From Nash to Cournot-Nash via Γ-convergence

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Abstract. In this work we study the question of when will Nash equilibria of N-player games converge to a suitable notion of equilibrium of a game with a continuum of players. This is a central question in Mean Field Games theory, but instead we consider the notion of Cournot-Nash equilibrium. When such a game has a so-called potential structure, we give a complete characterization of equilibria as the stationary points of a functional, which we call a potential function. In particular, its minimizers are equilibria. In the sequel, we focus on the case that the players interact pair-wise; in this case we show that the original sequence of N-player games also admit a potential structure and prove that their corresponding potential functions converge in the sense of Γ-convergence to the potential function of the game with a continuum of players. This allows us to answer positively to the question of convergence of equilibria.

Contents

1. Introduction

The goal of this work is to consider a class of N-player games with pairwise interactions and rigorously show that Nash equilibria of this class of games converge, as N goes to infinity, to Cournot-Nash equilibria (see Definition 1.1 below) of a game with a continuum of players.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Polish spaces, that is separable and metrizable, representing the space of types of players and of admissible strategies. Given a tuple $(x_i)_{i=1}^N \subset \mathcal{X}$, where x_i

corresponds to the type of player i, the goal of each player is to minimize a cost depending on their own type and the mean pair-wise interaction with other players. In the described situation, player i seeks to

$$\underset{y_i \in \mathcal{Y}}{\text{minimize}}\, g_N(y_i; y_{-i}) \overset{\text{def.}}{=} c(x_i, y_i) + L(y_i) + \frac{2}{N}\sum_{i \neq j} H(y_j, y_i), \tag{1}$$

where, as standard notation in game theory, $y_{-i} = (y_j)_{j \neq i}$. For the game in mixed strategies, we replace the $\mathcal{Y}$ with $\mathscr{P}(\mathcal{Y})$, the space of Radon probability measures over $\mathcal{Y}$, and each player seeks to

$$\underset{\nu_i \in \mathscr{P}(\mathcal{Y})}{\text{minimize}}\, g_N(\nu_i; \nu_{-i}) \overset{\text{def.}}{=} \int_{\mathcal{Y}} \left(c(x_i, y_i) + L(y_i)\right) \mathrm{d}\nu_i(y_i) + \frac{2}{N}\sum_{i \neq j} \int_{\mathcal{Y}\times\mathcal{Y}} H \mathrm{d}\nu_j \otimes \nu_i. \tag{2}$$

Such types of N-player games become rapidly intractable as N grows, hence one hopes that a suitable notion of equilibrium in a setting with infinitely many players will provide information for the behavior of equilibria when N is large but still finite. In many economic and social scenarios, the number of agents acting in the game reaches this regime of intractability, hence it is of great theoretical and practical importance to be able to rigorously describe the model with infinitely many players as the limit of a sequence of N-players games.

In the category of differential games, *i.e.* when players are constrained by controlled dynamics and hence solve an optimal control problem each, the notion of equilibrium in a setting with a continuum of players has been established by Larsy and Lions as *Mean Field Games* in their seminal papers [35–37]. In their formulation, an equilibrium becomes the solution of a pair of coupled PDEs, one describing the evolution of the probability distribution of players and another describing the optimality conditions of the underlying optimal control problem solved by each player. Although the convergence question was one of the primary motivations since the inception of the Mean Field Games theory in the lectures of Pierre-Louis Lions at Collège de France, see also the notes of Cardaliaguet [13], we still do not have a complete answer to this question.

On the other hand, the notion of games with a continuum of players is much older and has been studied in the economics literature since the 60's by Aumann [5, 6]. Although Aumann formulated his notion of equilibrium with preference relations, as was noted in [39], an equivalent way of defining equilibria in the continuous setting is to consider a cost function, indexed by the player and depending on the collective distribution of plays. The game then consists of finding an equilibrium between each player trying to minimize the cost despite the effects of the collective distribution. In [45], Schmeidler was interested in a similar model with a continuum of players, but more specifically he wanted to obtain existence of equilibria in pure strategies. He described a profile of strategies as a measurable function from the space of players to the space of admissible strategies. Later on, this notion of equilibrium was relaxed, for instance in [29, 39], defining equilibria as probability measures over the space of admissible strategies, introducing the notion that is now known as *Cournot-Nash* equilibrium, see Definition 1.1 below.

This relaxation also played an important part in the development of the optimal transportation problem, described as follows: given two Polish spaces $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$, a pair of Borel probability measures $\mu \in \mathscr{P}(\mathcal{X})$, $\nu \in \mathscr{P}(\mathcal{Y})$ and a transportation cost $c : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, one seeks to minimize the following

$$\mathcal{W}_c(\mu, \nu) \overset{\text{def.}}{=} \inf_{T_\sharp \mu = \nu} \int_{\mathcal{X}} c(x, T(x)) \mathrm{d}\mu = \min_{\gamma \in \Pi(\mu,\nu)} \int_{\mathcal{X}\times\mathcal{Y}} c(x, y) \mathrm{d}\gamma. \tag{3}$$

The infimum on the left is known as Monge's formulation [40] and is taken over all measurable maps T that transport μ to ν, in the sense that for all measurable sets $B \subset \mathcal{Y}$ it holds that $T_\sharp\mu(B) \overset{\text{def.}}{=} \mu(T^{-1}(B)) = \nu(B)$, where $T_\sharp\mu$ is called the push-forward measure.

The minimum on the right side is called Kantorovitch formulation [33], which is taken among all couplings of μ and ν,

$$\Pi(\mu,\nu) \stackrel{\text{def.}}{=} \{\gamma \in \mathscr{P}(\mathcal{X}\times\mathcal{Y}) : (\pi_{\mathcal{X}})_\sharp\gamma = \mu,\ (\pi_{\mathcal{Y}})_\sharp\gamma = \nu\}, \tag{4}$$

that is the probability measures in the product space whose marginals are μ and ν.

Back to our interpretation from games (1)(2), where $\mathcal{X}$ is the space of types of players, let $\mu \in \mathscr{P}(\mathcal{X})$ be a fixed distribution of players and $\mathcal{Y}$ is the space admissible strategies for said players, with distribution given by $\nu \in \mathscr{P}(\mathcal{Y})$. A coupling $\gamma \in \Pi(\mu,\nu)$ now becomes the joint distribution of players and strategies. In other words, given $A \times B \subset \mathcal{X}\times\mathcal{Y}$, the quantity $\gamma(A\times B)$ represents the probability that a player with type in A chooses a strategy in B.

Given a function $\Phi : \mathcal{X}\times\mathcal{Y}\times\mathscr{P}(\mathcal{Y}) \to \mathbb{R}\cup\{+\infty\}$, let $\Phi(x,y,\nu)$ denote the cost of a player of type x to choose the strategy y, in a mean field of strategies represented by the distribution ν, we obtain the notion of *Cournot-Nash* equilibrium.

Definition 1.1. A probability measure $\gamma \in \Pi(\mu,\nu)$ is a *Cournot-Nash equilibrium* for the game referent to the cost $\Phi : \mathcal{X}\times\mathcal{Y}\times\mathscr{P}(\mathcal{Y})$ if it satisfies the *equilibrium condition*

$$\gamma\left(\left\{(x,y)\in\mathcal{X}\times\mathcal{Y} : y \in \operatorname*{argmin}_{y'\in\mathcal{Y}} \Phi(x,y',\nu)\right\}\right) = 1, \tag{5}$$

it is called an *equilibrium of finite social cost* if

$$\int_{\mathcal{X}\times\mathcal{Y}} \Phi(x,y,\nu)\mathrm{d}\gamma < +\infty. \tag{6}$$

An equilibrium γ is called *pure* if it can be written as $\gamma = (\mathrm{id},T)_\sharp\mu$, where $T : \mathcal{X}\to\mathcal{Y}$ is a measurable map.

Results guaranteeing the existence of equilibria have been established with fixed point methods in the above-mentioned works. This approach relies strongly on the continuity of the cost function. In [9], whenever Φ has a potential structure, *i.e.* it can be written as

$$\Phi(x,y,\nu) = c(x,y) + \frac{\delta\mathcal{E}}{\delta\nu}[\nu](y), \tag{7}$$

the sum of an individual continuous cost $c(x,y)$ and the first variation of a functional $\mathcal{E}$, Blanchet and Carlier showed that if $\gamma\in\Pi(\mu,\nu)$ is an optimal transportation plan for the cost c and

$$\nu \in \operatorname*{argmin}_{\nu'\in\mathscr{P}(\mathcal{Y})} \mathcal{W}_c(\mu,\nu') + \mathcal{E}(\nu), \tag{8}$$

then γ is a Cournot-Nash equilibrium in the sense of Definition 1.1. The advantage of their approach is twofold: firstly, as their proof of existence is of variational nature it provides a natural approach to compute equilibria numerically, as done in [8,9], see also [10] for an approach using entropic regularization of the OT term. The characterization via optimal transport also gives information about the existence of pure Cournot-Nash equilibria, the original motivation of Schmeidler, that is at first glance abandoned when we define equilibria as couplings instead of maps. Since equilibria are described as optimal solutions to an optimal transport problem with first marginal given by μ, one can then use the well established conditions from OT to conclude that equilibria are pure, under suitable assumptions on μ and c, see for instance [16,27] or the recent monograph [43].

It is worth noting that definition 1.1 deviates from the literature, for instance from the one from [9], as we also require the finite social cost condition. It is not restrictive to the analysis from [9],where the cost c is assumed continuous and the underlying spaces compact, so that (6) holds trivially. In economic modeling we want to take into account that not all strategies are accessible to all types of individuals. Richer individuals have access to better education, health care and financial products and services. On the other

hand, it is not reasonable that they benefit from governmental aids. For such reasons it is pertinent to allow $c(x,y)$ to be $+\infty$ to model the fact that not all strategies are attainable for all players, so that (6) is no longer trivially satisfied.

1.1. **Contributions of this work.** In this paragraph we describe more precisely the setting and the results obtained in this work. Before passing to the convergence question, we give a full characterization of Cournot-Nash equilibria for costs having the potential structure from (7), where we assume only lower semi-continuity of c and $\mathcal{E}$. Instead of working with the energy $\nu \mapsto \mathcal{W}_c(\mu,\nu)+\mathcal{E}(\nu)$, used in [9], we use a lifted energy over the space of transportation plans with fixed marginal μ, which is defined as

$$\mathcal{J}(\gamma) \stackrel{\text{def.}}{=} \begin{cases} \displaystyle\int_{\mathcal{X}\times\mathcal{Y}} c(x,y)\mathrm{d}\gamma + \mathcal{E}(\nu), & \text{if } \gamma \in \Pi(\mu,\nu), \\ +\infty & \text{if } \gamma \notin \mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y}), \end{cases} \tag{9}$$

where

$$\mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y}) \stackrel{\text{def.}}{=} \left\{\gamma \in \mathscr{P}(\mathcal{X}\times\mathcal{Y}) : \quad (\pi_{\mathcal{X}})_\sharp \gamma = \mu\right\}. \tag{10}$$

We show in Theorem 3.3 that γ is a Cournot-Nash equilibrium if, and only if, it is an extremal point of the energy $\mathcal{J}$, see Definition 3.1. As the cost c can now be any l.s.c. function, we cannot resort to the same techniques from [9], as the latter relies strongly on the continuity of the cost to have uniqueness of solutions of the dual problem.

In the sequel we focus our attention to the case that $\mathcal{E}$ can be decomposed into a mean individual and interaction energies as follows

$$\mathcal{E}(\nu) = \mathcal{L}(\nu) + \mathcal{H}(\nu,\nu), \text{ where } \mathcal{L}(\nu) = \int_{\mathcal{Y}} L\mathrm{d}\nu \text{ and } \mathcal{H}(\nu,\nu) = \int_{\mathcal{Y}\times\mathcal{Y}} H\mathrm{d}\nu\otimes\nu, \tag{11}$$

where the first variation of the energy can be explicitly computed as

$$\frac{\delta\mathcal{E}}{\delta\nu}(\nu) = L + 2\int_{\mathcal{Y}} H(\cdot,y)\mathrm{d}\nu(y), \tag{12}$$

whenever H is symmetric. We make the following assumptions on these functionals

(H1) $\mu \in \mathscr{P}(\mathcal{X})$ is atomless.
(H2) $L\colon \mathcal{Y} \to \bar{\mathbb{R}}_+$ is lower semi-continuous and $H\colon \mathcal{Y}\times\mathcal{Y} \to \bar{\mathbb{R}}_+$ is Borel measurable.
(H3) H is symmetric, i.e., $H(y,\widetilde{y}) = H(\widetilde{y},y)$ for every $(y,\widetilde{y}) \in \mathcal{Y}\times\mathcal{Y}$.
(H4) The function $\mathcal{Y}\times\mathcal{Y} \ni (y,\widetilde{y}) \mapsto L(y)+L(\widetilde{y})+H(y,\widetilde{y}) \in \bar{\mathbb{R}}_+$ is lower semi-continuous.
(H5) L has compact level sets, *i.e.* for every $\kappa > 0$, the set $\{L \le \kappa\}$ is compact.

Under these conditions the characterization of Cournot-Nash equilibria holds, and we focus our attention to a stability result for the minimization of the energy $\mathcal{J}$ with respect to the marginal μ. In particular, under additional assumption (H6), in Theorem 3.10 we show a Lipschitz dependence of the value function for the 1-Wasserstein distance.

Once we understand the structure of the game with a continuum of players, we wish to answer the following question:

> *Given a sample of players following a continuous distribution, when will a sequence of Nash equilibria for the associated finite game will converge to a Cournot-Nash equilibrium?*

More precisely, let $(X_i)_{i\in\mathbb{N}}$ be an *i.i.d.* sequence of random variables with law $\mu \in \mathscr{P}(\mathcal{X})$, representing a sample of players, and we let the N first elements represent the type of the agents in our N-player game. Consider the probability space $(\Omega,\mathcal{F},\mathbb{P})$ induced by the sample $(X_i)_{i\in\mathbb{N}}$, so that Ω represents all the possible realizations of this sampling, $\mathcal{F}$ is the σ-algebra generated by the random variables X_i and, $\mathbb{P} = \mu^{\otimes\mathbb{N}}$. We can think of two types of *information structure* for our game.

Closed loop. The first and more natural one is a *closed loop information structure*, where each player knows their type before choosing their strategy. In other words, given an event $\omega = (X_i = x_i)_{i\in\mathbb{N}}$, each player seeks to

$$\begin{aligned}\underset{\nu_i\in\mathscr{P}(\mathcal{Y})}{\text{minimize}}\, J_{\omega,i}(\nu_i,\nu_{-i}) &\overset{\text{def.}}{=} \int_{\mathcal{Y}} c(x_i,y)\mathrm{d}\nu_i + \int_{\mathcal{Y}} L\mathrm{d}\nu_i + \frac{2}{N}\sum_{j\neq i}\int_{\mathcal{Y}\times\mathcal{Y}} H\mathrm{d}\nu_i\otimes\nu_j \\ &= \int_{\mathcal{Y}} c(x_i,y)\mathrm{d}\nu_i + \mathcal{L}(\nu_i) + \frac{2}{N}\sum_{j\neq i}\mathcal{H}(\nu_i,\nu_j).\end{aligned} \tag{13}$$

Open loop. The second one is an *open loop information structure*, where each player chooses a strategy before having the knowledge of the realization of the sample, in other words each player chooses an optimal execution strategy conditioned to the realization of their random variable. In this case, an admissible strategy is given by a measurable family $(\nu^x)_{x\in\mathcal{X}}\subset\mathscr{P}(\mathcal{Y})$, which is equivalent to considering a random probability measure $\boldsymbol{\nu}\in\mathscr{P}_\Omega(\mathcal{Y})$, see Section 2.2 or [19, 32], and the cost of player i becomes

$$\min_{\boldsymbol{\nu}_i\in\mathscr{P}_\Omega(\mathcal{Y})} \mathcal{J}_{\Omega,i}\left(\boldsymbol{\nu}_i,\boldsymbol{\nu}_{-i}\right) \overset{\text{def.}}{=} \mathbb{E}_{\mathbb{P}}\left[\int_{\mathcal{Y}} c(X_i,y)\mathrm{d}\nu_i^{X_i} + \mathcal{L}(\nu_i^{X_i}) + \frac{2}{N}\sum_{i\neq j}\mathcal{H}\left(\nu_i^{X_i},\nu_j^{X_j}\right)\right], \tag{14}$$

where $\boldsymbol{\nu}_i = \nu_i^{X_i}$ for some measurable family $(\nu_i^x)_{x\in\mathcal{X}}$.

In (13) and (14) we have excluded the cross terms $\mathcal{H}(\nu_i,\nu_i)$ for two reasons. From a modeling perspective, it makes sense that agents do not interact with themselves, which is exactly what this term represents. In addition, this formulation makes sense in pure strategies even in the case where the pairwise interaction diverges in the diagonal, $H(y,y)=+\infty$, for instance the case of an electrostatic interaction, see *e.g.* example 3.7. If we had kept the self-interaction in this case, any pure strategy would yield the player the value $+\infty$. We shall also consider the case that the pairwise interaction vanishes in the diagonal, *i.e.* $H(y,y)=0$. For the formulation in pure strategies this does not affect the cost functions of each player, but in mixed plays the diagonal terms $\mathcal{H}(\nu_i,\nu_i)$ can be included, see Proposition 4.9.

We prove in Propositions 4.9 and 4.2 that both formulations admit a potential functional, that is a functional whose minimizers give Nash equilibria to their corresponding game. In the closed loop case it is obtained by averaging the cost of each player

$$\mathcal{J}_{\omega,N}(\gamma_N) \overset{\text{def.}}{=} \begin{cases} \displaystyle\int_{\mathcal{X}\times\mathcal{Y}} c\mathrm{d}\gamma_N + \frac{1}{N}\sum_{i=1}^N \mathcal{L}\left(\nu^{x_i}\right) & \\ \qquad + \quad \displaystyle\frac{1}{N^2}\sum_{j\neq i}\mathcal{H}\left(\nu^{x_i},\nu^{x_j}\right), & \text{if } \gamma_N = \displaystyle\frac{1}{N}\sum_{i=1}^N \delta_{x_i}\otimes\nu^{x_i}, \\ +\infty, & \text{otherwise.}\end{cases} \tag{15}$$

For the open loop formulation, a potential functional can be analogously defined as

$$\mathcal{J}_{\Omega,N}(\boldsymbol{\gamma}_N) \overset{\text{def.}}{=} \begin{cases} \mathbb{E}_{\mathbb{P}}\left[\mathcal{J}_{\omega,N}(\boldsymbol{\gamma}_N)\right], & \text{if } \boldsymbol{\gamma}_N = \displaystyle\frac{1}{N}\sum_{i=1}^N \delta_{X_i}\otimes\nu^x, \\ +\infty, & \text{otherwise.}\end{cases} \tag{16}$$

To prove the convergence of Nash to Cournot-Nash equilibria, we show that both families of functionals converge in a variational sense to $\mathcal{J}$, the potential functional that yields Cournot-Nash equilibria. This variational sense is the so called notion of Γ-convergence, proposed by De Giorgi (see Section 2.3 and the references therein) for its fundamental property, which states that if a sequence of minimizers to some family of functionals that Γ converge has a limit, then this limit is a minimizer of the Γ-limit. This convergence combined with the potential structure of both the sequences of N-player games and that

of Cournot-Nash equilibria, immediately implies the desired convergence of equilibria obtained via the minimization of $\mathcal{J}_{\omega,N}$ and $\mathcal{J}_{\Omega,N}$. With slightly stronger assumptions, that H is continuous and bounded, we can prove that limits of any sequence of Nash equilibria satisfy the stationarity condition of the functional $\mathcal{J}$, which is equivalent to being a Cournot-Nash equilibrium.

We start with the open loop case in Section 4.1, since the Γ-convergence of $\mathcal{J}_{\Omega,N}$ to $\mathcal{J}$ is more straight forward. However, it has the additional difficulty of being the convergence of a functional over random measures to a functional over deterministic inputs. This issue is overcome with Lemma 4.3, by noticing that the stochasticity of the measures in the domain of (16) is restricted to the first marginal, which is an empirical measure and so converges with full probability to the common law of the sample. For the closed loop case, one can only hope to have a convergence result with full probability, hence in Lemma 4.10 we show a preliminary result that gives a weaker criterion for Γ-convergence with full probability. In the sequel, we show in Theorem 4.12, we conclude our convergence proof using this Lemma.

1.2. **Related work.** In this paragraph we describe previous results from the literature that are similar to ours thematically or in terms of the tools employed.

Convergence of Nash equilibria. As mentioned before, the convergence of Nash equilibria to equilibria in a continuous setting is a reoccurring idea in the Game Theory literature. Next we give a non-exhaustive list of advancements in this direction:

- in [15], the authors study the convergence question in the framework of MFGs by means of the Master equation, an equation in the space of probability measures that give global information on the optimality of the MFG system, which can then be viewed as the limit of the equilibrium conditions of N-player differential games. The analysis therein assumes that the coupling in the MFG system is non-local and promotes enough regularization to pass the gradients of the value function in the limit;
- the case of a local coupling was later treated in [14]. Both of these impressive works have a very fine analysis of the PDE system associated with the games, and therefore they concern the convergence of optimality conditions for the involved games and not necessarily the convergence of equilibria, which is largely open. This way, our approach allows to extract more information for a problem with more structure;
- the study of convergence of Nash to Cournot-Nash equilibria was conducted in [7], under the assumption that the sequence of functions defining the problem $\Phi_N : \mathcal{X} \times \mathcal{Y} \times \mathscr{P}(\mathcal{Y})$, as in Definition 1.1, are uniformy Lipschitz in the topology of $(d_{\mathcal{X}}, d_{\mathcal{Y}}, \mathcal{W}_1)$ and converge uniformly to Φ. While our problem enjoys the pairwise (and also potential) structure, our regularity assumptions are minimal and allow the treatment of many examples, c.f. subsection 1.3.

Γ-*convergence for points clouds.* Concerning the continuum limits of particle systems, our methods are conceptually close to those of Serfaty in [46] for the study the limits of particle systems under Coulomb type interactions. The major difference lies in the further complexity that the first marginal constraint, which is natural to our problem to fix the distribution of players μ (or μ_N in the N-players case), otherwise an elegant application of the probabilistic method[1] would give a simple proof of the Γ-convergence result as done in the recent lecture notes [47].

Concerning the probabilistic side of our Γ-convergence results we must mention [28], which concerns the continuum limit of the total variation functional defined on graphs

[1]Developed in the context of random graph theory [1]

induced by i.i.d. point clouds. Given is a rescaled convolution kernel η and an i.i.d. sample $(X_i)_{i\in\mathbb{N}}$ with continuous law ρ,the authors prove that the functional

$$GTV_{N,\varepsilon_N}(u) \overset{\text{def.}}{=} \frac{1}{\varepsilon_N}\frac{1}{N^2}\eta_{\varepsilon_N}(X_i - X_j)|u(X_i) - u(X_j)|,$$

converges with full probability, in the sense of Γ-convergence, to the weighted total variation functional $\mathrm{TV}(u;\rho)$, which for $u \in W^{1,1}$ is given by $\mathrm{TV}(u;\rho) = \int |\nabla u|\rho^2 \mathrm{d}x$. Their results are conditional to a good scaling of the parameter ε_N, so that $\varepsilon_N \ll W_\infty(\rho,\rho_N)$, that is the maximal distance of the optimal matching problem between the empirical measure ρ_N of a sample with common distribution ρ, which can be precisely estimated as $N \to \infty$ with full probability. This is the only probabilistic element in their analysis, and afterwards their methods become purely deterministic.

1.3. **Examples.** In this paragraph we discuss multiple examples that are covered by our model and their relevance in the literature.

Potential Cournot-Nash equilibria [9]. The first clear example is the model proposed by Blanchet and Carlier in [9]. As discussed above, they proposed a variational principle to find equilibria as in Definition 1.1 and gave plenty of examples of economic applications for this model as the holiday choice and technology choice models. The distinctions from ours is that, for them c had to be a continuous cost in order to give the variational characterization of equilibria using the OT problem as in equation (8). We do not need this assumption since we propose the lifting to the space of transportation plans $\gamma \in \mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})$. This lift is purely technical and the characterization via the value of the associated OT problem still holds in our case and is useful for numerical purposes, since one can use the dual formulation of the OT problem as a dimensionality reduction technique. On the other hand, taking c to be lower semi-continuous allows us to make a link with the next class of examples.

Abstract Lagrangian Mean Field Games [44]. Consider a crowd motion, where the starting point of each agent is distributed by a probability measure μ and the final goal of each agent is to reach a target set while minimizing a cost depending on their own trajectory and on the distribution of trajectories of all agents Q. One can think of the target set as the exit of a metro, for instance. In [44], Santambrogio and Shim propose a model where each agent chooses their trajectory among all possible continuous curves respecting their given initial condition. In this case, $\mathcal{X} = \Omega$ is a compact subset of $\mathbb{R}^d$ and $\mathcal{Y} = C^0([0,T];\Omega)$. Each agent then tries to find a curve σ, such that $\sigma(0) = x_0$, the given initial condition, while minimizing an energy of the form

$$F(\sigma,Q) \overset{\text{def.}}{=} \int_0^T \left(|\sigma'(t)|^2 + \int_{\mathcal{Y}} \eta(\sigma(t)-\tilde\sigma(t))|\sigma'(t)-\tilde\sigma'(t)|^2 \mathrm{d}Q(\tilde\sigma)\right)\mathrm{d}t + \Psi(\gamma(T))$$

Here Ψ is an end point cost and η is an interaction kernel of Cucker-Smale type in order to observe a phenomenon of consensus of the velocities as in the seminal paper [20]. The measure Q corresponds to the distribution of trajectories of all agents so that the integral term becomes a mean interaction cost and the initial condition is then imposed by the constraint $(e_0)_\sharp Q = \mu$. They defined equilibria as measures $Q \in \mathscr{P}(\mathcal{Y})$ such that $(e_0)_\sharp Q = \mu$ and

$$\int_{\mathcal{Y}} F(\sigma,Q)\mathrm{d}Q < \infty, \text{ and } F(\sigma,Q) = \inf_{\tilde\sigma(0)=\sigma(0)} F(\tilde\sigma,Q), \text{ for all } \sigma \in \operatorname{supp} Q. \tag{17}$$

In [42], this model was generalized into an abstract model, where F is given by

$$F(\sigma,Q) \overset{\text{def.}}{=} L(\sigma) + \int_{\mathcal{Y}} H(\sigma,\tilde\sigma)\mathrm{d}Q(\tilde\sigma),$$

where $L\colon \mathcal{Y} \to \mathbb{R}$ is an individual cost, $H(\cdot,\cdot)\colon \mathcal{Y}\times\mathcal{Y} \to \mathbb{R}$ is a symmetric cost of pairwise interaction, where $\mathcal{Y}$ is an abstract space of admissible strategies, being a Polish space. The players are now labelled by some $x \in \mathcal{X}$, another Polish space of types of players following a distribution $\mu \in \mathscr{P}(\mathcal{X})$. The initial condition map is now replaced by a continuous map $\pi : \mathcal{Y} \to \mathcal{X}$, and we consider measures $Q \in \mathscr{P}(\mathcal{Y})$ such that $\pi_\sharp Q = \mu$. Their notion of equilibrium is the same as in (17), but they show that equilibria are critical points of the following functional

$$Q \mapsto \int_{\mathcal{Y}} L\mathrm{d}Q + \int_{\mathcal{Y}\times\mathcal{Y}} H\mathrm{d}Q\otimes Q.$$

This suggests a link with the previous model of Cournot-Nash equilibria and indeed, for $c(x,y) = \chi_{\pi^{-1}(y)}(x)$ we can rewrite the constraints as

$$\pi_\sharp\nu = \mu \Longleftrightarrow \mathcal{W}_c(\mu,\nu) < \infty, \text{ since } \mathcal{W}_c(\mu,\nu) = \begin{cases} 0, & \text{if } \pi_\sharp\nu = \mu, \\ +\infty, & \text{otherwise.}\end{cases}$$

Conversely, if we propose the lifted energy to the space of transportation plans (9), the variational criterion for Cournot-Nash equilibria from Blanchet and Carlier is of the same form as the one for Lagrangian MFGs.

Wasserstein gradient flows (JKO schemes). For the final example, let $\mathcal{X} = \mathcal{Y} = \Omega$ be a compact subset of $\mathbb{R}^d$. We wish to discuss the case of Wasserstein gradient flows, also known as JKO schemes in reference to the seminal paper of Jordan, Kinderlehrer, and Otto [31], where the authors proposed a variational formulation of the Fokker–Planck equation. Their ideas were later generalized to other evolution equations, for instance in [3], see also [43, Chap. 8]. The scheme consists in solving the following variational problem iteratively

$$\min_{\rho\in\mathscr{P}(\Omega)} \frac{1}{2\tau}W_2^2(\rho_k,\rho) + \mathcal{F}(\rho) = \min_{\gamma\in\mathscr{P}_{\rho_k}(\Omega\times\Omega)} \frac{1}{2\tau}\int_{\Omega\times\Omega}|x-y|^2\mathrm{d}\gamma + \mathcal{F}(\pi_{2,\sharp}\gamma) \tag{18}$$

where W_2^2 corresponds to the value of the OT problem with $c(x,y) = |x-y|^2$. By solving this sequence of variational problems, one obtains a sequence $(\rho_k)_{k\in\mathbb{N}}$ and define an interpolation depending on the parameter τ as

$$\rho_\tau(t) \overset{\text{def.}}{=} \rho_k \text{ if } t \in [k\tau,(k+1)\tau).$$

For a variety of choices of $\mathcal{F}$, it can be shown that ρ_τ converges as $\tau \to 0$ to a solution of the evolution equation

$$\partial_t\rho + \operatorname{div}\left(\rho\nabla\frac{\delta\mathcal{F}}{\delta\rho}(\rho)\right) = 0, \quad \rho(0) = \rho_0, \tag{19}$$

with no-flux boundary conditions.

The case, see [17, 18],

$$\mathcal{F}(\rho) = \int_\Omega V(x)\mathrm{d}\rho(x) + \int_{\Omega\times\Omega} W(x-y)\mathrm{d}\rho\otimes\rho(x,y),$$

corresponds to an advection plus aggregation phenomenon, that is covered by our Γ-convergence results.

1.4. **The organization of this paper:** in Section 2 we review the major tools of measure theory, optimal transport and Γ-convergence that are used throughout the present manuscript. In Section 3 we prove the full characterization of Cournot-Nash equilibria in our potential setting and pass to stability question of the value function of our problem w.r.t. the distribution of players, which is of independent interest. Finally, in Section 4 we prove our main Γ-convergence results our sequences of open and closed loop N-player games. In Section 5 we give our concluding remarks and possible directions.

2. Preliminaries in measure theory and optimal transport

In this section we review the mathematical tools we require and also to fix the notation to be used throughout the text. The first two paragraphs are dedicated to the topologies of spaces of probability measures and the optimal transportation problem. Next, we introduce the space of random probability measures and the compactness properties à la Prokhorov that it enjoys. We finish with a brief discussion about Γ-convergence that is the main ingredient of our proofs.

2.1. Topologies on spaces of Radon measures.

In this paper, we shall work with general Polish spaces $(\mathcal{X}, d_{\mathcal{X}})$, that is a complete separable space equipped with a metric topology. We let $\mathcal{M}_b(\mathcal{X})$ denote the space of finite Radon measures over $\mathcal{X}$, the space of all Borel measures μ such that $\mu(K) < \infty$ whenever K is compact. We let $\mathscr{P}(\mathcal{X})$ denote the subspace of probability measures over $\mathcal{X}$, *i.e.* positive measures with unitary total mass.

It follows from Riesz' representation theorem that $\mathcal{M}_b(\mathcal{X})$ is the topological dual space of $C_0(\mathcal{X})$, the continuous functions converging to 0 at infinity, see [25, Chap.1]. This defines a norm in $\mathcal{M}_b(\mathcal{X})$, the total variation norm, and also its weak-$\star$ topology. We can define other notions of weak topology by changing the set of test functions.

Definition 2.1. Given a suitable space of functions $\mathcal{K}$, we say a sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ converges in the narrow topology to μ if

$$\int_{\mathcal{X}} \phi \mathrm{d}\mu_n \xrightarrow[n\to\infty]{} \int_{\mathcal{X}} \phi \mathrm{d}\mu_n \text{ for every } \phi \in C_b(\mathcal{X}),$$

where $C_b(\mathcal{X})$, is the set of continuous and bounded functions, and we write $\mu_n \xrightarrow[n\to\infty]{} \mu$.

Notice that if μ_n converges to μ in the weak-$\star$ topology, there is no guaranteeing that $\mu(\mathcal{X}) = \lim_{n\to\infty} \mu_n(\mathcal{X})$, the norm $|\cdot|(\mathcal{X})$ is only l.s.c. for this notion of convergence. For the narrow convergence, however, as it is in duality with $C_b(\mathcal{X})$, we can consider the constant 1 as test function and obtain the convergence of the total masses. This implies that $\mathscr{P}(\mathcal{X})$ is closed for the narrow topology, but not for the weak-$\star$, unless $\mathcal{X}$ is compact. Besides this, we also have a nice criterion of compactness for the narrow topology.

Theorem 2.2 (Prokhorov)**.** *Let $\mathcal{F} \subset \mathscr{P}(\mathcal{X})$ be a family of probability measures over $\mathcal{X}$. Then $\mathcal{F}$ is compact for the narrow topology, if and only if, it is a tight family, i.e. for all $\varepsilon > 0$ there is a compact set K such that*

$$\mu(\mathcal{X} \setminus K) < \varepsilon, \text{ for all } \mu \in \mathcal{F}.$$

Actually, the set $\mathcal{K} = C_b(\mathcal{X})$ is not the minimal set for which we can define a weak topology that yields the narrow convergence. This is clear since we can always approximate functions in $C_b(\mathcal{X})$ with Lipschitz functions, but we can even construct a countable set of test functions yielding the narrow convergence.

Proposition 2.3 ([3, Chapter 5])**.** *There exists a countable set $\mathcal{K} = (f_k)_{k\in\mathbb{N}}$ of Lipschitz functions, such that any sequence $(\mu_n)_{n\in\mathbb{N}}$ converges narrowly to μ, if and only if,*

$$\int_{\mathcal{X}} f_k \mathrm{d}\mu_n \xrightarrow[n\to\infty]{} \int_{\mathcal{X}} f_k \mathrm{d}\mu, \text{ for all } k \in \mathbb{N}.$$

Now consider a pair of Polish spaces $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ and let $\mathcal{X} \ni x \mapsto \nu^x \in \mathscr{P}(\mathcal{Y})$ be a measure-valued map.

Definition 2.4. We say $(\nu^x)_{x\in\mathcal{X}}$ is measurable if for any Borel set $B \subset \mathcal{Y}$, the function $x \mapsto \nu^x(B)$ is Borel measurable.

Now given $\mu \in \mathscr{P}(\mathcal{X})$ we can define a new probability measure $\gamma \in \mathscr{P}(\mathcal{X} \times \mathcal{Y})$ in the product space through duality as

$$\int_{\mathcal{X}\times\mathcal{Y}} f(x,y)\mathrm{d}\gamma(x,y) \overset{\text{def.}}{=} \int_{\mathcal{Y}} \left(\int_{\mathcal{X}} f(x,y)\mathrm{d}\nu^x(y) \right) \mu(x),$$

and we use the notation $\gamma = \mu \otimes \nu^x$. It turns out that all measures $\gamma \in \mathscr{P}(\mathcal{X} \times \mathcal{Y})$ can be written in this way as a consequence of the *disintegration theorem*, see [48, Thm. 1.1.6] for a proof in Polish spaces.

Theorem 2.5. *Let $\mathcal{X}_0$ and $\mathcal{X}_1$ be Polish spaces, and two probability measures $\mu_0 \in \mathscr{P}(\mathcal{X}_0)$ and $\mu_1 \in \mathscr{P}(\mathcal{X}_1)$. If $\pi : \mathcal{X}_0 \to \mathcal{X}_1$ is a measurable map such that $\pi_\sharp \mu_0 = \mu_1$, then there exists a μ_1-a.e. uniquely determined Borel family $(\mu_0^{x_1})_{x_1 \in \mathcal{X}_1} \subset \mathscr{P}(\mathcal{X}_0)$ such that*

$$\mu_0^{x_1}(\mathcal{X}_0 \setminus \pi^{-1}(x_1)) = 0 \text{ for } \mu_1\text{-a.e. } x_1 \in \mathcal{X}_1,$$

and for every measurable function $f : \mathcal{X}_0 \to [0, +\infty]$ if holds that

$$\int_{\mathcal{X}_0} f(x_0)\mathrm{d}\mu_0(x_0) = \int_{\mathcal{X}_1} \left(\int_{\pi^{-1}(x_1)} f(x_0)\mathrm{d}\mu_0^{x_1}(x_0) \right) \mathrm{d}\mu_1(x_1).$$

Any such $(\mu_0^{x_1})_{x_1 \in \mathcal{X}_1}$ is called a disintegration family and we write $\mu_0 = \mu_0^{x_1} \otimes \mu_1$.

Whenever $\gamma \in \Pi(\mu, \nu)$, we apply the previous theorem with $\mathcal{X}_0 = \mathcal{X} \times \mathcal{Y}$, $\mathcal{X}_1 = \mathcal{X}$ and $\pi = \pi_{\mathcal{X}}$ to write $\gamma = \mu \otimes \nu^x$. We could also consider the disintegration w.r.t. the second marginal, in which case we write $\gamma = \mu^y \otimes \nu$. One of the most useful, yet simple, applications of the disintegration theorem is the gluing lemma.

Lemma 2.6. *[3, Lemma 5.3.2] Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ be Polish spaces, $\gamma_{1,2} \in \mathscr{P}(\mathcal{X}_1 \times \mathcal{X}_2)$ and $\gamma_{1,3} \in \mathscr{P}(\mathcal{X}_1 \times \mathcal{X}_3)$ such that*

$$(\pi_{\mathcal{X}_1})_\sharp \gamma_{1,2} = (\pi_{\mathcal{X}_1})_\sharp \gamma_{1,3} = \mu_1.$$

Then there exists $\gamma_{1,2,3} \in \mathscr{P}(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)$ such that

$$(\pi_{\mathcal{X}_1,\mathcal{X}_2})_\sharp \gamma_{1,2,3} = \gamma_{1,2} \text{ and } (\pi_{\mathcal{X}_1,\mathcal{X}_3})_\sharp \gamma_{1,2,3} = \gamma_{1,3}.$$

Proof. The proof consists on taking the disintegration families $\gamma_{1,2} = \gamma_{1,2}^{x_1} \otimes \mu_1(x_1)$, $\gamma_{1,3} = \gamma_{1,3}^{x_1} \otimes \mu_1(x_1)$ and defining the new measure as

$$\gamma_{1,2,3} \overset{\text{def.}}{=} \int_{\mathcal{X}_1} \gamma_{1,2}^{x_1} \otimes \gamma_{1,3}^{x_1} \mathrm{d}\mu_1(x_1).$$

□

2.2. **Random probability measures and their weak topologies.** We will also use in this work, the notion of random probability measure. The simplest example of this kind of object is a sequence of empirical measures, that is, given an *i.i.d.* sample of random variables $(X_i)_{i\in\mathbb{N}}$ we define the measures

$$\boldsymbol{\mu}_N \overset{\text{def.}}{=} \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i}.$$

Clearly, for each realization of the random variables we obtain a different discrete measure. For a random sample of agents $(X_i)_{i\in\mathbb{N}}$, we will describe a profile of strategies with the measures

$$\boldsymbol{\gamma}_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i} \otimes \nu_i,$$

where $\nu_i \in \mathscr{P}(\mathcal{Y})$ represents the strategy, possibly in mixed plays, of player i. In general, a random measure is defined as follows.

Definition 2.7. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Polish space $\mathcal{X}$, a random measure $\boldsymbol{\mu}$ is a map from Ω into the space of Radon measures

$$\boldsymbol{\mu} : \Omega \ni \omega \mapsto \boldsymbol{\mu}(\omega) \in \mathcal{M}_b(\mathcal{X}),$$

which is measurable for the Borel σ-algebra defined with respect to the narrow topology, in duality with $C_b(\mathcal{X})$. We let $\mathcal{M}_\Omega(\mathcal{X})$ denote the space of all random measures, and $\mathscr{P}_\Omega(\mathcal{X})$ is the convex subset of $\mathcal{M}_\Omega(\mathcal{X})$ consisting of all $\mathscr{P}(\mathcal{X})$-valued random probability measures.

Given $\mu_b \in \mathscr{P}_\Omega(\mathcal{X})$, the map

$$C_b(\mathcal{X}) \ni \phi \mapsto \mathbb{E}\left[\int_{\mathcal{X}} \phi \mathrm{d}\boldsymbol{\mu}(\omega)\right],$$

is a bounded linear map over $C_b(\mathcal{X})$, so from Riesz' representation theorem this defines a non-random measure via duality, the *expectation measure* $\mathbb{E}\boldsymbol{\mu} \in \mathscr{P}(\mathcal{X})$ as

$$\int_{\mathcal{X}} \phi \mathrm{d}\mathbb{E}\boldsymbol{\mu} \overset{\text{def.}}{=} \mathbb{E}\left[\int_{\mathcal{X}} \phi \mathrm{d}\boldsymbol{\mu}(\omega)\right]. \tag{20}$$

In particular, a random measure can be identified by a non-random measure if, and only if, it coincides with its expectation almost surely.

The Glivenko-Cantelli law of large numbers, also known as the Glivenko-Cantelli theorem [22], states that the empirical measures $\boldsymbol{\mu}_N$ converge in the narrow topology to μ with probability 1. Hence, in order to give a topology to $\mathscr{P}_\Omega(\mathcal{X})$, the first naive candidate would be to consider $\mathbb{P}$-a.s. convergence of the random measures in the narrow topology. However, this topology would not be metrizable, and it also does not enjoy good compactness properties as Prokhorov's Theorem [23]. For these reasons, we consider the narrow topology in $\mathscr{P}_\Omega(\mathcal{X})$.

Definition 2.8. We say that an $f : \Omega \times \mathcal{X} \to \mathbb{R}$ is a random bounded continuous function, and we let $C_\Omega(\mathcal{X})$ denote the class of all such functions, if[2]

(1) $x \mapsto f(\omega, x) \in C_b(\mathcal{X})$ almost surely;
(2) $\omega \mapsto f(\omega, x)$ is $\mathcal{F}$-measurable for all $x \in \mathcal{X}$;
(3) $\omega \mapsto \|f(\omega, \cdot)\|_{L^\infty(\mathcal{X})}$ is integrable with respect to $\mathbb{P}$.

The *narrow topology of random measures* is then the weakest topology that makes

$$\mathscr{P}_\Omega(\mathcal{X}) \ni \boldsymbol{\mu} \mapsto \mathbb{E}_{\mathbb{P}}\left[\int_X f(\omega, x) d\boldsymbol{\mu}_\omega(x)\right] \text{ continuous for all } f \in C_\Omega(\mathcal{X}).$$

Since the functions of the form

$$\sum_{i=1}^N 1_{A_i}(\omega) f_i(x), \text{ for } A_i \quad \mathbb{P}\text{-measurable and } f_i \in C_b(\mathcal{X}),$$

are dense in $C_\Omega(\mathcal{X})$, it follows that $\mathbb{P}$-a.s. convergence implies convergence in the narrow topology of $\mathscr{P}_\Omega(\mathcal{X})$. The advantage is that the latter enjoys compactness properties analogous to Prokhorov's Theorem 2.2. This is extremely useful since, even if one can show that a sequence of random probability measures is tight almost surely and apply the classical version of Prokhorov's Theorem, for each event ω will be associated a subsequence where narrow convergence holds, but we cannot in general obtain a single subsequence that converges $\mathbb{P}$-almost surely. Hence the usefulness of the following result, see [19, Thm. 4.29].

Theorem 2.9 (Random Prokhorov's Theorem)**.** *A family of random measures $\mathcal{F} \subset \mathscr{P}_\Omega(\mathcal{X})$ is pre-compact for the narrow topology of random measures, if and only if, it is tight: for any $\varepsilon > 0$ there is a compact set K_ε such that*

$$\mathbb{E}\left[\boldsymbol{\mu}(\mathcal{X} \setminus K_\varepsilon)\right] \leq \varepsilon \text{ for every } \boldsymbol{\mu} \in \mathcal{F}.$$

[2] In [4] the random continuous functions are also called Carathéodory integrands.

2.3. **Γ-convergence.** The key idea in order to prove the convergence of certain Nash equilibria, associated with potential games, to Cournot-Nash equilibria is to exploit the fact that one can obtain such objects through the minimization of a family of energies indexed by the number of players. The main argument consists in showing that these energies converge in the sense of Γ convergence to the potential function that describes Cournot-Nash equilibria. This notion of convergence is defined as follows.

Definition 2.10. Let $(\mathcal{X}, d_{\mathcal{X}})$ be a complete metric space, a sequence of functionals $\mathscr{F}_n : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ Γ-converges to $\mathscr{F}$, if

- $\underline{\Gamma - \liminf}$**:** for every sequence $x_N \xrightarrow[N\to\infty]{d_{\mathcal{X}}} x$ in $\mathcal{X}$, it holds that
$$\mathscr{F}(x) \leq \liminf_{N\to\infty} \mathscr{F}_N(x_N).$$
- $\underline{\Gamma - \limsup}$**:** for every $x \in \mathcal{X}$, there is a sequence $x_N \xrightarrow[N\to\infty]{d_{\mathcal{X}}} x$ such that
$$\limsup_{N\to\infty} \mathscr{F}_N(x_N) \leq \mathscr{F}(x),$$
$(x_N)_{N\in\mathbb{N}}$ is called the recovery sequence of x.

The notion of Γ-convergence was introduced by De Giorgi in order to have good properties concerning the limits of minimizers of variational problems, see for instance [21]. In this sense, the fundamental property that makes it interesting is the fact that cluster points of minimizers of a sequence of minimizers of $\mathscr{F}_N$, which Γ-converges to $\mathscr{F}$, are minimizers of $\mathscr{F}$. Indeed, let $(x_N)_{N\in\mathbb{N}}$ be a sequence of minimizers of $(\mathscr{F}_N)_{N\in\mathbb{N}}$ converging to x. For an arbitrary $x' \in \mathcal{X}$, let x'_N be a corresponding recovery sequence, then it follows that

$$\begin{aligned}
\mathscr{F}(x) &\leq \liminf_{N\to\infty} \mathscr{F}_N(x_N) && \text{by the } \Gamma - \liminf \text{ inequality}\\
&\leq \liminf_{N\to\infty} \mathscr{F}_N(x'_N) && \text{by the minimality of } x_N\\
&\leq \limsup_{N\to\infty} \mathscr{F}_N(x'_N) \leq \mathscr{F}(x') && \text{since } x'_N \text{ is a recovery sequence.}
\end{aligned}$$

As x' was an arbitrary point of $\mathcal{X}$, it follows that x is a minimizer of $\mathscr{F}$.

Equivalently, given a sequence of functionals $\mathscr{F}_N$, we define the lower and upper Γ limits, respectively, as

$$\begin{aligned}
\Gamma\text{-}\liminf \mathscr{F}_N(x) &\overset{\text{def.}}{=} \inf\left\{\liminf_{N\to\infty} \mathscr{F}_N(x_N) :\ x_N \xrightarrow[N\to\infty]{} x\right\},\\
\Gamma\text{-}\limsup \mathscr{F}_N(x) &\overset{\text{def.}}{=} \inf\left\{\limsup_{N\to\infty} \mathscr{F}_N(x_N) :\ x_N \xrightarrow[N\to\infty]{} x\right\}.
\end{aligned} \tag{21}$$

From [11, Prop. 1.28], both Γ upper and lower limits are lower semi-continuous and $\mathscr{F}_N \xrightarrow[N\to\infty]{\Gamma} \mathscr{F}$ if and only if $\Gamma\text{-}\liminf \mathscr{F}_N = \Gamma\text{-}\limsup \mathscr{F}_N = \mathscr{F}$.

3. Potential structure and stability of the value function

In this section, our objective is twofold, first we extend the results of Blanchet and Carlier about the potential structure for Cournot-Nash equilibria, allowing for individual costs c that are l.s.c. instead of continuous. In the sequel we show a stability result of the value function w.r.t. the fixed marginal μ.

3.1. **Potential structure for Cournot-Nash equilibria.** The goal of this section is to characterize equilibria in the sense of Definition 1.1 as critical points of an energy functional. For now, we assume that the optimization problem a player of type $x \in \mathcal{X}$ tries to solve among a mean field of plays $\nu \in \mathscr{P}(\mathcal{Y})$ is given by

$$\min_{y\in\mathcal{Y}} \Phi(x,y,\nu) \stackrel{\text{def.}}{=} c(x,y) + \frac{\delta\mathcal{E}}{\delta\nu}(\nu)(y), \tag{22}$$

where c is l.s.c. and the second term can be written as the first variation of an energy $\mathcal{E} : \mathscr{P}(\nu) \to \mathbb{R}$, which is defined below.

Definition 3.1. We say that a functional $\mathcal{F}$ defined over the probability measures $\mathscr{P}(\mathcal{X})$ over a Polish space $\mathcal{X}$ admits a first variation at $\mu_0 \in \mathscr{P}(\mathcal{X})$ if there exists a measurable function $f : \mathcal{X} \to \mathbb{R}$ such that f is $\mu - \mu_0$ integrable for all μ in the domain of $\mathcal{F}$ and such that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0^+} \mathcal{F}(\mu_0 + \varepsilon(\mu-\mu_0)) = \langle f, \mu-\mu_0\rangle \stackrel{\text{def.}}{=} \int_{\mathcal{X}} f \mathrm{d}(\mu-\mu_0), \tag{23}$$

and we write $f = \dfrac{\delta\mathcal{F}}{\delta\mu}(\mu_0)$. In addition, we say that μ_0 is a critical point of $\mathcal{F}$ if

$$\left\langle \frac{\delta\mathcal{F}}{\delta\mu}(\mu_0), \mu-\mu_0 \right\rangle \geq 0 \text{ for all } \mu \in \operatorname{dom}\mathcal{F}.$$

It is clear that the first variation as in Definition 3.1 cannot be unique, since summing a constant to a function satisfying (23) will still satisfy the same relation, as the integration is taken against $\mu - \mu_0$, which integrates to 0. It is, however, unique up to a constant.

For the rest of this paragraph, we let $\mathcal{E}$ be an l.s.c. functional over $\mathscr{P}(\mathcal{Y})$, and we consider the energy

$$\mathcal{J}(\gamma) \stackrel{\text{def.}}{=} \begin{cases} \displaystyle\int_{\mathcal{X}\times\mathcal{Y}} c(x,y)\mathrm{d}\gamma + \mathcal{E}(\nu), & \text{if } \gamma \in \Pi(\mu,\nu), \\ +\infty & \text{if } \gamma \notin \mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y}), \end{cases} \tag{24}$$

with a general functional $\mathcal{E}$, so that

$$\Phi(x,y,\nu) = \frac{\delta\mathcal{J}}{\delta\gamma}(\gamma) \text{ for } \nu = (\pi_{\mathcal{Y}})_\sharp\gamma.$$

Our goal is to show that critical points of this energy are Cournot-Nash equilibria, notice however that satisfying the equilibrium condition (5) is independent of having a finite social cost (6), we can have bad equilibria that represents a society with infinite poverty, for instance if a non-negligible part of the population is infinitely poor. For this we make the following definition.

Definition 3.2. A measure $\varrho \in \mathscr{P}(\mathcal{Y})$ is a distribution of finite social cost for the distribution μ if there is a function $\kappa \in L^1(\mu)$ such that for μ-a.e. $x \in \mathcal{X}$ there is $y_x \in \mathcal{Y}$ satisfying

$$\Phi(x, y_x, \varrho) \leq \kappa(x).$$

The main result of this section is the following.

Theorem 3.3. *Assume that $0 \leq \mathcal{E}$ admits a first variation given by an l.s.c. function with compact sub-level sets over $\mathcal{Y}$ and let Φ be as (22). It follows that*

(i) $\gamma \in \mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})$ is a Cournot-Nash equilibrium in the sense of Definition 1.1, if and only if it is a critical point of $\mathcal{J}$ defined in (24). If in addition, $\nu = (\pi_{\mathcal{Y}})_\sharp\gamma$ is a distribution of finite social cost, then γ is a Cournot-Nash equilibrium of finite social cost.

(ii) if $\mathcal{J}$ admits a minimizer, then

$$\min_{\gamma\in\mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})} \mathcal{J}(\gamma) = \min_{\nu\in\mathscr{P}(\mathcal{Y})} \mathcal{W}_c(\mu,\nu) + \mathcal{E}(\nu),$$

and it is a Cournot-Nash equilibrium.

Proof. The proof is inspired by the arguments in [42, Thm. 4.5.1] for the case of an abstract Lagrangian Mean Field Game and [38, Appendix A].

First suppose that $\gamma \in \mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})$ is a critical point, and define the function

$$\phi(x) \overset{\text{def.}}{=} \inf_{\mathcal{Y}} \Phi(x,\cdot,\nu).$$

It follows that ϕ is Borel measurable since it is lower semi-continuous as we prove next. Take $x_k \xrightarrow[k\to\infty]{} x$ such that $\liminf \phi(x_k)$ is finite, otherwise there is nothing to prove, and assume up to the extraction of a subsequence that the lim inf is a limit. Consider $y_k \in \operatorname{argmin}\Phi(x_k,\cdot,\nu)$ so that $\Phi(x_k,\cdot,\nu)\le C$ is uniformly bounded. Therefore, as $c\ge 0$ it holds that

$$(y_k)_{k\in\mathbb{N}} \subset \left\{\frac{\delta\mathcal{E}}{\delta\nu}(y) \le C\right\},$$

which is a compact set. Up to another extraction, we may assume that $y_k \to y$, so that the lower semi-continuity of Φ gives

$$\phi(x) \le \Phi(x,y,\nu) \le \liminf_{k\to\infty} \Phi(x_k,y_k,\nu) = \liminf_{k\to\infty} \phi(x_k).$$

To prove item (i), if suffices to show that the set

$$A = \{(x,y)\in\mathcal{X}\times\mathcal{Y} : \phi(x) < \Phi(x,y,\nu)\}$$

is γ-negligible. Suppose this is not the case, and our goal is to construct a Borel measurable selection of the argmin operator, that is a Borel function $T:\mathcal{X}\to\mathcal{Y}$ such that

$$T(x) \in \operatorname*{argmin}_{\mathcal{Y}} \Phi(x,\cdot,\nu) \text{ for all } x\in X.$$

From [12, Thm. 1] it holds that if $E\subset\mathcal{X}\times\mathcal{Y}$ is a Borel set with the property that $E_x \overset{\text{def.}}{=} \{y\in\mathcal{Y} : (x,y)\in E\}$ is σ-compact for all $x\in\pi_{\mathcal{X}}(E)$, then there is a Borel measurable selection $T:\pi_{\mathcal{X}}(E)\to\pi_{\mathcal{Y}}(E)$. And from [12, Cor. 1], the measurable selection of the argmin operator can be obtained since A is a Borel set, as ϕ and Φ are Borel measurable, and the sub-level sets of the first variation of $\mathcal{E}$ are compact, so that

$$\mathcal{Y} = \bigcup_{n\in\mathbb{N}} \left\{y : \frac{\delta\mathcal{E}}{\delta\nu}(y) \le n\right\}, \text{ is } \sigma\text{-compact.}$$

In the sequel, we use it to define a transportation plan given by

$$\bar\gamma \overset{\text{def.}}{=} \gamma \llcorner (\mathcal{X}\times\mathcal{Y}\setminus A) + (\pi_{\mathcal{X}}, T\circ\pi_{\mathcal{X}})_\sharp \gamma \llcorner A.$$

Recalling that Φ is precisely the first variation of $\mathcal{J}$ evaluated at γ, we have

$$\begin{aligned} 0 \le \left\langle \frac{\delta\mathcal{J}}{\delta\gamma}(\gamma), \bar\gamma - \gamma\right\rangle &= \int_{\mathcal{X}\times\mathcal{Y}} \Phi(\bar x,\bar y,\nu)\mathrm{d}\bar\gamma - \int_{\mathcal{X}\times\mathcal{Y}} \Phi(x,y,\nu)\mathrm{d}\gamma \\ &= \int_A \underbrace{(\Phi(x,T(x),\nu) - \Phi(x,y,\nu))}_{<0}\,\mathrm{d}\gamma \le 0. \end{aligned}$$

This contradicts the fact that $\gamma(A) > 0$, and we conclude that γ is a Cournot-Nash equilibrium.

Conversely, suppose that γ is an equilibrium, from Def. (1.1) and it follows that

$$\int_{\mathcal{X}\times\mathcal{Y}} \phi(x)\mathrm{d}\gamma = \int_{\mathcal{X}\times\mathcal{Y}} \Phi(x,y,\nu)\mathrm{d}\gamma.$$

Hence, for any other admissible transportation plan $\bar{\gamma} \in \mathscr{P}_\mu(\mathcal{X} \times \mathcal{Y})$, it holds that

$$\int_{\mathcal{X}\times\mathcal{Y}} \Phi(x,y,\nu)\mathrm{d}\bar{\gamma} \geq \int_{\mathcal{X}\times\mathcal{Y}} \phi(x)\mathrm{d}\bar{\gamma} = \int_{\mathcal{X}} \phi(x)\mathrm{d}\mu = \int_{\mathcal{X}\times\mathcal{Y}} \Phi(x,y,\nu)\mathrm{d}\gamma.$$

From the fact that $\Phi(x,y,\nu) = \dfrac{\delta\mathcal{J}}{\delta\gamma}(\gamma)$, we conclude that γ is a critical point of $\mathcal{J}$.

Given $\gamma \in \Pi(\mu,\nu)$ that is a critical point of $\mathcal{J}$, hence is also a Cournot-Nash equilibrium, suppose in addition that ν is a distribution of finite social cost. By definition, μ-a.e. we have that $\phi(x) \leq \kappa(x)$ so that

$$\int_{\mathcal{X}\times\mathcal{Y}} \Phi(x,y,\nu)\mathrm{d}\gamma = \int_{\mathcal{X}} \phi(x)\mathrm{d}\mu \leq \int_{\mathcal{X}} \kappa(x)\mathrm{d}\mu < +\infty.$$

As any minimizer is a critical point, item (ii) follows. □

In the previous Theorem, the condition that the infimum is finite is non-trivial and should be verified for each problem. Imposing further conditions on $\mathcal{E}$, such as strict convexity, this can be verified as done in [9]. For the rest of this work, and specially for the proof of convergence of Nash to Cournot-Nash equilibria, we shall concentrate on a case where $\mathcal{E}$ is given as the sum of a linear and an interaction term, as in [9, 44] and Mazanti et al. That is, when $\mathcal{E}$ can be written as follows

$$\mathcal{E}(\nu) = \mathcal{L}(\nu) + \mathcal{H}(\nu,\nu), \text{ where } \mathcal{L}(\nu) = \int_{\mathcal{Y}} L\mathrm{d}\nu \text{ and } \mathcal{H}(\nu,\nu) = \int_{\mathcal{Y}\times\mathcal{Y}} H\mathrm{d}\nu\otimes\nu, \tag{25}$$

hence being the sum of an individual cost L and an interaction cost. In this case, the lifted energy from (24) becomes

$$\mathcal{J}(\gamma) \overset{\text{def.}}{=} \int_{\mathcal{X}\times\mathcal{Y}} c\mathrm{d}\gamma + \mathcal{L}(\nu) + \mathcal{H}(\nu,\nu) \tag{26}$$

and game with a continuum of players that we are interested is described by the cost

$$\Phi\colon \begin{cases} \mathcal{X}\times\mathcal{Y}\times\mathscr{P}(\mathcal{Y}) & \to \mathbb{R}_+ \cup \{+\infty\} \\ (x,y,\nu) & \mapsto c(x,y) + L(y) + 2\displaystyle\int_{\mathcal{Y}} H(y,y')\mathrm{d}\nu(y'). \end{cases} \tag{27}$$

In addition, we recall that we assume hypothesis (H1)-(H5) from the introduction. In particular, assumption (H1) that μ does not contain atoms is not restrictive, as discussed in Remark 3.4 bellow.

Remark 3.4. If μ has atoms, we can work in the lifted space

$$\mathcal{X}' = [0,1]\times\mathcal{X} \text{ and } \mu' \in \Pi(\mathcal{L}^1 \llcorner [0,1], \mu),$$

that is a coupling between the Lebesgue measure on the interval $[0,1]$ and μ. On the other hand, there is a map $T' : \mathcal{X}' \to \mathcal{X}$ such that $T'_\sharp\mu' = \mu$, simply given by the projection $T' = \pi_{\mathcal{X}}$. Then we can formulate a new game with c replaced by $c'(x',y) = c(\pi_{\mathcal{X}}(x'),y)$, which remains l.s.c. in the product space $\mathcal{X}'\times\mathcal{Y}$. This new game will then satisfy all hypothesis (H1)-(H5).

As the integral of l.s.c. functionals, both $\mathcal{L}$ and $\mathcal{H}$ are l.s.c. as functionals over $\mathscr{P}(\mathcal{Y})$, see for instance [43, Prop. 7.1]. Since the level sets of $\mathcal{L}$ are compact, we would be able to prove existence of minimizers for $\mathcal{J}$, were it not for the term $\mathcal{H}$ that can be $+\infty$, for instance if H diverges in the diagonal.

3.2. **On the finiteness of the infimum.** In the case of a potential of the form (27), we can characterize the cases where there the infimum is finite, and hence when we have existence, with a measure defined with the individual transportation cost c and the interaction energy H as follows: For $K \subset \mathcal{Y}$ compact, define

$$\operatorname{cap}_{c,H}(K) \overset{\text{def.}}{=} \left(\inf_{\varrho \in \mathscr{P}_{c,\mu}(K)} \int_{\mathcal{Y}\times\mathcal{Y}} H \mathrm{d}\varrho \otimes \varrho \right)^{-1}, \tag{28}$$

where $\mathscr{P}_{c,\mu}(K) \overset{\text{def.}}{=} \{\varrho \in \mathscr{P}(K) : \ \mathcal{W}_c(\mu,\varrho) < +\infty\}$. The capacity of an open set $U \subset \mathcal{Y}$ can then be defined through outer regularity

$$\operatorname{cap}_{c,H}(U) \overset{\text{def.}}{=} \sup\left\{ \operatorname{cap}_{c,H}(K) : \quad K \subset U \right\},$$

and for a general set A as the inf of the same quantity among all the open sets U containing A. This defines a monotone set function that can be used to characterize when the infimum of $\mathcal{J}$ is finite.

Lemma 3.5. *Under hypotheses (H2)–(H5), it holds that*

$$\inf \mathcal{J} < +\infty \iff \operatorname{cap}_{c,H}(\{L < +\infty\}) > 0.$$

Proof. Starting with the direct implication, suppose that there exists $\gamma \in \mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})$ such that $\mathcal{J}(\gamma) < +\infty$. In particular, letting ν denote the second marginal of γ, it follows that $\mathcal{W}_c(\mu,\nu) < +\infty$ and $\operatorname{supp}\nu \subset \{L < +\infty\}$. It then follows that

$$\operatorname{cap}_{c,H}(\{L < +\infty\}) \geq \mathcal{J}(\gamma)^{-1} > 0.$$

Conversely, if $\operatorname{cap}_{c,H}(\{L < +\infty\}) > 0$, there is some $N \in \mathbb{N}$ such that

$$\operatorname{cap}_{c,H}(\{L \leq N\}) > 0.$$

Hence there is a measure ϱ concentrated over the compact set $\{L \leq N\}$ such that $\mathcal{W}_c(\mu,\varrho) < +\infty$ and $\mathcal{H}(\varrho,\varrho) < +\infty$. Taking γ as an optimal transportation plan between μ and ϱ gives that $\mathcal{J}(\gamma) < +\infty$. □

The previous Lemma seems almost tautological, but in some particular cases there are strong results in the literature that characterize exactly which are the sets with positive capacity. In examples 3.6 and 3.6 we treat two models whose particular properties allow to verify the capacity criterion from Lemma 3.5.

Example 3.6. In the Lagrangian mean field game of Mazanti et.al. the interaction term is shown to be bounded by the individual cost, that is, there is a constant $C > 0$ such that for all $\nu \in \mathscr{P}(\mathcal{Y})$ it holds that $\mathcal{H}(\nu,\nu) \leq C(1 + 2\mathcal{L}(\nu))$, which trivializes the capacity condition since $\mathcal{L}$ is not identically $+\infty$.

Example 3.7. Consider now a simpler case where $\mathcal{X} = \mathcal{Y} = \mathbb{R}^d$,

$$c \in C_b(\mathbb{R}^d \times \mathbb{R}^d), \text{ and } H(y,\bar{y}) = |y-\bar{y}|^{-\alpha} \text{ for some } 0 < \alpha < d.$$

The condition that c is bounded implies that the set $\mathscr{P}_{c,\mu}(\mathbb{R}^d) = \mathscr{P}(\mathbb{R}^d)$ since the optimal transportation problem $\mathcal{W}_c(\mu,\varrho)$ is finite for any probability measure ϱ. This way, the capacity condition becomes

$$\operatorname{cap}_\alpha(\{L < +\infty\}) > 0,$$

where cap_s denotes the usual capacity, with $H(y,\bar{y}) = |y-\bar{y}|^{-\alpha}$. In this case, Frostman's Lemma, see [24, Chap. 4.3] and [41, Appendix B] or the original thesis of Frostman [26], gives a charaterization of sets with strictly positive α-capacity in $\mathbb{R}^d$. Indeed, for a general Borel set $A \subset \mathbb{R}^d$ it holds that

$$d_H(A) = \inf\{s \geq 0 : \ \operatorname{cap}_s(A) = 0\},$$

where $d_H(A)$ denotes the Haussdorff dimension of the set A.

It follows that, in order to satisfy the capacity condition, it suffices to verify that the set $\{L < +\infty\}$ is of dimension bigger than α.

Remark 3.8. Example 3.7 above motivates the characterization of sets with strictly positive $\mathrm{cap}_{c,H}$ for more general choices of c and H. As mentioned above the first difficulty is to choose a class of pairs (c, H) that do not make the infimum in the capacity $+\infty$. We have trivialized this question by considering c bounded, but it excludes the examples of Lagrangian Mean Field Games where $c(x, \sigma) = +\infty$ if $\sigma(0) \neq x$.

3.3. **Stability of the value function.** In this paragraph our primary goal is to show, assuming some additional hypothesis (H6), the following estimate

$$\left| \inf_{\gamma \in \mathscr{P}_{\mu_0}(\mathcal{X}\times\mathcal{Y})} \mathcal{J} - \inf_{\gamma \in \mathscr{P}_{\mu_1}(\mathcal{X}\times\mathcal{Y})} \mathcal{J} \right| \leq C W_1(\mu_0, \mu_1) \text{ for some } C > 0, \tag{29}$$

where $\mu_0, \mu_1 \in \mathscr{P}(\mathcal{X})$ are two distribution of agents and

$$W_1(\mu_0, \mu_1) \overset{\text{def.}}{=} \min_{\gamma \in \Pi(\mu_0, \mu_1)} \int_{\mathcal{X}\otimes\mathcal{X}} d_{\mathcal{X}}(x_0, x_1) \mathrm{d}\gamma,$$

is the 1-Wasserstein, or Kantorovitch-Rubinstein, distance, see [49].

The stability result (29) is of independent interest, in particular it gives some regularity for the value function in the Wasserstein topology and allows to estimate the value with arbitrary approximations of a given reference distribution μ, instead of for only empirical measures covered by our Γ-convergence results from Section 4. However, it is important to emphasize that, contrary to Γ-convergence, this result says nothing about the convergence of minimizers for the problems referent to a sequence μ_n converging to μ. Another potential application would be to the construction of ε-equilibria, which could be done with a suitable approximation of a given reference measure. However, this cannot be done for instance in an euclidean space with an empirical measure since we expect that

$$W_1(\mu_N, \mu) \approx N^{-1/d}$$

if $\mu \in \mathscr{P}(\mathbb{R}^d)$ and μ_N is a suitable sequence of empirical measures. Recalling that the cost minimized by each player behaves as $N \times \mathcal{J}(\gamma_N)$, this estimate does not give information on weather the minimization of $\mathcal{J}$ over $\mathscr{P}_{\mu_N}(\mathcal{X} \times \mathcal{Y})$ yield ε-Nash equilibria for the N-player game. But it might be useful in this direction with estimates of μ with better convergence rates.

To prove (29) we will exploit the *gluing method*, recently introduced in [38]. This method depends on the existence of a gluing operator as described in the following assumption:

(H6) There exists an operator $\mathcal{G} : \mathcal{X} \times \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ such that:

- $\mathcal{G}$ is consistent: for every $y \in \mathcal{Y}$ and $x \in \mathcal{X}$ it holds that $\mathcal{G}(x, x, y) = y$;
- there exists a positive constant $C > 0$ satisfying

$$\begin{aligned} c\left(x_1, \mathcal{G}(x_1, x_0, y)\right) &\leq c(x_0, y) + C d_{\mathcal{X}}(x_1, x_0), \\ L\left(\mathcal{G}(x_1, x_0, y)\right) &\leq L(y) + C d_{\mathcal{X}}(x_1, x_0), \\ H(\mathcal{G}(x_1, x_0, y), \mathcal{G}(\tilde{x}_1, \tilde{x}_0, \tilde{y})) &\leq H(y, \tilde{y}) + C\left(d_{\mathcal{X}}(x_1, x_0) + d_{\mathcal{X}}(\tilde{x}_1, \tilde{x}_0)\right). \end{aligned} \tag{30}$$

for any pairs $x_0, x_1 \in \mathcal{X}$ and $y, \tilde{y} \in \mathcal{Y}$.

Essentially, hypothesis (H6) says that there is an operator that given some player of type x_0 choosing play y, any other player of type x_1 can choose a strategy $\mathcal{G}(x_1, x_0, y)$ paying a perturbation, of order $d_{\mathcal{X}}(x_0, x_1)$, of the cost paid by the first player. With this assumption we can prove that

Lemma 3.9 (Gluing method)**.** *Let μ_0, μ_1 be probability measures in $\mathscr{P}(\mathcal{X})$ and $\gamma_0 \in \mathscr{P}_{\mu_0}(\mathcal{X} \times \mathcal{Y})$. Under the hypothesis (H6), there exists a measure $\gamma_1 \in \mathscr{P}_{\mu_1}(\mathcal{X} \times \mathcal{Y})$ such that*

$$\mathcal{J}(\gamma_1) \leq \mathcal{J}(\gamma_0) + 4C W_1(\mu_0, \mu_1),$$

where W_1 denotes the Kantorovitch-Rubinstein distance.

Proof. Given measures $\mu_0, \mu_1 \in \mathscr{P}(\mathcal{X})$ and let $\pi_{1,0} \in \Pi(\mu_1, \mu_0)$ be an optimal transportation plan between μ_0 and μ_1, *i.e.*

$$\int_{\mathcal{X}_1 \times \mathcal{X}_1} d_{\mathcal{X}}(x_1, x_0) \mathrm{d}\pi_{1,0}(x_1, x_0) = W_1(\mu_1, \mu_0),$$

where $\mathcal{X}_0$ and $\mathcal{X}_1$ are identical copies of the space $\mathcal{X}$.

Let $\Gamma \in \mathscr{P}(\mathcal{X}_1 \times \mathcal{X}_0 \times \mathcal{Y})$ denote the gluing of $\pi_{1,0}$ and γ_0, so that $(\pi_{\mathcal{X}_0,\mathcal{Y}})_\sharp \Gamma = \gamma_0$ and $(\pi_{\mathcal{X}_1,\mathcal{X}_0})_\sharp \Gamma = \pi_{1,0}$, see for instance Lemma 2.6 or Lemma 5.3.2 and Remark 5.3.3 of [3]. The measure γ_1 is then defined as $\gamma_1 := (\pi_{\mathcal{X}_1}, \mathcal{G})_\sharp \Gamma$. It follows from these definitions that

$$\begin{aligned}
\mathcal{J}(\gamma_0) &= \int_{\mathcal{X}_0 \times \mathcal{Y}} (c + L + H) \, \mathrm{d}\Gamma \otimes \Gamma \\
\mathcal{J}(\gamma_1) &= \int_{\mathcal{X}_1 \times \mathcal{Y}} (c + L + H) \, \mathrm{d}\left((\pi_{\mathcal{X}_1}, \mathcal{G})_\sharp \Gamma\right) \otimes \left((\pi_{\mathcal{X}_1}, \mathcal{G})_\sharp \Gamma\right).
\end{aligned}$$

Using the definition of the gluing operator from (H6), we get the following estimates

$$\begin{aligned}
\mathcal{J}(\gamma_1) &= \int_{\mathcal{X}_1 \times \mathcal{Y}} (c(x_1, \mathcal{G}(x_1, x_0, y)) + L(\mathcal{G}(x_1, x_0, y)) + \\
&\qquad H(\mathcal{G}(x_1, x_0, y), \mathcal{G}(\bar{x}_1, \bar{x}_0, \bar{y}))) \mathrm{d}\Gamma \otimes \Gamma \\
&\leq \int_{\mathcal{X}_0 \times \mathcal{Y}} (c + L + H) \, \mathrm{d}\Gamma \otimes \Gamma + C \int_{\mathcal{X}_1 \times \mathcal{X}_0 \times \mathcal{Y}} (3 d_{\mathcal{X}}(x_1, x_0) + d_{\mathcal{X}}(\tilde{x}_1, \tilde{x}_0)) \mathrm{d}\Gamma \otimes \Gamma \\
&= \mathcal{J}(\gamma_0) + 4C \int_{\mathcal{X}_1 \times \mathcal{X}_0} d_{\mathcal{X}}(x_1, x_0) \mathrm{d}\pi_{1,0} \\
&= \mathcal{J}(\gamma_0) + 4C W_1(\mu_0, \mu_1).
\end{aligned}$$

The result follows. □

The previous Lemma 3.9 will also be useful in the proof of Γ convergence in the open closed loop formulation. For now, we use it to prove the following:

Theorem 3.10. *Under the hypothesis (H6), the stability inequality* (29) *for the value function holds.*

Proof. Let $\gamma_0 \in \mathscr{P}_{\mu_0}(\mathcal{X} \times \mathcal{Y})$ optimal, so that

$$\mathcal{J}(\gamma_0) = \min_{\mathscr{P}_{\mu_0}(\mathcal{X} \times \mathcal{Y})} \mathcal{J}.$$

So let γ_1 be the measure obtained from the gluing method in Lemma 3.9. It then holds that

$$\inf_{\mathscr{P}_{\mu_1}(\mathcal{X} \times \mathcal{Y})} \mathcal{J} - \inf_{\mathscr{P}_{\mu_0}(\mathcal{X} \times \mathcal{Y})} \mathcal{J} \leq \mathcal{J}(\gamma_1) - \mathcal{J}(\gamma_0) \leq 4C W_1(\mu_0, \mu_1),$$

where C is the constant from (H6). Changing the roles of μ_0 and μ_1, we conclude. □

Example 3.11 (Back to example 3.6)**.** Let us now give further details for the Lagrangian MFG discussed in example 3.6. We consider a model where a population of agents tries to read a target set in minimal time under pair-wise interactions.

For simplicity, let $\Omega \subset \mathbb{R}^d$ be a convex set, and let $\Gamma \subset \Omega$ be the target set of the players. In this case $\mathcal{X} = \Omega$ and $\mathcal{Y} = C(\mathbb{R}_+; \Omega)$, the continuous functions with values in Ω. For $\sigma \in \mathcal{Y}$ we set

$$\tau(\sigma) \overset{\text{def.}}{=} \inf\{t \geq 0 : \sigma(t) \in \Gamma\},$$

the minimal time to reach the target and

$$c(x_0, \sigma) = \begin{cases} 0, & \text{if } \sigma(0) = x_0, \\ +\infty, & \text{otherwise.} \end{cases}$$

The individual and interaction energies are given by

$$L(\sigma) \stackrel{\text{def.}}{=} \int_0^{\tau(\sigma)} \ell(t, \sigma(t), \dot\sigma(t))\mathrm{d}t + \Psi(\sigma_\tau),$$
$$H(\sigma, \bar\sigma) \stackrel{\text{def.}}{=} \int_0^{\tau(\sigma)\wedge\tau(\bar\sigma)} h(t, \sigma(t), \dot\sigma(t), \bar\sigma(t), \dot{\bar\sigma}(t))\mathrm{d}t.$$

For simplicity, we assume that ℓ and h are bounded non-negative functions, that σ remains constant after reaching Γ for the first time and that if $\dot\sigma(t) = 0$, then $\ell(t, \sigma(t), \dot\sigma(t)) = h(t, \sigma(t), \dot\sigma(t), \bar\sigma(t), \dot{\bar\sigma}(t)) = 0$.

In order to have a small perturbation of the energies, the easiest way is to preserve the stopping time, hence given σ such that $\sigma(0) = x_0$ we search for a curve of the form

$$\sigma_{x_1}(t) \stackrel{\text{def.}}{=} \begin{cases} \left(1 - \frac{t}{t_0}\right) x_1 + \frac{t}{t_0}\sigma(t_0), & \text{if } t \in [0, t_0], \\ \sigma(t), & \text{otherwise.} \end{cases}$$

Therefore, choosing $t_0 \le \min\{\tau_\sigma, |x_0 - x_1|\}$ we obtain that

$$L(\sigma_{x_1}(t)) \le L(\sigma) + \int_0^{t_0} \ell(t, \sigma_{x_1}(t), \dot\sigma_{x_1}(t))\mathrm{d}t \le L(\sigma) + C|x_0 - x_1|.$$

An analogous reasoning for H gives the required result.

4. Convergence of Nash to Cournot-Nash

In this section we prove our convergence results of Nash to Cournot-Nash equilibria. For readability, we perform the analysis in the two cases separately, even if at the cost of some repetition. This way both results can be read independently. We start by recalling the definitions of each formulation described in the introduction and discuss then in more details as well.

First let us recall the definition of Nash-equilibria and introduce some notation.

Definition 4.1 (Nash equilibrium in N-player game)**.** An N-player game in pure strategies is a tuple $(g_i, S_i)_{i=1}^N$ where S_i denotes the space of admissible plays for player i and g_i is a function

$$g_i : S_i \times S_{-i} \ni (x_i, x_{-i}) \mapsto g_i(x_i, x_{-i}) \in \mathbb{R} \text{ where } S_{-i} \stackrel{\text{def.}}{=} \prod_{j\neq i} S_j.$$

Given an admissible profile of strategies $(x_j)_{j=1}^N$, x_{-i} corresponds to the tuple of strategies deprived of x_i and the quantity $g_i(x_i, x_{-i})$ represents the cost of player i choosing x_i given that the remaining players choose x_{-i}.

A game in mixed strategies, or mixed plays, is a tuple $(g_i, \mathscr{P}(S_i))_{i=1}^N$, such that

$$g_i(\nu_i, \nu_{-i}) \stackrel{\text{def.}}{=} \int_{S_i\times S_{-i}} g_i(x_i, x_{-i})\mathrm{d}\nu_i \otimes \nu_{-i}(x_i, x_{-i}),$$

where $\nu_{-i} \stackrel{\text{def.}}{=} \nu_1 \otimes \cdots \otimes \nu_{i-1} \otimes \nu_{i+1} \otimes \nu_N$.

Let us also recall the setting of our games. Given an i.i.d. sample of agents $(X_i)_{i\in\mathbb{N}}$ with common law $\mu \in \mathscr{P}(\mathcal{X})$, where the first N elements represent the type of the agents in our N-player game. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ induced by the sample $(X_i)_{i\in\mathbb{N}}$, so that $\Omega = \mathcal{X}^{\otimes\mathbb{N}}$ represents all the possible realizations of this sampling, $\mathcal{F}$ is the σ-algebra generated by the random variables X_i and, $\mathbb{P} = \mu^{\otimes\mathbb{N}}$.

4.1. **Convergence in the open loop formulation.**

4.1.1. *Potential struction in open loop.* In the open loop formulation, as each player chooses a strategy before having the knowledge of the realization of the sample, the type of player i is better described by the random variable X_i and an admissible strategy must be given by a measurable family $(\nu^x)_{x\in\mathcal{X}} \subset \mathscr{P}(\mathcal{Y})$, being uniquely described with a random probability measure, see Section 2.2 or [19, 32]. In other words, instead of choosing a deterministic strategy, given a state x a player chooses a distribution of strategies ν^x. The criterion that each player seeks to minimize is then

$$\min_{\boldsymbol{\nu}_i\in\mathscr{P}_\Omega(\mathcal{Y})} \mathcal{J}_{\Omega,i}\left(\boldsymbol{\nu}_i,\boldsymbol{\nu}_{-i}\right) \overset{\text{def.}}{=} \mathbb{E}_{\mathbb{P}}\left[\int_{\mathcal{Y}} c(X_i,y)\mathrm{d}\nu_i^{X_i} + \mathcal{L}(\nu_i^{X_i}) + \frac{2}{N}\sum_{i\neq j}\mathcal{H}\left(\nu_i^{X_i},\nu_j^{X_j}\right)\right], \tag{31}$$

where $\boldsymbol{\nu}_i = \nu_i^{X_i}$ for some measurable map $(\nu_i^x)_{x\in\mathcal{X}}$.

A profile $(\boldsymbol{\nu}_i)_{i=1}^N$ is pure if each $\boldsymbol{\nu}_i$ is a Dirac delta with full probability and can then be described with a map as measures of the form $\boldsymbol{\nu}_i = \delta_{T_i(X_i)}$. The formulation in pure strategies can then be expressed as

$$\min_{T_i} \mathcal{J}_{\Omega,i}(T_i,T_{-i}) = \mathbb{E}_{\mathbb{P}}\left[c(X_i,T_i(X_i)) + L(T_i(X_i)) + \frac{2}{N}\sum_{i\neq j}\mathcal{H}(T_i(X_i),T_j(X_j))\right]. \tag{32}$$

As introduced in Section 2.2, we let $\boldsymbol{\mu}_N \in \mathscr{P}_\Omega(\mathcal{X})$ be the random measure obtained via the sample of random variables $\boldsymbol{\mu}_N \overset{\text{def.}}{=} \frac{1}{N}\sum_{i=1}^N \delta_{X_i}$, and we define the space of random transportation plans

$$\mathscr{P}_{\Omega,\boldsymbol{\mu}_N}(\mathcal{X}\times\mathcal{Y}) \overset{\text{def.}}{=} \left\{\boldsymbol{\gamma}_N = \boldsymbol{\mu}_N\otimes\nu^x = \frac{1}{N}\sum_{i=1}^N \delta_{X_i}\otimes\nu^{X_i} :\ (\nu^x)_{x\in\mathcal{X}} \text{ is measurable}\right\}.$$

where we recall the definition of measurable family of measures from Def. 2.4. The potential function in open loop formulation is defined as the average of the costs of each player, assuming the form

$$\mathcal{J}_{\Omega,N}(\boldsymbol{\gamma}_N) \overset{\text{def.}}{=} \begin{cases} \mathbb{E}_{\mathbb{P}}\left[\displaystyle\int_{\mathcal{X}\times\mathcal{Y}} c\mathrm{d}\boldsymbol{\gamma}_N + \frac{1}{N}\sum_{i=1}^N \mathcal{L}\left(\nu^{X_i}\right) \right. \\ \qquad\qquad \left. + \dfrac{1}{N^2}\displaystyle\sum_{i\neq j}\mathcal{H}\left(\nu^{X_i},\nu^{X_j}\right)\right], & \text{if } \boldsymbol{\gamma}_N \in \mathscr{P}_{\Omega,\boldsymbol{\mu}_N}(\mathcal{X}\times\mathcal{Y}), \\ +\infty, & \text{otherwise.} \end{cases} \tag{33}$$

Observe that, by means of the disintegration Theorem (2.5) there is a canonical bijection between the set of random measures $\mathscr{P}_{\Omega,\mu}(\mathcal{X})$ and the set of symmetric strategy profiles, obtained through the disintegration theorem, so that this lifted energy is a suitable candidate to describe minimizers.

In addition, since c, L and H are l.s.c. and L has compact level sets, the potential function 33 admits minimizers, which we shall prove to yield Nash equilibria for the corresponding game.

Proposition 4.2. *Let*

$$\boldsymbol{\gamma}_N = \frac{1}{N}\sum_{i=1}^N \delta_{X_i}\otimes\nu_i^{X_i} \in \operatorname{argmin}\mathcal{J}_{\Omega,N}$$

then $\left(\nu_i^{X_i}\right)_{i=1}^N$ *induce Nash equilibria for the game* (31).

Proof. Let $\boldsymbol{\nu}_i = \nu_i^{X_i}$ and consider a deviation, so that player i chooses $\bar{\boldsymbol{\nu}}_i$ instead. Consider the new random transportation plan

$$\bar{\boldsymbol{\gamma}}_N \overset{\text{def.}}{=} \frac{1}{N}\sum_{j\neq i}\delta_{X_j}\otimes\boldsymbol{\nu}_j + \frac{1}{N}\delta_{X_i}\otimes\bar{\boldsymbol{\nu}}_i.$$

First notice that from the symmetry of H we have

$$\begin{aligned}\sum_{j\neq k}H(y_j,y_k) &= \sum_{j\neq k, j\neq i, k\neq i}H(y_j,y_k) + \sum_{k\neq i}H(y_i,y_k) + \sum_{j\neq i}H(y_j,y_i)\\ &= \sum_{j\neq k, j\neq i, k\neq i}H(y_j,y_k) + 2\sum_{j\neq i}H(y_j,y_i),\end{aligned}$$

so the minimality of $\boldsymbol{\gamma}_N$ gives

$$\begin{aligned}\mathcal{J}_{\Omega,N}(\boldsymbol{\gamma}_N) &= \frac{1}{N}\left(\mathbb{E}\left[\sum_{j\neq i}\int_{\mathcal{Y}}[c(x_j,y_j)+L(y_j)]\mathrm{d}\boldsymbol{\nu}_j + \frac{1}{N}\sum_{j\neq k, j,k\neq i}\mathcal{H}(\boldsymbol{\nu}_j,\boldsymbol{\nu}_k)\right] + \mathcal{J}_{\Omega,i}(\boldsymbol{\nu}_i,\boldsymbol{\nu}_{-i})\right)\\ &\leq \frac{1}{N}\left(\mathbb{E}\left[\sum_{j\neq i}\int_{\mathcal{Y}}[c(x_j,y_j)+L(y_j)]\mathrm{d}\boldsymbol{\nu}_j + \frac{1}{N}\sum_{j\neq k, j,k\neq i}\mathcal{H}(\boldsymbol{\nu}_j,\boldsymbol{\nu}_k)\right] + \mathcal{J}_{\Omega,i}(\bar{\boldsymbol{\nu}}_i,\boldsymbol{\nu}_{-i})\right)\\ &= \mathcal{J}_{\Omega,N}(\bar{\boldsymbol{\gamma}}_N).\end{aligned}$$

Canceling out the repeated terms we obtain that $\mathcal{J}_{\Omega,i}(\boldsymbol{\nu}_i,\boldsymbol{\nu}_{-i}) \leq \mathcal{J}_{\Omega,i}(\bar{\boldsymbol{\nu}}_i,\boldsymbol{\nu}_{-i})$, meaning that the profile $(\boldsymbol{\nu}_1,\dots,\boldsymbol{\nu}_N)$ is a Nash equilibria. □

4.1.2. Γ*-convergence for the open loop formulation.* The Γ convergence result relies on the characterization of the cluster points of random measures in the set $\mathscr{P}_{\Omega,\boldsymbol{\mu}_N}(\mathcal{X})$. Indeed, since the stochasticity of this class of measures is contained in the first marginal given by the empirical measure of an i.i.d. sample, we can expect that the limit will be a non-random measure. This is proved in the following

Lemma 4.3. *Let $\boldsymbol{\mu}_N$ be a sequence of empirical measures of an i.i.d. sample of law μ. Let $\boldsymbol{\gamma}_N$ be a sequence of random measures such that $\boldsymbol{\gamma}_N \in \mathscr{P}_{\Omega,\boldsymbol{\mu}_N}(\mathcal{X}\times\mathcal{Y})$ for all $N\in\mathbb{N}$ and converging in the narrow convergence of probability measures to a random measure $\boldsymbol{\gamma}$. Then $\boldsymbol{\gamma}$ is a deterministic measure in the sense that there is $\gamma\in\mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})$ such that $\boldsymbol{\gamma}(\omega)=\gamma$ almost surely.*

Remark 4.4. The major difficulty of the following proof comes from the fact that the conditional expectation is not continuous w.r.t. weak convergence in general. In order words, if a sequence of measures $(\gamma_N)_{N\in\mathbb{N}}$ converging weakly to γ has the following disintegration representation $\gamma_N = \mu\otimes\nu_N^x$ and $\gamma=\mu\otimes\nu^x$, it does not hold in general that $\nu_N^x \xrightarrow[N\to\infty]{} \nu^x$, not even for a.e. x.

Proof. If $\boldsymbol{\gamma}_N$ converges weakly to $\boldsymbol{\gamma}$ it is a priori just a random probability measure in $\mathscr{P}_\Omega(\mathcal{X}\times\mathcal{Y})$. Hence, we first need to show that $\boldsymbol{\gamma}(\omega)\in\mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})$ with probability 1. For any $f\in C_b(\mathcal{X})$, we have

$$\int_{\mathcal{X}} f(x)\mathrm{d}(\pi_{\mathcal{X}})_\sharp\boldsymbol{\gamma}(\omega) = \lim_{N\to\infty}\int_{\mathcal{X}} f\circ\pi_{\mathcal{X}}\mathrm{d}\boldsymbol{\gamma}_N(\omega) = \lim_{N\to\infty}\int_{\mathcal{X}} f\mathrm{d}\boldsymbol{\mu}_N(\omega) = \int_{\mathcal{X}} f\mathrm{d}\mu,$$

where the last limit is true almost surely from the Glivenko-Cantelli law of large numbers. As a consequence, by disintegration we have the following representation

$$\boldsymbol{\gamma}_N = \boldsymbol{\mu}_N\otimes\nu_N^x, \quad \boldsymbol{\gamma} = \mu\otimes\boldsymbol{\nu}^x,$$

where $(\nu_N^x)_{x\in\mathcal{X}}\subset\mathscr{P}(\mathcal{Y})$ is a sequence of measurable maps of deterministic measures and $(\boldsymbol{\nu}^y)_{y\in Y}$ is a family of random measures in $\mathscr{P}_\Omega(X)$. Notice that while the stochasticity of $\boldsymbol{\gamma}_N$ is concentrated in the $\mathcal{X}$-marginal, we cannot say for now that the same is true for

$\boldsymbol{\gamma}$ and our goal is precisely to show that the disintegration family $(\boldsymbol{\nu}^x)_{x\in\mathcal{X}}$ is a family of non-random probability measures.

Recall the definition of expectation measure in (20), for any $\varphi \in C_b(\mathcal{X}\times\mathcal{Y})$, we have

$$\int_{\mathcal{X}\times\mathcal{Y}} \varphi \mathrm{d}\mathbb{E}\boldsymbol{\gamma}_N = \mathbb{E}\left[\int_{\mathcal{X}\times\mathcal{Y}} \varphi \mathrm{d}\boldsymbol{\gamma}_N\right] \xrightarrow[N\to\infty]{} \mathbb{E}\left[\int_{\mathcal{X}\times\mathcal{Y}} \varphi \mathrm{d}\boldsymbol{\gamma}\right] = \int_{\mathcal{X}\times\mathcal{Y}} \varphi \mathrm{d}\mathbb{E}\boldsymbol{\gamma},$$

so that $\mathbb{E}\boldsymbol{\gamma}_N \xrightarrow[N\to\infty]{} \mathbb{E}\boldsymbol{\gamma}$. In the sequel, we check that $\mathbb{E}\boldsymbol{\gamma}_N = \mu\otimes\nu_N^x$. Indeed, still using duality, for $\varphi \in C_b(\mathcal{X}\times\mathcal{Y})$ we have that

$$\int_{\mathcal{X}\times\mathcal{Y}} \varphi \mathrm{d}\mathbb{E}\boldsymbol{\gamma}_N = \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\int_{\mathcal{Y}} \varphi(X_i,y)\mathrm{d}\nu_N^{X_i}\right] = \int_{\mathcal{X}\times\mathcal{Y}} \varphi \mathrm{d}(\mu\otimes\nu_N^x).$$

To finish the proof it suffices to show that for any real valued, bounded and $(\Omega,\mathcal{F},\mathbb{P})$-adapted random variable Θ and $\varphi \in C_b(\mathcal{X}\times\mathcal{Y})$, it holds that

$$\Delta_{N,\Theta} \overset{\text{def.}}{=} \left|\mathbb{E}\left[\Theta\int_{\mathcal{X}\times\mathcal{Y}} \varphi \mathrm{d}\boldsymbol{\gamma}_N\right] - \mathbb{E}\left[\Theta\right]\int_{\mathcal{X}\times\mathcal{Y}} \varphi \mathrm{d}\mathbb{E}\boldsymbol{\gamma}_N\right| \xrightarrow[N\to\infty]{} 0, \tag{34}$$

since then we will have that

$$\mathbb{E}\left[\Theta\left(\int_{\mathcal{X}\times\mathcal{Y}} \varphi \mathrm{d}(\boldsymbol{\gamma}-\mathbb{E}\boldsymbol{\gamma})\right)\right] = 0$$

for any bounded random variable Θ, meaning that $\boldsymbol{\gamma} = \mathbb{E}\boldsymbol{\gamma}$ almost surely.

For this, we will use *Hoeffding's inequality*, which states that if $Z_1,\dots,Z_N$ are i.i.d. real variables such that $a \le Z_i \le b$ for all i almost surely, then

$$\mathbb{P}\left(\left|\frac{1}{N}\sum_{i=1}^N Z_i - \mathbb{E}[Z_1]\right| \ge \varepsilon\right) \le 2\exp\left(-\frac{2N\varepsilon^2}{(b-a)^2}\right). \tag{35}$$

Notice that we can rewrite

$$\int_{\mathcal{X}\times\mathcal{Y}} \varphi \mathrm{d}\boldsymbol{\gamma}_N = \frac{1}{N}\sum_{i=1}^N \tilde{\varphi}_{i,N} \text{ where } \tilde{\varphi}_{i,N} \overset{\text{def.}}{=} \int_{\mathcal{Y}} \varphi(X_i,y)\mathrm{d}\nu_N^{X_i},$$

so that $(\tilde{\varphi}_{i,N})_{i=1}^N$ are i.i.d., $|\tilde{\varphi}_{i,N}| \le \|\varphi\|_{L^\infty}$ and

$$\mathbb{E}\tilde{\varphi}_{1,N} = \int_{\mathcal{X}\times\mathcal{Y}} \varphi \mathrm{d}\mu\otimes\nu_N^x = \int_{\mathcal{X}\times\mathcal{Y}} \varphi \mathrm{d}\mathbb{E}\boldsymbol{\gamma}_N.$$

So setting

$$A_\varepsilon \overset{\text{def.}}{=} \left\{\left|\frac{1}{N}\sum_{i=1}^N \tilde{\varphi}_{i,N} - \mathbb{E}\tilde{\varphi}_{1,N}\right| \ge \varepsilon\right\},$$

we can use Hoeffding's inequality to bound the LHS of (34)

$$\begin{aligned}
\Delta_{N,\Theta} &\le \mathbb{E}\left[|\Theta|\left|\frac{1}{N}\sum_{i=1}^N \tilde{\varphi}_{i,N} - \mathbb{E}\tilde{\varphi}_{1,N}\right|\right] \\
&\le \|\Theta\|_{L^\infty}\int_{A_\varepsilon}\left|\frac{1}{N}\sum_{i=1}^N \tilde{\varphi}_{i,N} - \mathbb{E}\tilde{\varphi}_{1,N}\right|\mathrm{d}\mathbb{P} + \|\Theta\|_{L^\infty}\varepsilon \\
&\le 2\|\Theta\|_{L^\infty}\|\varphi\|_{L^\infty}\mathbb{P}(A_\varepsilon) + \|\Theta\|_{L^\infty}\varepsilon \\
&\le 4\|\Theta\|_{L^\infty}\|\varphi\|_{L^\infty}\exp\left(-\frac{N\varepsilon^2}{2\|\varphi\|_{L^\infty}^2}\right) + \|\Theta\|_{L^\infty}\varepsilon
\end{aligned}$$

Choosing $\varepsilon = N^{-1/3}$, we get that $\Delta_{N,\Theta} \xrightarrow[N\to\infty]{} 0$. We conclude that $\boldsymbol{\gamma} = \mathbb{E}\boldsymbol{\gamma}$. □

Remark 4.5. In fact we have shown that $\boldsymbol{\gamma}_N$ has a subsequence converging to γ in the much stronger topology of narrow converge $\mathbb{P}$ almost surely.

The previous Lemma is the crucial observation that allows the passage of the limit of a sequence of stochastic variational problems to a deterministic one as we shall see in the following Γ-convergence result.

Theorem 4.6. *Given an i.i.d. sample $(X_i)_{i\in\mathbb{N}}$ with law μ, let $\boldsymbol{\mu}_N \in \mathscr{P}_\Omega(\mathcal{X})$ the associated sequence of empirical random measures. Let $\mathcal{J}_{\Omega,N}$ be the sequence of potential functionals defined in* (33)*, then it holds that*

$$\mathcal{J}_{\Omega,N} \xrightarrow[N\to\infty]{\Gamma} \mathcal{J}_\Omega(\boldsymbol{\gamma}) \overset{\text{def.}}{=} \begin{cases} \mathcal{J}(\gamma), & \text{if } \boldsymbol{\gamma} = \gamma \in \mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y}), \\ +\infty, & \text{otherwise,} \end{cases}$$

where the Γ-convergence is in $\mathscr{P}_\Omega(\mathcal{X}\times\mathcal{Y})$ equipped with the narrow topology of random probability measures.

Proof. Starting with Γ-lim inf, consider a sequence $(\boldsymbol{\gamma}_N)_{N\in\mathbb{N}}$ converging to $\boldsymbol{\gamma}$ in the narrow topology of random measures. From Lemma 4.3, it follows that $\boldsymbol{\gamma}$ is actually a non-random measure $\gamma \in \mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})$. Without loss of generality we assume that

$$\liminf_{N\to\infty} \mathcal{J}_{\Omega,N}(\boldsymbol{\gamma}_N) < \infty,$$

otherwise there is nothing to prove. Then, up to taking a subsequence attaining the lim inf, one can assume that $\mathcal{J}_{\Omega,N}(\boldsymbol{\gamma}_N) \le C$ for all $N \in \mathbb{N}$, so in particular $\boldsymbol{\gamma}_N \in \mathscr{P}_{\Omega,\mu_N}(\mathcal{X}\times\mathcal{Y})$.

For an arbitrary $M > 0$, define the truncated interaction energy as

$$\mathcal{H}^M(\nu,\nu) \overset{\text{def.}}{=} \int H^M \mathrm{d}\nu\otimes\nu, \text{ where } H^M \overset{\text{def.}}{=} H \wedge M$$

and the truncated total energies $\mathcal{J}^M$ and $\mathcal{J}^M_{\Omega,N}$ as in (26) and (33) by replacing $\mathcal{H}$ with $\mathcal{H}^M$. Then it follows from Fubini's Theorem that

$$\begin{aligned}
\mathcal{J}^M_{\Omega,N}(\boldsymbol{\gamma}_N) &= \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \int_{\mathcal{Y}} c(X_i,y) + L(y)\mathrm{d}\nu_N^{X_i} + \frac{1}{N^2}\sum_{i\neq j}\int_{\mathcal{Y}\times\mathcal{Y}} \mathcal{H}^M(\nu_N^{X_i},\nu_N^{X_j})\right] \\
&= \mathbb{E}\left[\int_{\mathcal{X}\times\mathcal{Y}} c + L\mathrm{d}\boldsymbol{\gamma}_N + \int_{\mathcal{Y}\times\mathcal{Y}} H^M \mathrm{d}\boldsymbol{\gamma}_N\otimes\boldsymbol{\gamma}_N\right] - \frac{1}{N^2}\sum_{i=1}^N \mathbb{E}\left[\mathcal{H}^M\left(\nu_N^{X_i},\nu_N^{X_j}\right)\right] \\
&= \mathcal{J}^M\left(\mathbb{E}\boldsymbol{\gamma}_N\right) - \frac{1}{N^2}\sum_{i=1}^N \underbrace{\mathbb{E}\left[\mathcal{H}^M(\nu_N^{X_i},\nu_N^{X_i})\right]}_{\le M} \ge \mathcal{J}^M\left(\mathbb{E}\boldsymbol{\gamma}_N\right) - \frac{M}{N},
\end{aligned}$$

where the last inequality was obtained from the fact that $\mathcal{H}^M$ is bounded by M. For any fixed $M > 0$, the sum on the right-hand side above vanishes as $N \to \infty$ and hence since $\mathbb{E}\boldsymbol{\gamma}_N \xrightarrow[N\to\infty]{} \gamma$, the lower semi-continuity of $\mathcal{J}$ gives that

$$\liminf_{N\to\infty} \mathcal{J}_{\Omega,N}(\boldsymbol{\gamma}_N) \ge \liminf_{N\to\infty} \mathcal{J}^M\left(\mathbb{E}\boldsymbol{\gamma}_N\right) \ge \mathcal{J}^M(\gamma).$$

Noticing that from the monotone convergence theorem $\mathcal{H}(\nu,\nu) = \sup_{M>0} \mathcal{H}^M(\nu,\nu)$, we get

$$\liminf_{N\to\infty} \mathcal{J}_{\Omega,N}(\boldsymbol{\gamma}_N) \ge \sup_{M>0} \mathcal{J}^M(\gamma) = \mathcal{J}(\gamma),$$

and the result follows.

To prove the Γ-limsup it suffices to construct recovery sequences only for non-random transportation plans $\gamma \in \mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})$. For any such measure, consider its disintegration representation as $\gamma = \mu\otimes\nu^x$, and define a recovery sequence as

$$\boldsymbol{\gamma}_N \overset{\text{def.}}{=} \boldsymbol{\mu}_N \otimes \nu^x,$$

where $(\boldsymbol{\mu}_N)_{N\in\mathbb{N}}$ is the family of empirical random measures built from the i.i.d. sample $(X_i)_{i\in\mathbb{N}}$ of law μ. Let us show that $\boldsymbol{\gamma}_N \xrightarrow[N\to\infty]{} \gamma$. We know from Lemma 4.3 that for any

cluster point $\tilde{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}_N$ it holds that $\tilde{\boldsymbol{\gamma}}$ is a deterministic measure, so for any convergent subsequence we have

$$\boldsymbol{\gamma}_{N_k} \xrightarrow[k\to\infty]{} \tilde{\boldsymbol{\gamma}} = \lim_{N\to\infty} \mathbb{E}\boldsymbol{\gamma}_N = \gamma,$$

so that the whole sequence must converge to γ.

Next, for each $N \in \mathbb{N}$, a simple computation yields

$$\begin{aligned}
\mathcal{J}_{\Omega,N}(\boldsymbol{\gamma}_N) =& \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\left(\int_{\mathcal{Y}} c(X_i,y)\mathrm{d}\nu^{X_i}(y) + \int_{\mathcal{Y}} L(y)\mathrm{d}\nu^{X_i}(y)\right) + \frac{1}{N^2}\sum_{i\neq j}\int H\mathrm{d}\nu^{X_i}\otimes\nu^{X_j}\right] \\
=& \frac{1}{N}\sum_{i=1}^{N}\int_{\mathcal{X}}\left(\int_{\mathcal{Y}} c(x_i,y)\mathrm{d}\nu^{x_i}(y) + \int_{\mathcal{Y}} L(y)\mathrm{d}\nu^{x_i}(y)\right)\mathrm{d}\mu(x_i) \\
&+ \frac{1}{N^2}\sum_{i\neq j}\int_{\mathcal{X}\times\mathcal{X}}\left(\int_{\mathcal{Y}\times\mathcal{Y}} H\mathrm{d}\nu^{x_i}\otimes\nu^{x_j}\right)\mathrm{d}\mu\otimes\mu(x_i,x_j) \\
=& \frac{1}{N}\sum_{i=1}^{N}\int_{\mathcal{X}\times\mathcal{Y}}(c+L)\mathrm{d}\gamma + \frac{1}{N^2}\sum_{i\neq j}\int H\mathrm{d}\nu\otimes\nu \\
=& \mathcal{J}(\gamma) - \frac{1}{N}\int_{\mathcal{Y}\times\mathcal{Y}} H\mathrm{d}\nu\otimes\nu \leq \mathcal{J}(\gamma).
\end{aligned}$$

Taking the lim sup as $N\to\infty$, the result follows. □

Now we use the properties of Γ convergence along with Prokhorov's compactness Theorem for random measures to show that cluster points of equilibria for the N-player game are Cournot-Nash equilibria in the sense of Definition 1.1.

Theorem 4.7. *Assume that* $\inf_{\mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})} \mathcal{J} < \infty$, *then if* $(\boldsymbol{\gamma}_N)_{N\in\mathbb{N}}$ *is a sequence of minimizers of* $\mathcal{J}_{\Omega,N}$, *then there exists a subsequence such that*

$$\boldsymbol{\gamma}_{N_k} \xrightarrow[N_k\to\infty]{} \gamma \in \mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y}),$$

in the narrow topology of $\mathscr{P}_\Omega(\mathcal{X}\times\mathcal{Y})$, *and in addition* γ *is a Cournot-Nash equilibrium in the sense of Definition 1.1.*

Assuming in addition that $H \in C_b(\mathcal{Y}\times\mathcal{Y})$, *for any sequence of Nash equilibria* $(\boldsymbol{\gamma}_N)_{N\in\mathbb{N}}$ *from game* (31), *that is*

$$\boldsymbol{\gamma}_N \overset{\text{def.}}{=} \frac{1}{N}\sum_{i=1}^{N}\delta_{X_i}\otimes\nu^x_{i,N} \in \mathscr{P}_{\Omega,\boldsymbol{\mu}_N}(\mathcal{X}\times\mathcal{Y}), \tag{36}$$

converging to γ *in the narrow topology of* $\mathscr{P}_\Omega(\mathcal{X}\times\mathcal{Y})$, *it holds that* $\gamma\in\mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})$, *and it is a Cournot-Nash equilibrium in the sense of Definition 1.1.*

Proof. To prove the first assertion, we know from the properties of Γ-convergence that

$$\inf_{\mathscr{P}_{\Omega,\boldsymbol{\mu}_N}(\mathcal{X}\times\mathcal{Y})} \mathcal{J}_{\Omega,N} \xrightarrow[N\to\infty]{} \inf_{\mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})} \mathcal{J} \overset{\text{def.}}{=} C < +\infty. \tag{37}$$

Hence, since the functionals $\mathcal{J}_{\Omega,N}$ are l.s.c. with compact level sets, for each $N\in\mathbb{N}$ it admits a minimizer $\boldsymbol{\gamma}_N$. So if this sequence has a cluster point, then it must also minimize $\mathcal{J}$, from Theorem 4.6. Hence, to finish the proof, it suffices to obtain such cluster point. This will be done with the version of Prokhorov's Theorem for random measures, see Theorem 2.9, which states that a sequence of random measures is sequentially compact in the narrow topology if and only if it is tight.

As $\boldsymbol{\mu}_N \xrightarrow[N\to\infty]{} \mu$ in the narrow topology of random measures it is a tight family, from the Prokhorov's Theorem, so for any $\varepsilon>0$ there exists a compact set $K_{\mathcal{X},\varepsilon}\subset\mathcal{X}$ such that

$$\mathbb{E}\left[\boldsymbol{\mu}_N(\mathcal{X}\setminus K_{\mathcal{X},\varepsilon})\right] < \frac{\varepsilon}{2}.$$

From (37) we get that, for N large enough,

$$\mathbb{E}\left[\int_{\mathcal{Y}} L\mathrm{d}\boldsymbol{\nu}_N\right] \leq 2C,$$

so that, for some $\varepsilon > 0$ we obtain from Markov's inequality that

$$\mathbb{E}\left[\boldsymbol{\nu}_N\left(\left\{L \leq \frac{4C}{\varepsilon}\right\}\right)\right] \leq \frac{\mathbb{E}\left[\int_{\mathcal{Y}} L\mathrm{d}\boldsymbol{\nu}_N\right]}{2C/\varepsilon} \leq \frac{\varepsilon}{2}.$$

Since L has compact level sets, we set $K_{\mathcal{Y},\varepsilon} = \{L \leq 4C/\varepsilon\}$ and set $K_\varepsilon \overset{\text{def.}}{=} K_{\mathcal{X},\varepsilon} \times K_{\mathcal{Y},\varepsilon}$, so that

$$\begin{aligned}\mathbb{E}\left[\boldsymbol{\gamma}_N\left(\mathcal{X}\times\mathcal{Y}\setminus K_\varepsilon\right)\right] &\leq \mathbb{E}\left[\boldsymbol{\gamma}_N\left(\left(\mathcal{X}\setminus K_{\mathcal{X},\varepsilon}\right)\times\mathcal{Y}\right)\right] + \mathbb{E}\left[\boldsymbol{\gamma}_N\left(\mathcal{X}\times\left(\mathcal{Y}\setminus K_{\mathcal{Y},\varepsilon}\right)\right)\right]\\ &= \mathbb{E}\left[\boldsymbol{\mu}_N\left(\left(\mathcal{X}\setminus K_{\mathcal{X},\varepsilon}\right)\times\mathcal{Y}\right)\right] + \mathbb{E}\left[\boldsymbol{\nu}_N\left(\mathcal{X}\times\left(\mathcal{Y}\setminus K_{\mathcal{Y},\varepsilon}\right)\right)\right] < \varepsilon.\end{aligned}$$

We conclude that the sequence of random measures $\boldsymbol{\gamma}_N$ is tight, and hence admits a convergent subsequence in the narrow topology of $\mathscr{P}_\Omega(\mathcal{X}\times\mathcal{Y})$. As discussed above, from Lemma 4.3 the limit of this subsequence belongs in $\mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})$ and minimizes $\mathcal{J}$. From the variational characterization of equilibria given in Theorem 3.3, γ is a Cournot-Nash equilibrium in the sense of Definition 1.1.

To prove the second assertion, let $\boldsymbol{\gamma}_N$ be defined as in (47) and γ a limit point. From Lemma 4.3, $\gamma \in \mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})$, our goal is to verify that γ is a critical point of $\mathcal{J}$, *i.e.* for any $\bar{\gamma} \in \mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})$ we verify that

$$\left\langle \frac{\delta\mathcal{J}}{\delta\gamma}, \bar{\gamma}-\gamma\right\rangle = \int_{\mathcal{X}\times\mathcal{Y}}\left(c(x,y)+L(y)+2\int_{\mathcal{Y}} H(y,\bar{y})\mathrm{d}\nu(\bar{y})\right)\mathrm{d}(\bar{\gamma}-\gamma)(x,y) \geq 0,$$

where $\nu = (\pi_{\mathcal{Y}})_\sharp\gamma$. From Thm. 3.3, this will show that γ is a Cournot-Nash equilibrium.

Fix some $\bar{\gamma} \in \mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})$, and recall the recovery sequence obtained from the Γ-convergence proof; consider a disintegration family $\bar{\gamma} = \mu\otimes\bar{\nu}^x$ so that

$$\bar{\boldsymbol{\gamma}}_N \overset{\text{def.}}{=} \frac{1}{N}\sum_{i=1}^N \bar{\nu}^{X_i} \xrightarrow[N\to\infty]{} \gamma.$$

We consider a unilateral deviation of player i with the alternative strategy $\bar{\nu}^{X_i}$, to the profile $\left(\nu_{1,N}^{X_1},\dots,\nu_{N,N}^{X_N}\right)$. Since the latter is a Nash equilibrium in mixed strategies, we get that $\mathcal{J}_{\Omega,i}(\bar{\nu}^{X_i},\nu_{-i,N}^{X_{-i}}) \geq \mathcal{J}_{\Omega,i}(\nu_{i,N}^{X_i},\nu_{-i,N}^{X_{-i}})$, for $\mathcal{J}_{\Omega,i}$ defined in (31). This can be rewritten as

$$\begin{aligned}&\mathbb{E}\left[\int_{\mathcal{Y}} c(X_i,y)+L(y)\mathrm{d}\bar{\nu}^{X_i} + \frac{2}{N}\sum_{j\neq i}\int_{\mathcal{Y}\times\mathcal{Y}} H\mathrm{d}\bar{\nu}^{X_i}\otimes\nu_{j,N}^{X_j}\right]\\ &\qquad\geq \mathbb{E}\left[\int_{\mathcal{Y}} c(X_i,y)+L(y)\mathrm{d}\nu_{i,N}^{X_i} + \frac{2}{N}\sum_{j\neq i}\int_{\mathcal{Y}\times\mathcal{Y}} H\mathrm{d}\nu_{i,N}^{X_i}\otimes\nu_{j,N}^{X_j}\right].\end{aligned}$$

Let us define the measures

$$\boldsymbol{\gamma}_{N,-i} \overset{\text{def.}}{=} \frac{1}{N}\sum_{j\neq i}\delta_{X_j}\otimes\nu_{j,N}^{X_j}, \text{ and } \boldsymbol{\nu}_{N,-i} \overset{\text{def.}}{=} (\pi_{\mathcal{Y}})_\sharp\boldsymbol{\gamma}_{N,-i},$$

so that evaluating the expectations, using the definition of the expectation measure we obtain

$$\int_{\mathcal{X}\times\mathcal{Y}}(c+L)\mathrm{d}\bar{\gamma}+2\int H\mathrm{d}\bar{\gamma}\otimes\mathbb{E}\boldsymbol{\gamma}_{N,-i} \geq \mathbb{E}\left[\int_{\mathcal{Y}} c(X_i,y)+L(y)\mathrm{d}\nu_{i,N}^{X_i}\right]+2\int H\mathrm{d}\mathbb{E}[\nu_{i,N}^{X_i}]\otimes\mathbb{E}\boldsymbol{\gamma}_{N,-i}.$$

Rewriting $\boldsymbol{\gamma}_{N,-i} = \boldsymbol{\gamma}_N - \frac{1}{N}\delta_{X_i} \otimes \boldsymbol{\nu}_{i,N}^{X_i}$ and averaging over all i, we get that

$$\int_{\mathcal{X}\times\mathcal{Y}} (c+L)\mathrm{d}\bar{\gamma} + 2\left(1-\frac{1}{N}\right)\int H\mathrm{d}\bar{\gamma}\otimes\mathbb{E}\boldsymbol{\gamma}_N$$
$$\geq \int_{\mathcal{X}\times\mathcal{Y}} (c+L)\mathrm{d}\mathbb{E}\boldsymbol{\gamma}_N + 2\int H\mathrm{d}\mathbb{E}\boldsymbol{\gamma}_N\otimes\mathbb{E}\boldsymbol{\gamma}_N - \frac{2}{N^2}\sum_{i=1}^N \int H\mathrm{d}\mathbb{E}\boldsymbol{\nu}_{i,N}^{X_i}\otimes\mathbb{E}\boldsymbol{\nu}_{i,N}^{X_i}$$

As $H \in C_b$, the last term is a $O(1/N)$ and hence vanishes as $N\to\infty$. In addition, since $\mathbb{E}\boldsymbol{\gamma}_N \xrightarrow[N\to\infty]{} \gamma$, from the convergence of $\boldsymbol{\gamma}_N$ and Lemma 4.3, we get that

$$0 \geq \liminf_{N\to\infty} \int_{\mathcal{X}\times\mathcal{Y}} c + L\mathrm{d}(\mathbb{E}\boldsymbol{\gamma}_N - \bar{\gamma}) + \frac{2(N-1)}{N}\int H\mathrm{d}\mathbb{E}\boldsymbol{\gamma}_N\otimes(\mathbb{E}\boldsymbol{\gamma}_N-\bar{\gamma})$$
$$\geq \liminf_{N\to\infty} \int_{\mathcal{X}\times\mathcal{Y}} c + L\mathrm{d}(\gamma-\bar{\gamma}) + 2\int H\mathrm{d}\gamma\otimes(\gamma-\bar{\gamma}) = \left\langle \frac{\delta\mathcal{J}}{\delta\gamma}, \gamma-\bar{\gamma}\right\rangle.$$

From Thm. 3.3, γ is a Cournot-Nash equilibrium. □

4.2. **Convergence in the closed loop formulation.**

4.2.1. *Potential structure for closed loop formulation.* We recall that the closed loop formulation described in the introduction can be described as follows: Given an event $\omega = (x_i)_{i\in\mathbb{N}}$, each player seeks to

$$\begin{aligned}\underset{\nu_i\in\mathscr{P}(\mathcal{Y})}{\text{minimize}}\, J_{\omega,i}(\nu_i,\nu_{-i}) &\overset{\text{def.}}{=} \int_{\mathcal{Y}} c(x_i,y)\mathrm{d}\nu_i + \int_{\mathcal{Y}} L\mathrm{d}\nu_i + \frac{2}{N}\sum_{j\neq i}\int_{\mathcal{Y}\times\mathcal{Y}} H\mathrm{d}\nu_i\otimes\nu_j \\ &= \int_{\mathcal{Y}} c(x_i,y)\mathrm{d}\nu_i + \mathcal{L}(\nu_i) + \frac{2}{N}\sum_{j\neq i}\mathcal{H}(\nu_i,\nu_j).\end{aligned} \tag{38}$$

Notice that we have written the relaxed formulation in mixed strategies and a profile in pure strategies is just a tuple $(\nu_i)_{i=1}^N$ such that $\nu_i = \delta_{y_i}$ for all players.

Under assumption (H1), that μ has no atoms, with full $\mathbb{P}$-probability, the event $\omega = (x_i)_{i\in\mathbb{N}}$ has distinct realizations, *i.e.* $x_i \neq x_j$ for all $i \neq j$. For every such event, there is a bijection between the strategy profiles $(\nu_i)_{i=1}^N$ and the measures $\gamma_N \in \mathscr{P}_{\mu_N(\omega)}(\mathcal{X}\times\mathcal{Y})$ by means of the disintegration theorem, which guarantees that each such measure is uniquely written as

$$\gamma_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}\otimes\nu^{x_i} \tag{39}$$

This representation can be seen as a lift of a profile of strategies $(\nu_i = \nu^{x_i})_{i=1}^N$ to the space of plans $\mathscr{P}(\mathcal{X}\times\mathcal{Y})$. We can define a potential function in the lifted space as

$$\mathcal{J}_{\omega,N}(\gamma_N) \overset{\text{def.}}{=} \begin{cases} \displaystyle\int_{\mathcal{X}\times\mathcal{Y}} c\mathrm{d}\gamma_N + \frac{1}{N}\sum_{i=1}^N \mathcal{L}\left(\nu^{x_i}\right) \\ \qquad + \quad \displaystyle\frac{1}{N^2}\sum_{j\neq i}\mathcal{H}\left(\nu^{x_i},\nu^{x_j}\right), & \text{if } \gamma_N\in\mathscr{P}_{\mu_N(\omega)}(\mathcal{X}\times\mathcal{Y}), \\ +\infty, & \text{otherwise,}\end{cases} \tag{40}$$

where $(\nu^{x_i})_{i=1}^N$ denotes the unique profile obtained though the representation (39).

The formulation in pure strategies can then be obtained by considering the following potential functional

$$J_{\omega,N}(y_1,\ldots,y_N) \overset{\text{def.}}{=} \mathcal{J}_{\omega,N}\left(\frac{1}{N}\sum_{i=1}^N \delta_{(x_i,y_i)}\right). \tag{41}$$

This is equivalent to restricting $\mathcal{J}_{\omega,N}$ to the set

$$\mathscr{P}^{\text{pure}}_{\mu_N}(\mathcal{X}\times\mathcal{Y}) \overset{\text{def.}}{=} \left\{\gamma_N \in \mathscr{P}_{\mu_N}(\mathcal{X}\times\mathcal{Y}) :\ \gamma_N = \frac{1}{N}\sum_{i=1}^N \delta_{(x_i,y_i)}\right\}.$$

Remark 4.8. Assumption (H1), that μ is atomless, is particularly relevant here in order to make the disintegration representation uniquely well-defined with full probability. As showed in Remark 3.4, this is not restrictive since we can replace the space $\mathcal{X}$ with $\mathcal{X}' = [0,1]\times\mathcal{X}$. In the context of the sampling, we would obtain an *i.i.d.* sequence $(X_i')_{i\in\mathbb{N}} = (T_i, X_i)_{i\in\mathbb{N}}$ with common law given by $\mu' = \mathcal{L}^1 \llcorner [0,1]\times\mu$, which has no atoms since the Lebesgue measure is non-atomic. Therefore, any event $\omega' = ((T_i,X_i) = (t_i,x_i))_{i\in\mathbb{N}}$ is such that $(t_i,x_i)\neq(t_j,x_j)$ with full probability.

As in the open loop case, the above potential functional admits minimizers since c, L and H are l.s.c. and L has compact level sets. We shall prove that minimizers for each potential functional yield Nash equilibria for the corresponding game and, in the case that H vanishes in the diagonal and is strictly positive elsewhere, we can prove that any minimizer induces a Nash equilibrium in pure strategies.

Proposition 4.9. *The following assertions hold:*

(i) It is equivalent to minimize $J_{\omega,N}$ and $\mathcal{J}_{\omega,N}$, minimizers of the latter are supported on the set of minimizers of the former and it holds that

$$\min_{\mathcal{Y}^N} J_{\omega,N} = \min_{\mathscr{P}^{pure}_{\mu_N}(\mathcal{X}\times\mathcal{Y})} \mathcal{J}_{\omega,N} = \min_{\mathscr{P}_{\mu_N}(\mathcal{X}\times\mathcal{Y})} \mathcal{J}_{\omega,N}. \tag{42}$$

(ii) Let

$$(y_i)_{i=1}^N \in \operatorname{argmin} J_{\omega,N}, \quad \gamma_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}\otimes\nu_{\omega,i} \in \operatorname{argmin}\mathcal{J}_{\omega,N}$$

then $(y_i)_{i=1}^N$ and $(\nu_{\omega,i})_{i=1}^N$ induce Nash equilibria for the game (38).

(iii) Suppose that H vanishes on the diagonal, that it is strictly positive outside it and that we allow for self interaction in our game, i.e. *we replace $\mathcal{L}$ with $\mathcal{L}_H(\gamma) = \mathcal{L}(\gamma) + \mathcal{H}(\gamma,\gamma)$. Then minimizers of $\mathcal{J}_{\omega,N}$ are of the form*

$$\gamma_N = \frac{1}{N}\sum_{i=1}^N \delta_{(x_i,y_i)},\ \textit{where } (y_1,\dots,y_N)\in\operatorname{argmin} J_{\omega,N}. \tag{43}$$

(iv) If $H = +\infty$ in the diagonal, any minimizer of $\mathcal{J}_{\omega,N}$ is atomless.

Proof. The first equality in (42) comes from the bijection between the set of pure equilibrium measures and $\mathcal{Y}^{\otimes N}$. The second is a direct consequence of the fact that the measures γ_N in the domain of $\mathcal{J}_{\omega,N}$ can be written as

$$\gamma_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}\otimes\nu_i,$$

so that we can write

$$\mathcal{J}_{\omega,N}(\gamma_N) = \int_{\mathcal{Y}^{\otimes N}} J_{\omega,N}(y_1,\dots,y_N)\mathrm{d}\nu_1\otimes\cdots\otimes\nu_N.$$

Then for any admissible γ_N, we have

$$\min_{\mathscr{P}_{\mu_N(\omega)}(\mathcal{X}\times\mathcal{Y})} \mathcal{J}_{\omega,N}(\gamma) \geq \min_{\mathcal{Y}^{\otimes N}} J_{\omega,N}.$$

Taking γ_N with second marginal supported on the set of minimizers of $J_{\omega,N}$ gives the result.

The proof of assertion (ii) is analogous to the proof of Proposition 4.2.

To prove (iii), notice that from item (i) and the fact that $H \geq 0$ it holds that

$$\inf \mathcal{J}_{\omega,N} \geq \inf J_{\omega,N} + \frac{1}{N^2} \sum_{i=1}^{N} \mathcal{H}(\nu_{\omega,i}, \nu_{\omega,i}) \geq \inf \mathcal{J}_{\omega,N}.$$

Which means that $\mathcal{H}(\nu_{\omega,i}, \nu_{\omega,i}) = 0$ for all $i = 1, \dots, N$, and since H only vanishes in the diagonal, it must hold that $\nu_{\omega,i} = \delta_{y_i}$. From (i) and the previous argument, any minimizer of $\mathcal{J}$ is of the form of (43).

With a dual reasoning, if γ_N has an atom, *i.e.* if there is a point where $\gamma_N(\{x_i, y_i\}) > 0$, and H explodes in the diagonal, the self interaction term gives $\mathcal{J}_{\omega,N}(\gamma_N) = +\infty$ and it cannot be a minimizer. □

4.2.2. Γ*-convergence for the closed loop formulation.* Now we move on to the question of the convergence of a sequence of Nash equilibria for the games in closed loop (38). In this case we have a family of games indexed by the sample $\omega = (x_i)_{i \in \mathbb{N}}$ of the players' state variables, therefore we can only expect a Γ convergence to hold with $\mathbb{P}$-probability 1. We start by showing a general Lemma that gives Γ-convergence with full probability.

Lemma 4.10. *Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a family of functionals $(\mathscr{F}_{\omega,N})_{\substack{N \in \mathbb{N} \\ \omega \in \Omega}}$ and a functional $\mathscr{F}$ over a Polish space $\mathcal{X}$ such that*

(1) there is a set Ω_0 with full $\mathbb{P}$-probability such that for any $x_N \to x$ the $\Gamma - \liminf$ inequality for $\mathscr{F}_{\omega,N}$ holds

$$\mathscr{F}(x) \leq \liminf_{N \to \infty} \mathscr{F}_{\omega,N}(x_N), \text{ for all } \omega \in \Omega_0$$

(2) for each $x \in \mathcal{X}$ there is a set Ω_x with full $\mathbb{P}$-probability for which we can construct recovery sequences of $\mathscr{F}_{\omega,N}$

$$\limsup_{N \to \infty} \mathscr{F}_{\omega,N}(x_N) \leq \mathscr{F}(x), \text{ for all } \omega \in \Omega_x.$$

Under these conditions, there is a set $\bar{\Omega}_0$ with full $\mathbb{P}$-probability such that for any $\omega \in \bar{\Omega}_0$ the sequence $\mathscr{F}_{\omega,N}$ Γ-converges to $\mathscr{F}$.

Proof. First we claim that there exists a countable and dense set $\mathscr{D} \subset \mathcal{X}$ which is dense in the energy $\mathscr{F}$, i.e. for each $x \in \mathcal{X}$ there is $(x_n)_{n \in \mathbb{N}} \subset \mathscr{D}$ such that

$$x_n \xrightarrow[n \to \infty]{} x \text{ and } \mathscr{F}(x_n) \xrightarrow[n \to \infty]{} \mathscr{F}(x). \tag{44}$$

See for instance [2, Lemma 11.12] for a constructive argument, a simple proof comes from the fact that $\mathbb{R} \times \operatorname{dom} \mathscr{F}$ is separable as an (arbitrary) subset of the separable space $\mathbb{R} \times \mathcal{X}$, since subsets of second countable spaces are second countable.

Hence we can define the set $\bar{\Omega}_0$ as

$$\bar{\Omega}_0 \stackrel{\text{def.}}{=} \Omega_0 \cap \bigcap_{x \in \mathscr{D}} \Omega_x,$$

where Ω_0 denotes the set where the Γ-lim inf holds for all points $x \in \mathcal{X}$ and Ω_x denotes the event in which we can construct recovery sequences for x. Since $\mathscr{D}$ is countable, it holds that $\mathbb{P}(\bar{\Omega}_0) = 1$.

To prove the Γ-convergence for each $\omega \in \bar{\Omega}_0$, we recall the notions of lower and upper Γ limits from Section 2.3, and to conclude it suffices to prove for all $\omega \in \bar{\Omega}_0$ that $\Gamma\text{-}\liminf \mathscr{F}_{\omega,N} = \Gamma\text{-}\limsup \mathscr{F}_{\omega,N} = \mathscr{F}$. Indeed, item (1) shows that

$$\mathscr{F} \leq \Gamma\text{-}\liminf \mathscr{F}_{\omega,N}, \text{ for all } \omega \in \bar{\Omega}_0.$$

On the other hand, from item (2), it follows for any $x \in \mathscr{D}$ that

$$\Gamma\text{-}\limsup \mathscr{F}_{\omega,N}(x) \leq \mathscr{F}(x), \text{ for all } \omega \in \bar{\Omega}_0.$$

Hence, for any $\omega \in \bar{\Omega}_0$ and an arbitrarily $x \in \mathcal{X}$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{D}$ satisfying (44), so that using the lower semi-continuity of the Γ upper limit we have that

$$\Gamma\text{-}\limsup_{N\to\infty} \mathscr{F}_{\omega,N}(x) \leq \liminf_{n\to\infty} \left(\Gamma\text{-}\limsup_{N\to\infty} \mathscr{F}_{\omega,N}(x_n)\right) \leq \liminf_{n\to\infty} \mathscr{F}(x_n) = \mathscr{F}(x)$$

which gives the Γ-convergence with full probability. □

To apply this Lemma, we know from the Glivenko-Cantelli law of large numbers that empirical measures converge $\mathbb{P}$ almost surely. Hence, we consider the set

$$\Omega_0 \overset{\text{def.}}{=} \left\{\omega = (x_i)_{i\in\mathbb{N}} \in \operatorname{supp}\mathbb{P} : \begin{array}{c} x_i \neq x_j, \text{ for } i \neq j \\ \mu_N(\omega) \xrightarrow[N\to\infty]{} \mu \end{array}\right\}. \tag{45}$$

The first condition above is so that the sequence of functionals $\mathcal{J}_{\omega,N}$ is well-defined for any $\omega \in \Omega_0$. From the fact that μ is atomless and the above discussion, $\mathbb{P}(\Omega_0) = 1$.

While the Γ-liminf argument will be similar to the open loop information structure, for the Γ-limsup we will use a construction depending on a sequence of random variations of the form

$$\begin{gathered} \frac{1}{N}\sum_{i=1}^{N} L_i + \frac{1}{N^2}\sum_{i\neq j} H_{i,j} \\ \text{where } L_i \overset{\text{def.}}{=} c(X_i, Y_i) + L(Y_i), \quad H_{i,j} \overset{\text{def.}}{=} H(Y_i, Y_j), \end{gathered} \tag{46}$$

where $(X_i, Y_i) \sim \gamma$. The first sum is fortunately an i.i.d. sequence, so that from the law of large numbers it must converge to its mean. The second term however is not i.i.d., but it is *exchangeable* as it can be written as a symmetric function of an i.i.d. sample. In the following Proposition, whose proof is a synthesis of the ideas from [34, Chap. 12], we show that such families of random variables also enjoy a law of large numbers.

Proposition 4.11. *Let $\left(\bar{H}_n\right)_{n\in\mathbb{N}}$ be a sequence of random variables obtained as the symmetric image of an i.i.d. sample, that is let it be the enumeration of the family of random variables*

$$(\Phi(X_i, X_j))_{i\neq j\in\mathbb{N}},$$

where $\Phi : \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ is a symmetric function and $(X_i)_{i\in\mathbb{N}}$ is an i.i.d. sample. Then

$$\frac{1}{N}\sum_{n=1}^{N} \bar{H}_n \xrightarrow[N\to\infty]{} \mathbb{E}[\bar{H}_1], \text{ with probability } 1.$$

For the sake of readability of the main ideas employed to prove the Γ-convergence result, we include the proof of the previous proposition in Appendix A.

Theorem 4.12. *With full $\mathbb{P}$-probability, the sequence of functionals $\mathcal{J}_{\omega,N}$ convergence to $\mathcal{J}$ in the sense of Γ convergence in the narrow topology of $\mathscr{P}(\mathcal{X}\times\mathcal{Y})$.*

Proof. It suffices to verify the hypothesis of Lemma 4.10. To prove (1), consider $\omega \in \Omega_0$ defined above in (45), and let $(\gamma_N)_{N\in\mathbb{N}}$ be a sequence such that $\gamma_N \in \mathscr{P}_{\mu_N}(\mathcal{X}\times\mathcal{Y})$ and converging to γ. So we can assume that γ_N can be written as $\gamma_N = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \delta_{x_i} \otimes \nu^{x_i}$, where $\nu^{x_i} \in \mathscr{P}(\mathcal{Y})$ and for any $\omega \in \Omega_0$, it follows from the continuity w.r.t. convergence of marginals that $\gamma \in \mathscr{P}_{\mu}(\mathcal{X}\times\mathcal{Y})$.

For an arbitrary $M > 0$, define $\mathcal{H}^M(\nu,\nu) \overset{\text{def.}}{=} \int H \wedge M \mathrm{d}\nu \otimes \nu$, and it follows that

$$\begin{aligned}\mathcal{J}_{\omega,N}(\gamma_N) &\geq \int_{\mathcal{X}\times\mathcal{Y}} c\mathrm{d}\gamma_N + \underbrace{\frac{1}{N}\sum_{i=1}^N \mathcal{L}(\nu^{x_i})}_{=\mathcal{L}(\nu_N)} + \underbrace{\frac{1}{N^2}\sum_{i,j}\mathcal{H}^M(\nu^{x_i},\nu^{x_j})}_{=\mathcal{H}^M(\nu_N,\nu_N)} - \frac{1}{N^2}\sum_{i=1}^N \mathcal{H}^M(\nu^{x_i},\nu^{x_i}) \\ &= \int_{\mathcal{X}\times\mathcal{Y}} c\mathrm{d}\gamma_N + \mathcal{L}(\nu_N) + \mathcal{H}^M(\nu_N,\nu_N) - \frac{1}{N^2}\sum_{i=1}^N \mathcal{H}^M(\nu^{x_i},\nu^{x_i}) \\ &\geq \int_{\mathcal{X}\times\mathcal{Y}} c\mathrm{d}\gamma_N + \mathcal{L}(\nu_N) + \mathcal{H}^M(\nu_N,\nu_N) - \frac{M}{N}.\end{aligned}$$

The sum on the RHS vanishes as $N \to \infty$ for each $M > 0$ and hence the lower semi-continuity of the remaining terms w.r.t. narrow convergence, as integrals of l.s.c. integrands, for every $M > 0$ gives

$$\begin{aligned}\liminf_{N\to\infty} \mathcal{J}_{\omega,N}(\gamma_N) &\geq \liminf_{N\to\infty} \int_{\mathcal{X}\times\mathcal{Y}} c\mathrm{d}\gamma_N + \mathcal{L}(\nu_N) + \mathcal{H}^M(\nu_N,\nu_N) \\ &\geq \int_{\mathcal{X}\times\mathcal{Y}} c\mathrm{d}\gamma + \mathcal{L}(\nu) + \mathcal{H}^M(\nu,\nu).\end{aligned}$$

Noticing that from the monotone convergence theorem $\mathcal{H}(\nu,\nu) = \sup_{M>0} \mathcal{H}^M(\nu,\nu)$, the Γ-lim inf follows.

To verify property (2) from Lemma 4.10, given some $\gamma \in \mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})$, let $\nu = (\pi_\mathcal{Y})_\sharp\gamma$. By an application of the disintegration theorem one can write $\gamma = \nu^x \otimes \mu$ for some Borel map $(\nu^x)_{x\in\mathcal{X}}$, *i.e.*

$$\int_{\mathcal{X}\times\mathcal{Y}} \varphi(x,y)\mathrm{d}\gamma = \int_\mathcal{X} \left(\int_\mathcal{Y} \varphi(x,y)\mathrm{d}\nu^x(y)\right)\mathrm{d}\mu(x), \text{ for all } \varphi \in C_b(\mathcal{X}\times\mathcal{Y}).$$

This disintegration family is only μ-a.e.uniquely defined, but we can fix one such family and define a new transportation plan as $\gamma_N \overset{\text{def.}}{=} \mu_N \otimes \nu^x$. Since we have fixed one disintegration family, $\gamma_N \in \mathscr{P}_{\mu_N}(\mathcal{X}\times\mathcal{Y})$ is well-defined for every event $\omega = (x_i)_{i\in\mathbb{N}}$. From the definition, it then holds that

$$\int_{\mathcal{X}\times\mathcal{Y}} \phi(x,y)\mathrm{d}\gamma_N \overset{\text{def.}}{=} \frac{1}{N}\sum_{i=1}^N \int \phi(x_i,y)\mathrm{d}\nu^{x_i}, \text{ for all } \phi \in C_b(\mathcal{X}\times\mathcal{Y}).$$

Hence $\gamma_N \in \Pi(\mu_N,\nu_N)$ where $\mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$, $\nu_N = \frac{1}{N}\sum_{i=1}^N \nu^{x_i}$.

Let us prove that γ_N converges narrowly to γ; indeed from Prop. 2.3 we know there is a countable set $\mathcal{K} \subset C_b(X)$, such that to prove narrow convergence it suffices to verify that

$$\int_{\mathcal{X}\times\mathcal{Y}} f(x,y)\mathrm{d}\gamma_N \xrightarrow[N\to\infty]{} \int_{\mathcal{X}\times\mathcal{Y}} f(x,y)\mathrm{d}\gamma, \text{ for all } f \in \mathcal{K}.$$

For each $f \in \mathcal{K}$, we compute

$$\int_{\mathcal{X}\times\mathcal{Y}} f(x,y)\mathrm{d}\gamma_N = \frac{1}{N}\sum_{i=1}^N \int_{\mathcal{X}\times\mathcal{Y}} f(x_i,y)d\nu^{x_i}(y).$$

Hence, each term of the sum on the right is a realization of the i.i.d. sequence of random variables $F_i \overset{\text{def.}}{=} \int_\mathcal{Y} f(X_i,\cdot)\mathrm{d}\nu^{X_i}$. From the strong law of large numbers, it holds with

probability 1 that

$$\int_{\mathcal{X}\times\mathcal{Y}} f\mathrm{d}\gamma_N = \frac{1}{N}\sum_{i=1}^{N} F_i(\omega) \xrightarrow[N\to\infty]{} \mathbb{E}_{\mathbb{P}}\left[F_1\right] = \int_X f\mathrm{d}\gamma.$$

Let $\Omega_{\gamma,f}$ denote the set of probability 1, which depends on γ and f, where the above converge holds. Then defining

$$\tilde{\Omega}_\gamma = \bigcap_{f\in\mathcal{K}} \Omega_{\gamma,f},$$

we have that $\mathbb{P}(\tilde{\Omega}_\gamma) = 1$ and for any $\omega \in \tilde{\Omega}_\gamma$ it holds that $\gamma_N \xrightarrow[N\to\infty]{} \gamma$.

We now apply a similar argument to the convergence of the energies. Indeed, writing

$$\mathcal{J}_{\omega,N}(\gamma_N) = \frac{1}{N}\sum_{i=1}^{N}\int_{\mathcal{Y}} \left(c(x_i,y)+L(y)\right)\mathrm{d}\nu^{x_i}(y) + \frac{1}{N^2}\sum_{j\neq i}\int_{\mathcal{Y}\times\mathcal{Y}} H\mathrm{d}\nu^{x_i}\otimes\nu^{x_j}.$$

We see that the first sum is the empirical average of the i.i.d. sequence of random variables $L_i \overset{\text{def.}}{=} \int_{\mathcal{Y}}(c(X_i,y)+L(y))\mathrm{d}\nu^{X_i}(y)$ while the double sum can be written in terms of the sequence $H_{i,j} \overset{\text{def.}}{=} \int H\mathrm{d}\nu^{X_i}\otimes\nu^{X_j}$. As a consequence, applying once again the strong law of large numbers, there is a set $\Omega_{L,\gamma}$ with probability 1, such that for any $\omega\in\Omega_{L,\gamma}$ it holds that

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N} L_i(\omega) \xrightarrow[N\to\infty]{} \mathbb{E}_{\mathbb{P}}[L_1] &= \int_{\mathcal{X}}\left[\int_{\mathcal{Y}}(c(x,y)+L(y))\mathrm{d}\nu^x(y)\right]\mathrm{d}\mu(x)\\ &= \int_{\mathcal{X}\times\mathcal{Y}}(c(x,y)+L(y))\mathrm{d}\gamma = \int_{\mathcal{X}\times\mathcal{Y}} c\mathrm{d}\gamma + \mathcal{L}(\nu).\end{aligned}$$

For the second term, the random variables $(H_{i,j})_{i\neq j}$ are no longer i.i.d., but satisfy the hypothesis of Thm. 4.11 with Φ given by

$$\Phi(x_1,x_2) \overset{\text{def.}}{=} \int_{\mathcal{Y}\times\mathcal{Y}} H\mathrm{d}\nu^{x_1}\otimes\nu^{x_2},$$

which is symmetric and measurable from the measurability of the family $(\nu^x)_{x\in\mathcal{X}}$. We conclude that there is another set $\Omega_{H,\gamma}$ with probability 1 such that for all $\omega\in\Omega_{H,\gamma}$ it holds that

$$\frac{1}{N^2}\sum_{j\neq i}\int_{\mathcal{Y}\times\mathcal{Y}} H\mathrm{d}\nu^{x_i}\otimes\nu^{x_j} \xrightarrow[N\to\infty]{} \mathbb{E}_{\mathbb{P}}[H_{1,2}] = \mathcal{H}(\nu,\nu).$$

Finally, the set $\Omega_\gamma \overset{\text{def.}}{=} \tilde{\Omega}_\gamma\cap\Omega_{L,\gamma}\cap\Omega_{H,\gamma}$ has probability 1 and satisfies all the properties of item (2).

From the thesis of Lemma 4.10, the Γ convergence with full $\mathbb{P}$-probability follows □

As in the closed loop case, with an analogous proof to the open loop case, we obtain a result assuring, with full $\mathbb{P}$-probability, the convergence of a particular sequence of Nash equilibria to equilibria of Cournot-Nash type, and whenever H is continuous the convergence of any sequence of Nash equilibria.

Theorem 4.13. *Assume that* $\inf_{\mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})} \mathcal{J} < \infty$, *then there are sequences of Nash equilibria for the game* (38), *described by transportation plans* $(\gamma_N)_{N\in\mathbb{N}}$ *such that, with full* $\mathbb{P}$*-probability, converge up to a subsequence in the narrow topology to a Cournot-Nash equilibrium* $\gamma\in\mathscr{P}_\mu(\mathcal{X}\times\mathcal{Y})$, *in the sense of Definition 1.1.*

Assuming in addition that $H \in C_b(\mathcal{Y} \times \mathcal{Y})$, with $\mathbb{P}$-full probability, for any sequence of Nash equilibria $(\gamma_N)_{N\in\mathbb{N}}$ from game (38), that is

$$\gamma_N \overset{\text{def.}}{=} \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i} \otimes \nu_{i,N} \in \mathscr{P}_{\mu_N(\omega)}(\mathcal{X} \times \mathcal{Y}), \tag{47}$$

converging to γ in the narrow topology of $\mathscr{P}(\mathcal{X} \times \mathcal{Y})$, it holds that $\gamma \in \mathscr{P}_\mu(\mathcal{X} \times \mathcal{Y})$, and it is a Cournot-Nash equilibrium in the sense of Definition 1.1.

5. Conclusion

If anything, the convergence of Nash to Cournot-Nash equilibria demonstrates how difficult the convergence question in the context of Mean Field Games is. The Γ-convergence approach relies entirely on the fact that a variational description of equilibria in provided in Thm. 3.3 and is not useful to other games, for which we have only fixed point techniques at disposal. The following questions then present themselves:

- Can we consider other types of energy? The analysis seems very specific to an energy that is the sum of an individual and a pair-wise interaction costs.
  - Using the characterization of convex functions as the envelope of all linear functions below it, one could try to adapt the arguments of the linear term to the case of an individual convex energy.
  - In principle, the arguments treating the pair-wise interaction term could be extended to a k-wise interaction, as long as the number of players interacting remains uniformly bounded, as in this case an analogous law of large numbers from Appendix A should hold.
- Another direction would be to derive a large deviations principle for the Gibbs measures associated with the potential function of the N-player games, whose Nash equilibria converge to Cournot-Nash equilibria, in accordance with the statistical mechanics intuition that motivated the original name Mean Field Games.
- In games where the underlying space $\mathcal{X}$ is euclidean and the cost c is a continuous and convex function, the connections with optimal transport described in the introduction allows us to conclude that minimizers of the potential function are equilibria in pure strategies. However, we cannot apply these results to the very natural case of Lagrangian MFGs, since c becomes the indicator function of satisfying prescribed initial conditions, see example 3.6. Can we obtain conditions guaranteeing that solutions are of pure strategies type?

Appendix A. The law of large numbers for symmetric functions of an i.i.d. sample

In this appendix we prove Proposition 4.11. The ideas are a minor modification of the presentation of [34], hence our goal is to make it as self-contained as possible to readers less familiarized with probability theory, but we hope it can be useful in other contexts as well. We also observe that this proof remains true if one considers $\Phi : \mathcal{X}^{\otimes k} \to \mathbb{R}$, for any $k \in \mathbb{N}$. With this we can now proceed with our Γ-convergence type result.

Proposition A.1. *Let $(H_{i,j})_{i\neq j\in\mathbb{N}}$ be a sequence of random variables obtained as*

$$H_{i,j} = (\Phi(X_i, X_j))_{i\neq j\in\mathbb{N}},$$

where $\Phi : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a symmetric function and $(X_i)_{i\in\mathbb{N}}$ is an i.i.d. sample. Then

$$\frac{1}{N^2}\sum_{\substack{1\leq i,j\leq N\\ i\neq j}} H_{i,j} \xrightarrow[N\to\infty]{} \mathbb{E}[H_{1,2}], \text{ with probability } 1.$$

Proof of Prop. 4.11. First, define the exchangeable σ-algebra as follows: we say a function $f : \mathbb{R}^{\otimes\mathbb{N}} \to \mathbb{R}$ is n-symmetric if it is symmetric w.r.t. permutations of at most n indexes. In other words, for any permutation $\sigma : \mathbb{N} \to \mathbb{N}$ swapping at most n indexes, then $f\left(\left(x_{\sigma(n)}\right)_{n\in\mathbb{N}}\right) = f\left((x_n)_{n\in\mathbb{N}}\right)$. We then define the exchangeable σ-algebra as

$$\mathcal{E}_\infty \overset{\text{def.}}{=} \bigcap_{n\in\mathbb{N}} \mathcal{E}_n, \text{ where } \mathcal{E}_n \overset{\text{def.}}{=} \sigma\left(\left\{ f\left((X_i)_{i\in\mathbb{N}}\right) : \begin{array}{l} f : \mathbb{R}^{\otimes\mathbb{N}} \to \mathbb{R} \\ \text{is } n\text{-symmetric and Borel} \end{array} \right\}\right),$$

where $\sigma\left(\{F_i\}_{i\in I}\right)$ is defined as the smallest σ-algebra that makes the hole family of random variables $(F_i)_{i\in I}$ measurable.

Take $g : \mathbb{R}^{\otimes\mathbb{N}} \to \mathbb{R}$, bounded and n-symmetric function, for all $i \leq n$, it holds from exchangeability that

$$\begin{aligned}
\mathbb{E}\left[H_{i,j} g\left(X_\cdot\right)\right] &= \mathbb{E}\left[\Phi(X_i, X_j) g\left(X_1, X_2, X_3, \dots, X_{i-1}, X_i, X_{i+1}, \dots, X_{j-1}, X_j, X_{j+1}, \dots\right)\right] \\
&= \mathbb{E}\left[\Phi(X_1, X_2) g\left(X_i, X_j, X_3, \dots, X_{i-1}, X_1, X_{i+1}, \dots, X_{j-1}, X_2, X_{j+1}, \dots\right)\right] \\
&= \mathbb{E}\left[H_{1,2} g\left(X_\cdot\right)\right]
\end{aligned}$$

In particular, taking $g = 1_A$ for an arbitrary set $A \in \mathcal{E}_n$ and averaging the above equality for all $1 \leq i, j \leq n$ with $i \neq j$, we obtain that

$$\mathbb{E}\left[\frac{1}{n(n-1)} \sum_{i\neq j} H_{i,j} 1_A\right] = \mathbb{E}\left[H_{1,2} 1_A\right], \text{ so that } \frac{1}{n(n-1)} \sum_{i\neq j} H_{i,j} = \mathbb{E}\left[H_{1,2} \middle| \mathcal{E}_n\right],$$

by the definition of conditional expectation for L^1 random variables. This means that $\frac{1}{n(n-1)} \sum_{i\neq j} H_{i,j}$ is a backwards martingale for the filtration $(\mathcal{E}_n)_{n\in\mathbb{N}}$ and a suitable martingale convergence Theorem, [34, Thm. 12.14], gives that

$$\frac{1}{n(n-1)} \sum_{i\neq j} H_{i,j} \xrightarrow[n\to\infty]{} \mathbb{E}\left[H_{1,2} \middle| \mathcal{E}_\infty\right] \text{ with convergence a.s. and in } L^1.$$

Since $(X_i)_{i\in\mathbb{N}}$ is i.i.d., the Hewitt-Savage $0-1$ law, see [34, Cor. 12.19] and [30], states that $\mathcal{E}_\infty$ is a trivial σ-algebra, so that for any set $A \in \mathcal{E}_\infty$, $\mathbb{P}(A)$ is either 0 or 1. Hence, as $\mathbb{E}\left[H_{1,2} \middle| \mathcal{E}_\infty\right]$ is an $\mathcal{E}_\infty$-adapted random variable, it must be given by a constant given by its mean $\mathbb{E}\left[\mathbb{E}\left[H_{1,2} \middle| \mathcal{E}_\infty\right]\right] = \mathbb{E}\left[H_{1,2}\right]$, and the result follows. □

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