



HAL
open science

Global existence for multi-dimensional partially diffusive systems

Jean-Paul Adogbo, Raphaël Danchin

► **To cite this version:**

Jean-Paul Adogbo, Raphaël Danchin. Global existence for multi-dimensional partially diffusive systems. 2025. hal-04863089

HAL Id: hal-04863089

<https://hal.science/hal-04863089v1>

Preprint submitted on 3 Jan 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

GLOBAL EXISTENCE FOR MULTI-DIMENSIONAL PARTIALLY DIFFUSIVE SYSTEMS

JEAN-PAUL ADOGBO & RAPHAËL DANCHIN

ABSTRACT. In this work, we explore the global existence of strong solutions for a class of partially diffusive hyperbolic systems within the framework of critical homogeneous Besov spaces. Our objective is twofold: first, to extend our recent findings on the local existence presented in [1], and second, to refine and enhance the analysis of Kawashima [15].

To address the distinct behaviors of low and high frequency regimes, we employ a hybrid Besov norm approach that incorporates different regularity exponents for each regime. This allows us to meticulously analyze the interactions between these regimes, which exhibit fundamentally different dynamics.

A significant part of our methodology is based on the study of a Lyapunov functional, inspired by the work of Beauchard and Zuazua [3] and recent contributions [8, 7, 6]. To effectively handle the high-frequency components, we introduce a parabolic mode with better smoothing properties, which plays a central role in our analysis.

Our results are particularly relevant for important physical systems, such as the magneto-hydrodynamics (MHD) system and the Navier-Stokes-Fourier equations.

INTRODUCTION

This work concerns global-in-time strong solutions and the study of the large time asymptotics for the following class of $n \times n$ systems of PDEs:

$$(0.1) \quad S^0(U)\partial_t U + \sum_{\alpha=1}^d S^\alpha(U)\partial_\alpha U - \sum_{\alpha,\beta=1}^d \partial_\alpha(Y^{\alpha\beta}(U)\partial_\beta U) = f(U, \nabla U),$$

with $\partial_t \stackrel{\text{def}}{=} \frac{\partial}{\partial t}$ and for all $\alpha = 1, \dots, d$, $\partial_\alpha \stackrel{\text{def}}{=} \frac{\partial}{\partial x_\alpha}$. These systems come into play in the description of various physical phenomena such as gas dynamics, fluid models with chemical reactions [13], fluids in the presence of a magnetic field . . . In System (0.1) the smooth matrix-valued functions S^α ($\alpha = 0, \dots, d$), $Y^{\alpha\beta}$ ($\alpha, \beta = 1, \dots, d$) and vector-valued function f are defined on some open subset \mathcal{U} of \mathbb{R}^n ($n \geq 1$) and the unknown $U \in \mathbb{R}^n$ depends on the time variable $t \in \mathbb{R}_+$ and on the space variable $x \in \mathbb{R}^d$ with $d \geq 2$. Additional structure assumptions will be specified in the next section.

It is well known that classical systems of conservation laws (that is, (0.1) with $f \equiv 0$ and $Y^{\alpha\beta} \equiv 0$) supplemented with smooth data admit local-in-time strong solutions that may develop singularities (shock waves) in finite time even if the initial data are small perturbations of a constant solution (see, for instance, the works by Majda in [19] and D. Serre in [23]). The picture changes when the second order operator in (0.1) is strongly elliptic. Then, the global existence for small and sufficiently regular perturbations of a constant solution $\bar{U} \in \mathcal{U}$ holds, as well as the convergence of the solution U to \bar{U} with the same decay rate as for the heat equation (see [15, 27]).

In many physical systems however, that condition is not verified, and it is more reasonable to assume that up to change of coordinates, the symbol $Y(\omega, U) \stackrel{\text{def}}{=} \sum_{\alpha,\beta=1}^d Y^{\alpha\beta}(U)\omega_\alpha\omega_\beta$ is a

2010 *Mathematics Subject Classification.* 35M11, 35Q30. 76N10.

Key words and phrases. Hyperbolic-parabolic systems, Partial diffusion, Critical regularity, global well-posedness .

block-diagonal and nonnegative matrix:

$$(0.2) \quad Y(\omega, U) = \begin{pmatrix} 0 & 0 \\ 0 & Z(\omega, U) \end{pmatrix}, \quad \omega \in \mathbb{S}^{d-1}, \quad U \in \mathcal{U},$$

where $Z(\omega, U)$ is a $n_2 \times n_2$ positive definite matrix, and $1 \leq n_2 \leq n$. In other words, $Y(\omega, U)$ acts on some components of the unknown, while the other components are unaffected since they satisfy conservation laws. An informative example is the MHD system, where the mass is conserved (see Section 5). As remarked by A. Matsumura and T. Nishida in their seminal paper [20] dedicated to the equations of motion of viscous and heat conductive gases, this indirect effect may be sufficient to ensure the global existence for small regular initial data. The resulting class of systems is named, depending on the authors and on the context, hyperbolic-parabolic or hyperbolic partially diffusive.

Hyperbolic partially diffusive systems have been extensively studied since the pioneering work by S. Kawashima in 1983 in his PhD thesis [15] which is the cornerstone of all theory. There, he pointed out sufficient conditions for local well-posedness for general (smooth enough) data, global existence for small data, and large time asymptotics. He also pointed out a condition on the structure of the matrices and functions involved in System (0.1) allowing one to obtain the *normal* form of the System (0.1). This condition enabled the author to prove the local well-posedness of the system (0.1) for initial data belonging to $H^s(\mathbb{R}^d)$, with $s > \frac{d}{2} + 2$.

A few years later, S. Kawashima and S. Shizuta in [16] exhibited two sufficient conditions to derive the normal form of system (0.1): the first one is the existence of a *dissipative entropy* which provides a symmetrization of the system that is compatible with the second order terms, the second condition stipulates that the null space of $Y(\omega, U)$ is independent of $U \in \mathcal{U}$ and $\omega \in \mathbb{S}^{d-1}$ (condition N in [16]). Later, D. Serre in [26] pointed out that this second condition is equivalent to the range of $Y(\omega, U)$ being independent of $U \in \mathcal{U}$ and $\omega \in \mathbb{S}^{d-1}$, which is simpler to verify. In [25], he further improved Kawashima's local result from [15] by enlarging the class of initial data to include those from inviscid conservation laws, specifically $H^s(\mathbb{R}^d)$, with $s > \frac{d}{2} + 1$.

In [1], we established the local well-posedness for System (0.1) supplemented with initial data in nonhomogeneous Besov space of type $B_{2,1}^s$. More precisely, the component of the initial data affected by the diffusion belongs to $B_{2,1}^s$ while the unaffected component is taken in $B_{2,1}^{s+1}$. The regularity index can be any real number $s \geq \frac{d}{2}$ (and even $s \geq \frac{d}{2} - 1$ for a particular class of systems) so that we can consider initial data which are not Lipschitz. Let us also emphasize that our structural assumptions are weaker than those made in [25].

In [15], the author exhibited a rather simple sufficient condition for global well-posedness of (0.1) supplemented with initial data which are in the neighborhood of linearly stable constant solutions. It is nowadays called the (SK) (meaning Shizuta-Kawashima) condition. In the case where S^0 is the Identity matrix, it exactly says that for the linearized system, the intersection between the kernel of the second order term (the symbol $Y(\cdot, \bar{U})$) and the set of all eigenvectors of the symmetric first order term is reduced to $\{0\}$ (see Section 1 for more details). In [15, 16], the equivalence between Condition (SK) and the existence of a *compensating function* (that allows to work out a Lyapunov functional equivalent to a suitable Sobolev norm of the solution) enabled the authors to exhibit the global-in-time L^2 integrability properties of all the components of the solution and to get global-in-time solutions for initial data belonging to $H^s(\mathbb{R}^d)$ with $s > 3 + \frac{d}{2}$.

In [27], S. Shizuta and S. Kawashima pointed out that the existence of a compensating function is equivalent to the strict dissipativity of the system. This means that, in Fourier space, the real parts of all eigenvalues of the matrix of the linearized system of (0.1) around the reference solution are strictly positive. In the same paper, the authors proved that if, in addition of being in a Sobolev space $H^s(\mathbb{R}^d)$ with large enough s , we assume the initial data U_0 to satisfy $(U_0 - \bar{U}) \in L^p$ for some $1 \leq p < 2$, then the decay rate of $U(t) - \bar{U}$ in L^2 for t

going to ∞ is the same as for the heat equation, namely $(1+t)^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{2})}$. Since then, more decay estimates have been proved under various assumptions; see, for example [22, 28, 29].

The condition (SK) is not sharp: several authors observed that it is not necessary for the existence of global strong solutions. For instance, in [21], P. Qu and Y. Wang established a global existence result in the case where exactly one eigenvector violates the condition (SK). In this respect, one can also mention the paper by C. Burtea, T. Crin-Barat, and J. Tan [4] dedicated to the relaxation limit of the Baer-Nunziato (BN) system, which does not verify the (SK) condition, and the recent work [12] by the second author and P. B. Mucha dedicated to the mathematical study of compressible Euler system with nonlocal pressure.

Although Condition (SK) provides a compensating function that ensures global existence, along with estimates on the entire solution and its decay, it is difficult to verify for concrete systems. Another limitation is that it does not provide more specific information on the part of the solution experiencing diffusion, even though it is expected to have better properties than the whole solution (see [10] for the case of the compressible Navier-Stokes equations).

For partially dissipative systems, K. Beauchard and E. Zuazua [3] pointed out a link between the (SK) condition and the Kalman maximal rank condition from control theory of ODEs (see [5, 30]) for the linearized system. They proved that for a symmetric hyperbolic partially dissipative system, the (SK) condition is equivalent to the Kalman maximal rank condition for the linearized system about a constant solution. They also introduced a Lyapunov functional that encodes sufficient information to recover most of the dissipative properties of the linearized system. This equivalence was generalized to any linear system in [11]. In [6, 7, 8], T. Crin-Barat and the second author leveraged the method of Beauchard and Zuazua to establish global existence and relaxation results in a *critical regularity setting* for symmetric hyperbolic partially dissipative systems with data having different regularities in low and high frequencies. An important new ingredient for achieving these improvements is to examine the time evolution of a *damped mode* which corresponds to the part of the solution that experiences maximal dissipation at low frequencies.

The primary aim of the present work is to extend the results of Kawashima by broadening the class of initial data and weakening the smallness condition. Following our recent work [1] dedicated to the local well-posedness and adapting the approach developed in [6] to the hyperbolic-parabolic setting, we shall propose a ‘critical’ functional framework that offers better control over the solution and, in particular, more accurate convergence rates to the equilibrium state as time goes to infinity. Specifically, we will prove the global existence for small initial data in functional spaces with *different* regularity exponents at low and high frequencies. As for partially dissipative systems, Beauchard and Zuazua’s approach will give us the information that the low frequencies (resp. high frequencies) of the solution of the linearized system behave like the heat flow (resp. are exponentially damped). To achieve maximal regularity for the component affected by diffusion, we will introduce the *parabolic mode*, analogous to the damped mode in the hyperbolic partially dissipative case [6].

This work is arranged as follows. The first section is dedicated to presenting the structure of the class of symmetric hyperbolic partially diffusive systems we shall consider, constructing a low and high frequencies Lyapunov functional that will play a key role in our global results and exhibiting a *parabolic mode* that will allow us to recover the full regularizing effect for the components of the solution experiencing the diffusion. In Section 2, we state the main results of the paper: global existence and decay estimates. Proving a first global existence result and time decay estimates for general partially diffusive systems satisfying the (SK) condition will be the goal of Section 3. In Section 4, under additional assumptions of structure (which are satisfied by the compressible Navier-Stokes system), we obtain a global existence result at the critical regularity level. In the last section, we apply the results to the MHD system. Some technical results are proved or recalled in Appendix.

Notation. As usual, C designates a generic harmless constant, the value of which depends on the context. We sometimes use the short notation $A \lesssim B$ to mean that $A \leq CB$.

For any normed space X , real number $\rho \in [1, \infty]$ and $T \in [0, \infty]$, we denote by $L^\rho(0, T; X)$ (or sometimes just $L_T^\rho(X)$) the set of measurable functions $z : (0, T) \rightarrow X$ such that $\|z\|_{L_T^\rho(X)} = \|\|z(t, \cdot)\|_X\|_{L^\rho(0, T)}$ is finite. The notation $\mathcal{C}_b(\mathbb{R}_+; X)$ stands for the set of continuous bounded functions from \mathbb{R}_+ to X .

1. ASSUMPTIONS AND APPROACHES

In this section, we specify the class of partially diffusive systems we aim to study, and present the tools allowing to get the main results.

1.1. Normal form. Let $\bar{U} \in \mathcal{U}$ be a constant solution of (0.1). Then, $V \stackrel{\text{def}}{=} U - \bar{U}$ satisfies

$$(1.1) \quad S^0(U)\partial_t V + \sum_{\alpha} S^\alpha(U)\partial_\alpha V - \sum_{\alpha, \beta} \partial_\alpha(Y^{\alpha\beta}(U)\partial_\beta V) = f(U, \nabla U).$$

In what follows, we assume that the matrices of System (0.1) and the external force f satisfy:

Assumption D.

(D1) The matrix $S^0(U)$ is symmetric, positive definite for every $U \in \mathcal{U}$, and has the form

$$S^0(U) = \begin{pmatrix} S_{11}^0(U) & 0 \\ 0 & S_{22}^0(U) \end{pmatrix},$$

where $S_{11}^0(U)$ (resp. $S_{22}^0(U)$) is a $n_1 \times n_1$ (resp. $n_2 \times n_2$) matrix, with $n_1 + n_2 = n$.

(D2) The matrices $S^\alpha(U)$ are symmetric for every $U \in \mathcal{U}$ and $\alpha = 1, \dots, d$.

(D3) For every $U \in \mathcal{U}$ and $\alpha, \beta = 1, \dots, d$, the matrices $Y^{\alpha\beta}(U)$ have the form

$$(1.2) \quad Y^{\alpha\beta}(U) = \begin{pmatrix} 0_{n_1} & 0 \\ 0 & Z^{\alpha\beta}(U) \end{pmatrix},$$

where $Z^{\alpha\beta}(U)$ is a $n_2 \times n_2$ matrix. Furthermore, the operator $\sum_{\alpha, \beta=1}^d Z^{\alpha\beta}(U)\partial_\alpha\partial_\beta$ is strongly elliptic in the sense:

$$(1.3) \quad \sum_{\alpha, \beta=1}^d \sum_{i, j \geq n_1+1} \xi_\alpha \lambda_i \xi_\beta \lambda_j Z_{ij}^{\alpha\beta}(U) \geq c_1(U) |\xi|^2 |\lambda|^2, \quad \forall \xi \in \mathbb{R}^d, \lambda \in \mathbb{R}^{n_2},$$

for some positive continuous function c_1 defined on \mathcal{U} .

(D4) f has components

$$(1.4) \quad f(U, \nabla U) \stackrel{\text{def}}{=} \begin{pmatrix} f^1(U, \nabla U) \\ f^2(U, \nabla U) \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} 0_{n_1} \\ Q(U, \nabla U, \nabla U) \end{pmatrix}$$

with Q quadratic in terms of ∇U (i.e. Q is a combination of terms $v(U)\nabla U \otimes \nabla U$).

System (0.1) under assumptions (i), (ii), (iii) and (iv) is said to be of the normal form ([16, 26]).

Remark 1.1. *The assumptions on f depend on the regularity framework: in [15, 27], since the solution is more regular, f^1 (resp. f^2) can be any function of $U, \nabla U^2$ (resp. $U, \nabla U$).*

Back to our setting, Theorems 2.1 and 2.2 below are in fact valid whenever

- (1) $f^1 = f^1(U, \nabla U^2)$ is a combination of terms $L^1 V^1 + v_1(U)V \otimes V + v_2(U)V \otimes \nabla V^2$, with L^1 a nonpositive matrix, and v_1, v_2 some smooth functions,
- (2) f^2 is a combination of terms of type $L^2 V^2 + v_3(U, \nabla V^1)\nabla V \otimes \nabla V + Q_2(V, \nabla V^1)$, where L^2 is a nonpositive matrix, Q_2 is quadratic and v_3 is a smooth function.

1.2. The Shizuta-Kawashima and Kalman rank conditions. Let us consider the following linearization of (1.1):

$$(1.5) \quad \bar{S}^0 \partial_t V + \sum_{\alpha=1}^d \bar{S}^\alpha \partial_\alpha V - \sum_{\alpha,\beta=1}^d \bar{Y}^{\alpha\beta} \partial_\alpha \partial_\beta V = F,$$

where, from now on, we use for any function S the notation $\bar{S} \stackrel{\text{def}}{=} S(\bar{U})$. Note that (1.1) corresponds to (1.5) with $F \stackrel{\text{def}}{=} F^1 + F^2 + F^t + f$, where

$$(1.6) \quad \begin{aligned} F^t &\stackrel{\text{def}}{=} (\bar{S}^0 - S(U)) \partial_t V, & F^1 &\stackrel{\text{def}}{=} (\bar{S}^\alpha - S^\alpha(U)) \sum_{\alpha=1}^d \partial_\alpha V, \\ F^2 &\stackrel{\text{def}}{=} \sum_{\alpha,\beta=1}^d \partial_\alpha (r^{\alpha\beta} \partial_\beta V) \quad \text{and} \quad r^{\alpha\beta} &\stackrel{\text{def}}{=}} (\bar{Y}^{\alpha\beta} - Y^{\alpha\beta}(U)). \end{aligned}$$

Let $\xi \in \mathbb{R}^d$ stand for the Fourier variable corresponding the space variable $x \in \mathbb{R}^d$. In the Fourier space, System (1.5) recasts into

$$(1.7) \quad \bar{S}^0 \partial_t \hat{V} + i \sum_{\alpha=1}^d \bar{S}^\alpha \xi_\alpha \hat{V} + \sum_{\alpha,\beta=1}^d \bar{Y}^{\alpha\beta} \xi_\alpha \xi_\beta \hat{V} = \hat{F}.$$

Set $\xi = \rho \omega$ where $\rho \geq 0$ and $\omega \in \mathbb{S}^{d-1}$. We define

$$(1.8) \quad A_\omega \stackrel{\text{def}}{=} i \sum_{\alpha=1}^d \bar{S}^\alpha \omega_\alpha \quad \text{and} \quad B_\omega \stackrel{\text{def}}{=} \sum_{\alpha,\beta=1}^d \bar{Y}^{\alpha\beta} \omega_\alpha \omega_\beta = \sum_{\alpha,\beta=1}^d \begin{pmatrix} 0_{n_1 \times n_1} & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & \bar{Z}^{\alpha\beta} \omega_\alpha \omega_\beta \end{pmatrix}.$$

With the above notation, taking $F = 0$ to simplify the presentation, System (1.5) becomes

$$(1.9) \quad \bar{S}^0 \partial_t \hat{V} + (\rho A_\omega + \rho^2 B_\omega) \hat{V} = 0.$$

Under Assumption **D**, we observe that for all $\omega \in \mathbb{S}^{d-1}$, the matrix A_ω is skew Hermitian, while the matrix B_ω is nonnegative.

System (1.5) is a particular case of the more general class of systems:

$$(1.10) \quad S \partial_t V + \mathcal{A}(D)V + \mathcal{B}(D)V = 0, \quad \text{where}$$

- S is a Hermitian positive definite matrix.
- $\mathcal{A}(D)$ is a homogeneous (matrix-valued) Fourier multiplier of degree a that satisfies

$$(1.11) \quad \operatorname{Re}(\mathcal{A}(\omega)\eta \cdot \eta) = 0, \quad \forall \omega \in \mathbb{S}^{d-1} \quad \text{and} \quad \eta \in \mathbb{C}^n$$

where \cdot designates the Hermitian scalar product in \mathbb{C}^n .

- $\mathcal{B}(D)$ is an homogeneous (matrix-valued) Fourier multiplier of degree b , such that, for some positive real number κ ,

$$(1.12) \quad \operatorname{Re}(\mathcal{B}(\omega)\eta \cdot \eta) \geq \kappa |\mathcal{B}(\omega)\eta|^2, \quad \forall \omega \in \mathbb{S}^{d-1} \quad \text{and} \quad \eta \in \mathbb{C}^n.$$

Observe that System (1.9) corresponds to the case $a = 1$ and $b = 2$.

Taking the real part of the inner product of (1.10) with \hat{V} , then using (1.11) yields

$$(1.13) \quad \frac{1}{2} \frac{d}{dt} (S \hat{V}(\xi) \cdot \hat{V}(\xi)) + \operatorname{Re}(\mathcal{B}(\xi) \hat{V}(\xi) \cdot \hat{V}(\xi)) = 0 \quad \text{for all} \quad \xi \in \mathbb{R}^d.$$

On the one hand, since the matrix S is Hermitian positive definite, we have

$$(1.14) \quad S \hat{V} \cdot \hat{V} \simeq |\hat{V}|^2.$$

On the other hand, the positivity (1.12) of \mathcal{B} guarantees that

$$(1.15) \quad \operatorname{Re}(\mathcal{B}(\xi) \hat{V}(\xi) \cdot \hat{V}(\xi)) \geq \kappa |\xi|^{-b} |\mathcal{B}(\xi) \hat{V}|^2 = \rho^b |\mathcal{B}_\omega \hat{V}|^2,$$

where, for $\omega \in \mathbb{S}^{d-1}$, we denoted

$$(1.16) \quad \mathcal{A}_\omega \stackrel{\text{def}}{=} \mathcal{A}(\omega) \quad \text{and} \quad \mathcal{B}_\omega \stackrel{\text{def}}{=} \kappa \mathcal{B}(\omega).$$

Hence (1.13) finally gives us some constant C_S depending only on S such that for all $t \in \mathbb{R}_+$,

$$(1.17) \quad \left\| \widehat{V}(t) \right\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^d} |\xi|^b |\mathcal{B}_\omega \widehat{V}|^2 d\xi \leq C_S \left\| \widehat{V}_0 \right\|_{L^2}^2.$$

If, for some $\omega \in \mathbb{S}^{d-1}$, the matrix \mathcal{B}_ω has a strictly lower rank than n , then the last inequality does not ensure dissipation on all components of V . We are going to make some assumptions that allow us to recover the decay of the missing components of the solution. Following Beauchard and Zuazua in [3]; Crin-Barat and the second author in [6, 11], we introduce a Lyapunov functional to track the diffusion of the solution to (1.9). We have the following result (we postpone the proof in the appendix):

Lemma 1.1. *Let $\mathcal{N}_\omega = S^{-1}\mathcal{A}_\omega$ and $\mathcal{M}_\omega = S^{-1}\mathcal{B}_\omega$ for all $\omega \in \mathbb{S}^{d-1}$. There exist positive parameters $\varepsilon_0, \dots, \varepsilon_{n-1}$ (that are defined inductively and can be taken arbitrarily small) and a Lyapunov functional*

$$(1.18) \quad L_{\rho,\omega}(\widehat{V}) \stackrel{\text{def}}{=} S\widehat{V} \cdot \widehat{V} + \min\left(\frac{1}{\kappa\rho^{a-b}}, \kappa\rho^{a-b}\right) \mathcal{I}_\omega$$

$$\text{with } \mathcal{I}_\omega \stackrel{\text{def}}{=} \sum_{k=1}^{n-1} \varepsilon_k \mathcal{R}e(\mathcal{M}_\omega \mathcal{N}_\omega^{k-1} \widehat{V} \cdot \mathcal{M}_\omega \mathcal{N}_\omega^k \widehat{V})$$

such that the following inequalities hold for some positive c and C , and all $\rho > 0$ and $\omega \in \mathbb{S}^{d-1}$:

$$(1.19) \quad \frac{d}{dt} \mathcal{I}_\omega + \frac{1}{2} \rho^a \sum_{k=1}^{n-1} \varepsilon_k |\mathcal{M}_\omega \mathcal{N}_\omega^k \widehat{V}|^2 \leq C \varepsilon_0 \max\left(\rho^a, \frac{\rho^{2b-a}}{\kappa^2}\right) |\mathcal{M}_\omega \widehat{V}|^2,$$

$$(1.20) \quad \frac{d}{dt} L_{\rho,\omega}(\widehat{V}) + c \min\left(\frac{\rho^b}{\kappa}, \kappa\rho^{2a-b}\right) \left(|\mathcal{M}_\omega \widehat{V}|^2 + \sum_{k=1}^{n-1} \varepsilon_k |\mathcal{M}_\omega \mathcal{N}_\omega^k \widehat{V}|^2 \right) \leq 0,$$

with, additionally,

$$(1.21) \quad C^{-1} |\widehat{V}|^2 \leq L_{\rho,\omega}(\widehat{V}) \leq C |\widehat{V}|^2.$$

The question now is whether the rate of dissipation in (1.20) can be compared to $|\widehat{V}|^2$. We get a positive answer if we work under the following (SK) (for Shizuta and Kawashima) condition. Let us recall this condition stated in [27].

Definition 1.1. *Let $n \in \mathbb{N}$, N, M be $n \times n$ matrices with complex coefficients. The pair (N, M) is said to satisfy the (SK) condition if we have at the same time $M\phi = 0$ and $\lambda\phi + N\phi = 0$ for some $\lambda \in \mathbb{R}$, if and only if $\phi = 0_{\mathbb{C}^n}$.*

In [3], K. Beauchard and E. Zuazua have highlighted an interesting connection between Condition (SK) and the Kalman criterion for observability in the theory of linear ODEs, that is recalled in the following lemma.

Lemma 1.2. *Let N and M be two $n \times n$ complex valued matrices. Then, the following properties are equivalent:*

(1) *For all positive $\varepsilon_0, \dots, \varepsilon_{n-1}$, we have $\sum_{l=0}^{n-1} \varepsilon_l |MN^l \eta|^2 > 0$ for all $\eta \in \mathbb{S}^{n-1}$.*

(2) *(N, M) satisfies the Kalman rank condition: the rank of the $n^2 \times n$ matrix*
$$\begin{pmatrix} M \\ MN \\ \dots \\ MN^{n-1} \end{pmatrix}$$

is equal to n .

(3) *(N, M) satisfies Condition (SK).*

It turns out that this connection had been noticed before by D. Serre in Chapter 6 of [24]. Here, Condition (SK) becomes apparent when investigating the question of convergence toward the equilibrium state of the system (1.10). More precisely, relying on iterated commutators (à la Hörmander in his theory of the hypoellipticity) between the hyperbolic and dissipative parts, D. Serre justifies, in the one-dimensional case, that $\mathfrak{N}_\omega = \{0\}$ (with \mathfrak{N}_ω being the intersection of the kernels of $\mathcal{M}_\omega, \mathcal{M}_\omega \mathcal{N}_\omega, \dots, \mathcal{M}_\omega \mathcal{N}_\omega^{n-1}$) is a necessary condition for decay, which is indeed equivalent to Condition (SK) and the Kalman rank condition.

Still considering System (1.10), assuming that the (SK) condition is satisfied by $(\mathcal{N}_\omega, \mathcal{M}_\omega)$ at every point of the unit sphere \mathbb{S}^{d-1} , we gather from the compactness of \mathbb{S}^{d-1} , (1.20) and (1.21) that there exists a constant $c_* > 0$ such that

$$(1.22) \quad \frac{d}{dt} L_{r,\omega}(\widehat{V}) + c_* \min\left(\frac{\rho^b}{\kappa}, \kappa \rho^{a-b}\right) L_{r,\omega}(\widehat{V}) \leq 0.$$

This implies, owing to (1.21), that

$$(1.23) \quad \widehat{V}(t, \xi) \leq C e^{-ct \min(\frac{|\xi|^b}{\kappa}, \kappa |\xi|^{a-b})} \widehat{V}_0(\xi), \quad t \in \dot{R}_+, \quad \xi \in \dot{R}^d$$

which reveals a different behavior of the solution in high and low frequencies. In particular, as it corresponds to the case $a = 1$ and $b = 2$, the linear system (1.5) behaves as a parabolic system for low frequencies, and has a damped behavior (but no gain of derivative) for high frequencies.

Note also that integrating (1.22) on \mathbb{R}^d , omitting the second term and using Fourier-Plancherel theorem ensures that

$$\mathcal{L}(V) \stackrel{\text{def}}{=} (2\pi)^{-d} \int_0^\infty \int_{\mathbb{S}^{d-1}} L_{r,\omega}(\widehat{V}) r^{d-1} d\sigma(\omega) dr$$

is a Lyapunov functional for System (1.10), that is equivalent to $\|V\|_{L^2}^2$.

1.3. Parabolic mode. Observe that under Assumption **D**, System (1.1) can be rewritten:

$$(1.24) \quad \begin{cases} S_{11}^0(U) \partial_t V^1 + \sum_{\alpha=1}^d (S_{11}^\alpha(U) \partial_\alpha V^1 + S_{12}^\alpha(U) \partial_\alpha V^2) = 0 \\ S_{22}^0(U) \partial_t V^2 + \sum_{\alpha=1}^d (S_{21}^\alpha(U) \partial_\alpha V^1 + S_{22}^\alpha(U) \partial_\alpha V^2) - \sum_{\alpha,\beta=1}^d \partial_\alpha (Z^{\alpha\beta}(U) \partial_\beta V^2) \\ \hspace{20em} = f^2(U, \nabla U) \end{cases}$$

with $V^1 : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^{n_1}$ and $V^2 : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^{n_2}$

$$(1.25) \quad S^0(U) = \begin{pmatrix} S_{11}^0(U) & 0 \\ 0 & S_{22}^0(U) \end{pmatrix} \quad \text{and} \quad S^\alpha(U) = \begin{pmatrix} S_{11}^\alpha(U) & S_{12}^\alpha(U) \\ S_{21}^\alpha(U) & S_{22}^\alpha(U) \end{pmatrix}.$$

Owing to Assumption (1.3), for fixed V^1 , the second equation of (1.24) is of parabolic type. Like for the heat equation, one can thus expect a gain of two space derivatives for V^2 , compared to e.g. the source term f^2 . From Beauchard-Zuazua's analysis however, we cannot track this smoothing property: we only get exponential decay for high frequencies of V^2 .

Introducing a suitable ‘‘parabolic mode’’ will help us to recover this information. To do so, let us isolate the linear part of the second equation of (1.24), and rewrite it as:

$$(1.26) \quad \overline{S}_{22}^0 \partial_t V^2 + \sum_{\alpha} \overline{S}_{21}^\alpha \partial_\alpha V^1 + \sum_{\alpha} \overline{S}_{22}^\alpha \partial_\alpha V^2 - \sum_{\alpha,\beta=1}^d \overline{Z}^{\alpha\beta} \partial_\alpha \partial_\beta V^2 = h$$

with $h = h^t + h^{22} + h^{21} + h^2 + f^2$ and

$$(1.27) \quad \begin{aligned} h^t &\stackrel{\text{def}}{=}} (\overline{S}_{22}^0 - S_{22}^0(U)) \partial_t V^2; & h^{21} &\stackrel{\text{def}}{=} \sum_{\alpha} (\overline{S}_{21}^\alpha - S_{21}^\alpha(U)) \partial_\alpha V^1; \\ h^{22} &\stackrel{\text{def}}{=} \sum_{\alpha} (\overline{S}_{22}^\alpha - S_{22}^\alpha(U)) \partial_\alpha V^2; & h^2 &\stackrel{\text{def}}{=} \sum_{\alpha,\beta} \partial_\alpha (r^{\alpha\beta} \partial_\beta V^2). \end{aligned}$$

In the Fourier space, if we set $\xi = \rho\omega$ with $\rho \geq 0$ and $\omega \in \mathbb{S}^{d-1}$, we can rewrite (1.26) as follows:

$$(1.28) \quad \overline{S}_{22}^0 \partial_t \widehat{V}^2 + i\rho(S_{21}(\omega)\widehat{V}^1 + S_{22}(\omega)\widehat{V}^2) + \rho^2 Z(\omega)\widehat{V}^2 = \widehat{h}$$

where

$$(1.29) \quad S_{21}(\omega) \stackrel{\text{def}}{=} \sum_{\alpha} \overline{S}_{21}^{\alpha} \omega_{\alpha}, \quad S_{22}(\omega) \stackrel{\text{def}}{=} \sum_{\alpha} \overline{S}_{22}^{\alpha} \omega_{\alpha} \quad \text{and} \quad Z(\omega) \stackrel{\text{def}}{=} \sum_{\alpha, \beta=1}^d \overline{Z}^{\alpha\beta} \omega_{\alpha} \omega_{\beta}.$$

After extending the above functions on $\mathbb{R}^d \setminus \{0\}$ by homogeneity of degree 0, we set:

$$(1.30) \quad S_{21}(D)z \stackrel{\text{def}}{=} \mathcal{F}^{-1}(S_{21}\widehat{z}), \quad S_{22}(D)z \stackrel{\text{def}}{=} \mathcal{F}^{-1}(S_{22}\widehat{z}) \quad \text{and} \quad Z(D)z \stackrel{\text{def}}{=} \mathcal{F}^{-1}(Z\widehat{z}).$$

For all $\omega \in \mathbb{S}^{d-1}$, $Z(\omega)$ is an invertible matrix (in fact from (1.3) the Kernel of $Z(\omega) \in \mathcal{M}_{n_2}(\mathbb{C})$ is $0_{\mathbb{C}^{n_2}}$), we define W from its Fourier transform as follows:

$$\begin{aligned} \widehat{W}(\xi) &\stackrel{\text{def}}{=} i\rho^{-1}(Z(\omega)^{-1}(S_{21}(\omega)\widehat{V}^1 + S_{22}(\omega)\widehat{V}^2) + \widehat{V}^2) \\ &= \rho^{-2}(Z(\omega))^{-1}(\widehat{h} - \overline{S}_{22}^0 \partial_t \widehat{V}^2). \end{aligned}$$

In other words, we have

$$(1.31) \quad \begin{aligned} W &\stackrel{\text{def}}{=} i|D|^{-1}(Z(D))^{-1}(S_{21}(D)V^1 + S_{22}(D)V^2) + V^2 \\ &= (-\Delta)^{-1}(Z(D))^{-1}(h - \overline{S}_{22}^0 \partial_t V^2). \end{aligned}$$

Then, we discover that W satisfies the following parabolic equation

$$(1.32) \quad \overline{S}_{22}^0 \partial_t W - \Delta Z(D)W = i|D|^{-1} \overline{S}_{22}^0 (Z(D))^{-1} \partial_t (S_{21}(D)V^1 + S_{22}(D)V^2) + h,$$

and thus expect W to have better regularity properties in high frequencies than the whole solution.

Although a similar idea has been used before in the context of the compressible Navier-Stokes equations (see [14] and the references therein), or for particular cases of symmetric hyperbolic partially dissipative system (see [6, 17]) it seems to be new for the general class that is considered here.

1.4. Transfer of integrability from the parabolic to the hyperbolic mode. Inserting $V^2 = W - i|D|^{-1}(Z(D))^{-1}(S_{21}(D)V^1 + S_{22}(D)V^2)$ in the first equation of (1.24), we observe that V^1 is solution of the following symmetric hyperbolic system:

$$(1.33) \quad S_{11}^0(U) \partial_t V^1 + \sum_{\alpha=1}^d S_{11}^{\alpha}(U) \partial_{\alpha} V^1 + S_{12}(D)Z(D)^{-1}S_{21}(D)V^1 = G^{11} + G^{12} + G^{13}$$

with

$$(1.34) \quad \begin{aligned} G^{11} &\stackrel{\text{def}}{=} \sum_{\alpha=1}^d (\overline{S}_{12}^{\alpha} - S_{12}^{\alpha}(U)) \partial_{\alpha} V^2, \\ G^{12} &\stackrel{\text{def}}{=} -S_{12}(D)Z(D)^{-1}S_{22}(D)V^2 \quad \text{and} \quad G^{13} \stackrel{\text{def}}{=} -i|D|S_{12}(D)W. \end{aligned}$$

Putting all the linear terms in the left-hand side, (1.33) reads

$$(1.35) \quad \overline{S}_{11}^0 \partial_t V^1 + \sum_{\alpha=1}^d \overline{S}_{11}^{\alpha} \partial_{\alpha} V^1 + S_{12}(D)Z(D)^{-1}S_{21}(D)V^1 = G^1 \stackrel{\text{def}}{=} \sum_{k=1}^5 G^{1k}$$

with $G^{14} \stackrel{\text{def}}{=} \sum_{\alpha=1}^d (\overline{S}_{11}^{\alpha} - S_{11}^{\alpha}(U)) \partial_{\alpha} V^1$ and $G^{15} \stackrel{\text{def}}{=}} (\overline{S}_{11}^0 - S_{11}^0(U)) \partial_t V^1.$

Given that $Z(\omega)$ is invertible, we have

$$\text{rank}(S_{12}(\omega)Z(\omega)^{-1}S_{21}(\omega)) = \text{rank}(S_{21}(\omega)) \quad \text{for all } \omega \in \mathbb{S}^{d-1}.$$

Since the 0–order operator $S_{12}(D)Z(D)^{-1}S_{21}(D)$ is responsible of the long time behavior of V^1 , the fact that $\text{rank}(S_{12}(\omega)Z(\omega)^{-1}S_{21}(\omega)) = n_1$ for all $\omega \in \mathbb{S}^{d-1}$ guarantees decay of V^1 . This is actually equivalent to asserting that $\text{rank}(S_{21}(\omega)) = n_1$ for all $\omega \in \mathbb{S}^{d-1}$. However, this assumption is not strictly necessary. In fact, under the assumption (1.3) and denoting

$$(1.36) \quad \begin{aligned} M_\omega &\stackrel{\text{def}}{=} (\overline{S^0})^{-1}B_\omega, \quad N_\omega = (\overline{S^0})^{-1}A_\omega, \quad \text{with } B_\omega, A_\omega \text{ defined in (1.8)} \\ \mathbf{N}_\omega &\stackrel{\text{def}}{=} i(\overline{S_{11}^0})^{-1}S_{11}(\omega) = i(\overline{S_{11}^0})^{-1} \sum_{\alpha=1}^d \overline{S_{11}^\alpha} \omega_\alpha \quad \text{and} \quad \mathbf{M}_\omega \stackrel{\text{def}}{=} (\overline{S_{11}^0})^{-1}S_{12}(\omega)Z(\omega)^{-1}S_{21}(\omega) \end{aligned}$$

we observe that the pair (M_ω, N_ω) satisfies the (SK) condition if and only if the pair $(\mathbf{M}_\omega, \mathbf{N}_\omega)$ does (see Lemma A.1 for more details). Moreover, the operator $\sum_{\alpha=1}^d \overline{S_{11}^\alpha} \partial_\alpha$ is skew-Hermitian in the sense of (1.11), while $S_{12}(D)Z(D)^{-1}S_{21}(D)$ is elliptic (not necessarily strictly) in the sense of (1.12). As a result, the Inequality (1.23) (with $a = 1$ and $b = 0$) ensures that

$$(1.37) \quad \widehat{V^1}(t, \xi) \leq C e^{-ct \min(1, |\xi|^2)} \widehat{V_0^1}(\xi).$$

where c and C are positive constants. In particular, the linear equation (1.33) (with $G^1 = 0$) exhibits damped behavior in high frequencies.

2. MAIN RESULTS

Our first goal is to prove a global existence result for System (1.1) supplemented with small initial data having different regularity in low and high frequencies. In order to find out a suitable functional framework for solving (1.1), we have to remember that a part of the system is first order hyperbolic. Hence conservation of regularity is closely related to the control of the norm of ∇V in $L_{loc}^1(\mathbb{R}_+; L^\infty)$ (see e.g. [2, Chap. 4]). Consequently, we have to assume that V_0 belongs to a Banach space X such that the corresponding solution to the linearized system (1.5) belongs to the set $L_{loc}^1(\mathbb{R}_+; C^{0,1})$. Following our recent work [1], we shall assume in a first time that the high frequencies of (V_0^1, V_0^2) belong to¹ $\dot{B}_{2,1}^{\frac{d}{2}+1} \times \dot{B}_{2,1}^{\frac{d}{2}}$. There is some freedom in the choice of the regularity index s for the low frequencies part V_0^l . A rather natural choice is $s = \frac{d}{2} - 1$ because, as seen in subsection 1.2, we have parabolic properties of the system in low frequencies. Hence starting from $V_0^l \in \dot{B}_{2,1}^{\frac{d}{2}-1}$ we expect that $\nabla V^l \in L_{loc}^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$, and thus, by embedding, $\nabla V^l \in L_{loc}^1(\mathbb{R}_+; L^\infty)$. Combining these properties and the dissipative properties of the system in high frequencies we get a control of the gradient of the solution, which is the key to preventing finite time blow-up (see [1]).

The following general global existence result for System (1.1) in this framework may be seen as an improvement of those in [15, 27, 13].

Theorem 2.1. *Let Assumption D be in force. Assume that $d \geq 2$ and that the pair (N_ω, M_ω) defined in (1.36) satisfies the (SK) condition, for all $\omega \in \mathbb{S}^{d-1}$. Then, there exists a positive constant α such that for all U_0 with range in \mathcal{U} and such that $V_0 = (V_0^1, V_0^2) \stackrel{\text{def}}{=} U_0 - \overline{U}$ with $V_0^1 \in \dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}$ and $V_0^2 \in \dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}}$ satisfies*

$$(2.1) \quad \|V_0^1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} + \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}}} \leq \alpha,$$

System (1.1) supplemented with initial data (V_0^1, V_0^2) admits a unique global solution V in

$$E := \left\{ V = (V^1, V^2) : V^l \in \mathcal{C}_b(\mathbb{R}_+, \dot{B}_{2,1}^{\frac{d}{2}-1}) \cap L^1(\mathbb{R}_+, \dot{B}_{2,1}^{\frac{d}{2}+1}), \right. \\ \left. V^{1,h} \in \mathcal{C}_b(\mathbb{R}_+, \dot{B}_{2,1}^{\frac{d}{2}+1}) \cap L^1(\mathbb{R}_+, \dot{B}_{2,1}^{\frac{d}{2}+1}), \quad V^{2,h} \in \mathcal{C}_b(\mathbb{R}_+, \dot{B}_{2,1}^{\frac{d}{2}}) \cap L^1(\mathbb{R}_+, \dot{B}_{2,1}^{\frac{d}{2}+2}) \right\},$$

¹The reader is referred to Appendix B for the definition of Besov spaces, and of the Littlewood-Paley decomposition.

where, for any tempered distribution Z , Z^l (resp. Z^h) denotes the low (resp. high) frequency part of Z (see the definition in (B.6)).

Moreover, there exists an explicit Lyapunov functional equivalent to

$$\|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} + \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}}}$$

and a constant C depending only on the matrices S^0, S^α and $Y^{\alpha\beta}$ such that

$$(2.2) \quad \mathbb{V}(t) \leq C\mathbb{V}_0, \quad t > 0$$

where (see (B.7) for the definition of $\|\cdot\|_{\dot{B}_{2,1}^s}^l$ and $\|\cdot\|_{\dot{B}_{2,1}^s}^h$):

$$(2.3) \quad \mathbb{V}(t) \stackrel{\text{def}}{=} \|V\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})}^l + \|V^1\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \|V^2\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h + \|V\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \|V^2\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}.$$

The following statement specifies the decay rates of the above solutions.

Theorem 2.2. *Under the hypotheses of Theorem 2.1 and if, additionally, $V_0 \in \dot{B}_{2,\infty}^{-\sigma_1}$ for some $1 - d/2 < \sigma_1 \leq d/2$ then, there exists a constant C depending only on σ_1 and such that*

$$(2.4) \quad \|V(t)\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \leq C \|V_0\|_{\dot{B}_{2,\infty}^{-\sigma_1}}, \quad \text{for all } t \geq 0.$$

Furthermore, denoting

$$\langle t \rangle \stackrel{\text{def}}{=} \sqrt{1+t^2} \quad \text{and} \quad C_0 \stackrel{\text{def}}{=} \|V_0\|_{\dot{B}_{2,\infty}^{-\sigma_1}}^l + \|V_0^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}$$

the following decay estimates hold true (where W has been defined according to (1.31)):

$$(2.5) \quad \sup_{t \geq 0} \left\| \langle t \rangle^{\frac{\sigma_1 + \sigma}{2}} V(t) \right\|_{\dot{B}_{2,1}^s}^l \leq CC_0 \quad \text{if } -\sigma_1 < \sigma \leq \frac{d}{2} - 1,$$

$$(2.6) \quad \sup_{t \geq 0} \left\| \langle t \rangle^{\frac{\sigma_1 + d/2 - 1}{2}} (W(t), V^2(t)) \right\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h \leq CC_0,$$

$$(2.7) \quad \sup_{t \geq 0} \left\| \langle t \rangle^{\frac{\sigma_1 + d/2 - 1}{2}} V^1(t) \right\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h \leq CC_0.$$

Remark 2.1. *Note that $\|V\|_{\dot{B}_{2,\infty}^{-\sigma_1}}$ can be arbitrarily large: only $\|V_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} + \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}}}$ has to be small.*

It is known since the second author in [10] on the compressible Navier-Stokes system that global well-posedness and time decay estimates hold for initial data less regular than those in Theorem 2.1: regularity $\dot{B}_{2,1}^{\frac{d}{2}-1}$ and $\dot{B}_{2,1}^{\frac{d}{2}}$ for the high frequencies of the velocity and of the density. However, this alternative framework requires stronger structure assumptions on the system. We here assume that $\mathcal{U} = \mathcal{U}^1 \times \mathbb{R}^{n_2}$ for some open subset \mathcal{U}^1 of \mathbb{R}^{n_1} and:

Assumption E.

- (E1) $S_{11}^0 = \text{Id}$ and S_{22}^0 depends only (and smoothly) on $U^1 \in \mathcal{U}^1$ and has range in the set of $n_2 \times n_2$ positive definite matrices,
- (E2) for all $\alpha = 1, \dots, d$, the matrix $S^\alpha(U)$ is symmetric, and
 - all the maps S_{11}^α are affine with respect to U^2 and independent of U^1 ,
 - all the maps S_{21}^α are independent of U^2 and depend smoothly on U^1 ,
 - the smooth maps $S_{11}^\alpha, S_{22}^\alpha : (U^1, U^2) \in \mathcal{U} \subset \mathbb{R}^n \rightarrow \mathcal{M}_{n_2}(\mathbb{R})$ are affine in U^2 ,
- (E3) all the maps $Y^{\alpha\beta}$ have the form (1.2), are independent of U^2 , smooth with respect to U^1 and the ellipticity property (1.3) is satisfied,
- (E4) $f \equiv 0$.

Let us emphasize that, as for our local existence result (see [1, Thm 1.6]), Assumption E is essentially needed to get estimates for the high frequencies of the solutions of (1.1). Note that these assumptions are fulfilled by the barotropic compressible Navier-Stokes equations.

Theorem 2.3. *Let Assumption **E** be in force and assume that the pair (N_ω, M_ω) defined in (1.36) satisfies the (SK) condition, for all $\omega \in \mathbb{S}^{d-1}$. Then, there exists a positive constant α such that for all $U_0 = (U_0^1, U_0^2)$ with range in $\mathcal{U}^1 \times \mathbb{R}^{n^2}$ and such that $V_0 = (V_0^1, V_0^2) \stackrel{\text{def}}{=} U_0 - \bar{U}$ is in $(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}}) \times \dot{B}_{2,1}^{\frac{d}{2}-1}$ and satisfies*

$$(2.8) \quad \|V_0^1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}}} + \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \leq \alpha,$$

System (1.1) supplemented with initial data V_0 admits a unique global-in-time solution in the subspace \mathcal{E} of functions $V = (V^1, V^2)$ of $\mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}}) \times \mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})$ such that

$$V^{1,l} \in L^1(\mathbb{R}_+, \dot{B}_{2,1}^{\frac{d}{2}+1}), \quad V^{1,h} \in L^1(\mathbb{R}_+, \dot{B}_{2,1}^{\frac{d}{2}}), \quad V^2 \in L^1(\mathbb{R}_+, \dot{B}_{2,1}^{\frac{d}{2}+1}) \quad \text{and} \quad \partial_t V^2 \in L^1(\mathbb{R}_+, \dot{B}_{2,1}^{\frac{d}{2}-1}).$$

Moreover, there exists an explicit Lyapunov functional equivalent to $\|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}}} + \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}$ and we have:

$$(2.9) \quad \tilde{V}(t) \leq C\tilde{V}(0) \quad \text{where} \quad \tilde{V}(t) \stackrel{\text{def}}{=} \|V\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})}^l + \|V^1\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h + \|V^2\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})}^h \\ + \|\partial_t V^2\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|V^1\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^l + \|V^1\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h.$$

3. PROOF OF THEOREM 2.1

This section is dedicated to proving the global existence and uniqueness of strong solutions for System (0.1), supplemented with initial data close to the reference solution \bar{U} , in the general case where Assumption **D** and the (SK) condition are satisfied.

The core of the proof involves establishing a priori estimates, with the remaining steps (proving existence and uniqueness) being more classical. As previously explained, our strategy revolves around deriving a Lyapunov functional in the style of Beauchard and Zuazua, which is equivalent to the norm we seek to control. This is then combined with the study of the *parabolic mode* W , defined in (1.31), to close the estimates.

3.1. Establishing the a priori estimates. Throughout this section, we fix some bounded open subset \mathcal{O} such that $\bar{\mathcal{O}} \subset \mathcal{U}$, and assume that we are given a smooth (and decaying) solution V of (1.1) on $[0, T] \times \mathbb{R}^d$ with $V_0 \stackrel{\text{def}}{=} U_0 - \bar{U}$ as initial data, satisfying

$$(3.1) \quad \sup_{t \in [0, T]} \|V(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \ll 1 \quad \text{and} \quad U([0, T] \times \mathbb{R}^d) \subset \bar{\mathcal{O}}.$$

We shall use repeatedly that, owing to the embedding $\dot{B}_{2,1}^{\frac{d}{2}} \hookrightarrow L^\infty$, we have also

$$(3.2) \quad \sup_{t \in [0, T]} \|V(t)\|_{L^\infty} \ll 1.$$

From now on, $(c_j)_{j \in \mathbb{Z}}$ stands for a positive sequence that verifies $\|(c_j)\|_{l^1(\mathbb{Z})} \leq 1$.

Since the system exhibits distinct behavior in the low- and high-frequency regimes at the linear level (see Subsection 1.2), it is natural to analyze it within these two contexts. But first, we need to prove the following bounds of time derivatives.

★ *Estimates for $\partial_t V$:*

Lemma 3.1. *Let V be a smooth and sufficiently decaying solution of (1.1) on $[0, T] \times \mathbb{R}^d$ satisfying (3.1), Assumption **D** and Condition (SK). Then, for all $\kappa \in]-\frac{d}{2}, \frac{d}{2}]$, we have:*

$$(3.3) \quad \|\partial_t V^1\|_{\dot{B}_{2,1}^\kappa} \lesssim \|\nabla V\|_{\dot{B}_{2,1}^\kappa}$$

$$(3.4) \quad \text{and} \quad \|\partial_t V^2\|_{\dot{B}_{2,1}^\kappa} \lesssim \|\nabla V\|_{\dot{B}_{2,1}^\kappa} + \|V^2\|_{\dot{B}_{2,1}^{\kappa+2}} + \|\nabla V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V\|_{\dot{B}_{2,1}^\kappa}.$$

Proof. Proving (3.3) relies on product and composition estimates (see Inequalities (B.11) and (B.14)), on the smallness condition (3.1) and on the fact that

$$\partial_t V^1 = - \sum_{\alpha} \left(\tilde{S}_{11}^{\alpha}(U) \partial_{\alpha} V^1 + \tilde{S}_{12}^{\alpha}(U) \partial_{\alpha} V^2 \right) \quad \text{with} \quad \tilde{S}_{1k}^{\alpha} \stackrel{\text{def}}{=} (S_{11}^0)^{-1} S_{1k}^{\alpha} \quad \text{for} \quad k = 1, 2.$$

Similarly, for proving (3.4), we use that denoting $\tilde{S}_{2k}^{\alpha} \stackrel{\text{def}}{=} (S_{22}^0)^{-1} S_{2k}^{\alpha}$ for $k = 1, 2$, we have

$$\begin{aligned} \partial_t V^2 &= - \sum_{\alpha} \left(\tilde{S}_{21}^{\alpha}(U) \partial_{\alpha} V^1 + \tilde{S}_{22}^{\alpha}(U) \partial_{\alpha} V^2 \right) \\ &\quad + \sum_{\alpha, \beta} (S_{22}^0)^{-1} (U) \partial_{\alpha} (Z^{\alpha\beta}(U) \partial_{\beta} V^2) + (S_{22}^0)^{-1} (U) f^2. \end{aligned}$$

So, using (B.11) and (B.14) yields

$$\|\partial_t V^2\|_{\dot{B}_{2,1}^{\kappa}} \lesssim \left(1 + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \left(\|\nabla V\|_{\dot{B}_{2,1}^{\kappa}} + \|f^2\|_{\dot{B}_{2,1}^{\kappa}} + \sum_{\alpha, \beta=1}^d \|\partial_{\alpha} (Z^{\alpha\beta}(U) \partial_{\beta} V^2)\|_{\dot{B}_{2,1}^{\kappa}} \right).$$

Using Leibniz formula in order to decompose the second order term as

$$\partial_{\alpha} \left(Z^{\alpha\beta}(U) \partial_{\beta} V^2 \right) = Z^{\alpha\beta}(U) \partial_{\alpha} \partial_{\beta} V^2 + \partial_{\alpha} \left((Z^{\alpha\beta}(U)) \partial_{\beta} V^2 \right),$$

and remembering Assumption **D** for f^2 yields

$$\begin{aligned} \|\partial_t V\|_{\dot{B}_{2,1}^{\kappa}} &\lesssim \left(1 + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \left(\|\nabla V\|_{\dot{B}_{2,1}^{\kappa}} + (1 + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|\nabla V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V\|_{\dot{B}_{2,1}^{\kappa}} \right. \\ &\quad \left. + (1 + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|V^2\|_{\dot{B}_{2,1}^{\kappa+2}} + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|V^2\|_{\dot{B}_{2,1}^{\kappa+1}} \right). \end{aligned}$$

We get then (3.4) by using the smallness condition (3.1). \square

★ *Low frequencies analysis:* Using from now on the notation $Z_j \stackrel{\text{def}}{=} \dot{\Delta}_j Z$, $F_j \stackrel{\text{def}}{=} \dot{\Delta}_j F$ and applying to (1.5) the operator $\dot{\Delta}_j$, we get

$$(3.5) \quad \bar{S}^0 \partial_t V_j + \sum_{\alpha} \bar{S}^{\alpha} \partial_{\alpha} V_j - \sum_{\alpha, \beta=1}^d \bar{Y}^{\alpha\beta} \partial_{\alpha} \partial_{\beta} V_j = F_j.$$

Taking the $L^2(\mathbb{R}^d, \mathbb{R}^n)$ scalar product with V_j , integrating by parts in the second term, and using the symmetry of matrices \bar{S}^{α} leads to:

$$\frac{d}{dt} \int_{\mathbb{R}^d} \bar{S}^0 V_j \cdot V_j - \sum_{\alpha, \beta=1}^d \int_{\mathbb{R}^d} \bar{Z}^{\alpha\beta} \partial_{\alpha} \partial_{\beta} V_j^2 \cdot V_j^2 = \int_{\mathbb{R}^d} F_j \cdot V_j.$$

Next, the strong ellipticity property (1.3) and Fourier-Plancherel formula ensure that

$$(3.6) \quad \frac{d}{dt} \int_{\mathbb{R}^d} \bar{S}^0 \widehat{V}_j \cdot \widehat{V}_j + \bar{c}_1 \int_{\mathbb{R}^d} |\xi|^2 |\widehat{V}_j^2|^2 d\xi \leq \left\| \widehat{F}_j \right\|_{L^2} \left\| \widehat{V}_j \right\|_{L^2} \quad \text{with} \quad \bar{c}_1 \stackrel{\text{def}}{=} c_1(\bar{U}).$$

Inequality (3.6) provides a L^2 -in-time control only for V^2 . In order to track diffusion effects for the whole solution, we proceed as explained in Subsection 1.2 introducing the functional $L_{\rho, \omega}$ defined in (1.18). After integration on \mathbb{R}^d , this amounts to considering

$$(3.7) \quad I_j \stackrel{\text{def}}{=} \sum_{k=1}^{n-1} \varepsilon_k \text{Re} \int_{\mathbb{R}^d} (M_{\omega} N_{\omega}^{k-1} \widehat{V}_j \cdot M_{\omega} N_{\omega}^k \widehat{V}_j)$$

(where N_{ω} and M_{ω} have been defined in (1.36)) and

$$L_{\rho, \omega}(\widehat{V}_j) \stackrel{\text{def}}{=} \bar{S}^0 \widehat{V}_j \cdot \widehat{V}_j + \min(\rho, \rho^{-1}) I_j.$$

Leveraging Lemma 1.1 with $a = 1$ and $b = 2$ and remembering that $\text{Supp } \widehat{V}_j \subset \{\frac{3}{4}2^j \leq |\xi| \leq \frac{8}{3}2^j\}$ (hence $|\xi| = \rho \simeq 2^j$ on $\text{Supp } \widehat{V}_j$), we get (note that the only change lies in the harmless additional source term F_j):

$$(3.8) \quad \frac{d}{dt} I_j + 2^j \sum_{k=1}^{n-1} \varepsilon_k \int_{\mathbb{R}^d} |M_\omega N_\omega^k \widehat{V}_j|^2 \leq C \varepsilon_0 \max(2^j, 2^{3j}) \int_{\mathbb{R}^d} |M_\omega \widehat{V}_j|^2 + C \left\| (\overline{S}^0)^{-1} \widehat{F}_j \right\|_{L^2} \left\| \widehat{V}_j \right\|_{L^2}$$

provided we choose $\varepsilon_1, \dots, \varepsilon_{n-1}$ small enough and, from (1.21), we have

$$(3.9) \quad \mathcal{L}_j^l \stackrel{\text{def}}{=} (2\pi)^{-d} \int_{\mathbb{R}^d} L_{\rho, \omega}(\widehat{V}_j) \simeq \|V_j\|_{L^2}^2, \quad \forall j \in \mathbb{Z}.$$

Putting together Inequalities (3.6), (3.8), and remembering that (3.9) holds true, we get

$$(3.10) \quad \frac{d}{dt} \mathcal{L}_j^l + 2^{2j} c \int_{\mathbb{R}^d} |\widehat{V}_j^2|^2 + \min(1, 2^{2j}) \sum_{k=1}^{n-1} \varepsilon_k \int_{\mathbb{R}^d} |M_\omega N_\omega^k \widehat{V}_j|^2 \leq C \varepsilon_0 2^{2j} \int_{\mathbb{R}^d} |M_\omega \widehat{V}_j|^2 + C \|F_j\|_{L^2} \sqrt{\mathcal{L}_j^l}.$$

Choosing $\varepsilon_0, \dots, \varepsilon_{n-1}$ according to Lemma 1.1 yields the following inequality:

$$\begin{aligned} \mathcal{N}_V &\stackrel{\text{def}}{=} 2^{2j} c \int_{\mathbb{R}^d} |\widehat{V}_j^2|^2 + \min(1, 2^{2j}) \sum_{k=1}^{n-1} \varepsilon_k \int_{\mathbb{R}^d} |M_\omega N_\omega^k \widehat{V}_j|^2 - C \varepsilon_0 2^{2j} \int_{\mathbb{R}^d} |M_\omega \widehat{V}_j|^2 \\ &\geq \min(1, 2^{2j}) \sum_{k=0}^{n-1} \varepsilon_k \int_{\mathbb{R}^d} |M_\omega N_\omega^k \widehat{V}_j|^2. \end{aligned}$$

Lemma 1.2 ensures that, since condition (SK) is satisfied, the quantity on the right-hand side of the above inequality is positive for any choice of positive parameters $\varepsilon_0, \dots, \varepsilon_{n-1}$. Consequently, if we use Fourier-Plancherel identity and the equivalence (3.9), we see that, for all $j \in \mathbb{Z}$,

$$(3.11) \quad \mathcal{N}_V \geq c \min(1, 2^{2j}) \mathcal{L}_j^l.$$

Proposition 3.1. *Let V be a smooth and sufficiently decaying solution of (1.1) on $[0, T] \times \mathbb{R}^d$ satisfying (3.1), Assumption D and Condition (SK). Then, for all $s \in]-\frac{d}{2}, \frac{d}{2}]$ and $j \in \mathbb{Z}$, we have:*

$$(3.12) \quad \frac{d}{dt} \mathcal{L}_j^l + \min(1, 2^{2j}) \mathcal{L}_j^l \leq c_j 2^{-js} \sqrt{\mathcal{L}_j^l} \left(\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|V\|_{\dot{B}_{2,1}^{s+1}} + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \left(\|V\|_{\dot{B}_{2,1}^{s+1}} + \|V^2\|_{\dot{B}_{2,1}^{s+2}} \right) \right).$$

Proof. For proving (3.12), we first remark that Inequalities (3.10) and (3.11) ensure that

$$(3.13) \quad \frac{d}{dt} \mathcal{L}_j^l + c \min(1, 2^{2j}) \mathcal{L}_j^l \leq C \|F_j\|_{L^2} \sqrt{\mathcal{L}_j^l}.$$

The rest of the proof consists in bounding the r.h.s of (3.13), which requires product estimates in homogeneous Besov spaces. Thanks to (B.11) and Proposition B.3, remembering the definition of F^1 in (1.6), we discover that for all $s \in]-\frac{d}{2}, \frac{d}{2}]$:

$$(3.14) \quad \begin{aligned} \|F^1\|_{\dot{B}_{2,1}^s} &\lesssim \sum_{\alpha=1}^d \left\| \overline{S}^\alpha - S^\alpha(U) \right\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V\|_{\dot{B}_{2,1}^s} \\ &\lesssim \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V\|_{\dot{B}_{2,1}^s}. \end{aligned}$$

To handle F^t , we use (3.3) and (3.4) with $\kappa = s$ and Inequality (B.11). We get:

$$(3.15) \quad \begin{aligned} \|F^t\|_{\dot{B}_{2,1}^s} &\lesssim \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\partial_t V\|_{\dot{B}_{2,1}^s} \\ &\lesssim \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \left(\|\nabla V\|_{\dot{B}_{2,1}^s} + \|V^2\|_{\dot{B}_{2,1}^{s+2}} + \|V\|_{\dot{B}_{2,1}^{s+1}} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right). \end{aligned}$$

To handle F^2 , we use Leibniz identity:

$$(3.16) \quad \partial_\alpha \left(r^{\alpha\beta}(U) \partial_\beta V^2 \right) = \partial_\alpha \left(r^{\alpha\beta}(U) \right) \partial_\beta V^2 + r^{\alpha\beta}(U) \partial_\alpha \partial_\beta V^2,$$

then Inequalities (B.11) and (B.16). We get

$$(3.17) \quad \|F^2\|_{\dot{B}_{2,1}^s} \lesssim \left(\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|V^2\|_{\dot{B}_{2,1}^{s+1}} + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V^2\|_{\dot{B}_{2,1}^{s+2}} \right).$$

The term f can be bounded from product estimate (B.11), and the form of f in (1.4):

$$(3.18) \quad \|f\|_{\dot{B}_{2,1}^s} \lesssim (1 + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|\nabla V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V\|_{\dot{B}_{2,1}^s} \lesssim \|\nabla V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V\|_{\dot{B}_{2,1}^s}.$$

Putting together (3.13), (3.14), (3.15) and (3.17) yields the desired estimate. \square

★ *High frequencies analysis:* As pointed out in Section 2, we need to control (at least the high frequencies part of) $\|\nabla V\|_{L_t^1(L^\infty)}$. In our Besov spaces setting, this amounts to bounding V in $L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})$. To achieve it, we have to exhibit the optimal smoothing properties of the system for V^2 . It turns out to be more convenient to rewrite the whole system as a coupling between the *parabolic unknown* W defined in (1.31) and the *hyperbolic unknown* V^1 .

Before proceeding, we introduce for any $q \times q$ symmetric positive definite matrix A and $L^2(\mathbb{R}^d; \mathbb{R}^q)$ function X the notation

$$(3.19) \quad \|X\|_{L_A^2} \stackrel{\text{def}}{=} \left(\int_{\mathbb{R}^d} AX \cdot X \right)^{1/2}.$$

• **Parabolic mode.** Our aim here is to establish the following result of parabolic maximal regularity for W .

Proposition 3.2. *Let V be a smooth (decaying) solution of (1.1) on $[0, T] \times \mathbb{R}^d$ satisfying (3.1). Assume that $\sigma \in]-\frac{d}{2}, \frac{d}{2}]$. There exists a constant \bar{c}_1 depending only on the coefficients of the system at \bar{U} such that, denoting $W_j \stackrel{\text{def}}{=} \Delta_j W$ we have for all $t \in [0, T]$ and $j \in \mathbb{Z}$,*

$$(3.20) \quad \begin{aligned} 2^{j\sigma} \|W_j(t)\|_{L_{\bar{S}_{22}^0}^2} + \bar{c}_1 2^{(\sigma+2)j} \int_0^t \|W_j\|_{L^2} \leq 2^{j\sigma} \|W_j(t)\|_{L_{\bar{S}_{22}^0}^2} \\ + C \int_0^t c_j \left(2^{-j} \left(\|V\|_{\dot{B}_{2,1}^{\sigma+1}} + \|V^2\|_{\dot{B}_{2,1}^{\sigma+2}} \right) + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \left(\|V\|_{\dot{B}_{2,1}^{\sigma+1}} + \|V^2\|_{\dot{B}_{2,1}^{\sigma+2}} + \|V\|_{\dot{B}_{2,1}^{\sigma+1}} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \right). \end{aligned}$$

Proof. Applying Δ_j to (1.32) and taking the scalar product with W_j yields (see (1.27) for the definition of h):

$$(3.21) \quad \begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} \bar{S}_{22}^0 W_j \cdot W_j - \int_{\mathbb{R}^d} \Delta Z(D) W_j \cdot W_j \\ \leq \left(\left\| \Delta_j |D|^{-1} \bar{S}_{22}^0 (Z(D))^{-1} \partial_t (S_{21}(D)V^1 + S_{22}(D)V^2) \right\|_{L^2} + \|h_j\|_{L^2} \right) \|W_j\|_{L^2}. \end{aligned}$$

Since Operator $|D|^{-1} \bar{S}_{22}^0 (Z(D))^{-1}$ is homogeneous of degree -1 , Bernstein inequality yields

$$\left\| \Delta_j |D|^{-1} \bar{S}_{22}^0 (Z(D))^{-1} \partial_t (S_{21}(D)V^1 + S_{22}(D)V^2) \right\|_{L^2} \lesssim 2^{-j} \left\| \Delta_j \partial_t (S_{21}(D)V^1 + S_{22}(D)V^2) \right\|_{L^2}.$$

Also, thanks to Fourier-Plancherel theorem and (1.3), we have:

$$-\int_{\mathbb{R}^d} \Delta Z(D)W_j \cdot W_j = (2\pi)^{-d} \int_{\mathbb{R}^d} |\xi|^2 Z(\xi) \widehat{W}_j(\xi) \cdot \widehat{W}_j(\xi) \geq 2^{2j} \bar{c}_1 \|W_j\|_{L^2}^2.$$

Hence, we deduce that

$$(3.22) \quad \begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} \overline{S}_{22}^0 W_j \cdot W_j + 2^{2j} \bar{c}_1 \|W_j\|_{L^2}^2 \\ \lesssim \left(2^{-j} \left\| \dot{\Delta}_j \partial_t (S_{21}(D)V^1 + S_{22}(D)V^2) \right\|_{L^2} + \|h_j\|_{L^2} \right) \|W_j\|_{L^2}. \end{aligned}$$

There remains to look at the right-hand side of (3.22). As (3.1) is satisfied, the inequalities (B.11), (3.3) and (3.4) give that for all $\sigma \in]-\frac{d}{2}, \frac{d}{2}]$

$$(3.23) \quad \begin{aligned} \left\| \dot{\Delta}_j h^t \right\|_{L^2} &\lesssim 2^{-j\sigma} c_j \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \left\| \partial_t V^2 \right\|_{\dot{B}_{2,1}^{\sigma}} \\ &\lesssim 2^{-j\sigma} c_j \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \left(\|\nabla V\|_{\dot{B}_{2,1}^{\sigma}} + \|\nabla V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V\|_{\dot{B}_{2,1}^{\sigma}} + \|V^2\|_{\dot{B}_{2,1}^{\sigma+2}} \right). \end{aligned}$$

For bounding h^{21} and h^{22} we use (B.11). We have for all $\sigma \in]-\frac{d}{2}, \frac{d}{2}]$:

$$(3.24) \quad \left\| \dot{\Delta}_j h^{21} \right\|_{L^2} + \left\| \dot{\Delta}_j h^{22} \right\|_{L^2} \lesssim 2^{-j\sigma} c_j \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V\|_{\dot{B}_{2,1}^{\sigma+1}}.$$

The control of h^2 relies on Inequality (B.11). Then, for all $\sigma \in]-\frac{d}{2}, \frac{d}{2}]$, we have

$$(3.25) \quad \left\| \dot{\Delta}_j h^2 \right\|_{L^2} \lesssim 2^{-j\sigma} c_j \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \left(\|V^2\|_{\dot{B}_{2,1}^{\sigma+2}} + \|V\|_{\dot{B}_{2,1}^{\sigma+1}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right).$$

Finally recalling that the operators $S_{21}(D)$ and $S_{22}(D)$ are 0-order, the term $\dot{\Delta}_j \partial_t (S_{21}(D)V^1 + S_{22}(D)V^2)$ may be bounded as follows:

$$(3.26) \quad \left\| \dot{\Delta}_j \partial_t (S_{21}(D)V^1 + S_{22}(D)V^2) \right\|_{L^2} \lesssim \left\| \partial_t \dot{\Delta}_j V \right\|_{L^2}$$

which becomes using (3.3), (3.4) and the smallness condition (3.1)

$$\left\| \dot{\Delta}_j \partial_t (S_{21}(D)V^1 + S_{22}(D)V^2) \right\|_{L^2} \lesssim 2^{-j\sigma} c_j \left(\|\nabla V\|_{\dot{B}_{2,1}^{\sigma}} + \|\nabla V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V\|_{\dot{B}_{2,1}^{\sigma}} + \|V^2\|_{\dot{B}_{2,1}^{\sigma+2}} \right).$$

The term f has been bounded in the proof of Proposition 3.1. In fact, from (3.18), we have,

$$(3.27) \quad \|f_j\|_{L^2} \lesssim c_j 2^{-j\sigma} \|\nabla V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V\|_{\dot{B}_{2,1}^{\sigma}}.$$

Putting all this information together we deduce that for all $\sigma \in]-\frac{d}{2}, \frac{d}{2}]$ and $j \in \mathbb{Z}$

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} \overline{S}_{22}^0 W_j \cdot W_j + 2^{2j} \bar{c}_1 \|W_j\|_{L^2}^2 \\ \lesssim 2^{-j\sigma} c_j \left(2^{-j} \left(\|V\|_{\dot{B}_{2,1}^{\sigma+1}} + \|V^2\|_{\dot{B}_{2,1}^{\sigma+2}} + \|V\|_{\dot{B}_{2,1}^{\sigma+1}} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \right. \\ \left. + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \left(\|V\|_{\dot{B}_{2,1}^{\sigma+1}} + \|V^2\|_{\dot{B}_{2,1}^{\sigma+2}} + \|V\|_{\dot{B}_{2,1}^{\sigma+1}} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \right) \|W_j\|_{L^2}. \end{aligned}$$

Using Lemma A.2 with $X = \overline{S}_{22}^0 W_j \cdot W_j$ gives us the desired estimate. \square

• **Estimate for V^1 .** Here we will look at V^1 as the solution of the symmetric hyperbolic equation (1.33) whose linearized part is given by the left-hand side of (1.35). Let $V_j^1 \stackrel{\text{def}}{=} \dot{\Delta}_j V^1$. Applying Operator $S_{11}^0(U) \dot{\Delta}_j (S_{11}^0(U))^{-1}$ to (1.33) yields

$$(3.28) \quad S_{11}^0(U) \partial_t V_j^1 + S_{11}^\alpha(U) \partial_\alpha V_j^1 + S_{12}(D) Z(D)^{-1} S_{21}(D) V_j^1 = R_j^{11} + \sum_{k=1}^2 H_j^k + \sum_{k=1}^3 G_j^{1k},$$

with

$$(3.29) \quad G_j^{1k} \stackrel{\text{def}}{=} S_{11}^0(U) \dot{\Delta}_j ((S_{11}^0(U))^{-1} G^{1k}), \quad R_j^{11} \stackrel{\text{def}}{=} S_{11}^0(U) \sum_{\alpha=1}^d \left[\dot{\Delta}_j, \tilde{S}_{11}^\alpha(U) \right] \partial_\alpha V^1,$$

where G^{11}, G^{12} and G^{13} are defined in (1.34), and

$$(3.30) \quad \begin{aligned} H_j^1 &\stackrel{\text{def}}{=} \left(\overline{S_{11}^0} - S_{11}^0(U) \right) \dot{\Delta}_j \left((S_{11}^0(U))^{-1} S_{12}(D) Z(D)^{-1} S_{21}(D) V^1 \right), \\ H_j^2 &\stackrel{\text{def}}{=} \overline{S_{11}^0} \dot{\Delta}_j \left(\left((\overline{S_{11}^0})^{-1} - (S_{11}^0(U))^{-1} \right) S_{12}(D) Z(D)^{-1} S_{21}(D) V^1 \right). \end{aligned}$$

Taking the L^2 inner product of (3.28) with V_j^1 , applying Leibniz' formula in the first term, integrating by parts in the second term, using the symmetry of the matrices S_{11}^0 and S_{11}^α and the fact that ${}^T S_{12}^\alpha = S_{21}^\alpha$ yields

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} S_{11}^0(U) V_j^1 \cdot V_j^1 + \int_{\mathbb{R}^d} Z(D)^{-1} S_{21}(D) V_j^1 \cdot S_{21}(D) V_j^1 \\ &= \frac{1}{2} \int_{\mathbb{R}^d} \left(\partial_t S_{11}^0(U) + \sum_{\alpha} \partial_\alpha (S_{11}^\alpha(U)) \right) V_j^1 \cdot V_j^1 + \int_{\mathbb{R}^d} (R_j^{11} + H_j^1 + H_j^2 + G_j^{11} + G_j^{12} + G_j^{13}) \cdot V_j^1. \end{aligned}$$

Next, the strong ellipticity of operator $Z(D)$ (see (1.3)) ensures that

$$(3.31) \quad \begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} S_{11}^0(U) V_j^1 \cdot V_j^1 + \bar{c}_1 \|S_{21}(D) V_j^1\|_{L^2}^2 \\ &\leq \frac{1}{2} \int_{\mathbb{R}^d} \left(\partial_t S_{11}^0(U) + \sum_{\alpha} \partial_\alpha (S_{11}^\alpha(U)) \right) V_j^1 \cdot V_j^1 + \int_{\mathbb{R}^d} (R_j^{11} + H_j^1 + H_j^2 + G_j^{11} + G_j^{12} + G_j^{13}) \cdot V_j^1. \end{aligned}$$

As $S^0(U)$ is a uniformly positive definite matrix on \mathcal{O} , using (3.2) gives

$$(3.32) \quad \mathcal{V}_j^1 \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} S_{11}^0(U) V_j^1 \cdot V_j^1 \simeq \|V^1\|_{L^2}^2,$$

and thus

$$(3.33) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \mathcal{V}_j^1 + \bar{c}_1 \|S_{21}(D) V_j^1\|_{L^2}^2 &\leq \frac{1}{2} \left(\|\partial_t S_{11}^0(U)\|_{L^\infty} \right. \\ &\quad \left. + \sum_{\alpha} \|\partial_\alpha (S_{11}^\alpha(U))\|_{L^\infty} \right) \mathcal{V}_j^1 + \|(R_j^{11}, H_j^1 + H_j^2, G_j^{11}, G_j^{12}, G_j^{13})\|_{L^2} \sqrt{\mathcal{V}_j^1}. \end{aligned}$$

If the rank of $S_{21}(\omega)$ is equal to n_1 for all $\omega \in \mathbb{S}^{d-1}$, then Inequality (3.33) guarantees decay for all components of V^1 . Since that rank assumption has not been made, we take advantage of Beauchard-Zuazua's method developed in subsection 1.2, this time for the equation obtained when applying the operator $\dot{\Delta}_j$ to (1.35). The only change lies in the (harmless) additional source terms G^1 defined in (1.34) and in (1.35).

Now, using the notation introduced in (1.36), it holds:

$$(3.34) \quad \partial_t \widehat{V}_j^1 + (\rho \mathbf{N}_\omega + \mathbf{M}_\omega) \widehat{V}_j^1 = (\overline{S_{11}^0})^{-1} \widehat{G}_j^1.$$

Let us further set:

$$(3.35) \quad \mathbf{I}_j \stackrel{\text{def}}{=} \sum_{k=1}^{n-1} \varepsilon_k \operatorname{Re} \int_{\mathbb{R}^d} (\mathbf{M}_\omega \mathbf{N}_\omega^{k-1} \widehat{V}_j^1 \cdot \mathbf{M}_\omega \mathbf{N}_\omega^k \widehat{V}_j^1) \quad \text{and} \quad \mathcal{L}_j^{1,h} \stackrel{\text{def}}{=} \mathcal{V}_j^1 + \min(2^j, 2^{-j}) \mathbf{I}_j.$$

Lemma 1.1 (with $a = 1$ and $b = 0$) gives for suitable $\varepsilon_1, \dots, \varepsilon_{n-1}$,

$$(3.36) \quad \mathcal{L}_j^{1,h} \simeq \|V_j^1\|_{L^2}^2$$

and, using Fourier-Plancherel theorem, for all $j \in \mathbb{Z}$:

$$(3.37) \quad \begin{aligned} & \frac{d}{dt} \mathcal{L}_j^{1,h} + \bar{c}_1 2^{2j} \|S_{21}(D)V_j^1\|_{L^2}^2 \\ & + \min(1, 2^{2j}) \sum_{k=1}^{n-1} \varepsilon_k \int_{\mathbb{R}^d} |\mathbf{M}_\omega \mathbf{N}_\omega^k \widehat{V}_j^1|^2 \leq C \left(\|\partial_t S_{11}^0(U)\|_{L^\infty} + \sum_{\alpha} \|\partial_\alpha(S_{11}^\alpha(U))\|_{L^\infty} \right. \\ & \left. + \|(R_j^{11}, G_j^{11}, G_j^{12}, G_j^{13})\|_{L^2} \right) \mathcal{L}_j^{1,h} + \left(\min(2^j, 2^{-j}) \|G_j^1\|_{L^2} \right) \sqrt{\mathcal{L}_j^{1,h}}. \end{aligned}$$

At this stage, we have to remember (see Lemma A.1) that if the pair (N_ω, M_ω) satisfies the (SK) condition then so does $(\mathbf{N}_\omega, \mathbf{M}_\omega)$, which guarantees that

$$(3.38) \quad \mathcal{N}_{V^1} \stackrel{\text{def}}{=} \bar{c}_1 2^{2j} \|S_{21}(D)V_j^1\|_{L^2}^2 + \min(1, 2^{2j}) \sum_{k=1}^{n-1} \varepsilon_k \int_{\mathbb{R}^d} |\mathbf{M}_\omega \mathbf{N}_\omega^k \widehat{V}_j^1|^2 \gtrsim \min(1, 2^{2j}) \mathcal{L}_j^{1,h}.$$

After suitably bounding the r.h.s. of (3.37), these considerations will lead to

Proposition 3.3. *Assume that V satisfies the assumptions of Proposition 3.1. Then the following inequality holds true for all $j \in \mathbb{Z}$:*

$$\begin{aligned} \frac{d}{dt} \mathcal{L}_j^{1,h} + \min(1, 2^{2j}) \mathcal{L}_j^{1,h} & \lesssim \left(\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \mathcal{L}_j^{1,h} \\ & + \|(V_j^2, 2^j W_j)\|_{L^2} \sqrt{\mathcal{L}_j^{1,h}} + 2^{-j(\frac{d}{2}+1)} c_j \left(\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^2 \right) \sqrt{\mathcal{L}_j^{1,h}} \\ & + c_j 2^{-j(\frac{d}{2}+1)} \left(\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \sqrt{\mathcal{L}_j^{1,h}} \\ & + \min(2^j, 2^{-j}) \left(\|(V_j^2, 2^j W_j)\|_{L^2} + c_j 2^{-j\frac{d}{2}} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \sqrt{\mathcal{L}_j^{1,h}}. \end{aligned}$$

Proof. Combining (3.37) and (3.38), we deduce that

$$(3.39) \quad \begin{aligned} \frac{d}{dt} \mathcal{L}_j^{1,h} + \min(1, 2^{2j}) \mathcal{L}_j^{1,h} & \leq C \left(\|\partial_t S_{11}^0(U)\|_{L^\infty} + \sum_{\alpha} \|\partial_\alpha(S_{11}^\alpha(U))\|_{L^\infty} \right) \mathcal{L}_j^{1,h} \\ & + \|(R_j^{11}, H_j^1, H_j^2, G_j^{11}, G_j^{12}, G_j^{13})\|_{L^2} \sqrt{\mathcal{L}_j^{1,h}} + \min(2^j, 2^{-j}) \|G_j^1\|_{L^2} \sqrt{\mathcal{L}_j^{1,h}}. \end{aligned}$$

Let us now look at the right-hand side of (3.39). For the first term, the embedding $\dot{B}_{2,1}^{\frac{d}{2}} \hookrightarrow L^\infty$ and (3.3) and (3.4) (with $\kappa = \frac{d}{2}$) allow us to get

$$(3.40) \quad \begin{aligned} \|\partial_t S_{11}^0(U)\|_{L^\infty} + \sum_{\alpha} \|\partial_\alpha(S_{11}^\alpha(U))\|_{L^\infty} & \lesssim \|\partial_t V\|_{L^\infty} + \|\nabla V\|_{L^\infty} \\ & \lesssim \|\nabla V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}. \end{aligned}$$

The other terms may be treated by using commutator and product estimates in homogeneous Besov spaces. First, combining (B.16) and Proposition B.1, one has

$$(3.41) \quad \|R_j^{11}\|_{L^2} \lesssim 2^{-(\frac{d}{2}+1)} c_j \|\nabla V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

Next, using (B.13) combined with (B.16) and the smallness condition (3.1), one gets

$$(3.42) \quad \|G_j^{11}\|_{L^2} \lesssim 2^{-j\sigma} c_j \left(\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V^2\|_{\dot{B}_{2,1}^\sigma} + \|V\|_{\dot{B}_{2,1}^\sigma} \|\nabla V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \quad \text{for all } \sigma > 0.$$

Moreover, from Bernstein's Inequality, we have

$$(3.43) \quad \|(G_j^{12}, G_j^{13})\|_{L^2} \lesssim \|(V_j^2, 2^j W_j)\|_{L^2}.$$

Bounding G_j^{14} relies on product estimate (B.13) and (B.16). We get

$$(3.44) \quad \|G_j^{14}\|_{L^2} \lesssim c_j 2^{-j\frac{d}{2}} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

Next, we write that, by virtue of Proposition B.2 and Inequality (B.16)

$$\|G_j^5\|_{L^2} \lesssim c_j 2^{-j\frac{d}{2}} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\partial_t V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}$$

which, combined with (3.3) with $\kappa = \frac{d}{2}$ and (3.1) leads to

$$(3.45) \quad \|G_j^{15}\|_{L^2} \lesssim c_j 2^{-j\frac{d}{2}} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}.$$

Taking $\sigma = \frac{d}{2}$ in (3.42) and combining with (3.43), (3.44), (3.45), we get that

$$(3.46) \quad \|G_j^1\|_{L^2} \lesssim \|(V_j^2, 2^j W_j)\|_{L^2} + c_j 2^{-j\frac{d}{2}} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

Taking advantage of Bernstein's inequality, we can bound the terms H_j^1 and H_j^2 in L^2 . On the one hand, using Hölder inequality combined with the embedding $\dot{B}_{2,1}^{\frac{d}{2}} \hookrightarrow L^\infty$ one may write

$$\|H_j^1\|_{L^2} \lesssim \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \left\| \dot{\Delta}_j \left((S_{11}^0(U))^{-1} S_{12}(D) Z(D)^{-1} S_{21}(D) V^1 \right) \right\|_{L^2}.$$

Now, the product estimate (B.13), the composition estimate (B.14) and the fact that the operator $S_{12}(D)Z(D)^{-1}S_{21}(D)$ is homogeneous of degree 0 ensure that: for all $\sigma > 0$

$$\|H_j^1\|_{L^2} \lesssim c_j 2^{-j\sigma} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \left((1 + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|V^1\|_{\dot{B}_{2,1}^\sigma} + \|V\|_{\dot{B}_{2,1}^\sigma} \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right).$$

which combined with the smallness condition (3.1) implies that

$$(3.47) \quad \|H_j^1\|_{L^2} \lesssim c_j 2^{-j\sigma} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \left(\|V^1\|_{\dot{B}_{2,1}^\sigma} + \|V\|_{\dot{B}_{2,1}^\sigma} \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right).$$

On the other hand, the product estimate (B.13), the composition estimate (B.14) and the fact that the operator $S_{12}(D)Z(D)^{-1}S_{21}(D)$ is homogeneous of degree 0 provide, for all $\sigma > 0$

$$(3.48) \quad \|H_j^2\|_{L^2} \lesssim c_j 2^{-j\sigma} (\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V^1\|_{\dot{B}_{2,1}^\sigma} + \|V\|_{\dot{B}_{2,1}^\sigma} \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}).$$

Finally, plugging (3.40), (3.41), (3.43), (3.42), (3.47), (3.48) (with $\sigma = \frac{d}{2} + 1$), (3.46) into (3.39), yields the desired inequality. \square

★ *Global a priori estimates.* Propositions 3.1, 3.2 and 3.3 will enable us to derive bounds on V in the space E (defined in Theorem 2.1) in terms of the initial data.

Before proceeding, introducing the following notation (with $N_0 \in \mathbb{N}$) is in order:

$$(3.49) \quad \begin{aligned} \mathcal{L} &\stackrel{\text{def}}{=} \sum_{j \leq N_0} 2^{j(\frac{d}{2}-1)} \sqrt{\mathcal{L}_j^l} + \sum_{j > N_0} 2^{j(\frac{d}{2}+1)} \sqrt{\mathcal{L}_j^{1,h}} + \sum_{j > N_0} 2^{j\frac{d}{2}} \|W_j\|_{L^2_{\mathbb{S}_{22}^0}}, \\ \mathcal{H} &\stackrel{\text{def}}{=} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{l,N_0} + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} + \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0} \end{aligned}$$

with

$$(3.50) \quad \|z\|_{\dot{B}_{2,1}^{l,N_0}} \stackrel{\text{def}}{=} \sum_{j \leq N_0} 2^{js} \|\dot{\Delta}_j z\|_{L^2} \quad \text{and} \quad \|z\|_{\dot{B}_{2,1}^{h,N_0}} \stackrel{\text{def}}{=} \sum_{j > N_0} 2^{js} \|\dot{\Delta}_j z\|_{L^2}.$$

Note that (3.9), (3.36) and the fact that $\bar{S}_{22}^0 \simeq I_{n_2}$ guarantee that

$$(3.51) \quad \mathcal{L} \simeq \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^{l,N_0} + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} + \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{h,N_0}.$$

Furthermore, (1.31) and Bernstein inequalities give

$$(3.52) \quad \|W_j\|_{L^2} \lesssim \|V_j\|_{L^2}, \text{ for all } j > 0 \quad \text{and} \quad \|W_j\|_{L^2} \lesssim 2^{-j} \|V_j\|_{L^2}, \text{ for all } j \leq 0,$$

and for all $s \in \mathbb{R}$

$$\begin{aligned} \|V^2\|_{\dot{B}_{2,1}^{s,N_0}}^{h,N_0} &\leq \|W\|_{\dot{B}_{2,1}^{s,N_0}}^{h,N_0} + \||D|^{-1}(Z(D))^{-1}(S_{21}(D)V^1 + S_{22}(D)V^2)\|_{\dot{B}_{2,1}^{s,N_0}}^{h,N_0} \\ &\leq \|W\|_{\dot{B}_{2,1}^{s,N_0}}^{h,N_0} + C \|V^1\|_{\dot{B}_{2,1}^{s-1}}^{h,N_0} + C 2^{-N_0} \|V^2\|_{\dot{B}_{2,1}^{s,N_0}}^{h,N_0}. \end{aligned}$$

Therefore for N_0 satisfying

$$(3.53) \quad 2^{-N_0} C \leq \frac{1}{2},$$

one has

$$(3.54) \quad \begin{aligned} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0} &\lesssim \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0} + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} \lesssim \mathcal{H}, \\ \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{h,N_0} &\lesssim \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{h,N_0} + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^{h,N_0} \lesssim \mathcal{L}. \end{aligned}$$

Note that in particular, we have

$$(3.55) \quad \mathcal{L} \simeq \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^l + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h + \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h$$

so that is enough to bound the Lyapunov functional \mathcal{L} .

Let us further recall the following inequalities:

$$(3.56) \quad \|z\|_{\dot{B}_{2,1}^{s,N_0}}^{l,N_0} \leq C 2^{N_0(s-s')} \|z\|_{\dot{B}_{2,1}^{s'}}^{l,N_0} \quad \text{and} \quad \|z\|_{\dot{B}_{2,1}^{s'}}^{h,N_0} \leq C 2^{N_0(s'-s)} \|z\|_{\dot{B}_{2,1}^s}^{h,N_0} \quad \text{for } s' \leq s,$$

and the following interpolation inequalities for all $\varsigma \in [0, 1]$:

$$(3.57) \quad \|V^2\|_{\dot{B}_{2,1}^{3\varsigma+\frac{d}{2}-1}} \leq \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^{1-\varsigma} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{\varsigma} \quad \text{and} \quad \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq \sqrt{\|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}}.$$

Our goal is to bound $\mathcal{L}(t)$ and $\int_0^t \mathcal{H}(t)$ for all $t \in \mathbb{R}_+$, in terms of the initial data. Let us start with the low frequencies. As $d \geq 2$, taking $s = \frac{d}{2} - 1$ in Inequality (3.12), applying lemma A.2, multiplying by $2^{j(\frac{d}{2}-1)}$, summing up on $j \leq N_0$ and denoting

$$\mathfrak{V}^{l,N_0} \stackrel{\text{def}}{=} \sum_{j \leq N_0} 2^{j(\frac{d}{2}-1)} \sqrt{\mathcal{L}_j^l},$$

we arrive for some positive constants c_0, C and all $t \in \mathbb{R}_+$ at

$$\mathfrak{V}^{l,N_0}(t) + c_0 \int_0^t \|V(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{l,N_0} \leq \mathfrak{V}^{l,N_0}(0) + C \int_0^t \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}}.$$

Next, after using the interpolation inequalities (3.57) combined with (3.56), (3.54) and (3.51), the previous inequality gets simplified into

$$(3.58) \quad \mathfrak{V}^{l,N_0}(t) + c_0 \int_0^t \|V(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{l,N_0} \leq \mathfrak{V}^{l,N_0}(0) + \int_0^t \mathcal{L} \mathcal{H}.$$

Let us next focus on the high frequencies. Let

$$\mathfrak{V}^{1,h,N_0} \stackrel{\text{def}}{=} \sum_{j > N_0} 2^{j(\frac{d}{2}+1)} \sqrt{\mathcal{L}_j^{1,h}}.$$

Combining Proposition 3.3, Lemma A.2, multiplying by $2^{j(\frac{d}{2}+1)}$ and summing up on $j > N_0$ gives (for some positive constants c'_0 and C)

$$\begin{aligned} \mathfrak{V}^{1,h,N_0}(t) + c'_0 \int_0^t \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} &\leq \mathfrak{V}^{1,h,N_0}(0) + C \int_0^t \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \\ &+ C \int_0^t (\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}) \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} \\ &+ 2^{-N_0} C \int_0^t \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0} + C \int_0^t \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0} + C \int_0^t (\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^2) \\ &+ C \int_0^t (\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}). \end{aligned}$$

Hence, since (3.54) is satisfied, using (3.56), (3.57) and (3.51) enables us to simplify the previous Inequality as

$$\begin{aligned} \mathfrak{V}^{1,h,N_0}(t) + c'_0 \int_0^t \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} &\leq \mathfrak{V}^{1,h,N_0}(0) + C \int_0^t \mathcal{L}\mathcal{H} + C \int_0^t \mathcal{L}^2\mathcal{H} \\ &+ 2^{-N_0} C \int_0^t \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0} + C \int_0^t \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0}. \end{aligned}$$

Using again (3.54) for the penultimate term in the right-hand side of the previous Inequality, we end up with,

$$(3.59) \quad \mathfrak{V}^{1,h,N_0}(t) + c'_0 \int_0^t \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} \leq \mathfrak{V}^{1,h,N_0}(0) + C \int_0^t \mathcal{L}\mathcal{H} + C \int_0^t \mathcal{L}^2\mathcal{H} + 2^{-N_0} C \int_0^t \mathcal{H} + C \int_0^t \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0}.$$

Next, let us simplify as much as possible the estimate given by Proposition 3.2. Taking $\sigma = \frac{d}{2}$ in Inequality (3.20), summing up on $j > N_0$ and setting

$$\mathfrak{W}^{h,N_0} \stackrel{\text{def}}{=} \sum_{j>N_0} 2^{\frac{d}{2}j} \|W_j\|_{L^2_{S_{22}^0}}$$

provides

$$\begin{aligned} \mathfrak{W}^{h,N_0}(t) + \tilde{c}_0 \int_0^t \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0} &\leq \mathfrak{W}^{h,N_0}(0) + 2^{-N_0} C \int_0^t \left(\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} \right) + C \int_0^t \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^2 \\ &+ C \int_0^t \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \left(\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^2 \right), \end{aligned}$$

where \tilde{c}_0 and C are positive constants. Next, using (3.56), (3.57) combined with (3.54) and eventually the smallness condition (3.1), we can eliminate some redundant terms. We get

$$(3.60) \quad \mathfrak{W}^{h,N_0}(t) + \tilde{c}_0 \int_0^t \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0} \leq \mathfrak{W}^{h,N_0}(0) + C \int_0^t (\mathcal{L} + \mathcal{L}^2)\mathcal{H} + 2^{-N_0} C \int_0^t \mathcal{H}.$$

For some $0 < \varepsilon < 1$ that will be fixed hereafter, denote

$$(3.61) \quad \tilde{\mathcal{L}} \stackrel{\text{def}}{=} \mathfrak{W}^{l,N_0} + \varepsilon \mathfrak{V}^{1,h,N_0} + \mathfrak{W}^{h,N_0}.$$

Then, putting together (3.58), (3.59) and (3.60) we can find a constant κ_0 such that

$$(3.62) \quad \tilde{\mathcal{L}}(t) + \kappa_0 \int_0^t \|V(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{l,N_0} + \varepsilon \kappa_0 \int_0^t \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} + \kappa_0 \int_0^t \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0} \leq \tilde{\mathcal{L}}(0) \\ + C \int_0^t (\mathcal{L} + \mathcal{L}^2) \mathcal{H} + 2^{-N_0} C \int_0^t \mathcal{H} + \varepsilon C \int_0^t \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0}.$$

Choosing ε and N_0 so that

$$(3.63) \quad \varepsilon C < \kappa_0/2 \quad \text{and} \quad 2^{-N_0} C \leq \frac{\kappa_0}{4},$$

the last two terms of (3.62) may be absorbed by left-hand side. We deduce that

$$(3.64) \quad \tilde{\mathcal{L}}(t) + \frac{\kappa_0}{2} \int_0^t \|V(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{l,N_0} + \frac{\kappa_0}{4} \int_0^t \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} + \frac{\kappa_0}{2} \int_0^t \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0} \\ \leq \tilde{\mathcal{L}}(0) + C \int_0^t (\mathcal{L} + \mathcal{L}^2) \mathcal{H}.$$

We claim that there exists $\alpha > 0$ such that if $\tilde{\mathcal{L}}(0) < \alpha$ then, for all $t \in [0, T]$, the left-hand side of Inequality (3.64) is smaller than $\tilde{\mathcal{L}}(0)$. Indeed, let us choose $\alpha < \min(1, \frac{\kappa_0}{16C})$ so that $\mathcal{L}(0) < \alpha$ implies that (3.1) is satisfied, and set

$$(3.65) \quad T_0 \stackrel{\text{def}}{=} \sup \left\{ T_1 \in [0, T], \text{ such that } \sup_{0 \leq t \leq T_1} \tilde{\mathcal{L}}(t) \leq \alpha \right\}.$$

The above set is nonempty (as 0 is in it) and contains its supremum since $\tilde{\mathcal{L}}$ is continuous (remember that we assumed that V is smooth). Hence, starting from (3.64), we have

$$\tilde{\mathcal{L}}(t) + \frac{\kappa_0}{2} \int_0^t \|V(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{l,N_0} + \frac{\kappa_0}{4} \int_0^t \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} + \frac{\kappa_0}{2} \int_0^t \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0} \leq \tilde{\mathcal{L}}(0) + 2C\alpha \int_0^t \mathcal{H}.$$

Using the smallness of $\tilde{\mathcal{L}}(0)$, one may conclude that $\tilde{\mathcal{L}} < \alpha$ on $[0, T_0]$. As $\tilde{\mathcal{L}}$ is continuous, we must have $T_0 = T$ and thus the following estimate holds on $[0, T]$:

$$(3.66) \quad \tilde{\mathcal{L}}(t) + \frac{\kappa_0}{8} \int_0^t \|V(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{l,N_0} + \frac{\kappa_0}{8} \int_0^t \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} + \frac{\kappa_0}{8} \int_0^t \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0} \leq \tilde{\mathcal{L}}(0).$$

Clearly, time $t = 0$ does not play any particular role, and one can apply the same argument on any sub-interval of $[0, T]$ which leads for $0 \leq t_0 \leq t \leq T$ to

$$(3.67) \quad \tilde{\mathcal{L}}(t) + \frac{\kappa_0}{8} \int_{t_0}^t \|V(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{l,N_0} + \frac{\kappa_0}{8} \int_{t_0}^t \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} + \frac{\kappa_0}{8} \int_{t_0}^t \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0} \leq \tilde{\mathcal{L}}(t_0).$$

Note that the estimates (3.66), (3.54) and the fact that $2^{js} \simeq 2^{js'}$ for all $s, s' \in [\frac{d}{2} - 1, \frac{d}{2} + 2]$ and $0 \leq j \leq N_0$ ensure that

$$(3.68) \quad \|V\|_{E_t} \leq C \left(\|V(t_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^l + \|V^2(t_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h + \|V^1(t_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h \right) \quad \text{for all } 0 \leq t_0 \leq t \leq T$$

where E_t is the space defined in Theorem 2.1 pertaining to the interval $[0, t]$.

Owing to (3.52) and (3.68) the low frequencies of W also satisfy, for all $0 \leq t_0 \leq t \leq T$

$$(3.69) \quad \|W(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^l + \int_{t_0}^t \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^l \leq C \left(\|V(t_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^l + \|V^2(t_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h + \|V^1(t_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h \right).$$

3.2. Proving the existence and uniqueness parts of Theorem 2.1. Having the a priori estimate (3.66) at hand, constructing a global solution obeying Inequality (2.2) for any data V_0 satisfying (3.1) follows from rather standard arguments. First, in order to benefit from the classical theory of symmetric partially hyperbolic diffusive systems, we remove the very low and very high frequencies of Z_0 so as to have initial data belonging to nonhomogeneous Besov spaces. More precisely, we set for all $p \in \mathbb{N}$,

$$(3.70) \quad V_{0,p} \stackrel{\text{def}}{=} (\dot{S}_p - \dot{S}_{-p})V_0 \quad \text{with} \quad \dot{S}_q \stackrel{\text{def}}{=} \chi(2^{-q}D), \quad q \in \mathbb{Z}.$$

For each $p \in \mathbb{N}$, and $s \geq \frac{d}{2} + 2$, the data $V_{0,p} = (V_{0,p}^1, V_{0,p}^2)$ belong to the *nonhomogeneous* Besov space $B_{2,1}^{s+1} \times B_{2,1}^s$ (see the definition in (B.5)). Consequently, [1, Theorem 1.2] provides us with a unique maximal solution $(V_p^1, V_p^2) \in \mathcal{C}([0, T^p[; B_{2,1}^{s+1} \times B_{2,1}^s)$. Moreover $(V_p^1, V_p^2) \in \mathcal{C}^1([0, T^p[; B_{2,1}^{s-1} \times B_{2,1}^{s-2})$. Taking advantage of the a priori estimates (3.69) and (3.68), and denoting by \mathbb{V}_p the function \mathbb{V} defined in (2.3) pertaining to V_p , we get $\mathbb{V}_p \leq C\mathbb{V}_{0,p}$ as long as V_p satisfies the smallness condition (3.1). Owing to the definition of $V_{0,p}$, we have $\mathbb{V}_{0,p} \leq C\mathbb{V}_0$ and, obviously, $\|V_p\|_{L_{T^p}^\infty(B_{2,1}^{\frac{d}{2}})} \leq \mathbb{V}_p(T^p)$. Hence, using a classical bootstrap argument, one can conclude that, if \mathbb{V}_0 is small enough, then

$$(3.71) \quad \mathbb{V}_p(t) \leq C\mathbb{V}_0 \quad \text{for all } t \in [0, T^p[.$$

In order to show that the solution V_p is global (that is $T^p = \infty$), we take advantage of the continuation criterion in [1, Rem. 1.3]. In fact the interpolation inequality (3.57), embedding Inequality (3.56) and eventually the estimate (3.71) imply that if, for some $T < \infty$

$$(3.72) \quad \begin{aligned} & \bullet U_p([0, T) \times \mathbb{R}^d) \text{ is contained in a compact subset of } \mathcal{U}, \\ & \bullet \int_0^T \left(\|\nabla V_p\|_{L^\infty}^2 + \|\partial_t V_p\|_{L^\infty} \right) < \infty, \end{aligned}$$

then the solution V_p may be continued beyond $[0, T^p[$. Note that (3.72) is satisfied, owing to (3.71) and embedding. Hence $T^p = \infty$ and (3.71) is thus satisfied for all time.

To prove the convergence of (V_p) to some V , one may for instance show that for all $T > 0$, (V_p) is a Cauchy sequence in the space

$$\mathcal{F}_T \stackrel{\text{def}}{=} \{(V^1, V^2) \in L^\infty(0, T; \dot{B}_{2,1}^{\frac{d}{2}} \times \dot{B}_{2,1}^{\frac{d}{2}-1}) \text{ and } V^2 \in L^1(0, T; \dot{B}_{2,1}^{\frac{d}{2}})\}.$$

Adapting the proof of [1, section 2.4] to our case where instead of nonhomogeneous Besov space, we use the corresponding homogeneous Besov spaces ensures that (V_p) is a Cauchy sequence in \mathcal{F}_T and thus has a limit V in this space, and then passing to the limit in (1.1) is straightforward. Compactness arguments can also be used (see [2, Chap. 4]).

Furthermore, time continuity of the solution and $\mathbb{V}(T) \leq C\mathbb{V}_0$ (for \mathbb{V}), for all $T > 0$ may be obtained by adapting the arguments of [2, Chap. 4]. This completes the proof of the existence part of Theorem 2.1. As for uniqueness, it suffices to adapt the proof of [1, section 2.4].

3.3. Proof of Theorem 2.2. The overall strategy is inspired by the joint work of the second author with T. Crin-Barat in [6, Theorem 2.2], and Z. Xin and J. Xu in [28].

■ *First step: Uniform bounds in $\dot{B}_{2,\infty}^{-\sigma_1}$.* The proof of Inequality (2.4) starts from estimate (3.13) on the Lyapunov functional \mathcal{L}_j^l that has been defined on (3.9). After using Lemma A.2 omitting the second (nonnegative) term of the left-hand side, then multiplying by $2^{-j\sigma_1}$ and taking the supremum on \mathbb{Z} , we end up for all $t \geq 0$ with

$$(3.73) \quad \|V(t)\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \leq \|V_0\|_{\dot{B}_{2,\infty}^{-\sigma_1}} + \int_0^t \|(f, F^1, F^2, F^t)\|_{\dot{B}_{2,\infty}^{-\sigma_1}},$$

where F^1 , F^2 and F^t have been defined in (1.6), and f satisfies Assumption D.

In order to bound the term F^1 , we use Inequality (B.12). It holds for $\frac{d}{2} < \sigma_1 \leq \frac{d}{2}$,

$$\|F^1\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \lesssim \sum_{\alpha=1}^d \|S^\alpha(U) - \bar{S}^\alpha\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \|\nabla V\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

In order to bound $S^\alpha(U) - \bar{S}^\alpha$ in $\dot{B}_{2,\infty}^{-\sigma_1}$, one cannot use directly Inequality (B.16) as $-\sigma_1$ may be negative. However, applying (B.15) we can still obtain if (3.1) is satisfied,

$$(3.74) \quad \|S^\alpha(U) - \bar{S}^\alpha\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \lesssim \|V\|_{\dot{B}_{2,\infty}^{-\sigma_1}},$$

whence

$$\|F^1\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \lesssim \|V\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}.$$

Next, remembering the form of F^2 , then combining Inequality (B.12) (recall that $1 - d/2 < \sigma_1 \leq d/2$) and the same composition estimate (applied to $r^{\alpha\beta}(U)$) as in (3.74) yields

$$\begin{aligned} \|F^2\|_{\dot{B}_{2,\infty}^{-\sigma_1}} &\lesssim \sum_{\beta=1}^d \|r^{\alpha\beta}(U) \partial_\beta V^2\|_{\dot{B}_{2,\infty}^{-\sigma_1+1}} \\ &\lesssim \|V\|_{\dot{B}_{2,\infty}^{1-\sigma_1}} \|\nabla V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \lesssim (\|V\|_{\dot{B}_{2,\infty}^{-\sigma_1}}^l + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h) \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}. \end{aligned}$$

Concerning F^t , we have, keeping (3.3) and (3.4) in mind that

$$\begin{aligned} \|F^t\|_{\dot{B}_{2,\infty}^{-\sigma_1}} &\lesssim \|V\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \|\partial_t V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \\ &\lesssim \|V\|_{\dot{B}_{2,\infty}^{-\sigma_1}} (\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^2). \end{aligned}$$

Similarly, the term f may be bounded as follows:

$$\begin{aligned} \|f\|_{\dot{B}_{2,\infty}^{-\sigma_1}} &\lesssim (1 + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|\nabla V \otimes \nabla V\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \\ &\lesssim \|\nabla V\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \|\nabla V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \lesssim (\|V\|_{\dot{B}_{2,\infty}^{-\sigma_1}}^l + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h) \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}. \end{aligned}$$

Thus, regrouping all those estimates, we obtain

$$\|V(t)\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \leq \|V_0\|_{\dot{B}_{2,\infty}^{-\sigma_1}} + \int_0^t \mathcal{Y}_1 + \int_0^t \|V\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \mathcal{Y}_2$$

with

$$\mathcal{Y}_1 \stackrel{\text{def}}{=} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \quad \text{and} \quad \mathcal{Y}_2 \stackrel{\text{def}}{=} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^2 + \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}.$$

Since (3.67) is satisfied, one can prove that $\mathcal{Y}_1, \mathcal{Y}_2 \in L^1(\mathbb{R}_+)$. Applying Gronwall inequality completes the proof of (2.4).

Let us highlight that one has to justify that if V_0 is in $\dot{B}_{2,\infty}^{-\sigma_1}$ (in addition to (2.1)), then the solution constructed in Theorem (2.1) is in $\dot{B}_{2,\infty}^{-\sigma_1}$ for all time. This may be checked by following the construction scheme of the previous subsection.

■ *Second step : proof of generic decay estimates.* The functional

$$\tilde{\mathcal{L}} : t \in \mathbb{R}_+ \mapsto \mathfrak{V}^{l,N_0}(t) + \varepsilon \mathfrak{W}^{1,h,N_0}(t) + \mathfrak{W}^{h,N_0}(t),$$

defined in (3.61), with ε and N_0 satisfying (3.63), is nonincreasing, as Inequality (3.67) is satisfied. Consequently $\tilde{\mathcal{L}}$ is differentiable almost everywhere. It is worth noting that, as, (3.51), the Inequalities (3.9), (3.36) and the fact that $\bar{S}_{22}^0 \simeq I_{n_2}$ imply that

$$(3.75) \quad \tilde{\mathcal{L}} \simeq \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^{l,N_0} + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} + \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{h,N_0}.$$

Furthermore, one proved in (3.67) that there exists a positive constant $\kappa_0 > 0$ such that for all $0 \leq t_0 \leq t$.

$$\tilde{\mathcal{L}}(t) + \frac{\kappa_0}{8} \int_{t_0}^t \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{l,N_0} + \frac{\kappa_0}{8} \int_{t_0}^t \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} + \frac{\kappa_0}{8} \int_{t_0}^t \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0} \leq \tilde{\mathcal{L}}(t_0).$$

From this, we can deduce that

$$\frac{d}{dt} \tilde{\mathcal{L}} + \frac{\kappa_0}{8} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{l,N_0} + \frac{\kappa_0}{8} \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} + \frac{\kappa_0}{8} \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0} \leq 0 \quad \text{a. e. on } \mathbb{R}_+.$$

Granted with this information and (2.4), one can prove the decay estimates of Theorem (2.2) by following the interpolation argument of [28]. The starting point is that, provided, $-\sigma_1 < d/2 - 1$, we have the following interpolation inequality:

$$\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^{l,N_0} \lesssim \left(\|V\|_{\dot{B}_{2,1}^{-\sigma_1}}^{l,N_0} \right)^{\theta_0} \left(\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{l,N_0} \right)^{1-\theta_0} \quad \text{with } \theta_0 \stackrel{\text{def}}{=} \frac{2}{d/2 + 1 + \sigma_1}.$$

Inequality (2.4) thus implies that

$$\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{l,N_0} \gtrsim \left(\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^{l,N_0} \right)^{\frac{1}{1-\theta_0}} \left(\|V_0\|_{\dot{B}_{2,1}^{-\sigma_1}}^{l,N_0} \right)^{-\frac{\theta_0}{1-\theta_0}}.$$

Similarly, one has, as $1 - \sigma < \frac{d}{2}$,

$$\|W\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^{h,N_0} \gtrsim \left(\|W\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{h,N_0} \right)^{\frac{1}{1-\theta_0}} \left(\|W_0\|_{\dot{B}_{2,1}^{1-\sigma_1}}^{h,N_0} \right)^{-\frac{\theta_0}{1-\theta_0}} \gtrsim \left(\|W\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{h,N_0} \right)^{\frac{1}{1-\theta_0}} \left(\|W_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{h,N_0} \right)^{-\frac{\theta_0}{1-\theta_0}}.$$

For the high frequencies term $\|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0}$, using the estimate of Theorem 2.1, one can just write:

$$\|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} \gtrsim \left(\|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} \right)^{\frac{1}{1-\theta_0}} \left(\|V_0^1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} \right)^{-\frac{\theta_0}{1-\theta_0}}.$$

Hence, remembering (3.75), we see that there exists a (small) constant c such that

$$\frac{d}{dt} \tilde{\mathcal{L}} + c C_0^{-\frac{\theta_0}{1-\theta_0}} \tilde{\mathcal{L}}^{\frac{1}{1-\theta_0}} \leq 0 \quad \text{with } C_0 \stackrel{\text{def}}{=} \|V_0\|_{\dot{B}_{2,1}^{-\sigma_1}}^{l,N_0} + \|V_0^1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} + \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{h,N_0}.$$

Integrating, this gives us

$$\tilde{\mathcal{L}}(t) \leq \left(1 + c \frac{\theta_0}{1-\theta_0} \left(\frac{\mathcal{L}(0)}{C_0} \right)^{\frac{\theta_0}{1-\theta_0}} t \right)^{1-\frac{1}{\theta_0}} \tilde{\mathcal{L}}(0),$$

whence, using again (3.75),

$$(3.76) \quad \|V(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^{l,N_0} + \|W(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{h,N_0} + \|V^1(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h,N_0} \lesssim (1+t)^{-\alpha_1} \mathbb{V}_0 \quad \text{with } \alpha_1 \stackrel{\text{def}}{=} \frac{d/2 - 1 + \sigma_1}{2}.$$

The previous Inequality combined with (3.54) provides (2.6) and (2.7). Inequality (2.5) follows from Inequalities (3.76) and (2.4), and interpolation.

4. PROOF OF THEOREM 2.3

This section is dedicated to the proof of a refinement of Theorem 2.1 in the case where System (1.1) meets the additional conditions outlined in Assumption **E**. Let $V = (V^1, V^2)$ be a smooth (and decaying) solution of System (1.1) under Assumption **E**, satisfying $(\bar{U}^1 + V^1) \in \mathcal{O}^1$ for some bounded open subset \mathcal{O}^1 such that $\bar{\mathcal{O}}^1 \subset \mathcal{U}^1$. We assume in addition that

$$(4.1) \quad \sup_{0 \leq t \leq T} \|V^1(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \ll 1, \quad \text{for all } T < T^*,$$

which owing to the Besov embedding $\dot{B}_{2,1}^{\frac{d}{2}} \hookrightarrow L^\infty$ implies that

$$(4.2) \quad \sup_{0 \leq t \leq T} \|V^1(t)\|_{L^\infty} \ll 1 \quad \text{for all } T < T^*.$$

Let us start with the following result.

♠ *Estimate for $\partial_t V$:*

Lemma 4.1. *Let $-\frac{d}{2} < \kappa \leq \frac{d}{2}$. Under the assumptions of Theorem 2.3 and (4.1), we have for all $j \in \mathbb{Z}$,*

$$(4.3) \quad \begin{aligned} & \bullet 2^{j\kappa} \|\partial_t V_j^1\|_{L^2} \lesssim 2^{j\kappa} \|\nabla V_j\|_{L^2} + c_j \left(\|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V^1\|_{\dot{B}_{2,1}^\kappa} + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V^2\|_{\dot{B}_{2,1}^\kappa} \right), \\ & \bullet 2^{j\kappa} \|\partial_t V_j^2\|_{L^2} \lesssim 2^{j\kappa} \|\nabla V_j\|_{L^2} + c_j \left(\|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V^1\|_{\dot{B}_{2,1}^\kappa} + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V^2\|_{\dot{B}_{2,1}^\kappa} \right. \\ & \qquad \qquad \qquad \left. + \|V^2\|_{\dot{B}_{2,1}^{\kappa+2}} + \|V^1\|_{\dot{B}_{2,1}^{\kappa+1}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right). \end{aligned}$$

Proof. The second inequality of (4.3) relies on the following identity:

$$\begin{aligned} \partial_t V^2 &= -(S_{22}^0(U^1))^{-1} \sum_{\alpha} (S_{21}^\alpha(U^1) \partial_\alpha V^1 + S_{22}^\alpha(U) \partial_\alpha V^2) \\ &\quad + (S_{22}^0(U^1))^{-1} \sum_{\alpha, \beta} \partial_\alpha (Z^{\alpha\beta}(U^1)) \partial_\beta V^2 + \sum_{\alpha, \beta} (S_{22}^0(U^1))^{-1} (Z^{\alpha\beta}(U^1) \partial_\alpha \partial_\beta V^2). \end{aligned}$$

Combining Inequalities (4.1), (B.11) and (B.16) we discover that for all α, β ,

$$\begin{aligned} & \bullet \left\| \left((S_{22}^0(U^1))^{-1} S_{21}^\alpha(U^1) - (\bar{S}_{22}^0)^{-1} \bar{S}_{21}^\alpha \right) \partial_\alpha V^1 \right\|_{\dot{B}_{2,1}^\kappa} \lesssim \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V^1\|_{\dot{B}_{2,1}^\kappa}, \\ & \bullet \left\| (S_{22}^0(U^1))^{-1} \partial_\alpha (Z^{\alpha\beta}(U^1)) \partial_\beta V^2 \right\|_{\dot{B}_{2,1}^\kappa} \lesssim \|V^1\|_{\dot{B}_{2,1}^{\kappa+1}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}, \\ & \bullet \left\| (S_{22}^0(U^1))^{-1} (Z^{\alpha\beta}(U^1) \partial_\alpha \partial_\beta V^2) \right\|_{\dot{B}_{2,1}^\kappa} \lesssim \|V^2\|_{\dot{B}_{2,1}^{\kappa+2}}. \end{aligned}$$

In order to handle the term $(S_{22}^0(U^1))^{-1} S_{22}^\alpha(U) \partial_\alpha V^2$, instead of Inequality (B.16) we use (B.18) since $(S_{22}^0(U^1))^{-1} S_{22}^\alpha(U)$ is at most linear wth respect to U^2 . It holds, for $\alpha = 1, \dots, d$,

$$\begin{aligned} & \left\| \left((S_{22}^0(U^1))^{-1} S_{22}^\alpha(U) - (\bar{S}_{22}^0)^{-1} \bar{S}_{22}^\alpha \right) \partial_\alpha V^2 \right\|_{\dot{B}_{2,1}^\kappa} \\ & \qquad \qquad \qquad \lesssim \left((1 + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \|\nabla V^2\|_{\dot{B}_{2,1}^\kappa}. \end{aligned}$$

Combining those estimates and using (4.1) gives the second inequality of (4.3). Similar arguments lead to the first one. \square

♠ *Low frequencies analysis:* The refinement of Proposition 3.1 is the following

Lemma 4.2. *Under the assumptions of Theorem 2.3 and (4.1), we have for all $s \in]-\frac{d}{2}, \frac{d}{2}]$ and $j \in \mathbb{Z}$,*

$$(4.4) \quad \frac{d}{dt} \mathcal{L}_j^l + \min(1, 2^{2j}) \mathcal{L}_j^l \leq c_j 2^{-js} \sqrt{\mathcal{L}_j^l} \left(\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V\|_{\dot{B}_{2,1}^{s+1}} + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V^2\|_{\dot{B}_{2,1}^{s+1} \cap \dot{B}_{2,1}^{s+2}} \right. \\ \left. + \|V^1\|_{\dot{B}_{2,1}^{s+1}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right).$$

Proof. Starting from (3.13), the proof of (4.4) is just a matter of re-estimating F_j^1, F_j^t and F_j^2 in L^2 , for all $j \in \mathbb{Z}$, by taking into account Assumption **E**. Now, using (B.11) and (B.16) we get for all $\frac{d}{2} < s \leq \frac{d}{2}$ and $\alpha = 1, \dots, d$

$$\|\overline{S}_{21}^\alpha - S_{21}^\alpha(U^1)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \lesssim \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

Since $S_{21}^\alpha = {}^T S_{12}^\alpha$ we get the same estimate for $\|\overline{S}_{12}^\alpha - S_{12}^\alpha(U^1)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}$. Next taking into account the structure of S_{11}^α and S_{22}^α and using the composition estimate (B.18) yields

$$(4.5) \quad \left\| \left(\overline{S}_{11}^\alpha - S_{11}^\alpha(U), \overline{S}_{22}^\alpha - S_{22}^\alpha(U) \right) \right\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \lesssim \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}},$$

whence

$$\|F_j^1\|_{L^2} \lesssim c_j 2^{-js} \sum_{\alpha=1}^d \left\| \left(\overline{S}^\alpha - S^\alpha(U) \right) \right\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V\|_{\dot{B}_{2,1}^s} \lesssim c_j 2^{-js} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V\|_{\dot{B}_{2,1}^s}.$$

Keeping inequalities (4.3) (with $\kappa = s$) and (4.1) in mind, we can write

$$\begin{aligned} \|F_j^t\|_{L^2} &\lesssim c_j 2^{-js} \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\partial_t V\|_{\dot{B}_{2,1}^s} \\ &\lesssim c_j 2^{-js} \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \left(\|\nabla V^1\|_{\dot{B}_{2,1}^s} (1 + \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}}) + \|\nabla V^2\|_{\dot{B}_{2,1}^s} \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|V^2\|_{\dot{B}_{2,1}^{s+1} \cap \dot{B}_{2,1}^{s+2}} \right). \end{aligned}$$

Combining the decomposition (3.16) (recall that $r^{\alpha\beta}(U)$ depend only on U^1 , with $r^{\alpha\beta}(\overline{U}^1) = 0$) and the product estimate (B.11) and the composition law (B.16) (if $s+1 > 0$) or (B.14) and (4.1) (if $1 - d/2 < s+1 \leq 0$), one has

$$(4.6) \quad \|F_j^2\|_{L^2} \lesssim c_j 2^{-js} \left(\|V^1\|_{\dot{B}_{2,1}^{s+1}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V^2\|_{\dot{B}_{2,1}^{s+2}} \right).$$

Putting together those estimates gives us (4.4). \square

♠ *High frequencies analysis: Parabolic mode.*

Proposition 4.1. *Under the assumptions of Theorem 2.3 and using the notation (3.19), we have the following estimate for the parabolic mode W defined in (1.31): for all $j \geq 0$,*

$$(4.7) \quad 2^{j(\frac{d}{2}-1)} \|W_j(t)\|_{L_{\overline{S}_{22}}^2} + 2^{j(\frac{d}{2}+1)} \overline{c}_1 \int_0^t \|W_j\|_{L^2} \leq 2^{j(\frac{d}{2}-1)} \|W_j(0)\|_{L_{\overline{S}_{22}}^2} + C c_j \int_0^t \left(\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} \right) + C 2^{j(\frac{d}{2}-1)} \int_0^t \|V_j\|_{L^2}.$$

Proof. Starting from (3.22), using Lemma A.2 with $X = \int_{\mathbb{R}^d} \overline{S}_{22}^0 W_j \cdot W_j$ then reverting to (3.26) we get

$$(4.8) \quad \|W_j(t)\|_{L_{\overline{S}_{22}}^2} + 2^{2j} \overline{c}_1 \int_0^t \|W_j\|_{L^2} \leq \|W_j(0)\|_{L_{\overline{S}_{22}}^2} + \int_0^t (2^{-j} \|\partial_t V_j\|_{L^2} + \|h_j\|_{L^2})$$

with $h = h^t + h^{21} + h^{22} + h^2$ defined in (1.27). Taking $\kappa = \frac{d}{2} - 1$ in (4.3) and using the smallness condition (4.1), we discover that for all $j \in \mathbb{Z}$

$$\begin{aligned} 2^{j(\frac{d}{2}-1)} \|\partial_t V_j\|_{L^2} &\lesssim 2^{j\frac{d}{2}} \|V_j\|_{L^2} + c_j (\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}) + \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \\ &\lesssim 2^{j\frac{d}{2}} \|V_j\|_{L^2} + c_j (\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 + \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}). \end{aligned}$$

Bearing in mind Assumption **E** and using Inequality (B.11) combined with (B.16) (or (B.17)), we have the following outcome:

$$\begin{aligned} \|h^t\|_{L^2} &\lesssim 2^{-j(\frac{d}{2}-1)}c_j \left(\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} \right), \\ \|h^{21}\|_{L^2} &\lesssim 2^{-j(\frac{d}{2}-1)}c_j \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}, \\ \|h^{22}\|_{L^2} &\lesssim 2^{-j(\frac{d}{2}-1)}c_j (\|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|\nabla V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}, \\ \|h^2\|_{L^2} &\lesssim 2^{-j(\frac{d}{2}-1)}c_j \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}. \end{aligned}$$

Plugging the above inequalities in (4.8), multiplying by $2^{j(\frac{d}{2}-1)}$ and using the fact that $2^{-j} \leq 1$ for all $j \geq 0$ completes the proof of (4.7). \square

♠ *High frequencies analysis: Estimates of V^1 .* The substitute of Proposition 3.3 is the following.

Proposition 4.2. *Under the assumptions of Theorem 2.3, the following inequality holds true:*

$$\begin{aligned} \frac{d}{dt} \mathcal{L}_j^{1,h} + \min(1, 2^{2j}) \mathcal{L}_j^{1,h} &\lesssim \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \mathcal{L}_j^{1,h} + \left(\|(V_j^2, 2^j W_j)\|_{L^2} \right. \\ &\quad \left. + 2^{-j\frac{d}{2}} c_j \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + 2^{-j\frac{d}{2}} c_j \min(2^{2j}, 1) \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \sqrt{\mathcal{L}_j^{1,h}}, \end{aligned}$$

with $\mathcal{L}_j^{1,h}$ defined in (3.9).

Proof. As $S_{11}^0 = \text{Id}$, performing the method leading to (3.39), we discover that

$$\begin{aligned} \frac{d}{dt} \mathcal{L}_j^{1,h} + \min(1, 2^{2j}) \mathcal{L}_j^{1,h} &\leq C \sum_{\alpha} \|\partial_{\alpha}(S_{11}^{\alpha}(U))\|_{L^{\infty}} \mathcal{L}_j^{1,h} \\ &\quad + \|(R_j^{11}, G_j^{11}, G_j^{12}, G_j^{13})\|_{L^2} \sqrt{\mathcal{L}_j^{1,h}} + \min(2^j, 2^{-j}) \|G_j^1\|_{L^2} \sqrt{\mathcal{L}_j^{1,h}}. \end{aligned}$$

Since, from Assumption **E**, $S_{11}^{\alpha}(U)$ is affine with respect to U^2 , we get

$$\|\partial_{\alpha}(S_{11}^{\alpha}(U))\|_{L^{\infty}} \lesssim \|\nabla V^2\|_{L^{\infty}} \lesssim \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}.$$

Next, taking advantage of (B.8) the term $\|R_j^{11}\|_{L^2}$ can be bounded as follows:

$$\|R_j^{11}\|_{L^2} \lesssim 2^{-j\frac{d}{2}} c_j \|\nabla V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

The fact that $S_{12}^{\alpha}(U)$ depends only on U^1 combined with the Inequalities (B.11), (B.16) and (4.1) ensures that

$$\|G_j^{11}\|_{L^2} \lesssim 2^{-j\frac{d}{2}} c_j \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \quad \text{and} \quad \|G_j^{11}\|_{L^2} \lesssim 2^{-j(\frac{d}{2}-1)} c_j \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}.$$

Similarly, keeping in mind that $S_{11}^{\alpha}(U)$ is at most linear with respect to U^2 one has

$$\|G_j^{14}\|_{L^2} \lesssim 2^{-j(\frac{d}{2}-1)} c_j \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}.$$

Combining the previous inequalities and (3.43) completes the proof of the proposition. \square

♠ *Conclusion.* Taking $s = \frac{d}{2} - 1$ in (4.4), then applying Lemma A.2, we get for all $j \leq N_0$:

$$\sqrt{\mathcal{L}_j^l(t)} + c2^{2j} \int_0^t \sqrt{\mathcal{L}_j^l} \leq \sqrt{\mathcal{L}_j^l(0)} + c_j 2^{-j(\frac{d}{2}-1)} \int_0^t \left(\|V\|^2_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} \right).$$

At the same time, applying Lemma A.2 to the inequality of Proposition 4.2 and taking advantage of the fact that $\mathcal{L}_j^1 \simeq \|V_j^1\|_{L^2}^2$ yields for all $j \geq N_0 > 0$

$$\begin{aligned} \sqrt{\mathcal{L}_j^{1,h}(t)} + \int_0^t \sqrt{\mathcal{L}_j^{1,h}} &\leq \sqrt{\mathcal{L}_j^{1,h}(0)} + \int_0^t \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|V_j^1\|_{L^2} \\ &\quad + c_j 2^{-j\frac{d}{2}} \int_0^t \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \left(\|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) + \|(V_j^2, 2^j W_j)\|_{L^2}. \end{aligned}$$

For $0 < \varepsilon < 1$, let us set

$$(4.9) \quad \begin{aligned} \tilde{L} &\stackrel{\text{def}}{=} \sum_{j \leq N_0} 2^{j(\frac{d}{2}-1)} \sqrt{\mathcal{L}_j^l} + \sum_{j > N_0} \left(2^{j(\frac{d}{2}-1)} \|W\|_{L^2_{\tilde{s}_{22}}} + \varepsilon 2^{j\frac{d}{2}} \sqrt{\mathcal{L}_j^{1,h}} \right) \quad \text{and} \\ \tilde{H} &\stackrel{\text{def}}{=} \sum_{j \leq N_0} 2^{j(\frac{d}{2}+1)} \|V_j\|_{L^2} + \sum_{j > N_0} \left(2^{j(\frac{d}{2}+1)} \|W_j\|_{L^2} + \varepsilon 2^{j\frac{d}{2}} \|V_j^1\|_{L^2} \right). \end{aligned}$$

Putting together the previous two inequalities and (4.7) and using the notation $\|\cdot\|_{\dot{B}_{2,1}^{h,N_0}}$ defined in (3.50), we end up with

$$(4.10) \quad \begin{aligned} \tilde{L}(t) + c \int_0^t \tilde{H} &\leq \tilde{L}(0) + \int_0^t \left(\|V\|^2_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \\ &\quad + 2^{-N_0} \int_0^t \left(\|V^2\|_{\dot{B}_{2,1}^{h,N_0}} + \|V^1\|_{\dot{B}_{2,1}^{h,N_0}} \right) + \varepsilon \int_0^t \|W\|_{\dot{B}_{2,1}^{h,N_0}}. \end{aligned}$$

In order to close our estimate we need to exhibit a bound of $\int_0^t \|V^2\|_{\dot{B}_{2,1}^{h,N_0}}$. This can be accomplished by utilizing Inequality (3.54) (with index $\frac{d}{2} + 1$ instead of $\frac{d}{2} + 2$). First by choosing $\varepsilon > 0$ sufficiently small, the last term on the r.h.s. of the previous Inequality can be absorbed by the left-hand side. Next, by selecting N_0 large enough and using (3.54), the second integral on the r.h.s. of the previous Inequality can also be absorbed into the left-hand side. Owing to the smallness condition (4.1) and interpolation inequality, the term $\int_0^t \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}$ can likewise be bounded by the left hand side of (4.10). Therefore, (4.10) transforms into:

$$(4.11) \quad \tilde{L}(t) + \frac{c}{2} \int_0^t \tilde{H} \leq \tilde{L}(0) + C \int_0^t \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2.$$

Owing to interpolation inequalities, one can prove that the integral on the right-hand side of (4.11) is dominated by $\int_0^t \tilde{L}\tilde{H}$. Hence one can conclude exactly as in the previous section that if $\|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|V_0^1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}}}$ is small enough, then there exist some (new) positive real numbers c_0 and C such that

$$(4.12) \quad \tilde{L}(t) + c_0 \int_0^t \tilde{H} \leq \tilde{L}(0) \quad \text{for all } 0 \leq t \leq T.$$

Using (3.54) and (4.12) and the fact that $2^{j\frac{d}{2}} \simeq 2^{j(\frac{d}{2}-1)} \simeq 2^{j(\frac{d}{2}+1)}$ for all $0 \leq j \leq N_0$, we deduce that for all $0 \leq t \leq T$,

$$(4.13) \quad \|V(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^l + \|(W(t), V^2(t))\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^h + \|V^1(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h \\ + \int_0^t \left(\|V\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^l + \|(W, V^2)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h \right) \leq C \left(\|V(0)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^l + \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^h + \|V_0^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h \right).$$

Since, according to (1.31), we have

$$(4.14) \quad S_{22}^0(U) \partial_t V^2 = \Delta Z(D)W + h^{21} + h^{22} + h^2$$

with the functions h^{21} , h^{22} and h^2 defined in (1.27), taking advantage of the usual product estimates and composition laws ensures that $\|\partial_t V^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})}$ is also bounded by the right-hand side of (4.13).

From this point, the rest of the proof follows standard methods. We begin by regularizing the initial data as in (3.70). Then, each regularized initial data $V_{0,p}$ is in the *nonhomogeneous* Besov space $B_{2,1}^{\frac{d}{2}} \times B_{2,1}^{\frac{d}{2}+1}$ and [1, Theorem 1.2] provides us with a unique smooth maximal solution $V_p = (V_p^1, V_p^2)$ on $[0, T^p)$ such that for all $T < T^p$,

$$(4.15) \quad V_p^1 \in \mathcal{C}([0, T]; B_{2,1}^{\frac{d}{2}+1}), \quad V_p^2 \in \mathcal{C}([0, T]; B_{2,1}^{\frac{d}{2}}) \cap L_T^1(B_{2,1}^{\frac{d}{2}+2}) \quad \text{and} \quad \partial_t V_p \in L^1([0, T]; B_{2,1}^{\frac{d}{2}}).$$

Since this solution is (relatively) smooth and (2.8) holds for all $p \in \mathbb{N}$, it satisfies (4.13) for all $t < T^p$. This implies in particular that the smallness condition (4.1) holds on $[0, T^p[$ and that

$$(4.16) \quad \int_0^{T^p} \|\nabla V_p^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} dt < \infty.$$

In order to prove that $T^p = \infty$, we need the following Lemma.

Lemma 4.3. *Under Assumption E, let $V = (V^1, V^2) \in \mathcal{C}([0, T^*]; B_{2,1}^{\frac{d}{2}+1} \times B_{2,1}^{\frac{d}{2}})$ be a solution of (1.1) with the regularity described at (4.15) for all $T < T^*$. There exists a (small) positive constant η such that if*

$$(4.17) \quad \int_0^{T^*} \|\nabla V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \int_0^{T^*} \|\partial_t V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} < \infty,$$

$$(4.18) \quad \text{and} \quad \sup_{0 \leq t < T^*} \|V^1(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq \eta,$$

then V may be continued beyond T^* as a solution with the regularity corresponding to (4.15).

Proof. The proof is very close to that of [1, Theorem 1.2]. Arguing as for [1, Proposition 2.1], we can establish that for all $t < T^*$ and $j \in \mathbb{Z}$, we have:

$$\|\Delta_j V^1(t)\|_{L^2} \leq \|\Delta_j V_0^1\|_{L^2} + C \int_0^t \|\nabla V^2\|_{L^\infty} \|\Delta_j V^1\|_{L^2} \\ + \int_0^t \|(R_j^{11}, \sum_{\alpha=1}^d S_{12}^\alpha(U^1) \partial_\alpha V^2)\|_{L^2} \quad \text{with} \quad R_j^{11} \stackrel{\text{def}}{=} \sum_{\alpha=1}^d [(S_{11}^\alpha(U^2), \Delta_j](\partial_\alpha V^1).$$

To bound R_j^{11} we use the commutator estimate (A.6) in [1] while the last term in the r.h.s. of the previous inequality can be bounded from (B.14). Owing to (4.18), we finally get

$$(4.19) \quad \|V^1(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \leq \|V_0^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + C \int_0^t \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|\nabla V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + C \int_0^t \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}.$$

Next, remembering that V^2 satisfies (1.26) (with h^t, h^{21}, h^{22}, h^2 defined in (1.27) and $f^2 = 0$) and following the lines of the proof of [1, Prop. B.2 or prop. 2.2], we discover that for all $0 < t < T$,

$$(4.20) \quad \begin{aligned} \|V^2(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \int_0^t \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} + \int_0^t \|\partial_t V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \\ &+ \sum_{\alpha=1}^d \int_0^t \left\| \left(\overline{S}_{21}^\alpha \partial_\alpha V^1, \overline{S}_{22}^\alpha \partial_\alpha V^2 \right) \right\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \int_0^t \|(V^2, h^t, h^{21}, h^{22}, h^2)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}. \end{aligned}$$

Using the product estimate (B.10) for h^2 and estimate (A.15) in [1] for h^t , it holds that

$$\begin{aligned} \|h^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}, \\ \|h^t\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\partial_t V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|\partial_t V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}. \end{aligned}$$

Next, since S_{21}^α and S_{22}^α are affine with respect to U^2 , they are of the form $\mathfrak{S}_1(U^1)V^2 + \mathfrak{S}_2(U^1)$ where \mathfrak{S}_1 and \mathfrak{S}_2 are smooth. Taking advantage of (B.10), we thus have for $k = 1, 2$

$$\|S_{2k}^\alpha(U) - \overline{S}_{2k}^\alpha\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \lesssim \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + (1 + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}},$$

which implies, owing to (4.18), that

$$\|h^{21}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|h^{22}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \lesssim \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

Plugging all this information into (4.20) and using (4.18) yields

$$\begin{aligned} \|V^2(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \int_0^t \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} + \int_0^t \|\partial_t V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \\ &+ \int_0^t \left(\|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\nabla V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} (\|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|\partial_t V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}) + \|V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right). \end{aligned}$$

Using again (4.18) and combining with Inequality (4.19), we end up for all $t \in [0, T[$ with

$$\begin{aligned} \|\nabla V^1(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|V^2(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \int_0^t \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} &\lesssim \|\nabla V_0^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \\ &+ \int_0^t \left(1 + \|\nabla V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\partial_t V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \right) \left(\|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) + \int_0^t \|\nabla V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}. \end{aligned}$$

Remembering (4.17) and using Gronwall Lemma enables us to say that

$$\sup_{t \in [0, T[} \left(\|\nabla V^1(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|V^2(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) + \int_0^T \|\nabla^2 V^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} < \infty,$$

and applying [1, Rem. 1.4] and embedding allows to continue the solution beyond T . \square

Combining this Lemma with the global estimates obtained in (4.13) and the smallness assumption (2.8), we conclude that the lifespan T^p of V^p is infinite for all p .

To complete the proof of existence, one can argue as in the previous section. For uniqueness, see [1, Theorem 1.6]. In fact, there, it is shown that uniqueness holds true (without smallness condition) in the set of functions (V^1, V^2) such that $V^1 \in C([0, T]; \dot{B}_{2,1}^{\frac{d}{2}})$ and $V^2 \in C([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1}) \cap L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})$.

5. APPLICATION TO THE MHD SYSTEM

As explained in e.g. [15, Chapter 6] or [18, Chapter 3], the motion of a viscous, compressible, and heat conducting magnetohydrodynamic (abbreviated as MHD) flow in \mathbb{R}^3 can be described by the following full compressible MHD equations²:

$$(5.1) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \rho \partial_t u + \rho u \cdot \nabla u - \operatorname{div}(2\mu D(u) + \lambda I_3 \operatorname{div}(u)) + \nabla p = -\frac{1}{\mu_0}(\operatorname{rot} B) \times B, \\ \rho e_\theta(\partial_t \theta + u \cdot \nabla \theta) + \theta p_\theta \operatorname{div}(u) - \operatorname{div}(k \nabla \theta) = \mathbb{T} + \frac{1}{\sigma \mu_0^2} |\operatorname{rot} B|^2, \\ \partial_t B + u \cdot \nabla B + B \operatorname{div} u - B \cdot \nabla u - \frac{1}{\sigma \mu_0} \Delta B = -\nabla \left(\frac{1}{\sigma \mu_0} \right) \times \operatorname{rot} B. \end{cases}$$

Here $u(t, x) \in \mathbb{R}^3$ denotes the velocity field, $\rho = \rho(t, x) \in \mathbb{R}_+$ the density, $p = p(t, x) \in \mathbb{R}$ the pressure, $e = e(t, x) \in \mathbb{R}$ the internal energy by unit mass, $\theta = \theta(t, x) \in \mathbb{R}_+$ the absolute temperature and $B(t, x) \in \mathbb{R}^3$ the magnetic induction. We denote by $D(u) \stackrel{\text{def}}{=} \frac{1}{2}(\nabla u + {}^T \nabla u)$ the deformation tensor and \mathbb{T} is the viscous dissipation function, given by

$$(5.2) \quad \mathbb{T} \stackrel{\text{def}}{=} \frac{\mu}{2} \sum_{i,j=1}^3 (\partial_{x_j} u^i + \partial_{x_i} u^j)^2 + \lambda (\operatorname{div} u)^2.$$

The real numbers λ and μ are the viscosity coefficients depending on the thermodynamic quantities ρ and θ ; $\sigma = \sigma(\rho, \theta)$ is called the coefficient of electrical conductivity and μ_0 the magnetic permeability.

In order to reduce (5.1) to a closed system of 8 equations with the 8 unknowns (ρ, u, θ, B) , we make the following (physically relevant):

Assumption G.

- (1) The thermodynamic quantities p and e are smooth functions of $\rho > 0$ and $\theta > 0$, satisfying

$$(5.3) \quad p_\rho(\rho, \theta) = \frac{\partial p}{\partial \rho} > 0 \quad \text{and} \quad e_\theta(\rho, \theta) = \frac{\partial e}{\partial \theta} > 0.$$

- (2) The coefficients μ , λ and k are smooth functions of (ρ, θ) that satisfy

$$(5.4) \quad \mu > 0, \quad \nu \stackrel{\text{def}}{=} 2\mu + \lambda > 0 \quad \text{and} \quad k > 0.$$

- (3) The coefficient σ is a smooth function of (ρ, θ) satisfying

$$(5.5) \quad \sigma > 0.$$

System (5.1) is supplemented with initial data ρ_0 , u_0 , θ_0 and B_0 such that $\operatorname{div} B_0 = 0$. Then, for smooth enough solutions, applying the operator div on (5.1)₄ and using the identities:

$$\begin{aligned} & - (u \cdot \nabla) B - B \operatorname{div} u + (B \cdot \nabla) u = \operatorname{rot}(u \times B) - u \operatorname{div} B, \\ & \frac{1}{\sigma \mu_0} \Delta B - \nabla \left(\frac{1}{\sigma \mu_0} \right) \times \operatorname{rot} B = -\operatorname{rot} \left(\frac{1}{\sigma \mu_0} \operatorname{rot} B \right) + \frac{1}{\sigma \mu_0} \nabla \operatorname{div} B, \end{aligned}$$

we discover that $\operatorname{div} B$ is solution of the following transport-diffusion equation

$$(\operatorname{div} B)_t + \operatorname{div}(u \operatorname{div} B) = \operatorname{div} \left(\left(\frac{1}{\sigma \mu_0} \right) \nabla \operatorname{div} B \right).$$

Hence, MHD system has the following property:

- (P) A solution of (5.1) such that $\operatorname{div} B_0 = 0$ satisfies $\operatorname{div} B = 0$ as long as it remains smooth.

²Below, for any differentiable function $S = S(\rho, \theta)$ we use the notation $S_\rho \stackrel{\text{def}}{=} \frac{\partial S}{\partial \rho}$ and $S_\theta \stackrel{\text{def}}{=} \frac{\partial S}{\partial \theta}$.

In terms of $U \stackrel{\text{def}}{=} (\rho, u, \theta, B)$, (5.1) may be seen as a symmetrizable hyperbolic partially diffusive system on the phase space $\mathcal{U} \stackrel{\text{def}}{=} \{(\rho, u, \theta, B) \in \mathbb{R}^8 / \rho > 0, \theta > 0\}$. Indeed, we have:

$$(5.6) \quad S^0(U) \frac{d}{dt} U + \sum_{\alpha=1}^3 S^\alpha(U) \partial_\alpha U - \sum_{\alpha, \beta=1}^3 \partial_\alpha (Y^{\alpha\beta} \partial_\beta V) = f(U)$$

where

$$S^0(U) \stackrel{\text{def}}{=} \begin{pmatrix} \frac{p_\rho}{\rho} & 0 & 0 & 0 \\ 0 & \rho I_3 & 0 & 0 \\ 0 & 0 & \frac{\rho e_\theta}{\theta} & 0 \\ 0 & 0 & 0 & \frac{1}{\mu_0} I_3 \end{pmatrix}, \quad f(U) \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\theta} (\mathbb{T} + \frac{1}{\sigma \mu_0} |\text{rot } B|^2) - k \nabla \theta \cdot \nabla (\frac{1}{\theta}) \\ -\nabla (\frac{1}{\sigma \mu_0^2}) \text{rot } B - \left(\nabla (\frac{1}{\sigma \mu_0^2}) \cdot \nabla \right) B \end{pmatrix},$$

$$\sum_{\alpha=1}^3 S^\alpha(U) \xi_\alpha \stackrel{\text{def}}{=} \begin{pmatrix} \frac{p_\rho}{\rho} u \cdot \xi & p_\rho \xi & 0 & 0 \\ p_\rho {}^T \xi & \rho (u \cdot \xi) I_3 & p_\theta {}^T \xi & \frac{1}{\mu_0} ({}^T \xi B - (B \cdot \xi) I_3) \\ 0 & p_\theta \xi & \frac{\rho e_\theta}{\theta} (u \cdot \xi) & 0 \\ 0 & \frac{1}{\mu_0} ({}^T B \xi - (B \cdot \xi) I_3) & 0 & \frac{1}{\mu_0} (u \cdot \xi) I_3 \end{pmatrix}$$

and $Y^{\alpha\beta} \stackrel{\text{def}}{=}} \begin{pmatrix} 0 & 0 \\ 0 & Z^{\alpha\beta} \end{pmatrix}$ with $\sum_{\alpha, \beta=1}^3 Z^{\alpha\beta} \xi_\alpha \xi_\beta \stackrel{\text{def}}{=} \begin{pmatrix} \mu |\xi|^2 I_3 + (\mu + \lambda) \xi \otimes \xi & 0 & 0 \\ 0 & \frac{k}{\theta} |\xi|^2 & 0 \\ 0 & 0 & \frac{1}{\mu_0^2 \sigma} |\xi|^2 I_3 \end{pmatrix}$

where the relation $B \times \text{rot } B = \frac{1}{2} \nabla (|B|^2) - (B \cdot \nabla) B$ has been used.

It is clear that $S^0(U)$ is a positive definite matrix for all $U \in \mathcal{U}$, and that the matrices $S^\alpha(U)$ are real symmetric. Furthermore a simple calculation reveals that for all $A = (X_1, Y, X_2)$ with $X_1 \in \mathbb{C}^3$, $X_2 \in \mathbb{C}^3$ and $Y \in \mathbb{R}$, and all $\xi \in \mathbb{R}^3$ with $|\xi| = 1$, we have

$$(5.7) \quad \sum_{\alpha, \beta=1}^d \langle Z^{\alpha\beta} \xi_\alpha \xi_\beta A, A \rangle \geq \min(\mu, \nu) |X_1|^2 + \frac{k}{\theta} Y^2 + \frac{1}{\mu_0^2 \sigma} |X_2|^2$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian product in \mathbb{C}^7 . As for the right-hand side f of (5.6), it can be regarded as a lower order (quadratic) term satisfying **(D4)**.

We claim that Condition **(SK)** is satisfied at any $U \in \mathcal{U}$. Indeed, let $\phi = (a, A)$ with $a \in \mathbb{C}$ and consider $\omega \in \mathbb{S}^2$. As $S^0(U)$ is invertible, if $(S^0)^{-1}(U) (\sum_{\alpha, \beta=1}^3 Y^{\alpha\beta}(U) \omega_\alpha \omega_\beta) \phi = 0_{\mathbb{C}^8}$, then

$$\left(\sum_{\alpha, \beta=1}^3 Z^{\alpha\beta}(U) \omega_\alpha \omega_\beta \right) A = 0_{\mathbb{C}^7},$$

which, owing to (5.7), (5.4) and (5.5) implies that $A = 0_{\mathbb{C}^7}$. Now, if there exists $\lambda \in \mathbb{C}$ such that $(S^0)^{-1}(U) (\sum_{\alpha=1}^3 S^\alpha(U) \omega_\alpha) \phi = \lambda \phi$, then $a \frac{p_\rho}{\rho} {}^T \omega = 0_{\mathbb{C}^2}$. Since $p_\rho > 0$ (Assumption **G**). Then, owing to $\omega \neq 0$, we have $a = 0$ and thus $\phi = 0_{\mathbb{C}^8}$.

Consequently, we have the following result which is a direct application of Theorem 2.1:

Theorem 5.1 (Global existence for (MHD) system). *Let Assumption **G** be in force and fix some positive real numbers $\bar{\theta}$ and \bar{B} . Assume that the initial data $(\rho_0, u_0, \theta_0, B_0)$ are such that $\text{div } B_0 = 0$. There exist positive constants c and C depending only on the coefficients of the system such that if*

$$(5.8) \quad \|\rho_0 - \bar{\rho}\|_{\dot{B}_{2,1}^{\frac{1}{2}} \cap \dot{B}_{2,1}^{\frac{5}{2}}} + \|(u_0, \theta_0 - \bar{\theta}, B_0 - \bar{B})\|_{\dot{B}_{2,1}^{\frac{1}{2}} \cap \dot{B}_{2,1}^{\frac{3}{2}}} \leq c$$

then System (5.1) supplemented with initial data $(\rho_0, u_0, \theta_0, B_0)$ admits a unique global-in-time solution (ρ, u, θ, B) such that $V = (V^1, V^2)$ with $V^1 \stackrel{\text{def}}{=}} \rho - \bar{\rho}$ and $V^2 \stackrel{\text{def}}{=} (u, \theta - \bar{\theta}, B - \bar{B})$ belongs to the space E of Theorem 2.1, and Inequality (2.2) holds.

If, in addition, $(\rho_0 - \bar{\rho}, u_0, \theta_0 - \bar{\theta}, B_0 - \bar{B})$ belongs to $\dot{B}_{2,\infty}^{-\sigma_1}$ for some $-1/2 < \sigma_1 \leq 3/2$ then the solution V satisfies (2.4) and the decay estimates of Theorem 2.2.

If we put $\theta = 0$ and $B = 0$ in (5.1) and keep only the first two equations, we get the so-called barotropic compressible Navier-Stokes system:

$$(5.9) \quad \begin{cases} \partial_t \rho + u \cdot \nabla \rho + \rho \operatorname{div}(u) = 0 \\ \rho \partial_t u + \rho u \cdot \nabla u - \operatorname{div}(2\mu(\rho)Du + \lambda(\rho)I_d \operatorname{div}(u)) + \nabla p(\rho) = 0. \end{cases}$$

Without difficulties, we can see that, if

$$(5.10) \quad \mu(\rho) > 0, \quad \nu(\rho) > 0 \quad \text{and} \quad p_\rho(\rho) > 0 \quad \text{for all } \rho > 0,$$

then the system (5.9) satisfies both Assumption **E** and the (SK) condition. Then, Theorem 2.3 can be applied and we recover the following result first proved by the second author in [9]:

Theorem 5.2. *Let $d \geq 2$ and (5.10) be assumed. Fix some constant and positive reference density $\bar{\rho}$. There exist positive constants c and C depending only on μ , λ , $\bar{\rho}$ and p such that if*

$$(5.11) \quad \|\rho_0 - \bar{\rho}\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}}} + \|u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \leq c$$

then System (5.9) supplemented with initial data (ρ_0, u_0) admits a unique global-in-time solution (ρ, u) with $\rho > 0$. Moreover (ρ, u) belongs to the class \mathcal{E} , defined in Theorem 2.3, and satisfies the estimate (2.9).

Acknowledgments. *The first author has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement N° 945332.*

APPENDIX A. SOME INEQUALITIES

The goal of this part is twofold: first, we give an outline of the proof of Lemma 1.1 and establish a basic linear algebra result which ensures that if the pair (N_ω, M_ω) defined in (1.36) satisfies Condition (SK), then so does $(\mathbf{N}_\omega, \mathbf{M}_\omega)$. Then, we recall a result about a differential inequality that has been used repeatedly in the paper.

Let us start with the justification of Lemma 1.1. Let V satisfy Equation (1.10). Then, performing the change of variable

$$(A.1) \quad \tau \stackrel{\text{def}}{=} \frac{t\rho^b}{\kappa} \quad \text{and} \quad r \stackrel{\text{def}}{=} \kappa\rho^{a-b} \quad \text{with} \quad \rho \stackrel{\text{def}}{=} |\xi|,$$

we discover that $v(\tau) \stackrel{\text{def}}{=} \widehat{V}(t, \xi)$ is solution of

$$v' + (r\mathcal{N}_\omega + \mathcal{M}_\omega)v = 0 \quad \text{with} \quad \mathcal{M}_\omega = S^{-1}\mathcal{B}_\omega, \quad \mathcal{N}_\omega = S^{-1}\mathcal{A}_\omega \quad \text{and} \quad \mathcal{A}_\omega, \mathcal{B}_\omega \quad \text{defined in (1.16)}.$$

According to [6, 11], we can find arbitrarily small positive parameters $\varepsilon_1, \dots, \varepsilon_{n-1}$ such that

$$\mathcal{J}'_\omega + \frac{r}{2} \sum_{k=1}^{n-1} \varepsilon_k |\mathcal{M}_\omega \mathcal{N}_\omega^k v|^2 \leq \varepsilon_0 \max(r, r^{-1}) |\mathcal{M}_\omega v|^2 \quad \text{with} \quad \mathcal{J}_\omega \stackrel{\text{def}}{=} \sum_{k=1}^{n-1} \varepsilon_k \operatorname{Re}(\mathcal{M}_\omega \mathcal{N}_\omega^{k-1} v \cdot \mathcal{M}_\omega \mathcal{N}_\omega^k v).$$

Taking the real part of the Hermitian product in \mathbb{C}^n of the following equation

$$Sv' + (r\mathcal{A}_\omega + \mathcal{B}_\omega)v = 0$$

with v and using the properties (1.11) and (1.12), we readily get

$$\frac{1}{2} \frac{d}{dt} (Sv \cdot v) + |\mathcal{B}_\omega v|^2 \leq 0,$$

whence

$$\frac{1}{2} \frac{d}{dt} \left(Sv \cdot v + 2 \min(r, r^{-1}) \mathcal{J}_\omega \right) + |\mathcal{B}_\omega v|^2 + \frac{\min(1, r^2)}{2} \sum_{k=1}^{n-1} \varepsilon_k |\mathcal{M}_\omega \mathcal{N}_\omega^k v|^2 \leq \varepsilon_0 |\mathcal{M}_\omega v|^2.$$

Because $|\mathcal{M}_{\omega v}| \simeq |\mathcal{B}_{\omega v}|$, using the rescaling (A.1) completes the proof of Lemma 1.1.

We also have the following result, which forms the foundation of Subsection 1.4.

Lemma A.1. *Let the nonnegative integers n_1 and n_2 satisfy $n = n_1 + n_2$. Let $M_{22} \in \mathcal{M}_{n_2}(\mathbb{C})$, $N_{11} \in \mathcal{M}_{n_1}(\mathbb{C})$, $N_{22} \in \mathcal{M}_{n_2}(\mathbb{C})$, $N_{12} \in \mathcal{M}_{n_1, n_2}(\mathbb{C})$ and $N_{21} \in \mathcal{M}_{n_2, n_1}(\mathbb{C})$ such that ${}^T \overline{N_{21}} = N_{12}$. Let us set*

$$M \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 \\ 0 & M_{22} \end{pmatrix} \quad \text{and} \quad N \stackrel{\text{def}}{=} \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}.$$

Finally, let $Q \in \mathcal{M}_{n_2}(\mathbb{C})$ satisfy

$$(A.2) \quad \operatorname{Re}(Q\eta_2 \cdot \eta_2) \neq 0 \quad \text{for all } \eta_2 \in \mathbb{C}^{n_2} \setminus \{0\}.$$

Then, if the pair (N, M) satisfies Condition (SK), so does $(N_{11}, N_{12}QN_{21})$. The converse is true if, furthermore, M_{22} is invertible.

Proof. Let $\phi_1 \in \mathbb{C}^{n_1}$ satisfy $N_{12}QN_{21}\phi_1 = 0$ and $N_{11}\phi_1 = \lambda\phi_1$ for some $\lambda \in \mathbb{C}$. Then, we have

$$0 = N_{12}QN_{21}\phi_1 \cdot \phi_1 = QN_{21}\phi_1 \cdot N_{21}\phi_1,$$

whence $N_{21}\phi_1 = 0$, due to (A.2). This implies that $\phi \stackrel{\text{def}}{=} (\phi_1, 0) \in \mathbb{C}^n$ satisfies $N\phi = \lambda\phi$. Since obviously $\phi \in \ker M$, and as (N, M) satisfies Condition (SK), one may conclude that $\phi = 0$.

Conversely, if $\phi = (\phi_1, \phi_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ satisfies $M\phi = 0$ with M_{22} invertible, then we must have $\phi_2 = 0$. If we assume in addition that $N\phi = \lambda\phi$ for some $\lambda \in \mathbb{C}$, then we have $N_{11}\phi_1 = \lambda\phi_1$ and $N_{21}\phi_1 = 0$, and thus $N_{12}QN_{21}\phi_1 = 0$. As $(N_{11}, N_{12}QN_{21})$ satisfies Condition (SK), we conclude that $\phi_1 = 0$. \square

The following classical result has been used a number of times in this text (see the proof in e.g. the Appendix of [1]).

Lemma A.2. *Let $X : [0, T] \rightarrow \mathbb{R}_+$ be a continuous function such that X is differentiable. Assume that there exist a constant $B \geq 0$ and a measurable function $A : \mathbb{R}_+ \rightarrow [0, T]$ such that*

$$\frac{1}{2} \frac{d}{dt} X + BX \leq AX^{\frac{1}{2}} \quad \text{a.e on } [0, T].$$

Then, for all $t \in [0, T]$, we have

$$X^{\frac{1}{2}}(t) + B \int_0^t X^{\frac{1}{2}} \leq X^{\frac{1}{2}}(0) + \int_0^t A.$$

APPENDIX B. SOME PROPERTIES OF BESOV SPACES

For the reader's convenience, we here recall some results on the Littlewood-Paley decomposition and Besov spaces, the source of which can be found in [2, Chap. 2].

To define the Littlewood-Paley decomposition, we fix some smooth radial non increasing function χ with $\operatorname{Supp} \chi \subset B(0, \frac{4}{3})$ and $\chi \equiv 1$ on $B(0, \frac{3}{4})$, then set $\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$ so that

$$\chi + \sum_{j \geq 0} \varphi(2^{-j} \cdot) = 1 \text{ on } \mathbb{R}^d \quad \text{and} \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \cdot) = 1 \text{ on } \mathbb{R}^d \setminus \{0\}.$$

For S a function and z a tempered distribution, we (formally) define the pseudo-differential operator $S(D)$ by:

$$(B.1) \quad S(D)z \stackrel{\text{def}}{=} \mathcal{F}^{-1} (S(\cdot) \mathcal{F}z).$$

Next, we introduce the following low frequency cut-off:

$$(B.2) \quad \dot{S}_j = \chi(2^{-j} D) \text{ for all } j \in \mathbb{Z} \quad \text{and} \quad S_j \stackrel{\text{def}}{=} \dot{S}_j \text{ for all } j \geq 0, \quad S_j = 0 \text{ for all } j \leq -1,$$

and define the homogeneous dyadic block $\dot{\Delta}_j$ and nonhomogeneous dyadic block Δ_j as

$$(B.3) \quad \begin{aligned} \dot{\Delta}_j &\stackrel{\text{def}}{=} \varphi(2^{-j}D) \text{ for all } j \in \mathbb{Z} \\ \Delta_j &= \dot{\Delta}_j \text{ for all } j \geq 0, \quad \Delta_{-1} = \dot{S}_0 \quad \text{and} \quad \Delta_j = 0 \text{ for all } j < -1. \end{aligned}$$

We also consider the subset \mathcal{S}'_h of tempered distributions z such that

$$(B.4) \quad \lim_{j \rightarrow -\infty} \dot{S}_j z = 0.$$

We introduce the homogeneous Besov semi-norms (resp. nonhomogeneous Besov norms):

$$(B.5) \quad \|z\|_{\dot{B}_{p,r}^s} \stackrel{\text{def}}{=} \|2^{js} \|\dot{\Delta}_j z\|_{L^p}\|_{l^r} \quad (\text{resp. } \|z\|_{B_{p,r}^s} \stackrel{\text{def}}{=} \|2^{js} \|\dot{\Delta}_j z\|_{L^p}\|_{l^r}).$$

Then, for any $s \in \mathbb{R}$ and $(p, r) \in [1, \infty]$ we define the homogeneous Besov spaces $\dot{B}_{p,r}^s$ (resp. nonhomogeneous Besov spaces $B_{p,r}^s$) to be the subset of those z in \mathcal{S}'_h (resp. the subset of those z in the tempered distribution space \mathcal{S}') such that $\|z\|_{\dot{B}_{p,r}^s}$ (resp. $\|z\|_{B_{p,r}^s}$) is finite. Any $z \in \mathcal{S}'$ can be decomposed in terms of high and low frequencies parts, as follows:

$$(B.6) \quad z^l \stackrel{\text{def}}{=} \sum_{j \leq 0} \dot{\Delta}_j z \quad \text{and} \quad z^h \stackrel{\text{def}}{=} \sum_{j > 0} \dot{\Delta}_j z.$$

We constantly used the following Besov semi-norms for low and high frequencies:

$$(B.7) \quad \|z\|_{\dot{B}_{2,1}^s}^l \stackrel{\text{def}}{=} \sum_{j \leq 0} 2^{js} \|\dot{\Delta}_j z\|_{L^2} \quad \text{and} \quad \|z\|_{\dot{B}_{2,1}^s}^h \stackrel{\text{def}}{=} \sum_{j > 0} 2^{js} \|\dot{\Delta}_j z\|_{L^2}.$$

Even though most of the functions considered here have range in the set of vectors or matrices, we keep the same notation for Besov spaces pertaining to this case.

In order to bound the commutator terms, we used the following results:

Proposition B.1. *The following inequalities hold true:*

$$(B.8) \quad \left\| [a, \dot{\Delta}_j] b \right\|_{L^2} \leq C c_j 2^{-j\sigma} \|\nabla a\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|b\|_{B_{2,1}^{\sigma-1}} \quad \text{with} \quad \sum_j c_j = 1 \quad \text{if} \quad -d/2 < \sigma \leq 1 + d/2,$$

$$(B.9) \quad \sup_{j \in \mathbb{Z}} \left\| [a, \dot{\Delta}_j] b \right\|_{L^2} \leq C 2^{-j\sigma} \|\nabla a\|_{\dot{B}_{2,\infty}^{\frac{d}{2}} \cap L^\infty} \|b\|_{\dot{B}_{2,\infty}^{\sigma-1}} \quad \text{if} \quad -d/2 \leq \sigma < 1 + d/2.$$

The following product laws in Besov spaces have been used repeatedly.

Proposition B.2. *Let $(s, r) \in]0, \infty[\times [1, \infty]$. Then $B_{2,r}^s \cap L^\infty$ is an algebra and we have*

$$(B.10) \quad \|ab\|_{\dot{B}_{2,r}^s} \leq C (\|a\|_{L^\infty} \|b\|_{\dot{B}_{2,r}^s} + \|b\|_{L^\infty} \|a\|_{\dot{B}_{2,r}^s}).$$

If, furthermore $-d/2 < s \leq d/2$, then the following inequalities hold:

$$(B.11) \quad \|ab\|_{\dot{B}_{2,1}^s} \leq C \|a\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|b\|_{\dot{B}_{2,1}^s}$$

and if $-d/2 \leq s < d/2$

$$(B.12) \quad \|ab\|_{\dot{B}_{2,\infty}^s} \leq C \|a\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|b\|_{\dot{B}_{2,\infty}^s}.$$

Inequality (B.10) is often combined with the embedding $\dot{B}_{2,1}^{\frac{d}{2}} \hookrightarrow L^\infty$. Then for $s > 0$ and $1 \leq r \leq \infty$, Inequality (B.10) becomes

$$(B.13) \quad \|ab\|_{\dot{B}_{2,r}^s} \leq C \|a\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|b\|_{\dot{B}_{2,r}^s} + \|a\|_{\dot{B}_{2,r}^s} \|b\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

Among the results necessary to prove Theorems 2.1, 2.3 and 2.2, we have the following one.

Proposition B.3. *Let f be a function in $C^\infty(\mathbb{R})$. Then, the following inequalities hold:*

- If $-\frac{d}{2} < s \leq \frac{d}{2}$, then

$$(B.14) \quad \|f \circ u - f \circ v\|_{\dot{B}_{2,1}^s} \leq C(f', \|u, v\|_{L^\infty})(1 + \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|u - v\|_{\dot{B}_{2,1}^s}.$$

- If $-\frac{d}{2} \leq s < \frac{d}{2}$, then (B.14) remains valid for $r = \infty$, that is,

$$(B.15) \quad \|f \circ u - f \circ v\|_{\dot{B}_{2,\infty}^s} \leq C(f', \|u, v\|_{L^\infty})(1 + \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|u - v\|_{\dot{B}_{2,\infty}^s}.$$

- If $f(0) = 0$ and $s > 0$, then

$$(B.16) \quad \|f \circ u\|_{\dot{B}_{2,r}^s} \leq C(f', \|u\|_{L^\infty}) \|u\|_{\dot{B}_{2,r}^s}.$$

Proof. The proof of (B.14) and (B.16) can be found in [2, pages 94 and 104] while (B.15) and (B.14) can be obtained by adapting the proof of first inequality of [2, page 449]. \square

Let us finally state some other composition results:

Proposition B.4. *Let $0 \leq n_1 \leq n$ be two integers. Let $f : (X, Y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n-n_1} \mapsto f(X, Y)$ be a smooth function on \mathbb{R}^n vanishing at 0. Assume that f is linear with respect to Y .*

Then, the following inequalities hold true:

$$(B.17) \quad \|f(u, v)\|_{\dot{B}_{2,1}^s} \leq C(f', \|u\|_{L^\infty})(\|v\|_{\dot{B}_{2,1}^s} (1 + \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) + \|u\|_{\dot{B}_{2,1}^s}) \quad \text{for any } 0 < s \leq \frac{d}{2}.$$

Furthermore if $-\frac{d}{2} < s \leq \frac{d}{2}$ we have

$$(B.18) \quad \|f(u_1, v_1) - f(u_2, v_2)\|_{\dot{B}_{2,1}^s} \leq C \|v_2 - v_1\|_{\dot{B}_{2,1}^s} (1 + \|u_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \\ + C(1 + \|u_1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|u_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \left(\|u_2 - u_1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|v_1\|_{\dot{B}_{2,1}^s} + \|u_1 - u_2\|_{\dot{B}_{2,1}^s} \right).$$

Finally, if $-\frac{d}{2} \leq s < \frac{d}{2}$ then we have

$$(B.19) \quad \|f(u_1, v_1) - f(u_2, v_2)\|_{\dot{B}_{2,\infty}^s} \leq C \|v_2 - v_1\|_{\dot{B}_{2,\infty}^s} (1 + \|u_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \\ + C(1 + \|u_1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|u_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \left(\|u_2 - u_1\|_{\dot{B}_{2,\infty}^s} \|v_1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|u_1 - u_2\|_{\dot{B}_{2,\infty}^s} \right).$$

In the above two inequalities, $C = C(f', \|u_1, u_2\|_{L^\infty})$.

Proof. The reader can refer to [1] for the proof in the nonhomogeneous Besov spaces case, the adaptation to our framework being straightforward. \square

REFERENCES

- [1] J.-P. Adogbo and R. Danchin. Local well-posedness in the critical regularity setting for hyperbolic systems with partial diffusion. *arXiv:2307.05981*, 2024.
- [2] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*, volume 343. Springer, 2011.
- [3] K. Beauchard and E. Zuazua. Large time asymptotics for partially dissipative hyperbolic system. *Arch. Rational Mech. Anal.*, 199:177–227, 2011.
- [4] C. Burtea, T. Crin-Barat, and J. Tan. Pressure-relaxation limit for a one-velocity Baer-Nunziato model to a Kapila model. *Math. Models Methods Appl. Sci.*, 33(4):687–753, 2023.
- [5] J.-M. Coron. *Control and nonlinearity*. Number 136 in Mathematical Surveys and Monographs. American Mathematical Soc., 2007.
- [6] T. Crin-Barat and R. Danchin. Partially dissipative hyperbolic systems in the critical regularity setting: the multi-dimensional case. *J. Math. Pures Appl. (9)*, 165:1–41, 2022.
- [7] T. Crin-Barat and R. Danchin. Partially dissipative one-dimensional hyperbolic systems in the critical regularity setting, and applications. *Pure and Applied Analysis*, 4(1):85–125, 2022.
- [8] T. Crin-Barat and R. Danchin. Global existence for partially dissipative hyperbolic systems in the lp framework, and relaxation limit. *Mathematische Annalen*, 386(3):2159–2206, 2023.
- [9] R. Danchin. Global existence in critical spaces for compressible Navier-Stokes equations. *Invent. Math.*, 141(3):579–614, 2000.

- [10] R. Danchin. Global existence in critical spaces for compressible viscous and heat conductive gases. *Archive for Rational Mechanics and Analysis*, 160:1–39, 2001.
- [11] R. Danchin. Partially dissipative systems in the critical regularity setting, and strong relaxation limit. *EMS Surv. Math. Sci.*, 9(1):135–192, 2022.
- [12] R. Danchin and P. B. Mucha. The compressible euler system with nonlocal pressure: global existence and relaxation. *Calculus of Variations and Partial Differential Equations*, 63(6):148, 2024.
- [13] V. Giovangigli and L. Matuszewski. Structure of entropies in dissipative multicomponent fluids. *Kinet. Relat. Models*, 6(2):373–406, 2013.
- [14] B. Haspot. Existence of global strong solutions in critical spaces for barotropic viscous fluids. *Archive for Rational Mechanics and Analysis*, 202(2):427–460, 2011.
- [15] S. Kawashima. *Systems of a hyperbolic parabolic type with applications to the equations of magnetohydrodynamics*. PhD thesis, Kyoto University, 1983.
- [16] S. Kawashima and Y. Shizuta. On the normal form of the symmetric hyperbolic-parabolic systems associated with the conservation laws. *Tohoku Math. J. (2)*, 40(3):449–464, 1988.
- [17] V. Lemarié. Parabolic-elliptic keller-segel’s system. *Tunisian Journal of Mathematics*, To appear.
- [18] T. Li and T. Qin. Physics and partial differential equations. *Higher Education Press*, 1:543–575, 2021.
- [19] A. Majda. Compressible fluid flow and systems of conservation laws in several space variable. *Springer*, 1984.
- [20] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.*, 20:67–104, 1980.
- [21] P. Qu and Y. Wang. Global classical solutions to partially dissipative hyperbolic systems violating the kawashima condition. *Journal de Mathématiques Pures et Appliquées*, 109:93–146, 2018.
- [22] B. Hanouzet S. Bianchini and R. Natalini. Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy. *Comm. Pure and Appl. Math.*, 60:1559–1622, 2007.
- [23] D. Serre. *Systèmes de lois de conservation. I*. Fondations. Diderot Editeur, Paris, 1996. Hyperbolicité, entropies, ondes de choc.
- [24] D. Serre. System of conservation laws with dissipation. lecture at the SISSA, 2008.
- [25] D. Serre. Local existence for viscous system of conservation laws: H^s -data with $s > 1 + d/2$. In *Nonlinear partial differential equations and hyperbolic wave phenomena*, volume 526 of *Contemp. Math.*, pages 339–358. Amer. Math. Soc., Providence, RI, 2010.
- [26] D. Serre. The structure of dissipative viscous system of conservation laws. *Phys. D*, 239(15):1381–1386, 2010.
- [27] S. Shizuta and S. Kawashima. Systems of equations of hyperbolic-parabolic type with applications to the discrete boltzmann equation. *Hokkaido Math J.*, 14:249–275, 1985.
- [28] Z. Xin and J. Xu. Optimal decay for the compressible Navier-Stokes equations without additional smallness assumptions. *Journal of Differential Equations*, 274:543–575, 2021.
- [29] J. Xu and S. Kawashima. The optimal decay estimates on the framework of Besov spaces for generally dissipative systems. *Arch. Rational Mech. Anal*, 218:275–315, 2015.
- [30] E. Zuazua. Propagation, observation, and control of waves approximated by finite difference methods. *SIAM Rev.*, 47(2):197–243, 2005.