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► **To cite this version:**

B Costacèque, L Decreusefond. Convergence rate for the coupon collector's problem with Stein's method. 2025. hal-04865020

**HAL Id: hal-04865020**

**<https://hal.science/hal-04865020v1>**

Preprint submitted on 5 Jan 2025

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# Convergence rate for the coupon collector's problem with Stein's method

B. Costacèque      L. Decreusefond

2024

In this paper, we consider the classical coupon collector problem with uniform probabilities. Since the seminal paper by P. Erdős and A. Rényi [8], it is well-known that the renormalized number of attempts required to complete a collection of  $n$  items distributed with uniform probability tends to a Gumbel distribution when  $n$  goes to infinity. We propose to determine how fast this convergence takes place for a certain distance to be introduced by using the so-called generator approach of Stein's method. To do so, we introduce a semi-group similar to the classical Ornstein-Uhlenbeck semi-group and whose stationary measure is the standard Gumbel distribution. We give its essential properties and apply them to prove that the renormalized number of attempts converges to the Gumbel distribution at rate  $\log n/n$ .

Keywords: Coupon collector's problem, Gumbel distribution, Stein's method, generator approach

Math Subject Classification: 60F05,60E07

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## 1 Introduction

Stein's method in its modern acceptance (see [6, 5]) is based upon an idea the physicists call *stochastic quantization*. Given a probability measure  $\mu$  on a space  $(E, \mathcal{E})$ , this procedure established in [14] consists in constructing a Markov semi-group  $(P_t)_{t \geq 0}$  whose stationary distribution is  $\mu$ . In physics, this approach is, for instance, used for numerical simulations of

gauge theories with fermions [13]. In the context of the Stein's method, we use this construction to establish the representation formula:

$$\int_E f \, d\mu - \int_E f \, d\nu = \int_E \int_0^\infty LP_t f \, dt \, d\nu, \quad (1)$$

where  $L$  is the generator of the semi-group and  $\nu$  is any other probability measure on  $(E, \mathcal{E})$ . The historical presentation of the Stein's method relies on the requirement that

$$\int_E Lf \, d\nu = 0, \quad \forall f \in \text{Dom } L \iff \nu = \mu \quad (2)$$

where  $\text{Dom } L$  is the domain of  $L$  (see [9]). If we are lucky enough to find such an operator  $L$  and that this operator generates a Markov process of semi-group  $(P_t)_{t \geq 0}$ , then it remains to evaluate the regularity of the solution of the so-called Stein's equation

$$Lg = f - \int f \, d\mu$$

whose solution is given by

$$\int_0^\infty P_t f \, dt. \quad (3)$$

The stochastic quantization procedure suggests starting with the definition of the Markov semi-group instead of the generator. This avoids proving the often difficult equivalence (2) and *a priori* estimating the regularity of the function given by (3). In certain situations, such as [3], it may be necessary to separate the integral from 0 to infinity into two parts: one from 0 to a variable threshold  $\varepsilon$ , and the other from  $\varepsilon$  to infinity. Each term should be approached differently so that we do not end up with a simple upper-bound of the Lipschitz continuity, for instance, of the solution of the Stein's equation.

Shall we start with the operator  $L$  or with the Markov semi-group, one of the main obstacle to the generalization of the Stein's method to measures beyond the Gaussian and the Poisson measures is to construct one of these elements. There is no magical recipe to achieve such a goal for an arbitrary probability measure. See [11] and references therein for a thorough survey of how to construct an operator satisfying (2) for a wide range of probability measures. For the standard Gaussian measure on  $\mathbf{R}^n$ , the usual operator is

$$Lf(x) = -\langle x, \nabla f(x) \rangle + \Delta f(x),$$

for which the so-called Ornstein-Uhlenbeck semi-group is given by

$$P_t f(x) = \int_{\mathbf{R}^n} f\left(e^{-t}x + \sqrt{1 - e^{-2t}}y\right) \, d\mu(y).$$

If  $\mu$  denotes the law of a Poisson process  $N$  of control measure  $\sigma$ , the so-called Glauber semi-group is defined by

$$P_t f(\omega) = \mathbb{E}\left[f\left(e^{-t} \circ \omega + (1 - e^{-t}) \circ N\right)\right]$$

where  $p \circ \omega$  is the  $p$ -thinning of the configuration  $\omega$  (see [7]). The crucial remark is that the semi-group property of the operators so defined is a mere rewriting of the stability of the

Gaussian and Poisson measures: The standard Gaussian measure on  $\mathbf{R}^n$  is the unique probability law on  $\mathbf{R}^n$  such that

$$e^{-t} Z' + \sqrt{1 - e^{-2t}} Z'' \stackrel{d}{=} Z \quad (4)$$

where  $Z', Z''$  are two independent copies of  $Z$ . The Poisson process of control measure  $\sigma$  is characterized by the fact that

$$e^{-t} \circ N' + (1 - e^{-t}) \circ N'' \stackrel{d}{=} N \quad (5)$$

where  $N', N''$  are two independent copies of  $N$ . This invariance also entails that the Gaussian (respectively the Poisson) measure is the stationary measure of the semi-group  $(P_t)_{t \geq 0}$ . Remark that at a formal level, equations (4) and (5) can be written as

$$D_a Z' \oplus D_b Z'' \stackrel{d}{=} Z$$

where  $a^2 + b^2 = 1$  in (4),  $a + b = 1$  in (5) and  $\stackrel{d}{=}$  denotes an equality in distribution. The operator  $D_a$  is the multiplication (respectively the thinning) of real numbers (respectively of point processes) and the sign  $\oplus$  represents the addition of real numbers (respectively the superposition of point processes) in (4) (respectively (5)). This means that the algebraic structure at play here is that of a semi-group (the plus operation) with a family  $(D_a)_{a \in T}$  of automorphisms which satisfy the identity  $D_a \circ D_b = D_{ab}$ ; the transformation  $D_a$  is called a *multiplication* (by  $a$ ). In the seminal paper [4], such a structure is called a *convex cone*. It is shown in [4], that there exist numerous other examples of stable distributions provided that we change the meaning of the  $\oplus$  sign and the group of automorphisms. These distributions are often interesting because their stability implies that they appear in limit theorems akin to the central limit theorem.

In this paper, we focus on the Gumbel distribution which is characterized by the stability identity

$$\max(Z' + \log a, Z'' + \log(1 - a)) \stackrel{d}{=} Z, \quad (6)$$

for any  $a \in (0, 1)$ . This falls in the previously introduced framework where  $x \oplus y = \max(x, y)$  and  $D_a x = x + \log a$ . We then take advantage of this identity to define a Markov semi-group which leaves the Gumbel distribution invariant. We then compute its generator  $L$  and show that the equivalence (2) holds.

Let  $I_n := \{1, \dots, n\}$  and consider  $(X_k)_{k \geq 1}$  a sequence of independent random variables, uniformly distributed over  $I_n$ . We denote by

$$T_n = \inf \{k \geq 1, \{X_1, \dots, X_k\} = I_n\}.$$

The analysis of the properties of  $T_n$  is the so-called coupon collector problem. It is well known that

$$Z_n := \frac{T_n}{n} - \log n \stackrel{d}{\rightarrow} G(0, 1), \quad (7)$$

where  $G(0, 1)$  denotes the standard Gumbel distribution, with c.d.f.  $\exp(-\exp(-x))$  on  $\mathbf{R}$ . We use our semi-group and generator, which are characteristic to the Gumbel distribution, to

develop Stein's method in order to assess the order of convergence in (7). The proof is a bit trickier than usual and requires several intermediate technical steps.

This work is organized as follows. In the second Section, we introduce a new Markov semi-group  $(\mathbf{P}_t^0)_{t \geq 0}$  whose stationary measure is the Gumbel distribution and give its essential properties. In particular, we obtain a Stein identity characterizing the Gumbel distribution. The third Section contains the main result of this paper, that is, a rate of convergence for the standard coupon collector problem in smooth Wasserstein distance. The last Section is dedicated to the proof of that result.

## 2 Stochastic quantization of the Gumbel distribution

The Gumbel distribution is one of the three so-called extreme value distributions. It arises in the study of extreme events, for instance when one aims at predicting record floods or drought. The most famous occurrence of the Gumbel distribution is given by the celebrated Fisher-Tippett-Gnedenko [16], which states that a renormalized maximum of independent and identically distributed (iid) random variables can have only three possible limits - if any: the Fréchet distribution, the (negative) Weibull distribution and the Gumbel distribution. Which one appears depends on the right tail of the initial distribution. The Fréchet distribution corresponds to regularly varying tails, the Weibull distribution to bounded right tails, and the Gumbel distribution to light tails. The domain of attraction of the latter thus includes many fundamental distributions, like the Gaussian distribution, the log-normal distribution, the exponential distribution, the logistic distribution, *etc.*

In the sequel, we will work with the following test-function sets: the class  $\text{Lip}(\mathbf{R})$  of 1-Lipschitz functions over  $\mathbf{R}$  and the class  $\text{Lip}_{[2]}(\mathbf{R})$  of differentiable Lipschitz functions whose first derivative is also 1-Lipschitz over  $\mathbf{R}$ :

$$\text{Lip}_{[2]}(\mathbf{R}) := \{f \in \mathcal{C}^1(\mathbf{R}), f \text{ and } f' \text{ are 1-Lipschitz}\}.$$

Notice that unlike the smooth Wasserstein distance  $d_{W_1}$  (see for instance [1] p.11), we do not assume that our test-functions are bounded.

**DEFINITION 1.**— Let  $\mu$  be the Gumbel distribution, with p.d.f.  $x \mapsto e^{-(x+e^{-x})}$  on  $\mathbf{R}$  and  $Z$  a random variable with that distribution. Let  $f \in \text{Lip}_{[2]}(\mathbf{R})$ . Set for all  $x \in \mathbf{R}$ :

$$D_a x := x + \log a, \quad a \geq 0$$

with  $D_0 x = -\infty$ , and define a family of linear operators on  $\text{Lip}_{[2]}(\mathbf{R})$  by setting:

$$\mathbf{P}_t^0 f(x) := \mathbb{E}[f(D_{e^{-t}} x \oplus D_{1-e^{-t}} Z)].$$

for all  $t \in \mathbf{R}_+$  and  $x \in \mathbf{R}$

This expression also writes

$$\mathbf{P}_t^0 f(x) = \mathbb{E}\left[f((x-t) \oplus (Z + \log(1-e^{-t})))\right]$$

and defines a semi-group whose stationary measure is the Gumbel distribution.

**THEOREM 1.**– The family of operators  $(\mathbf{P}_t^0)_{t \geq 0}$  is an ergodic semi-group. It is stationary for the Gumbel distribution  $\mu$ .

*Proof.* It is clear that  $\mathbf{P}_0^0 = \text{Id}$  and  $\mathbf{P}_t^0$  is a linear operator well defined on  $\text{Lip}_{[2]}(\mathbf{R})$  (recall that functions in this set are Lipschitz and that the Gumbel distribution admits a mean) for all non-negative  $t$ . Now we check the semi-group property. Thanks to equation (6)  $(X + \alpha) \oplus (Y + \beta)$  has the same distribution as  $X + \log(e^\alpha + e^\beta)$ , i.e.  $D_\alpha X \oplus D_\beta Y \stackrel{d}{=} D_{\alpha+\beta} X$ . As a result, we find that  $\mathbf{P}_t^0 \circ \mathbf{P}_s^0 = \mathbf{P}_{t+s}^0$  for all  $t, s \geq 0$ :

$$\begin{aligned} (\mathbf{P}_t^0 \circ \mathbf{P}_s^0)f(x) &= \mathbf{P}_t^0(\mathbf{P}_s^0 f)(x) \\ &= \mathbb{E} \left[ \mathbf{P}_t^0 f((x-s) \oplus (Z + \log(1 - e^{-s}))) \right] \\ &= \mathbb{E} \left[ f((x-s-t) \oplus (Z + \log(1 - e^{-s}) - t) \oplus (Z' + \log(1 - e^{-t}))) \right] \\ &= \mathbb{E} \left[ f((x-s-t) \oplus (Z + \log(e^{-t}(1 - e^{-s}))) \oplus (Z' + \log(1 - e^{-t}))) \right] \\ &= \mathbb{E} \left[ f((x-s-t) \oplus (Z + \log(1 - e^{-s-t}))) \right] \\ &= \mathbf{P}_{t+s}^0 f(x), \end{aligned}$$

where  $Z, Z'$  are i.i.d. and have the Gumbel distribution  $\mathcal{G}(0, 1)$ . To prove ergodicity, we start from the following equality, which stems from the definition of the maximum:

$$\mathbf{P}_t^0 f(x) = f(x-t) \mathbb{P}(Z + \log(1 - e^{-t}) \leq x-t) + \mathbb{E} \left[ f(Z + \log(1 - e^{-t})) \mathbb{1}_{\{Z + \log(1 - e^{-t}) \geq x-t\}} \right]. \quad (8)$$

From this we deduce

$$\mathbf{P}_t^0 f(x) = f(x-t) e^{-\gamma_t e^{-x}} + \mathbb{E} \left[ f(Z + \log(1 - e^{-t})) \mathbb{1}_{\{Z \geq x - \log \gamma_t\}} \right],$$

where we set  $\gamma_t := e^t - 1$ . We then dominate the term in the expectation by:

$$|f(Z + \log(1 - e^{-t})) \mathbb{1}_{\{Z \geq x - \log \gamma_t\}}| \leq |f(0)| + |Z|,$$

and conclude by dominated convergence that  $\mathbf{P}_t^0 f(x)$  tends to  $\mathbb{E}_\mu[f]$  when  $t$  goes to  $+\infty$  for all  $x \in \mathbf{R}$ . Finally, using the same notations as before, the max-stability property of the Gumbel distribution readily gives that  $\mu$  is a stationary measure for  $(\mathbf{P}_t^0)_{t \geq 0}$ :

$$\mathbb{E}[\mathbf{P}_t^0 f(Z)] = \mathbb{E} \left[ f((Z-t) \oplus (Z + \log(1 - e^{-t}))) \right] = \mathbb{E} \left[ f(Z + \log(e^{-t} + 1 - e^{-t})) \right] = \mathbb{E}[f(X)].$$

□

**THEOREM 2.**– The generator  $\mathcal{L}_0$  of the semi-group  $(\mathbf{P}_t^0)_{t \geq 0}$  is given for all  $f \in \text{Lip}_{[2]}(\mathbf{R})$  by:

$$\mathcal{L}_0 f(x) = -f'(x) + e^{-x} \int_0^\infty e^{-y} (f(x+y) - f(x)) dy = -f'(x) + e^{-x} \mathbb{E}[f(x+Y) - f(x)],$$

where  $Y$  is a random variable with exponential distribution  $\mathcal{E}(1)$ . It admits the alternative

expressions:

$$\mathcal{L}_0 f(x) = -f'(x) + \int_x^\infty e^{-y} f'(y) dy = -f'(x) + e^{-x} \mathbb{E}[f'(x+Y)].$$

*Proof.* For starters, we make explicit (8):

$$\begin{aligned} \mathbf{P}_t^0 f(x) &= f(x-t)e^{-\gamma_t e^{-x}} + \int_{x-\log \gamma_t}^\infty f(z + \log \gamma_t - t) e^{-(z+e^{-z})} dz \\ &= f(x-t)e^{-\gamma_t e^{-x}} + \gamma_t \int_x^\infty f(z-t) e^{-(z+\gamma_t e^{-z})} dz. \end{aligned} \quad (9)$$

Differentiating with respect to  $t$ , we find:

$$\begin{aligned} \frac{d}{dt} \mathbf{P}_t^0 f(x) &= -f'(x-t)e^{-\gamma_t e^{-x}} - e^t f(x-t) e^{-x} e^{-\gamma_t e^{-x}} + e^t \int_x^\infty f(z-t) e^{-(z+\gamma_t e^{-z})} dz \\ &\quad + \gamma_t \int_x^\infty \frac{d}{dt} (f(z-t) e^{-(z+\gamma_t e^{-z})}) dz \\ &= -f'(x-t)e^{-\gamma_t e^{-x}} - e^t f(x-t) e^{-x} e^{-\gamma_t e^{-x}} + e^t \int_x^\infty f(z-t) e^{-(z+\gamma_t e^{-z})} dz \\ &\quad + R_t. \end{aligned} \quad (10)$$

The remainder  $R_t$  is equal to

$$R_t = -\gamma_t \left( \int_x^\infty f'(z-t) e^{-(z+\gamma_t e^{-z})} dz + e^t \int_x^\infty f(z-t) e^{-(2z+\gamma_t e^{-z})} dz \right).$$

Since  $f$  is Lipschitz, its derivative is bounded. Thus the dominated convergence theorem justifies the previous computation. The remainder is bounded by:

$$\begin{aligned} |R_t| &\leq \gamma_t \int_x^\infty e^{-(z+\gamma_t e^{-z})} dz + \gamma_t e^t \int_x^\infty (|f(0)| + |z| + t) e^{-(2z+\gamma_t e^{-z})} dz \\ &= 1 - e^{-\gamma_t e^{-x}} + \gamma_t e^t (|f(0)| + t) \int_{x_+}^\infty e^{-(2z+\gamma_t e^{-z})} dz + \gamma_t e^t \int_{x_+}^\infty |z| e^{-(2z+\gamma_t e^{-z})} dz \\ &\leq \gamma_t e^{-x} + \gamma_t e^t (|f(0)| + t) \int_{x_+}^\infty e^{-(z+\gamma_t e^{-z})} dz + \gamma_t e^t \int_{x_+}^\infty e^{-(z+\gamma_t e^{-z})} dz \\ &= \gamma_t e^{-x} (1 + e^t (1 + |f(0)| + t)). \end{aligned}$$

We have used that  $x$  is always less than  $x_+ = x \oplus 0$  as well as the basic inequalities  $e^{-z} \leq 1$  and  $|z| \leq e^z$  whenever  $z$  is non-negative. As a result, the remainder  $R_t$  vanishes as  $t$  goes to  $0^+$ . On the other hand, the first three terms in (10) have a non-trivial limit:

$$\begin{aligned} \mathcal{L}_0 f(x) &= -f'(x) - e^{-x} f(x) + \int_x^\infty e^{-z} f(z) dz \\ &= -f'(x) + \int_x^\infty e^{-z} (f(z) - f(x)) dz \\ &= -f'(x) + e^{-x} \int_0^\infty e^{-y} (f(x+y) - f(x)) dy, \end{aligned}$$

which is the announced expression for the generator of  $(\mathbf{P}_t^0)_{t \geq 0}$ . Furthermore, an integration by parts in the first equality yields:

$$\begin{aligned}\mathcal{L}_0 f(x) &= -f'(x) + \int_x^\infty e^{-y} f'(y) \, dy \\ &= -f'(x) + e^{-x} \int_0^\infty e^{-y} f'(x+y) \, dy \\ &= -f'(x) + e^{-x} \mathbb{E}[f'(x+Y)],\end{aligned}$$

by recognizing the density of a random variable  $Y$  with exponential distribution  $\mathcal{E}(1)$ . □

The next result is an analogue for the Gumbel distribution of the so-called *Stein's lemma*. The latter states that the generator  $\mathcal{N}$  of the standard Ornstein-Uhlenbeck semi-group characterizes the normal distribution  $\gamma$  in the sense that if  $\nu$  is a probability measure:

$$\nu = \gamma \iff \mathbb{E}_\nu[f] = 0$$

for all  $f$  in a large enough class of functions.

**THEOREM 3.**– Let  $X$  be a random variable on  $\mathbf{R}$  such that  $\mathbb{E}[e^{-X}]$  is finite. Then  $X$  has the Gumbel distribution  $\mathcal{G}(0, 1)$  if and only if

$$\mathbb{E}[f'(X)] = \mathbb{E}[e^{-X} f'(X+Y)], \tag{11}$$

for all  $f \in \text{Lip}_{[2]}(\mathbf{R})$ , where  $Y$  has the exponential distribution  $\mathcal{E}(1)$  and is independent of  $X$ .

*Proof.* The reciprocal means that if  $X$  has the Gumbel distribution, then  $\mathbb{E}[\mathcal{L}_0 f(X)] = 0$ . This can be proved by a generic argument regarding semi-groups:

$$\begin{aligned}\mathbb{E}[\mathcal{L}_0 f(X)] &= \mathbb{E}\left[\lim_{t \rightarrow 0} \frac{d}{dt} \mathbf{P}_t^0 f(X)\right] \\ &= \lim_{t \rightarrow 0} \frac{d}{dt} \mathbb{E}[\mathbf{P}_t^0 f(X)] \\ &= \lim_{t \rightarrow 0} \frac{d}{dt} \mathbb{E}[\mathbf{P}_0^0 f(X)] \\ &= 0,\end{aligned}$$

because the Gumbel distribution is a stationary measure for  $(\mathbf{P}_t^0)_{t \geq 0}$ .

Now we turn to the direct implication: we first prove that the Laplace transform  $L_X : \lambda \mapsto \mathbb{E}[e^{-\lambda X}]$  of  $X$  is defined on a neighbourhood of 0. By assumption, we already know that  $L_X(1)$  is finite, and thus  $L_X$  is well-defined on  $[0, 1]$ . Now take  $f = x \mapsto e^{tx}$ , for some  $t \in [0, 1)$ . Then the previous identity gives:

$$L_X(-t) = \mathbb{E}[e^{-(1-t)X} e^{tY}] = L_X(1-t) \frac{1}{1-t}.$$



Because  $t$  is less than 1, the right-hand side is well defined, so the left-hand side is also well-defined. Thus so is  $L_X$  is well defined (at least) on  $(-1, 1]$ . Actually, by taking  $f = x \mapsto e^{-x}$ , it is easily seen that  $L_X$  is in fact defined on  $(-1, 2]$ . By recurrence, one shows that  $L_X$  is well-defined on  $] - 1, \infty)$ .

To determine  $L_X$ , we apply the Bohr-Mollerup theorem (see [2]) to  $f : \lambda \mapsto L_X(\lambda - 1)$ , which is well-defined on  $\mathbf{R}_+^*$  thanks to the previous discussion. This theorem states that if  $f$  is defined on  $\mathbf{R}_+^*$ , is log-convex, satisfies  $f(1) = 1$  and  $f(x + 1) = xf(x)$  for all positive  $x$ , then  $f$  is the Gamma function. Clearly,  $f(1) = L_X(0) = 1$ . It is also well-known that the logarithm of the Laplace transform of a probability measure is convex (this is an easy consequence of Hölder's inequality). Finally, the assumption gives for all positive  $\lambda$ :

$$\begin{aligned} f(\lambda) &= L_X(\lambda - 1) \\ &= \mathbb{E}[e^{-\lambda X} e^{-(\lambda-1)Y}] \\ &= L_X(\lambda) \mathbb{E}[e^{-(\lambda-1)Y}] \\ &= f(\lambda + 1) \frac{1}{\lambda}, \end{aligned}$$

which is the required functional equation. Therefore

$$L_X(\lambda) = \Gamma(\lambda + 1), \quad \lambda \in (-1, +\infty).$$

An easy computation shows that this is indeed the Laplace transform of the Gumbel distribution. Since they coincide on a neighbourhood of 0, this proves that  $X$  has the Gumbel distribution. □

Next we wish to determine how much  $(\mathbf{P}_t^0)_{t \geq 0}$  preserves the regularity of  $f$ , an element of  $\text{Lip}_{[2]}(\mathbf{R})$ . In the sequel we will mainly need of  $f$  to be of class  $\mathcal{C}^1$  and 1-Lipschitz, but at some point it will be necessary to bound uniformly the fluctuations of  $f'$ , thus the third assumption seems inevitable. This is a consequence of the fact that  $(\mathbf{P}_t^0)_t$  has no regularization properties, unlike the original Ornstein-Uhlenbeck semi-group. Furthermore, functions  $f$  in this set have their first derivatives bounded:  $\|f'\|_\infty < \infty$ . This fact ensure that  $\mathcal{L}_0 \mathbf{P}_t^0 f$  behaves well. But first we state an easy lemma which will be very useful in what follows and which can be seen as a pseudo-commutation rule between the derivative and  $\mathbf{P}_t^0$ :

**LEMMA 4.**– Let  $f \in \text{Lip}_{[2]}(\mathbf{R})$ . Then  $x \mapsto \mathbf{P}_t^0 f(x)$  is differentiable for all non-negative  $t$  and:

$$(\mathbf{P}_t^0 f)'(x) = f'(x - t) e^{-\gamma_t e^{-x}}, \quad x \in \mathbf{R}. \tag{12}$$

*Proof.* It suffices to differentiate this alternative expression of  $\mathbf{P}_t^0 f$  with respect to  $x$ :

$$\mathbf{P}_t^0 f(x) = f(x - t) e^{-\gamma_t e^{-x}} + \gamma_t \int_x^\infty f(z - t) e^{-(z + \gamma_t e^{-z})} dz.$$

□

REMARK 1.– The original commutation rule for the standard Ornstein-Uhlenbeck states that  $\nabla \mathbf{P}_t^0 = e^{-t} \mathbf{P}_t^0 \nabla$ . A true commutation rule exists between our semi-group and another kind of 'gradient'. With the notations of theorem 2, one can indeed check that the linear operator

$$\mathbf{D}_0 f(x) := \int_{-\infty}^{\infty} (f(x \oplus y) - f(x)) e^{-y} dy = \int_x^{\infty} f'(y) e^{-y} dy = e^{-x} \mathbb{E}[f'(x + Y)],$$

satisfies the relation  $\mathbf{D}_0 \mathbf{P}_t^0 = e^{-t} \mathbf{P}_t^0 \mathbf{D}_0$ . However we will not make use of it in this paper.

COROLLARY 5.– Let  $f \in \text{Lip}_{[2]}(\mathbf{R})$ . Then for all non-negative  $t$ ,  $\mathbf{P}_t^0 f$  is 1-Lipschitz and its derivative is 2-Lipschitz. Furthermore  $\mathcal{L}_0 \mathbf{P}_t^0 f$  is well defined and, if  $t$  is positive, then  $\mathcal{L}_0 \mathbf{P}_t^0 f$  is bounded:

$$\|\mathcal{L}_0 \mathbf{P}_t^0 f\|_{\infty} \leq \frac{1}{1 - e^{-t}} \|f'\|_{\infty}.$$

*Proof.* Thanks to the previous lemma, we know that  $(\mathbf{P}_t^0 f)'(x) = f'(x - t) e^{-\gamma t e^{-x}}$ . From this we deduce that  $\mathbf{P}_t^0 f$  is continuously differentiable, with bounded and 2-Lipschitz derivative (since a product of 1-Lipschitz functions is 2-Lipschitz). Moreover, for all  $x \in \mathbf{R}$

$$\begin{aligned} \int_x^{\infty} e^{-z} |(\mathbf{P}_t^0 f)'(z)| dz &\leq \|f'\|_{\infty} \int_x^{\infty} e^{-z} e^{-\gamma t e^{-z}} dz \\ &\leq \|f'\|_{\infty} \int_{\mathbf{R}} e^{-z} e^{-\gamma t e^{-z}} dz \\ &= \frac{1}{\gamma t} \|f'\|_{\infty}. \end{aligned}$$

This shows that  $\mathcal{L}_0 \mathbf{P}_t^0 f$  is defined, and even bounded as soon as  $t$  is positive.  $\square$

### 3 Rate of convergence to the Gumbel distribution

Using the results exposed in the previous part, we can initiate the generator approach of Stein's method. Let  $Z$  be a random variable with Gumbel distribution. Because  $\mathbf{P}_0^0 = \text{Id}$  and the Gumbel distribution is stationary for  $(\mathbf{P}_t^0)_{t \geq 0}$ , we can write for all  $f \in \text{Lip}_{[2]}(\mathbf{R})$

$$\begin{aligned} \mathbb{E}[f(Z_n)] - \mathbb{E}[f(Z)] &= \mathbb{E}[\mathbf{P}_0^0 f(Z_n)] - \mathbb{E}[\mathbf{P}_{\infty}^0 f(Z_n)] \\ &= -\mathbb{E}\left[\int_0^{\infty} \frac{d}{dt} \mathbf{P}_t^0 f(Z_n) dt\right] \\ &= -\mathbb{E}\left[\int_0^{\infty} \mathcal{L}_0 \mathbf{P}_t^0 f(Z_n) dt\right] \\ &= -\int_0^{\infty} \mathbb{E}[\mathcal{L}_0 \mathbf{P}_t^0 f(Z_n)] dt. \end{aligned}$$

As a result, to get a rate of convergence of  $Z_n = (T_n - n \log n)/n$  to the Gumbel distribution, we need to study  $\mathbb{E}[\mathcal{L}_0 \mathbf{P}_t^0 f(Z_n)]$  and determine how fast this term goes to 0 when  $n$  tends to infinity. To do so, we will have to compare  $Z_n$  and  $Z_{n-1}$  at several occasions. This involves defining those two random variables on the same probability space. We do so by coupling

$T_n$  and  $T_{n-1}$ . We will need the following random variables: let  $\tau_i^n$  be the number of weeks needed to get the  $i$ -th new item while  $i-1$  distinct ones have already been collected. With this definition:

$$T_n = \sum_{i=1}^n \tau_i^n.$$

Furthermore, it can be easily proved that the  $\tau_i^n$  are independent and have a geometric distribution with parameter  $(n-i+1)/n$  respectively.

Thus we need to define the random sequences  $(\tau_i^n)_{1 \leq i \leq n}$  and  $(\tau_i^{n-1})_{1 \leq i \leq n-1}$  in such a way that they take close values a.s. while having the desired distributions. Our coupling is the following. Put simply, the  $\tau_{i-1}^{n-1}$  are built using the same exponential random variables  $Y_i$  as  $\tau_i^n$ , for  $i \geq 2$  (the case  $i=1$  is trivial:  $\tau_1^n = 1$  a.s.). This coupling itself relies on the well-known inversion method:

LEMMA 6.– Let  $Y_2, \dots, Y_n$  be i.i.d. random variables with exponential distribution  $\mathcal{E}(1)$ . Set  $\tau_1^n = \tau_1^{n-1} = 1$  and:

$$\tau_i^n := \left\lceil -\frac{Y_i}{\log(1-p_i^n)} \right\rceil \text{ and } \tau_{i-1}^{n-1} := \left\lceil -\frac{Y_i}{\log(1-p_{i-1}^{n-1})} \right\rceil, \quad 2 \leq i \leq n$$

where we set:

$$p_i^n := \frac{n-i+1}{n}, \quad n \in \mathbb{N}^*, \quad i \in I_n.$$

Then  $\tau_i^n \sim \mathcal{G}(p_i^n)$  and  $\tau_{i-1}^{n-1} \sim \mathcal{G}(p_{i-1}^{n-1})$ . Furthermore:

$$\tau_{i-1}^{n-1} \leq \tau_i^n, \quad 2 \leq i \leq n.$$

*Proof.* This is simply a consequence of the fact that if  $Y \sim \mathcal{E}(1)$ , then:

$$\left\lceil \frac{Y}{\lambda} \right\rceil \sim \mathcal{G}(e^{-\lambda}), \quad \lambda > 0.$$

The second part of the lemma is proved by noticing that  $p \mapsto \lceil -y/\log(1-p) \rceil$  is non-increasing on  $(0, 1)$  for all  $y \in \mathbf{R}_+$  and that  $p_i^n \leq p_{i-1}^{n-1}$ :

$$p_i^n = \frac{n-i+1}{n} \leq \frac{n-i+1}{n-1} = p_{i-1}^{n-1}.$$

□

Now we can compare  $Z_n$  and  $Z_{n-1}$ . We will need in particular to know how much  $Z_n$  is close to  $Z_{n-1}$  in norm  $\mathbf{L}^1$ :

LEMMA 7.– There exists a positive constant  $C$  such that:

$$\|Z_n - Z_{n-1}\|_{\mathbf{L}^1} \leq C \frac{\log n}{n}, \quad n \geq 2.$$

*Proof.* Notice that since  $\tau_1^n = 1$  a.s., we have for all  $n \geq 2$ :

$$\begin{aligned} Z_n - Z_{n-1} &= \frac{1}{n} \sum_{i=1}^n \tau_i^n - \log n - \frac{1}{n-1} \sum_{i=1}^{n-1} \tau_i^{n-1} + \log(n-1) \\ &= \frac{1}{n} \sum_{i=1}^{n-1} \tau_{i+1}^n - \frac{1}{n-1} \sum_{i=1}^{n-1} \tau_i^{n-1} + \frac{1}{n} + \log\left(1 - \frac{1}{n}\right) \\ &= \frac{1}{n} \sum_{i=1}^{n-1} (\tau_{i+1}^n - \tau_i^{n-1}) - \frac{1}{n(n-1)} \sum_{i=1}^{n-1} \tau_i^{n-1} + \frac{1}{n} + \log\left(1 - \frac{1}{n}\right) \end{aligned}$$

By the triangle inequality, and because  $\tau_{i+1}^n \geq \tau_i^{n-1}$  a.s., we find:

$$\begin{aligned} \|Z_n - Z_{n-1}\|_{\mathbf{L}^1} &\leq \frac{2}{n-1} + \frac{1}{n} \sum_{i=1}^{n-1} \|\tau_{i+1}^n - \tau_i^{n-1}\|_{\mathbf{L}^1} + \frac{1}{n(n-1)} \sum_{i=1}^{n-1} \|\tau_i^{n-1}\|_{\mathbf{L}^1} \\ &= \frac{2}{n-1} + \frac{1}{n} \sum_{i=1}^{n-1} \mathbb{E}[\tau_{i+1}^n - \tau_i^{n-1}] + \frac{1}{n(n-1)} \sum_{i=1}^{n-1} \mathbb{E}[\tau_i^{n-1}] \\ &= \frac{2}{n-1} + \frac{1}{n} \sum_{i=1}^{n-1} \left( \frac{n}{n-i} - \frac{n-1}{n-i} \right) + \frac{1}{n(n-1)} \sum_{i=1}^{n-1} \frac{n-1}{n-i} \\ &= \frac{2}{n-1} + \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{i} + \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{i} \\ &= \frac{2}{n-1} + \frac{2}{n} H_{n-1}, \end{aligned}$$

where  $H_n := \sum_{i=1}^n i^{-1}$  denotes the  $n$ -th harmonic number. □

**Notation 1:** Set  $\alpha_n := 1 - 1/n$  and  $\beta_n := -(n-1) \log \alpha_n$ , as well as:

$$\delta_n := \log\left(1 - \frac{1}{n}\right) - \frac{1}{n} \log(n-1).$$

We will make constant use of the following inequalities in the sequel for  $n \geq 2$ :

$$-1 \leq \delta_n \leq \frac{1}{n} \leq 0 \leq \alpha_n < \beta_n < 1. \quad (13)$$

The next proposition is basically a Stein's identity for  $Z_n$ , which is expressed using both of  $Z_n$  and  $Z_{n-1}$ . Observe that when  $n$  goes to  $\infty$ , it approaches the characteristic relation (11) of the Gumbel distribution.

**THEOREM 8.**— Let  $f$  be a function in  $\mathcal{C}^1(\mathbf{R}, \mathbf{R})$  and  $n \geq 2$ . We have the identity :

$$\mathbb{E}[f'(Z_n)] = C_n \mathbb{E}\left[e^{-\beta_n Z_{n-1}} f'\left(\alpha_n x + \frac{G_n}{n} + \delta_n\right)\right], \quad (14)$$

where the  $\tau_i^{n-1}$  are i.i.d., respectively with distribution  $\text{Geom}\left(\frac{n-i}{n-1}\right)$  for each  $i \in I_{n-1}$ , the random variable  $G_n$  has distribution  $\text{Geom}\left(\frac{1}{n}\right)$  and is independent of the  $\tau_i^{n-1}$  as well as of the

$\tau_i^n$  for all  $i$ , and

$$C_n := n \left(1 - \frac{1}{n}\right)^{(n-1)\log(n-1)} \geq \frac{n}{n-1} \geq 1.$$

*Proof.* The idea is to do a change of probability: we replace the distribution of  $(\tau_1^n, \dots, \tau_{n-1}^n, \tau_n^n)$  by the distribution of  $(\tau_1^{n-1}, \dots, \tau_{n-1}^{n-1}, \tau_n^n)$ . Doing so makes appear a density function which will behave as  $e^{-Z_n} \stackrel{d}{=} e^{-Z}$  when  $n$  goes to infinity, where  $Z \sim \mathcal{G}(0, 1)$ . As for  $\tau_n^n$ , this term is singled out because of the following well-known result:

$$\frac{\tau_n^n}{n} \stackrel{d}{=} \frac{1}{n} \text{Geom}\left(\frac{1}{n}\right) \xrightarrow[n \rightarrow \infty]{d} \mathcal{E}(1).$$

This will give us the exponential random variable which appears in the generator of the Gumbel distribution. Now denote by  $P_j^n$  the joint distribution of  $(\tau_1^n, \dots, \tau_j^n)$  for all  $j \in I_n$  and  $(t_i)_{1 \leq i \leq j} \in (\mathbb{N}^*)^j$ :

$$\begin{aligned} P_j^n(t_1, \dots, t_j) &:= \mathbb{P}\left(\bigcap_{i=1}^j \{\tau_i^n = t_i\}\right) \\ &= \prod_{i=1}^j \left(\frac{n-i+1}{n}\right) \left(1 - \frac{n-i+1}{n}\right)^{t_i-1} \\ &= \prod_{i=1}^j \left(\frac{n-i+1}{n}\right) \left(\frac{i-1}{n}\right)^{t_i-1}. \end{aligned}$$

Furthermore, the density of  $(\tau_1^n, \dots, \tau_{n-1}^n)$  with respect to the distribution of  $(\tau_1^{n-1}, \dots, \tau_{n-1}^{n-1})$  is equal to:

$$\begin{aligned} \frac{P_{n-1}^n(t_1, \dots, t_{n-1})}{P_{n-1}^{n-1}(t_1, \dots, t_{n-1})} &= \frac{\prod_{i=1}^{n-1} \left(\frac{n-i+1}{n}\right) \left(\frac{i-1}{n}\right)^{t_i-1}}{\prod_{i=1}^{n-1} \left(\frac{n-i}{n-1}\right) \left(\frac{i-1}{n-1}\right)^{t_i-1}} \\ &= \left(1 - \frac{1}{n}\right)^{n-1} \frac{n!}{(n-1)!} \frac{\prod_{i=1}^{n-1} \left(\frac{1}{n}\right)^{t_i-1}}{\prod_{i=1}^{n-1} \left(\frac{1}{n-1}\right)^{t_i-1}} \\ &= n \left(1 - \frac{1}{n}\right)^{\sum_{i=1}^{n-1} t_i} \\ &= n \left(1 - \frac{1}{n}\right)^{(n-1) \left(\frac{1}{n-1} \sum_{i=1}^{n-1} t_i\right)} \\ &= n \left(1 - \frac{1}{n}\right)^{(n-1)\log(n-1)} \left(1 - \frac{1}{n}\right)^{(n-1) \left(\frac{1}{n-1} \sum_{i=1}^{n-1} (t_i - \log(n-1))\right)} \end{aligned}$$

Using that equality, we perform a change of probability to get rid of the  $(n-1)$  first  $\tau_i^n$  and replace them by the  $\tau_i^{n-1}$ . Since  $\tau_n^n$  is independent of  $\tau_i^n$  for all  $i \leq n-1$ , we can replace  $\tau_n^n$  by a random variable  $G_n$  with distribution  $\text{Geom}\left(\frac{1}{n}\right)$ . This substitution is essential, as  $\tau_n^n$  is not independent of  $\tau_{n-1}^{n-1}$ , since both have been constructed by using the same exponential random variable  $Y_n$ . In other words, we have used two couplings of  $\tau_n^n$  and  $\tau_{n-1}^{n-1}$ : the first one, described in lemma 6, and a trivial one, with independent geometric random variables.

The purpose of the first was to compare  $Z_n$  and  $Z_{n-1}$ , while the second will make appear the random variable  $Y$  of lemma 11 at the limit.

$$\begin{aligned}
\mathbb{E}[f'(Z_n)] &= \mathbb{E}\left[f'\left(\frac{1}{n} \sum_{i=1}^{n-1} \tau_i^n - \log n + \frac{\tau_n^n}{n}\right)\right] \\
&= \mathbb{E}\left[f'\left(\frac{1}{n} \sum_{i=1}^{n-1} \tau_i^n - \log n + \frac{G_n}{n}\right)\right] \\
&= \mathbb{E}\left[\frac{P_{n-1}^n(\tau_1^{n-1}, \dots, \tau_{n-1}^{n-1})}{P_{n-1}^{n-1}(\tau_1^{n-1}, \dots, \tau_{n-1}^{n-1})} f'\left(\frac{1}{n} \sum_{i=1}^{n-1} \tau_i^{n-1} - \log n + \frac{G_n}{n}\right)\right] \\
&= n\left(1 - \frac{1}{n}\right)^{(n-1)\log(n-1)} \mathbb{E}\left[\left(1 - \frac{1}{n}\right)^{(n-1)Z_{n-1}} f'\left(\frac{1}{n} \sum_{i=1}^{n-1} \tau_i^{n-1} - \log n + \frac{G_n}{n}\right)\right] \\
&= C_n \mathbb{E}\left[\left(1 - \frac{1}{n}\right)^{(n-1)Z_{n-1}} f'\left(\frac{1}{n} \sum_{i=1}^{n-1} \tau_i^{n-1} - \log n + \frac{G_n}{n}\right)\right] \\
&= C_n \mathbb{E}\left[\left(1 - \frac{1}{n}\right)^{(n-1)Z_{n-1}} f'\left(\left(1 - \frac{1}{n}\right)Z_{n-1} + \frac{G_n}{n} + \log\left(1 - \frac{1}{n}\right) - \frac{1}{n} \log(n-1)\right)\right].
\end{aligned}$$

Notice that when  $n$  goes to infinity, the term  $C_n$  before the expectation converges to 1 (as fast as  $\log n/n$  goes to 0). □

**Notation 2:** The following notation will be convenient in the sequel:

$$g_n(x) := e^{-\beta_n x} \mathbb{E}\left[f'\left(\alpha_n x + \frac{G_n}{n} + \delta_n\right)\right]. \quad (15)$$

We will apply identity (14) to  $f = \mathbf{P}_t^0 h$ , which is indeed of class  $\mathcal{C}^1$ . By using the previous results, we will find a bound to control the rate of convergence of  $\mathbb{E}[\mathcal{L}_0 f(Z_n)] = \mathbb{E}[\mathcal{L}_0 \mathbf{P}_t^0 h(Z_n)]$  to 0. Because that bound will be integrated over  $[0, +\infty)$ , we must ensure it is an integrable function of  $t$ . This problem also appears with the standard Ornstein-Uhlenbeck semi-group, at least in dimension 2 or higher (see [6]). A simple way to fix it consists in finding two different bounds, one for  $t$  in  $[0, \varepsilon]$  and the other for  $t$  in  $[\varepsilon, \infty)$ . Optimizing in  $\varepsilon$  then concludes the proof.

**THEOREM 9.**– Let  $h \in \text{Lip}_{[2]}(\mathbf{R})$ . There exists a constant  $C > 0$  such that:

$$|\mathbb{E}[\mathcal{L}_0 \mathbf{P}_t^0 h(Z_n)]| \leq C \frac{1 + |\log \gamma_t| \log n}{\gamma_t n}, \quad t \in [\varepsilon, \infty), \quad n \in \mathbb{N}^*.$$

On the other hand, we have the following bound for  $\mathbb{E}[\mathcal{L}_0 \mathbf{P}_t^0 F(M_n)]$  in the neighborhood of 0:

**THEOREM 10.**– Let  $h \in \text{Lip}_{[2]}(\mathbf{R})$ . There exists a constant  $C > 0$  such that:

$$|\mathbb{E}[\mathcal{L}_0 \mathbf{P}_t^0 h(Z_n)]| \leq C \frac{\log n}{n}, \quad t \in [0, \varepsilon], \quad n \in \mathbb{N}^*.$$

The proofs of these two theorems are postponed to the last section. Using those results, we can conclude:

**THEOREM 11.**– Let  $Z$  be a random variable with the standard Gumbel distribution  $\mathcal{G}(0, 1)$ . Then there exists a constant  $C > 0$  such that for all integer  $n \geq 2$ , the following inequality holds:

$$d_{[2]}(Z_n, Z) \leq C \frac{\log n}{n}.$$

*Proof.* We already know that for every  $h \in \text{Lip}_{[2]}(\mathbf{R})$

$$\mathbb{E}[h(Z_n)] - \mathbb{E}[h(Z)] = - \int_0^\infty \mathbb{E}[\mathcal{L}_0 \mathbf{P}_t^0 h(Z_n)] dt.$$

Let  $\varepsilon > 0$ . Thanks to the two previous bounds (9) and (10), we have:

$$\begin{aligned} |\mathbb{E}[h(Z_n)] - \mathbb{E}[h(Z)]| &\leq \int_0^\infty |\mathbb{E}[\mathcal{L}_0 \mathbf{P}_t^0 h(Z_n)]| dt \\ &\leq \int_0^\varepsilon |\mathbb{E}[\mathcal{L}_0 \mathbf{P}_t^0 h(Z_n)]| dt + \int_\varepsilon^\infty |\mathbb{E}[\mathcal{L}_0 \mathbf{P}_t^0 h(Z_n)]| dt \\ &\leq C \frac{\log n}{n} \varepsilon + C \frac{\log n}{n} \int_\varepsilon^\infty \frac{1 + |\log \gamma_t|}{\gamma_t} dt \\ &= C \frac{\log n}{n} \left( \varepsilon + \int_\varepsilon^\infty \frac{1 + |\log \gamma_t|}{\gamma_t} dt \right). \end{aligned}$$

Since the rate of convergence is the same for  $t \in [0, \varepsilon]$  and for  $t \in [\varepsilon, +\infty)$ , optimize in  $\varepsilon$  only changes the constant. Let us just mention that the function  $\varepsilon \mapsto 1 - |(1 + \log(\gamma_\varepsilon))|/\gamma_\varepsilon$  admits one unique zero around  $\varepsilon = 0.7$ , and it is a global minimum. This concludes the proof.  $\square$

**REMARK 2.**– A result by Gaunt and Li ([10]) together with theorem 11 imply that  $d_K(Z_n, Z)$  is less than  $C(\log n/n)^{1/3}$  (for a different constant  $C$ ). Actually the optimal rate of convergence for the Kolmogorov distance is  $\log n/n$  too. This result is proved in [17] by using Fourier analysis arguments.

The rest of this paper is subdivided into two sections, one for each bound.

## 4 Proofs of theorems 9 and 10

### 4.1 Bounding the error on $[0, \varepsilon]$

The goal of this subsection is to prove theorem 9. For that purpose, we introduce the decomposition upon which relies this subsection and the next:

$$\begin{aligned}
 \mathbb{E}[\mathcal{L}_0 f(Z_n)] &= -\mathbb{E}[f'(Z_n)] + \mathbb{E}[e^{-Z_n} f'(Z_n + Y)] \\
 &= -C_n \mathbb{E}[g_n(Z_{n-1})] + \mathbb{E}[g(Z_n)] \\
 &= (1 - C_n) \mathbb{E}[g_n(Z_{n-1})] + \mathbb{E}[g(Z_n)] - \mathbb{E}[g_n(Z_{n-1})] \\
 &= (1 - C_n) \mathbb{E}[g_n(Z_{n-1})] + (\mathbb{E}[g(Z_n)] - \mathbb{E}[g(Z_{n-1})]) + (\mathbb{E}[g(Z_{n-1})] - \mathbb{E}[g_n(Z_{n-1})]) \\
 &= (1) + (2) + (3), \tag{16}
 \end{aligned}$$

where the second equality is a consequence of proposition 11. Next we deal with (1) and (2). To do so, we give some properties of  $g$  and  $\mathbb{E}[g_n(Z_{n-1})]$ :

LEMMA 12.- Let  $h_t = \mathbf{P}_t^0 h$ , where  $h \in \text{Lip}_{[2]}(\mathbf{R})$ , so that

$$g(x) = e^{-x} \mathbb{E}[e^{-\gamma_t} e^{-(x+Y)} h'(x+Y-t)],$$

where  $Y \sim \mathcal{E}(1)$ . We couple  $G_n$  with  $Y$  in the following way

$$G_n := \frac{1}{n} \left\lceil \frac{-Y}{\log a_n} \right\rceil \sim \text{Geom}\left(\frac{1}{n}\right).$$

Then the followings hold true.

- The function  $g$  is  $(e\gamma_t)^{-1}$ -Lipschitz.
- The sequence  $(\mathbb{E}[g_n(Z_{n-1})])_{n \geq 1}$  is bounded by  $C\gamma_t^{-1}$  for some  $C > 0$ . As a result,

$$(C_n - 1) |\mathbb{E}[g_n(Z_{n-1})]| \leq \frac{C}{\gamma_t} (C_n - 1) = \frac{1}{\gamma_t} O\left(\frac{\log n}{n}\right), \quad t \geq 0, \quad n \in \mathbb{N}^*.$$

*Proof.* - It is enough to notice that  $g'$  admits a simple expression thanks to the change of variable  $y' = y + x$ :

$$g(x) = e^{-x} \int_0^\infty e^{-\gamma_t e^{-(x+z)}} h'(x+z-t) e^{-z} dz = \int_x^\infty e^{-z} e^{-\gamma_t e^{-z}} h'(z-t) dz.$$

Differentiating with respect to  $x$ , we find that  $g'(x) = -e^{-x} e^{-\gamma_t e^{-x}} h'(x-t)$ , so that

$$|g'(x)| \leq e^{-x} e^{-\gamma_t e^{-x}} \leq \frac{e^{-1}}{\gamma_t}.$$

- Recall that  $g_n$  is given by (15), and that the constants  $\alpha_n, \beta_n$  and  $\delta_n$  have been defined just before (13). We know that the double exponential term is a non-decreasing



function, and that  $\lceil y \rceil \leq y + 1$ , for every non-negative  $y$ . Besides, as  $-n \log \alpha_n \geq 1$  and  $\delta_n \leq -1/n$ , we have:

$$-\frac{Y}{n \log \alpha_n} + \delta_n \leq \frac{G_n}{n} + \delta_n \leq -\frac{Y}{n \log \alpha_n} + \delta_n + \frac{1}{n} \leq Y. \quad (17)$$

Moreover, since  $Z_{n-1} \geq 1 - \log(n-1) \geq -\log n$  a.s., and as  $x$  will be replaced by  $Z_{n-1}$  in the sequel, we only need to work with  $x \geq -\log n$ :

$$\begin{aligned} |g_n(x)| &\leq e^{-\beta_n x} \mathbb{E} \left[ e^{-\gamma_t e^{-(\alpha_n x + \frac{G_n}{n} + \delta_n)}} \right] \\ &\leq e^{-\beta_n x} \mathbb{E} \left[ e^{-\gamma_t e^{-(\alpha_n x + Y)}} \right] \\ &= \frac{1}{\gamma_t} e^{-(\beta_n - \alpha_n)x} (1 - e^{-\gamma_t e^{-\alpha_n x}}) \\ &\leq \frac{1}{\gamma_t} e^{-(\beta_n - \alpha_n)x} \\ &\leq \frac{1}{\gamma_t} n^{\beta_n - \alpha_n}. \end{aligned}$$

as for all positive  $\gamma$  we have  $\mathbb{E}[e^{-\gamma e^{-Y}}] = (1 - e^{-\gamma})/\gamma$  if  $Y$  has exponential distribution  $\mathcal{E}(1)$ . This last term is bounded with respect to  $n$ , because  $n^{\beta_n - \alpha_n} \underset{n \rightarrow \infty}{\sim} 1 + \log n / (2n)$ .

Finally, notice that:

$$0 \leq C_{n-1} = n \left(1 - \frac{1}{n}\right)^{(n-1) \log(n-1)} - 1 \leq n e^{-(1 - \frac{1}{n}) \log(n-1)} - 1 = e^{\log(1 + \frac{1}{n-1}) + \frac{1}{n} \log n} - 1 \underset{n \rightarrow \infty}{\sim} \frac{\log n}{n}.$$

The intermediary result, available for  $x$  greater than  $-\log n$

$$e^{-\beta_n x} \mathbb{E} \left[ e^{-\gamma_t e^{-(\alpha_n x + \frac{G_n}{n} + \delta_n)}} \right] \leq \frac{C}{\gamma_t}, \quad (18)$$

will be useful on its own in the sequel. □

This lemma has two immediate conclusions: the first item furnishes a bound for (1), while the second one directly settles the case of (2) in (16):

$$|\mathbb{E}[g(Z_n)] - \mathbb{E}[g(Z_{n-1})]| \leq \frac{e^{-1}}{\gamma_t} \|Z_n - Z_{n-1}\|_{\mathbf{L}^1} \leq \frac{C \log n}{\gamma_t n}.$$

As for (3), we will need to decompose it further:

$$\begin{aligned} &\mathbb{E}[g(Z_{n-1})] - \mathbb{E}[g_n(Z_{n-1})] \\ &= \left( \mathbb{E}[g(Z_{n-1})] - \mathbb{E} \left[ e^{-\beta_n Z_{n-1}} e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)}} h'(Z_{n-1} + Y - t) \right] \right) \\ &\quad + \left( \mathbb{E} \left[ e^{-\beta_n Z_{n-1}} e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)}} h'(Z_{n-1} + Y - t) \right] - \mathbb{E}[g_n(Z_{n-1})] \right) \\ &= (3.1) + (3.2). \end{aligned}$$

Let us start with (3.2). Recall that  $\delta_n$  is non-positive:

$$\begin{aligned}
& \left| \mathbb{E} \left[ e^{-\beta_n Z_{n-1}} e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)}} h'(Z_{n-1} + Y - t) \right] - \mathbb{E}[g_n(Z_{n-1})] \right| \\
& \leq \mathbb{E} \left[ e^{-\beta_n Z_{n-1}} e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)}} \left| h'(Z_{n-1} + Y - t) - h' \left( \alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n - t \right) \right| \right] \\
& \leq \mathbb{E} \left[ e^{-\beta_n Z_{n-1}} e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)}} \left| (1 - \alpha_n) Z_{n-1} + Y - \frac{G_n}{n} - \delta_n \right| \right] \\
& \leq \mathbb{E} \left[ e^{-\beta_n Z_{n-1}} e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)}} \left( \frac{1}{n} |Z_{n-1}| + \left| Y - \frac{G_n}{n} \right| - \delta_n \right) \right].
\end{aligned}$$

That last expression is also equal to:

$$\begin{aligned}
& \frac{1}{n} \mathbb{E} \left[ e^{-\beta_n Z_{n-1}} e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)}} (|Z_{n-1}| - \delta_n) \right] + \mathbb{E} \left[ e^{-\beta_n Z_{n-1}} e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} - \delta_n)}} \left| Y - \frac{G_n}{n} \right| \right] \\
& \leq \frac{C}{n\gamma_t} (\mathbb{E}[|Z_{n-1}|] - \delta_n) + \left( \mathbb{E} \left[ e^{-2\beta_n Z_{n-1}} e^{-2\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)}} \right] \right)^{\frac{1}{2}} \left\| Y - \frac{G_n}{n} \right\|_{\mathbf{L}^2} \\
& \leq \frac{C}{n\gamma_t} (\mathbb{E}[|Z_{n-1}|] - \delta_n) + \frac{C}{\gamma_t} (\mathbb{E}[e^{-\beta_n Z_{n-1}}])^{\frac{1}{2}} \left\| Y - \frac{G_n}{n} \right\|_{\mathbf{L}^2}.
\end{aligned}$$

We have used successively inequality (18), Cauchy-Schwarz inequality and once more inequality (18). The sequences  $(\mathbb{E}[|Z_{n-1}|])_{n \geq 1}$  and  $(\mathbb{E}[e^{-Z_{n-1}}])_{n \geq 1}$  are bounded. This is easily seen for the first one, for instance by recalling that:

$$\mathbb{V}(Z_n) = \frac{\pi^2}{6} + o(1).$$

As for the second sequence, notice that since  $\beta_n \leq 1$ , we have:

$$\mathbb{E}[e^{-\beta_n Z_{n-1}}] \leq (\mathbb{E}[e^{-Z_{n-1}}])^{\beta_n} \leq 1,$$

thanks to the next statement:

**LEMMA 13.**– For all  $\lambda \geq 0$ , the sequence  $(\mathbb{E}[e^{-\lambda Z_n}])_{n \geq 1}$  is bounded. Moreover, we have the asymptotic expansion:

$$\mathbb{E}[e^{-Z_n}] - 1 \underset{n \rightarrow \infty}{\sim} -\frac{\log n}{2n}. \quad (19)$$

*Proof.* For all  $n \geq 1$ , we have by definition of  $Z_n$ :

$$\mathbb{E}[e^{-\lambda Z_n}] = n^\lambda \prod_{i=1}^n \frac{1 - \frac{i-1}{n}}{e^{\frac{\lambda}{n}} - \frac{i-1}{n}} = n^\lambda n! \prod_{i=1}^n \frac{1}{n e^{\frac{\lambda}{n}} - i + 1} \leq n^\lambda n! \prod_{i=1}^n \frac{1}{n - i + 1 + \lambda} = n^\lambda n! \prod_{i=1}^n \frac{1}{i + \lambda},$$

thanks to the inequality  $n e^{\lambda/n} \geq n + \lambda$ . Therefore:

$$\mathbb{E}[e^{-\lambda Z_n}] = n^\lambda \prod_{i=1}^n \frac{i}{i + \lambda} \leq e^{-\lambda \left( \sum_{i=1}^n \frac{1}{i + \lambda} - \log n \right)} = e^{-\lambda \left( \sum_{i=1}^n \frac{1}{i} - \log n - \lambda \sum_{i=1}^n \frac{1}{i(i + \lambda)} \right)},$$

and using the well-known asymptotic expansion  $\sum_{i=1}^n i^{-1} = \log n + \gamma + o(1)$ , where  $\gamma = 0.577\dots$  is the so-called gamma constant, we conclude that the right-hand side converges for every non-negative  $\lambda$ . As for the second part of the statement

$$\begin{aligned}
\mathbb{E}[e^{-Z_n}] &= nn! \prod_{i=1}^n \frac{1}{ne^{1/n} - i + 1} \\
&= \frac{nn!}{(n+1)!} \prod_{i=1}^n \frac{n-i+2}{ne^{1/n} - i + 1} \\
&= \frac{n}{n+1} \prod_{i=1}^n \frac{n-i+2}{ne^{1/n} - i + 1} \\
&= \frac{n}{n+1} \prod_{i=1}^n \frac{n-i+2}{(n(e^{1/n} - 1) - 1) + n - i + 2} \\
&= \frac{n}{n+1} \prod_{i=1}^n \left(1 + \frac{n(e^{1/n} - 1) - 1}{i+1}\right)^{-1} \\
&= \frac{n}{n+1} \exp\left(-\sum_{i=1}^n \log\left(1 + \frac{n(e^{1/n} - 1) - 1}{n-i+2}\right)\right)
\end{aligned}$$

Recall that:

$$n(e^{1/n} - 1) - 1 \underset{n \rightarrow \infty}{\sim} \frac{1}{2n}.$$

Therefore, a standard result about positive divergent series yields:

$$\mathbb{E}[e^{-Z_n}] - 1 \underset{n \rightarrow \infty}{\sim} \frac{n}{n+1} e^{-\frac{1}{2n}H_n} - 1 \underset{n \rightarrow \infty}{\sim} e^{-\frac{1}{2n}\log n} - 1 \underset{n \rightarrow \infty}{\sim} -\frac{1}{2n}\log n,$$

which concludes the proof.  $\square$

As a result, we need to compare  $Y$  and  $G_n/n$ . We already know that the latter converges weakly to  $Y$ . To get a rate of convergence, we use the fact that  $G_n$  depends of  $Y$  in a simple way:

**LEMMA 14.**– Let  $Y \sim \mathcal{E}(1)$  and  $G_n$  defined as before. Then  $(G_n/n)_{n \geq 1}$  converges to  $Y$  in  $\mathbf{L}^2$  norm, with rate of convergence at least  $\sqrt{2}/n$ :

$$\left\| Y - \frac{G_n}{n} \right\|_{\mathbf{L}^2} \leq \frac{\sqrt{2}}{n}, \quad n \geq 2.$$

*Proof.* To determine the rate of convergence in  $\mathbf{L}^2$  norm, we will need to compute  $\mathbb{E}[Y[Y/\lambda]]$

for some  $\lambda > 0$ . Set  $u_n := \int_0^{n\lambda} ye^{-y} dy$

$$\begin{aligned}
\mathbb{E}\left[Y\Gamma\left[\frac{Y}{\lambda}\right]\right] &= \int_0^\infty y\Gamma\left[\frac{y}{\lambda}\right]e^{-y} dy \\
&= \sum_{n=0}^\infty (n+1) \int_{n\lambda}^{(n+1)\lambda} ye^{-y} dy \\
&= \sum_{n=0}^\infty (n+1)(u_n - u_{n+1}) \\
&= \sum_{n=0}^\infty u_n + (nu_n - (n+1)u_{n+1}) \\
&= \sum_{n=0}^\infty u_n \\
&= \frac{\lambda e^{-\lambda}}{(1 - e^{-\lambda})^2} + \frac{1}{1 - e^{-\lambda}}.
\end{aligned}$$

Evaluating this identity at  $\lambda = -\log(1 - 1/n) > 0$ , we get that

$$\begin{aligned}
\mathbb{E}\left[\left|Y - \frac{G_n}{n}\right|^2\right] &= 2 + 2 - \frac{1}{n} - \frac{2}{n}\mathbb{E}\left[Y\Gamma\left[\frac{Y}{\lambda}\right]\right] \\
&= 4 - \frac{1}{n} + 2(n-1)\log\left(1 - \frac{1}{n}\right) - 2 \\
&= 2 - \frac{1}{n} + 2(n-1)\log\left(1 - \frac{1}{n}\right) \\
&\leq \frac{2}{n^2}.
\end{aligned}$$

The last inequality comes the well-known inequality:

$$\log\left(1 - \frac{1}{x}\right) \leq -\frac{1}{x} - \frac{1}{2x^2}, \quad x \geq 1.$$

□

And so (3.2) is bounded by  $C/(\gamma_t n)$ , for some constant  $C > 0$  independent of  $t$ . Now we move onto (3.1). Before bounding this term, we show briefly how we can replace  $e^{-\beta_n Z_{n-1}}$  by  $e^{-Z_{n-1}}$ .

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(e^{-\beta_n Z_{n-1}} - e^{-Z_{n-1}}\right)e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)}} h'(Z_{n-1} + Y - t)\right]\right| \\
&\leq \mathbb{E}\left[\left|e^{-\beta_n Z_{n-1}} - e^{-Z_{n-1}}\right|e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)}}\right] \\
&\leq \mathbb{E}\left[\left|e^{-\beta_n Z_{n-1}} - e^{-Z_{n-1}}\right|e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + Y)}}\right] \\
&= \mathbb{E}\left[\left|e^{-\beta_n Z_{n-1}} - e^{-Z_{n-1}}\right|\varphi_n(Z_{n-1})\right],
\end{aligned}$$

where we set  $\varphi_n(x) := \mathbb{E}\left[e^{-\gamma_t e^{-(\alpha_n x + Y)}}\right]$  for all  $x \geq -\log n$ . The change of variable  $y' = y - \log \gamma_t$  shows that:

$$\varphi_n(x) = \frac{1}{\gamma_t} \int_{-\log \gamma_t}^\infty e^{-y} e^{-e^{-(\alpha_n x + y)}} dy \leq \frac{1}{\gamma_t} \int_{\mathbf{R}} e^{-y} e^{-e^{-(\alpha_n x + y)}} dy = \frac{1}{\gamma_t} e^{\alpha_n x}.$$

Thus, noticing that  $z \mapsto e^{-Z_{n-1}z}$  is Lipschitz on  $[\beta_n - \alpha_n, 1 - \alpha_n]$ , with Lipschitz constant less than  $|Z_{n-1}|(e^{-(\beta_n - \alpha_n)Z_{n-1}} + e^{-(1 - \alpha_n)Z_{n-1}})$  almost surely, we can replace  $\beta_n$  by 1:

$$\begin{aligned} \left| \mathbb{E} \left[ (e^{-\beta_n Z_{n-1}} - e^{-Z_{n-1}}) \varphi_n(Z_{n-1}) \right] \right| &\leq \frac{C}{\gamma_t} \mathbb{E} \left[ |e^{-(\beta_n - \alpha_n)Z_{n-1}} - e^{-(1 - \alpha_n)Z_{n-1}}| \right] \\ &\leq \frac{C}{\gamma_t} (1 - \beta_n) \mathbb{E} \left[ |Z_{n-1}| (e^{-(\beta_n - \alpha_n)Z_{n-1}} + e^{-(1 - \alpha_n)Z_{n-1}}) \right]. \end{aligned}$$

We have used Cauchy-Schwarz inequality and lemma 13 to bound the remaining expectation. As a result, we can exchange (3.1) for

$$\begin{aligned} \mathbb{E} \left[ e^{-Z_{n-1}} (e^{-\gamma_t e^{-(Z_{n-1}+Y)}} - e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)}}) h'(Z_{n-1} + Y - t) \right] \\ \leq \mathbb{E} \left[ e^{-Z_{n-1}} |e^{-\gamma_t e^{-(Z_{n-1}+Y)}} - e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)}}| \right]. \end{aligned} \quad (20)$$

As tempting as it may be, we *cannot* exploit (yet) the fact that  $x \mapsto e^{-\gamma_t e^{-x}}$  is Lipschitz, because doing so would yield an expression without the term  $\gamma_t^{-1}$ , which is crucial to find an integrable bound with respect to  $t$  on  $[\varepsilon, +\infty)$ . Set  $\varepsilon_n := -n \log \alpha_n \geq 1$ . Then the above expression is bounded by:

$$\begin{aligned} &\mathbb{E} \left[ e^{-Z_{n-1}} |e^{-\gamma_t e^{-(Z_{n-1}+Y)}} - e^{-\gamma_t e^{-(\alpha_n Z_{n-1}+Y)}}| \right] \\ &\quad + \mathbb{E} \left[ e^{-Z_{n-1}} (e^{-\gamma_t e^{-(\alpha_n Z_{n-1}+Y)}} - e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)}}) \right] \\ &= \mathbb{E} \left[ e^{-Z_{n-1}} (e^{-\gamma_t e^{-(\alpha_n Z_{n-1}+Y)}} - e^{-\gamma_t e^{-(Z_{n-1}+Y)}}) \mathbb{1}_{\{Z_{n-1} \leq 0\}} \right] \\ &\quad + \mathbb{E} \left[ e^{-Z_{n-1}} (e^{-\gamma_t e^{-(Z_{n-1}+Y)}} - e^{-\gamma_t e^{-(\alpha_n Z_{n-1}+Y)}}) \mathbb{1}_{\{Z_{n-1} > 0\}} \right] \\ &\quad + \mathbb{E} \left[ e^{-Z_{n-1}} (e^{-\gamma_t e^{-(\alpha_n Z_{n-1}+Y)}} - e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)}}) \right] \\ &\leq \mathbb{E} \left[ e^{-Z_{n-1}} (e^{-\gamma_t e^{-(\alpha_n Z_{n-1}+Y)}} - e^{-\gamma_t e^{-(Z_{n-1}+Y)}}) \mathbb{1}_{\{Z_{n-1} \leq 0\}} \right] \\ &\quad + \mathbb{E} \left[ e^{-Z_{n-1}} (e^{-\gamma_t e^{-(Z_{n-1}+Y)}} - e^{-\gamma_t e^{-(\alpha_n Z_{n-1}+Y)}}) \mathbb{1}_{\{Z_{n-1} > 0\}} \right] \\ &\quad + \mathbb{E} \left[ e^{-Z_{n-1}} (e^{-\gamma_t e^{-(\alpha_n Z_{n-1}+Y)}} - e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{Y}{\varepsilon_n} + \delta_n)}}) \right]. \end{aligned}$$

We have used the inequality (17) at the last line, which amounts to saying that  $G_n/n \geq Y/\varepsilon_n$ . Unfortunately, in spite of the monotony of the double exponential and the inequality  $G_n/n + \delta_n \leq Y$ , we could not remove the absolute values in the first expression, because it is not true that  $\alpha_n x \leq x$  if  $x < 0$ . This explains the need for the above inequality. With the absolute value out of the picture, the two first terms become easy to estimate as we know that for  $\alpha \in \{\alpha_n, 1\}$ ,

$$e^{-x} \mathbb{E} \left[ e^{-\gamma_t e^{-(\alpha x + Y)}} \right] = \frac{1}{\gamma_t} e^{-(1-\alpha)x} (1 - e^{-\gamma_t e^{-\alpha x}}), \quad x \in \mathbf{R}$$

so that for all  $x \in \mathbf{R}$ :

$$\begin{aligned} \mathbb{E} \left[ e^{-x} (e^{-\gamma_t e^{-(\alpha_n x + Y)}} - e^{-\gamma_t e^{-(x+Y)}}) \right] &= \frac{1}{\gamma_t} \left( (e^{-(1-\alpha_n)x} - 1) + (e^{-\gamma_t e^{-x}} - e^{-(1-\alpha_n)x} e^{-\gamma_t e^{-\alpha_n x}}) \right) \\ &= \frac{1}{\gamma_t} \left( (e^{-(1-\alpha_n)x} - 1)(1 - e^{-\gamma_t e^{-\alpha_n x}}) + (e^{-\gamma_t e^{-x}} - e^{-\gamma_t e^{-\alpha_n x}}) \right) \end{aligned}$$

Now we can bound the first two terms by putting the indicator functions together again:

$$\begin{aligned} \mathbb{E} \left[ e^{-Z_{n-1}} |e^{-\gamma_t e^{-(Z_{n-1}+Y)}} - e^{-\gamma_t e^{-(\alpha_n Z_{n-1}+Y)}}| \right] \\ \leq \frac{1}{\gamma_t} \mathbb{E} \left[ |e^{-(1-\alpha_n)Z_{n-1}} - 1| + |e^{-\gamma_t e^{-Z_{n-1}}} - e^{-\gamma_t e^{-\alpha_n Z_{n-1}}}| \right]. \end{aligned}$$

Controlling these two expectations is simple: first the function  $z \mapsto e^{-Z_{n-1}z}$  is Lipschitz on  $[0, 1 - \alpha_n]$ , with Lipschitz constant less than  $|Z_{n-1}|(1 + e^{-(1-\alpha_n)Z_{n-1}})$  almost surely. Second, remark that the function  $z \mapsto e^{-\gamma_t e^{-z}}$  is  $e^{-1}$  Lipschitz. Thanks to Cauchy-Schwartz inequality, we know that

$$\sup_{n \in \mathbb{N}^*} \mathbb{E} [ |Z_{n-1}| e^{-\lambda Z_{n-1}} ] < +\infty, \lambda \geq 0.$$

Therefore, we can write:

$$\mathbb{E} \left[ e^{-Z_{n-1}} |e^{-\gamma_t e^{-(Z_{n-1}+Y)}} - e^{-\gamma_t e^{-(\alpha_n Z_{n-1}+Y)}}| \right] \leq \frac{C}{\gamma_t} \frac{1}{n}.$$

The third term is a source of troubles because  $Y$  is divided by  $\varepsilon_n = -n \log \alpha_n \geq 1$ . A change of variable gives:

$$e^{-x} \mathbb{E} \left[ e^{-\gamma_t e^{-(\alpha_n x + \frac{Y}{\varepsilon_n} + \delta_n)}} \right] = \varepsilon_n e^{-x} \int_0^\infty e^{-\varepsilon_n y} e^{-\gamma_t e^{-(\alpha_n x + y + \delta_n)}} dy, x \in \mathbf{R}.$$

We need to compare this integral to  $e^{-x} \int_0^\infty e^{-y} e^{-\gamma_t e^{-(\alpha_n x + y + \delta_n)}} dy$ :

LEMMA 15.– There exists a constant  $C > 0$  such that for all  $n \geq 2$ , we have for all positive  $t$  and for all real numbers  $x$ :

$$\begin{aligned} e^{-x} \left| \int_0^\infty e^{-y} e^{-\gamma_t e^{-(\alpha_n x + y)}} dy - \varepsilon_n \int_0^\infty e^{-\varepsilon_n y} e^{-\gamma_t e^{-(\alpha_n x + y + \delta_n)}} dy \right| \\ \leq C \frac{|x| + |\log \gamma_t| \log n}{\gamma_t n} e^{-\frac{1}{n}x}. \end{aligned}$$

In particular

$$\begin{aligned} \mathbb{E} \left[ e^{-Z_{n-1}} \left| \int_0^\infty e^{-y} e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + y)}} dy - \varepsilon_n \int_0^\infty e^{-\varepsilon_n y} e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + y + \delta_n)}} dy \right| \right] \\ \leq C \frac{1 + |\log \gamma_t| \log n}{\gamma_t n}. \end{aligned}$$

*Proof.* The arguments are similar to those given above, but certain new difficulties arise nonetheless.

- First we replace  $\varepsilon_n = -n \log(1 - 1/n)$  by 1 by using that  $\varepsilon_n \geq 1$ . Recall that  $\delta_n$  is negative, so that:

$$0 \leq (\varepsilon_n - 1) \int_0^\infty e^{-\varepsilon_n y} e^{-\gamma_t e^{-(\alpha_n x + y + \delta_n)}} dy \leq (\varepsilon_n - 1) \int_0^\infty e^{-y} e^{-\gamma_t e^{-(\alpha_n x + y)}} dy \leq \frac{e^{\alpha_n x}}{\gamma_t} (\varepsilon_n - 1).$$

Next we take care of the  $e^{-\varepsilon_n y}$  part in the integral. As before, observe that  $z \mapsto e^{-y z}$  is Lipschitz on  $[1, \varepsilon_n]$ , with Lipschitz constant  $y e^{-y}$ :

$$\begin{aligned} 0 &\leq \int_0^\infty (e^{-y} - e^{-\varepsilon_n y}) e^{-\gamma_t e^{-(\alpha_n x + y + \delta_n)}} dy \\ &\leq (\varepsilon_n - 1) \int_0^\infty y e^{-y} e^{-\gamma_t e^{-(\alpha_n x + y + \delta_n)}} dy = (\varepsilon_n - 1) e^{-\tau} \mathbb{E}[(Z + \tau) \mathbb{1}_{\{Z + \tau \geq 0\}}], \end{aligned}$$

where  $\tau := \log \gamma_t - \alpha_n x - \delta_n \in \mathbf{R}$  and  $Z$  is a random variable with standard Gumbel distribution. The right-hand side makes appear  $|\log \gamma_t|$  in the bound:

$$\begin{aligned} 0 &\leq \int_0^\infty (e^{-y} - e^{-\varepsilon_n y}) e^{-\gamma_t e^{-(\alpha_n x + y + \delta_n)}} dy \\ &\leq e^{-\tau} (C + |\log \gamma_t| - \alpha_n |x|) (\varepsilon_n - 1) = \frac{e^{\alpha_n x + \delta_n}}{\gamma_t} (C + |\log \gamma_t| + \alpha_n |x|) (\varepsilon_n - 1) \end{aligned}$$

- As for the final term, recall once more that  $\delta_n = \log(1 - 1/n) - \log(n - 1)/n \leq 0$ :

$$\begin{aligned} 0 &\leq \int_0^\infty e^{-\varepsilon_n y} e^{-\gamma_t e^{-(\alpha_n x + y)}} dy - \int_0^\infty e^{-\varepsilon_n y} e^{-\gamma_t e^{-(\alpha_n x + y + \delta_n)}} dy \\ &= \int_0^\infty e^{-\varepsilon_n y} e^{-\gamma_t e^{-(\alpha_n x + y)}} dy - e^{\varepsilon_n \delta_n} \int_{\delta_n}^\infty e^{-\varepsilon_n y} e^{-\gamma_t e^{-(\alpha_n x + y)}} dy \\ &\leq \int_0^\infty e^{-\varepsilon_n y} e^{-\gamma_t e^{-(\alpha_n x + y)}} dy - e^{\varepsilon_n \delta_n} \int_0^\infty e^{-\varepsilon_n y} e^{-\gamma_t e^{-(\alpha_n x + y)}} dy \\ &= (1 - e^{\varepsilon_n \delta_n}) \int_0^\infty e^{-\varepsilon_n y} e^{-\gamma_t e^{-(\alpha_n x + y)}} dy \\ &\leq \frac{e^{\alpha_n x}}{\gamma_t} (1 - e^{\varepsilon_n \delta_n}). \end{aligned}$$

The second inequality of the statement is obtained by replacing  $x$  by  $Z_{n-1}$  and integrating with respect to this random variable. Recall that the sequence  $(\mathbb{E}[|Z_{n-1}| e^{-\lambda Z_{n-1}}])_{n \geq 1}$  is bounded for all non-negative  $\lambda$ . □

## 4.2 Bounding the error on $[0, \varepsilon]$

Theorem 10 is proved in this subsection.

We have already found a bound depending on  $\gamma_t^{-1}$ , so we have to obtain another bound, constant with respect to  $t$ . This boils down to bounding the same four terms as before, but independently of  $t$ , which makes the task much faster and easier. First, we have

$$\begin{aligned} (1 - C_n) |\mathbb{E}[g_n(Z_{n-1})]| &\leq \mathbb{E} \left[ e^{-\beta_n Z_{n-1}} e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)}} \right] \\ &\leq (1 - C_n) \mathbb{E}[e^{-\beta_n Z_{n-1}}] \\ &\lesssim (1 - C_n), \end{aligned}$$

and we already know that this last term goes to 0 as fast as  $\log n/n$ . The second term is dealt with in the next lemma:

**LEMMA 16.**– Let  $g : x \mapsto e^{-x} \mathbb{E}[e^{-\gamma t} h'(x + Y - t)]$ , where  $h \in \text{Lip}_{[2]}(\mathbf{R})$  and  $Y$  is a random variable with exponential distribution  $\mathcal{E}(1)$ . Then there exists  $C > 0$  such that:

$$|\mathbb{E}[g(Z_n)] - \mathbb{E}[g(Z_{n-1})]| \leq C \frac{\log n}{n}, \quad t \geq 0, n \geq 2.$$

*Proof.* We have that:

$$\begin{aligned} & |\mathbb{E}[g(Z_n)] - \mathbb{E}[g(Z_{n-1})]| \\ & \leq \mathbb{E} \left[ e^{-Z_n} |e^{-\gamma t} e^{-(Z_n+Y)} h'(Z_n + Y - t) - e^{-\gamma t} e^{-(Z_{n-1}+Y)} h'(Z_{n-1} + Y - t)| \right] \\ & \quad + \mathbb{E} \left[ |e^{-Z_n} - e^{-Z_{n-1}}| e^{-\gamma t} e^{-(Z_{n-1}+Y)} |h'(Z_{n-1} + Y - t)| \right] \\ & \leq \mathbb{E}[e^{-Z_n} |Z_n - Z_{n-1}|] + \mathbb{E}[|e^{-Z_n} - e^{-Z_{n-1}}|]. \end{aligned}$$

To exploit the fact that  $\tau_{i+1}^n \geq \tau_i^{n-1}$ , recall that  $\tau_1^n = 1$  a.s. and introduce:

$$Z'_n := \frac{1}{n-1} \sum_{i=1}^n \tau_i^n - \log(n-1) = \frac{1}{n-1} \sum_{i=1}^{n-1} \tau_{i+1}^n + \frac{1}{n-1} - \log(n-1), \quad n \geq 2.$$

Clearly  $Z'_n \geq Z_{n-1}$  and  $Z'_n \geq Z_n$  since

$$Z_n - Z'_n = -\frac{1}{n(n-1)} \sum_{i=1}^n \tau_i^n + \log\left(1 - \frac{1}{n}\right) \leq 0.$$

Therefore:

$$\begin{aligned} |\mathbb{E}[g(Z_n)] - \mathbb{E}[g(Z_{n-1})]| & \leq \mathbb{E}[e^{-Z_n} |Z_n - Z_{n-1}|] + \mathbb{E}[|e^{-Z_n} - e^{-Z_{n-1}}|] \\ & \leq \mathbb{E}[e^{-Z_n} (Z'_n - Z_n)] + \mathbb{E}[e^{-Z_n} (Z'_n - Z_{n-1})] \\ & \quad + \mathbb{E}[e^{-Z_n} - e^{-Z'_n}] + \mathbb{E}[e^{-Z_{n-1}} - e^{-Z'_n}] \\ & = (1) + (2) + (3) + (4). \end{aligned}$$

We start with (1). Introduce the notation:

$$Z_{n \setminus i} := \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^n \tau_j^n - \log n, \quad i \in I_n.$$

Now conditioning on  $Y_i$ , the  $i$ -th exponential random variable in the definitions of  $\tau_{i+1}^n$  and



$\tau_i^{n-1}$ , we can write:

$$\begin{aligned}
\mathbb{E}[e^{-Z_n}(Z'_n - Z_n)] &= \frac{1}{n(n-1)} \sum_{i=1}^n \mathbb{E}[e^{-Z_n} \tau_i^n] - \log\left(1 - \frac{1}{n}\right) \mathbb{E}[e^{-Z_n}] \\
&\leq \frac{1}{n(n-1)} \sum_{i=1}^n \mathbb{E}[e^{-Z_n} \tau_i^n] + \frac{1}{n-1} \\
&= \frac{1}{n(n-1)} \sum_{i=1}^n \mathbb{E}[e^{-Z_{n \setminus i}}] \mathbb{E}[e^{-\frac{1}{n} \tau_i^n} \tau_i^n] + \frac{1}{n-1} \\
&\leq \frac{1}{n(n-1)} \sum_{i=1}^n \mathbb{E}[e^{-Z_{n \setminus i}}] \mathbb{E}[\tau_i^n] + \frac{1}{n-1} \\
&= \frac{1}{n-1} \sum_{i=1}^n \mathbb{E}[e^{-Z_{n \setminus i}}] \frac{1}{n-i+1} + \frac{1}{n-1}.
\end{aligned}$$

That last expectation is bounded with respect to  $n$ :

$$\begin{aligned}
\mathbb{E}[e^{-Z_{n \setminus i}}] &= n \prod_{\substack{j=1 \\ j \neq i}}^n \frac{n-j+1}{ne^{1/n} - j + 1} \\
&= \frac{nn!}{n-i+1} \prod_{\substack{j=1 \\ j \neq i}}^n \frac{1}{ne^{1/n} - j + 1} \\
&\leq \frac{nn!}{n-i+1} \prod_{\substack{j=1 \\ j \neq i}}^n \frac{1}{n-j+2} \\
&= \frac{nn!}{n-i+1} \times \frac{n-i+2}{(n+1)!} \\
&= \frac{n-i+2}{n-i+1} \times \frac{n}{n+1} \\
&\leq 2.
\end{aligned}$$

Consequently we have

$$\begin{aligned}
\mathbb{E}[e^{-Z_n}(Z'_n - Z_n)] &\leq \frac{1}{n-1} + \frac{1}{n(n-1)} \sum_{i=1}^n \mathbb{E}[e^{-Z_{n \setminus i}}] \mathbb{E}[e^{-\frac{1}{n} \tau_i^n} \tau_i^n] \\
&\leq \frac{1}{n-1} + \frac{2}{n-1} H_n.
\end{aligned}$$

Those arguments apply again to (2) and give (almost) the same bound:

$$\begin{aligned}
\mathbb{E}[e^{-Z_n}(Z'_n - Z_{n-1})] &= \frac{1}{n-1} + \frac{1}{n-1} \sum_{i=1}^{n-1} \mathbb{E}[e^{-Z_n}(\tau_{i+1}^n - \tau_i^{n-1})] \\
&= \frac{1}{n-1} + \frac{1}{n-1} \sum_{i=1}^{n-1} \mathbb{E}[e^{-Z_n \wedge i}] \mathbb{E}[e^{-\tau_{i+1}^n}(\tau_{i+1}^n - \tau_i^{n-1})] \\
&\leq \frac{1}{n-1} + \frac{2}{n-1} \sum_{i=1}^{n-1} \frac{1}{n-i} \\
&= \frac{1}{n-1} + \frac{2}{n-1} H_{n-1}.
\end{aligned}$$

Finally we deal with (3) and (4). This last part essentially boils down to determining how fast  $\mathbb{E}[e^{-Z_n}]$  goes to 1, a question whose answer has already been given in lemma 19, where we found that

$$\mathbb{E}[e^{-Z_n}] - 1 \underset{n \rightarrow \infty}{\sim} -\frac{\log n}{2n},$$

which concludes the proof.  $\square$

As for the third term, we rely once more on the fact that  $z \mapsto e^{-Z_{n-1}z}$  is  $|Z_{n-1}|(e^{-\beta_n Z_{n-1}} + e^{-Z_{n-1}})$ -Lipschitz almost surely, when  $z \in [\beta_n, 1]$ :

$$\begin{aligned}
&\left| \mathbb{E}\left[ (e^{-\beta_n Z_{n-1}} - e^{-Z_{n-1}}) e^{-\gamma_t} e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)} h'(Z_{n-1} + Y - t) \right] \right| \\
&\leq \mathbb{E}\left[ |e^{-\beta_n Z_{n-1}} - e^{-Z_{n-1}}| \right] \\
&\leq (1 - \beta_n) \mathbb{E}\left[ |Z_{n-1}| (e^{-\beta_n Z_{n-1}} + e^{-Z_{n-1}}) \right] \\
&\leq C(1 - \beta_n).
\end{aligned}$$

Now, as before, we use that  $x \mapsto e^{-\gamma_t e^{-x}}$  is  $e^{-1}$ -Lipschitz on  $\mathbf{R}$ , so that the fourth and final term satisfies:

$$\begin{aligned}
&\left| \mathbb{E}\left[ e^{-Z_{n-1}} \left( e^{-\gamma_t e^{-(Z_{n-1}+Y)}} - e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)}} \right) h'(Z_{n-1} + Y - t) \right] \right| \\
&\leq \mathbb{E}\left[ e^{-Z_{n-1}} \left| e^{-\gamma_t e^{-(Z_{n-1}+Y)}} - e^{-\gamma_t e^{-(\alpha_n Z_{n-1} + \frac{G_n}{n} + \delta_n)}} \right| \right] \\
&\leq e^{-1} \mathbb{E}\left[ e^{-Z_{n-1}} \left( (1 - \alpha_n) |Z_{n-1}| + (Y - \frac{G_n}{n} - \delta_n) \right) \right].
\end{aligned}$$

That concludes the proof of theorem 10.

REMARK 3.— To the best of our knowledge, very few papers so far have given similar uniform bounds for the coupon collector problem: [17]. The author finds the same rate as ours, but in the Kolmogorov distance and through a different method. Sándor's PhD student has developed this topic in her doctoral thesis [15]. She has made use of Stein-Chen method, among other techniques, to find rates of convergence in several limit theorems related to the coupon collector problem. A much more recent reference regarding that topic is [12].

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