

Variational principles for fully coupled stochastic fluid dynamics across scales

Arnaud Debussche, Etienne Mémin

▶ To cite this version:

Arnaud Debussche, Etienne Mémin. Variational principles for fully coupled stochastic fluid dynamics across scales. 2025. hal-04865618

HAL Id: hal-04865618 https://hal.science/hal-04865618v1

Preprint submitted on 6 Jan 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Variational principles for fully coupled stochastic fluid dynamics across scales

Arnaud Debussche¹ and Etienne Mémin¹

¹Univ. Rennes, Inria, ENS Rennes, IRMAR UMR 6625, 35042 Rennes, France

September 19, 2024

Abstract

This work investigates variational frameworks for modeling stochastic dynamics in incompressible fluids, focusing on large-scale fluid behavior alongside small-scale stochastic processes. The authors aim to develop a coupled system of equations that captures both scales, using a variational principle formulated with Lagrangians defined on the full flow, and incorporating stochastic transport constraints. The approach smooths the noise term along time, leading to stochastic dynamics as a regularization parameter approaches zero. Initially, fixed noise terms are considered, resulting in a generalized stochastic Euler equation, which becomes problematic as the regularization parameter diminishes. The study then examines connections with existing stochastic frameworks and proposes a new variational principle that couples noise dynamics with large-scale fluid motion. This comprehensive framework provides a stochastic representation of large-scale dynamics while accounting for fine-scale components. The evolution of the small-scale velocity component is governed by a linear Euler equation with random coefficients, influenced by large-scale transport, stretching, and pressure forcing.

1 Introduction

Today and in the foreseeable future, no numerical simulation can realistically model the entire array of interacting multiple scales present in fully developed turbulent flows. This is particularly evident in geophysical flow circulation, which spans from the scale of the Sun's heating (approximately 10,000 km) down to the turbulence dissipation scale (approximately 1 mm). Energy transfers occur towards smaller scales, or conversely, in the opposite direction, influenced by rotation and stratification [48, 47, 52]. The overlooked sub-grid processes in turbulent flow dynamical systems must be

1 INTRODUCTION

adequately considered to (i) faithfully represent at coarse resolution (relatively to the dissipation scale) the spread of an ensemble of realizations from a set of imperfect or unknown initial conditions, (ii) achieve accurate energy transfers, and (iii) define stable numerical simulations.

The first point is dynamical and potentially impacts severely the chaotic nature of the system. The second one is physical and impedes the representation of key processes in the energy transfer across scales, while the third is numerical and can lead without caution to numerical instabilities, potentially causing blow-ups, the generation of spurious artifacts or enforcing over-smooth solutions. All three of these effects are crucial in large-scale numerical models for forecasting, data assimilation, and data analysis. Although they may seem quite different in nature, they are challenging to disentangle in practice.

Foremost, the impact of sub-grid scale parameterization on large-scale transport is one of the primary sources of error and uncertainty in simulations of geophysical flows. Sub-grid modeling addresses fundamental issues ranging from the effects of turbulence to the practical design of numerical schemes for computational simulations. This is inevitable since simulating geophysical flows on large bassin at the Kolmogorov scale is entirely unachievable.

In the case of geophysical flows, intermittent flow-coupled forcing and small-scale processes resulting from thermodynamic effects, species mixing, or biogeochemistry create highly complex systems that are exceedingly difficult to model deterministically. Probabilistic modeling appears to be the most viable approach, especially when priorities include reducing resolution and computational costs, as well as accurately representing uncertainties and their dynamics.

Furthermore, concerning data assimilation and ensemble forecasting, a narrow spread of realizations can pose challenges in coupling data with the system's dynamics. In such cases, observations that deviate significantly from the ensemble cannot effectively correct the ensemble evolution. Given the increasing prominence of ensemble methods and the necessity to simulate multiple realizations covering plausible scenarios, there is a growing need to explore stochastic modeling capable of accurately representing dynamics uncertainties.

Several strategies for introducing randomness into flow dynamics or climate models have been proposed in the literature in recent years [4, 26]. In this work, we focus on a specific approach derived from the stochastic transport of fluid parcels [40]. This approach, termed modeling under location uncertainty (LU), has been demonstrated to be versatile and enables the derivation of flow dynamics from classical conservation laws [3, 5, 8, 9, 10, 30, 43, 44]. Furthermore, unlike settings constructed with empirical random forcing, LU incorporates an inherent mechanism to prevent uncontrolled variance growth. This energy conservation property is advan-

tageous in creating a stochastic system that accurately represents large-scale versions of the deterministic dynamical system [2, 10].

From the point of view of Statistical Physics, this balance can be interpreted as an instance of a fluctuation-dissipation relationship between the noise term and stochastic diffusion. Recently, a mathematical analysis of LU Navier-Stokes equations has been performed [15]. Beyond demonstrating the existence of weak (probabilistic) solutions in 3D – with uniqueness in 2D – it was shown that this stochastic model tends toward the deterministic equation as the noise vanishes. This provides physical consistency to this formal setting.

In essence, LU shares similarities with another approach known as stochastic advection by Lie transport (SALT), derived from a variational formulation, as initially proposed in [29]. Readers may also refer to [49] for extensions of such variational principles driven by martingale. Originally, LU is derived from a distribution form of Newton's second law. In this paper, one of the goals is to investigate whether LU may also be related to a variational formalism.

The primary aim of this study is to propose a variational principle that enables the derivation of dynamics for the small-scale components of the flow. While several strategies have been proposed within the LU setting to incorporate noise explicitly dependent on the solution [2, 5, 45, 51], or driven by dynamics extracted from data [36], no direct derivation of *a priori* dynamics for the small scales has been yet achieved in the different stochastic frameworks proposed in the literature. We will show that such a variational principle can be formulated to derive dynamics for the set of functions on which the noise is decomposed. This dynamics takes the form of a linear Euler equation involving transport and stretching by the large-scale flow accompanied with a small-scale pressure forcing. In this derivation, we will demonstrate that a Stratonovich-like noise is essential to take into account the correlations between the small and large scales. However, we need to pass to an equivalent Itô form, which is better suited for explicit computation.

Before describing in details the different variational principles investigated, a description of the LU setting is first briefly recalled. More specifically, the formulation of LU is presented in terms of both the Itô and Stratonovich stochastic integrals.

2 Location uncertainty

Like many large scale flow dynamics representations that describe the flow in terms of a large-scale smooth velocity component and a fluctuation component (with respect to an averaging that must be specified) the LU formalism decomposes the flow in terms of a smooth-in-time Lagrangian velocity com-

ponent and a highly oscillating zero-mean random component,

$$d\mathbf{x}_t = \boldsymbol{u}(\mathbf{x}_t, t) dt + \boldsymbol{\sigma}_t(\mathbf{x}_t, t) d\boldsymbol{W}_t.$$
(1)

The random fluctuations are modelled on a canonical stochastic basis defined as the quadruple $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where Ω is a set, \mathcal{F} is a sigma algebra, $(\mathcal{F}_t)_{t\geq 0}$ is a right-continuous filtration and \mathbb{P} is a probability measure. In this decomposition $\mathbf{x} : \mathbb{R}^+ \times \Omega \to \mathcal{S}$ is the Lagrangian displacement defined within the bounded domain $\mathcal{S} \subset \mathbb{R}^d$ (d = 2 or 3) with smooth boundary, and $\mathbf{u} : \mathbb{R}^+ \times \mathcal{S} \times \Omega \to \mathcal{S}$ denotes the large-scale velocity that is both spatially and temporally correlated, while $\boldsymbol{\sigma} d \boldsymbol{W}$ is the unresolved component, which is assumed uncorrelated in time and correlated in space. This latter term is built from a cylindrical Wiener process \boldsymbol{W} on $H = L^2(\mathcal{S}, \mathbb{R}^d)$, the space of square integrable functions on \mathcal{S} with values in \mathbb{R}^d [41] and through a time dependent integral covariance operator $\boldsymbol{\sigma}_t$ defined for each $\omega \in \Omega$ from a bounded and symmetric positive kernel $\hat{\boldsymbol{\sigma}}$:

$$(\boldsymbol{\sigma}_t \boldsymbol{f})(x) := \int_{\mathcal{S}} \widehat{\boldsymbol{\sigma}}(x, y, t) \boldsymbol{f}(y) \, \mathrm{d}y, \ \boldsymbol{f} \in H.$$

Since the correlation kernel is bounded in x; y and t, the operator σ_t maps H into itself and is Hilbert-Schmidt. From the spectral theorem of compact, self-adjoint operator the noise component can be conveniently written as the spectral decomposition – with explicit dependence on the parameters (x, t, ω) , where ω denotes randomness:

$$\boldsymbol{\sigma}_t \boldsymbol{W}_t(x,\omega) = \sum_{i \in \mathbb{N}} \beta_t^i(\omega) \boldsymbol{\varphi}_i(x),$$

where $(\beta_i)_{i\in\mathbb{N}}$ is a sequence of independent standard Brownian motions on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and $(\varphi_i)_{i\in\mathbb{N}}$ are the correlation operator eigenfunctions scaled by their eigenvalues, $\lambda_i^{1/2}$. In addition, we assume that the operator-valued process $\{\sigma_t(\cdot)\}_{0\leq t\leq T}$ is stochastically integrable, that is $\mathbb{E}[\sum_{j\in\mathbb{N}}\lambda_j(\omega,t)] < \infty$. As such, under the probability measure \mathbb{P} , the stochastic integral $\{\int_0^t \sigma_s dW_s(\cdot)\}_{0\leq t\leq T}$ is a *H*-valued Gaussian process of zero mean, $\mathbb{E}_{\mathbb{P}}\int_0^t \sigma_s dW_s(\cdot) = 0$, and of bounded variance, $\mathbb{E}_{\mathbb{P}}[\|\int_0^t \sigma_s dW_s(\cdot)\|^2] < \infty$. We denote by $L^2(\Omega, L^2(\mathcal{S}))$ the Hilbert space of functions of $L^2(\mathcal{S})$ of bounded variance, equipped with inner product $(f,g)_{L^2(\Omega)} = \mathbb{E}(f,g)_{L^2(\mathcal{S})}$. Divergence-free or divergent noise can be considered [2]. The former case yields a divergence free constraint on the kernel:

$$\nabla \cdot \hat{\boldsymbol{\sigma}}(x, y, t) = 0, \ x, y \in \mathcal{S}, \ t \ge 0.$$

Associated with σ_t , we define the (matrix) tensor $a_{ij}(x,t)$ as

$$\int_{\Omega} \check{\boldsymbol{\sigma}}^{ik}(x, x', t) \check{\boldsymbol{\sigma}}^{kj}(x', x, t) \mathrm{d}x' = \sum_{k=0}^{\infty} \varphi_k^i(x, t) \varphi_k^j(x, t).$$
(2)

This quantity corresponds in the general case to the quadratic variation of the noise

$$\boldsymbol{a}_{ij}(x,t)\mathrm{d}t = \mathrm{d}_t \left\langle \int_0^{\cdot} \boldsymbol{\sigma}_s \mathrm{d}\boldsymbol{W}_s^i(x), \int_0^{\cdot} \boldsymbol{\sigma}_s \mathrm{d}\boldsymbol{W}_s^j(x) \right\rangle_t^{\cdot}.$$
 (3)

The quadratic variation (whose definition is briefly recap in Appendix-A) is a finite variation process when the correlation operator, σ_t , is random. For deterministic operator, it can be understood as the one-point covariance tensor:

$$\boldsymbol{a}_{ij}(x,t)\mathrm{d}t = \mathbb{E}\left(\left(\boldsymbol{\sigma}_t\mathrm{d}\boldsymbol{W}_t\right)^i(x)\left(\boldsymbol{\sigma}_t\mathrm{d}\boldsymbol{W}_t\right)^j(x)\right),\tag{4}$$

and for that reason we refer to a as the variance tensor by abuse of language.

Stochastic Reynolds transport theorem Given the form of the stochastic flow (1), the derivation of stochastic dynamics within the LU setting relies on a stochastic representation of the Reynolds transport theorem (SRTT) [40, 42]. This theorem provides the rate of change of a random scalar q within a volume V(t), transported by the stochastic flow (1). For general unresolved flows the SRTT reads

$$d\left(\int_{V(t)} q(x,t) \, dx\right) = \int_{V(t)} \left(\mathbb{D}_t q + q \nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_{\boldsymbol{a}}) dt\right) dx,\tag{5a}$$

$$\mathbb{D}_t q = \mathrm{d}_t q + (\boldsymbol{u} - \boldsymbol{u}_{\boldsymbol{a}}) \cdot \nabla q \, \mathrm{d}t + \boldsymbol{\sigma}_t \mathrm{d} \boldsymbol{W}_t \cdot \nabla q - \frac{1}{2} \nabla \cdot (\boldsymbol{a} \nabla q) \, \mathrm{d}t, \qquad (5\mathrm{b})$$

where $d_t q(x,t) = q(x,t+dt) - q(x,t)$ stands for the forward time-increment of q at a fixed point x. The operator \mathbb{D}_t is introduced as the stochastic transport operator [40, 42], and plays the role of the material derivative. Recall that u is the large-scale velocity used in (1) and a is defined in (2).

In this expression, the resulting advection is characterized by an effective transport component, modified by a drift

$$\boldsymbol{u}_{\boldsymbol{a}} = \frac{1}{2} \nabla \cdot \boldsymbol{a} + \boldsymbol{\sigma}_t \nabla \cdot \boldsymbol{\sigma}_t, \tag{6}$$

referred to as the Itô-Stokes drift (ISD) in [2]. It represents the resulting statistical effect of the small-scale inhomogeneity action on the transported quantity. For homogeneous divergence-free noises, this quantity is null, i.e. the variance tensor a is constant in space due to homogeneity. The third term in (5b), is intuitive, and corresponds to the advection of scalar q by the small-scale velocity component. This term continuously backscatters energy into the tracer energy through its quadratic variation. This gain of energy brought by the noise is then exactly compensated by the loss associated with the last diffusion term. This diffusion term is reminiscent of a generalized Boussinesq eddy viscosity – i.e the variance noise tensor plays the role of a viscosity (note that it has the unit of a viscosity in m^2/s). In

the stochastic transport operator expression (5b) the stochastic integral has to be interpreted in Itô form. For the following, it is insightful to move to a Stratonovich integral.

LU Stratonovich expression The Stratonovich integral has the great advantage to be associated to classical (deterministic) calculus rule, and is practical for that reason. In the other hand, they do not have the martingale property of the Itô integral. In particular, they are not of zero mean. Both stochastic integral are nevertheless equivalent in the sense that one can move from one to the other under the hypothesis of regular enough stochastic processes for the Stratonovich integral to exist. Itô integral is defined for very mild conditions (adaptability with respect to a filtration) while the Stratonovich integral requires stronger regularity conditions. As described in [2] the stochastic flow (1) and the transport operator can be turned in terms of the following equivalent expressions (with some regularity conditions on the transported quantity)

$$\mathrm{d}\mathbf{x}_t = \boldsymbol{u}\mathrm{d}t - \frac{1}{2}\nabla \cdot \boldsymbol{a}\mathrm{d}t + \boldsymbol{\sigma}_t \circ \mathrm{d}\boldsymbol{W}_t, \tag{7}$$

which involves a Stratonovich expression of the stochastic integral, specifically indicated by the \circ symbol. Note that in the Stratonovich case the noise term is not anymore of zero expectation. With this expression of the flow the transport operator corresponds now to the Lagrangian (material) derivative associated to the effective transport velocity [2]:

$$\mathbb{D}_{t}^{\circ}q = \mathrm{d}_{t} \circ q + (\boldsymbol{u} - \boldsymbol{u}_{\boldsymbol{a}}^{\circ}) \cdot \nabla q \, \mathrm{d}t + \boldsymbol{\sigma}_{t} \circ \mathrm{d}\boldsymbol{W}_{t} \cdot \nabla q, \qquad (8)$$

where $d_t \circ q \stackrel{\triangle}{=} \theta(x, t + (dt/2)) - \theta(x, t - (dt/2))$ stands for the centered time increment. In Statonovich form the Itô-Stokes drift reads [2]

$$\boldsymbol{u}_{\boldsymbol{a}}^{\circ} = \frac{1}{2} \nabla \cdot \boldsymbol{a} - \frac{1}{2} \boldsymbol{\sigma}_{t} \nabla \cdot \boldsymbol{\sigma}_{t}.$$
(9)

It can be noticed that for incompressible noise $(\nabla \cdot \boldsymbol{\sigma}_t = 0)$ the Itô form and the Stratonovich form of the Itô-Stokes drift ((6) and (9), respectively) reduce to the same expression.

The analysis of such type of noise, called transport noise, is the subject of a very intense research work in the literature. The aptitude of such noise to regularize partial differential equations (PDE's) has been explored in different setting, the justification of stochastic representation of flow dynamics models as well as their analysis are currently investigated by several groups (see for instance [1, 6, 7, 13, 15, 16, 18, 19, 20, 21, 23, 24, 25, 34] and references therein for recent publications).

3 Stochastic variational principle for fluid flow

In order to derive a stochastic variational principle, we introduce the regularized flow velocity we work with:

$$\boldsymbol{v}(x,t) = \boldsymbol{u}^*(x,t) + \int_{t-\epsilon}^{t+\epsilon} h_{\epsilon}(t-s)\boldsymbol{\xi}_i(x,t)d\beta_s^i = \boldsymbol{u}(x,t) - \boldsymbol{u}_{\boldsymbol{a}}(x,t) + \boldsymbol{u}^{\epsilon}(x,t). \quad (10)$$

The noise term $\boldsymbol{u}^{\epsilon}$ is defined through a regularization with regular functions h_{ϵ} of compact support. Function $h_{\epsilon}(t) = 1/\epsilon h(t/\epsilon)$ is positive on the support $[-\epsilon, +\epsilon]$ and null otherwise. The functions $\boldsymbol{\xi}_i$ in (10) are velocity basis functions such that $\mathbb{E}(|\boldsymbol{u}^{\epsilon}(t)|^2_{L^2(\mathcal{S})}) < \infty$. In the limit of a zero time-scale correlation parameter this noise leads to a Stratonovich transport expression. This known fact will be further justified later on in section 33.2 and Appendix-B.

The effective velocity component $u^* = u - u_a^\circ$ incorporates the largescale velocity component, u, and the Itô-Stokes drift [2]. The Itô-Stokes drift depends only on the noise and is here expressed in its Stratonovich expression. Denoting by γ the scaling of the noise variance, the noise expectation scales as

$$\mathbb{E}\|\boldsymbol{u}_t^{\epsilon}(x)\|^2 \sim \frac{\gamma}{\epsilon} \text{ and thus } \boldsymbol{u}_t^{\epsilon}(x) \sim \epsilon^{-1/2} \gamma^{1/2}, \tag{11}$$

as a consequence we have thus,

$$\partial_t \boldsymbol{u}_t^{\epsilon}(x) \sim \epsilon^{-\frac{3}{2}} \gamma^{1/2} \text{ and } \partial_{x_i} \boldsymbol{u}_t^{\epsilon}(x) \sim \epsilon^{-\frac{1}{2}} \gamma^{1/2}.$$
 (12)

The velocity component, \boldsymbol{u} , should encompass solutions of a large-scale stochastic representation of the Navier-Stokes equations. In particular at the limit of vanishing noise, such a solution should further converge toward a deterministic solution of the Navier-Stokes equation (the unique solution in 2D and a weak solution for a 3D domain). Such a requirement is similar to constrain large eddy simulations (LES) to match direct numerical simulations (DNS) of the Navier-Stokes equations obtained at refined resolution. Such a property has been rigorously demonstrated for LU in [15].

As outlined previously, LU is derived from the Reynolds transport theorem and Newton's second law. It does not initially ensue from a variational principle, unlike other settings [12, 29, 49]. All these settings propose slightly different stochastic parameterizations to account for rapidly evolving unresolved-scale effects. The variational approach proposed in [29], referred to as stochastic advection by Lie transport (SALT), combines stochastic transport advection constraints in the form of (8) with a Lagrangian related to the large-scale kinetic energy.

Due to their respective differences, these two schemes exhibit different conservation properties: SALT preserves helicity [29], while LU conserves energy [2, 42]. Interestingly, neither of these two schemes carries the whole set of Euler equation invariants. In that sense, they likely both correspond to approximations of what should be an ideal stochastic representation of the Euler equations.

As highlighted below, the energy cost functions associated with the variational principles considered here and the one employed in SALT are slightly different. For SALT, only the large-scale kinetic energy, with no energy interaction with the small scale, is considered, while the energy over the whole flow will be considered in our case. Through this, we will show that SALT and LU inherit from different approximations. LU corresponds to an approximation of a generalized stochastic regularized Euler system associated with a variational principle taking into account the full kinetic energy, while SALT results from a variational principle expressed solely on the large-scale kinetic energy with no interaction between the small-scale and the large-scale components.

Stochastic transport will be introduced as constraints in the variational models derived in this work. Constraints both on density and scalar transport must be considered. They are usually referred to as Clebsh constraints. In this study, we will explore a simple fluid model only.

3.1 Fluid large-scale variational formulation

In the fluid setting associated with pure passive transport of scalar, we assume the following energy functional $S(\boldsymbol{u}, \rho, \lambda)$ defined as

$$S(\boldsymbol{u},\rho,\lambda) = \int_{t_1}^{t_2} \left(\ell(\boldsymbol{v},\rho) + \left\langle \lambda, \partial_t \rho + \boldsymbol{\nabla} \cdot (\boldsymbol{v}\rho) \right\rangle_{L^2(\mathcal{S})} \right) \mathrm{d}t,$$

with $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_{\boldsymbol{a}} + \boldsymbol{u}^{\epsilon}.$ (13)

In the pathwise expression above λ is a scalar Lagrange multiplier that enforces the density ρ to be transported. By duality a constraint on the transport by the flow of any scalar (associated with a density multiplier) could be added. The angle brackets $\langle \cdot, \cdot \rangle_{L^2(S)}$ denote the L^2 pairing over the domain S and t_1 and t_2 denote two arbitrary times. In this very first case we will consider the same condition on the noise term as the one assumed for the SALT variationnal setting [29, 49]. The noise term is specified a priori and does not depend on the solution. Consequently, it is not considered as a variable of the system. Let us note that such assumption excludes immediately any formulations where the noise is defined from the current state of the system such as in [35]. It can be stressed out that the (regularized) noise term remains in the cost function term $\ell(\boldsymbol{v}, \rho)$.

We are now ready to consider the principle of least action for the energy

functional (13). The differential $\delta S(\boldsymbol{u}, \rho, \lambda)$ reads

$$\delta S(\boldsymbol{u},\rho,\lambda) = \int_{t_1}^{t_2} \left(\left\langle \frac{\delta \ell}{\delta \boldsymbol{u}} - \rho \boldsymbol{\nabla} \lambda, \delta \boldsymbol{u} \mathrm{d} t \right\rangle_{L^2(\mathcal{S})} + \left\langle \partial_t \rho + \boldsymbol{\nabla} \cdot \left(\boldsymbol{v} \rho \right), \delta \lambda \mathrm{d} t \right\rangle_{L^2(\mathcal{S})} \right. \\ \left. + \left\langle \frac{\delta \ell}{\delta \rho}, \delta \rho \mathrm{d} t \right\rangle_{L^2(\mathcal{S})} - \left\langle \partial_t \lambda + \left(\boldsymbol{v} \cdot \boldsymbol{\nabla} \right) \lambda, \delta \rho \mathrm{d} t \right\rangle_{L^2(\mathcal{S})} \right).$$
(14)

The variations in the above expression are arbitrary and vanish at the times t_1 and t_2 and we have explicitly used the fact that the stochastic integrals are regularized and obey the product rule. We have also used the fact that the variations vanish at the endpoints in time, so no initial or terminal terms appear. Therefore, to satisfy the variational principle, the following equations must hold almost surely for any perturbation, δu , $\delta \rho$ and $\delta \lambda$:

$$\delta \boldsymbol{u}: \quad \frac{\delta \ell}{\delta \boldsymbol{u}} - \rho \boldsymbol{\nabla} \lambda = 0, \tag{15a}$$

$$\delta \rho : \qquad \frac{\delta \ell}{\delta \rho} - \partial_t \lambda - \boldsymbol{v} \cdot \boldsymbol{\nabla} \lambda = 0, \tag{15b}$$

$$\delta \lambda : \quad \partial_t \rho + \boldsymbol{\nabla} \cdot (\boldsymbol{v} \rho) = 0. \tag{15c}$$

The next step is to eliminate the Lagrange multiplier λ . To do so, we compute

$$\partial_t (\rho \nabla \lambda) = (\partial_t \rho) \nabla \lambda + \rho \nabla \partial_t \lambda$$

= $-\nabla \cdot (\boldsymbol{v}\rho) \nabla \lambda + \rho \nabla (\frac{\delta \ell}{\delta \rho}) - \rho \nabla (\boldsymbol{v} \cdot \nabla \lambda),$ (16)

and with equation (15a) we have finally:

$$\partial_t \left(\frac{\delta\ell}{\delta \boldsymbol{u}}\right) + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \left(\frac{\delta\ell}{\delta \boldsymbol{u}}\right) = \rho \boldsymbol{\nabla} \left(\frac{\delta\ell}{\delta \rho}\right) - \partial_{x_i} \boldsymbol{v}^j \left(\frac{\delta\ell}{\delta \boldsymbol{u}}\right)^j - \nabla \cdot \boldsymbol{v} \left(\frac{\delta\ell}{\delta \boldsymbol{u}}\right).$$
(17)

This Euler-Lagrange equation provides a general stochastic expression for the large-scale velocity component evolution. Let us now take the concrete exemple of the Euler equation to infer precisely a system of associated stochastic partial differential equations (SPDE).

3.1.1 Euler equations

The Lagrangian for the Euler equations in a 3-dimensional domain S with the Euclidean metric and Cartesian coordinates is given by the pathwise L^2 -kinetic energy

$$\ell(\boldsymbol{v},\rho) = \int_{\mathcal{S}} \left(\frac{1}{2} |\boldsymbol{v}|^2 \rho\right) \mathrm{d}\mathcal{S}.$$
 (18)

To enforce incompressibility, we include a constraint with a pressure Lagrange multiplier, p, that sets the density ρ to a constant. The constrained action $S(\boldsymbol{v}, \rho, p, \lambda)$ that we want to minimize is given by

$$S(\boldsymbol{v},\rho,p,\lambda) = \int_{t_1}^{t_2} \left(\ell(\boldsymbol{v},\rho) - \langle p,\rho-1 \rangle_{L^2(\mathcal{S})} + \langle \lambda,\partial_t \rho + \boldsymbol{\nabla} \cdot (\boldsymbol{v}\,\rho) \rangle_{L^2(\mathcal{S})} \right) \mathrm{d}t.$$
(19)

3 STOCHASTIC VARIATIONAL PRINCIPLE FOR FLUID FLOW 10

The variational derivatives of the Lagrangian are

$$\frac{\delta \ell}{\delta \boldsymbol{u}} = \rho \boldsymbol{v}, \text{ and } \frac{\delta \ell}{\delta \rho} = \frac{1}{2} |\boldsymbol{v}|^2.$$
 (20)

The variational derivatives of the Lagrangian with respect to the velocity variable defines the momentum. The variational derivative with respect to the density gives the Bernoulli function. Inserting these expressions into the stochastic Euler-Lagrange equation (17) complemented with the pressure constraint provides the momentum equation. We obtain formally the system of coupled SPDE's

$$\partial_t(\rho \boldsymbol{v}) + (\boldsymbol{v} \cdot \boldsymbol{\nabla})(\rho \boldsymbol{v}) + \boldsymbol{\nabla} \boldsymbol{v} \cdot (\rho \boldsymbol{v}) + (\boldsymbol{\nabla} \cdot \boldsymbol{v})\rho \boldsymbol{v} = \rho \boldsymbol{\nabla} \left(\frac{1}{2}|\boldsymbol{v}|^2\right) - \boldsymbol{\nabla} p,$$

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\boldsymbol{v}\rho) = 0,$$

$$\rho = 1.$$
(21)

The incompressibility condition implies that the continuity equation reduces to

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \tag{22}$$

while the momentum equation reads

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{v} = -\boldsymbol{\nabla} p. \tag{23}$$

The resulting stochastic system has exactly the same form as the classical Euler equations and shares formally their properties, including the preservation of both energy and helicity. Mathematically, it becomes ill-defined at the zero limit of the regularization parameter ϵ , making numerical handling challenging for small values of this parameter. Furthermore, the system lacks clear scale separation and does not correspond to neither the LU nor SALT solutions. However, it can be observed that for a constant large-scale velocity, the Euler equation boils down to the balance

$$-\gamma \partial_t \boldsymbol{u}_{\boldsymbol{a}}^{\boldsymbol{o}} + \epsilon^{-\frac{3}{2}} \gamma^{\frac{1}{2}} \partial_t \boldsymbol{u}^{\boldsymbol{\epsilon}} + \left((\boldsymbol{u} - \gamma \, \boldsymbol{u}_{\boldsymbol{a}}^{\boldsymbol{o}} + \epsilon^{-\frac{1}{2}} \gamma^{\frac{1}{2}} \boldsymbol{u}^{\boldsymbol{\epsilon}}) \cdot \boldsymbol{\nabla} \right) (-\gamma \, \boldsymbol{u}_{\boldsymbol{a}}^{\boldsymbol{o}} + \epsilon^{-\frac{1}{2}} \gamma^{\frac{1}{2}} \boldsymbol{u}^{\boldsymbol{\epsilon}}) = -\boldsymbol{\nabla} p^{\boldsymbol{\epsilon},\boldsymbol{a}}, \quad (24)$$

which consists in a Euler type equation on the noise term and on the Itô-Stokes drift with v as the velocity transport. Assuming this balance still holds for any large-scale velocity component, u, we obtain the following system

$$\partial_t \boldsymbol{u} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{u} = -\boldsymbol{\nabla}(p - p^{\epsilon, \boldsymbol{a}}),$$
(25a)

$$\partial_t (-\boldsymbol{u}_a^{\circ} + \boldsymbol{u}^{\epsilon}) + \left((\boldsymbol{u} - \boldsymbol{u}_a^{\circ} + \boldsymbol{u}^{\epsilon}) \cdot \boldsymbol{\nabla} \right) (\boldsymbol{u}_a^{\circ} + \boldsymbol{u}^{\epsilon}) = -\boldsymbol{\nabla} p^{\epsilon, \boldsymbol{a}}, \quad (25b)$$

$$\nabla \cdot \boldsymbol{u} - \frac{1}{2} \nabla \cdot \nabla \cdot \boldsymbol{a} + \nabla \cdot \boldsymbol{u}^{\epsilon} = 0.$$
(25c)

The extension of the balance (24) is justifiable assuming there is a time scale separation where the large scale velocity is much slower and smoother than the fine scales. Besides, we can notice that although these equations are formally perfectly defined for regularized noise, the last equation is badly defined at the limit of $\epsilon \to 0$ because of the terms $\partial_t u^{\epsilon}$ and $(u^{\epsilon} \cdot \nabla u^{\epsilon})$ that correspond to "Brownian acceleration" terms.

Recalling that the noise correlation functions are assumed to be specified and considering all the elements gathered so far, the system can now be finally written in terms of the evolution of the large-scale component with incompressibility constraints:

$$\partial_t \boldsymbol{u} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{u} = -\boldsymbol{\nabla}(p - p^{\boldsymbol{\epsilon}, \boldsymbol{a}}),$$

$$\nabla \cdot \boldsymbol{u} - \frac{1}{2} \nabla \cdot \nabla \cdot \boldsymbol{a} + \nabla \cdot \boldsymbol{u}^{\boldsymbol{\epsilon}} = 0.$$
 (26)

For ϵ tending to zero, h tends to the Dirac evaluation function and we get at the limit the following system:

$$d_{t}\boldsymbol{u} + \left((\boldsymbol{u}^{*}dt + \boldsymbol{\varphi}_{i} \circ d\beta^{i}) \cdot \boldsymbol{\nabla} \right) \boldsymbol{u} = -\boldsymbol{\nabla}(dp - dp_{\sigma}), \nabla \cdot \boldsymbol{u} - \nabla \cdot \boldsymbol{u}_{\boldsymbol{a}}^{\circ} = \nabla \cdot \boldsymbol{\varphi}_{i} = 0,$$
(27)

where the stochastic integral should be understood in a Stratonovich sense. The functions φ_i corresponds to displacement $\xi^i dt$ and dp_{σ} is a semi-martingale pressure term. The above system corresponds exactly to the LU representation of the Euler system. Through Itô integration by part formulae it can be immediately checked that this system preserves energy in the same way as the deterministic Euler equation (for adequate boundary conditions) but looses helicity conservation [2, 42]. Hence, LU corresponds to an approximation of the initial regularized stochastic Euler equations. It cannot be directly obtained from the considered variational principle.

Remark: Comparison with SALT [29] The SALT Euler equations are obtained from a different variational principle in which the following action functional $S(\boldsymbol{u}, \rho, p, \lambda)$ is considered:

$$S(\boldsymbol{u},\rho,p,\lambda) = \int_{t_1}^{t_2} \left(\frac{1}{2} \|\boldsymbol{u}\|^2 - \langle p,\rho-1 \rangle_{L^2(\mathcal{S})} + \langle \lambda,\partial_t \rho + \boldsymbol{\nabla} \cdot (\boldsymbol{v}\,\rho) \rangle_{L^2(\mathcal{S})} \right) \mathrm{d}t.$$
(28)

Here, the kinetic energy corresponds to the norm of the large-scale velocity only, while the density is transported by the entire random flow. The interaction term $\langle \boldsymbol{u}, -\boldsymbol{u}_a^{\circ} + \boldsymbol{u}^{\epsilon} \rangle_{L^2(S)}$ between the small-scale and the large-scale components is not considered in the Lagrangian. The transport constraints are on the contrary expressed through a noise term correlated to the large scale component, leading to a Stratonovich noise at the decorrelation limit. It is worth noting that considering directly a Itô-like uncorrelated noise (without the balancing diffusion term of the stochastic transport operator 5b) in the functional above would make no sense as it would lead to stochastic advection equations that are not well-posed at the zero limit of ϵ .

The associated variational derivatives of the Lagrangian are

$$\frac{\delta\ell}{\delta \boldsymbol{u}} = \rho \boldsymbol{u}, \text{ and } \frac{\delta\ell}{\delta\rho} = \frac{1}{2}|\boldsymbol{v}|^2.$$
 (29)

Injecting these variational derivatives in (17) yields the SALT momentum equation:

$$\partial_t \boldsymbol{u} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{u} + \boldsymbol{\nabla} \boldsymbol{u}^{\epsilon} \cdot \boldsymbol{u} = -\boldsymbol{\nabla} \left(p - \frac{1}{2} |\boldsymbol{u}|^2 - \frac{1}{2} |\boldsymbol{u}^{\epsilon}|^2 \right), \qquad (30)$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0,$$

where the notation $\nabla u \cdot v$ stands for $\sum_{\ell} (\partial_{x_k} u^{\ell}) v^{\ell}$.

With the assumption that the noise lives at a small spatial scale, $\eta^{-d/2}$, with d the flow dimension, the stretching term,

$$\boldsymbol{\nabla} \boldsymbol{u}^{\boldsymbol{\epsilon}} \boldsymbol{\cdot} \boldsymbol{u} = \partial_{\boldsymbol{x}_i} \boldsymbol{u}^{\boldsymbol{\epsilon},j} \boldsymbol{u}^j,$$

corresponds to a higher order term of order $\eta^{-d/2-1}$, which comes necessary with a balancing pressure term, if it is non zero. Removing these balanced terms, this system reduces to the LU equations. This provides another way of specifying the LU equations. However, note that this corresponds to a restrictive form of LU as it is associated with the assumption that the noise does not depend on the solution, which is fully allowed in the original LU derivation based on SRTT. It is interesting to note that one transitions then from a system that preserves helicity (SALT) to a system that conserves energy (LU).

In the following we explore how these solutions can be complemented through a variational formulation in which the noise component depends now on the large scale component and constitutes a variable of the system.

3.2 Variational formulation for the noise correlation dynamics with a decorrelation assumption

In this section we aim to construct a variational principle for specifying a proper dynamics for the noise basis functions $\{\boldsymbol{\xi}_i, i \in \mathbb{N}\}$. This second variational principle will be built upon the previous variational principle leading, for the Euler Lagrangian, to generalized stochastic Euler equations. It is important to stress out that this previous variational principle was expressed pathwise, allowing us to characterize the large-scale dynamics. However, it is not suitable for estimating noise correlation functions. To address this

limitation, we must transition towards defining a variational principle expressed in expectation. Incorporating both a pathwise setting and a principle in expectation within a common general variational framework appears challenging. From this perspective, we propose a two-stage process. First, the dynamics of the large-scale component will be assumed to arise from a pathwise variational principle with an action functional of the form of (13), specified on the full regularized velocity, (10), for given Itô-Stokes drift and time-correlated noise.

Subsequently, we will construct a variational principle for the noise correlations conditionally to this large-scale dynamics. The large-scale dynamics will, in turn, be enriched by the noise functions, which are solutions to this conditional variational problem. This approach resembles the classical decomposition into large scales and small scales as commonly performed in turbulence modeling.

In the second variational principle expressed in expectation, we will need to express all the correlations between the correlated regularized noise considered and the different large-scale components of all the functions involved. To that end, we will need to express correlated noise expressions, tending at the limit to Stratonovich representation, to decorrelated, Itô forms of these noise terms. In other words, this comes back to express this second variational principle within a LU philosophy, where the small-scale components are decorrelated in time and the transport expressed through the Itô form of the stochastic transport operator.

More precisely, for the noise functions we will consider in the following an action functional, $\mathbb{S}(\rho, \lambda, \boldsymbol{\xi})$ defined as:

$$\mathbb{S}(\rho,\lambda,\boldsymbol{\xi}) = \mathbb{E}\int_{t_1}^{t_2} \left(\ell(\boldsymbol{u},\rho,\boldsymbol{\xi}) + \left\langle\lambda, \mathbb{D}_t^{\epsilon}\rho + \rho\boldsymbol{\nabla}\cdot\boldsymbol{v}\right\rangle_{L^2(\mathcal{S})}\right) \mathrm{d}t, \qquad (31)$$

which relies on a regularized stochastic density transport defined as:

$$\mathbb{D}_{t}^{\epsilon}\rho + \rho \nabla \cdot \boldsymbol{v} = \partial_{t}\rho + \nabla \cdot (\boldsymbol{u}\rho) + \sum_{i} \nabla \cdot \left(\rho(t)\boldsymbol{\xi}_{i}(t) \int_{t-\epsilon}^{t+\epsilon} h_{\epsilon}(t-s) \mathrm{d}\beta_{s}^{i}\right) = 0. \quad (32)$$

This regularized transport expression involves a correlated noise that will be shown to converge toward a Stratonovich transport expression with Stratonovich noise $\sigma_t \circ dW_t^\epsilon$. Let us note that for simplicity the large-scale component of the transport, which is assumed to be provided by the first variational principle, includes here directly the Itô-Stokes drift in its expression. This assumption is supported by the fact that the Itô-Stokes drift is much slower the noise components. In the following, to fully compute the correlation between the noise and the large-scale velocity component, \boldsymbol{u} , we will need to consider an equivalent regularized expression converging toward a transport

3 STOCHASTIC VARIATIONAL PRINCIPLE FOR FLUID FLOW 14

in Itô form and involving hence an Itô regularized noise with increments decorrelated from $\boldsymbol{u}(t)$ (where $\int_t^{t+\epsilon} \tilde{h}_{\epsilon}(s) ds = 1$):

$$\sigma_t \boldsymbol{W}_t^{\epsilon} = \sum_i \int_t^{t+\epsilon} \tilde{h}_{\epsilon}(t-s) \boldsymbol{\xi}_i(t) \mathrm{d}\beta_s^i.$$
(33)

In full generality the noise correlation functions, $\{\boldsymbol{\xi}_i, i \in \mathbb{N}\}$ will be assumed to be stochastic processes of the form

$$d\boldsymbol{\xi}_{i} = \boldsymbol{\mu}_{i}dt + \sum_{j} \int_{t}^{t+\epsilon} \tilde{h}_{\epsilon}(t-s)\boldsymbol{\Lambda}_{j}^{i}(t)d\beta_{s}^{j}.$$
(34)

We also notice that in the noise terms of (32) and (33) considered above, the noise correlation functions, $\boldsymbol{\xi}_i$ and $\boldsymbol{\Lambda}_j^i$, are for simplicity not regularized. They can come outside of the convolution integral.

It can be remarked that the stochastic density transport in Stratonovich form

$$\mathbb{D}_{t}^{\circ}\rho + \rho\nabla \cdot \boldsymbol{v} = \partial_{t}\rho + \nabla \cdot \left(\rho(\boldsymbol{u} + \sigma_{t} \circ \boldsymbol{W}_{t}^{\epsilon})\right) = 0, \qquad (35)$$

reads in Itô form as

$$\partial_t \rho + \nabla \cdot \left(\rho(\boldsymbol{u} + \frac{1}{2} \sum_i \boldsymbol{\Lambda}_i^i) + \rho \sigma_t \boldsymbol{W}_t^\epsilon \right) - \frac{1}{2} \boldsymbol{\nabla} \cdot \left(\boldsymbol{\xi}_i \nabla \cdot (\boldsymbol{\xi}_i \rho) \right) = 0.$$
(36)

Two different possibilities can then be considered. It is possible either to work with the regularized form of the Itô transport formulation (where $\int_t^{t+\epsilon} \tilde{h}_{\epsilon}(s) ds = 1$):

$$\partial_{t}\rho + \nabla \cdot (\boldsymbol{u}\rho) + \frac{1}{2} \sum_{i} \nabla \cdot (\boldsymbol{\Lambda}_{i}^{i}\rho) - \frac{1}{2} \nabla \cdot \left(\boldsymbol{\xi}_{i} \nabla \cdot (\boldsymbol{\xi}_{i}\rho)\right) + \sum_{i} \nabla \cdot \left(\rho \boldsymbol{\xi}_{i}(t)\right) \int_{t}^{t+\epsilon} \tilde{h}_{\epsilon}(t-s) \mathrm{d}\beta_{s}^{i}, \quad (37)$$

or with a regularized Stratonovich transport, (32), (with $\int_{t-\epsilon}^{t+\epsilon} h_{\epsilon}(s) ds =$ 1) transformed in a Itô form with a decorrelated Itô noise. As shown in Appendix-B both options are equivalent.

As mentioned previously, for the variables $(\boldsymbol{u}, \rho, \lambda)$ we consider a pathwise action functional (13), for which the associated functional derivatives, given in (15 a,b,c), yields the Euler-Lagrange equations, (17), which provides a large-scale momentum equation.

In order to infer a dynamics for the noise basis, we need to exhibit supplementary constraints and to transition to a variational principle in expectation. All the variables, g, will be decomposed in terms of a smooth component and a decorrelated noise component, expressed as the sum of correlation functions g_i :

$$g(x,t) = \bar{g}(x,t) + \int_{t}^{t+\epsilon} \tilde{h}_{\epsilon}(t-s)g_{i}(x,t)\mathrm{d}\beta_{s}^{i} = \bar{g}(x,t) + g_{t}^{\epsilon}(x), \qquad (38)$$

where the functions \bar{g} and g_i are of finite variation.

Relying now on the regularized approximation of the transport expression in its Itô form (with a decorrelated regularized noise, (33)), the action, (31), considered for the noise correlation functions, is more precisely rewritten for all $j \in \mathbb{N}$ as $\mathbb{S}(\bar{\rho}, \rho_j, \bar{\lambda}, \lambda_j, \boldsymbol{\xi}_j) =$

$$\mathbb{E}\!\!\int_{t_1}^{t_2}\!\!\left(\ell(\boldsymbol{u},\rho,\boldsymbol{\xi}) + \left\langle\lambda, \mathbb{D}_t^{\epsilon}\rho + \rho\boldsymbol{\nabla}\cdot\boldsymbol{v}\right\rangle_{L^2(\mathcal{S})}\right) \mathrm{d}t.$$
(39)

The Clebsch constraint related to the density transport can be split and rewritten as :

$$\mathbb{E}\left\langle\bar{\lambda},\partial_{t}\bar{\rho}+\boldsymbol{\nabla}\cdot\left(\boldsymbol{u}\bar{\rho}+\Upsilon_{\epsilon}\sum_{j}\boldsymbol{\xi}_{j}\rho_{j}\right)+\frac{1}{2}\sum_{i}\nabla\cdot\left(\boldsymbol{\Lambda}_{i}^{i}\bar{\rho}\right)-\frac{1}{2}\sum_{i}\boldsymbol{\nabla}\cdot\left(\boldsymbol{\xi}_{i}\nabla\cdot\left(\boldsymbol{\xi}_{i}\bar{\rho}\right)\right)\right\rangle_{L^{2}(\mathcal{S})}+\sum_{j}\mathbb{E}\left\langle\Upsilon_{\epsilon}\lambda_{j},\partial_{t}\rho_{j}+\boldsymbol{\nabla}\cdot\left(\boldsymbol{u}\rho_{j}+\boldsymbol{\xi}_{j}\bar{\rho}\right)+\frac{1}{2}\sum_{\ell}\nabla\cdot\left(\boldsymbol{\Lambda}_{\ell}^{\ell}\rho_{j}\right)-\frac{1}{2}\sum_{\ell}\boldsymbol{\nabla}\cdot\left(\boldsymbol{\xi}_{\ell}\nabla\cdot\left(\boldsymbol{\xi}_{\ell}\rho_{j}\right)\right)\right\rangle_{L^{2}(\mathcal{S})},$$
(40)

where $\Upsilon^{\epsilon} = \int_{t}^{t+\epsilon} \tilde{h}_{\epsilon}^{2}(t-s) \mathrm{d}s.$

In the above equation, the noise decorrelation and the Itô isometry have been used. We also recall that third order moment of Gaussian random variables cancels. It should be noticed that the temporal derivative term on the third line has been simplified as

$$\begin{split} \mathbb{E} \Big\langle \lambda_j \int_t^{t+\epsilon} \tilde{h}_{\epsilon}(t-s) \mathrm{d}\beta_s^j, \, \rho_j \partial_t \Big(\int_t^{t+\epsilon} \tilde{h}_{\epsilon}(t-s) \mathrm{d}\beta_s^j \Big) \Big\rangle_{L^2(\mathcal{S})} \\ &= \mathbb{E} \Big\langle \lambda_j \rho_j, \, \frac{1}{2} \partial_t \int_t^{t+\epsilon} (\tilde{h}_{\epsilon}(t-s))^2 \mathrm{d}t \Big\rangle_{L^2(\mathcal{S})} \\ &= \mathbb{E} \Big\langle \lambda_j \rho_j, \, \frac{1}{2} \partial_t \Upsilon_{\epsilon} \Big\rangle_{L^2(\mathcal{S})} = 0. \end{split}$$

Let us recall that the noise basis functions are not regularized and the integration constant $\Upsilon^{\epsilon} = \int_{t}^{t+\epsilon} \tilde{h}_{\epsilon}^{2}(t-s) \mathrm{d}s \sim C \epsilon^{-1} \gg 1$.

Applying the least action principle and cancelling out the variational derivatives with respect to the different variables, we obtain the following constraints (without summation over repeated indices):

$$\mathbb{E} \left\langle \frac{\delta \ell}{\delta \bar{\rho}} - \partial_t \bar{\lambda} - \boldsymbol{u} \cdot \boldsymbol{\nabla} \bar{\lambda} - \Upsilon_{\epsilon} \sum_j \boldsymbol{\xi}_j \cdot \boldsymbol{\nabla} \lambda_j - \frac{1}{2} \sum_i \boldsymbol{\nabla} \bar{\lambda} \cdot \boldsymbol{\Lambda}_i^i - \frac{1}{2} \sum_j \boldsymbol{\xi}_j \cdot \boldsymbol{\nabla} (\boldsymbol{\xi}_j \cdot \boldsymbol{\nabla} \bar{\lambda}), \delta \bar{\rho} \right\rangle_{L^2(\mathcal{S})} = 0,$$
(41a)

$$\mathbb{E} \left\langle \frac{\delta \ell}{\delta \rho_{j}} - \Upsilon_{\epsilon} \partial_{t} \lambda_{j} - \Upsilon_{\epsilon} \boldsymbol{u} \cdot \boldsymbol{\nabla} \lambda_{j} - \Upsilon_{\epsilon} \boldsymbol{\xi}_{j} \cdot \boldsymbol{\nabla} \bar{\lambda} - \Upsilon_{\epsilon} \frac{1}{2} \sum_{\ell} \boldsymbol{\nabla} \lambda_{j} \cdot \boldsymbol{\Lambda}_{\ell}^{\ell} - \Upsilon_{\epsilon} \frac{1}{2} \sum_{\ell} \boldsymbol{\xi}_{\ell} \cdot \boldsymbol{\nabla} \left(\boldsymbol{\xi}_{\ell} \cdot \boldsymbol{\nabla} \lambda_{j} \right), \delta \rho_{j} \right\rangle_{L^{2}(\mathcal{S})} = 0, \quad (41b)$$

$$\mathbb{E} \Big\langle \partial_t \bar{\rho} + \boldsymbol{\nabla} \cdot \left(\boldsymbol{u} \bar{\rho} + \Upsilon_\epsilon \sum_j \boldsymbol{\xi}_j \rho_j \right) + \frac{1}{2} \sum_j \boldsymbol{\nabla} \cdot \left(\boldsymbol{\Lambda}_j^j \bar{\rho} \right) \\ - \frac{1}{2} \sum_j \boldsymbol{\nabla} \cdot \left(\boldsymbol{\xi}_j \boldsymbol{\nabla} \cdot \left(\boldsymbol{\xi}_j \boldsymbol{\nabla} \bar{\rho} \right) \right), \delta \bar{\lambda} \Big\rangle_{L^2(\mathcal{S})} = 0, \quad (41c)$$

$$\mathbb{E} \Big\langle \partial_t \rho_j + \nabla \cdot \left(\boldsymbol{u} \rho_j + \boldsymbol{\xi}_j \bar{\rho} \right) + \frac{1}{2} \sum_{\ell} \nabla \cdot \left(\boldsymbol{\Lambda}_{\ell}^{\ell} \rho_j \right) \\ - \frac{1}{2} \sum_{\ell} \nabla \cdot \left(\boldsymbol{\xi}_{\ell} \nabla \cdot \left(\boldsymbol{\xi}_{\ell} \rho_j \right) \right), \Upsilon_{\epsilon} \delta \lambda_j \Big\rangle_{L^2(\mathcal{S})} = 0,, \quad (41d)$$

$$\mathbb{E} \left\langle \frac{\delta \ell}{\delta \boldsymbol{\xi}_{j}} - \Upsilon_{\epsilon} \bar{\rho} \, \boldsymbol{\nabla} \lambda_{j} - \Upsilon_{\epsilon} \rho_{j} \, \boldsymbol{\nabla} \bar{\lambda} + \underbrace{\frac{1}{2} \sum_{i} \boldsymbol{\nabla} \cdot (\bar{\rho} \boldsymbol{\xi}_{i}) \boldsymbol{\nabla} \bar{\lambda} - \boldsymbol{\nabla} (\boldsymbol{\nabla} \bar{\lambda} \cdot \boldsymbol{\xi}_{i}) \bar{\rho}}_{A} + \underbrace{\Upsilon_{\epsilon} \frac{1}{2} \sum_{i} \left(\boldsymbol{\nabla} \cdot (\rho_{j} \boldsymbol{\xi}_{i}) \boldsymbol{\nabla} \lambda_{j} - \boldsymbol{\nabla} (\boldsymbol{\nabla} \lambda_{j} \cdot \boldsymbol{\xi}_{i}) \rho_{j} \right)}_{B}, \delta \boldsymbol{\xi}_{j} \right\rangle_{L^{2}(S)} = 0, \quad (41e)$$

$$\mathbb{E} \left\langle \frac{1}{2} \left(\bar{\rho} \, \boldsymbol{\nabla} \bar{\lambda} + \Upsilon_{\epsilon} \sum_{i} \rho_{i} \boldsymbol{\nabla} \lambda_{i} \right), \delta \mathbf{\Lambda}_{j}^{j} \right\rangle_{L^{2}(\mathcal{S})} = 0.$$
(41f)

Taking the time derivative of (41e) we have:

$$\mathbb{E} \left\langle \Upsilon_{\epsilon}^{-1} \partial_{t} \left(\frac{\delta \ell}{\delta \boldsymbol{\xi}_{j}} \right) - \partial_{t} \bar{\rho} \nabla \lambda_{j} - \bar{\rho} \nabla \partial_{t} \lambda_{j} - \partial_{t} \rho_{j} \nabla \bar{\lambda} - \rho_{j} \nabla \partial_{t} \bar{\lambda} + \Upsilon_{\epsilon}^{-1} \frac{1}{2} \partial_{t} \sum_{i} \left(\nabla \cdot (\bar{\rho} \boldsymbol{\xi}_{i}) \nabla \bar{\lambda} - \nabla (\nabla \bar{\lambda} \cdot \boldsymbol{\xi}_{i}) \bar{\rho} \right) + \frac{1}{2} \partial_{t} \sum_{i} \left(\nabla \cdot (\rho_{j} \boldsymbol{\xi}_{i}) \nabla \lambda_{j} - \nabla (\nabla \lambda_{j} \cdot \boldsymbol{\xi}_{i}) \rho_{j} \right), \delta \boldsymbol{\xi}_{j} \right\rangle_{L^{2}(S)} = 0.$$
(42)

The expression of the temporal derivatives can now be injected in this expression to get an evolution equation of the noise correlation functions. In the next section we infer more precisely the evolution equations associated to specific case of the Euler equations.

3.2.1 Euler equations

As previously, the Euler pathwise action, $S(\boldsymbol{u}, \rho, p, \lambda)$, is defined by the L^2 kinetic energy of the whole flow, $\ell(\boldsymbol{v}, \rho)$, together with density transport and incompressibility constraints, (18). We complement the pathwise action with the action in expectation $\mathbb{S}(\boldsymbol{\xi}_i, \bar{\rho}, \rho_i, \bar{p}, p_i, \bar{\lambda}, \lambda_i)$, (39), along with an incompressibility constraint. We consider hence the couple of actions:

$$S(\boldsymbol{u},\rho,p,\lambda) = \int_{t_1}^{t_2} \left(\ell(\boldsymbol{v},\rho) - \left\langle p,\rho-1 \right\rangle_{L^2(\mathcal{S})} + \left\langle \mathbb{D}_t^{\epsilon}\rho + \rho \boldsymbol{\nabla} \cdot \boldsymbol{v} \right\rangle_{L^2(\mathcal{S})} \right) \mathrm{d}t,$$
(43)

and

$$\mathbb{S}(\boldsymbol{\xi}_{i},\bar{\rho},\rho_{i},\bar{p},p_{i},\bar{\lambda},\lambda_{i}) = \mathbb{E}\!\!\int_{t_{1}}^{t_{2}} \left(\ell(\boldsymbol{v},\rho) - \left\langle p,\rho-1\right\rangle_{L^{2}(\mathcal{S})} + \left\langle \mathbb{D}_{t}^{\epsilon}\rho + \rho\boldsymbol{\nabla}\cdot\boldsymbol{v}\right\rangle_{L^{2}(\mathcal{S})}\right) \mathrm{d}t.$$

$$\tag{44}$$

Let us recall that the variational derivatives of the Lagrangian with respect to the large-scale velocity and density are

$$\frac{\delta \ell}{\delta \boldsymbol{u}} = \rho \boldsymbol{v}, \text{ and } \frac{\delta \ell}{\delta \rho} = \frac{1}{2} |\boldsymbol{v}|^2,$$
(45)

while for the noise correlation functions we have

$$\mathbb{E} \langle \frac{\delta \ell}{\delta \boldsymbol{\xi}_{i}}, \, \delta \boldsymbol{\xi}_{i} \rangle_{L^{2}(\mathcal{S})} = \mathbb{E} \langle \Upsilon_{\epsilon}(\bar{\rho} \, \boldsymbol{\xi}_{j} + \rho_{j} \boldsymbol{u}), \, \delta \boldsymbol{\xi}_{j} \rangle_{L^{2}(\mathcal{S})}. \tag{46}$$

The pressure variables give two additional constraints:

$$\mathbb{E} \left\langle \bar{\rho} - 1, \delta \bar{p} \right\rangle_{L^2(\mathcal{S})} = 0, \tag{47a}$$

$$\mathbb{E} \left\langle \Upsilon_{\epsilon} \rho_{i}, \delta p_{i} \right\rangle_{L^{2}(\mathcal{S})} = 0.$$
(47b)

With these two constraints, it comes out a constant density $\bar{\rho} = 1$ and null correlation functions, ρ_j , for the noise density. From the constraints on density transport (41c) and (41d) stem the incompressibility conditions: $\nabla \cdot \boldsymbol{u} = 0, \nabla \cdot \boldsymbol{\xi}_i = 0$. Note that as a consequence of the divergence-free condition on the noise shape functions, the function $\boldsymbol{\mu}_i$ and $\boldsymbol{\Lambda}_i^j$ are also divergence-free.

From the pathwise action and its associated stochastic Euler-Lagrange equation, (17), we obtain the system of regularized stochastic Euler equations:

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{v} = -\boldsymbol{\nabla} \left(p - \frac{1}{2} |\boldsymbol{v}|^2 \right),$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0.$$
(48)

A shown previously, this system can be approximated at the limit of small ϵ by the LU dynamics.

As for the noise correlation functions, as $\bar{\rho} = 1$ and $\rho_i = 0$, from constraint (15a) and (45) we get $\bar{\rho} \nabla \lambda_j = \bar{\rho} \boldsymbol{\xi}_j$ and $\bar{\rho} \nabla \bar{\lambda} = \bar{\rho} \boldsymbol{u}$. From equation (41f), a non zero solution for \boldsymbol{u} imposes $\Lambda_i^j = 0$, which means the basis function $\boldsymbol{\xi}_j$ are of finite variation. In equation (42) we insert the expression of time derivative inferred from constraints (41a, 41b, 41c, 41d). We then keep only terms in $\bar{\rho}$, and eliminate terms in ρ_i , $\partial_t \bar{\rho}$ and $\nabla \bar{\rho}$. We also apply the incompressibility condition on \boldsymbol{u} and $\boldsymbol{\xi}_j$. Equation (42) simplifies as

$$\mathbb{E}\Big\langle \Upsilon_{\epsilon}^{-1}\partial_t \Big(\frac{\delta\ell}{\delta\boldsymbol{\xi}_i}\Big) - \bar{\rho}\,\boldsymbol{\nabla}\partial_t\lambda_i - \Upsilon_{\epsilon}^{-1}\frac{1}{2}\partial_t\sum_j \Big(\boldsymbol{\nabla}(\boldsymbol{\nabla}\bar{\lambda}\cdot\boldsymbol{\xi}_j)\bar{\rho}\,,\delta\boldsymbol{\xi}_i\Big)_{L^2(\mathcal{S})} = 0. \tag{49}$$

Including the expressions of $\partial_t \lambda_i$ (41b):

$$\partial_t \lambda_j = \frac{1}{\Upsilon_\epsilon} \frac{\delta \ell}{\delta \rho_j} - \boldsymbol{u} \cdot \boldsymbol{\nabla} \lambda_j - \boldsymbol{\xi}_j \cdot \boldsymbol{\nabla} \bar{\lambda} - \frac{1}{2} \sum_{\ell} \boldsymbol{\xi}_\ell \cdot \boldsymbol{\nabla} \big(\boldsymbol{\xi}_\ell \cdot \boldsymbol{\nabla} \lambda_j \big), \quad (50)$$

and as $\mathbb{E}\langle \frac{\delta \ell}{\delta \rho_j}, \delta \boldsymbol{\xi}_j \rangle_{L^2(\mathcal{S})} = \mathbb{E}\langle \Upsilon_{\epsilon} \boldsymbol{u} \cdot \boldsymbol{\xi}_j, \delta \boldsymbol{\xi}_j \rangle_{L^2(\mathcal{S})}$, we obtain finally a linear Euler type equation for the noise functions dynamics:

$$\partial_t \boldsymbol{\xi}_i + (\boldsymbol{u} \cdot \boldsymbol{\nabla}) \boldsymbol{\xi}_i + (\boldsymbol{\xi}_i \cdot \boldsymbol{\nabla}) \boldsymbol{u} = -\boldsymbol{\nabla} p^{\epsilon}, \qquad (51)$$

where the pressure term reads

$$p^{\epsilon} = -\boldsymbol{\xi}_{i} \cdot \boldsymbol{u} + \frac{1}{2} \sum_{\ell} \nabla \cdot \left(\boldsymbol{\xi}_{\ell} (\boldsymbol{\xi}_{\ell} \cdot \boldsymbol{\xi}_{i}) \right) - \Upsilon_{\epsilon}^{-1} \frac{1}{2} \sum_{j} \partial_{t} (\boldsymbol{u} \cdot \boldsymbol{\xi}_{j}).$$
(52)

We note that the last pressure term is very small, and can be neglected.

The noise functions are transported and deformed by the large-scale components and a pressure term. The transport of the small scales by the large scales corresponds to Kraichnan's random sweeping hypothesis [11, 33]. This assumption, also exploited by Tennekes [50] for turbulent boundary layer models put forward that small eddies (i.e. much smaller than the main energy containing eddies) are transported by an Eulerian large-scale field without any dynamical deformation [31]. The large-scale flow involved is not the mean flow (which would correspond to Taylor frozen turbulence hypothesis of purely passive turbulence transport) and the absence of dynamical distortion ensues from missing interactions between the small scales and large-scale component, as well as interactions between the small-scales modes. The pressure term consists first of a forcing term depending on the evolution of the angle between the small-scale modes and the large-scale velocity and second, of a turbulent transport term, which is a third-order product term of the small-scale modes' noise. It is noticeable that this latter term is dominant for spatially small-scale noise correlation functions.

3 STOCHASTIC VARIATIONAL PRINCIPLE FOR FLUID FLOW 19

The pressure term can be explicitly computed or estimated in the usual way by considering the incompressibility condition on the noise basis functions $\boldsymbol{\xi}_j$. More precisely, we obtain the noise pressure p_i^{ϵ} through the elliptic equations:

$$\Delta p_i^{\epsilon} = \boldsymbol{\nabla} \cdot \left((\boldsymbol{u} \cdot \boldsymbol{\nabla}) \boldsymbol{\xi}_i + (\boldsymbol{\xi}_i \cdot \boldsymbol{\nabla}) \boldsymbol{u} \right).$$
(53)

Note that also a further simplified equation can be settled through the Leray projector, \mathcal{P} , onto the space of divergence-free vectors. We get in that case:

$$\partial_t \boldsymbol{\xi}_i + \mathcal{P}(\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{\xi}_i) + \mathcal{P}(\boldsymbol{\xi}_i \cdot \boldsymbol{\nabla} \boldsymbol{u}) = 0.$$
(54)

This equation has the advantage of not requiring any boundary condition for the pressure term, which can often be cumbersome to set.

To sum up we obtain the following large scale Euler equation, with an explicit expression of the small-scale component evolution.

$$\partial_{t}\boldsymbol{u} + \left((\boldsymbol{u} - \boldsymbol{u}_{\boldsymbol{a}} + \boldsymbol{u}^{\epsilon}) \cdot \boldsymbol{\nabla}\right)\boldsymbol{u} = -\boldsymbol{\nabla}p,$$

$$\nabla \cdot \boldsymbol{u} - \frac{1}{2}\nabla \cdot \nabla \cdot \boldsymbol{a} = 0,$$

$$\boldsymbol{u}^{\epsilon} = \int_{t-\epsilon}^{t+\epsilon} h_{\epsilon}(t-s)\boldsymbol{\xi}_{i}(x,t)d\beta_{s}^{i}, \text{ with }$$

$$\partial_{t}\boldsymbol{\xi}_{i} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\xi}_{i} + (\boldsymbol{\xi}_{i} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla}p^{\epsilon},$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_{i} = 0.$$
(55)

Letting $\epsilon \to 0$ we then obtain the stochastic LU system:

$$d_{t}\boldsymbol{u} + \left((\boldsymbol{u}^{*}dt + \boldsymbol{\xi}_{i} \circ d\beta^{i}) \cdot \boldsymbol{\nabla} \right) \boldsymbol{u} = -\boldsymbol{\nabla}(dp - dp_{\sigma}),$$

$$\nabla \cdot \boldsymbol{u} - \nabla \cdot \boldsymbol{u}_{a}^{\circ} = \nabla \cdot \boldsymbol{\xi}_{i} = 0,$$

$$\partial_{t}\boldsymbol{\xi}_{i} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\xi}_{i} + (\boldsymbol{\xi}_{i} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla}q.$$
 (56)

The limit $(55) \rightarrow (56)$ can be justified rigorously (see [14, 16]). Physically, (55) may be more interesting than (56) since it does not assume a complete decorrelation between the small and large scales.

3.3 Ornstein-Uhlenbeck process for the regularized noise

One simple solution consists in considering an Ornstein-Uhlenbeck (OU) process for the noise, denoted Z_t^{ϵ} in the following. In that case the regularized noise is defined through the Ornstein-Uhlenbeck semi-group generator. A regularized noise with an OU process reads:

$$\boldsymbol{u}^{\epsilon} = \sigma_t \boldsymbol{Z}_t^{\epsilon} = \sum_i \boldsymbol{\xi}_i(t) Z_t^{\epsilon,i} = \sum_i \boldsymbol{\xi}_i(t) \int_t^{t+\epsilon} e^{\frac{1}{\epsilon}(t-s)} \mathrm{d}\beta_s^i.$$
(57)

The exponential functional does not have a compact support but this regularization also fully enters within the set of the regularized noises considered in this work. For this noise, the dynamics of the small-scale velocity is then given by:

$$d_{t}\boldsymbol{u}^{\epsilon} = -\left(\sum_{i} \left((\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\xi}_{i} + (\boldsymbol{\xi}_{i} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{\nabla}p^{\epsilon} \right) \boldsymbol{Z}_{t}^{\epsilon} dt - \frac{1}{\epsilon} \sum_{i} \boldsymbol{\xi}_{i} \boldsymbol{Z}_{t}^{\epsilon,i} + \sum_{i} \boldsymbol{\xi}_{i} d\beta_{t}^{i}.$$
 (58)

In the model described above, the nonlinear small-scale self-interaction is represented by an Ornstein-Uhlenbeck process, and the dynamics of the small-scale velocity components are governed by a linear SPDE. This approach is similar to the scheme proposed in [37, 38], where nonlinear self-interactions are modeled using a linear stochastic operator in the form of an Ornstein-Uhlenbeck process, composed of a damping term and an additive noise term. Our framework fully justifies this form, thereby validating the assumptions made in [37, 38] regarding the representation of nonlinear self-interactions. We remark that if the noise functions are assumed stationary ($\partial_t \boldsymbol{\xi}_i = 0$), we are left with an Ornstein-Uhlenbeck process for the unresolved components of the form:

$$d_t \boldsymbol{u}^{\epsilon} = -\frac{1}{\epsilon} \sum_i \boldsymbol{\xi}_i Z_t^{\epsilon,i} + \sum_i \boldsymbol{\xi}_i d\beta_t^i = -\frac{1}{\epsilon} \boldsymbol{u}^{\epsilon} + \sum_i \boldsymbol{\xi}_i d\beta_t^i.$$
(59)

Such models have been intensively used in ocean modelling and climate science for representing intermittent small-scale boundary layer processes in ocean-atmosphere interactions [28, 39, 46]. They have been successful in explaining the ubiquitous red spectrum of sea surface temperature [22, 28], El Niño Southern Oscillation variability [32] as well as thermohaline circulation variability [27] and mean current-fluctuations interactions [17]. The framework derived here can be seen as a generalization of these models, in which we provide and justify a dynamics for the noise modal functions.

4 Discussion and conclusion

This study addresses several research issues in turbulence fluid modeling, particularly the accurate representation of multi-scale interactions, including the influence of small-scale turbulence on large-scale ocean dynamics and vice versa. Traditional models often rely on phenomenological or heuristic approaches, which can lack rigor and fail to capture the complexity of these bidirectional interactions. Here, we provide a rigorous derivation that goes beyond these traditional methods.

One significant contribution of this work is the introduction of a stochastic partial differential equation (SPDE) framework with regularized noise terms for modeling both the large-scale velocity components and the dynamics of the small-scale noise terms. The random sweeping hypothesis, initially proposed by Kraichnan [33] and further developed by Tennekes [50] for the atmospheric boundary layer, assumes a passive transport of small scales by large scales. These models extend Taylor's "frozen" turbulence scheme, where turbulence is advected by a mean field, by incorporating randomness in the transport velocity field. The coupled variational framework proposed in this study goes further by deriving a stochastic partial differential equation (SPDE) describing the evolution of the large-scale velocity component in tandem with the dynamics of small-scale noise terms. This latter, consisting of a linear Euler-type equation, justifies the description of small-scale dynamics through large-scale random advection as proposed in Kraichnan's sweeping hypothesis. However, it also incorporates additional deformation terms associated with pressure and stretching.

Our approach also relates to the MTV model [37, 38] for stochastic climate modeling and simpler models where noise modal functions are stationary, aligning then with Hasselmann's 1976 model [28]. By proposing a formal variational principle, we provide a systematic method to derive governing equations for both large-scale flows and their stochastic perturbations, potentially leading to more accurate and reliable models.

To our knowledge, this is the very first time a formal principle is proposed to derive a stochastic dynamics with an explicit evolution for the small-scale component. Future works will focus on extending this framework to the primitive equations, fundamental in describing ocean dynamics, and to wave-current interactions. This extension is crucial for accurately modeling coastal dynamics, energy dissipation, and nutrient mixing, which are inherently complex due to small-scale features and nonlinear wave interactions.

Overall, our approach bridges a critical gap in the large-scale modeling of turbulent flows, setting the stage for future research that could lead to more accurate predictions by inherently accounting for uncertainties associated with unresolved small-scale nonlinear phenomena.

In future studies, we will undertake a mathematical analysis of the proposed stochastic coupled system with additional friction terms, transforming it into a coupled Navier-Stokes system. The goal will be to demonstrate the existence of (probabilistic) weak solutions, similar to the approach used for the large-scale Navier-Stokes LU equation [15].

Acknowledgements: The authors acknowledge the support of the ERC EU project 856408-STUOD and benefit from the support of the French government "Investissements d'Avenir" program ANR-11-LABX-0020-01.

A Quadratic (co-)variation

In stochastic calculus, quadratic covariation (or cross-variance) of two realvalued processes X and Y play a fundamental role. Quadratic variation is a bounded variation process defined as:

$$\langle X, Y \rangle_t = \lim_{n \to 0} \sum_{i=1}^{p_n} (X_i^n - X_{i-1}^n) (Y_i^n - Y_{i-1}^n),$$
 (60)

where $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ is a partition of the interval [0, t] and this limit, if it exists, is defined in the sense of convergence in probability.

Assuming that X and Y are two real-valued continuous semimartingales, defined as $X_t = X_0 + A_t + M_t$, $Y_t = Y_0 + B_t + N_t$ with M, N martingales and A, B finite variation processes, then their quadratic covariation (60) exists, and is given by

$$\langle X, Y \rangle_t = \langle M, N \rangle_t. \tag{61}$$

In particular, the quadratic variation of a standard Brownian motion B (as a martingale) is given by $\langle B \rangle_t = t$, the quadratic variation of two bounded variation processes f, and g (such as deterministic functions) can be shown to be zero ($\langle f, g \rangle_t = 0$), as well as the covariation between a martingale and a bounded variation process ($\langle f, M \rangle_t = 0$).

The quadratic (co-)variations play an important role in the Itô calculus and its generalization of the chain rule. In particular, they are involved in the Itô integration by parts formula:

$$d(XY) = XdY + YdX + d\langle X, Y \rangle_t.$$
(62)

The quadratic variation of the Itô integrals of two adapted processes with respect to martingale, M and N, respectively, is provided by the following important formula:

$$\left\langle \int_{0}^{\cdot} \Theta_{s} \mathrm{d}M_{s}, \int_{0}^{\cdot} \Theta_{s}^{\prime} \mathrm{d}N_{s} \right\rangle_{t} = \int_{0}^{t} \Theta_{s} \Theta_{s}^{\prime} \mathrm{d}\langle M, N \rangle_{t}$$
(63)

This property is involved in the Itô isometry, enabling to express the covariance of two Itô integrals:

$$\mathbb{E}\Big[\Big(\int_0^t f \mathrm{d}M_s\Big)\Big(\int_0^t g \mathrm{d}N_s\Big)\Big] = \mathbb{E}\Big[\int_0^t f g \,\mathrm{d}\langle M, N\rangle_s\Big],\tag{64}$$

where f and g are two adapted processes such that $\int_0^t f^2 d\langle M, M \rangle_s$ and $\int_0^t g^2 d\langle N, N \rangle_s$ are integrable.

B Itô form of the density transport with a regularized correlated noise

In this appendix we show that the density transport with regularized correlated noise term (i.e. of Stratonovich type)

$$\mathbb{D}_{t}^{\epsilon}\rho + \rho \nabla \cdot \boldsymbol{v} = \partial_{t}\rho + \nabla \cdot (\boldsymbol{u}\rho) + \sum_{i} \nabla \cdot \left(\rho(t)\boldsymbol{\xi}_{i}(t) \int_{t-\epsilon}^{t+\epsilon} h_{\epsilon}(t-s)\mathrm{d}\beta_{s}^{i}\right) = 0, \quad (65)$$

up to negligible terms, reads as

$$\partial_{t}\rho + \nabla \cdot (\boldsymbol{u}\rho) + \frac{1}{2} \sum_{i} \nabla \cdot (\boldsymbol{\Lambda}_{i}^{i}\rho) - \frac{1}{2} \nabla \cdot \left(\boldsymbol{\xi}_{i} \nabla \cdot (\boldsymbol{\xi}_{i}\rho)\right) + \sum_{i} \nabla \cdot \left(\rho \boldsymbol{\xi}_{i}(t)\right) \int_{t}^{t+\epsilon} \tilde{h}_{\epsilon}(t-s) \mathrm{d}\beta_{s}^{i}, \quad (66)$$

for (Itô type) regularized noise with increments decorrelated from u(t):

$$\sigma_t \boldsymbol{W}_t^{\epsilon} = \sum_i \int_t^{t+\epsilon} \tilde{h}_{\epsilon}(t-s) \boldsymbol{\xi}_i(t) \mathrm{d}\beta_s^i, \tag{67}$$

where the functions, $\{\boldsymbol{\xi}_i, i \in \mathbb{N}\}$ are defined in full generality as stochastic processes of the form

$$\mathrm{d}\boldsymbol{\xi}_{i} = \boldsymbol{\mu}_{i}\mathrm{d}t + \sum_{j} \int_{t}^{t+\epsilon} \tilde{h}_{\epsilon}(t-s)\boldsymbol{\Lambda}_{j}^{i}(t)\mathrm{d}\beta_{s}^{j}.$$
(68)

More precisely, we show below that the following approximation holds

$$\mathbb{D}_{t}^{\epsilon}\rho + \rho \nabla \cdot \boldsymbol{v} \approx \widetilde{\mathbb{D}}_{t}^{\epsilon}\rho + \rho \nabla \cdot \boldsymbol{v}, \qquad (69)$$

where the approximated density transport $\widetilde{\mathbb{D}}_t^{\epsilon} \rho + \rho \nabla \cdot \boldsymbol{v}$ reads

$$d_{t}\rho + \nabla \cdot (\boldsymbol{u}\rho)dt + \frac{1}{2}\sum_{ij} \nabla \cdot (\boldsymbol{\Lambda}_{i}^{j}b_{ij}^{\epsilon}\rho)dt - \frac{1}{2}\sum_{ij} \nabla \cdot (\delta_{ij}^{\epsilon}\boldsymbol{\xi}_{i}\boldsymbol{\nabla} \cdot (\boldsymbol{\xi}_{j}\rho))dt + \sum_{i} \nabla \cdot (\boldsymbol{\xi}_{i}\rho) \int_{t}^{t+\epsilon} \tilde{h}_{\epsilon}(t-s)d\beta_{s}^{i}, \quad (70)$$

with $\delta_{ij}^{\epsilon}(t) \xrightarrow{a.s} \frac{1}{2} \delta_{ij}$ and $b_{ij}^{\epsilon} \xrightarrow{a.s} \frac{1}{2} \delta_{ij}$ and consequently

$$\mathbb{D}_{t}^{\epsilon} \approx \widetilde{\mathbb{D}}_{t}^{\epsilon} \quad \text{and} \quad \lim_{\epsilon \to 0} \widetilde{\mathbb{D}}_{t}^{\epsilon} \longrightarrow \mathbb{D}_{t}.$$
(71)

B ITÔ FORM OF THE DENSITY TRANSPORT WITH A REGULARIZED CORRELATED NOIS

In order to show this, the noise term in (65) is written for each i as:

$$\int_{0}^{T} \left(\nabla \cdot \left(\boldsymbol{\xi}_{i}(t)\rho(t) \right) \int_{t-\epsilon}^{t+\epsilon} h_{\epsilon}(t-s) d\beta_{s}^{i} \right) dt = \underbrace{\int_{0}^{T} \left(\nabla \cdot \left(\boldsymbol{\xi}_{i}(t)\rho(t) \right) \int_{t}^{t+\epsilon} h_{\epsilon}(t-s) d\beta_{s}^{i} \right) dt + \underbrace{\int_{A1}^{T} \left(\left(\nabla \cdot \left(\boldsymbol{\xi}_{i}(t-\epsilon)\rho(t-\epsilon) \right) \int_{t-\epsilon}^{t} h_{\epsilon}(t-s) d\beta_{s}^{i} \right) dt + A2}_{A2} \underbrace{\int_{0}^{T} \left(\nabla \cdot \left(\boldsymbol{\xi}_{i}(t)\rho(t) - \boldsymbol{\xi}_{i}(t-\epsilon)\rho(t-\epsilon) \right) \int_{t-\epsilon}^{t} h_{\epsilon}(t-s) d\beta_{s}^{i} \right) dt}_{B}. \quad (72)$$

The two first term $A = A_1 + A_2$ on the right-hand side can be approximated as:

$$A \approx \int_{0}^{T} \nabla \cdot \left(\boldsymbol{\xi}_{i}(t) \rho(t) \right) \int_{t}^{t+\epsilon} \underbrace{\left(h_{\epsilon}(t-s) + h_{\epsilon}(t+\epsilon-s) \right)}_{\tilde{h}_{\epsilon}(t-s)} \mathrm{d}\beta_{s}^{i} \mathrm{d}t, \qquad (73)$$

which corresponds to a smoothing of the Itô integral, with kernel, \tilde{h}_{ϵ} , such that $\int_{0}^{\epsilon} \tilde{h}_{\epsilon} = 1$. The second right-hand side term of (72), is written $B = B_1 + B_2$ with

$$B_{1} = \int_{0}^{T} \left(\nabla \cdot \left(\boldsymbol{\xi}_{i}(t) \left(\rho(t) - \rho(t - \epsilon) \right) \right) \int_{t-\epsilon}^{t} h_{\epsilon}(t-s) \mathrm{d}\beta_{s}^{i} \right) \mathrm{d}t \qquad (74)$$
$$\approx -\int_{0}^{T} \left(\nabla \cdot \left(\boldsymbol{\xi}_{i}(t) \int_{t-\epsilon}^{t} \left[\sum_{i} \left(\int_{s-\epsilon}^{s+\epsilon} h_{\epsilon}(s-r) \mathrm{d}\beta_{r}^{j} \right) \right] \right) \right]$$

$$\nabla \cdot \left(\boldsymbol{\xi}_{j}(s)\rho(s)\right) ds \int_{t-\epsilon}^{t} h_{\epsilon}(t-\tau) d\beta_{\tau}^{i} dt$$
(75)

$$\approx -\sum_{j} \int_{0}^{T} \left(\int_{t-\epsilon}^{t} \int_{s-\epsilon}^{s+\epsilon} h_{\epsilon}(s-r) \mathrm{d}\beta_{r}^{j} \mathrm{d}s \right)$$
$$\nabla \cdot \left(\boldsymbol{\xi}_{i}(t) \nabla \cdot \left(\boldsymbol{\xi}_{i}(t) \rho(t) \right) \right) \int_{0}^{t} h_{\epsilon}(t-\tau) \mathrm{d}\beta_{\tau}^{i} \mathrm{d}t \qquad (76)$$

$$\nabla \cdot \left(\boldsymbol{\xi}_i(t) \nabla \cdot \left(\boldsymbol{\xi}_j(t) \rho(t) \right) \right) \int_{t-\epsilon} h_{\epsilon}(t-\tau) \mathrm{d}\beta_{\tau}^i \right) \mathrm{d}t \tag{76}$$

$$\approx -\sum_{j} \int_{0}^{T} \delta_{ij}^{\epsilon}(t) \nabla \cdot \left(\boldsymbol{\xi}_{j}(t) \nabla \cdot \left(\boldsymbol{\xi}_{i}(t) \rho(t) \right) \right) \mathrm{d}t = \tilde{B}_{1}, \tag{77}$$

where

$$\delta_{ij}^{\epsilon}(t) = \int_{t-\epsilon}^{t} \int_{s-\epsilon}^{s+\epsilon} h_{\epsilon}(s-r) \mathrm{d}\beta_{r}^{j} \mathrm{d}s \int_{t-\epsilon}^{t} h_{\epsilon}(t-\tau) \mathrm{d}\beta_{\tau}^{i} \xrightarrow{a.s} \frac{1}{2} \delta_{ij}, \qquad (78)$$

REFERENCES

and hence

$$\tilde{B}_1 \xrightarrow{a.s} -\frac{1}{2} \nabla \cdot \left(\boldsymbol{\xi}_i \nabla \cdot \left(\boldsymbol{\xi}_i \rho \right) \right). \tag{79}$$

The other term B_2 is

$$B_2 = \int_0^T \left[\nabla \cdot \left(\left(\boldsymbol{\xi}_i(t) - \boldsymbol{\xi}_i(t-\epsilon) \right) \rho(t-\epsilon) \right) \int_{t-\epsilon}^t h_\epsilon(t-s) \mathrm{d}\beta_s^i \right] \mathrm{d}t$$
(80)

$$\approx \int_0^T \left[\nabla \cdot \left(\sum_j \int_{t-\epsilon}^t h_\epsilon(t-s) \mathbf{\Lambda}_j^i(t) \mathrm{d}\beta_s^j \,\rho(t-\epsilon) \right) \int_{t-\epsilon}^t h_\epsilon(t-s) \mathrm{d}\beta_s^i \right] \mathrm{d}t \quad (81)$$

$$\approx \sum_{j} \int_{0}^{T} \nabla \cdot \left(\Lambda_{j}^{i}(t) \rho(t) \right) \underbrace{\left(\int_{t}^{t+\epsilon} (h_{\epsilon}(t-s) \mathrm{d}\beta_{s}^{j}) \int_{t}^{t+\epsilon} h_{\epsilon}(t+\epsilon-s) \mathrm{d}\beta_{s}^{i} \right)}_{b_{ij}^{\epsilon}(t)} \mathrm{d}t \quad (82)$$

$$\approx \sum_{j} \int_{0}^{T} b_{ij}^{\epsilon}(t) \nabla \cdot \left(\mathbf{\Lambda}_{j}^{i} \rho(t) \right) \mathrm{d}t, \tag{83}$$

with

$$b_{ij}^{\epsilon} \xrightarrow{a.s} \frac{1}{2} \delta_{ij}.$$
 (84)

The approximation of density transport $\mathbb{D}_t^{\epsilon} \rho + \rho \nabla \cdot v$ reads thus as:

$$d_{t}\rho + \nabla \cdot (\boldsymbol{u}\rho)dt + \frac{1}{2} \sum_{ij} \nabla \cdot (\boldsymbol{\Lambda}_{i}^{j} b_{ij}^{\epsilon} \rho)dt - \frac{1}{2} \sum_{ij} \nabla \cdot (\delta_{ij}^{\epsilon} \boldsymbol{\xi}_{i} \nabla \cdot (\boldsymbol{\xi}_{j} \rho))dt + \sum_{i} \nabla \cdot (\boldsymbol{\xi}_{i} \rho) \int_{t}^{t+\epsilon} \tilde{h}_{\epsilon}(t-s)d\beta_{s}^{i}.$$
(85)

and we have consequently:

$$\mathbb{D}_{t}^{\epsilon} \approx \widetilde{\mathbb{D}}_{t}^{\epsilon} \quad \text{and} \quad \lim_{\epsilon \to 0} \widetilde{\mathbb{D}}_{t}^{\epsilon} \longrightarrow \mathbb{D}_{t}.$$
(86)

References

- [1] A. Agresti, M. Hieber, A. Hussein, and M. Saal. The stochastic primitive equations with transport noise and turbulent pressure. *Stochastics and Partial Differential Equations: Analysis and Computations*, 2022.
- [2] Werner Bauer, Pranav Chandramouli, Bertrand Chapron, Long Li, and Etienne Mémin. Deciphering the role of small-scale inhomogeneity on geophysical flow structuration: A stochastic approach. *Journal of Physical Oceanography*, 50(4):983 – 1003, 01 Apr. 2020.

- [3] Werner Bauer, Pranav Chandramouli, Long Li, and Etienne Mémin. Stochastic representation of mesoscale eddy effects in coarse-resolution barotropic models. *Ocean Modelling*, 151:101646, 2020.
- [4] Judith Berner, Ulrich Achatz, Lauriane Batté, Lisa Bengtsson, Alvaro de la Cámara, Hannah M Christensen, Matteo Colangeli, Danielle RB Coleman, Daan Crommelin, Stamen I Dolaptchiev, et al. Stochastic parameterization: Toward a new view of weather and climate models. Bulletin of the American Meteorological Society, 98(3):565–588, 2017.
- [5] Rüdiger Brecht, Long Li, Werner Bauer, and Etienne Mémin. Rotating shallow water flow under location uncertainty with a structurepreserving discretization. *Journal of Advances in Modeling Earth Sys*tems, 13(12):e2021MS002492, 2021.
- [6] Zdzisław Brzeźniak and Jakub Slavík. Well-posedness of the 3D stochastic primitive equations with multiplicative and transport noise. *Journal* of Differential Equations, 296:617–676, 2021.
- [7] Giulia Carigi and Eliseo Luongo. Dissipation properties of transport noise in the two-layer quasi-geostrophic model. *Journal of Mathematical Fluid Mechanics*, 25(2):28, 2023.
- [8] Pranav Chandramouli, Dominique Heitz, Sylvain Laizet, and Etienne Mémin. Coarse large-eddy simulations in a transitional wake flow with flow models under location uncertainty. *Computers & Fluids*, 168:170– 189, 2018.
- [9] Pranav Chandramouli, Etienne Mémin, and Dominique Heitz. 4D large scale variational data assimilation of a turbulent flow with a dynamics error model. *Journal of Computational Physics*, 412:109446, 2020.
- [10] Bertrand Chapron, Pierre Dérian, Etienne Mémin, and Valentin Resseguier. Large-scale flows under location uncertainty: a consistent stochastic framework. *Quarterly Journal of the Royal Meteorological Society*, 144(710):251–260, 2018.
- [11] Shiyi Chen and Robert H. Kraichnan. Sweeping decorrelation in isotropic turbulence. *Physics of Fluids A: Fluid Dynamics*, 1(12):2019– 2024, 6/23/2024 1989.
- [12] Xin Chen, Ana Bela Cruzeiro, and Tudor S. Ratiu. Stochastic variational principles for dissipative equations with advected quantities. *Journal of Nonlinear Science*, 33(1):5, 2022.
- [13] Dan Crisan, Franco Flandoli, and Darryl D Holm. Solution properties of a 3D stochastic Euler fluid equation. *Journal of Nonlinear Science*, 29(3):813–870, 2019.

- [14] Arnaud Debussche and Martina Hofmanová. Rough analysis of two scale systems. ArXiv, 2306.15781, 2023.
- [15] Arnaud Debussche, Berenger Hug, and Etienne Mémin. A consistent stochastic large-scale representation of the Navier–Stokes equations. *Journal of Mathematical Fluid Mechanics*, 25(1):19, 2023.
- [16] Arnaud Debussche and Umberto Pappalettera. Second order perturbation theory of two-scale systems in fluid dynamics. ArXiv, 2206.07775, 2023.
- [17] Timothy Delsole and Brian F. Farrell. The quasi-linear equilibration of a thermally maintained, stochastically excited jet in a quasigeostrophic model. *Journal of Atmospheric Sciences*, 53(13):1781–1797, 1996.
- [18] Franco Flandoli, Lucio Galeati, and Dejun Luo. Delayed blow-up by transport noise. Communications in Partial Differential Equations, 46(9):1757–1788, 2021.
- [19] Franco Flandoli and Dejun Luo. High mode transport noise improves vorticity blow-up control in 3D Navier–Stokes equations. *Probability Theory and Related Fields*, 180(1):309–363, 2021.
- [20] Franco Flandoli and Umberto Pappalettera. 2D Euler equations with stratonovich transport noise as a large-scale stochastic model reduction. *Journal of Nonlinear Science*, 31(1):24, 2021.
- [21] Franco Flandoli and Francesco Russo. Reduced dissipation effect in stochastic transport by Gaussian noise with regularity greater than 1/2. ArXiv, 2305.19293, 2023.
- [22] Claude Frankignoul and Klaus Hasselmann. Stochastic climate models. part II. application to sea-surface temperature anomalies and thermocline variability. *Tellus*, 29:289–305, 1977.
- [23] Lucio Galeati. On the convergence of stochastic transport equations to a deterministic parabolic one. Stochastics and Partial Differential Equations: Analysis and Computations, 8(4):833–868, 2020.
- [24] Lucio Galeati and Dejun Luo. Weak well-posedness by transport noise for a class of 2D fluid dynamics equations. ArXiv, 2305.08761, 2023.
- [25] Daniel Goodair, Dan Crisan, and Oana Lang. Existence and uniqueness of maximal solutions to spdes with applications to viscous fluid equations. ArXiv, 2209.09137, 2022.
- [26] Georg Gottwald, Daan Crommelin, and Christian Franzke. Stochastic climate theory. In *Nonlinear and Stochastic Climate Dynamics*, pages 209–240. Cambridge University Press, 2017.

- [27] Stephen M. Griffies and Eli Tziperman. A linear thermohaline oscillator driven by stochastic atmospheric forcing. *Journal of Climate*, 8(10):2440–2453, 1995.
- [28] Klaus Hasselmann. Stochastic climate models part I. theory. Tellus, 28(6):473-485, 2024/06/25 1976.
- [29] Darryl D Holm. Variational principles for stochastic fluid dynamics. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 471(2176):20140963, 2015.
- [30] Souleymane Kadri Harouna and Etienne Mémin. Stochastic representation of the Reynolds transport theorem: revisiting large-scale modeling. *Computers and Fluids*, 156:456–469, 2017.
- [31] Gabriel Katul, Marc Parlange, John Albertson, and Chia-Ren Chu. The random sweeping decorrelation hypothesis in stratified turbulent flows. *Fluid Dynamics Research*, 16(5):275–295, 1995.
- [32] Richard Kleeman. Spectral analysis of multidimensional stochastic geophysical models with an application to decadal enso variability. *Journal* of the Atmospheric Sciences, 68(1):13–25, 2011.
- [33] Robert H. Kraichnan. Kolmogorov's hypotheses and eulerian turbulence theory. The Physics of Fluids, 7(11):1723–1734, 6/23/2024 1964.
- [34] Oana Lang, Dan Crisan, and Étienne Mémin. Analytical properties for a stochastic rotating shallow water model under location uncertainty. *Journal of Mathematical Fluid Mechanics*, 25(2):29, 2023.
- [35] Long Li, Bruno Deremble, Noé Lahaye, and Etienne Mémin. Stochastic data-driven parameterization of unresolved eddy effects in a baroclinic quasi-geostrophic model. *Journal of Advances in Modeling Earth Sys*tems, 15(2):e2022MS003297, 2023.
- [36] Long Li, Etienne Mémin, and Gilles Tissot. Stochastic parameterization with dynamic mode decomposition. In *Stochastic Transport in Upper Ocean Dynamics*. Springer Nature, 2022.
- [37] Andrew J. Majda, Ilya Timofeyev, and Eric Vanden Eijnden. Models for stochastic climate prediction. *Proceedings of the National Academy* of Sciences, 96(26):14687–14691, 2024/06/22 1999.
- [38] Andrew J. Majda, Ilya Timofeyev, and Eric Vanden Eijnden. A mathematical framework for stochastic climate models. *Communications on Pure and Applied Mathematics*, 54(8):891–974, 2024/06/22 2001.
- [39] James C. McWilliams and Edward Huckle. Ekman layer rectification. Journal of Physical Oceanography, 36(8):1646–1659, 2006.

- [40] Etienne Mémin. Fluid flow dynamics under location uncertainty. Geophysical & Astrophysical Fluid Dynamics, 108(2):119–146, 2014.
- [41] Giuseppe Da Prato and Jerzy Zabczyk. Stochastic equations in infinite dimensions. Cambridge University Press, 1992.
- [42] Valentin Resseguier, Etienne Mémin, and Bertrand Chapron. Geophysical flows under location uncertainty, Part I Random transport and general models. *Geophys. & Astro. Fluid Dyn.*, 111(3):149–176, 2017.
- [43] Valentin Resseguier, Etienne Mémin, and Bertrand Chapron. Geophysical flows under location uncertainty, Part II Quasi-geostrophy and efficient ensemble spreading. *Geophys. & Astro. Fluid Dyn.*, 111(3):177– 208, 2017.
- [44] Valentin Resseguier, Etienne Mémin, and Bertrand Chapron. Geophysical flows under location uncertainty, Part III SQG and frontal dynamics under strong turbulence conditions. *Geophys. & Astro. Fluid Dyn.*, 111(3):209–227, 2017.
- [45] Valentin Resseguier, Wei Pan, and Baylor Fox-Kemper. Data-driven versus self-similar parameterizations for stochastic advection by lie transport and location uncertainty. *Nonlinear Processes in Geophysics*, 27(2):209–234, 2020.
- [46] R. Saravanan and James C. McWilliams. Advective ocean-atmosphere interaction: An analytical stochastic model with implications for decadal variability. *Journal of Climate*, 11(2):165–188, 1998.
- [47] Leslie M. Smith, Jeffrey R. Chasnov, and Fabian Waleffe. Crossover from two- to three-dimensional turbulence. *Phys. Rev. Lett.*, 77:2467– 2470, Sep 1996.
- [48] Leslie M. Smith and Fabian Waleffe. Transfer of energy to twodimensional large scales in forced, rotating three-dimensional turbulence. *Physics of Fluids*, 11(6):1608–1622, 1999.
- [49] Oliver Street and Dan Crisan. Semi-martingale driven variational principles. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 477(2247):20200957, 2021.
- [50] Henk Tennekes. Eulerian and lagrangian time microscales in isotropic turbulence. *Journal of Fluid Mechanics*, 67(3):561–567, 1975.
- [51] Francesco L Tucciarone, Etienne Mémin, and Long Li. Primitive equations under location uncertainty: Analytical description and model development. In *Stochastic Transport in Upper Ocean Dynamics II*. Springer Nature, 2023.

REFERENCES

[52] Hua Xia, David Byrne, Gregory Falkovich, and Mickael Shats. Upscale energy transfer in thick turbulent fluid layers. *Nature Physics*, 7(4):321– 324, 2011.