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OPTIMISATION OF SPACE-TIME PERIODIC EIGENVALUES

BENIAMIN BOGOSEL, IDRIS MAZARI-FOUQUER, AND GRÉGOIRE NADIN

ABSTRACT. The goal of this paper is to provide a qualitative analysis of the optimisation of space-time periodic principal eigenvalues. Namely, considering a fixed time horizon T and the d -dimensional torus \mathbb{T}^d , let, for any $m \in L^\infty((0, T) \times \mathbb{T}^d)$, $\lambda(m)$ be the principal eigenvalue of the operator $\partial_t - \Delta - m$ endowed with (time-space) periodic boundary conditions. The main question we set out to answer is the following: how to choose m so as to minimise $\lambda(m)$? This question stems from population dynamics. We prove that in several cases it is always beneficial to rearrange m with respect to time in a symmetric way, which is the first comparison result for the rearrangement in time of parabolic equations. Furthermore, we investigate the validity (or lack thereof) of Talenti inequalities for the rearrangement in time of parabolic equations. The numerical simulations which illustrate our results were obtained by developing a framework within which it is possible to optimise criteria with respect to functions having a prescribed rearrangement (or distribution function).

Keywords: Non-symmetric operator, Principal eigenvalue, Rearrangements, Talenti inequalities, Spectral optimisation.

MSC Classification: 35P05, 35Q93, 49M41, 65N35.

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1. INTRODUCTION & MAIN RESULTS

1.1. Setting and objectives.

1.1.1. *Scope of the paper.* Let \mathbb{T}^d be the d -dimensional torus and $T > 0$ be a time horizon. The primary focus of this paper is the optimisation, with respect to a function m in $L^\infty((0, T) \times \mathbb{T}^d)$ naturally seen as \mathbb{T}^d -periodic in space and T -periodic in time, of the principal parabolic eigenvalue $\lambda(m)$ associated with the following system:

$$(1.1) \quad \begin{cases} \frac{\partial u_m}{\partial t} - \Delta u_m = \lambda(m)u_m + mu_m & \text{in } (0, T) \times \mathbb{T}^d, \\ u_m(T, \cdot) = u_m(0, \cdot), \\ u_m \geq 0, \neq 0, \iint_{(0, T) \times \mathbb{T}^d} u_m^2 = 1. \end{cases}$$

Here we use the notation $f_{\mathbb{T}^d} = \frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d}$. This eigenvalue problem is well-studied [7, 13]; its well-posedness follows from the Krein-Rutman theorem when m is Hölder continuous [25, Chapter 2], or from a Perron-Frobenius approach [33]. The main question under consideration is the following: *how to choose m so as to minimise the eigenvalue λ ?* In other words, we are interested in the problem

$$\min_m \lambda(m)$$

where m is constrained to belong to a certain admissible set. Naturally we would need to specify the precise constraints, but, at a qualitative level, there are two main questions. The first one is the *symmetry* of optimisers: is it true that it always better to replace m with another potential (that satisfies the same constraints) but that is also symmetric in time and in space? The second one has to do with the *monotonicity* of the optimisers: provided the answer to the first question is positive, is it true that the optimiser is not only symmetric, but also monotone? Let us emphasise that these questions are answered positively when considering only the symmetry and monotonicity with respect to the space variable; this result is due to Nadin [33] and relies on a class of elliptic Talenti inequalities (we refer to section 1.1.2). On the other hand, to the best of our knowledge, no results exist for the symmetry and monotonicity in time of optimisers.

In this paper, we do not provide a complete answer to these questions, but we offer a detailed analysis of several cases. To be more specific:

- (1) We first prove a general symmetry result: all optimisers of the first eigenvalue are symmetric in time. This is Theorem I.
- (2) Second, we investigate the question of monotonicity in a variety of regimes. It is important to note that here we restrict our investigation to the following case: we assume that $m(t, x) = c(t)V(x)$, where V is a fixed potential, and c is the function to be optimised. This case will be referred to as "multiplicative potentials".
 - (a) A first core result deals with the case where the eigenvalue problem is set in all of \mathbb{R}^d and V is a quadratic function. In this case we prove that it is always better to replace c with its symmetric decreasing rearrangement (see Definition 1); this is Theorem II.
 - (b) A second class of results deals with the problem in the torus. In that case we show that in a singular perturbation limit and in a large heat operator limit, it is always better to replace c with its symmetric decreasing rearrangement. These are Theorems III-IV.
 - (c) We then show that, contrary to what happens in the elliptic case, these results can not be linked to Talenti inequalities, which goes to highlight the deep difficulties posed by this endeavour. This is Theorem V.
 - (d) Finally, we develop a numerical framework within which it is possible to optimise eigenvalues (or other criteria) with respect to functions having a fixed rearrangement – this allows to go beyond the usual thresholding algorithm, and the numerical simulations we obtain back both our results and conjectures. We refer to section 7.

1.1.2. *State of the art.* The questions at hand are at the crossroads of multiple active research lines; we briefly summarise the relevant works below.

Time-space periodic principal eigenvalues. The theory of time-space periodic principal eigenvalues is well established. To the best of our knowledge, the investigation of this spectral problem was first initiated by Beltramo & Hess [7], who used it to study bifurcation results for non-linear parabolic equations. Observe that [7] uses the Krein-Rutman approach to establish the existence of $\lambda(m)$. The standard work here is the monograph of Hess [18], which assumes some regularity on m . Furthermore, this periodic eigenvalue can be seen as the principal eigenvalue of a degenerate, non-symmetric elliptic operator. Indeed, introduce, for any $\varepsilon > 0$, the operator $L_\varepsilon := -\varepsilon\partial_{tt}^2 - \Delta + \partial_t - m$. One can show using elliptic methods

that L_ε has a principal eigenvalue $\lambda_\varepsilon(m)$, and it is readily seen that one can write $\lambda(m) = \lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon(m)$. With this interpretation in mind it is important to note that Pinsky [36] studied the existence and uniqueness of a principal periodic eigenvalue $\lambda_\varepsilon(m)$ and, additionally, investigated the sensitivity of $\lambda_\varepsilon(m)$ under perturbations of the coefficients of L_ε . In this contribution as well, the coefficient of L (in particular, m) are assumed to have some Hölder continuity. The question of existence of $\lambda(m)$ when m is merely assumed to be L^∞ was later revisited by Godoy, Lami & Paczka [14] and Daners [12, 13] using, for [14], a semi-group approach and, for [13], an adaptation of the approach of Beltramo & Hess [7]. Regarding applications to population dynamics, Cantrell & Cosner [10] and Berestycki, Hamel & Roques [9] showed that the study of principal eigenvalues was crucial in understanding the invasion of a species in space periodic environments, especially so for characterising the invasion speed. Their approach was later shown by Nolen, Rudd & Xin [35] and Nadin [32] to hold in the case of time-space periodic equations, and these methods were adapted to accommodate the study of space-time periodic solutions of reaction-diffusion equations in bounded domains by Nadin [34]. Hutson, Mischaikow & Poláčik [20] and Hutson, Shen & Vickers [21] investigated the qualitative dynamics of space-time periodic reaction-diffusion equations by using the interpretation of $\lambda(m)$ as a criterion for extinction or survival of a species; we will go back to a discussion of some of their results in a subsequent paragraph. We refer to [25, Chapter 2] for a self-contained introduction to the topic, and we conclude by mentioning that a related approach that recently garnered a lot of interest from the mathematical community is based on the study of the principal Floquet bundle. We refer to [25, Chapter 5].

Optimisation of eigenvalues of (non-)symmetric operators. Regarding the main qualitative questions of this paper, it is important to review some key contributions in spectral optimisation of elliptic and parabolic operators. A lot of work has been devoted to the understanding of the following optimisation problem:

Minimise $\mu(D, m)$

$$\text{with respect to } m \in M := \left\{ m \in L^\infty(\Omega), -1 \leq m \leq 1, \int_{\Omega} m = m_0 \right\}$$

where $\mu(D, m)$ is the first eigenvalue of the operator $-D\Delta - m$ endowed with Dirichlet, Neumann or Robin boundary conditions in a smooth bounded domain Ω . Similar to $\lambda(m)$, $\mu(D, m)$ provides a natural survival or extinction criterion for various classes of reaction-diffusion equations, and minimising μ amounts to answering the following question: what is the best distribution of resources to ensure the optimal survival of a population? This question emerged in the works of Cantrell & Cosner [10] and subsequently attracted much attention from several points of view. The first one is the question of the geometry of optimisers, while the second one is the influence of the diffusion coefficient D . As regards the first question, [10] provided a first qualitative result: in the one-dimensional case, it is better, in the case of periodic boundary conditions, to place resources at the center of the interval. Later contributions by Berestycki, Hamel & Roques [8] and Kao, Lou & Yanagida [22] showed more general symmetry properties: in \mathbb{T}^d , the optimal resources distribution for $\mu(D, \cdot)$ is symmetric decreasing in every direction. There are several proofs available, the most flexible one relying on rearrangement inequalities. We refer to [8] for further details, as well as to [26] for more general

symmetry properties. A key aspect of these contributions is that $\mu(D, m)$ admits a variational formulation; this is due to the symmetry of the operator of $-D\Delta - m$. A natural question was then to understand the properties of spectral optimisation problems for non-symmetric operators, typically of the first eigenvalue $\Lambda(X, m)$ of

$$-\Delta + \langle X, \nabla \cdot \rangle - m,$$

in which case the eigenvalue has to be optimised with respect to both the drift X and the potential m . This problem was analysed by Hamel, Nadirashvili & Russ [15, 16] (see also [2]). Nevertheless, their techniques can not be applied to parabolic optimisation problem, as it involves modifying the drift term. In other words, $\Lambda(X, m)$ can be compared with $\Lambda(X', m^\#)$ where $m^\#$ and X' are suitably symmetrised versions of m and X respectively. However, for parabolic problems (seen as degenerate elliptic problems), the drift is the vector field $X = (1, 0, \dots, 0)$, the modified vector field would be $X' = (\mathbb{1}_{[0, T/2]} - \mathbb{1}_{[T/2, 2]}, 0, \dots, 0)$ and the modified operator is no longer parabolic. Nadin [31] showed a spatial symmetry result for the optimisation of the time-periodic eigenvalue $\lambda(m)$ (see (1.1)) in the one-dimensional case: letting for any $f := f(t, x)$, $f^*(t, \cdot)$ be the symmetric decreasing rearrangement of f in x (i.e. the only symmetric non-increasing function with the same repartition function as $f(t, \cdot)$, see Definition 1) then

$$\lambda(m^*) \leq \lambda(m).$$

His proof relies on a combination of Talenti inequalities and of the Perron-Frobenius construction of $\lambda(m)$. His proof does not extend to symmetry in time results.

Regarding the second question, that is, the monotonicity property of $\mu(D, m)$ with respect to the diffusivity D , this is settled immediately using the variational formulation of $\mu(D, m)$. In the case of non-symmetric elliptic and parabolic operators, the question is more subtle. For the sake of simplicity, let us consider the first eigenvalue $\lambda(\tau, A, D)$ of the operator $\tau \partial_t - D\Delta + A \langle X, \nabla \rangle - m$. The monotonicity with respect to D is false in general, as established by Nadin & Carrère [11]. The monotonicity with respect to τ is correct when $X \equiv 0$; this result is due to Liu, Lou, Peng & Zhou [28].

1.2. Mathematical statement of the problem. Throughout this paper, we use the notations

$$\int_{\mathbb{T}^d} f = \frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d} f, \quad \int_0^T f = \frac{1}{T} \int_0^T f, \quad \iint_{(0, T) \times \mathbb{T}^d} f = \frac{1}{T \cdot |\mathbb{T}^d|} \iint_{(0, T) \times \mathbb{T}^d} f.$$

We let $(u_m, \lambda(m))$ denote the unique eigenpair associated with (1.1). The potential m , unless otherwise stated, belongs to the set of T -periodic in time, \mathbb{T}^d -periodic in space, bounded measurable functions. While we will investigate rearrangement inequalities for the map $m \mapsto \lambda(m)$ we will also study optimisation problems. To properly state the admissible classes and questions we need to introduce basic notions of symmetric rearrangements.

1.2.1. Symmetric rearrangement. We first recall the definition of symmetric rearrangement of functions: for $f \in L^\infty((0, T))$ there exists a unique $f^\# \in L^\infty((0, T))$ with the following three properties:

- (1) $f^\#$ is symmetric with respect to $\frac{T}{2}$,
- (2) $f^\#$ is non-increasing in $[T/2, T]$,

- (3) $f^\#$ has the same distribution function as f : for any $\xi \in \mathbb{R}$, $|\{f \geq \xi\}| = |\{f^\# \geq \xi\}|$.

Definition 1 (Symmetric decreasing rearrangement). *For a function $f : [0, T] \rightarrow \mathbb{R}$, the function $f^\#$ is called the symmetric decreasing rearrangement of f . When f is T -periodic, $f^\#$ is also understood to be extended by T -periodicity.*

We refer for further details on rearrangements to [24]. In the sequel, for a function $m \in L^\infty((0, T) \times \mathbb{T}^d)$, we will denote by $m^\#$ the rearrangement of the function $t \mapsto m(t, x)$, with $x \in (0, T) \times \mathbb{T}^d$ given.

1.2.2. *Admissible classes and the optimisation problems.* We will consider two admissible classes. The first one is

$$(1.2) \quad \mathcal{M} := \left\{ m \in L^\infty((0, T) \times \mathbb{T}^d), -\kappa \leq m \leq 1 \text{ a.e.}, \iint_{(0, T) \times \mathbb{T}^d} m = m_0 \right\},$$

where $\kappa \geq 0$, and $m_0 \in (-\kappa, 1)$. The first optimisation problem that will be of importance to us is

$$(1.3) \quad \min_{m \in \mathcal{M}} \lambda(m).$$

The existence of an optimiser is a consequence of the direct method in the calculus of variations. Regarding this problem we will show (Theorem 1) that any optimiser m_{\min} is symmetric in time and in space. Nevertheless, observe that it is not clear that $m_{\min}^\# = m_{\min}$, and we thus devote some effort to understanding the following question: is it always true that $\lambda(m^\#) \leq \lambda(m)$?

For this purpose, we need to introduce the following comparison relation between functions: for two T -periodic functions $c_1, c_2 : [0, T] \rightarrow \mathbb{R}$, we say that c_1 is dominated by c_2 and we write

$$c_1 \preceq c_2$$

if, and only if,

$$(1.4) \quad \forall s \in [0, T], \sup_{|E|=s} \int_E c_1 \leq \sup_{|E|=s} \int_E c_2 \text{ and } \int_0^T c_1 = \int_0^T c_2.$$

As for any $s \in [0, T]$ and any $c \in L^\infty((0, T))$ we have

$$\sup_{|E|=s} \int_E c = \int_{\frac{T-s}{2}}^{\frac{T+s}{2}} c^\#$$

we have the following lemma:

Lemma 2. *For any $c_1, c_2 \in L^\infty((0, T))$, $c_1 \preceq c_2$ if, and only if,*

$$\forall r \in [0, T], \int_{\frac{T-r}{2}}^{\frac{T+r}{2}} c_1^\# \leq \int_{\frac{T-r}{2}}^{\frac{T+r}{2}} c_2^\#$$

and

$$\int_0^T c_1^\# = \int_0^T c_2^\#.$$

We are now in position to introduce our second admissible class. Let $L_{\text{per}}^{\infty}((0, T))$ be the set of T -periodic bounded measurable functions. Let $\bar{c} \in L_{\text{per}}^{\infty}((0, T))$ be such that

$$\bar{c} = \bar{c}^{\#}.$$

We define

$$(1.5) \quad \mathcal{C} := \{c \in L_{\text{per}}^{\infty}((0, T)), c \preceq \bar{c}\}.$$

Remark 3. *An important case is $\bar{c} = a + (b - a)\mathbb{1}_{[\frac{T-\ell}{2}, \frac{T+\ell}{2}]}$ for $a < b$ two real numbers. In this case the class \mathcal{C} can be described more explicitly as $\mathcal{C} = \{c : [0, T] \rightarrow \mathbb{R}, a \leq c \leq b, \int_0^T c = aT + \ell(b - a)\}$. This is not important for the mathematical analysis, but it can be in the numerical schemes. In section 7, we refer to this case as the “classical” one, as this is the one most often considered in the literature. The core of section 7 is to develop efficient methods when \bar{c} is not a characteristic function.*

For a fixed potential $V = V(x)$, we let $\lambda(cV)$ be the first eigenvalue of (1.1) with $m(t, x) = c(t)V(x)$. When no confusion is possible, $\lambda(c)$ denotes $\lambda(cV)$. The second problem under consideration is

$$(1.6) \quad \min_{c \in \mathcal{C}} \lambda(c).$$

Here again, the existence of an optimal c_{\min} is a consequence of the direct method of the calculus of variations.

Remark 4. *The topological properties of \mathcal{C} are well-known: \mathcal{C} is convex and compact for the weak $L^{\infty} - *$ topology; we refer to [5]. Furthermore, the extreme points of \mathcal{C} are exactly the functions c that satisfy $c^{\#} = \bar{c}$. This has an important consequence: as it is straightforward to see that $c \mapsto \lambda(c)$ is strictly concave, any solution c_{\min} of (1.6) satisfies $c_{\min}^{\#} = \bar{c}$.*

While studying (1.6) we will also be led to proving in some regimes that

$$\lambda(c^{\#}) \leq \lambda(c).$$

It is important to note that a core result of the paper is the study of the Gaussian case, when V is a quadratic potential and the equation is set in all of \mathbb{R}^d , where we can prove more precise results.

1.3. Main results.

1.3.1. *Symmetry in time of optimal potentials for (1.3).* We are interested in symmetry and monotonicity (in t and x) of the solutions m_{\min} of (1.3). As mentioned earlier, the symmetry in x of the optimiser can be obtained fairly easily using rearrangement arguments [31, Theorem 3.9].

Remark 5. *As, for any $\tau > 0$, $\lambda(m(\cdot + \tau, \cdot)) = \lambda(m)$, so that symmetry in time has to be understood as: symmetry up to a translation.*

Theorem I. *Any solution m_{\min} of (1.3) is symmetric in time: up to a translation, there holds, for any $t \in [0, T/2]$, for any $x \in \Omega$, $m_{\min}(t, x) = m_{\min}(T - t, x)$. Similarly, for a fixed V , any solution c_{\min} of (1.6) is symmetric in time: up to a translation, there holds, for any $t \in [0, T/2]$, $c_{\min}(t) = c_{\min}(T - t)$.*

This theorem will be obtained as a corollary of the following variational formulation of the eigenvalue which in turn relies on a Heinze-type formula [33, Proposition 2.6].

Proposition 6. *For any $m \in L^\infty((0, T) \times \Omega)$, the following holds:*

$$(1.7) \quad \lambda(m) = \min_{\alpha \in L^2(0, T; W^{1,2}(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega)), \iint_{(0, T) \times \mathbb{T}^d} \alpha^2 = 1} \left(\iint_{(0, T) \times \Omega} |\nabla_x \alpha|^2 - \iint_{(0, T) \times \mathbb{T}^d} m \alpha^2 - \min_{\beta \in L^2(0, T; W^{1,2}(\mathbb{T}^d))} \iint_{(0, T) \times \mathbb{T}^d} (-\beta \partial_t \alpha^2 + |\nabla_x \beta|^2 \alpha^2) \right).$$

This proposition, coupled with an elementary concavity property shows that the solution m_{\min} of (1.3) satisfies the so-called bang-bang property or, in other words, that there exists $E \subset (0, T) \times \mathbb{T}^d$ such that

$$m_{\min}(t, x) = (1 + \kappa) \mathbf{1}_E(t, x) - \kappa,$$

and the set E is symmetric in time.

1.3.2. The influence of time rearrangement for multiplicative potentials: Gaussian potentials. In this section we investigate in more details the question of the influence of time rearrangements: is it always beneficial to replace c with $c^\#$? We first study the Gaussian case, before investigating several limiting regimes in the initial framework of $(0, T) \times \mathbb{T}^d$ periodic functions.

We first study a related problem set in all of \mathbb{R}^d , which we dub the Gaussian case, as eigenfunctions are always of Gaussian type.

Theorem II. *Let $A \in S_d^{++}(\mathbb{R})$ be a symmetric positive definite matrix and $c \in L^\infty_{\text{per}}((0, T))$. Let $\bar{\lambda}(c), \bar{c} \geq 0$ be the first eigenvalue associated with the problem*

$$(1.8) \quad \begin{cases} \partial_t w_0 - \Delta w_0 = \bar{\lambda}(c) w_0 - \frac{1}{2} c(t) \langle x, Ax \rangle w_0 & \text{in } [0, T] \times \mathbb{R}^d, \\ w_0(T, \cdot) = w_0(0, \cdot), \\ w_0 \neq 0 \\ 0 \leq w_0 \leq w_0(0, 0) = 1. \end{cases}$$

Then there holds

$$\bar{\lambda}(c) \geq \bar{\lambda}(c^\#),$$

with equality if and only if $c = c^\#$ up to translation.

We could not locate the proof of existence of a solution to (1.8) in the literature; we provide it. Based on this case, we conjecture that optimisers should always be monotone decreasing but we are not able to prove it at the moment. Note however that this result provides some justification for our conjecture that in general rearranging c should be beneficial for the eigenvalue. Another intuitive justification behind this conjecture is to be found in the monotonicity in frequency of λ . To be more precise, consider a profile c that is not constant in time and, for any $k \in \mathbb{N}$, let $c_k(t) := c(kt)$. A theorem of Liu, Lou, Peng & Zhou [28] asserts that $(\lambda(c_k V))_{k \in \mathbb{N}}$ is an increasing sequence for $V \in L^\infty((0, T) \times \mathbb{T}^d)$. Consequently, oscillations are detrimental to spectral optimisation, whence a natural conjecture is that the optimiser should be “as monotonic” as possible. Nevertheless, while our results lends

credence to the link between monotonicity in frequency and monotonicity of optimisers in this case, we will see that, unlike monotonicity in space, monotonicity in time can not a priori be related to an isoperimetric type-property, see Theorem V.

1.3.3. *Monotonicity in time for two scaling limits of the equation.* We are able to show that c_{\min} , the solution of (1.6) is monotonic in time or, put otherwise, that $c_{\min} = c_{\min}^{\#}$, in two scaling limits of the equation.

First scaling limit: large heat operator We first replace (1.1) with

$$(1.9) \quad \begin{cases} \mu \partial_t u_{c,\mu} - \mu \Delta u_{c,\mu} = cV u_{c,\mu} + \lambda(c, \mu) u_{c,\mu} & \text{in } (0, T) \times \mathbb{T}^d, \\ u_{c,\mu}(0, \cdot) = u_{c,\mu}(T, \cdot), \\ u_{c,\mu} \geq 0, \neq 0, \end{cases}$$

and we are interested in

$$(1.10) \quad \min_{c \in \mathcal{C}} \lambda(c, \mu).$$

Theorem III. *Let $V \in L^\infty(\mathbb{T}^d)$ be fixed and, for any $\mu > 0$, c_{\min}^μ be a solution of (1.10). Assume that V is not constant. Then there exists $\mu_0 > 0$ such that, for any $\mu \geq \mu_0$, up to a translation:*

$$c_{\min}^\mu = (c_{\min}^\mu)^{\#} = \bar{c}.$$

Second scaling limit: singular perturbations In the next result we assume the following:

(\mathbf{H}_V)

$V \in \mathcal{C}^2(\mathbb{T}^d)$, $x^* = 0$ is the only local maximiser of V in \mathbb{T}^d and $\nabla^2 V(0) \in S_d^{--}(\mathbb{R})$.

Here, $S_d^{--}(\mathbb{R})$ denotes the class of symmetric negative definite matrix. For any $\varepsilon > 0$, let $u_{c,\varepsilon}$ be the eigenfunction of

$$(1.11) \quad \begin{cases} \varepsilon \partial_t u_{c,\varepsilon} - \varepsilon^2 \Delta u_{c,\varepsilon} = cV u_{c,\varepsilon} + \lambda_\varepsilon(c) u_{c,\varepsilon} & \text{in } (0, T) \times \mathbb{T}, \\ u_{c,\varepsilon}(0, \cdot) = u_{c,\varepsilon}(T, \cdot), \\ u_{c,\varepsilon} \geq 0, \neq 0. \end{cases}$$

We consider

$$(1.12) \quad \min_{c \in \mathcal{C}} \lambda_\varepsilon(c).$$

Theorem IV. *Assume $\bar{c} \geq 0$. Let V be fixed and satisfy (\mathbf{H}_V) and, for any $\mu > 0$, c_{\min}^ε be a solution of*

$$(1.13) \quad \min_{c \in \mathcal{C}} \lambda_\varepsilon(c).$$

Up to a translation,

$$c_{\min}^\varepsilon \xrightarrow{\varepsilon \rightarrow 0^*} \bar{c} = \bar{c}^{\#} \text{ in } L^p((0, T)) \text{ for any } p \in [1; +\infty).$$

1.4. **Some comments on our proofs – Can our results be linked to time-periodic Talenti inequalities?** Both Theorems IV and III ultimately rely on elliptic Talenti inequalities; we show that the optimisation problems can be recast as an energy minimisation for an elliptic problem. Nevertheless, the two limit problems obtained are different in nature: in Theorem III, the limit problem is a linear heat equation that can be dealt with using Fourier series, and the standard Talenti inequalities. In Theorem IV, the main tool is the Hopf-Cole transform,

which allows to relate (1.6) to a problem set in all of \mathbb{R}^d using the theory of singular perturbations; we build on some geometric estimates of [19]. As is often the case in singular perturbation problems, assumptions such as (\mathbf{H}_V) are required. It is worth observing that in proving Theorem IV we will use Theorem II, which also relies on elliptic Talenti inequalities. This begs the question of the validity of Talenti inequalities for time rearrangements of parabolic equations.

To be more specific, Theorem III relies on the following fact: we let, for any $c \in \mathcal{C}$, φ_c be the unique solution of

$$(1.14) \quad \begin{cases} \partial_t \varphi_c - \Delta \varphi_c = cV - \left(f_0^T c \cdot f_{\mathbb{T}^d} V \right), \\ \varphi_c(T, \cdot) = \varphi_c(0, \cdot). \end{cases}$$

Then we prove (see the proof of Proposition 18) that

$$\iint_{(0,T) \times \mathbb{T}^d} \varphi_c^2 \leq \iint_{(0,T) \times \mathbb{T}^d} \varphi_{\bar{c}}^2,$$

using a Fourier series decomposition and applying elliptic Talenti inequalities. A natural question is whether this inequality holds for norms other than the L^2 norm, in which case this might give hope to derive a general monotonicity result using an isoperimetric inequality approach. We investigate the optimisation problem

$$(1.15) \quad \max_{c \in \mathcal{C}} J_k(c) := \iint_{(0,T) \times \mathbb{T}^d} \varphi_c^{2k}.$$

We will in fact show that $J_k(c) \leq J_k(c^\#)$ for all c and V if, and only if, $k = 1$.

Before stating our precise result, let us briefly comment on Talenti inequalities in the elliptic case: the original elliptic Talenti inequality [37] reads as follows: letting, for any f , f^* be its symmetric decreasing rearrangement and Ω be the unit ball, there holds, for any increasing function J ,

$$\int_{\Omega} J(u_f) \leq \int_{\Omega} J(u_{f^*})$$

where u_f solves $-\Delta u_f = f$, $u_f \in W_0^{1,2}(\Omega)$. Vazquez [38] extended the approach of Talenti to heat equations set in the ball and proved that, retaining the notation f^* for the rearrangement in space of a function $f = f(t, x)$, the following comparison holds: for any increasing convex function J , there holds

$$\iint_{(0,T) \times \Omega} J(u_f) \leq \iint_{(0,T) \times \Omega} J(u_{f^*})$$

where $\partial_t u_f - \Delta u_f = f$, $u(0, \cdot) = 0$, $u(t, \cdot) \in W_0^{1,2}(\Omega)$. In fact, using the domination relation defined in (1.4), he proves that

$$u_f(t, \cdot) \preceq u_{f^*}(t, \cdot)$$

for any t . Alvino, Lions & Trombetti proposed systematic approaches to this question [1, 2, 3, 4, 5, 6] and established various comparison results, albeit always comparing the solution of a parabolic equation with the solution of a similar equation whose coefficients are suitably symmetrised in space. The question of symmetry in time, and of whether it was possible to have such strong results (namely, that there should be a univocal notion of time symmetrisation that would provide comparison principles) was partially settled in [29] where the second author shows that it is in general hopeless. Nevertheless, [29] deals with initial value problems rather

than with time-periodic solutions, and so this analysis can not be satisfactory for our needs. In this sense, Theorem V provides a conclusive (negative) answer and highlights the exceptionality of the L^2 norm in parabolic equations.

Our main theorem is the following:

Theorem V. *We let $T = 2\pi$, $V(x) = \cos(x)$ and $\bar{c} = \cos(\pi + t)$. \bar{c} solves (1.15) if, and only if, $k = 1$.*

Remark 7. *We can not prove theoretically a symmetry breaking result for “classical” constraints on c (see Remark 3). It seems from numerical simulation that \bar{c} might actually be a maximiser. We refer to section 7.*

2. PROOF OF THEOREM I

The proof of Theorem I relies on Proposition 6. Observe that a version of (1.7) was previously obtained in [33, Proposition 2.6], and uses the Heinze change of variables [17].

Proof of Proposition 6. Let u_m be the solution of (1.1) and v_m be the principal eigenfunction of the adjoint operator, that is:

$$(2.1) \quad \begin{cases} -\partial_t v_m - \Delta v_m = \lambda(m)v_m + m v_m & \text{in } (0, T) \times \mathbb{T}^d, \\ v_m(0, \cdot) = v_m(T, \cdot), \\ v_m \geq 0, \neq 0. \end{cases}$$

We work with the normalisation

$$(2.2) \quad \iint_{(0, T) \times \mathbb{T}^d} u_m^2 = \iint_{(0, T) \times \mathbb{T}^d} u_m v_m = 1.$$

Define $\alpha_m := \sqrt{u_m v_m}$, $\beta_m := \frac{1}{2} (\ln(u_m) - \ln(v_m))$. We use the Hopf-Cole transform by writing $u_m = e^{-\varphi_m + a_m}$, $v_m = e^{-\psi_m + b_m}$, where we assume that $\min_{(0, T) \times \mathbb{T}^d} \varphi_m = \min_{(0, T) \times \mathbb{T}^d} \psi_m = 0$ and a_m, b_m are chosen to satisfy (2.2). Then (φ_m, ψ_m) satisfy

$$(2.3) \quad \begin{cases} \partial_t \varphi_m - \Delta \varphi_m + |\nabla \varphi_m|^2 = -\lambda(m) - m & \text{in } (0, T) \times \mathbb{T}^d, \\ -\partial_t \psi_m - \Delta \psi_m + |\nabla \psi_m|^2 = -\lambda(m) - m & \text{in } (0, T) \times \mathbb{T}^d, \\ \varphi_m(0, \cdot) - \varphi_m(T, \cdot) = \psi_m(T, \cdot) - \psi_m(0, \cdot) = 0. \end{cases}$$

With these notations, we have $\alpha_m = e^{-\frac{\varphi_m + \psi_m}{2}}$, $\beta_m = \frac{\psi_m - \varphi_m}{2}$. Subtracting the two equations of (2.3) we obtain

$$\partial_t \frac{\varphi_m + \psi_m}{2} - \Delta \frac{\varphi_m - \psi_m}{2} + \left\langle \nabla \frac{\varphi_m - \psi_m}{2}, \nabla (\varphi_m + \psi_m) \right\rangle = 0.$$

This rewrites

$$(2.4) \quad -\frac{1}{2} \partial_t \alpha_m^2 - \nabla \cdot (\alpha_m^2 \nabla \beta_m) = 0.$$

Similarly, summing the equations in (2.3) and dividing by 2 we deduce that

$$\begin{aligned} \partial_t \frac{\varphi_m - \psi_m}{2} - \Delta \frac{\varphi_m + \psi_m}{2} + \left| \nabla \frac{\varphi_m + \psi_m}{2} \right|^2 \\ + \frac{|\nabla \varphi_m|^2 + |\nabla \psi_m|^2 - 2 \langle \nabla \varphi_m, \nabla \psi_m \rangle}{4} = -\lambda(m) - m \end{aligned}$$

or, using a Hopf-Cole transform,

$$-\Delta\alpha_m = (\lambda(m) + m + \partial_t\beta_m + |\nabla\beta_m|^2) \alpha_m.$$

We deduce the following from (2.4): for any $t \in [0, T]$, β_m is the unique minimiser of

$$W^{1,2}(\mathbb{T}^d) \ni \beta \mapsto \int_{\mathbb{T}^d} \alpha_m^2 |\nabla\beta|^2 - \frac{1}{2} \int_{\mathbb{T}^d} \beta \partial_t \alpha_m^2,$$

whence β_m is the unique minimiser of

$$L^2(0, T; W^{1,2}(\mathbb{T}^d)) \ni \beta \mapsto E(\beta, \alpha_m) := \frac{1}{2} \iint_{(0, T) \times \mathbb{T}^d} \alpha_m^2 |\nabla\beta|^2 - \frac{1}{2} \iint_{(0, T) \times \mathbb{T}^d} \beta \partial_t \alpha_m^2.$$

To alleviate notations, define

$$A((0, T) \times \mathbb{T}^d) := L^2(0, T; W^{1,2}(\mathbb{T}^d)) \cap W^{1,2}(0, T; L^2(\mathbb{T}^d)).$$

Since $\beta_m \in W^{1,2}(0, T; L^2(\mathbb{T}^d))$ by construction, this shows that β_m solves

$$\min_{\beta \in A((0, T) \times \mathbb{T}^d)} \frac{1}{2} \iint_{(0, T) \times \mathbb{T}^d} (\partial_t \beta + |\nabla\beta|^2) \alpha_m^2.$$

Similarly, as $\alpha_m > 0$ by construction, α_m is the unique minimiser of

$$\min_{\alpha \in A((0, T) \times \mathbb{T}^d), \iint_{(0, T) \times \mathbb{T}^d} \alpha^2 = 1} \iint_{(0, T) \times \mathbb{T}^d} |\nabla\alpha|^2 - \iint_{(0, T) \times \mathbb{T}^d} (m + \partial_t \beta_m + |\nabla\beta_m|^2) \alpha^2.$$

Consequently, (α_m, β_m) is a solution of the saddle point problem

$$\begin{aligned} & \min_{\alpha \in A((0, T) \times \mathbb{T}^d), \iint_{(0, T) \times \mathbb{T}^d} \alpha^2 = 1} \\ & \max_{\beta \in L^2(0, T; W^{1,2}(\mathbb{T}^d)) \cap W^{1,2}(0, T; L^2(\mathbb{T}^d))} \iint_{(0, T) \times \mathbb{T}^d} |\nabla\alpha|^2 - \iint_{(0, T) \times \mathbb{T}^d} (m + \partial_t \beta_m + |\nabla\beta_m|^2) \alpha^2. \end{aligned}$$

Conversely, if $(\tilde{\alpha}, \tilde{\beta})$ is a solution of this saddle point problem with associated value Λ , direct computations show that, setting $\gamma := -\ln(\alpha)$, we have

$$\partial_t(\gamma + \tilde{\beta}) - \Delta(\gamma + \tilde{\beta}) + |\nabla(\gamma + \tilde{\beta})|^2 = -\Lambda - m$$

whence $e^{-(\gamma + \tilde{\beta})} > 0$ solves (1.1) with eigenvalue Λ . The uniqueness of solutions to (1.1) allows to conclude. \square

Proof of Theorem I. We just prove the result for (1.6) as the proof for (1.3) is similar. Introduce the notation $B((0, T) \times \mathbb{T}^d) := L^2(0, T; L^2(\mathbb{T}^d))$. Let c_{\min} be an optimiser for (1.6). Up to a translation, we can assume that

$$\int_{\frac{T}{2}}^T c_{\min} = \int_{\frac{T}{2}}^T \bar{c}.$$

Recall (see (1.5)) that $\bar{c}^\# = \bar{c}$. From Proposition 6 we know that

$$\begin{aligned} \lambda(c_{\min}) = & \min_{\alpha \in A((0, T) \times \mathbb{T}^d), \iint_{(0, T) \times \mathbb{T}^d} \alpha^2 = 1} \left(\iint_{(0, T) \times \mathbb{T}^d} |\nabla_x \alpha|^2 - \iint_{(0, T) \times \mathbb{T}^d} c_{\min} V \alpha^2 \right. \\ & \left. - \min_{\beta \in B((0, T) \times \mathbb{T}^d)} \iint_{(0, T) \times \mathbb{T}^d} (-\beta \partial_t \alpha^2 + |\nabla\beta|^2 \alpha^2) \right). \end{aligned}$$

Let (α^*, β^*) be the unique solution of this saddle-point problem and let (I_1, I_2) be defined, respectively, as

$$I_1 := \frac{\iint_{[0; \frac{T}{2}] \times \mathbb{T}^d} |\nabla_x \alpha^*|^2 - \iint_{[0; \frac{T}{2}] \times \mathbb{T}^d} c_{\min} V(\alpha^*)^2 - \min_{\beta} \iint_{[0; \frac{T}{2}] \times \mathbb{T}^d} (\partial_t \beta(\alpha^*)^2 + |\nabla \beta|^2(\alpha^*)^2)}{\iint_{[0; \frac{T}{2}] \times \mathbb{T}^d} (\alpha^*)^2}$$

$$I_2 := \frac{\iint_{[\frac{T}{2}; T] \times \mathbb{T}^d} |\nabla_x \alpha^*|^2 - \iint_{[\frac{T}{2}; T] \times \mathbb{T}^d} c_{\min} V(\alpha^*)^2 - \min_{\beta} \iint_{[\frac{T}{2}; T] \times \mathbb{T}^d} (\partial_t \beta(\alpha^*)^2 + |\nabla \beta|^2(\alpha^*)^2)}{\iint_{[\frac{T}{2}; T] \times \mathbb{T}^d} (\alpha^*)^2}.$$

It follows from (1.7) and the inequality $\frac{a+b}{c+d} \geq \min(\frac{a}{c}, \frac{b}{d})$ that

$$\lambda(c_{\min}) \geq \min(I_1, I_2).$$

Assume without loss of generality that $I_2 \leq I_1$. In that case, define

$$\tilde{c}_{\min}(t) = \begin{cases} c_{\min}(t) & \text{if } t \in [\frac{T}{2}; T], \\ c_{\min}(T-t) & \text{else} \end{cases}$$

and

$$\tilde{\alpha}^*(t, \cdot) := \begin{cases} \alpha^*(t, \cdot) & \text{if } t \in [\frac{T}{2}; T], \\ \alpha^*(T-t, \cdot) & \text{else,} \end{cases} \quad \tilde{\beta}^*(t, \cdot) := \begin{cases} \beta^*(t, \cdot) & \text{if } t \in [\frac{T}{2}; T], \\ -\beta^*(T-t, \cdot) & \text{else.} \end{cases}$$

By construction, we have

$$\min_{\beta \in B((0, T) \times \mathbb{T}^d)} \iint_{(0, T) \times \mathbb{T}^d} -\beta \partial_t (\tilde{\alpha}^*)^2 + |\nabla \beta|^2 (\tilde{\alpha}^*)^2 = \iint_{(0, T) \times \mathbb{T}^d} -\tilde{\beta}^* \partial_t (\tilde{\alpha}^*)^2 + |\nabla \tilde{\beta}^*|^2 (\tilde{\alpha}^*)^2$$

and

$$I_2 = \frac{\iint_{(0, T) \times \mathbb{T}^d} |\nabla \tilde{\alpha}^*|^2 - \iint_{(0, T) \times \mathbb{T}^d} \tilde{c}_{\min} V(\tilde{\alpha}^*)^2 - \iint_{(0, T) \times \mathbb{T}^d} -\tilde{\beta}^* \partial_t (\tilde{\alpha}^*)^2 + |\nabla \tilde{\beta}^*|^2 (\tilde{\alpha}^*)^2}{\iint_{(0, T) \times \mathbb{T}^d} (\tilde{\alpha}^*)^2}.$$

so that

$$\lambda(c_{\min}) \geq I_2 \geq \lambda(\tilde{c}_{\min}).$$

This establishes that

$$\lambda(c_{\min}) \geq \lambda(\tilde{c}_{\min}).$$

Two points remain to be checked: first, that $\tilde{c}_{\min} \in \mathcal{C}$ and, second, that if $\lambda(c_{\min}) = \lambda(\tilde{c}_{\min})$, then $c_{\min} = \tilde{c}_{\min}$. Regarding the first point, it is a direct consequence of the construction of \tilde{c}_{\min} . Indeed, let $s \in (0, T/2)$ and let E_s^* be optimal for the problem

$$\max_{E \subset [\frac{T}{2}, T], |E|=s} \int_{\frac{T}{2}}^T \mathbf{1}_E c_{\min}.$$

Let \tilde{E}_s be the set obtained by symmetrising E_s^* with respect to $t = \frac{T}{2}$; clearly \tilde{E}_s is a solution of

$$\max_{E \subset [0, T], |E|=s} \int_0^T \mathbf{1}_E \tilde{c}_{\min}.$$

However, by the Hardy-Littlewood inequality for the decreasing rearrangement, we know that

$$\int_{\frac{T}{2}}^T \mathbf{1}_E c_{\min} \leq \int_{\frac{T}{2}}^T \mathbf{1}_{E^\#} (c_{\min})^\# \leq \int_{\frac{T}{2}}^T \mathbf{1}_{E^\#} \bar{c}$$

by definition of \mathcal{C} . We thus conclude that

$$\max_{E \subset [0, T], |E|=s} \int_0^T \mathbf{1}_E \tilde{c}_{\min} \leq \int_{\frac{T}{2}}^T \mathbf{1}_{\tilde{E}_s^\#} \bar{c}$$

and, as $\int_0^T \tilde{c}_{\min} = \int_0^T \bar{c}$, consequently, $\tilde{c}_{\min} \preceq \bar{c}$. Now, assume that

$$\lambda(c_{\min}) = \lambda(\tilde{c}_{\min}).$$

We deduce from (1.7) that

$$\tilde{\alpha}^* = \alpha_{\tilde{c}_{\min}}, \tilde{\beta}^* = \beta_{c_{\min}}.$$

We already observed, in the proof of Proposition 6, that

$$\ln(\alpha_m) + \beta_m = \varphi_m.$$

Let $\varphi^* := -\ln(u_{c_{\min}})$, so that, defining

$$\tilde{\varphi}^*(t, \cdot) := \begin{cases} \varphi^*(t, \cdot) & \text{if } t \geq \frac{T}{2}, \\ \varphi^*(T-t, \cdot) & \text{else,} \end{cases} \quad \tilde{\psi}^*(t, \cdot) := \begin{cases} \psi^*(t, \cdot) & \text{if } t \geq \frac{T}{2}, \\ \psi^*(T-t, \cdot) & \text{else,} \end{cases}$$

it follows that

$$\ln(\tilde{\alpha}^*) + \beta^* = \varphi_{\tilde{c}_{\min}} = \tilde{\varphi}^* \text{ and, similarly, } \psi_{\tilde{c}_{\min}} = \tilde{\psi}^*.$$

As \tilde{c}_{\min} is symmetric, we obtain

$$\varphi_{\tilde{c}_{\min}}(t, \cdot) = \psi_{\tilde{c}_{\min}}(T-t, \cdot),$$

which in turn implies that

$$\tilde{\varphi}^*(t, \cdot) = \tilde{\psi}^*(T-t, \cdot).$$

Turning back to the definitions of $\tilde{\varphi}^*, \tilde{\psi}^*$, we finally conclude that, for any $t \in [\frac{T}{2}; T]$,

$$\varphi^*(t, \cdot) = \psi^*(T-t, \cdot), \varphi^*(T-t, \cdot) = \psi^*(t, \cdot),$$

from which we deduce, from (2.3), that $c_{\min} = \tilde{c}_{\min}$. \square

3. PROOF OF THEOREM II

We begin with a study of the eigenpair given by (1.8).

3.1. Study of (1.8).

3.1.1. *Preliminary on periodic solutions of non-linear ODEs.* In this subsection, given $f \in L_{\text{per}}^\infty((0, T))$, we are interested in the existence and uniqueness of a solution γ_f to

$$(3.1) \quad \begin{cases} \gamma_f' + \gamma_f^2 = f & \text{in } (0, T), \\ \gamma_f(0) = \gamma_f(T). \end{cases}$$

We introduce the following auxiliary eigenvalue problem: for any $\alpha \in \mathbb{R}$, let $\theta_f(\alpha)$ be the first eigenvalue associated with the problem

$$(3.2) \quad \begin{cases} -y_{\alpha, f}'' - 2\alpha y_{\alpha, f}' - \alpha^2 y_{\alpha, f} = -f y_{\alpha, f} + \theta_f(\alpha) y_{\alpha, f} & \text{in } [0, T], \\ y_{\alpha, f}(0) = y_{\alpha, f}(T), \\ y_{\alpha, f} \geq 0. \end{cases}$$

In this section, for the sake of notational compactness, we use the notation

$$\langle f \rangle_{[0, T]} = \frac{1}{T} \int_0^T f.$$

The main result here is the following:

Theorem VI. *For any $f \geq 0, f \neq 0$, there exist exactly two solutions γ_f^1, γ_f^2 to (3.1). Up to relabelling, we have*

$$\gamma_f^1 > 0, \gamma_f^2 < 0 \text{ and } \langle \gamma_f^1 \rangle_{[0,T]} = -\langle \gamma_f^2 \rangle_{[0,T]}.$$

Finally, with the notations of (3.2),

$$\theta_f(\langle \gamma_f^1 \rangle_{[0,T]}) = \theta_f(\langle \gamma_f^2 \rangle_{[0,T]}) = 0.$$

The proof of this theorem is lengthy and relies on several intermediate results. We begin with the following proposition.

Proposition 8. *For any $f \in L_{\text{per}}^\infty((0, T))$ there exists a solution γ_f to (3.1) if, and only if, there exists $\alpha_f > 0$ such that*

$$\theta_f(\alpha_f) = 0.$$

Moreover, in this case, one has $\alpha_f = \langle \gamma_f \rangle_{[0,T]}$.

Proof of Proposition 8. First of all, assume that there exists a solution γ_f to (3.1) and define $y_f : t \mapsto e^{\int_0^t (\gamma_f - \langle \gamma_f \rangle)}$. y_f is periodic, and direct computations show that $-y_f'' - 2\langle \gamma_f \rangle_{[0,T]} y_f' - \langle \gamma_f \rangle_{[0,T]}^2 y_f = -f y_f$. As $y_f > 0$ this rewrites $\theta_f(\langle \gamma_f \rangle_{[0,T]}) = 0$. Conversely, assume that for a given α we have $\theta_f(\alpha) = 0$. In that case, define $\gamma_f := \alpha + y_f'/y_f$. Direct computations show $\gamma_f' + \gamma_f^2 = f$. \square

We continue with a basic estimate on (3.2)

Lemma 9. *For any $f \in L_{\text{per}}^\infty((0, T)), f \geq 0, f \neq 0$, there holds*

$$\theta_f(\alpha) \xrightarrow{\alpha \rightarrow \pm\infty} -\infty, \theta_f(0) > 0.$$

Furthermore, there exists a unique $\alpha > 0$ such that $\theta_f(\alpha) = 0$ and a unique $\alpha' < 0$ such that $\theta_f(\alpha') = 0$.

Proof of Lemma 9. For $\alpha = 0$ the eigenvalue $\theta_f(0)$ has the variational formulation

$$\theta_f(0) = \min_{y \in W_{\text{per}}^{1,2}((0,T)), \int_0^T y^2 = 1} \left(\int_0^T (y')^2 + \int_0^T y^2 f \right).$$

$W_{\text{per}}^{1,2}((0, T))$ stands for the set of T -periodic $W^{1,2}$ functions. As $f \geq 0, \neq 0$ we deduce that $\theta_f(0) > 0$. Furthermore, a direct computation proves that $\frac{\partial \theta_f(\alpha)}{\partial \alpha} = -2\alpha$. The conclusion follows. \square

We now establish the following lemma:

Lemma 10. *Let $f \geq 0, f \neq 0$ and assume γ_1, γ_2 are two distinct solutions of (3.1). Then there holds (up to relabelling) $0 < \langle \gamma_1 \rangle_{[0,T]} = -\langle \gamma_2 \rangle_{[0,T]}$.*

Proof of Lemma 10. Define $\tilde{\gamma} := \gamma_1 - \gamma_2$. Then $\tilde{\gamma}$ solves $\tilde{\gamma}' + \tilde{\gamma}(\gamma_1 + \gamma_2) = 0$. Upon integration, this implies $\tilde{\gamma}(T) = \tilde{\gamma}(0)e^{-T\langle \gamma_2 \rangle_{[0,T]} - T\langle \gamma_1 \rangle_{[0,T]}}$, whence, as $\tilde{\gamma} \neq 0$, $\langle \gamma_1 \rangle_{[0,T]} = -\langle \gamma_2 \rangle_{[0,T]}$. As $\theta_f(0) > 0$, $\langle \gamma_1 \rangle_{[0,T]} \neq 0$ and the conclusion follows. \square

Corollary 11. *There exists at most one non-negative and one non-positive solution of (3.1).*

Lemma 12. *Let $f \geq 0, \neq 0$ and γ be a solution of (3.1) with $\langle \gamma \rangle_{[0,T]} > 0$. Then $\gamma > 0$ in $(0, T)$. Similarly, if $\langle \gamma \rangle_{[0,T]} < 0$, then $\gamma < 0$ in $(0, T)$.*

Proof of Lemma 12. Assume that γ solves (3.1) and satisfies $\langle \gamma \rangle_{[0,T]} > 0$. Up to a translation, we can assume that $\gamma(0) > 0$. Since $f \geq 0$, we deduce that $\gamma' + \gamma^2 \geq 0$ whence, from the comparison principle for ODEs, we deduce $\gamma(t) \geq \frac{1}{t + \frac{1}{\gamma(0)}}$ and the conclusion follows. If, on the other hand, $\langle \gamma \rangle_{[0,T]} < 0$, it suffices to observe that $\tilde{\gamma} := -\gamma(T-t)$ solves $\tilde{\gamma}' + \tilde{\gamma}^2 = f(T-\cdot)$, $\langle \tilde{\gamma} \rangle_{[0,T]} > 0$. We are reduced to the previous case. \square

Proof of Theorem VI. From Lemma 9, let $\alpha_2 < 0 < \alpha_1$ be such that $\theta_f(\alpha_i) = 0$, $i = 1, 2$, and let γ_f^i ($i = 1, 2$) be two solutions of (3.1) with $\langle \gamma_f^i \rangle_{[0,T]} = \alpha_i$. From Lemma 10 γ_f^i is the unique solution of (3.1) with $\langle \gamma_f^1 \rangle_{[0,T]} > 0$ and $\langle \gamma_f^2 \rangle_{[0,T]} < 0$. From Lemma 12, we have $\gamma_f^1 > 0$, $\gamma_f^2 < 0$ in $(0, T)$, which concludes the proof. \square

3.1.2. *Analysis of (1.8).* We begin with the following proposition:

Proposition 13. *If $c \geq 0$, there exists a unique $(\bar{\lambda}(c), w_0)$ of (1.8).*

After building a particular solution to (1.8), the difficult part in this proposition is the uniqueness, which ultimately relies on the Donsker-Varadhan formula. Observe that Lions implicitly uses this approach to derive uniqueness results for spectral problems in [27, Lecture 7]. We begin with the construction of a particular solution to (1.8).

Lemma 14. *Let P be an orthogonal matrix such that*

$$A = P \text{diag}(\mu_1, \dots, \mu_d) P^T,$$

where $0 < \mu_1 \leq \dots \leq \mu_d$ are the eigenvalues of A . An explicit solution of (1.8) is given by

$$\bar{\lambda}(c) = \frac{1}{T} \int_0^T \text{Tr}(B_c), \quad w_0(t, x) = e^{\beta(t) - \frac{1}{2} \langle x, B_c(t)x \rangle}$$

with

$$B_c(t) = P \text{diag}(\xi_1(t), \dots, \xi_d(t)) P^T \text{ where, for any } i, \forall i \in \{1, \dots, d\} \begin{cases} \frac{\xi_i'(t)}{2} + \xi_i(t)^2 = \frac{\mu_i}{2} c(t), \\ \xi_i(0) = \xi_i(T), \\ \xi_i > 0 \end{cases}$$

and

$$\beta(t) = \int_0^t (\bar{\lambda}(c) - \text{Tr}(B_c(t))) dt.$$

Proof of Lemma 14. We first look for a solution $(\bar{\lambda}(c), w_0)$ of (1.8) with w_0 writing $w_0 = e^{\beta(t) - \frac{1}{2} \langle x, B_c(t)x \rangle}$ for γ a function to be determined and B_c a symmetric positive definite matrix-valued function. Direct computations show that, for any such w_0 and for any $\bar{\lambda}$, $(\bar{\lambda}, w_0)$ solves (1.8) if, and only if,

$$(3.3) \quad \begin{cases} \beta'(t) - \frac{1}{2} \langle x, B_c'(t)x \rangle + \text{Tr}(B_c(t)) - \langle x, B_c(t)^2 x \rangle = \bar{\lambda} - \frac{c(t)}{2} \langle x, Ax \rangle, \\ \beta, B_c \text{ are } T\text{-periodic}, B_c(t) \in S_d^{++}(\mathbb{R}). \end{cases}$$

In particular, if we can find a solution B_c of the differential equation

$$(3.4) \quad \frac{B_c'(t)}{2} + B_c(t)^2 = \frac{c(t)}{2} A, \quad B_c(T) = B_c(0), \quad B_c(\cdot) \in S_d^{++}(\mathbb{R})$$

then, defining

$$\beta : t \mapsto \int_0^t ((\text{Tr}(B_c))_{[0,T]} - \text{Tr}(B_c)), \quad \bar{\lambda} := \int_0^T \text{Tr}(B_c)$$

it follows that $(\bar{\lambda}, w_0)$ is a solution of (1.8). Let us prove that a solution to (3.4) exists. We look for B_c in the form $B_c = P \text{diag}(\xi_1(t), \dots, \xi_d(t)) P^T$ so that (3.4) rewrites

$$(3.5) \quad \forall i \in \{1, \dots, d\} \begin{cases} \frac{\xi_i'(t)}{2} + \xi_i(t)^2 = \frac{\mu_i}{2} c(t), \\ \xi_i(0) = \xi_i(T), \\ \xi_i > 0. \end{cases}$$

From Theorem VI, ξ_i exists, and is uniquely defined, whence the conclusion. \square

Proof of Proposition 13. We let $(\bar{\lambda}_0, w_0)$ be the solution of (1.8) provided by Lemma 14. First of all observe that, if $(\bar{\lambda}, w)$ is a solution of (1.8) then, since $\|w_0\|_{L^\infty} \leq A e^{-A\|x\|^2}$ by construction, it follows from integration by parts that $\bar{\lambda} = \bar{\lambda}_0$. Second, let $\bar{\eta}_0$ be defined as a solution of

$$(3.6) \quad \begin{cases} -\partial_t \eta_0 - \Delta \eta_0 - \nabla \cdot \left(\frac{\nabla_x w_0}{w_0} \eta_0 \right) = 0 & \text{in } (0, T) \times \mathbb{R}^d, \\ \forall t \in (0, T), \int_{\mathbb{R}^d} \eta_0(t, \cdot) = 1, \\ \eta_0 \geq 0, \eta_0(T, \cdot) = \eta_0(0, \cdot). \end{cases}$$

That such an η_0 exists follows from the same construction as in Lemma 14: define \bar{z}_0 as the solution of

$$(3.7) \quad \begin{cases} -\partial_t z_0 - \Delta z_0 = \bar{\lambda}_0 z_0 - \frac{1}{2} c(t) \langle x, Ax \rangle z_0 & \text{in } [0, T] \times \mathbb{R}^d, \\ z_0(T, \cdot) = z_0(0, \cdot), \\ z_0(0, 0) = 1 = \|z_0\|_{L^\infty}, \\ z_0 \geq 0, \end{cases}$$

which can be done exactly in the same way as in Lemma 14, and set $\eta_0 := w_0 z_0$. Observe that $\|\eta_0(t, x)\| \lesssim C e^{-A\|x\|^2}$ for some constant A . Now, consider $\varphi_0 := -\ln(w_0)$ and define the functional

$$\mathcal{E} : \varphi \mapsto \iint_{(0,T) \times \mathbb{R}^d} \left(\partial_t \varphi - \Delta \varphi + \|\nabla \varphi\|^2 - \frac{c}{2} \langle x, Ax \rangle \right) d\eta_0.$$

Since \mathcal{E} is strictly convex in φ and since φ_0 is one of its critical points we deduce that

$$\forall \varphi, \mathcal{E}(\varphi) \geq \mathcal{E}(\varphi_0) = \bar{\lambda}_0.$$

Furthermore, observe that, for any φ , if we set $\bar{\varphi} := \varphi - \varphi_0$,

$$\mathcal{E}(\varphi) - \mathcal{E}(\varphi_0) = \iint_{(0,T) \times \mathbb{R}^d} \|\nabla \bar{\varphi}\|^2 d\eta_0.$$

In particular, let $(\bar{\lambda}_0, w)$ be a solution of (1.8). Then, setting $\varphi = -\ln(w)$ we have $0 = \mathcal{E}(\varphi) - \mathcal{E}(\varphi_0) = \iint_{(0,T) \times \mathbb{R}^d} \|\nabla \bar{\varphi}\|^2 d\eta_0$. We deduce that there exists a constant a such that $\varphi = \varphi_0 + a$. Since $\varphi(0, 0) = \varphi_0(0, 0) = 1$ we have $a = 0$ and the conclusion follows. \square

3.2. Proof of Theorem II.

Proof of Theorem II. We know from Lemma 14 that

$$\bar{\lambda}(c) = \sum_{i=1}^d \langle \xi_{i,c} \rangle_{[0,T]}$$

where, for every $i \in \{1, \dots, d\}$,

$$\begin{cases} \frac{\xi'_{i,c}(t)}{2} + \xi_{i,c}(t)^2 = \frac{\mu_i}{2} c(t), \\ \xi_{i,c}(0) = \xi_{i,c}(T), \\ \xi_{i,c} > 0. \end{cases}$$

Recall that the μ_i are the eigenvalues of $A = -\nabla^2 V(0)$. If we can prove that

$$(3.8) \quad \forall i \in \{1, \dots, d\},$$

$\langle \xi_{i,c} \rangle_{[0,T]} \geq \langle \xi_{i,c^\#} \rangle_{[0,T]}$ with equality if, and only if, up to translation: $c = c^\#$

then the conclusion follows. We thus prove (3.8). With the notations of Theorem VI we know that $\theta_{\mu_i c/2}(\langle \xi_{i,c} \rangle_{[0,T]}) = 0$ and that, for any given c , the map $\theta_{\mu_i c/2}$ is strictly decreasing on $(0; +\infty)$. Furthermore we know from [33, Theorem 1.2] that, for any f such that, for any τ , $f(\cdot + \tau) \not\equiv f^\#$, for any $\alpha > 0$, $\theta_f(\alpha) > \theta_{f^\#}(\alpha)$. Consequently, $\theta_{\mu_i c^\#/2}(\langle \xi_{i,c} \rangle_{[0,T]}) < 0$ and thus (3.8) holds. \square

4. PROOF OF THEOREM III

4.1. Asymptotic behaviour of $(\lambda(c, \mu), u_{c,\mu})$ and limit problem. For any $m \in L^\infty((0, T) \times \mathbb{T}^d)$, we let $\bar{\varphi}_m$ be the unique solution of

$$(4.1) \quad \begin{cases} \partial_t \bar{\varphi}_m - \Delta \bar{\varphi}_m = m - \iint_{(0,T) \times \mathbb{T}^d} m & \text{in } (0, T) \times \mathbb{T}^d, \\ \bar{\varphi}_m(0, \cdot) = \bar{\varphi}_m(T, \cdot), \\ \iint_{(0,T) \times \mathbb{T}^d} \bar{\varphi}_m = 0, \end{cases}$$

and we define

$$\Lambda(m) := \iint_{(0,T) \times \mathbb{T}^d} |\nabla \bar{\varphi}_m|^2.$$

The main result of this section is the following:

Proposition 15. *There exists a constant M that only depends on $\|m\|_{L^\infty((0,T) \times \mathbb{T}^d)}$, T and d such that*

$$(4.2) \quad \left| \lambda(m, \mu) + \iint_{(0,T) \times \mathbb{T}^d} m - \frac{\Lambda(m)}{\mu} \right| \leq M \frac{\Lambda(m)}{\mu^{\frac{3}{2}}}.$$

Remark 16 (On the exponent $3/2$). *We could improve the exponent to 2 at the cost of greater technicality. Nevertheless, this suboptimal exponent is sufficient for our purposes.*

Proof of Proposition 15. We work with a fixed m so that we write u_μ for the solution of

$$(4.3) \quad \mu(\partial_t u_\mu - \Delta u_\mu) = \lambda(m, \mu)u_\mu + m u_\mu$$

endowed with periodic boundary conditions and normalised with $\iint_{(0,T) \times \mathbb{T}^d} u_\mu^2 = 1$.

Uniform bounds on $\lambda(m, \mu)$ First of all, observe that, by writing $u_\mu := e^{-\varphi_\mu}$, then φ_μ solves

$$\mu \partial_t \varphi_\mu - \mu \Delta \varphi_\mu + \mu |\nabla \varphi_\mu|^2 = -\lambda(m, \mu) - m,$$

endowed with time-space periodic boundary conditions. Integrating this equation on $(0, T) \times \mathbb{T}^d$, we deduce that there holds $\lambda(m) = -\iint_{(0,T) \times \mathbb{T}^d} m - \mu \iint_{\mathbb{T}^d} |\nabla \varphi_\mu|^2 \leq -\iint_{(0,T) \times \mathbb{T}^d} m \leq \|m\|_{L^\infty((0,T) \times \mathbb{T}^d)}$. Second, taking a point of minimum of φ_μ in $(0, T) \times \mathbb{T}^d$ we obtain $\lambda(m) \geq -\|m\|_{L^\infty((0,T) \times \mathbb{T}^d)}$. We deduce

$$(4.4) \quad \forall \mu > 0, |\lambda(m, \mu)| \leq \|m\|_{L^\infty((0,T) \times \mathbb{T}^d)}.$$

Uniform convergence of u_μ

Lemma 17. *There exists M that only depends on $\|m\|_{L^\infty((0,T) \times \mathbb{T}^d)}$ such that for all μ large enough*

$$(4.5) \quad \|u_\mu - 1\|_{L^\infty((0,T) \times \mathbb{T}^d)} \leq \frac{M}{\mu}.$$

Proof of Lemma 17. We first prove that there exists M_0 such that, for any μ large enough

$$(4.6) \quad \sup_{t \in [0, T]} \int_{\mathbb{T}^d} u_\mu(t, \cdot) \leq M_0.$$

Proof of (4.6). Integrate the equation on u_μ in space, pass to the absolute value, integrate in time and use the Cauchy-Schwarz inequality and (4.4) to obtain

$$\mu \int_0^T \left| \frac{d}{dt} \int_{\mathbb{T}^d} u_\mu \right| \leq 2 \|m\|_{L^\infty((0,T) \times \mathbb{T}^d)}.$$

Thus $I : t \mapsto \int_{\mathbb{T}^d} \mu u_\mu$ is uniformly bounded in $BV(0, T)$, whence

$$(4.7) \quad \sup_{t \in [0, T]} \left| \int_{\mathbb{T}^d} u_\mu(t, \cdot) - \iint_{(0,T) \times \mathbb{T}^d} u_\mu \right| \leq \frac{M_1}{\mu}.$$

Using the Cauchy-Schwarz inequality again, we deduce that there exists M_0 such that (4.6) holds. \square

Observe that (4.6) implies, once again integrating (4.3) in \mathbb{T}^d , that there exists $M > 0$ such that:

$$(4.8) \quad \sup_{t \in [0, T]} \left| \frac{d}{dt} \int_{\mathbb{T}^d} u_\mu \right| \leq \frac{M}{\mu}.$$

Let us now prove

$$(4.9) \quad \text{for any } p \in [1; \infty) \text{ there holds } \sup_{t \in [0, T]} \mu \|\nabla_x u_\mu\|_{L^p(\mathbb{T}^d)} \leq M_p < \infty.$$

Consider the function $\hat{u}_\mu = u_\mu - \int_{\mathbb{T}^d} u_\mu$. From (4.6) and (4.8), there holds

$$\partial_t(\mu \hat{u}_\mu) - \Delta(\mu \hat{u}_\mu) - m \hat{u}_\mu - \lambda \hat{u}_\mu \in L^\infty((0, T) \times \mathbb{T}^d).$$

From standard parabolic regularity and a bootstrap argument, (4.9) holds.

In particular, choosing p large enough, we conclude from the Sobolev embedding $W^{1,p}(\mathbb{T}^d) \hookrightarrow \mathcal{C}^0(\mathbb{T}^d)$ that

$$(4.10) \quad \sup_{t \in [0, T]} \mu \left\| u_\mu(t, \cdot) - \int_{\mathbb{T}^d} u_\mu(t, \cdot) \right\|_{L^\infty(\mathbb{T}^d)} \leq M.$$

From (4.6) and up to modifying M we derive

$$(4.11) \quad \|u_\mu\|_{L^\infty((0, T) \times \mathbb{T}^d)} \leq M.$$

Finally, let us prove that

$$(4.12) \quad \sup_{t \in [0, T]} \left| \int_{\mathbb{T}^d} u_\mu(t, \cdot) - 1 \right| \leq \frac{M}{\mu}.$$

As, by parabolic regularity,

$$\iint_{(0, T) \times \mathbb{T}^d} (\partial_t u_\mu)^2 \leq \frac{M}{\mu^2}$$

the estimate (4.11) implies that the map $I_2 : t \mapsto \int_{\mathbb{T}^d} u_\mu^2$ satisfies

$$\int_0^T (I_2'(t))^2 dt \leq \frac{M}{\mu^2}.$$

Since $\iint_{(0, T) \times \mathbb{T}^d} u_\mu^2 = 1$ we deduce from the Sobolev embedding $W^{1,2}(0, T) \hookrightarrow \mathcal{C}^0(0, T)$ that

$$\sup_{t \in [0, T]} \left| \int_{\mathbb{T}^d} u_\mu^2 - 1 \right| \leq \frac{M}{\mu}.$$

However, by the Poincaré inequality

$$(4.13) \quad \left| \int_{\mathbb{T}^d} u_\mu^2 - \left(\int_{\mathbb{T}^d} u_\mu \right)^2 \right| \leq \int_{\mathbb{T}^d} |\nabla_x u_\mu|^2 \leq \frac{M}{\mu^2}.$$

This implies (4.12). Going back to (4.10) we finally obtain

$$\sup_{t \in [0, T]} \|u_\mu(t, \cdot) - 1\|_{L^\infty(\mathbb{T}^d)} \leq \frac{M}{\mu}.$$

□

Uniform convergence of the invariant measure associated with the system

Introduce, for any $\mu > 0$, the solution η_μ of

$$(4.14) \quad \begin{cases} \mu \left(-\partial_t \eta_\mu - \Delta \eta_\mu - 2\nabla \cdot \left(\eta_\mu \frac{\nabla_x u_\mu}{u_\mu} \right) \right) = 0 & \text{in } (0, T) \times \mathbb{T}^d, \\ \eta_\mu \geq 0, \iint_{(0, T) \times \mathbb{T}^d} \eta_\mu = 1. \end{cases}$$

Let v_μ be the solution of the adjoint problem

$$(4.15) \quad \begin{cases} \mu (-\partial_t v_\mu - \Delta v_\mu) = \lambda(m, \mu) v_\mu + m v_\mu & \text{in } (0, T) \times \mathbb{T}^d, \\ v_\mu \geq 0, \iint_{(0, T) \times \mathbb{T}^d} v_\mu u_\mu = 1. \end{cases}$$

Then direct computations show that

$$\eta_\mu = u_\mu v_\mu.$$

Adapting *mutatis mutandis* the proof of (4.5), there holds

$$\|v_\mu - 1\|_{L^\infty((0,T) \times \mathbb{T}^d)} \leq \frac{M}{\mu}.$$

Consequently we obtain

$$(4.16) \quad \|\eta_\mu - 1\|_{L^\infty((0,T) \times \mathbb{T}^d)} \leq \frac{M}{\mu}.$$

Using the Donskers-Varadhan formula Use again the Hopf-Cole transform by writing $u_\mu = e^{-\varphi_\mu}$, where φ_μ solves

$$\mu (\partial_t \varphi_\mu - \Delta \varphi_\mu + |\nabla \varphi_\mu|^2) = -\lambda(m, \mu) - m.$$

Then it appears that $\lambda(m, \mu)$ admits the variational formulation

$$-\lambda(m, \mu) = \max_{\eta \in \mathbb{P}((0,T) \times \mathbb{T}^d)} \min_{\phi \in L^2(0,T; W^{1,2}(\Omega))} \iint_{(0,T) \times \mathbb{T}^d} (\mu (\partial_t \phi - \Delta \phi + |\nabla \phi|^2) + m) d\eta$$

where $\mathbb{P}((0,T) \times \mathbb{T}^d)$ is the set of probability measures on $(0,T) \times \mathbb{T}^d$. Indeed, we can check immediately that (φ_μ, η_μ) is a solution of this saddle point problem and the uniqueness of a solution follows from the same arguments as in the proof of Theorem I. We also refer to [30]. In particular, as (φ_μ, η_μ) is a solution of this saddle point problem, we deduce, using $(\bar{\eta} \equiv \text{Vol}((0,T) \times \mathbb{T}^d), \varphi_\mu)$ and $(\eta_\mu, \frac{1}{\mu} \bar{\varphi}_m)$ (recall that $\bar{\varphi}_m$ is defined by (4.1)) as test couples, that we have the estimate

$$\mu \iint_{(0,T) \times \mathbb{T}^d} |\nabla \varphi_\mu|^2 \leq -\lambda(m, \mu) - \iint_{(0,T) \times \mathbb{T}^d} m \leq \frac{1}{\mu} \iint_{(0,T) \times \mathbb{T}^d} |\nabla \bar{\varphi}_m|^2 \eta_\mu.$$

Using (4.16) this implies that

$$(4.17) \quad -\lambda(m, \mu) - \iint_{(0,T) \times \mathbb{T}^d} m - \frac{\Lambda(m)}{\mu} \leq \frac{1}{\mu^2} \iint_{(0,T) \times \mathbb{T}^d} |\nabla \bar{\varphi}_m|^2 \leq M \frac{\Lambda(m)}{\mu^2}.$$

To conclude the proof, we thus need to show that

$$(4.18) \quad \left| \mu \iint_{(0,T) \times \mathbb{T}^d} |\nabla \varphi_\mu|^2 - \frac{\Lambda(m)}{\mu} \right| \leq M \frac{\Lambda(m)}{\mu^{\frac{3}{2}}}.$$

Observe that, introducing $\bar{\varphi}_\mu := \mu \varphi_\mu$, (4.18) is equivalent to proving that

$$(4.19) \quad \left| \iint_{(0,T) \times \mathbb{T}^d} |\nabla \bar{\varphi}_\mu|^2 - \iint_{(0,T) \times \mathbb{T}^d} |\nabla \bar{\varphi}_m|^2 \right| \leq M \frac{\Lambda(m)}{\sqrt{\mu}}.$$

A first observation is that $\bar{\varphi}_\mu = -\mu \ln(u_\mu)$ so that (4.5) implies

$$(4.20) \quad \|\bar{\varphi}_\mu\|_{L^\infty((0,T) \times \mathbb{T}^d)} \leq M.$$

Define $z_\mu := \bar{\varphi}_\mu - \bar{\varphi}_m$, which solves

$$\partial_t z_\mu - \Delta z_\mu + \frac{1}{\mu} |\nabla \bar{\varphi}_\mu|^2 = -\lambda(m, \mu) - \iint_{(0,T) \times \mathbb{T}^d} m.$$

Up to replacing z_μ with $z_\mu + A$ where A is independent of μ we can without loss of generality assume that

$$0 \leq z_\mu \leq M$$

for some constant M independent of μ thanks to (4.20). Multiplying by z_μ and integrating by parts gives

$$\begin{aligned} \iint_{\mathbb{T}^d} |\nabla z_\mu|^2 &\leq \iint_{(0,T) \times \mathbb{T}^d} |\nabla z_\mu|^2 + \frac{1}{\mu} \iint_{(0,T) \times \mathbb{T}^d} z_\mu |\nabla \bar{\varphi}_\mu|^2 \\ &\leq \left(-\lambda(m, \mu) - \iint_{(0,T) \times \mathbb{T}^d} m \right) \left(\int_0^T \left\| z_\mu - \int_{\mathbb{T}^d} z_\mu \right\|_{L^2(\mathbb{T}^d)} + \|z_\mu\|_{L^\infty((0,T) \times \mathbb{T}^d)} \right). \end{aligned}$$

The Jensen inequality, the Poincaré inequality and (4.17) yield that, by setting $X_\mu := \left(\iint_{(0,T) \times \mathbb{T}^d} |\nabla z_\mu|^2 \right)^{\frac{1}{2}}$, we have

$$X_\mu^2 \leq \frac{\Lambda(m)}{\mu} X_\mu + \frac{M\Lambda(m)}{\mu}.$$

We deduce that for μ large enough there holds (for a constant still denoted by M)

$$|X_\mu| \leq M \sqrt{\frac{\Lambda(m)}{\mu}}.$$

Thus

$$\begin{aligned} (4.21) \quad \left(\iint_{(0,T) \times \mathbb{T}^d} |\nabla \bar{\varphi}_\mu|^2 \right)^{\frac{1}{2}} &\leq M \sqrt{\frac{\Lambda(m)}{\mu}} + \left(\iint_{\mathbb{T}^d} |\nabla \bar{\varphi}_m|^2 \right)^{\frac{1}{2}} \\ &\leq M \left(\sqrt{\frac{\Lambda(m)}{\mu}} + \sqrt{\Lambda(m)} \right) \leq M \sqrt{\Lambda(m)}. \end{aligned}$$

Going back to (4.19),

$$\begin{aligned} \left| \iint_{(0,T) \times \mathbb{T}^d} |\nabla \bar{\varphi}_\mu|^2 - \iint_{(0,T) \times \mathbb{T}^d} |\nabla \bar{\varphi}_m|^2 \right| &= \left| \iint_{(0,T) \times \mathbb{T}^d} \langle \nabla z_\mu, \nabla(\bar{\varphi}_\mu + \bar{\varphi}_m) \rangle \right| \\ &\leq \left(\iint_{(0,T) \times \mathbb{T}^d} |\nabla z_\mu|^2 \right)^{\frac{1}{2}} \left(\iint_{(0,T) \times \mathbb{T}^d} |\nabla \bar{\varphi}_\mu + \nabla \bar{\varphi}_m|^2 \right)^{\frac{1}{2}} \\ &\leq M \sqrt{\frac{\Lambda(m)}{\mu}} \sqrt{\Lambda(m)} \\ &\leq M \frac{\Lambda(m)}{\sqrt{\mu}}. \end{aligned}$$

This concludes the proof of (4.19) and, consequently, the proof of Proposition 15. \square

4.2. Analysis of the asymptotic problem. In this section, we go back to our initial problem, so that the potential m writes as cV for a fixed potential V . With a slight abuse of notation, we denote by $\lambda(c, \mu), \Lambda(c), \bar{\varphi}_c$ the quantities $\lambda(cV, \mu), \Lambda(cV), \bar{\varphi}_{cV}$. As we know from Proposition 15 that

$$\lambda(c, \mu) = - \iint_{(0,T) \times \mathbb{T}^d} m - \frac{\Lambda(c)}{\mu} + \Lambda(c) \underset{\mu \rightarrow \infty}{o} \left(\frac{1}{\mu} \right)$$

we first investigate the maximisation problem

$$(4.22) \quad \max_{c \in \mathcal{C}} \Lambda(c).$$

Recall that

$$\Lambda(c) = \iint_{(0,T) \times \mathbb{T}^d} |\nabla \bar{\varphi}_c|^2 \text{ with } \begin{cases} \partial_t \bar{\varphi}_c - \Delta \bar{\varphi}_c = c(t)V(x) - \int_0^T c \cdot f_{\mathbb{T}^d} V & \text{in } (0,T) \times \mathbb{T}^d, \\ \bar{\varphi}_c(0, \cdot) = \bar{\varphi}_c(T, \cdot). \end{cases}$$

The main proposition is the following:

Proposition 18. *Assume V is not constant. For any $c \in \mathcal{C}$ (see Eq. (1.5)), there holds*

$$\Lambda(c) \leq \Lambda(\bar{c}) \text{ with equality if, and only if, there exists } \tau_0 \text{ such that } c(\tau_0 + \cdot) = \bar{c}.$$

Proof of Proposition 18. Decompose V in the basis $(\lambda_k, \psi_k)_{k \in \mathbb{N}}$ of Laplace eigenfunctions of \mathbb{T}^d as

$$V(x) = \sum_{k=0}^{\infty} a_k \psi_k.$$

The function $\bar{\varphi}_c$ writes

$$\bar{\varphi}_c = \sum_{k=0}^{\infty} a_k \alpha_{k,c}(t) \psi_k(x) \text{ where } \begin{cases} \alpha_{0,c} : t \mapsto \int_0^t (c - \langle c \rangle), \\ \forall k \geq 1, \alpha'_{k,c}(t) + \lambda_k \alpha_{k,c}(t) = c(t) & \text{in } [0, T], \\ \alpha_{k,c}(0) = \alpha_{k,c}(T). \end{cases}$$

In particular,

$$\Lambda(c) = \sum_{k=1}^{\infty} \lambda_k a_k^2 \int_0^T \alpha_{k,c}^2(t) dt.$$

We introduce, for any $k \geq 1$,

$$\Lambda_k(c) := \int_0^T \alpha_{k,c}^2(t) dt$$

and we investigate the optimisation problem

$$\max_{c \in \mathcal{C}} \Lambda_k(c).$$

Let us show that, for any $k \in \mathbb{N}$,

$$(4.23)$$

$$\Lambda_k(c) \leq \Lambda_k(\bar{c}) \text{ with equality if, and only if, there exists } \tau_0 \text{ such that } c(\tau_0 + \cdot) = \bar{c}.$$

Proof of (4.23). We rewrite Λ_k as the Dirichlet energy of a certain elliptic problem. For any $k \in \mathbb{N}$, let $\beta_{k,c}$ be the solution of

$$(4.24) \quad \begin{cases} -\beta'_{k,c}(t) + \lambda_k \beta_{k,c}(t) = \alpha_{k,c}(t) & \text{in } [0, T], \\ \beta_{k,c}(0) = \beta_{k,c}(T). \end{cases}$$

Multiplying (4.24) by $\alpha_{k,c}$ and integrating by parts we obtain

$$\Lambda_k(c) = \int_0^T \alpha_{k,c}^2(t) = \int_0^T \alpha_{k,c}(-\beta'_{k,c} + \lambda_k \beta_{k,c}) = \int_0^T \beta_{k,c}(\alpha'_{k,c} + \lambda_k \alpha_{k,c}) = \int_0^T \beta_{k,c} c.$$

Now, observe that

$$-\beta''_{k,c} + \lambda_k^2 \beta_{k,c} = \left(\frac{d}{dt} + \lambda_k \right) \left(-\frac{d}{dt} + \lambda_k \right) \beta_{k,c} = c.$$

Consequently, $\beta_{k,c}$ solves

$$\min_{\beta \text{ } T\text{-periodic}, \beta \in W^{1,2}(0,T)} \left(E(\beta, c) := \frac{1}{2} \int_0^T (\beta')^2 + \frac{\lambda_k^2}{2} \int_0^T \beta^2 - \int_0^T \beta c \right)$$

and

$$\int_0^T \beta_{k,c} c = -2E(\beta_{k,c}, c).$$

Thus, (4.23) is equivalent to

$$(4.25) \quad \min_c E(\beta_{k,c}, c) = \min_{\beta, c} E(\beta, c).$$

Introduce the increasing periodic rearrangement $f_{\#}$ of a function f (it is the same notion as the one given in Definition 1 with the requirement that $f^{\#}$ be decreasing replaced with the requirement that it should be increasing), and that the Polyá-Szegő and Hardy-Littlewood inequalities are also valid for the periodic rearrangement: for any $f \in W^{1,2}(0, T)$ and any $f, g \in W^{1,2}(0, T)$, $f, g \geq 0$, there holds

$$(4.26) \quad \int_0^T (f')^2 \geq \int_0^T ((f_{\#})')^2, \int_0^T fg \geq \int_0^T f_{\#} g^{\#}.$$

In particular, this implies, for any (β, c)

$$2E(\beta, c) \geq 2E(\beta_{\#}, c^{\#}) \geq -\Lambda_k(c^{\#}).$$

This gives

$$\forall c \in \mathcal{C}, \Lambda_k(c) \leq \Lambda_k(c^{\#}).$$

Now observe that

$$\Lambda_k(c) = \Lambda_k(c^{\#}),$$

implies equality in (4.26) for $f = \beta_{k,c}$ and that $\beta_{k,c}^{\#} = \beta_{k,\bar{c}}$. Since the Polyá-Szegő inequality is rigid, this implies that there exists $\tau_0 > 0$ such that

$$\beta_{k,c}(\tau_0 + \cdot) = \beta_{k,c}^{\#}.$$

Using the equation satisfied by $\beta_{k,c}$, this proves that $c(\tau_0 + \cdot) = c^{\#}$, thereby yielding (4.23). \square

Back to the proof of Proposition 18 We know that

$$\Lambda(c) = \sum_{k=1}^{\infty} \lambda_k a_k^2 \Lambda_k(c).$$

From (4.23) we deduce that

$$\Lambda(c) \geq \Lambda(c^{\#}).$$

If we have equality, then there holds, for any $k \in \mathbb{N}$,

$$a_k^2 \Lambda_k(c) = a_k^2 \Lambda_k(c^{\#}).$$

Since V is not constant, there exists $k \geq 1$ such that $a_k \neq 0$, and, for this k , (4.23) implies that, up to a translation by τ_0 (which, without loss of generality, we take to be zero), $c = c^{\#}$. \square

5. PROOF OF THEOREM IV

Recall that we work with the following eigenvalue problem (where, up to a translation we assume that the eigenfunction reaches its maximum at $(t^*, x^*) = (0, 0)$)

$$\begin{cases} \varepsilon \partial_t u_{\varepsilon, V} - \varepsilon^2 \Delta u_{\varepsilon, V} = \lambda_\varepsilon(V) u_{\varepsilon, V} + c(t) V(x) u_{\varepsilon, V} & \text{in } (0, T) \times \mathbb{T}^d, \\ u_{\varepsilon, V}(0, 0) = 1 = \|u_{\varepsilon, V}\|_{L^\infty((0, T) \times \mathbb{T}^d)}, \\ u_{\varepsilon, V}(T, \cdot) = u_{\varepsilon, V}(0, \cdot). \end{cases}$$

We assume that $\bar{c} \geq 0$ so that, for any $c \in \mathcal{C}$, we have $c \geq 0$. Our goal is to investigate the asymptotic behaviour of $\lambda_\varepsilon(V)$ as $\varepsilon \rightarrow 0$ and, more specifically, to prove the following proposition:

Proposition 19. *Let $A := -\nabla^2 V(0) \in S_d^{++}(\mathbb{R})$ and, for any $c \in \mathcal{C}$, let $\bar{\lambda}(c)$ be the principal parabolic, periodic in time eigenvalue of $\partial_t - \Delta + \frac{c}{2} \langle \cdot, A \cdot \rangle$ in \mathbb{R}^d (see (1.8)). Then*

$$\frac{\lambda_\varepsilon(c) + \langle c \rangle V(0)}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \bar{\lambda}(c).$$

We refer to Proposition 13 for the well-posedness of (1.8). We single Proposition 19 out as it is in itself interesting; however, as we will be working with a sequence of minimisers $\{c_\varepsilon\}_{\varepsilon \rightarrow 0}$ we also need a uniform version of this proposition (the proof of which is a minor adaptation of the proof of Proposition 19, see Remark 22).

Proposition 20. *With the same notations as in Proposition 19, let $\{c_\varepsilon\}_{\varepsilon \rightarrow 0}$ be a sequence of elements in \mathcal{C} that weakly converges to some c_∞ . Then*

$$\frac{\lambda_\varepsilon(c_\varepsilon) + V(0) \int_0^T c_\varepsilon}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \bar{\lambda}(c_\infty)$$

where $\bar{\lambda}(\cdot)$ is defined as the first eigenvalue of (1.8).

To prove Proposition 19, we rely on a blow-up result for eigenvalues of elliptic operators [19].

Proposition 21. [19, Proposition 2] *Let $\Phi \in \mathcal{C}^2(\mathbb{T}^d)$ be such that any local minimiser is non-degenerate and, for any $\varepsilon > 0$, let $\bar{\lambda}_\varepsilon(\Phi)$ be the first eigenvalue of the operator $-\varepsilon^2 \Delta + \Phi$ on \mathbb{T}^d . There exists a constant Λ that only depends on $\|\Phi\|_{\mathcal{C}^2}$ and \mathbb{T}^d such that there holds*

$$\min_{\mathbb{T}^d} \Phi \leq \bar{\lambda}_\varepsilon(\Phi) \leq \min_{\mathbb{T}^d} \Phi + \Lambda \varepsilon.$$

Proof of Proposition 19. Let us first show that we can assume, without loss of generality, that $V(0) = \max_{\mathbb{T}^d} V = 0$. Using the Hopf-Cole transform $\varphi_{\varepsilon, V} := -\ln(u_{\varepsilon, V})$, the function $\varphi_{\varepsilon, V}$ satisfies

$$\varepsilon \partial_t \varphi_{\varepsilon, V} - \varepsilon^2 \Delta \varphi_{\varepsilon, V} + \varepsilon^2 |\nabla \varphi_{\varepsilon, V}|^2 = -\lambda_\varepsilon(V) - c(t) V(x).$$

Let γ_ε be the unique solution of

$$\begin{cases} \varepsilon \gamma'_\varepsilon(t) = c(t) V(0) - \int_0^T c \cdot V(0), \\ \gamma_\varepsilon(T) = \gamma_\varepsilon(0). \end{cases}$$

Setting $\tilde{\varphi}_{\varepsilon, V} := \varphi_{\varepsilon, V} + \gamma_\varepsilon$, the function $\tilde{\varphi}_{\varepsilon, V}$ solves the Hamilton-Jacobi equation

$$\varepsilon \partial_t \tilde{\varphi}_{\varepsilon, V} - \varepsilon^2 \Delta \tilde{\varphi}_{\varepsilon, V} + \varepsilon^2 |\nabla \tilde{\varphi}_{\varepsilon, V}|^2 = -\lambda_\varepsilon(V) - \int_0^T c \cdot V(0) - c(t)(V(x) - V(0))$$

and up to replacing $\lambda_\varepsilon(V)$ with $\lambda_\varepsilon(V) + \int_0^T c \cdot V(0)$ we can thus assume that:

$$(5.1) \quad V(x) \leq V(0) = 0, \nabla^2 V(0) \in S_d^{++}(\mathbb{R}).$$

We now prove the following: if (5.1) holds, there exists a constant C such that

$$(5.2) \quad \forall \varepsilon > 0 \text{ small enough, } 0 \leq \frac{\lambda_\varepsilon(V)}{\varepsilon} \leq C.$$

The fact that $\lambda_\varepsilon(V) \geq 0$ follows from the maximum principle: at a point (t^*, x^*) of maximum of $u_{\varepsilon, V} > 0$ we obtain $\lambda_\varepsilon(V) + c(t^*)V(x^*) \geq 0$ and the conclusion follows since $V \leq 0$. In order to obtain the upper-bound on $\lambda_\varepsilon(V)/\varepsilon$ we use the fact that $V \leq 0$ so that $c(t)V(x) \geq \|c\|_{L^\infty} V(x)$. As a consequence $\lambda_\varepsilon(c) \leq \lambda_\varepsilon(\|c\|_{L^\infty})$. Without loss of generality, let us assume that $\|c\|_{L^\infty} = 1$. Thus if we define $\bar{\lambda}_\varepsilon(V)$ as the first eigenvalue of $-\varepsilon^2 \Delta - V$ we obtain $0 \leq \lambda_\varepsilon(c) \leq \bar{\lambda}_\varepsilon(V)$. As $V \leq V(0) = 0$ and as 0 is a non-degenerate critical point we deduce from Proposition 21 that $0 \leq \lambda_\varepsilon(V) \leq \Lambda \varepsilon$ for some constant Λ that only depends on $\|V\|_{\mathcal{C}^2}$. (5.2) follows.

Blow-up procedure We introduce the rescaled function $w_\varepsilon : (t, x) \mapsto u_{\varepsilon, V}(t, \sqrt{\varepsilon}x)$. Letting $\Omega \subset]-\pi; \pi[{}^d$ be a neighbourhood of 0, w_ε solves:

$$\begin{cases} \partial_t w_\varepsilon - \Delta w_\varepsilon = \frac{1}{\varepsilon} \lambda_\varepsilon(c) w_\varepsilon + c \frac{V(\sqrt{\varepsilon} \cdot)}{\varepsilon} w_\varepsilon & \text{in } (0, T) \times \frac{1}{\varepsilon} \Omega, \\ w_\varepsilon(T, \cdot) = w_\varepsilon(0, \cdot), \\ w_\varepsilon(0, 0) = 1. \end{cases}$$

Observe that locally uniformly

$$\frac{V(\sqrt{\varepsilon} \cdot)}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \left(x \mapsto \frac{1}{2} \langle x, \nabla^2 V(0)x \rangle \right).$$

Recall that $A := -\nabla^2 V(0) \in S_d^{++}(\mathbb{R})$. Let $\bar{\lambda}(c)$ be a closure point of the sequence $\frac{\lambda_\varepsilon(c)}{\varepsilon}$ as $\varepsilon \rightarrow 0$. As $0 \leq w_\varepsilon \leq 1$, proceeding as in [18, Chapter 2] we deduce from parabolic estimates that w_ε converges locally uniformly as $\varepsilon \rightarrow 0$ to a bounded function $w_0 \in \mathcal{C}^{1,2}(\mathbb{R}^d)$ that solves

$$(5.3) \quad \begin{cases} \partial_t w_0 - \Delta w_0 = \bar{\lambda}(c) w_0 - \frac{1}{2} c(t) \langle x, Ax \rangle w_0 & \text{in } [0, T] \times \mathbb{R}^d, \\ w_0(T, \cdot) = w_0(0, \cdot), \\ w_0(0, 0) = 1, \\ w_0 \geq 0. \end{cases}$$

As w_0 is bounded, non-zero and non-negative, the conclusion follows. \square

Remark 22. Observe that, at this stage, if we work with a weakly converging sequence $\{c_\varepsilon\}_{\varepsilon \rightarrow 0}$ rather than with a fixed c , the potential $(c_\varepsilon V(\sqrt{\varepsilon} \cdot)/\varepsilon)$ converges weakly to $-c_\infty \langle Ax, x \rangle$ and the same argument allows to conclude and to prove Proposition 20.

We can finally conclude the proof of the main theorem.

Proof of Theorem IV. We let, for any $\varepsilon > 0$, c_{\min}^ε be a solution of (1.12). Up to a subsequence, we know that $c_{\min}^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} c_0$ for some $c_0 \in \mathcal{C}$. We deduce from Proposition 19 that

$$\frac{\lambda_\varepsilon(c_{\min}^\varepsilon) + \langle c_{\min}^\varepsilon \rangle V(0)}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \bar{\lambda}(c_0).$$

Passing to the limit in

$$\lambda_\varepsilon(c_{\min}^\varepsilon) \leq \lambda_\varepsilon(c) \text{ for all } c \in \mathcal{C}$$

we deduce that

$$\forall c \in \mathcal{C}, \bar{\lambda}(c_0) \leq \bar{\lambda}(c).$$

From Theorem II we deduce that $c_0 = \bar{c}$ and thus we have that $\{c_{\min}^\varepsilon\}_{\varepsilon \rightarrow 0}$ converges weakly in $L^\infty - *$ to \bar{c} as $\varepsilon \rightarrow 0$. As \bar{c} is an extreme point of \mathcal{C} (see Remark 4), this convergence is strong in all $L^p((0, T))$. \square

6. PROOF OF THEOREM V

We let $H : p \mapsto p^{2k}$. The proof of Theorem V boils down to proving that \bar{c} does not satisfy first order optimality conditions for (1.15) if $k \neq 1$. We thus first investigate these optimality conditions.

Optimality conditions for (1.15) Introduce, for any $c \in \mathcal{C}$, the solution ψ_c of

$$(6.1) \quad \begin{cases} -\partial_t \psi_c - \Delta \psi_c = H'(\varphi_c) - \iint_{(0, T) \times \mathbb{T}^d} H'(\varphi_c), \\ \psi_c(T, \cdot) = \psi_c(0, \cdot), \\ \iint_{(0, T) \times \mathbb{T}^d} \psi_c = 0. \end{cases}$$

The first order optimality conditions read:

Lemma 23. *Let c_{\min} be a solution of (1.15) and define*

$$\Psi_{c_{\min}} : t \mapsto \int_{\mathbb{T}^d} \psi_{c_{\min}}(t, x) m(x) dx.$$

Then

$$\forall c \in \mathcal{C}, \int_0^T \Psi_{c_{\min}} c_{\min} \geq \int_0^T \Psi_{c_{\min}} c.$$

In particular, if \bar{c} is optimal, $\Psi_{\bar{c}}$ is symmetric with respect to $\frac{T}{2} = \pi$.

Proof of Lemma 23. The Gateaux differentiability of $J_H(c) := \iint_{(0, T) \times \mathbb{T}^d} H(\varphi_c)$ with respect to c is a consequence of parabolic regularity and, letting $\dot{\varphi}_c$ denote the Gateaux derivative of the solution mapping $\mathcal{C} \ni c \mapsto \varphi_c$ at c in a given direction $h \in L_{\text{per}}^\infty((0, T))$ with $\langle h \rangle = 0$, $\dot{\varphi}_c$ satisfies

$$(6.2) \quad \begin{cases} \partial_t \dot{\varphi}_c - \Delta \dot{\varphi}_c = h(t) m(x) \in (0, T) \times \mathbb{T}, \\ \dot{\varphi}_c(0, \cdot) = \dot{\varphi}_c(T, \cdot), \\ \langle \dot{\varphi}_c \rangle = 0. \end{cases}$$

Similarly, the Gateaux derivative of the map J_H admits the expression

$$\dot{J}_H(c)[h] = \iint_{(0, T) \times \mathbb{T}} \dot{\varphi}_c H'(\varphi_c).$$

Multiplying (6.2) by the solution ψ_c of (6.1) and integrating by parts, this gives

$$\dot{J}_H(c)[h] = \iint_{(0, T) \times \mathbb{T}} h(t) m(x) \psi_c(t, x) dt dx = \int_0^T h(t) \Psi_{c_{\min}}(t) dt.$$

In particular, if c_{\min} is a solution of (1.15), we deduce that, if we set $\Psi_{c_{\min}} : t \mapsto \int_{\mathbb{T}^d} m(x) \psi_{c_{\min}}(t, x) dx$, there holds

$$(6.3) \quad \forall c \in \mathcal{C}, \int_0^T \Psi_{c_{\min}}(t) c_{\min}(t) dt \geq \int_0^T \Psi_{c_{\min}}(t) c(t) dt.$$

Now, let us assume that \bar{c} is optimal and let us write, for the sake of convenience, Ψ for $\Psi_{\bar{c}}$. We know that for any $c \in \mathcal{C}$ we have

$$\int_0^T \Psi \bar{c} \geq \int_0^T \Psi c.$$

Now, assume by contradiction that Ψ is not symmetric. In order to alleviate notations, for any subset $E \subset (0, T/2)$, we let \tilde{E} be the symmetric of E with respect to $T/2$ and $\tilde{\Psi}$ be defined as

$$\tilde{\Psi} : t \mapsto \begin{cases} \Psi(T/2 - t) & \text{if } t \in (0, T/2), \\ \Psi(t) & \text{if } t \in (T/2, T). \end{cases}$$

Since Ψ is not symmetric, there exists an interval $I = (a, b) \subset (0, T/2)$ such that

$$\sup_{(a,b)} \tilde{\Psi} < \inf_{(a,b)} \Psi \quad \text{or} \quad \inf_{(a,b)} \tilde{\Psi} > \sup_{(a,b)} \Psi \quad \text{in } (a, b).$$

Since both cases are treated similarly, assume that

$$\sup_{(a,b)} \tilde{\Psi} < \inf_{(a,b)} \Psi.$$

The interval \tilde{I} is explicitly given by $\tilde{I} = (T - b, T - a)$. We first assume that $\Psi > 0$ in (a, b) . Observe that \bar{c} is decreasing. We let $d := (a + b)/2$ and we define c as

$$c : t \mapsto \begin{cases} \bar{c}(t) & \text{if } t \in (0, T) \setminus (I \cup \tilde{I}), \\ \bar{c}(t) & \text{if } t \in ((a + b)/2, b) \cup (T - (a + b)/2, T - a), \\ \bar{c}(t + T - b - a) & \text{if } t \in (a, (a + b)/2), \\ \bar{c}(t + (a + b) - T) & \text{if } t \in (T - b, T - (a + b)/2). \end{cases}$$

Observe that as \bar{c} is strictly increasing in $(0, T/2)$ there exists a constant $\delta > 0$ such that for any $t \in ((a + b)/2, b)$

$$\bar{c}(t) - \bar{c}(t - (a + b)/2) \geq \delta,$$

whence a direct comparison shows

$$\int_0^T \Psi(\bar{c} - c) < 0,$$

a contradiction. □

We now prove the following proposition (recall that $m(x) = \cos(x)$ and $\bar{c}(t) = \cos(\pi + t)$):

Proposition 24. *\bar{c} does not satisfy first-order optimality conditions for (1.15).*

Proof of Proposition 24. We can solve (1.14) explicitly. As $\int_{(0,T) \times \mathbb{T}^d} \bar{c} m = 0$, $\varphi_{\bar{c}}$ solves

$$\partial_t \varphi_{\bar{c}} - \Delta \varphi_{\bar{c}} = \cos(\pi + t) \cos(x).$$

Thus,

$$\varphi_{\bar{c}} = \omega(t) \cos(x) \quad \text{with} \quad \begin{cases} \omega'(t) + \omega(t) = \cos(\pi + t), \\ \omega(2\pi) = \omega(0). \end{cases}$$

The function ω admits the explicit expression

$$\omega(t) = \frac{\cos(t + \pi) + \sin(t + \pi)}{2}.$$

Consequently the function $\psi = \psi_{c_{\min}}$ solves

$$-\partial_t \psi - \Delta \psi = H'(\omega(t) \cos(x)) = (2k)\omega(t)^{2k-1} \cos(x)^{2k-1}.$$

Since

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos(x)^{2k-1} \cos(x) dx = \frac{1}{2^{2k-1}}$$

it follows that the function $\Psi = \int_{-\pi}^{\pi} \psi \cos(\cdot)$ solves

$$-\Psi' + \Psi = \frac{2k\pi}{2^{2k-1}} \omega^{2k-1} = a_k \omega^{2k-1} \text{ with } a_k = \frac{2k\pi}{2^{2k-1}}.$$

Applying the operator $(d/dt + \text{Id})$ to the previous equation we obtain

$$\frac{1}{a_k} (-\Psi'' + \Psi) = \omega(t)^{2k-2} ((2k-1)\omega'(t) + \omega(t)).$$

If Ψ were symmetric with respect to $t = \pi$, then so would the map

$$F : t \mapsto \omega(t)^{2k-2} ((2k-1)\omega'(t) + \omega(t)).$$

However, observe that

$$F'(t) = (2k-1)(2k-2)\omega'(t)^2 \omega(t)^{2k-3} + (2k-1)\omega''(t)\omega(t)^{2k-2} + (2k-1)\omega'(t)\omega(t)^{2k-2}.$$

As

$$\omega(\pi) = \omega'(\pi) = -\omega''(\pi) = \frac{1}{2}$$

we conclude that

$$F'(\pi) = (2k-1)(2k-2) \frac{1}{2^{2k-1}} - \frac{2k-1}{2^{2k-1}} + (2k-1) \frac{1}{2^{2k-1}} = \frac{(2k-1)(2k-2)}{2^{2k-1}}.$$

In particular, $F'(\pi) = 0$ if and only if $k = 1$. \square

7. NUMERICAL SIMULATIONS

In the following we illustrate the theoretical results from previous sections through a series of numerical simulations. These help us formulate several conjectures in the conclusion of the article.

7.1. General structure of the section. In this section, we work in the one-dimensional (in space) setting: we identify \mathbb{T} with the interval $[0, 1]$. We proceed in steps of increasing complexity: after explaining how to handle, at a numerical level, admissible classes of the form (1.5) (which is non standard when compared to the existing literature, see for instance [23] and the references therein), we study, successively, the following problems (the order of the problems corresponds to their intrinsic mathematical and computational difficulties):

- (1) First, we investigate Talenti-like problems, that is, we focus on (1.15). We give a numerical illustration of the symmetry breaking phenomenon of Theorem V.
- (2) Second, we focus on (1.6), and our numerical simulations back up Theorems III–IV: it seems likely that, in general, an optimiser c_{\min} of (1.6) satisfies $c_{\min} = c_{\min}^{\#}$.
- (3) Finally, we turn to (1.3) where we also obtain, numerically, the monotonicity in time of optimisers.

7.2. Generalities on the discretisation of the equations and of the constraints.

7.2.1. *Discretisation of the PDE and of the objective functional.* We discretise the time interval $[0, T]$ using a uniform partition

$$0 = t_0 < t_1 < \dots < t_N = T, t_i = i/N.$$

Similarly, the space interval $[0, 1]$ is discretised using a uniform partition of $[0, 1]$

$$0 = x_0 < x_1 < \dots < x_M = 1, x_i = i/M.$$

The time and space steps are denoted by $(\Delta t) = T/N$, $(\Delta x) = 1/M$ respectively. The periodicity conditions in time and space are implemented by identifying $t_{i+N} = t_i$ and $x_{i+M} = x_i$. Therefore, indices are extended by periodicity in time and space whenever necessary.

Given a function $u : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ consider the matrix $U \in \mathbb{R}^{M \times N}$ approximating the values of u at the nodes of a finite difference grid on $[0, T] \times \mathbb{T}$ with nodes given above by $u(t_j, x_i) \approx U_{i,j}$. Denote by U_j , $j = 0, \dots, N$, the vector approximating $U(t_j, \cdot)$ at the nodes of the space grid.

The Laplace operator $(-\Delta)$ is approximated using second order finite differences, through the matrix

$$A = \frac{1}{\Delta x^2} \begin{pmatrix} 2 & -1 & \dots & \dots & -1 \\ -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & \dots & \dots & -1 & 2 \end{pmatrix}.$$

Thus, for a fixed $0 \leq j \leq N - 1$

$$-\left(\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta x^2} \right)_{i=0}^{M-1} = AU_j.$$

The time derivative is approximated using first order finite differences. The parabolic equation is discretised implicitly for the diffusion term, and explicitly for the remaining terms.

The direct problem. We first study the discretisation of problem (1.14). Without loss of generality, assume that $\int_{\mathbb{T}} V(x) dx = 0$. The discrete approximation for (1.14) is given by

$$\frac{U_{n+1} - U_n}{(\Delta t)} + AU_{n+1} = c(t_i)V, \text{ for every } 0 \leq n \leq N - 1$$

where we denote we a slight abuse of notation V the discretisation of V at space points x_i . Re-factoring gives

$$(I + (\Delta t)A)U_{n+1} - U_n = (\Delta t)c_i(t)V, \text{ for every } 0 \leq n \leq N - 1.$$

Taking into account the periodicity in time we find that the discrete problem is equivalent to

$$(7.1) \quad \mathcal{A}\mathbf{U} = \mathbf{R},$$

where $\mathbf{U} \in \mathbb{R}^{M \times N}$ is the vector obtained by concatenating $\mathbf{U} = (U_0, \dots, U_{N-1})$ and $B = (I + (\Delta t)A)$, \mathcal{A} is the block matrix

$$(7.2) \quad \mathcal{A} = \begin{pmatrix} -I & B & 0 & \dots & 0 \\ 0 & -I & B & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ B & 0 & \dots & \dots & -I \end{pmatrix}$$

and the right hand side is given by

$$\mathbf{R} = \begin{pmatrix} c_0 V \\ c_1 V \\ \vdots \\ c_{N-1} V \end{pmatrix}.$$

Note that the system $\mathcal{A}\mathbf{U} = \mathbf{R}$ discretises simultaneously the time and space aspects of the equation. This allows us to treat the periodicity in time in a straightforward way, without resorting to iterative approximations. Up to a constant factor, the discrete version of (1.15) is

$$\max_c \|\mathbf{U}\|_{\ell^p}$$

for some $p \geq 2$. In order to solve (1.15), we need to discretise the adjoint, which corresponds to another parabolic equation. To avoid complications we differentiate the discrete system directly.

It is standard to compute derivatives for $J(\mathbf{U}) = \sum_{i,j} j(U_{i,j})$ under the constraint (7.1). Indeed, if $h \in \mathbb{R}^N$ is a perturbation for c then

$$J'(\mathbf{U}(c))(h) = \sum_{i,j} j'(U_{i,j}) U_{i,j}(c)'(h),$$

where $\mathbf{U}'(h)$ satisfies

$$\mathcal{A}\mathbf{U}'(h) = \begin{pmatrix} h_0 V \\ h_1 V \\ \vdots \\ h_{N-1} V \end{pmatrix}.$$

We consider the solution \mathbf{P} to the adjoint system

$$\mathcal{A}^T \mathbf{P} = j'(\mathbf{U}),$$

which implies that

$$J'(\mathbf{U}(c))(h) = \mathbf{P} \cdot \begin{pmatrix} h_0 V \\ h_1 V \\ \vdots \\ h_{N-1} V \end{pmatrix}.$$

This expression yields

$$(7.3) \quad \frac{\partial J(c)}{\partial c_i} = p_i \cdot V.$$

The two types of constraints considered for c are introduced below. Each one of them is preserved during the gradient descent/ascent algorithm using projections, which will be detailed below.

7.2.2. Discretisation of the classical constraints. We first study constraints that we call classical (see Remark 3), that is, corresponding to $a \leq c \leq b$, $\int_0^T c(t) dt = c_0 \in [a, b]$. To construct the projection on this constraint, observe that the mapping

$$\tau \mapsto \int_0^T \max\{a, \min\{c(t) + \tau, b\}\} dt$$

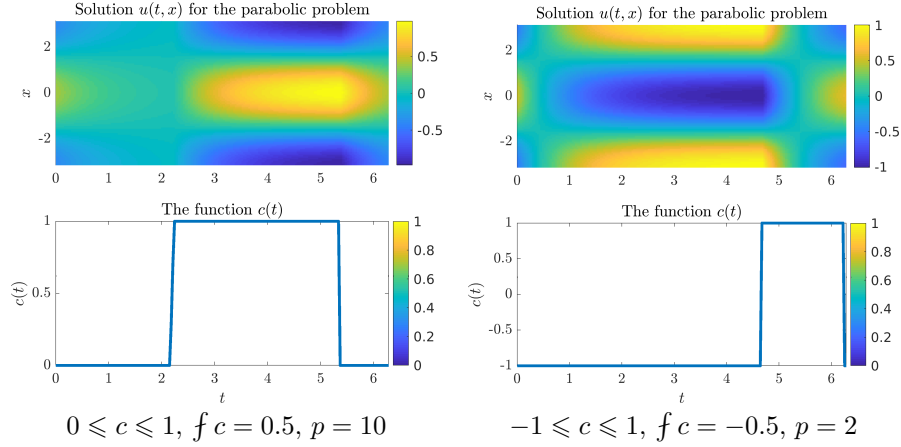


FIGURE 1. Numerical solutions for problem (1.15) for various parameters, using bound and average constraints on the function c .

is increasing in τ . Therefore, using a dichotomy algorithm we find a threshold τ_0 such that

$$\int_0^T \max\{a, \min\{c(t) + \tau_0, b\}\} dt = c_0$$

where the equality is understood up to a fixed tolerance chosen beforehand. The projection operator is

$$(7.4) \quad P(c) = \max\{a, \min\{c(t) + \tau_0, b\}\}.$$

While the projection operator (7.4) is defined in the continuous context, the discrete version is obtained in a straightforward way by replacing the integral with the average (a and b can also vary with t ; in this case replace a, b with $a(t)$ and $b(t)$ in the truncation operator above).

Numerical simulations. We solve (1.15) numerically with the aid of a gradient ascent algorithm with projection. The vector c is updated using the usual formula $c^{k+1} = P(c^k + \gamma_k \nabla J(c^k))$ (P is the projection operator defined in (7.4)) where $\gamma_k > 0$ is an ascent step chosen such that the objective function increases: $J(c^{k+1}) > J(c^k)$. The step γ_k is updated at every iteration: if the objective function increases, the step is slightly increased; if the objective function does not increase, the iteration is repeated with a smaller step. Simulations are ran for various bounds and average constraints for c . The variable c is initialised randomly, then projected onto the constraints using (7.4). In the numerical computations we choose $V(x) = \cos x$. Interestingly enough, at a numerical level and for such constraints, we do not observe the symmetry breaking phenomenon of Theorem V: c is equal to its maximal value on a connected interval and is equal to its minimal value on the complement. Simulations are performed using 100 discretisation points for the spatial variable and 200 discretisation points for the time variable. The final time is $T = 2\pi$ and the periodic x variable belongs to $[-\pi, \pi]$. Numerical simulations generally converge in under 100 iterations for the parameters recalled above. Examples are shown in Figure 1.

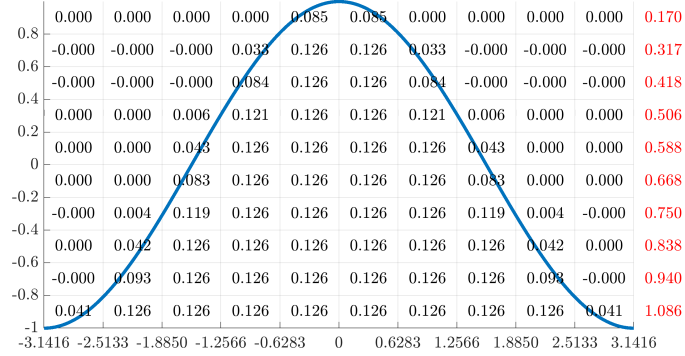


FIGURE 2. The region $[-\pi, \pi] \times [-1, 1]$ containing the graph of the cosine function is divided into $N \times K$ rectangles. The covered area in each small rectangle is indicated in the figure. The area of a small rectangle is $(\Delta t) \times (\Delta c) \approx 0.126$. A function having the same discrete rearrangement will be characterised by a $K \times N$ matrix with entries decreasing on columns, having fixed sum on lines (rightmost column above).

7.2.3. *Discretisation of rearrangement constraints.* In this case we handle the case of functions c having a fixed rearrangement $c^\# = \bar{c}$. Let c_{\min}, c_{\max} be the minimum and maximum values of c on $[0, T]$. Consider the domain $[0, T] \times [c_{\min}, c_{\max}]$ divided into the discrete grid given by t_0, \dots, t_N , the time discretisation instances and $c_{\min} = v_0, \dots, v_K = c_{\max}$, an equi-distributed discretisation of the image space of c on $[0, T]$. We set $\Delta c := (c_{\max} - c_{\min})/K$. Let $F_{i,j}$ be the area of the graph of c contained in the cell $[t_j, t_{j+1}] \times [v_i, v_{i+1}]$ for $0 \leq i \leq K-1$, $0 \leq j \leq N-1$. See Figure 2 for an illustration containing the proportions $F_{i,j}$ for the cosine function and $K = N = 10$.

The numerical discretisation of the rearrangement constraint we propose is inspired from two natural observations.

- If c is a function then $F_{i,j} \geq F_{i+1,j}$: the "quantity" of c contained in a given interval, on a given slice is larger than what lies in a superior slice.
- If c has a prescribed rearrangement then $\sum_{j=0}^{N-1} F_{i,j}$ is fixed. This follows from the Cavalieri principle.

We propose the following discrete rearrangement constraint. If \bar{c} is a symmetric decreasing function with respect to $T/2$ then for any $i \in \{0, \dots, K-1\}$ consider

$$q_i = |\{(t, y) : t \in [0, T], y \in [v_i, v_{i+1}], y \leq c(t)\}|.$$

The sequence q_i is decreasing.

Consider a matrix $F_{i,j}$ of size $(K \times N)$ and entries verifying for all $0 \leq j \leq N-1$, $0 \leq i \leq K-1$ (extended by periodicity):

$$F_{i,j} \geq F_{i+1,j}, \quad \sum_{j=0}^{N-1} F_{i,j} = q_i.$$



FIGURE 3. An example of a discrete function defined on the $N \times K$ rectangular grid having the required properties to be a discrete rearrangement of the cosine function: sum of values of lines equals to the same quantity for the cosine, values are decreasing on vertical columns. The area of a small rectangle is $(\Delta t) \times (\Delta c) \approx 0.126$.

We reconstruct the discrete function c by summing on the columns of F :

$$(7.5) \quad c_j = c_{\min} + \frac{1}{\Delta x} \sum_{i=0}^{K-1} F_{i,j}.$$

In Figure 3 an example of a different matrix F corresponding to a discrete rearrangement of the cosine function is given. Together with the values, the cumulative sum on columns computed with (7.5) is also illustrated.

The cumulative sum is not the most representative illustration, since one discrete interval $[t_i, t_{i+1}]$ may not be fully covered at some of the intermediary levels. Another way of interpreting the discrete rearrangement proposed is shown in Figure 4 where in each cell a rectangle corresponding to the fraction covered by the discrete function in the current cell is shown. Of course, when N and K are large enough, the description of the rearrangement becomes more and more precise. In the numerical simulations we use $N \geq 100$ and $K = 100$ vertical slices to characterize the discrete rearrangement. Note that in order to characterize a one dimensional function with prescribed rearrangement, a two dimensional array is required.

Similar to our previous computations, we obtain

$$\frac{\partial J}{\partial F_{i,j}} = \frac{\partial J}{\partial c_j} \frac{\partial c_j}{\partial F_{i,j}} = \frac{1}{\Delta t} p_i \cdot V.$$

This constraint is also preserved in a gradient algorithm using projections. Indeed, note that by construction $F_{i,j}$ is an area proportion in a rectangle of area $(\Delta t) \times (\Delta c)$, where $\Delta c = \frac{c_{\max} - c_{\min}}{K}$. Therefore $0 \leq F_{i,j} \leq (\Delta t) \times (\Delta c)$. This leads to the following iterative algorithm:

- **First slice:** only bound and average constraints. Find $F_{0,j}$, $0 \leq j \leq N-1$ such that

$$0 \leq F_{0,j} \leq (\Delta t) \times (\Delta c), \quad \sum_{j=0}^{N-1} F_{i,j} = q_0.$$

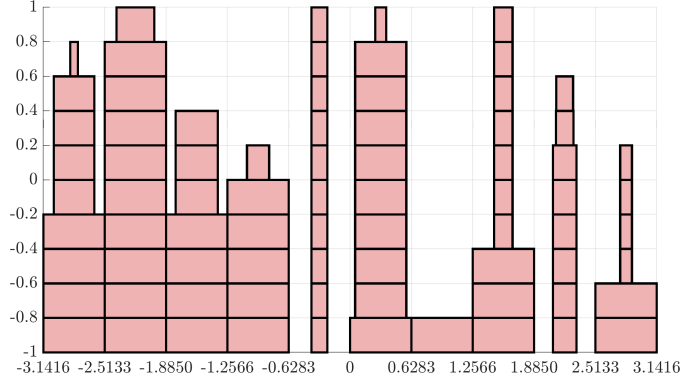


FIGURE 4. An alternative illustration for the discrete rearrangement of the cosine illustrated in Figure 3. In each one of the $N \times K$ cells a rectangle is plotted having area proportional to the area covered by the rearrangement in that cell.

To achieve this, simply apply the projection algorithm stated in (7.4).

- **Subsequent slices:** fixed average, smaller than previous slice. Given $i \geq 1$, find $F_{i,j}$, $0 \leq j \leq N-1$ such that

$$0 \leq F_{i,j} \leq F_{i-1,j}, \quad \sum_{j=0}^{N-1} F_{i,j} = q_i.$$

To achieve this, apply the projection algorithm stated in (7.4), with variable bounds.

- Recover the discrete function c using (7.5).

Numerical Simulations. Like in the previous case, we take $T = 2\pi$ and $V(x) = \cos(x)$ for $x \in [-\pi, \pi]$. Assume that $c : [0, T] \rightarrow [-1, 1]$ has the same symmetric rearrangement as $\cos(t)$. The rearrangement constraint is implemented using $K = 100$ vertical slices, using the same notations indicated above. Two simulations are shown where (1.15) is maximised for $p = 2$ and $p = 6$. In the first case, for $p = 2$ the optimal c is identically equal to \cos . For $p = 6$ the symmetry is lost, as indicated by the theoretical results of Theorem V. Results are illustrated in Figure 5.

7.3. The eigenvalue problem. The discretisation of the eigenvalue problem (1.1) follows the same lines as the one used for the direct problem. The resulting discrete problem has the form

$$(7.6) \quad (\mathcal{A} - (\Delta t) \text{diag}(m))\mathbf{U} = (\Delta t)\lambda(m)\mathbf{U},$$

where \mathcal{A} is the same matrix as in (7.1), (7.2) and $\text{diag}(m)$ denotes the diagonal matrix with entries given by $m_{i,j} \approx m(x_i, t_j)$. We are interested in the principal eigenvalue for problem (7.6) for which we have $\mathbf{U} > 0$ pointwise. More precisely, the elements on the diagonal of $\text{diag}(m)$ are, in order, given by the columns of the matrix $m_{i,j}$: $(m_{0,j})_{j=0}^{M-1}$, $(m_{1,j})_{j=0}^{M-1}$, ..., $(m_{N-1,j})_{j=0}^{M-1}$.

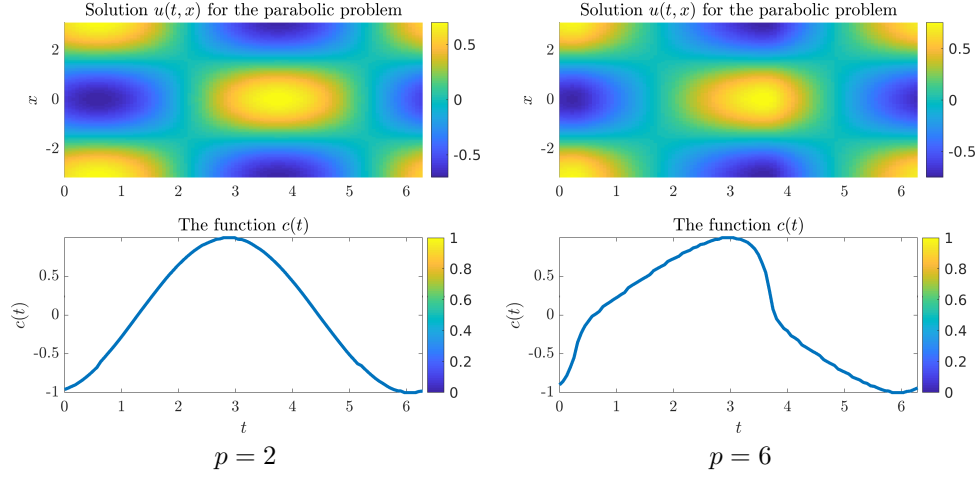


FIGURE 5. Numerical optimisation for the functional (1.15) for functions c having a prescribed rearrangement. Numerical results show that there is a different behaviour of the numerical optimiser with respect to the parameter p . For $p = 2$ the solution is symmetric, while for $p = 6$ the symmetry is lost. This confirms the results of Theorem V.

We observe that adding a constant δm to m and setting $\tilde{m} = m + \delta m$ simply shifts the eigenvalue: $\lambda(\tilde{m}) = \lambda(m) - \delta m$. Since the matrix \mathcal{A} already has negative eigenvalues, to guarantee that the eigenvalue of the smallest absolute value found numerically (using standard libraries like `eigs` in Matlab) has a constant sign eigenvector, we shift m with a constant such that all its components are strictly positive and, change $\lambda(m)$ accordingly to find the discrete eigenvalue (7.6).

Our objective is to optimise the eigenvalue $\lambda(m)$ with respect to m . Like in the previous case, to avoid technical difficulties, we differentiate the discrete problem (7.6) in order to optimise $\lambda(m)$ numerically. Considering $h \in \mathbb{R}^{M \times N}$ a perturbation of m and differentiating (7.6) gives

$$(7.7) \quad (\mathcal{A} - (\Delta t) \text{diag}(m))\mathbf{U}' - (\Delta t) \text{diag}(h)\mathbf{U} = (\Delta t)\lambda'(m)(h)\mathbf{U} + (\Delta t)\lambda(m)\mathbf{U}'.$$

The adjoint problem for (7.6) is the same eigenvalue problem, but for the transpose matrix (remember that \mathcal{A} defined in (7.2) is not symmetric)

$$(\mathcal{A}^T - (\Delta t) \text{diag}(m))\mathbf{V} = (\Delta t)\lambda(m)\mathbf{V},$$

in other words, \mathbf{V} is a left eigenvector for the same eigenvalue $\lambda(m)$, which also satisfies $\mathbf{V} > 0$ pointwise. Multiplying (7.7) with \mathbf{V}^T and simplifying leads to an explicit formula for the derivative of $\lambda(m)$

$$(7.8) \quad \lambda'(m)(h) = -\frac{1}{\mathbf{V}^T \mathbf{U}} \mathbf{V}^T (\text{diag}(h)\mathbf{U}).$$

Since $\mathbf{U}, \mathbf{V} > 0$ the scalar product $\mathbf{V}^T \mathbf{U}$ is strictly positive. More precisely we find that

$$\frac{\partial \lambda(m)}{\partial m_{i,j}} = -\frac{1}{\mathbf{V}^T \mathbf{U}} U_{i,j} V_{i,j}.$$

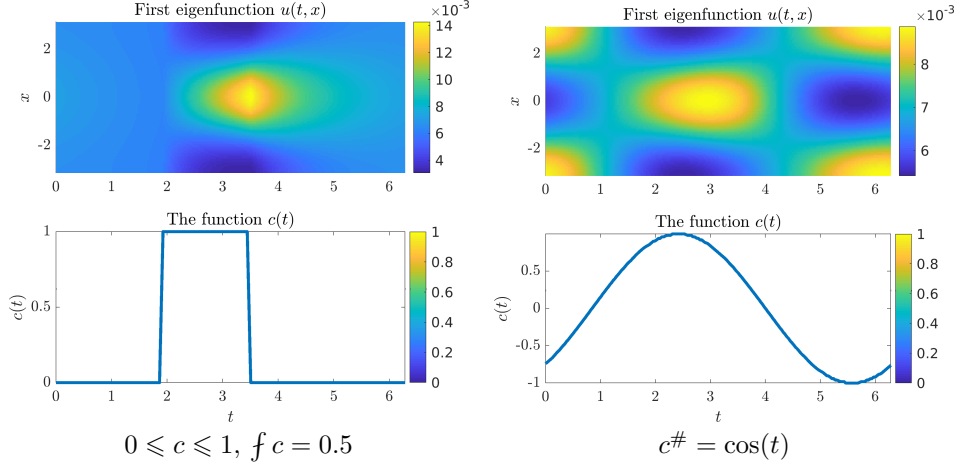


FIGURE 6. Numerical minimisation of the principal eigenvalue in (1.1) with respect to potentials of the form $c(t)m(x)$ with $m(x) = \cos(x)$ and $c(t)$ verifying classical or rearrangement constraints.

Numerical simulations. (a) Separated time/space potentials: We choose $T = 2\pi$, 100 discretisation points for the space variable and 200 discretisation points for the time variable. For potentials of the form $c(t)m(x)$ we choose $m(x) = \cos(x)$ and c verifying either classical constraints (bounds, fixed average) or rearrangement constraints. The eigenvalues are computed in Matlab using the `eigs` routine. We refer to Figure 6.

(b) Two dimensional potentials with rearrangement constraints. The discrete framework presented in Section 7.2.3 for the rearrangement constraint can be extended *ad literam* in higher dimensions.

To parametrize potentials $m : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ which have a fixed rearrangement consider the following:

- Divide the region $[0, T] \times \mathbb{T} \times [\min m, \max m]$ into $N \times M \times K$ boxes labeled $B_{i,j,k}$ using equidistributed points along each dimension. Denote by h_0, \dots, h_K a discretisation of $[\min m, \max m]$
- Consider $F_{i,j,k}$ the proportion of the volume of the graph of the potential m contained in $B_{i,j,k}$.
- Take $F_{i,j,k}$ as optimisation variables assuming that $k \mapsto F_{i,j,k}$ is decreasing and $\sum_{i,j} F_{i,j,k} = q_k$ is fixed and equal to the integral of m on $\{(t, x, z) : z \in [h_i, h_{i+1}]\}$.

Given $F_{i,j,k}$, satisfying the constraints above, re-construct m using the formula

$$(7.9) \quad m_{i,j} = m_{\min} + \frac{1}{\Delta t \cdot \Delta x} \sum_{k=0}^{K-1} F_{i,j,k}.$$

The eigenvalue can become degenerate if one is not careful with the time scaling and diffusivity of the equation, in the sense that the gradient will be extremely low. To overcome this numerical hurdle, we consider the eigenvalues of the operator

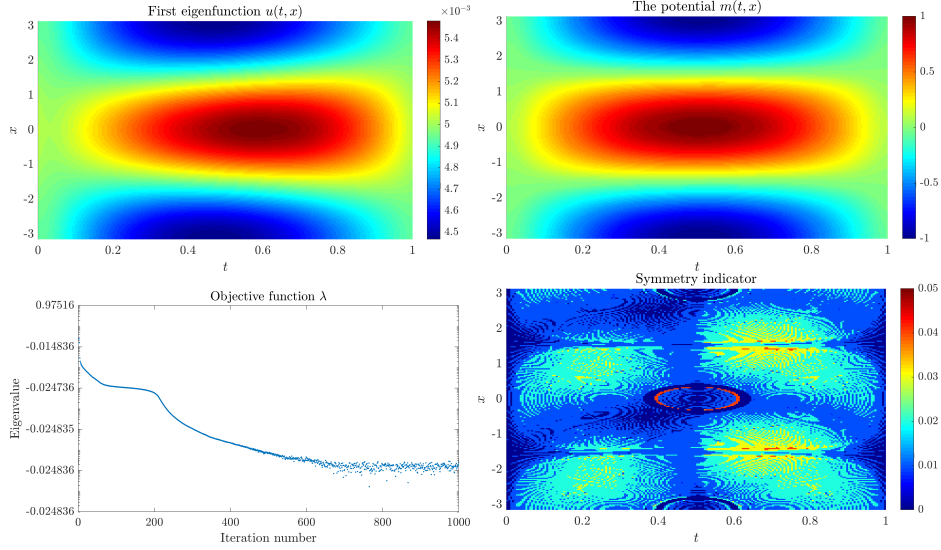


FIGURE 7. Rearrangement constraint $m^\# = \cos(x) \cos(2\pi t)$, $\alpha = 0.1$, $\beta = 10$. The eigenfunction, optimal potential and objective function are plotted. The fourth image plots $\max\{|m(t, x) - m(T/2 - t, x)|, |m(t, x) - m(t, -x)|\}$: small values indicate closeness to a symmetric potential.

$\alpha \partial_t - \beta \Delta - mI$. This amounts to a scaling of the time-space domain. Simulations presented below support the conjecture that the optimal potential under a rearrangement constraint is symmetric in time and in space. Simulations are made for $(t, x) \in [0, T] \times [-\pi, \pi]$. The optimisation algorithm is a gradient descent algorithm with projection on the discrete rearrangement constraint. The initialisation is chosen randomly. Precise details are given for each computation.

Computation 1. Parameters: $\alpha = 0.1$, $\beta = 10$, $m^\#(t, x) = \cos(x) \cos(2\pi t)$ on $[0, 1] \times [-\pi, \pi]$. The discretisation uses 201 time steps, 201 space steps and 200 slices for the rearrangement constraint. The initialisation is chosen at random and projected onto the rearrangement constraint. The gradient descent algorithm is ran for 1000 iterations.

Results are illustrated in Figure 7 where the optimal potential, corresponding eigenvalue and the evolution of the eigenvalue are shown. To investigate the symmetry of the numerical minimiser we shift the discrete solution such that the maximal value is at the center of the grid. The optimal potential is compared with the symmetrisation with respect to the vertical and horizontal direction. The maximal difference between m and its symmetrisation is plotted. The values obtained are small enough to conjecture that the minimiser $m(t, x)$ is symmetric decreasing in time and space. The precise formula is: $\max\{|m(t, x) - m(T/2 - t, x)|, |m(t, x) - m(t, -x)|\}$.

Computation 2. In a second computation we consider a rearrangement constraint where the graph of $m^\#$ is a cone: functions taking values in $[0, 1]$ with slice areas equal to $(1 - h)^2 \pi \delta h$, where δh is $1/K$ and $K = 200$ is the number of horizontal slices used in the rearrangement constraint. The scaling parameters are

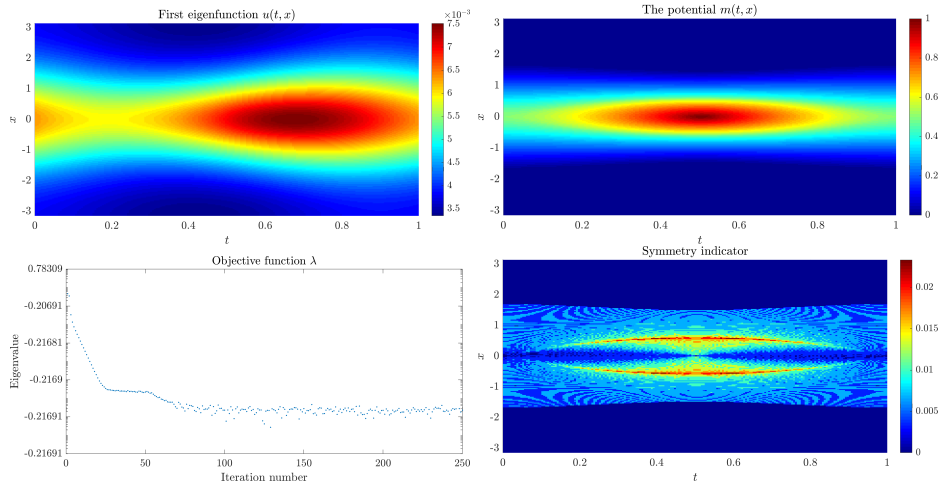


FIGURE 8. Rearrangement constraint: $m^\#$ is a cone, $\alpha = 0.1$, $\beta = 1$. The eigenfunction, optimal potential and objective function are plotted. The fourth image plots $\max\{|m(t, x) - m(T/2 - t, x)|, |m(t, x) - m(t, -x)|\}$: small values indicate closeness to a symmetric potential.

$\alpha = 0.1$, $\beta = 1$ and the number of iterations is fixed to 250. The numerical result obtained is shown in Figure 8 and is close to being symmetric decreasing in time and space.

8. CONCLUSION

In this article, we were able to derive symmetry and monotonicity properties of optimisers of space-time periodic eigenvalues in various particular frameworks. Based on our results and on our numerical simulations, we conjecture that the optimiser should always be symmetric decreasing in time and space, but the tools required to prove it seem out of reach at the moment.

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