



HAL
open science

Bernoulli's principle in hydrodynamics

Laure Saint-Raymond

► **To cite this version:**

Laure Saint-Raymond. Bernoulli's principle in hydrodynamics. St. Petersburg mathematicians and their discoveries, 2024, 978-5-4439-4703-7. <hal-04866149>

HAL Id: hal-04866149

<https://hal.science/hal-04866149v1>

Submitted on 10 Jan 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Bernoulli's principle in hydrodynamics

Daniel Bernoulli is often regarded as the father of hydrodynamics. “It seems that the first person to use this term was M. Daniel Bernoulli, who gave this title to his *Treatise of fluid motions*, printed in Strasbourg in 1738 [6]. If the title was new, it must be admitted that the work was also new. M. Daniel Bernoulli seems to be the first who reduced the laws of fluid motion to safe and non-arbitrary principles, which none of the authors of hydraulics had done before him. The same author had already given in 1727, in the *Memoirs of the Academy of Petersburg* [5], an essay on his new theory.” (D’Alembert, [12]).

As a mathematician and physicist, Bernoulli used analysis, particularly differential calculus, to formalize some fundamental laws of fluid mechanics and put them into equations.

1 A problem of vessels and drain

His memoir *Hydrodynamica* [6], dedicated to the Duke of Courland and Semigallia Biron, brings together various theoretical contributions and comments on the subject of fluid forces and movements, corroborated by numerous experiments.

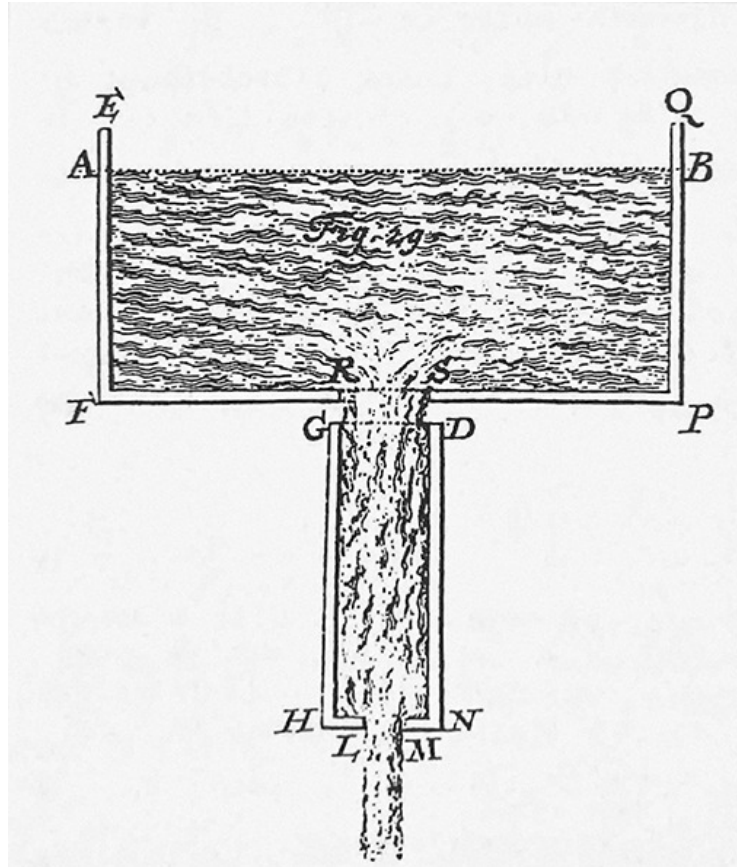
These include fluids flowing through vessels with various shapes of openings or pipes, driving effect by rotation or translation of the vessels, and even an (unrealistic) application to navigation.

These include fluids flowing through vessels with various shapes of openings or pipes, driving effect by rotation or translation of the vessels, and even an (unrealistic) application to navigation.

1.1 A counter-intuitive result. The starting point of all his computations, now referred to as Bernoulli's principle, relates the velocity of a fluid to its pressure at a given point. It can be stated as follows

Bernoulli's principle. *In a horizontally flowing fluid, the pressure of the fluid at points where its velocity is high is lower than the pressure of the fluid at points where its velocity is low.*

Thus, in a horizontal pipe with sections of different diameters, the pressure of the water in the sections where the water is flowing fast is lower than in the sections where the water is flowing slowly. This may seem counter-intuitive,



as one would tend to associate high velocity with high pressure. However, this principle simply translates to the fact that water will accelerate if there is more pressure behind it than in front of it.

1.2 A counterpart of Huygens principle in mechanics. Bernoulli derived this principle by analogy with the living force principle of Huygens in mechanics [20]. In his own words, the proof is the following :

“The potential elevation of the system, each part of which is moved by any velocity, shows the vertical height which the center of gravity of this system reaches, if each particle, by the upward movement raised by its velocity, is understood to rise as far as it can.

The vertical height marks the actual descent by which the center of gravity descends after each particle has been at rest.

The potential elevation is necessarily equal to the actual descent when all motion remains in the spread matter.” (Bernoulli, [6])

The potential elevation is therefore connected to the velocity, which is itself computed in terms of the flux: “The movement of fluids is very close to such that everywhere the velocity is reciprocally proportional to the corresponding size of the vessel” (Bernoulli, [6]). The actual descent is expressed in terms of the pressure using the laws of hydrostatics.

Bernoulli was then able to use these conservation laws on infinitesimal volumes and introduced geometric considerations to compute, for instance, the drain rate of a vessel.

2 Modern formulation

Bernoulli's original text is quite difficult to access today: the terminology no longer corresponds to current standards, the hypotheses are not always clearly formulated (except in the form of rules for applications), and the notations are very confusing.

The principle is nevertheless explained nowadays in a simple form in all fluid mechanics courses.

2.1 The pressure, a macroscopic observation of internal interactions. The pressure involved in Bernoulli's principle is the internal pressure of the fluid, which is exerted in all directions as the fluid flows. Note that this pressure is not the same as the pressure the fluid would exert on an object that would get in its way and stop its movement.

It is interesting to note that Bernoulli's memoir actually contains the first elements of the kinetic theory of gases: fluid particles have a state of thermal agitation that is all the more vivid the higher the pressure. The pressure is, in fact, the result of the many shocks between these fluid particles.

2.2 A global energy balance. It is the pressure of the neighboring portions of the fluid that is the source of the force providing the fluid with the work necessary for its acceleration.

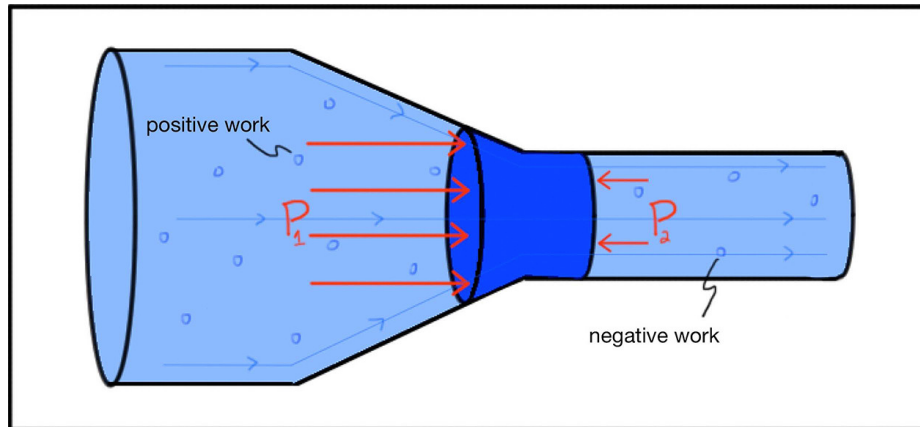
Consider a pipe in which water is flowing in a laminar way from left to right. When the volume of water, shown in dark blue in the figure below, reaches the narrow part of the pipe, its speed increases. The pressure force generated by the pressure on the left of the volume of water in question pushes it to the right and provides it with positive work since it pushes it in the direction of its movement. The pressing force generated by the pressure to the right of the volume of water under consideration pushes it to the left and provides it with negative work since it pushes it in the opposite direction to its displacement.

It is known that the water must accelerate (for conservation of the volume flow), and therefore the total work supplied by the pressure forces to the portion of fluid considered must be positive.

The work W_F of a force F (parallel to the motion) is written Fd , d being the distance traveled when the force is applied. The pressure force is given by PS , P being the pressure and S being the area to which the pressure is applied. Thus, we write :

$$W_{\text{pressure}} = PSd.$$

The area S_ℓ on the left is larger than the area S_r on the right. But when the volume of the water under consideration enters the narrow part of the pipe, it



One of the experiments from Bernoulli's *Hydrodynamica* book.

deforms and lengthens: the pressure force due to P_ℓ on the right-hand side is therefore applied over a greater distance d_ℓ than the pressure force due to P_r on the left-hand side. The total volume of the section of water studied remains constant $S_\ell d_\ell = S_r d_r$. Therefore, the pressure P_ℓ of the water on the left must be greater than the pressure P_r of the water on the right for the sum of the work done on the volume of the water studied to be positive overall.

3 Euler equations versus Bernoulli equation

Bernoulli's principle can be made more quantitative by writing a detailed energy balance. This leads, in particular, to clarifying the assumptions on the flow.

3.1 An inviscid incompressible flow. Bernoulli's work mainly concerns water, which is considered incompressible. Section 8 of *Hydrodynamica* proposes actually an extension to the case of "elastic fluids," taking into account the effects of compressibility (which was a very original contribution), but we will not treat this case here.

Bernoulli also specifies in his manuscript that he neglects all the effects of viscosity, and in particular the adhesion to the boundary (which he calls "friction" or "tenacity" of the fluid).

For an incompressible inviscid fluid of density ρ , Euler's (later) work [15] shows that the dynamic equation governing the flow can be written as

$$\left\{ \begin{array}{l} \underbrace{\nabla_x \cdot u = 0}_{\text{incompressibility constraint}} \\ \rho \cdot \underbrace{(d_t u + (u \cdot \nabla_x) u)}_{\text{acceleration along the flow}} = \underbrace{-\rho g e_z - \nabla_x P}_{\text{forces}} \end{array} \right. \quad (1)$$

where $x = (x_1, x_2, z)$ is the 3D vector of spatial coordinates, $u = u(t, x)$ is a 3D vector field representing the bulk velocity of the fluid at time t at a given

position x , g is the gravity and $P = P(t, x)$ is the (scalar) pressure. From the mathematical point of view, P is just the Lagrange multiplier associated with the incompressibility constraint. Indeed, taking the divergence of the second equation, we get $\nabla_x \cdot ((u \cdot \nabla_x)u) = -\Delta_x P$, meaning that P depends in a non-local way on u and its derivatives and on the boundary conditions.

Note that the Eulerian description of the fluid with u and P is, in a way, the point of view of an observer who stands at a given position (as opposed to the Lagrangian description following the fluid particles).

3.2 The Bernoulli equation. In his work, Bernoulli also restricts his attention to a steady state flow, which implies that the Eulerian velocity field u does not depend on time $u = u(x)$. Recall that this does not mean that a given fluid particle has a constant velocity, but that at any given point in the domain occupied by the fluid, an observer always sees the same flow. The fluid particles, on the other hand, are advected by the flow: they move along the streamlines with a velocity that will be precisely given by Bernoulli's relation.

The last assumption, which is not formulated in a very precise and rigorous way, concerns the geometry of the flow. Bernoulli assumes for simplicity that the fluid is "divided into layers perpendicular to the direction of motion", and that "the particles of fluid [are] moved with the same velocity, so that everywhere the velocity of the fluid is reciprocally proportional to the corresponding magnitude of the vessel." This assumption can be relaxed by considering a small tube of fluid around a field line, but a necessary condition is that these lines remain approximately parallel, i.e. that the flow is laminar.

The definition of laminar flow is actually not completely consensual. We will limit ourselves here to the case of irrotational flows $\text{rot } u = 0$. Since u is also divergence-free, the convection term can be rewritten

$$(u \cdot \nabla_x)u = \nabla_x \frac{|u|^2}{2} - u \wedge (\text{rot } u) - u(\nabla_x \cdot u) = \nabla_x \frac{|u|^2}{2}.$$

which is also a potential term. We therefore end up with Bernoulli's equation

$$\nabla_x \left(\rho \frac{|u|^2}{2} + \rho g z + P \right) = 0.$$

For general laminar incompressible flows, the conservation holds only along the flow lines :

$$u \cdot (u \cdot \nabla_x)u = (u \cdot \nabla_x) \frac{|u|^2}{2},$$

so that, given two points l and r on the same flow line

$$\rho \frac{|u_l|^2}{2} + \rho g z_l + P_l = \rho \frac{|u_r|^2}{2} + \rho g z_r + P_r.$$

This implies, in particular, Bernoulli's principle (with $z_l = z_r$).

However, the assumption of a permanent laminar flow in the geometry considered by Bernoulli is questionable. It is briefly discussed in his disserta-

tion, but the physical phenomena involved are extremely complicated and are actually still poorly understood.

4 Eddies and turbulence

This final section provides an overview of the complexity mentioned above.

4.1 Creation of vorticity. As shown previously, an irrotational flow satisfies the Bernoulli relation. On the other hand, it is easy to verify that an incompressible inviscid flow (with impermeability condition $u \cdot n = 0$ on the walls) does not create vorticity. Indeed, in two dimensions of space, the vorticity $\omega = \nabla_x^\perp \cdot u$ is simply transported

$$d_t \omega + (u \cdot \nabla_x) \omega = 0,$$

and therefore conserved. In three dimensions of space, the vorticity $\Omega = \text{rot } u$ satisfies

$$d_t \Omega + (u \cdot \nabla_x) \Omega = \Omega \cdot \nabla_x u.$$

There is an additional “stretching” term, but if the vorticity is initially zero, it remains so (provided that the solution is smooth enough to make sense of the stability estimate).

Bernoulli indicated in his memoir that his model is locally flawed, especially near the opening and the boundaries of the vessel, due to viscosity (which is never exactly zero) and wall adhesion. The Euler equations (1) were generalized for viscous fluids by Navier [24] and Stokes [27]

$$\left\{ \begin{array}{l} \nabla_x \cdot u = 0 \\ \underbrace{d_t u + (u \cdot \nabla_x) u}_{\text{acceleration}} = \underbrace{-g e_z - \nabla_x \frac{P}{\rho}}_{\text{forces}} + \underbrace{\nu \Delta_x u}_{\text{viscosity}} \end{array} \right. \quad (2)$$

where the viscosity term (with $\nu > 0$) accounts for the non-conservative internal forces (friction between neighboring fluid particles having different speeds). These equations do not seem at first sight to be very different from (1), especially when the Reynolds number (proportional to $1/\nu$) is very large. In particular, we have the transport-diffusion equation for vorticity Ω

$$d_t \Omega + (u \cdot \nabla_x) \Omega - \nu \Delta_x \Omega = \Omega \cdot \nabla_x u,$$

which should preserve the irrotationality of the flow.

However, as the Navier–Stokes equations are of order 2 (the Laplacian involves second derivatives), one must impose more conditions on the boundaries, which allow in particular to model the effect of adhesion to the wall ($u = 0$ on the walls). This condition is, in general, different from the prediction given by Euler’s equations for an inviscid fluid. It is therefore expected that even

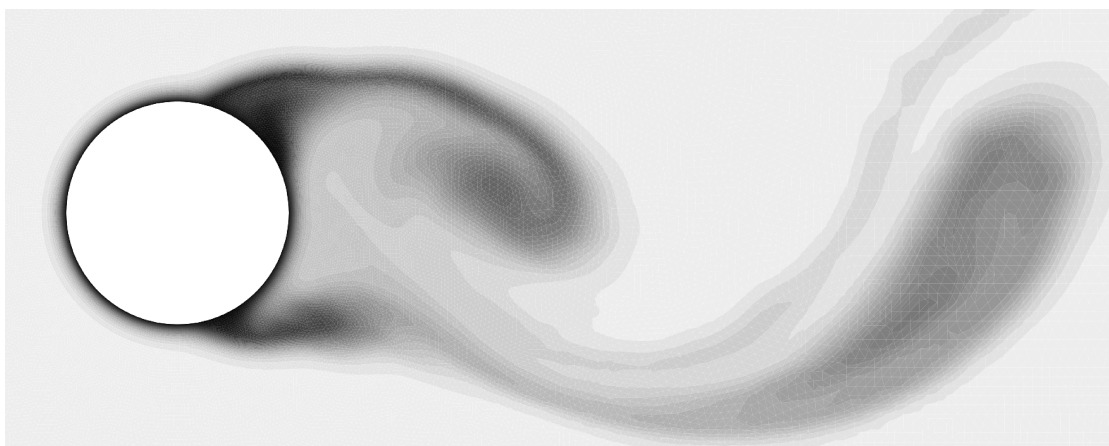
weakly viscous fluids ($\nu \ll 1$) behave very differently from ideal fluids ($\nu = 0$), especially in the vicinity of the walls. Observations show that small vortices are generated near the walls and can potentially destabilize the whole fluid.

4.2 Transition to turbulence. Prandtl was the first to study these boundary effects in more detail, introducing the notion of boundary layer and characterizing it by means of multi-scale expansions [26]. The main idea is that far from the boundary the viscous flow should resemble the idealized flow u_E governed by (1), which is then connected to the boundary condition by a fairly steep profile defined on a layer of size $\sqrt{\nu}$. To understand this connection, the idea is to zoom in on the direction orthogonal to the boundary at scale $\sqrt{\nu}$. For a stationary flow, in the simple geometry where the domain is 2D and the boundary is $x_2 = 0$, the equation for this profile $u = (v_1(x_1, x_2/\sqrt{\nu}), \sqrt{\nu}v_1(x_1, x_2/\sqrt{\nu}))$ reads

$$\begin{cases} \nabla_x \cdot v = 0 \\ (v \cdot \nabla_x)v_1 - d_{x_2 x_2}^2 v_1 = -d_{x_1} p_E(x_1, 0) \\ v(x_1, 0) = 0, \quad \lim_{x_2 \rightarrow \infty} v(x_1, x_2) = u_E(x_1, 0) \end{cases} \quad (3)$$

where (x_1, x_2) are the tangential and normal coordinates and u_E, p_E are the velocity and pressure obtained by the Euler equation (1).

Unfortunately, this equation is not always mathematically well posed. The stability of the boundary layer (and thus the validity of the two-scale expansion) is conditional on the fact that the gradient of pressure imposed by the internal flow $d_{x_1} p_E(x_1, 0)$ does not become positive and too large. Physically, in the case of flow around an obstacle, it is observed that the boundary layers that form upstream of the obstacle progressively lose their momentum and



A numerical simulation of a flow around an obstacle (credit : Bertrand Maury), so we see the mesh. The flow is laminar ahead of the obstacle (on the left), and becomes turbulent (with Von Karman alleys) past the obstacle. The grey level corresponds to the vorticity intensity.

cannot climb the adverse (positive) pressure gradient downstream, so they separate from the wall. In other words, the interaction with the boundary is expected to create vorticity, and this vorticity is then propagated within the fluid in the form of small vortices at all scales. This has two extremely important consequences. On the one hand, even if the viscosity is small, the presence of these small structures makes the flow quite strongly dissipative. On the other hand, the velocity field loses its regularity, and the Navier–Stokes equations (2) themselves can become unstable.

Understanding this transition mechanism to turbulence is a major challenge for physicists and mathematicians. Without making an exhaustive list of mathematical contributions on this subject, we can summarize the state of the art as follows

- the multi-scale expansion proposed by Prandtl is justified in some situations when the boundary layer is stable, for instance, with a negative gradient of pressure [25, 22, 17] (see also [8, 9, 18] and the references therein). The separation of the boundary layer has been described formally by Goldstein [19], and recently studied mathematically in [13].

- the instability of the flow in presence of small-scale structures is so strong that one even loses the uniqueness of solutions for the Euler equations [14] as well as for the Navier–Stokes equations [7, 1]. This means that, for flows of this complexity, it is very difficult (if not impossible) to follow precisely all the details of the velocity field.

- the enhanced dissipation and inviscid damping due to the combination of the small viscosity and the presence of small-scale structures are established for particular flows (typically shear flows) in the absence of boundaries (see the review [2] for a simple presentation of these mechanisms).

- in regimes when turbulence is fully developed, Kolmogorov proposed in 1941 a statistical approach [23], predicting, in particular, the distribution of eddies at different scales (see [16] for instance). Although completely heuristic, these predictions are consistent with the critical regularity for the Navier–Stokes equations (the threshold for the uniqueness [10, 21]).

However the prediction of the emission of vortices at the boundary layer (destabilization of the laminar flow), the probabilistic study of the Navier–Stokes equations to rigorously derive the Kolmogorov model (fully turbulent regime), or the mathematical description of the transition between these two regimes remain very challenging open problems. They probably require a very different approach from the usual point of view adopted in PDEs, where one could take advantage of the instability and mixing properties as is the case for example in the KAM theory of dynamical systems (see [3, 4] for results in this direction).

Acknowledgements

The author thanks L. Chevillard, A.-L. Dalibard, I. Gallagher, and B. Maury for their very useful comments on a preliminary version of this manuscript.

Laure Saint-Raymond

Bibliography

- [1] Albritton, D., Brué, E. and Colombo, M. (2021) *Non-uniqueness of Leray solutions of the forced Navier–Stokes equations*. arXiv:2112.03116.
- [2] Bedrossian, J., Germain, P. and Masmoudi, N. (2019) Stability of the Couette flow at high Reynolds number in 2D and 3D, *Bull. American Math. Soc.* 56(3), pp. 373–414.
- [3] Bedrossian, J., Blumenthal, A. and Punshon-Smith, S. (2019) *Almost-sure exponential mixing of passive scalars by the stochastic Navier–Stokes equations*. arXiv:1905.03869.
- [4] Bedrossian, J. and Punshon-Smith, S. (2021) *Chaos in stochastic 2d Galerkin–Navier–Stokes*. arXiv:2106.13748.
- [5] Bernoulli, D. (1727) *Theoria nova de motu aquarum per canales quocunque fluentes*, *Commentarii Academiae scientiarum imperialis Petropolitanae*. Vol. 2. (Printed in 1929).
- [6] Bernoulli, D. (1738) *Hydrodynamica, sive de Viribus et Motibus Fluidorum commentarii*. Opus Academicum, Strasbourg Dulsecker.
- [7] Buckmaster, T. and Vicol, V. (2019) Non-uniqueness of weak solutions to the Navier–Stokes equation, *Annals of Mathematics*, 189(1), pp. 101–144.
- [8] Caffisch, R.E. and Sammartino, M. (1998) Zero Viscosity Limit for Analytic Solutions, of the Navier–Stokes Equation on a Half-Space. Part I: Existence for Euler and Prandtl Equations, *Communications in mathematical physics*, 192(2), pp. 433–461.
- [9] Caffisch, R.E. and Sammartino, M. (1998) Zero Viscosity Limit for Analytic Solutions of the Navier–Stokes Equation on a Half-Space. Part II: Construction of the Navier–Stokes Solution, *Communications in mathematical physics*, 192(2), pp. 463–491.
- [10] Constantin, P., Weinan, E. and Titi, E.S. (1994) Onsager’s conjecture on the energy conservation for solutions of Euler’s equation, *Comm. Math. Phys.*, 165(1), pp. 207–209.
- [11] d’Alembert, J. Le R. (1744) *Traité de l’équilibre et du mouvement des fluides: pour servir de suite au Traité de dynamique*.
- [12] d’Alembert, J. Le R. Hydrodynamique, in *Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers*.
- [13] Dalibard, A.-L. and Masmoudi, N. (2019) Separation for the stationary Prandtl equation, *Publications mathématiques de l’IHES*, 130(1), pp. 187–297.
- [14] De Lellis, C. and Székelyhidi, L. (2013) Dissipative continuous Euler flows, *Inventiones Mathematicae*, 193(2), pp. 377–407.
- [15] Euler, L. (1757) Principes généraux du mouvement des fluides, in *Mémoires de l’Académie royale des sciences et des belles lettres de Berlin*. Vol. 11.

- [16] Frisch, U. (1995) *Turbulence : the Legacy of A.N. Kolmogorov*. Cambridge University Press.
- [17] Gerard-Varet, D. and Maekawa, Y. (2019) Sobolev stability of Prandtl expansions for the steady Navier–Stokes equations, *Archive for Rational Mechanics and Analysis*, 233(3), pp. 1319–1382.
- [18] Gerard-Varet, D. and Masmoudi, N. (2015) Well-posedness for the Prandtl system without analyticity or monotonicity, *Ann. Sci. Ec. Norm. Supér. (4)*, 48(6), pp. 1273–1325.
- [19] Goldstein, S. (1948) On laminar boundary-layer flow near a position of separation, *Quart. J. Mech. Appl. Math.*, 1, pp. 43–69.
- [20] Huygens, C. (1673) *Horologium Oscillationum*.
- [21] Isett, P. (2018) A proof of Onsager's conjecture, *Annals of Math*, 188(3), pp. 871–963.
- [22] Iyer, S. and Masmoudi, N. (2020) *Global-in-x Stability of Steady Prandtl Expansions for 2D Navier–Stokes Flows*. arXiv:2008.12347.
- [23] Kolmogorov, A.N. (1941) The local structure of turbulence in incompressible viscous fluids at very large Reynolds numbers, *Dokl. Akad. Nauk. SSSR* 30, pp. 299–303; Reprinted in 1991 *Proc. R. Soc. London A*, 434, pp. 9–13.
- [24] Navier, H. (1823) Mémoire sur les lois du mouvement des fluides, *Académie royale des Sciences*, 6, pp. 389–416.
- [25] Oleinik, O.A. and Samokhin, V.N. (1999) *Mathematical models in boundary layer theory* (Applied Mathematics and Mathematical Computation, Vol. 15). Boca Raton, FL: Chapman & Hall/CR.
- [26] Prandtl, L. (1905) Über flüssigkeitsbewegung bei sehr kleiner reibung, Verhandl, in *3rd Int. Math. Kongr.*, Heidelberg, 1904, pp. 484–491.
- [27] Stokes, G. (1845) *Transactions of Cambridge Philosophical Society*. Vol. 8, pp. 287–305.