i} \in \Omega_{X/Z}^1(X).$$

The 1-form ω naturally defines the differential

$$\nabla_\omega := d_\alpha + \omega \wedge : \Omega_{X/Z}^p(X) \rightarrow \Omega_{X/Z}^{p+1}(X).$$

The cohomology space

$$H^n(X/Z, \omega) := \frac{\Omega_{X/Z}^n(X)}{\text{Im} \left(\nabla_\omega : \Omega_{X/Z}^{n-1}(X) \rightarrow \Omega_{X/Z}^n(X) \right)}$$

is naturally equipped with a structure of left D_Z -module: for a vector field ∂_z on Z we set

$$\nabla_{\partial_z}^{\text{GM}}[\xi(z)] := \left[\frac{\partial \xi}{\partial z}(z) - \sum_{i=0}^\ell s_i \frac{1}{f_i(\alpha; z)} \frac{\partial f_i}{\partial z}(\alpha; z) \xi(z) \right], \quad [\xi(z)] \in H^n(X/Z, \omega).$$

We call the D_Z -module $H^n(X/Z, \omega)$ the (n -th) Gauss-Manin connection.

Now we consider the case when M is given by the Gauss-Manin connection. In the language of D -modules, it is the so-called (0-th) direct image functor:

$$H^n(X/Z, \omega) \simeq H^0 \int_\pi \mathcal{O}_X f(\alpha; z)^{-s} \alpha^\nu =: M. \quad (5.2)$$

Here, $\mathcal{O}_X f(\alpha; z)^{-s} \alpha^\nu$ denotes the D_X -module associated to an integrable connection $\nabla : \mathcal{O}_X \rightarrow \Omega_X^1$ given by

$$\nabla g := dg - \sum_{i=0}^{\ell} s_i g \frac{df_i}{f_i} + \sum_{j=1}^n \nu_j g \frac{d\alpha_j}{\alpha_j} \quad (g \in \mathcal{O}_X),$$

where d denotes the exterior derivative on X . An integral of the form

$$\int_{\Gamma} f(\alpha; z)^{-s} \alpha^\nu \omega(\alpha; z) \quad (5.3)$$

for some cycle Γ and $\omega(\alpha; z) \in \Omega_{X/Z}^n(X)$ is a solution to the D_Z -module $H^n(X/Z, \omega)$. The integral in (5.3) is called *Euler integral*, see [1] and [31].

Lemma 5.2. *For generic (s, ν) , one has the vanishing theorem $L^j \iota_z^* M = 0$ for all $j \neq 0$.*

Proof. By the base change formula and the isomorphism in (5.2), it is enough to prove

$$H^{n+j}(X_z, \omega) = 0 \quad \text{for all } j \neq 0. \quad (5.4)$$

We first construct the sequence of subvarieties $Z = Z_n \supset Z_{n-1} \supset \cdots \supset Z_0$ such that $\dim_{\mathbb{C}} Z_i \leq i$, $Z_i \setminus Z_{i-1}$ is smooth and there exists a smooth variety X_i such that there exists an inclusion $i : \pi^{-1}(Z_i \setminus Z_{i-1}) \rightarrow X_i$, which gives a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(Z_i \setminus Z_{i-1}) & \xrightarrow{\pi} & Z_i \setminus Z_{i-1} \\ \downarrow & \nearrow & \\ X_i & & \end{array} \quad (5.5)$$

Here, X_i is equipped with a simple normal crossing divisor $D_i = \sum_j D_{ij}$ such that any finite intersection of D_{ij} is transversal to a fiber of π and $X_i \setminus D_i = \pi^{-1}(Z_i \setminus Z_{i-1})$. Such a sequence can be constructed as in [10, 6.15]. Indeed, one can construct Z_{n-1} and X_n with the desired property. Then, we take the smooth locus of Z_{n-1} and proceed inductively. Moreover, the construction of [10, 6.15] combined with the argument of [23, 2.3] proves that there is a linear form $\ell_{ij}(s, \nu) \neq 0$ so that $\exp\{2\pi\sqrt{-1} \cdot \ell_{ij}(s, \nu)\}$ is the local monodromy around D_{ij} of the local system that $f(\alpha; z)^{-s} \alpha^\nu$ defines on a fiber X_z . The identity (5.4) follows from the same argument as [1, Appendix]. The genericity assumption for the parameters is that $\ell_{ij}(s, \nu) \notin \mathbb{Z}$. \square

It follows from Lemma 5.2 that $\nabla_{\chi}(Z) = \nabla_{\chi}(Z, M)$. Indeed, one has

$$\mathbb{R}\mathrm{Hom}_{D_Z}(M, \mathcal{O}_z) = \mathbb{R}\mathrm{Hom}_{D_Z}(M, \widehat{\mathcal{O}}_z) = \mathbb{R}\mathrm{Hom}_{D_Z}(\mathbb{L}\iota_z^* M, \mathbb{C}),$$

where the first equality is a consequence of regular holonomicity of M and the comparison theorem [22, Proposition 7.3.1]. Thus, it follows that

$$\chi_z(M) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}_{D_Z}^i(M, \mathcal{O}_z) = \dim_{\mathbb{C}}(\iota_z^* M, \mathbb{C}) = \dim_{\mathbb{C}} H^n(X_z, \omega) = \chi_z. \quad (5.6)$$

Theorem 5.3. *For generic (s, ν) , we have $\nabla_\chi^1(Z) = \text{Sing}^1(M)$.*

Proof. From Theorem 5.1, we have $\nabla_\chi^1(Z) \supset \text{Sing}^1(M)$. The general theory of D -module shows that a D -module is an integrable connection outside its singular locus. This means that the Euler characteristics does not change on the nonsingular locus. This proves the inclusion $\nabla_\chi(Z) \subset \text{Sing}(M)$. \square

5.3 Arrangement case

Now let us assume that the homogeneous degrees of polynomials f_i all equal to one.

Theorem 5.4. *For generic (s, ν) , one has an identity*

$$\text{Sing}(M) = \nabla_\chi(Z). \quad (5.7)$$

Proof. Since $\nabla_\chi(Z)$ is purely one codimensional, it is enough to prove the inclusion $\text{Sing}(M) \subset \nabla_\chi(Z)$ in view of Theorem 5.3. Let us define an open subvariety $U \subset Z$ by $U = Z \setminus \nabla_\chi(Z)$. We prove the inclusion $\text{Ch}(M|_U) \subset T_U^*U$ where $M|_U$ is the restriction of M onto U . We construct a relative compactification of $\pi^{-1}(U)$ with respect to the projection $\pi : \pi^{-1}(U) \rightarrow U$. We consider the process of iterated blowing-ups ([35, Theorem 4.2.4], [40, §10.8]). The result is a blowing-up $\widetilde{\pi^{-1}(U)}$ of $\pi^{-1}(U)$ equipped with a normal crossing divisor D . The associated morphism $\widetilde{\pi^{-1}(U)} \rightarrow U$ is transversal with respect to the stratification that D defines. The result now follows from a standard lemma on the direct image. The following lemma follows from, e.g., [26, Theorem 4.27]. \square

Lemma 5.5. *Let Y, Z be smooth varieties and $f : Y \rightarrow Z$ be a proper morphism. Assume that a normally crossing divisor $D = \cup_{j=1}^N D_j$ of Y is given and f is transversal to the stratification that D defines. Let N be a D_Y -module. For a subset J of $\{1, \dots, n\}$, we set $D_J := \cap_{j \in J} D_j$. If*

$$\text{Char}(N) \subset \cup_{J \subset \{1, \dots, N\}} T_{D_J}^* Y,$$

then $\int_f N$ is an integrable connection. Here, $T_{D_J}^ Y$ denotes the conormal bundle of D_J .*

A Cosmological integrals in a nutshell

Theoretical cosmology aims at understanding the origin, evolution, and large-scale structure of the universe. An essential part of cosmology is knowing how matter and energy are distributed throughout the universe at any given time. A remarkable example of this is the *Cosmic Microwave Background* (CMB). *Cosmological correlators* are the statistical quantities that model such distributions. There are different ways of calculating cosmological correlators; the one we are interested in is the *wavefunction formalism*. It consists of two main steps: computing the wavefunction and then using it to calculate the correlators. In perturbation theory, it is sufficient to compute a finite set of *wavefunction coefficients* that characterize the wavefunction completely up to the perturbative order one is interested in.

This appendix gives a self-contained introduction to the *wavefunction coefficients* in flat space, aiming at explaining how they lead to physically meaningful examples of Euler integrals of products of linear forms [4].

Let $\mathcal{G}_{L,n}$ denote the set of connected graph with n external edges and L loops. Then, in perturbation theory, the n -th wavefunction coefficient can be expanded in an infinite sum over all possible Feynman diagrams $G \in \mathcal{G}_{L,n}$ given by

$$\Psi_n = \sum_{\substack{L \geq 0 \\ G \in \mathcal{G}_{L,n}}} \psi_G, \quad (\text{A.1})$$

where the ψ_G are integrals that become some expression of the physical parameters after the integration. In cosmology, when drawing Feynman diagrams, the n external edges connect the vertices in V to a horizontal line representing the *boundary*. See [7, Section 1] for some illustrations of Feynman diagrams for $n = 2, 3, 4, 5$. The method to associate an integral to each diagram is given by the *Feynman rules*. For an explicit definition of these rules we refer to [6, Section 2.2]. For physicists, it is important to note that we work with *conformally coupled scalar fields*. This enables the mathematical approach to capture the essential behavior of the system while remaining computationally feasible.

Under these assumptions, it is convenient to draw a simplified version of the Feynman diagrams where we only keep internal vertices and edges and truncate the n edges connecting to the boundary. In the following, let $G = (V, E)$ be a connected undirected graph, where V is the set of n vertices of G and E is its finite collection of edges. We write $V(G)$ and $E(G)$ when we want to emphasize that V and E are, respectively, the vertex and edge set of G .

To the vertices and edges of G , we associate complex parameters representing energies, $X = (X_i)_{i \in V}$ and $Y = (Y_{ij})_{ij \in E}$. Figure 6 illustrates some of the graphs we will use in our examples. In physics, the (X, Y) -variables are real and positive, but for practical applications it is useful to think of them in larger domains.

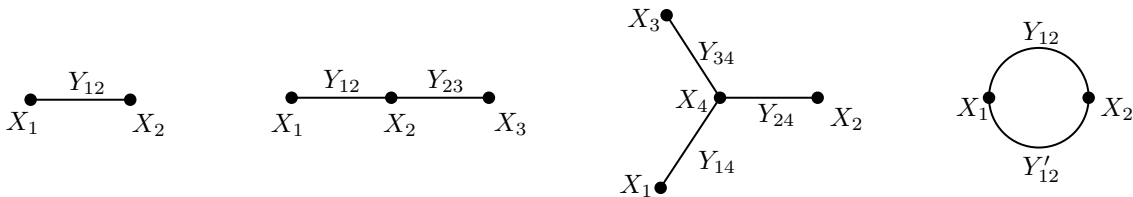


Figure 6: Two-site chain, three-site chain, four-site star, and one-loop bubble.

The first ingredient towards introducing Euler integrals arising in cosmology are the integrands of the terms in the expansion (A.1). By abusing notation, we will still refer to these as wavefunction coefficients in flat space and denote them ψ_G . However, in what follows, the ψ_G will be rational functions of the parameters (X, Y) . The next subsections provide a concise review of two alternative methods to derive the flat space wavefunction coefficients: the *recursion formula* and the *cosmological polytope* [7, 24, 28].

Recursion Formula The recursion formula, introduced in [7], represents the wavefunction coefficient ψ_G associated with a graph G as a combination of wavefunction coefficients

of subgraphs obtained by sequentially deleting individual edges from G . The core idea is to reduce to computing only wavefunction coefficients of graphs that consist of a single vertex and no edges. In this case, in fact, if X is the vertex's energy variable, the wavefunction coefficient is $1/X$, as explained in [7, Equation (2.12)]. Then, the recursion formula is

$$\psi_G(X, Y) = \frac{1}{\sum_{k \in V(G)} X_k} \cdot \sum_{ij \in E(G)} (\psi_{H_i} \cdot \psi_{H_j}), \quad (\text{A.2})$$

where H_i, H_j are the subgraphs containing the vertex v_i , (resp. v_j), obtained from G by deleting the edge $e = ij$. Furthermore, the variable X_i (resp. X_j) attached to v_i in H_i (resp. v_j in H_j) becomes $X_i + Y_{ij}$ (resp. $X_j + Y_{ij}$). The derivation of this formula is explicitly given in [7, Section 2.3].

Example A.1. Let G be the two-site chain illustrated in Figure 6. We have $n = 2$, and the energies associated to vertices and edges are (X_1, X_2, Y_{12}) . We denote ψ_2 the associated wavefunction coefficient. The recursion formula is resolved in one step:

$$\psi_2 = \frac{1}{X_1 + X_2} \left(\psi \left(\begin{array}{c} \bullet \\ X_1 + Y_{12} \end{array} \right) \cdot \psi \left(\begin{array}{c} \bullet \\ X_2 + Y_{12} \end{array} \right) \right) = \frac{1}{(X_1 + X_2)(X_1 + Y_{12})(X_2 + Y_{12})}$$

◇

Example A.2. In this example, we show an application of the recursion formula (A.2) to compute the flat space wave function ψ_3 associated with the three-site chain showed in Figure 6. Let $1/X := \frac{1}{X_1 + X_2 + X_3}$, then we have

$$\begin{aligned} \psi_3 &= \frac{1}{X} \left(\psi \left(\begin{array}{c} \bullet \\ X_1 + Y_{12} \end{array} \right) \cdot \psi \left(\begin{array}{c} \bullet \quad Y_{23} \\ X_2 + Y_{12} \quad X_3 \end{array} \right) + \psi \left(\begin{array}{c} \bullet \quad Y_{12} \\ X_1 \quad X_2 + Y_{23} \end{array} \right) \cdot \psi \left(\begin{array}{c} \bullet \\ X_3 + Y_{23} \end{array} \right) \right) \\ &= \frac{1}{X} \left(\frac{1}{X_1 + Y_{12}} \cdot \psi \left(\begin{array}{c} \bullet \quad Y_{23} \\ X_2 + Y_{12} \quad X_3 \end{array} \right) + \psi \left(\begin{array}{c} \bullet \quad Y_{12} \\ X_1 \quad X_2 + Y_{23} \end{array} \right) \cdot \frac{1}{X_3 + Y_{23}} \right) \\ &= \frac{1}{X} \left(\frac{1}{(X_1 + Y_{12})(X_2 + X_3 + Y_{12})} \cdot \psi \left(\begin{array}{c} \bullet \\ X_2 + Y_{12} + Y_{23} \end{array} \right) \psi \left(\begin{array}{c} \bullet \\ X_3 + Y_{23} \end{array} \right) \right. \\ &\quad \left. + \frac{1}{X_1 + X_2 + Y_{23}} \cdot \psi \left(\begin{array}{c} \bullet \\ X_1 + Y_{12} \end{array} \right) \psi \left(\begin{array}{c} \bullet \\ X_2 + Y_{12} + Y_{23} \end{array} \right) \cdot \frac{1}{X_3 + Y_{23}} \right) \\ &= \frac{1}{X} \left(\frac{1}{(X_1 + Y_{12})(X_2 + X_3 + Y_{12})(X_2 + Y_{12} + Y_{23})(X_3 + Y_{23})} \right. \\ &\quad \left. + \frac{1}{(X_1 + X_2 + Y_{23})(X_1 + Y_{12})(X_2 + Y_{12} + Y_{23})(X_3 + Y_{23})} \right). \end{aligned}$$

By finding a common denominator, we obtain the following rational function:

$$\frac{X_1 + X_3 + 2X_2 + Y_{12} + Y_{23}}{(X_1 + X_2 + X_3)(X_1 + Y_{12})(X_2 + Y_{12} + Y_{23})(X_3 + Y_{23})(X_1 + X_2 + Y_{23})(X_2 + X_3 + Y_{12})}$$

The use of different colors for the linear forms in the denominator will enhance the clarity of Example A.7 in the next section. ◇

Cosmological Polytope An alternative way to introduce the wavefunction coefficients is via the canonical form of a *positive geometry* [5]. For the definition of positive geometry and its canonical form we refer to [29, Definition 1]. The cosmological polytope \mathcal{P}_G of a graph G as above plays the role of a positive geometry which is built ad hoc to have as canonical form the differential form

$$\Omega_{\mathcal{P}_G} = \psi_G(X, Y) dX \wedge dY,$$

where $dX = dX_1 \wedge \dots \wedge dX_n$ and $dY = \wedge_{ij \in E} dY_{ij}$. For the general definition of this polytope as the convex hull of its vertices we refer to [28]. The facets of the cosmological polytope were characterized by Arkani-Hamed, Benincasa, and Postnikov in [7]. Such a characterization is resumed in [28, Theorem 2.1], where a complete description of the higher codimensional faces is also given.

The cosmological polytope \mathcal{P}_G associated with the graph $G = (V, E)$ lives in the space $\mathbb{R}^{|E|+|V|-1}$, with standard basis vectors $X = (X_1, \dots, X_n)$ and $Y = (Y_{ij})_{ij \in E}$. The facets of \mathcal{P}_G are in bijection with the connected subgraphs $H = (V(H), E(H))$, where a subgraph is another graph formed from a subset of the vertices and edges of G where all endpoints of the edges of H are in the vertex set of H . In particular, the facet F_H is the intersection of \mathcal{P}_G with the hyperplane

$$\sum_{v \in V(H)} X_v + \sum_{\substack{e = ij, \\ i \in V(H), j \notin V(H)}} Y_{ij} + \sum_{\substack{e = ij \notin E(H), \\ i \in V_H, j \in V(H)}} 2Y_{ij} = 0. \quad (\text{A.3})$$

The facet F_G associated with the full graph G and determined by the hyperplane defined by $\sum_{i=1}^n X_i = 0$ is called the *scattering facet*. One can immediately check that the hyperplanes appearing in the denominator of the rational function from Example A.2 are precisely the ones obtained by computing (A.3) for each subgraph of the three-site chain. It is an open problem to prove that the flat space wavefunction coefficient of a cosmological graph G , or more in general a graph, is a rational function with poles on the facet hyperplanes of the cosmological polytope. However, a challenge that arises when computing the canonical form of the cosmological polytope is the computation of the adjoint hypersurface. This is the polynomial that encodes the zeros of the flat space wave function. For more details about the adjoint of a polytope see [27]. This challenge motivates the study of triangulations of the cosmological polytope [24].

Example A.3. The cosmological polytope of the two-site chain is a triangle in \mathbb{R}^3 with coordinates $[X_1, X_2, Y]$ cut out by the hyperplanes $\{X_1 + X_2, X_1 + Y_{12}, X_2 + Y_{12}\}$. The adjoint hypersurface in this case is 1. \diamond

The f -vector of the cosmological polytope can be computed via the recursive formulas in Theorem 4.5 and Corollary 4.6 in [28]. In this way, one can compute the number of hyperplanes contributing in the arrangement that we will discuss in the next section.

Cosmological Integrals

Euler integrals of linear arise in the study of cosmological integrals. Closely analogous to Feynman integrals in particle physics, these objects were introduced in [6] for trees, and later

their definition was also extended to graphs with cycles in [21, 8]. Starting from the flat space wavefunction coefficients ψ_G , introduced in the previous section, we shift the variables X_i associated to the vertices by new complex variables α_i for $i = 1, \dots, n$ which will be the integration variables. Furthermore, an additional factor $\alpha^\varepsilon = \alpha_1^\varepsilon \cdots \alpha_n^\varepsilon$, where $\varepsilon \in \mathbb{C}$, together with the product of the edge variables are added to the integrand. The choice of ε parametrizes the cosmology in which the process takes place. This procedure gives the following integral

$$I_G(X, Y, \varepsilon) = \int_{\Gamma} 2^{n-1} \cdot \prod_{ij \in E} Y_{ij} \cdot \alpha^\varepsilon \cdot \psi_G(X_1 + \alpha_1, \dots, X_n + \alpha_n, Y) d\alpha, \quad (\text{A.4})$$

where $d\alpha = d\alpha_1 \wedge \cdots \wedge d\alpha_n$, and Γ is a twisted cycle on the complement of hyperplanes arrangement in $(\mathbb{C}^*)^n$ defined by the linear forms appearing in the denominators of the function $\psi_G(X + \alpha, Y)$, with $X + \alpha = (X_1 + \alpha_1, \dots, X_n + \alpha_n)$. In the context of cosmology, Γ is the positive orthant. More explicitly, given

$$\psi_G(X + \alpha, Y) = \frac{P(X + \alpha, Y)}{L_1(X + \alpha, Y) \cdots L_r(X + \alpha, Y)},$$

where $P, L_i \in \mathbb{C}[X, Y]$ are respectively the adjoint hypersurface and the facets of the cosmological polytope, as defined in the previous section, the twisted cycle Γ lives in the top-dimensional twisted de Rham homology group defined on the complement of hyperplanes

$$\begin{aligned} \mathcal{H}_G^{(X, Y)} &= (\mathbb{C}^*)^n \setminus V_{(\mathbb{C}^*)^n}(L_1(X + \alpha, Y) \cdots L_r(X + \alpha, Y)) \\ &= \{\alpha \in (\mathbb{C}^*)^n \mid L_i(X + \alpha, Y) \neq 0, i = 1, \dots, r\}, \end{aligned} \quad (\text{A.5})$$

for a generic choice of parameters $(X, Y) \in \mathbb{C}^{|V|+|E|}$. For the more detailed physics motivation behind the definition of (A.4) we refer to [6, Section 2.4].

Example A.4. For $n = 2$, the function $\psi_2(X_1, X_2, Y_{12})$ gives the integral

$$I_2(X_1, X_2, Y_{12}, \varepsilon) = \int_{\Gamma} \frac{2Y_{12} \cdot \alpha_1^\varepsilon \alpha_2^\varepsilon}{(X_1 + X_2 + \alpha_1 + \alpha_2)(X_1 + Y_{12} + \alpha_1)(X_2 + Y_{12} + \alpha_2)} d\alpha_1 \wedge d\alpha_2.$$

The associated hyperplane arrangement is represented in [6, Figure 1]. ◇

By construction, the integrals described in (A.4) are Euler integrals of product of linear forms. Therefore, we can use the machinery of Sections 3 and 4 to investigate their singular locus. Given a graph G , we denote z_G , or z_n if the number of vertices determines G , the matrix in (3.2) associated to the hyperplanes in (A.5). Then, z_G is a matrix of size $(n+1) \times r$, and we denote B_G the associated bipartite graph introduced in Section 3.

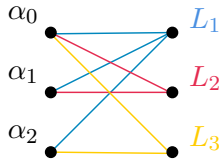
Lemma A.5. *Given a connected undirected graph G , then the bipartite graph B_G is also connected.*

Proof. Consistently with Section 3, we denote V_1 and V_2 , respectively the left and right vertex set of the bipartite graph B_G . The set V_1 contains $n + 1$ vertices representing the homogeneous coordinates $\alpha_0, \dots, \alpha_n$, while the set V_2 has size r with each vertex representing an hyperplane L_i . We distinguish the following two cases:

- For any pair of vertices $v_i, v_j \in V_1$, a path between them is given by two edges connecting each v_i to the vertex representing the hyperplane $L_1 = X_1 + \dots + X_n + \alpha_1 + \dots + \alpha_n$ given by shifting the scattering facet;
- For any pair of vertices $v_i, v_j \in V_2$, a path between them is given by the edges corresponding to the non-zero coefficient of the variable α_0 of the hyperplanes L_i, L_j .

Therefore, given any pair of vertices $v_i, v_j \in V_1 \cup V_2$ there exists a path among them. \square

Example A.6. This example illustrates in details the computation of the principal A -determinant of the sparse arrangement induced by the integral of the two-site chain (Figure 6, left) from Example A.4 and its Euler discriminant. In the notation of Section 3, we have $k = 2$ and $n = 5$. The matrix z_2 , the bipartite graph B_G , and the matrix A , whose columns give the vertices of the edge polytope P_G are illustrated in Figure 7.

$$z_2 = \begin{bmatrix} z_{03} & z_{04} & z_{05} \\ z_{13} & z_{14} & 0 \\ z_{23} & 0 & z_{25} \end{bmatrix}$$


$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 7: The matrix z_2 , the bipartite graph B_G , and the matrix A for the two-site chain. The colors for the edges of B_G matches the non-zero coefficients in the corresponding hyperplanes L_1, L_2, L_3 in the integral from Example A.4.

The polytope $P_G \subset \mathbb{R}^6$ is 4-dimensional, it has f-vector $(7, 17, 18, 8, 1)$, and normalized volume 4. Theorem 3.9 provides a closed formula to compute the principal A -determinant of the sparse arrangement:

$$E_A(z_2) = z_{03}z_{04}^2z_{05}^2z_{13}^2z_{14}^2z_{23}^2z_{25}^2(z_{03}z_{14} - z_{04}z_{13})(z_{03}z_{25} - z_{05}z_{23})(z_{03}z_{14}z_{25} - z_{05}z_{14}z_{23} - z_{04}z_{13}z_{25}).$$

The expected degree (see Proposition (2.4)) is $\deg(E_A(z_2)) = (\dim(P_G) + 1) \cdot \text{vol}(P_G) = (4 + 1) \cdot 4 = 20$. The multiplicities of each factor are computed using the methods from Section 3.4. Let us point out that each factor of $E_A(z_2)$ indeed corresponds to a singularity of the Euler integral

$$I_\Gamma(z_2, s, \nu) = \int_\Gamma (z_{03} + z_{13}\alpha_1 + z_{23}\alpha_2)^{-s_1} (z_{04} + z_{14}\alpha_1)^{-s_2} (z_{05} + z_{25}\alpha_2)^{-s_3} \alpha_1^{\nu_1} \alpha_2^{\nu_2} \frac{d\alpha_1}{\alpha_1} \wedge \frac{d\alpha_2}{\alpha_2}.$$

For instance, let us pick the factor z_{03} of $E_A(z_2)$. For simplicity, we assume that the z_{ij} are all real and generic and $\text{Re}(s_1) < 1, \text{Re}(\nu_1) > -1, \text{Re}(\nu_2) > -1$. Let $\Delta(z_2)$ be the bounded chamber specified by three lines $\{\alpha_1 = 0\}, \{\alpha_2 = 0\}$ and $\{L_1 = 0\}$. By a change of

coordinates given by $\beta_i = -z_{i3}/z_{03}\alpha_i$ ($i = 1, 2$), we obtain that

$$\int_{\Delta(z_2)} L_1^{-s_1} L_2^{-s_2} L_3^{-s_3} \alpha_1^{\nu_1} \alpha_2^{\nu_2} \frac{d\alpha_1}{\alpha_1} \wedge \frac{d\alpha_2}{\alpha_2} = z_{03}^{\nu_1+\nu_2-s_1} (-z_{13})^{-\nu_1} (-z_{23})^{-\nu_2} \int_{\Delta} (1 - \beta_1 - \beta_2)^{-s_1} f(\beta; z_2) \beta_1^{\nu_1} \beta_2^{\nu_2} \frac{d\beta_1}{\beta_1} \wedge \frac{d\beta_2}{\beta_2},$$

where $\Delta = \{(\beta_1, \beta_2) \in \mathbb{R}^2 \mid \beta_1, \beta_2, 1 - \beta_1 - \beta_2 > 0\}$ and $f(\beta; z_2)$ is a holomorphic function in β_1, β_2 with holomorphic parameters z_2 coming from a power series expansion of $L_2^{-s_2} L_3^{-s_3}$. By expanding $f(\beta; z_2)$ as $\sum_{m_1, m_2 \geq 0} c_{m_1, m_2}(z_2) \beta_1^{m_1} \beta_2^{m_2}$, the integral is expanded into a function

$$z_{03}^{\nu_1+\nu_2-s_1} (-z_{13})^{-\nu_1} (-z_{23})^{-\nu_2} \sum_{m_1, m_2 \geq 0} c_{m_1, m_2}(z_2) \frac{\Gamma(1 - s_1) \Gamma(1 + \nu_1 + m_1) \Gamma(1 + \nu_2 + m_2)}{\Gamma(3 - s_1 + \nu_1 + m_1 + \nu_2 + m_2)}.$$

This function is clearly singular for generic values of $s_1, s_2, s_3, \nu_1, \nu_2$ when $z_{03} \rightarrow 0$.

To compute the singular locus of the physical integral we need to further restrict to the subspace where the coefficients in z_2 are parametrized by the variables X, Y which represent energies. Therefore, the matrix z_2 becomes

$$z_2(X, Y) = \begin{bmatrix} X_1 + X_2 & X_1 + Y_{12} & X_2 + Y_{12} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Using Theorem 4.1 we can determine the defining equation of the Euler discriminant:

$$E_\chi(X, Y) = (X_1 + X_2)(X_1 + Y_{12})^2(X_2 + Y_{12})^2(X_2 - Y_{12})(X_1 - Y_{12})Y_{12}$$

When describing the singularities of the differential equations annihilating the integral from Example A.4, the singularity $Y_{12} = 0$ is discarded, see [6, Section 3]. This is due to the normalization factor Y_{12} appearing in the numerator of the integral. \diamond

Example A.7. This example illustrates the principal A -determinant and the Euler discriminant for the three-site chain. The matrix z_3 and the bipartite graph are shown in Figure 8.

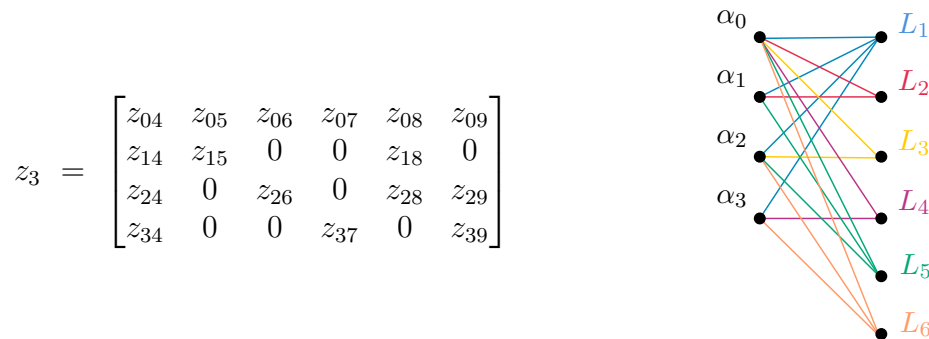


Figure 8: The coefficients matrix and the bipartite graph for the three-site chain. The colors refer those of the hyperplanes in Example A.2 after introducing the α -variables.

Using Theorem 3.9 and the method presented in Section 3.4, we have that $E_A(z_3)$ equals

$$\begin{aligned}
& z_{04} z_{05}^{11} z_{06}^9 z_{07}^{11} z_{08}^3 z_{09}^3 z_{14}^8 z_{15}^{11} z_{18}^8 z_{24}^4 z_{26}^9 z_{28}^6 z_{29}^6 z_{34}^8 z_{37}^{11} z_{39}^8 (z_{04} z_{15} - z_{05} z_{14})(z_{04} z_{18} - z_{08} z_{14}) \\
& (z_{04} z_{26} - z_{06} z_{24})(z_{04} z_{28} - z_{08} z_{24})(z_{04} z_{29} - z_{09} z_{24})(z_{04} z_{37} - z_{07} z_{34})(z_{04} z_{39} - z_{09} z_{34}) \\
& (z_{08} z_{29} - z_{09} z_{28})(z_{14} z_{28} - z_{18} z_{24})^2 (z_{24} z_{39} - z_{29} z_{34})^2 (z_{05} z_{18} - z_{08} z_{15})^3 (z_{06} z_{28} - z_{08} z_{26})^3 \\
& (z_{06} z_{29} - z_{09} z_{26})^3 (z_{07} z_{39} - z_{09} z_{37})^3 (z_{05} z_{18} z_{26} + z_{06} z_{15} z_{28} - z_{08} z_{15} z_{26})^3 \\
& (z_{06} z_{29} z_{37} + z_{07} z_{26} z_{39} - z_{09} z_{26} z_{37})^3 (z_{04} z_{15} z_{26} - z_{05} z_{14} z_{26} - z_{06} z_{15} z_{24}) \\
& (z_{04} z_{26} z_{37} - z_{06} z_{24} z_{37} - z_{07} z_{26} z_{34})(z_{04} z_{15} z_{29} - z_{05} z_{14} z_{29} - z_{09} z_{15} z_{24}) \\
& (z_{04} z_{15} z_{37} - z_{05} z_{14} z_{37} - z_{07} z_{15} z_{34})(z_{04} z_{15} z_{39} - z_{05} z_{14} z_{39} - z_{09} z_{15} z_{34}) \\
& (z_{04} z_{18} z_{37} - z_{07} z_{18} z_{34} - z_{08} z_{14} z_{37})(z_{04} z_{18} z_{39} - z_{08} z_{14} z_{39} - z_{09} z_{18} z_{34}) \\
& (z_{04} z_{28} z_{37} - z_{07} z_{28} z_{34} - z_{08} z_{24} z_{37})(z_{07} z_{28} z_{39} + z_{08} z_{29} z_{37} - z_{09} z_{28} z_{37}) \\
& (z_{05} z_{18} z_{29} - z_{08} z_{15} z_{29} + z_{09} z_{15} z_{28})(z_{14} z_{28} z_{39} - z_{18} z_{24} z_{39} + z_{18} z_{29} z_{34}) \\
& (z_{04} z_{26} z_{39} - z_{06} z_{24} z_{39} + z_{06} z_{29} z_{34} - z_{09} z_{26} z_{34})(z_{04} z_{18} z_{29} - z_{08} z_{14} z_{29} + z_{09} z_{14} z_{28} - z_{09} z_{18} z_{24}) \\
& (z_{04} z_{18} z_{26} + z_{06} z_{14} z_{28} - z_{06} z_{18} z_{24} - z_{08} z_{14} z_{26})(z_{04} z_{15} z_{28} - z_{05} z_{14} z_{28} + z_{05} z_{18} z_{24} - z_{08} z_{15} z_{24}) \\
& (z_{04} z_{28} z_{39} - z_{08} z_{24} z_{39} + z_{08} z_{29} z_{34} - z_{09} z_{28} z_{34})(z_{04} z_{29} z_{37} + z_{07} z_{24} z_{39} - z_{07} z_{29} z_{34} - z_{09} z_{24} z_{37}) \\
& (z_{05} z_{18} z_{29} z_{37} - z_{07} z_{15} z_{28} z_{39} - z_{08} z_{15} z_{29} z_{37} + z_{09} z_{15} z_{28} z_{37}) \\
& (z_{04} z_{15} z_{26} z_{37} - z_{05} z_{14} z_{26} z_{37} - z_{06} z_{15} z_{24} z_{37} - z_{07} z_{15} z_{26} z_{34}) \\
& (z_{04} z_{15} z_{26} z_{39} - z_{05} z_{14} z_{26} z_{39} - z_{06} z_{15} z_{24} z_{39} + z_{06} z_{15} z_{29} z_{34} - z_{09} z_{15} z_{26} z_{34}) \\
& (z_{04} z_{15} z_{28} z_{37} - z_{05} z_{14} z_{28} z_{37} + z_{05} z_{18} z_{24} z_{37} - z_{07} z_{15} z_{28} z_{34} - z_{08} z_{15} z_{24} z_{37}) \\
& (z_{04} z_{15} z_{29} z_{37} - z_{05} z_{14} z_{29} z_{37} + z_{07} z_{15} z_{24} z_{39} - z_{07} z_{15} z_{29} z_{34} - z_{09} z_{15} z_{24} z_{37}) \\
& (z_{04} z_{18} z_{26} z_{37} + z_{06} z_{14} z_{28} z_{37} - z_{06} z_{18} z_{24} z_{37} - z_{07} z_{18} z_{26} z_{34} - z_{08} z_{14} z_{26} z_{37}) \\
& (z_{04} z_{18} z_{26} z_{39} + z_{06} z_{14} z_{28} z_{39} - z_{06} z_{18} z_{24} z_{39} + z_{06} z_{18} z_{29} z_{34} - z_{08} z_{14} z_{26} z_{39} - z_{09} z_{18} z_{26} z_{34}) \\
& (z_{04} z_{15} z_{28} z_{39} - z_{05} z_{14} z_{28} z_{39} + z_{05} z_{18} z_{24} z_{39} - z_{05} z_{18} z_{29} z_{34} - z_{08} z_{15} z_{24} z_{39} + z_{08} z_{15} z_{29} z_{34} - \\
& z_{09} z_{15} z_{28} z_{34})(z_{04} z_{18} z_{29} z_{37} - z_{07} z_{14} z_{28} z_{39} + z_{07} z_{18} z_{24} z_{39} - z_{07} z_{18} z_{29} z_{34} - z_{08} z_{14} z_{29} z_{37} + \\
& z_{09} z_{14} z_{28} z_{37} - z_{09} z_{18} z_{24} z_{37}).
\end{aligned}$$

The degree is $\deg(E_A(z_3)) = (\dim(P_G) + 1) \cdot \text{vol}(P_G) = (8 + 1) \cdot 30 = 270$. The factors in cerulean are the ones that do not identically vanish when restricting to the subspace arising in physics. The restriction of the coefficients to the physical parameter produces the matrix

$$\begin{bmatrix}
X_1 + X_2 + X_3 & X_1 + Y_{12} & X_2 + Y_{12} + Y_{23} & X_3 + Y_{23} & X_1 + X_2 + Y_{23} & X_2 + X_3 + Y_{12} \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1
\end{bmatrix},$$

whose non-vanishing maximal minors give the factors of the Euler discriminant:

$$\begin{aligned}
E_X(X, Y) &= (X_1 + X_2 + X_3)(X_1 + Y_{12})^{10}(X_2 + Y_{12} + Y_{23})^9(X_3 + Y_{23})^{10}(X_1 + X_2 + Y_{23})^3 \\
& (X_2 + X_3 + Y_{12})^3(X_2 + X_3 - Y_{12})(X_3 - Y_{23})^6(X_1 + X_3 - Y_{12} - Y_{23})(X_1 - Y_{12})^6 \\
& (X_1 + X_2 - Y_{23})(X_1 - X_3 - Y_{12} + Y_{23})(X_2 - Y_{12} + Y_{23})^3(X_2 + Y_{12} - Y_{23})^3 Y_{12}^6 Y_{23}^6 \\
& (X_3 - 2Y_{12} - Y_{23})(X_1 - Y_{12} - 2Y_{23})(X_2 - Y_{12} - Y_{23})(X_1 - Y_{12} + 2Y_{23}) \\
& (X_3 + 2Y_{12} - Y_{23})(Y_{12} - Y_{23})(Y_{12} + Y_{23}),
\end{aligned}$$

which has degree 77. The exponents here are computed using the built-in function `EulerDiscriminantQ` from the `julia` package `PLD.jl` as explained in Example 4.2.

◇

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