



HAL
open science

The NLS4 equation of the NLS hierarchy and its rational solutions

Pierre Gaillard

► **To cite this version:**

Pierre Gaillard. The NLS4 equation of the NLS hierarchy and its rational solutions. 2025. hal-04862567

HAL Id: hal-04862567

<https://hal.science/hal-04862567v1>

Preprint submitted on 3 Jan 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The NLS4 equation of the NLS hierarchy and its rational solutions

Pierre Gaillard,

Université Bourgogne Europe,
Institut de mathématiques de Bourgogne,
9 avenue Alain Savary BP 47870
21078 Dijon Cedex, France :
E-mail : Pierre.Gaillard@u-bourgogne.fr

Abstract

This study concerns the research of rational solutions of the hierarchy of the nonlinear Schrödinger equation. In particular, we are interested in the equation of order 4.

Rational solutions to the fourth equation of the NLS hierarchy are constructed and explicit expressions of these solutions are given for the first orders. These solutions depend on multi real parameters. We study the associated patterns of these solutions in the (x, t) plane according to the different values of parameters.

This work allows to highlight the phenomenon of rogue waves as in the case of lower order equations such as the nonlinear Schrödinger equation, the mKdV equation or the Hirota equation.

Key Words : NLS hierarchy, rational solutions, NLS equation, mKdV equation, Hirota equation, rogue waves.

PACS numbers :

33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

1 Introduction

We consider the fourth equation of the NLS hierarchy of order 4 (*NLS4*) which can be written as

$$u_t + u_{5x} + 10|u|^2 u_{3x} + 20u_{2x} u_x \bar{u} + 10(|u_x|^2 u)_x + 30|u|^4 u_x = 0, \quad (1)$$

with as usual the subscript meaning the partial derivatives and \bar{u} the complex conjugate of u .

As the NLS equation [92, 3, 4, 5, 12, 7, 30, 15], the first equation of this hierarchy, the mKdV equation [99, 81, 82, 83, 84] which is the second one, the

LPD equation [85, 86, 87, 88, 89, 90, 91] which is the third one, the (NLS4) equation is the fourth equation of this NLS hierarchy.

Recent works have been done on this subject. We can quote [103] where the global existence of smooth solutions to the initial value problem in the Lax hierarchy of the nonlinear Schrödinger equation is established. We can also mention [104] where rogue wave solutions on the periodic background for the fourth-order nonlinear Schrödinger (NLS) equation were constructed in terms of Jacobi elliptic functions.

We recall some basic facts about the first three equations of this NLS hierarchy and the phenomenon of the rogue waves. The first equation is the well known, one dimensional focusing nonlinear Schrödinger equation (NLS) which can be written in the following form

$$iv_t + v_{xx} + 2|v|^2v = 0, \quad (2)$$

with the subscripts have the same meaning as described in the beginning.

We do not give details on the history of research, in impressive numbers on this equation, but we limit ourselves to precursors.

Zakharov and Shabat [92, 93] solved it in 1972 using the inverse scattering method. Periodic and almost periodic algebro-geometric solutions were first constructed by Its and Kotlyarov [94, 95] in 1976.

Concerning the phenomenon of the rogue waves, relatively recent results can be cited. Different types of representations of the solutions have been given, in terms of Fredholm determinants, in terms of wronskians and in terms of degenerate determinants of order N . Families of quasi-rational solutions v_N have been explicitly constructed until order $N = 13$ [105].

These solutions appear as deformations of the Peregrine breather P_N as the last one can be obtained when all parameters are equal to 0, and so we also obtain in particular three representations of the Peregrine breather P_N . The P_N breather can be expressed as a quotient of two polynomials of degree $N(N + 1)$ in the variables x and t ; the maximum of the modulus of the P_N breather is equal to $2N + 1$. The classification of patterns of the modulus of the deformations of the P_N breather in function of the parameters have been realized highlighting the phenomenon of rogue waves [105].

The second equation of the hierarchy is the modified Korteweg-de Vries (mKdV) equation which can be written in the form

$$u_t - 6u^2u_x + u_{xxx} = 0, \quad (3)$$

with $u_t = \partial_t u$, $u_x = \partial_x u$ and $u_{xxx} = \partial_x^3 u$.

The mKdV equation has many applications in various fields as to model supercontinuum generation in optical fibres [96] or to describe pulses consisting of a few optical cycles [97].

A lot of study have been made to construct solutions to the mKdV equation.

In particular, Hirota [98] constructed the exact soliton in 1972 of this equation. The inverse scattering method was used by Tanaka [99] to solve the mKdV equation. Rational solutions were constructed by using particular types of polynomials and the bilinear Hirota method in [59].

The third equation of the NLS hierarchy is the Lakshmanan-Porsezian-Daniel (LPD) which can be written as

$$iu_t + u_{xxxx} + 8|u|^2u_{xx} + 2u^2\bar{u}_{xx} + 6u_x^2\bar{u} + 4u|u_x|^2 + 6|u|^4u = 0, \quad (4)$$

where the subscripts mean the partial derivatives.

This model was introduced in the context of Heisenberg spin chain [100]. This equation with spatio-temporal dispersion as well as group velocity dispersion, is used to get and describe solitons which are some of the main focal points in the field of mathematical physics, optical fibers and nonlinear optics.

Many mathematical methods have been used to study this equation, in particular among others, the method of undetermined coefficients is applied in [101] to look for bright, dark and singular soliton solutions.

Here, we are interested in the equation of order 4 and we construct the first orders of solutions which to the best of our knowledge were not yet found.

In this paper, explicit solutions to the fourth equation of the NLS hierarchy have been completely expressed, depending on multiple real parameters. At order N , these solutions depend on $2N - 2$ real parameters. The evolution of these solutions in function of the different real parameters has been studied for the first three orders. These solutions are important because they allow to highlight the phenomenon of appearance of rogue waves precisely in function of these parameters as in the case of the simple nonlinear Schrödinger equation.

The general result will be published at a later date as it is not yet proven. We therefore limit ourselves to give the expression of a few solutions which, on the other hand, are correct and have been verified.

The formulation of these solutions is further enriched by utilizing techniques found in special function theory, particularly those discussed in the works such as [102]. These contributions to generalized forms of special functions and their applications in solving differential equations are significant in this context, providing insights into how these mathematical structures aid in constructing solutions to integrable systems.

2 Rational solutions of order 1 to the NLS4 equation

Theorem 2.1 *The function $v(x, t)$ defined by*

$$v(x, t) = 1 - \frac{4}{1 + (2x - 60t)^2} \quad (5)$$

is a solution to the (NLS4) equation (1)

$$u_t + u_{5x} + 10|u|^2 u_{3x} + 20u_{2x} u_x \bar{u} + 10(|u_x|^2 u)_x + 30|u|^4 u_x = 0.$$

Three views of the solutions with different areas of representation are given.

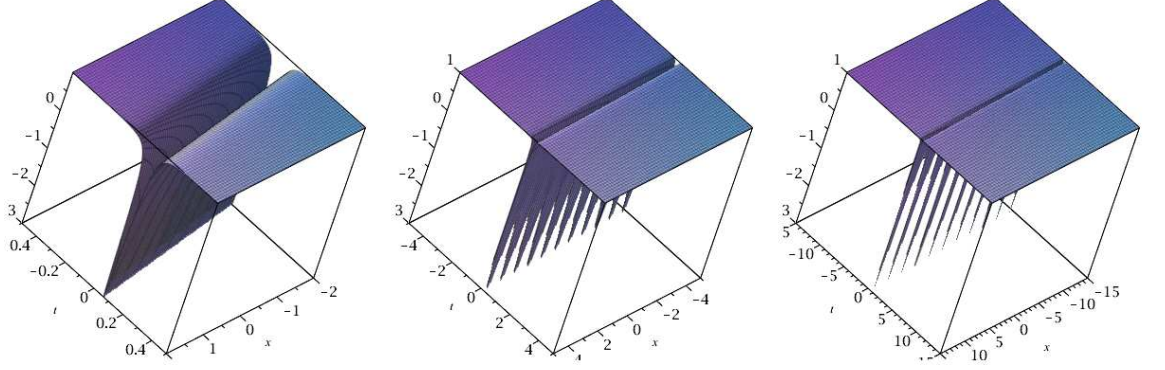


Figure 1. Solution of order 1 to (NLS4).

We get a smooth solution of the equation (1).

On the line $\{(x, t)/x - 6t = 0\}$, the solution u is constant equal to -3 .

When $x \rightarrow \infty$ or $t \rightarrow \infty$, the solution u is such that $u(x, t) \rightarrow 1$.

3 Rational solutions of order 2 of the NLS4 equation depending on 2 real parameters

Theorem 3.1 *The function $v(x, t)$ defined by*

$$v(x, t) = \frac{n(x, t)}{d(x, t)} \quad (6)$$

with

$$\begin{aligned} n(x, t) = & -(-64x^6 + 2304b_1x^5 + 11520tx^5 + 768ia_1x^4 - 864000t^2x^4 + 144x^4 - \\ & 768a_1^2x^4 - 345600b_1tx^4 - 34560b_1^2x^4 - 4992b_1x^3 + 276480b_1^3x^3 + 34560000t^3x^3 + \\ & 20736000b_1t^2x^3 - 92160ia_1tx^3 - 18432ia_1x^3b_1 + 4147200b_1^2tx^3 + 92160ta_1^2x^3 - \\ & 32640tx^3 + 18432b_1a_1^2x^3 - 777600000t^4x^2 - 3072a_1^4x^2 - 186624000b_1^2t^2x^2 - \\ & 165888b_1^2a_1^2x^2 + 58752b_1^2x^2 + 2160000t^2x^2 + 4147200ia_1t^2x^2 - 4147200t^2a_1^2x^2 - \\ & 24883200b_1^3tx^2 + 165888ia_1b_1^2x^2 - 622080000b_1t^3x^2 + 6144ia_1^3x^2 - 1244160b_1^4x^2 - \\ & 1152ia_1x^2 + 5760a_1^2x^2 + 725760b_1tx^2 + 1658880ia_1b_1tx^2 + 180x^2 - 1658880b_1ta_1^2x^2 + \\ & 3732480000b_1^2t^3x - 73728ia_1^3xb_1 - 368640ia_1^3tx + 36864b_1a_1^4x + 74649600b_1^4tx - \\ & 5184000b_1^2tx - 663552ia_1xb_1^3 - 161280ta_1^2x - 290304b_1^3x + 82944000t^3a_1^2x + \\ & 184320ta_1^4x + 2985984b_1^5x - 4608ia_1xb_1 - 49766400ia_1xb_1t^2 + 9331200000t^5x + \\ & 746496000b_1^3t^2x - 5616b_1x - 57024000t^3x + 9953280b_1^2ta_1^2x - 115200ia_1tx - \\ & 45360tx + 663552b_1^3a_1^2x - 50688b_1a_1^2x + 49766400b_1t^2a_1^2x - 82944000ia_1t^3x - \end{aligned}$$

$$\begin{aligned}
& 9953280 ia_1 x b_1^2 t + 9331200000 b_1 t^4 x - 30067200 b_1 t^2 x - 89579520 b_1^5 t - 27993600000 b_1^2 t^4 - \\
& 110592 b_1^2 a_1^4 - 995328 b_1^4 a_1^2 - 7464960000 b_1^3 t^3 + 221184 ia_1^3 b_1^2 + 531360000 t^4 - \\
& 1119744000 b_1^4 t^2 + 12288 ia_1^5 + 383616000 b_1 t^3 + 1872 a_1^2 + 12026880 b_1^3 t + \\
& 102643200 b_1^2 t^2 - 622080000 t^4 a_1^2 - 45 + 96768 b_1^2 a_1^2 - 345600 t^2 a_1^2 + 414720 b_1 t a_1^2 - \\
& 46656000000 t^6 + 5529600 ia_1^3 t^2 - 720 ia_1 - 2985984 b_1^6 - 4096 a_1^6 + 8448 a_1^4 + \\
& 518400 b_1^4 - 19906560 b_1^3 t a_1^2 - 497664000 b_1 t^3 a_1^2 - 1105920 b_1 t a_1^4 - 149299200 b_1^2 t^2 a_1^2 + \\
& 277200 t^2 + 995328 ia_1 b_1^4 + 1536 ia_1^3 + 19906560 ia_1 b_1^3 t + 149299200 ia_1 b_1^2 t^2 + \\
& 497664000 ia_1 b_1 t^3 + 69120 ia_1 b_1^2 + 4492800 ia_1 t^2 + 2211840 ia_1^3 b_1 t + 18000 b_1^2 - \\
& 55987200000 b_1 t^5 + 1244160 ia_1 b_1 t - 2764800 t^2 a_1^4 + 622080000 ia_1 t^4 + 191520 b_1 t) e^{2 ia_1}
\end{aligned}$$

and

$$\begin{aligned}
d(x, t) = & -(-64 x^6 + 2304 b_1 x^5 + 11520 t x^5 + 768 ia_1 x^4 - 864000 t^2 x^4 + 144 x^4 - \\
& 768 a_1^2 x^4 - 345600 b_1 t x^4 - 34560 b_1^2 x^4 - 4992 b_1 x^3 + 276480 b_1^3 x^3 + 34560000 t^3 x^3 + \\
& 20736000 b_1 t^2 x^3 - 92160 ia_1 t x^3 - 18432 ia_1 x^3 b_1 + 4147200 b_1^2 t x^3 + 92160 t a_1^2 x^3 - \\
& 32640 t x^3 + 18432 b_1 a_1^2 x^3 - 777600000 t^4 x^2 - 3072 a_1^4 x^2 - 186624000 b_1^2 t^2 x^2 - \\
& 165888 b_1^2 a_1^2 x^2 + 58752 b_1^2 x^2 + 2160000 t^2 x^2 + 4147200 ia_1 t^2 x^2 - 4147200 t^2 a_1^2 x^2 - \\
& 24883200 b_1^3 t x^2 + 165888 ia_1 b_1^2 x^2 - 622080000 b_1 t^3 x^2 + 6144 ia_1^3 x^2 - 1244160 b_1^4 x^2 - \\
& 1152 ia_1 x^2 + 5760 a_1^2 x^2 + 725760 b_1 t x^2 + 1658880 ia_1 b_1 t x^2 + 180 x^2 - 1658880 b_1 t a_1^2 x^2 + \\
& 3732480000 b_1^2 t^3 x - 73728 ia_1^3 x b_1 - 368640 ia_1^3 t x + 36864 b_1 a_1^4 x + 74649600 b_1^4 t x - \\
& 5184000 b_1^2 t x - 663552 ia_1 x b_1^3 - 161280 t a_1^2 x - 290304 b_1^3 x + 82944000 t^3 a_1^2 x + \\
& 184320 t a_1^4 x + 2985984 b_1^5 x - 4608 ia_1 x b_1 - 49766400 ia_1 x b_1 t^2 + 9331200000 t^5 x + \\
& 746496000 b_1^3 t^2 x - 5616 b_1 x - 57024000 t^3 x + 9953280 b_1^2 t a_1^2 x - 115200 ia_1 t x - \\
& 45360 t x + 663552 b_1^3 a_1^2 x - 50688 b_1 a_1^2 x + 49766400 b_1 t^2 a_1^2 x - 82944000 ia_1 t^3 x - \\
& 9953280 ia_1 x b_1^2 t + 9331200000 b_1 t^4 x - 30067200 b_1 t^2 x - 89579520 b_1^5 t - 27993600000 b_1^2 t^4 - \\
& 110592 b_1^2 a_1^4 - 995328 b_1^4 a_1^2 - 7464960000 b_1^3 t^3 + 221184 ia_1^3 b_1^2 + 531360000 t^4 - \\
& 1119744000 b_1^4 t^2 + 12288 ia_1^5 + 383616000 b_1 t^3 + 1872 a_1^2 + 12026880 b_1^3 t + \\
& 102643200 b_1^2 t^2 - 622080000 t^4 a_1^2 - 45 + 96768 b_1^2 a_1^2 - 345600 t^2 a_1^2 + 414720 b_1 t a_1^2 - \\
& 46656000000 t^6 + 5529600 ia_1^3 t^2 - 720 ia_1 - 2985984 b_1^6 - 4096 a_1^6 + 8448 a_1^4 + \\
& 518400 b_1^4 - 19906560 b_1^3 t a_1^2 - 497664000 b_1 t^3 a_1^2 - 1105920 b_1 t a_1^4 - 149299200 b_1^2 t^2 a_1^2 + \\
& 277200 t^2 + 995328 ia_1 b_1^4 + 1536 ia_1^3 + 19906560 ia_1 b_1^3 t + 149299200 ia_1 b_1^2 t^2 + \\
& 497664000 ia_1 b_1 t^3 + 69120 ia_1 b_1^2 + 4492800 ia_1 t^2 + 2211840 ia_1^3 b_1 t + 18000 b_1^2 - \\
& 55987200000 b_1 t^5 + 1244160 ia_1 b_1 t - 2764800 t^2 a_1^4 + 622080000 ia_1 t^4 + 191520 b_1 t) e^{2 ia_1}
\end{aligned}$$

is a solution to the (NLS₄) equation (1).

Contrary to the previous section, in all the following, we represent the modules of the solutions in the plane $(x; t)$ of the coordinates.

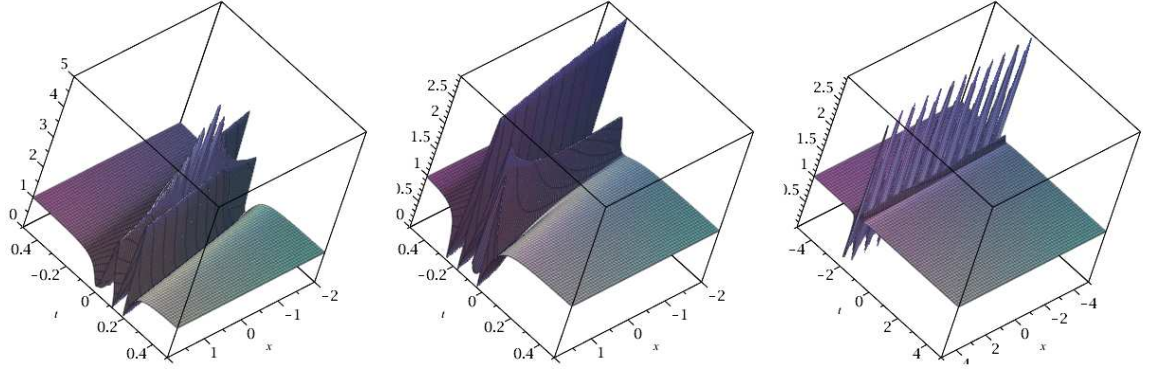


Figure 2. Solution of order 2 to the equation (1); to the left $a_1 = 0, b_1 = 0$; in the center $a_1 = 0, b_1 = 1$; to the right $a_1 = 0, b_1 = 10$.

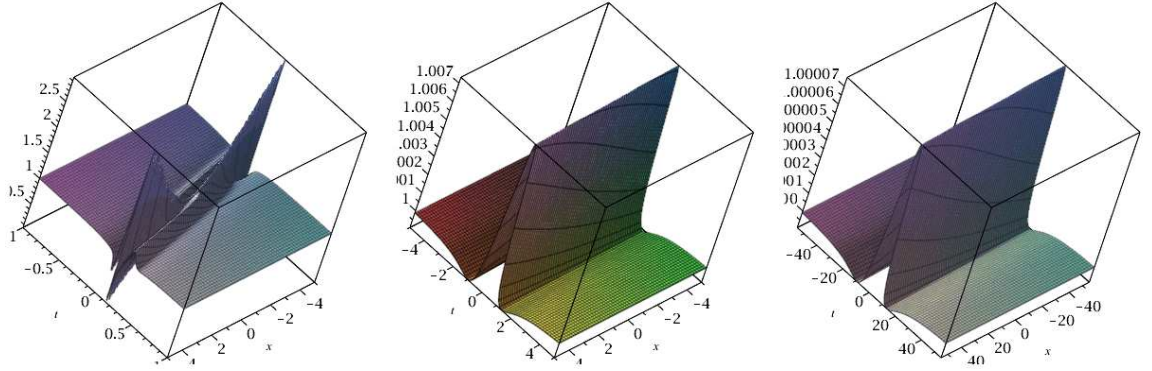


Figure 3. Solution of order 2 to the equation (1); to the left $a_1 = 1, b_1 = 0$; in the center $a_1 = 10, b_1 = 1$; to the right $a_1 = 100, b_1 = 100$.

When $x \rightarrow \infty$ or $t \rightarrow \infty$, the solution u is such that $u(x, t) \rightarrow e^{2ia_1}$ or 1 if all parameters are equal to 0.

4 Rational solutions of order 3 of the NLS4 equation

The solution depending on 4 real parameters being too long, we give it in the appendix. We present here the solution without parameters.

Theorem 4.1 *The function $v(x, t)$ defined by*

$$v(x, t) = \frac{n(x, t)}{d(x, t)} \quad (7)$$

with

$$n(x, t) = 4096 x^{12} - 1474560 t x^{11} - 18432 x^{10} + 243302400 t^2 x^{10} + 10444800 t x^9 -$$

$$\begin{aligned}
& 24330240000 t^3 x^9 + 164229120000 t^4 x^8 - 57600 x^8 - 2073600000 t^2 x^8 - 78829977600000 t^5 x^7 + \\
& 7925760 t x^7 + 218972160000 t^3 x^7 - 172800 x^6 - 14282956800000 t^4 x^6 + 671846400 t^2 x^6 + \\
& 2759049216000000 t^6 x^6 - 183638016000 t^3 x^5 + 614515507200000 t^5 x^5 - 70946979840000000 t^7 x^5 + \\
& 18938880 t x^5 - 17871114240000000 t^6 x^4 - 14000256000 t^2 x^4 + 226800 x^4 + 14251852800000 t^4 x^4 + \\
& 1330255872000000000 t^8 x^4 + 349360128000000000 t^7 x^3 - 566590464000000 t^5 x^3 - \\
& 17736744960000000000 t^9 x^3 + 1248998400000 t^3 x^3 - 137808000 t x^3 + 17448566400 t^2 x^2 - \\
& 39822451200000 t^4 x^2 - 441403084800000000 t^8 x^2 + 113400 x^2 + 12583683072000000 t^6 x^2 + \\
& 159630704640000000000 t^{10} x^2 - 1070915904000 t^3 x - 81972000 t x + 477570816000000 t^5 x + \\
& 32651735040000000000 t^9 x - 149015531520000000 t^7 x - 87071293440000000000 t^{11} x - \\
& 4419424800 t^2 - 107629793280000000000 t^{10} + 2176782336000000000000 t^{12} + 736175692800000000 t^8 + \\
& 41194539360000 t^4 - 14175 - 1203911424000000 t^6
\end{aligned}$$

and

$$\begin{aligned}
d(x, t) = & 4096 x^{12} - 1474560 t x^{11} + 6144 x^{10} + 243302400 t^2 x^{10} + 3072000 t x^9 - \\
& 24330240000 t^3 x^9 + 164229120000 t^4 x^8 + 34560 x^8 - 1078272000 t^2 x^8 - 78829977600000 t^5 x^7 - \\
& 14192640 t x^7 + 139345920000 t^3 x^7 + 149760 x^6 - 10102579200000 t^4 x^6 + 2994278400 t^2 x^6 + \\
& 2759049216000000 t^6 x^6 - 322983936000 t^3 x^5 + 464021913600000 t^5 x^5 - 70946979840000000 t^7 x^5 - \\
& 30274560 t x^5 - 14108774400000000 t^6 x^4 + 2298240000 t^2 x^4 + 54000 x^4 + 19477324800000 t^4 x^4 + \\
& 1330255872000000000 t^8 x^4 + 2848628736000000000 t^7 x^3 - 692001792000000 t^5 x^3 - \\
& 177367449600000000000 t^9 x^3 - 438082560000 t^3 x^3 + 37756800 t x^3 - 4054665600 t^2 x^2 + \\
& 33371481600000 t^4 x^2 - 3688436736000000000 t^8 x^2 + 48600 x^2 + 14464852992000000 t^6 x^2 + \\
& 159630704640000000000 t^{10} x^2 + 825598656000 t^3 x - 20023200 t x - 966898944000000 t^5 x + \\
& 278144409600000000000 t^9 x - 165139845120000000 t^7 x - 870712934400000000000 t^{11} x + \\
& 5359413600 t^2 - 93117911040000000000 t^{10} + 2176782336000000000000 t^{12} + 796641868800000000 t^8 - \\
& 948270240000 t^4 + 2025 + 95657863680000000 t^6
\end{aligned}$$

is a solution to the (NLS4) equation (1).

In the following, we give the patterns of the modules of the solutions according to different values of the parameters.

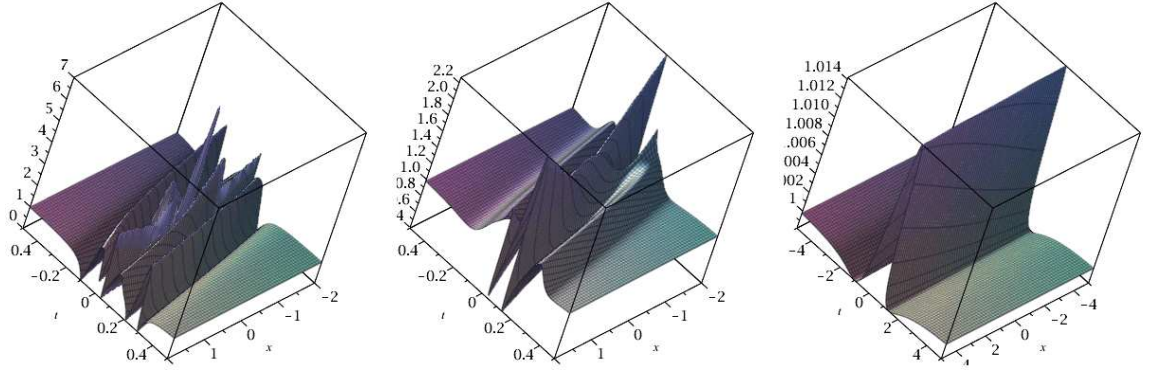


Figure 4. Solution of order 3 to (1); to the left $a_1 = 0, b_1 = 0, a_2 = 0, b_2 = 0$; in the center $a_1 = 1, b_1 = 0, a_2 = 0, b_2 = 0$; to the right $a_1 = 10, b_1 = 0, a_2 = 0, b_2 = 0$.

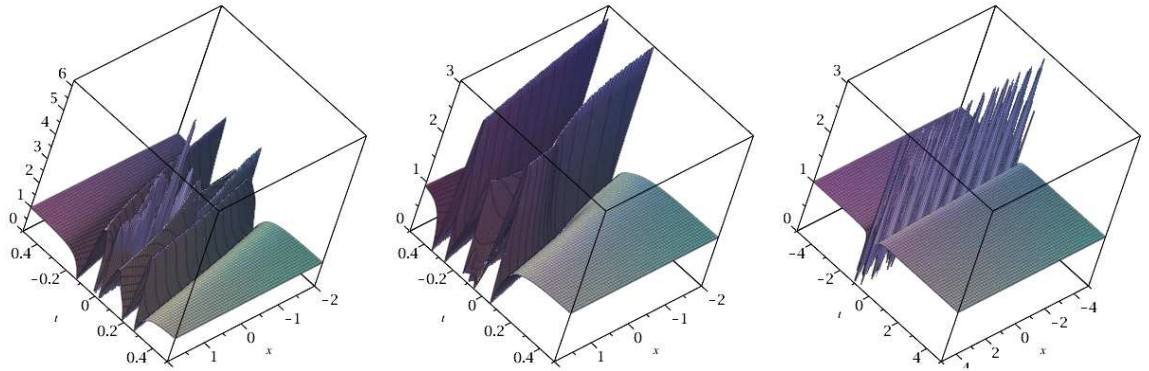


Figure 5. Solution of order 3 to (1); to the left $a_1 = 0, b_1 = 0, 1, a_2 = 0, b_2 = 0$; in the center $a_1 = 0, b_1 = 1, a_2 = 0, b_2 = 0$; to the right $a_1 = 0, b_1 = 5, a_2 = 0, b_2 = 0$.

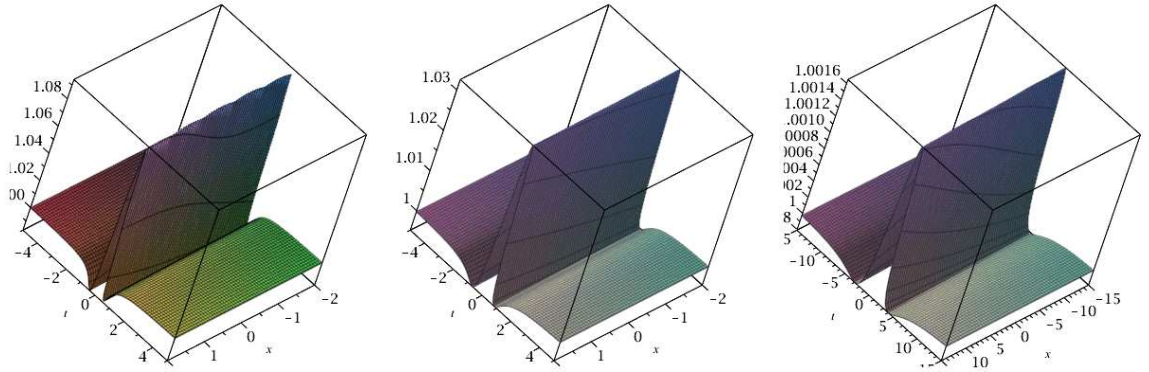


Figure 6. Solution of order 3 to (1); to the left $a_1 = 0, b_1 = 0, a_2 = 0, 5, b_2 = 0$; in the center $a_1 = 0, b_1 = 0, a_2 = 1, b_2 = 0$; to the right $a_1 = 0, b_1 = 5, a_2 = 10, b_2 = 0$.

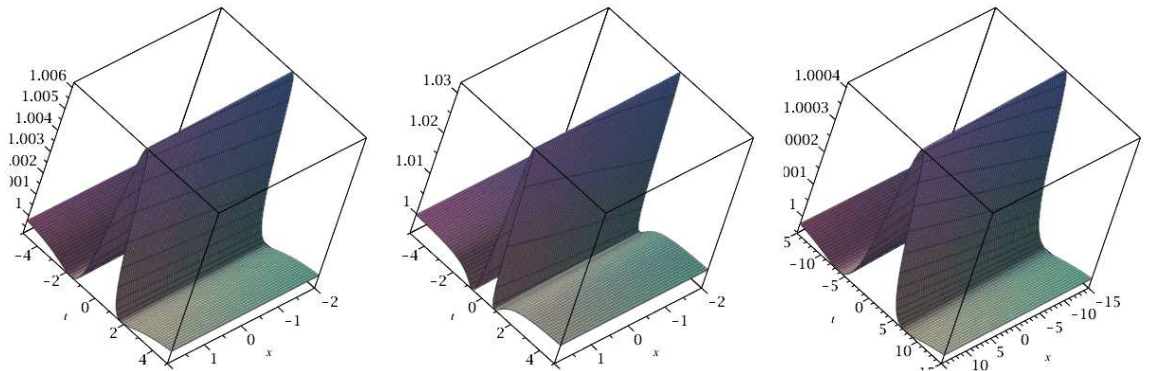


Figure 7. Solution of order 3 to (1); to the left $a_1 = 0, b_1 = 0, a_2 = 0, b_2 = 0, 5$; in the center $a_1 = 0, b_1 = 0, a_2 = 0, b_2 = 1$; to the right $a_1 = 0, b_1 = 5, a_2 = 0, b_2 = 10$.

In the case where all parameters are equal to 0, the solution u can be written in the form $u(x, t) = 1 - \frac{N(x, t)}{D(x, t)}$ with

$$N(x, t) = 16200 + 11578982400 t^2 + 24(-675 - 69120000 t^2)(2x - 60t)^2 + 24(1209600 t + 22118400000 t^3)(2x - 60t) + 360(2x - 60t)^8 + 5040(2x - 60t)^6 + 276480 t(2x - 60t)^5 + 24(-450 + 34560000 t^2)(2x - 60t)^4 + 19353600 t(2x - 60t)^3 + 24(2x - 60t)^{10}$$

and

$$D(x, t) = 2024 + 4802457600 t^2 + 21233664000000 t^4 + (12144 - 91238400 t^2)(2x - 60t)^2 + 9600 t(2x - 60t)^9 - 46080 t(2x - 60t)^7 + (-8553600 t + 345047040000 t^3)(2x - 60t) + 120(2x - 60t)^8 + (2320 + 13824000 t^2)(2x - 60t)^6 - 103680 t(2x - 60t)^5 + (3360 - 13824000 t^2)(2x - 60t)^4 + (5529600 t - 44236800000 t^3)(2x - 60t)^3 + (1 + (2x - 60t)^2)^6$$

When $x \rightarrow \infty$ or $t \rightarrow \infty$, the solution u is such that $u(x, t) \rightarrow e^{2i(a_1 - 3a_2 + 10b_2)}$ or 1 if all parameters are equal to 0.

Contrary to other equations belonging to this NLS hierarchy, as the NLS equation [105], the mKdV equation [59], or the Lakshmanan Porsezian Daniel equation [62], there are no specific configurations which appear in function of the different values of the parameters.

5 Conclusion

Rational solutions to the NLS4 equation have been given for the first orders. For the first order, we get moreover smooth solutions. We observe the appearance of lines $\{(x; t)/2x - 60t = c\}$ in these solutions.

When $x \rightarrow \infty$ or $t \rightarrow \infty$, all these solutions u are such that $u(x, t) \rightarrow 1$.

In all these N-order solutions we get quotient of a polynomial of degree $N(N + 1)$ in x and t for the numerator by a polynomial of degree $N(N + 1)$ in x and t for the denominator.

In particular, this explains the asymptotic behavior of these rational solutions.

References

- [1] V. E. Zakharov, Stability of periodic waves of finite amplitude on a surface of a deep fluid, *J. Appl. Tech. Phys*, V. 9, 86-94, (1968)
- [2] N. Akhmediev, V. Eleonski, N. Kulagin, Generation of periodic trains of picosecond pulses in an optical fiber : exact solutions, *Sov. Phys. J.E.T.P.*, V. 62, 894-899, (1985)
- [3] N. Akhmediev, A. Ankiewicz, J.M. Soto-Crespo, Rogues waves and rational solutions of nonlinear Schrödinger equation, *Phys. Rev. E*, V. 80, 026601-1-9, 2009
- [4] N. Akhmediev, A. Ankiewicz, First-order exact solutions of the nonlinear Schrödinger equation in the normal-dispersion regime, *Phys. Rev. A*, V. 47, N. 4, 3213-3221, 2009
- [5] A. Ankiewicz, D. J. Kedziora, N. Akhmediev, Rogue wave triplets, *Phys Lett. A*, V. 375, 2782-2785, 2011
- [6] P. Dubard, P. Gaillard, C. Klein, V. B. Matveev, On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation, *Eur. Phys. J. Spe. Top.*, V. 185, 247-258, 2010
- [7] V. Eleonskii, I. Krichever, N. Kulagin, Rational multisoliton solutions of nonlinear Schrödinger equation, *Dokl. Math. Phy*, V. 287, 606-610, 1986
- [8] P. Gaillard, V.B. Matveev, Wronskian addition formula and its applications, *Max-Planck-Institut für Mathematik, MPI 02-31*, V. 161, 2002
- [9] P. Gaillard, A new family of deformations of Darboux-Pöschl-Teller potentials, *Lett. Math. Phys.*, V. 68, 77-90, 2004
- [10] P. Gaillard, V.B. Matveev, New formulas for the eigenfunctions of the two-particle Calogero-Moser system, *Lett. Math. Phys.*, V. 89, 1-12, 2009
- [11] P. Gaillard, V.B. Matveev, Wronskian and Casorati determinant representations for Darboux-Pöschl-Teller potentials and their difference extensions, *J. Phys A : Math. Theor.*, V. 42, 404409-1-16, 2009
- [12] P. Dubard, P. Gaillard, C. Klein, V. B. Matveev, On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation, *Eur. Phys. J. Spe. Top.*, V. 185, 247-258, 2010
- [13] P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, *J. Phys. A : Meth. Theor.*, V. 44, 435204-1-15, 2011
- [14] P. Gaillard, Wronskian representation of solutions of the NLS equation and higher Peregrine breathers, *J. Math. Sciences : Adv. Appl.*, V. 13, N. 2, 71-153, 2012

- [15] P. Gaillard, Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves, *J. Math. Phys.*, V. 54, 013504-1-32, 2013
- [16] P. Gaillard, Wronskian representation of solutions of NLS equation and seventh order rogue waves, *J. Mod. Phys.*, V. 4, N. 4, 246-266, 2013
- [17] P. Gaillard, V.B. Matveev, Wronskian addition formula and Darboux-Pöschl-Teller potentials, *J. Math.*, V. 2013, ID 645752, 1-10, 2013
- [18] P. Gaillard, Two-parameters determinant representation of seventh order rogue waves solutions of the NLS equation, *J. Theor. Appl. Phys.*, V. 7, N. 45, 1-6, 2013
- [19] P. Gaillard, Deformations of third order Peregrine breather solutions of the NLS equation with four parameters, *Phys. Rev. E*, V. 88, 042903-1-9, 2013
- [20] P. Gaillard, Two parameters deformations of ninth Peregrine breather solution of the NLS equation and multi rogue waves, *J. Math.*, V. 2013, 1-111, 2013
- [21] P. Gaillard, Six-parameters deformations of fourth order Peregrine breather solutions of the NLS equation, *J. Math. Phys.*, V. 54, 073519-1-22, 2013
- [22] P. Gaillard, The fifth order Peregrine breather and its eight-parameters deformations solutions of the NLS equation, *Commun. Theor. Phys.*, V. 61, 365-369, 2014
- [23] P. Gaillard, Ten parameters deformations of the sixth order Peregrine breather solutions of the NLS equation, *Phys. Scripta*, V. 89, 015004-1-7, 2014
- [24] P. Gaillard, Higher order Peregrine breathers, their deformations and multi-rogue waves, *J. Of Phys. : Conf. Ser.*, V. 482, 012016-1-7, 2014
- [25] P. Gaillard, M. Gastineau, Eighteen parameter deformations of the Peregrine breather of order ten solutions of the NLS equation, *Int. J. Mod. Phys. C*, V. 26, N. 2, 1550016-1-14, 2014
- [26] P. Gaillard, Two parameters wronskian representation of solutions of nonlinear Schrödinger equation, eight Peregrine breather and multi-rogue waves, *J. Math. Phys.*, V. 5, 093506-1-12, 2014
- [27] P. Gaillard, Hierarchy of solutions to the NLS equation and multi-rogue waves, *J. Phys. : Conf. Ser.*, V. 574, 012031-1-5, 2015
- [28] P. Gaillard, Tenth Peregrine breather solution of the NLS, *Ann. Phys.*, V. 355, 293-298, 2015

- [29] P. Gaillard, M. Gastineau, The Peregrine breather of order nine and its deformations with sixteen parameters solutions of the NLS equation *Phys. Lett. A*, V. 379, 1309-1313, 2015
- [30] P. Gaillard, Other $2N-2$ parameters solutions to the NLS equation and $2N+1$ highest amplitude of the modulus of the N -th order AP breather, *J. Phys. A: Math. Theor.*, V. 48, 145203-1-23, 2015
- [31] P. Gaillard, Multi-parametric deformations of the Peregrine breather of order N solutions to the NLS equation and multi-rogue waves, *Adv. Res.*, V. 4, N. 5, 346-364, 2015
- [32] P. Gaillard, Higher order Peregrine breathers solutions to the NLS equation, *Jour. Phys. : Conf. Ser.*, V. 633, 012106-1-6, 2016
- [33] P. Gaillard, M. Gastineau Patterns of deformations of Peregrine breather of order 3 and 4, solutions to the NLS equation with multi-parameters, *Journal of Theoretical and Applied Physics*, V. 10,1-7, 2016
- [34] P. Gaillard, M. Gastineau Twenty parameters families of solutions to the NLS equation and the eleventh Peregrine breather, *Commun. Theor. Phys*, V. 65, N. 2, 136-144, 2016
- [35] P. Gaillard, Rational solutions to the KPI equation and multi rogue waves, *Annals Of Physics*, V. 367, 1-5, 2016
- [36] P. Gaillard, M. Gastineau Twenty two parameters deformations of the twelfth Peregrine breather solutions to the NLS equation, *Adv. Res.*, V. 10, 83-89, 2016
- [37] P. Gaillard, Towards a classification of the quasi rational solutions to the NLS equation, *Theor. And Math. Phys.*, V. 189, N. 1, 1440-1449, 2016
- [38] P. Gaillard, Fredholm and Wronskian representations of solutions to the KPI equation and multi-rogue waves, *Jour. of Math. Phys.*, V. 57, 063505-1-13, doi: 10.1063/1.4953383, 2016
- [39] P. Gaillard, M. Gastineau Families of deformations of the thirteenth Peregrine breather solutions to the NLS equation depending on twenty four parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. 21, N. 3, 130-139, 2017
- [40] P. Gaillard, From Fredholm and Wronskian representations to rational solutions to the KPI equation depending on $2N - 2$ parameters, *Int. Jour. of Appl. Sci. And Math.*, V. 4, N. 3, 60-70, 2017
- [41] P. Gaillard, Families of rational solutions of order 5 to the KPI equation depending on 8 parameters, *New Hor. in Math. Phys.*, V. 1, N. 1, 26-31, 2017

- [42] P. Gaillard, 6-th order rational solutions to the KPI equation depending on 10 parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. 21, N. 2, 92-98, 2017
- [43] P. Gaillard, N -Order rational solutions to the Johnson equation depending on $2N - 2$ parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. 4, N. 9, 19-37, 2017
- [44] P. Gaillard, Families of rational solutions to the KPI equation of order 7 depending on 12 parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. 4, N. 11, 24-30, 2017
- [45] P. Gaillard, Rational solutions to the Johnson equation and rogue waves, *Int. Jour. of Inn. In Sci. and Math.*, V. 6, N. 1, 14-19, 2018
- [46] P. Gaillard, Multiparametric families of solutions of the KPI equation, the structure of their rational representations and multi-rogue waves, *Theo. And Mat. Phys.*, V. 196, N. 2, 1174-1199, 2018
- [47] P. Gaillard, The Johnson equation, Fredholm and wronskian representations of solutions and the case of order three, *Adv. In Math. Phys.*, V. 2018, 1-18, 2018
- [48] P. Gaillard, Families of solutions of order 5 to the Johnson equation depending on 8 parameters, *NHIMP*, V. 2, N. 4, 53-61, 2018
- [49] P. Gaillard, Multiparametric families of solutions to the Johnson equation, *J. Phys. : Conf. Series*, V. 1141, 012102-1-10, 2018
- [50] P. Gaillard, Rational solutions to the Boussinesq equation, *Fund. Jour. Of Math. And Appl.*, V. 2, 109-112, 2019
- [51] P. Gaillard, Differential relations for the solutions to the NLS equation and their different representations, *Comm. In Adv. Math. Sci.*, V. 2, N. 4, 1-4, 2019
- [52] P. Gaillard, Multi-parametric families of solutions of order N to the Boussinesq and KP equations and the degenerate rational case, *UJMA*, V. 3, N. 2, 44-52, 2020
- [53] P. Gaillard, The mKdV equation and multi-parameters rational solutions, *Wave Motion*, V. 100, 102667-1-9, 2021
- [54] P. Gaillard, Degenerate Riemann theta functions, Fredholm and wronskian representations of the solutions to the KdV equation and the degenerate rational case, *Jour. Of Geom. And Phys.*, V. 161, 104059-1-12, 2021
- [55] P. Gaillard, $2N$ parameter solutions to the Burgers equation, *Jour. Of Appl. Nonlin. Dyn.*, V. 1, N. 1, 1-6, 2021

- [56] P. Gaillard, Multi-Parametric rational Solutions to the KdV equation, *Asi. Jour. Of Res. and Rev. In Phys.*, V. 3, N. 3, 14-21, 2021
- [57] P. Gaillard, Multiparametric solutions to the Gardner equation and the degenerate rational case, *Jour. Of Appl. Ana. And Appl.*, V. 11, N.4, 2102-2113, 2021
- [58] P. Gaillard, Other families of rational solutions to the KPI equation, *Asi. Res. Jour Of Math.*, V. 17, N. 6, 27-34, 2021
- [59] P. Gaillard, Rational solutions to the mKdV equation associated to particular polynomials, *Wave Motion*, V. 107, 102824-1-11, 2021
- [60] P. Gaillard, Multiparametric rational solutions of order N to the KPI equation and the explicit case of order 3, *Arch. Of cur. Res. Int.*, V. 21, N . 6, 58-71, 2021
- [61] P. Gaillard, Rational solutions to the KPI equation from particular polynomials, *Wave Motion*, V. 108, 102828-1-9, 2022
- [62] P. Gaillard, Rogue waves of the Lakshmanan Porsezian Daniel equation depending on multi-parameters, *As. Jour. Of Adv. Res. And Rep.*, V 16, N. 3, 32-40, 2022
- [63] P. Gaillard, Quasi-rational and rational solutions to the defocusing NLS equation, *Jour. Of Math. Sci. And Mod. Res. And Rep.*, V 5, N. 1, 24-34, 2022
- [64] P. Gaillard, Rational solutions to the KdV equation in terms of particular polynomials, *Jour. Of Nonlin. Mod. and Ana.*, V. 4, N. 4, 615-626, 2022
- [65] P. Gaillard, Rational solutions to the Painlevé II equation from particular polynomials, *Int. J. Of Appl. Nonlin. Sci.*, V. 3, N. 3, 189-196, 2022
- [66] P. Gaillard, Multi-lump solutions to the KPI equation with a zero degree of derivation, *Jour. Of Theor. And Appl. Phys.*, V. 16, N. 4, 1-8, 2022
- [67] P. Gaillard, Families of solutions to the KPI equation given by an extended Darboux transformation, *Part. Diff. Equ. And Appl.*, V. 3, N. 80, 1-14, 2022
- [68] P. Gaillard, Rogue waves of the Hirota equation in terms of quasi-rational solutions depending on multi-parameters, *WSEAS Trans. On Math.*, DOI: 10.37394/23206.2023.22.24, V. 22, 190-203, 2023
- [69] P. Gaillard, Different representations of the solutions to the cylindrical nonlinear Schrödinger equation, *Dis. Nonlin. And Compl.*, V. 12, N. 3, 539-553, 2023
- [70] P. Gaillard, Rational solutions to the KdV equation from Riemann theta functions, *Jour. Of Theor. And Appl. Phys.*, V. 17, N. 3, 1-5, 2023

- [71] P. Gaillard, On Hadamard's coefficients for two dimensional Schrödinger operators, *Int. Jour. Of Theor. Phys.*, V. 62, N. 205, 1-16, 2023
- [72] P. Gaillard, Rational solutions to the Burgers' equation from particular polynomials, *WSEAS Trans. On Math*, DOI: 10.37394/23206.2023.22.94, 867-873, 2023
- [73] P. Gaillard, Rational solutions to the Gardner equation from particular polynomials, *WSEAS Equ.*, DOI: 10.37394/232021.2023.3.2, V. 3, 1-24, 2023
- [74] P. Gaillard, Rational solutions to the KPI equation as multi-lumps with a one degree of summation, *Int. J. Appl. Comput. Math*, doi.org/10.1007/s40819-024-01694-9, 1-14, 2024
- [75] P. Gaillard, Rational solutions to the sixth equation of the NLS hierarchy, *ISRG Journal Of Multi. Stud.*, V. 2, N. 6, 1-14, 2024
- [76] P. Gaillard, Wronskian representation of the solutions to the KdV Equation, *Jour. Of App. Phys. Sci. Int.*, V. 16, N. 2, 1-11, 2024
- [77] P. Gaillard, Quasi-rational solutions to the seventh equation of the NLS hierarchy, *Asi. Jour. Of Adv. Res. And Rep.*, V. 18, N. 7, 256-265, 2024
- [78] P. Gaillard, Solutions of the functional difference Toda equation from centered Darboux transformation, *Aequat. Math.*, 1-21, <https://doi.org/10.1007/s00010-024-01097-7>, 2024
- [79] P. Gaillard, From Algebro geometric solutions of the Toda equation to Sato formulas, *App. Math.*, V1, 1-11, 2024
- [80] S. Tanaka, Modified Korteweg. de Vries equation and scattering theory, *Proc. Japan Acad.*, V. 48, 466-469, 1972
- [81] M. Wadati, The exact solution of the modified Korteweg-de Vries equation, *J. Phys. Soc. Jpn.*, V. 32, 1681-1681, 1972
- [82] Y. Ono, Algebraic soliton of the modified Korteweg-de Vries equation, *Jour. Phys. Soc. Japan*, V. 41, N. 5, 1817-1818, 1976
- [83] A. Chowdury, A. Ankiewicz, N. Akhmediev, Periodic and rational solutions of modified Korteweg-de Vries equation, *Eur. Phys. J. D*, V. 70, N. 104, 1-7, 2016
- [84] A. Chowdury, A. Ankiewicz, N. Akhmediev, Periodic and rational solutions of mKdV equation, *Eur. Phys. J. D*, V. 70, N. 104, 1-7, 2016
- [85] M. Lakshmanan, K. Porsezian, M. Daniel, Effect of discreteness on the continuum limit of the Heisenberg spin chain, *Phys. Lett. A*, V. 133, N. 9, 483488, 1998

- [86] M. Lakshmanan, M. Daniel, K. Porsezian, On the integrability aspects of the one-dimensional classical continuum isotropic biquadratic Heisenberg spin chain, *J. Math. Phys.*, V. 33, N. 5, 1807-1816, 1992
- [87] M. Daniel, K. Porsezian, M. Lakshmanan, On the integrable models of the higher order water wave equation, *Phys. Lett. A*, V. 174, N. 3, 237-240, 1993
- [88] G. Akram, M. Sadaf, M. Dawood, D. Baleanu, Optical solitons for Lakshmanan Porsezian Daniel equation with Kerr law non-linearity using improved tan expansion technique, *Res. In Phys.*, V. 29, 104758-1-13, 2021
- [89] A.A. Al Qarni et al., Optical solitons for Lakshmanan Porsezian Daniel model by Riccati equation approach, *Optik*, V. 182, 922-929, 2019
- [90] R.T. Alqahtani, M.M. Babatin, A. Biswas, Bright optical solitons for Lakshmanan Porsezian Daniel model by semi-inverse variational principle, *Optik*, V. 154, 109-114, 2018
- [91] S. Arshed et al., Optical solitons in birefringent fibers for Lakshmanan Porsezian Daniel model using $\exp(-i\phi)$ -expansion method, *Optik*, V. 172, 651-656, 2018
- [92] V. E. Zakharov, Stability of periodic waves of finite amplitude on a surface of a deep fluid, *J. Appl. Tech. Phys*, V. 9, 86-94, 1968
- [93] V. E. Zakharov, A.B. Shabat Exact theory of two dimensional self focusing and one dimensional self modulation of waves in nonlinear media, *Sov. Phys. JETP*, V. 34, 62-69, 1972
- [94] A.R. Its, V.P. Kotlyarov, Explicit expressions for the solutions of nonlinear Schrödinger equation, *Dokl. Akad. Nauk. SSSR, S. A*, V. 965, N. 11, 1976
- [95] A.R. Its, A.V. Rybin, M.A. Salle, Exact integration of nonlinear Schrödinger equation, *Teore. i Mat. Fiz.*, V. 74, N. 1, 29-45, 1988
- [96] H. Leblond, P. Grellu, D. Mihalache, Models for supercontinuum generation beyond the slowly-varying-envelope approximation, *Phys. Rev. A*, V. 90, 053816-1-9, 2014
- [97] H. Leblond, D. Mihalache, Few-optical-cycle solitons: Modified Korteweg-de Vries sine-Gordon equation versus other non-slowly-varying-envelope-approximation models, *Phys. Rev. A*, V. 79, 063835-1-7, 2009
- [98] R. Hirota, Exact envelope-soliton solutions of a nonlinear wave equation, *Jour. Of Math. Phys.*, V. 14, N. 7, 805-809, 1973
- [99] S. Tanaka, Modified Korteweg de Vries equation and scattering theory, *Proc. Japan Acad.*, V. 48, 466-469, 1972

- [100] M. Lakshmanan, K. Porsezian, M. Daniel, Effect of discreteness on the continuum limit of the Heisenberg spin chain, *Phys. Lett. A*, V. 133, N. 9, 483488, 1998
- [101] J. Vega-Guzman, R.T. Alqahtani, Q. Zhou, M.F. Mahmood, S.P. Moshokoa, M.Z. Ullah, A. Biswas, M. Belic, Optical solitons for Lakshmanan Porsezian Daniel model with spatiotemporal dispersion using the method of undetermined coefficients, *Optik*, V. 144, 115-123, 2017
- [102] M. Zayed, S.A Wani, W. Ramirez, F.F. Gandara, Properties and applications of Bell polynomials of two variables, *J. Math. Computer Sci.*, V. 35, 291-303, 2024
- [103] X. Feng, The global solutions for a fourth order NLS equation, *Act. Math. Sci.*, V. 15, N. 2, 196-206, 1995
- [104] H.Q. Zhang, F. Chen, Rogue waves for the fourth-order nonlinear Schrödinger equation on the periodic background, *Chaos*, V. 31, 023129, 2021
- [105] P. Gaillard, Towards a classification of the quasi rational solutions to the NLS equation, *Theor. And Math. Phys.*, V. **189**, N. **1**, 1440-1449, 2016

Appendix

Solution of order 3 to the (NLS4) equation depending on 4 real parameters :
The function $v(x, t)$ defined by

$$v(x, t) = \left(1 - 24 \frac{n(x, t)}{d(x, t)} \right) e^{i(2a_1 - 6a_2 + 20b_2)} \quad (8)$$

with

$$\begin{aligned} n(x, t) = & 675 + 88473600 t^2 + i(-120 (4 a_1 - 24 a_2 + 120 b_2)^6 (80 a_2 - 1248 b_2) + \\ & 900 (4 a_1 - 24 a_2 + 120 b_2)^4 (464 a_2 - 4000 b_2) + 90 (4 a_1 - 24 a_2 + 120 b_2)^5 (-107 + \\ & 28 (8 b_1 + 80 t)^2 + 12 (16 a_2 - 160 b_2)^2) - 21600 (8 b_1 + 80 t)^2 (16 a_2 - 160 b_2) + \\ & 5400 (4 a_1 - 24 a_2 + 120 b_2)^2 (176 a_2 - 2464 b_2 + 4 (8 b_1 + 80 t)^2 (16 a_2 - 160 b_2) + \\ & 4 (16 a_2 - 160 b_2)^3) - 225 (4 a_1 - 24 a_2 + 120 b_2)^3 (11 + 80 (8 b_1 + 80 t)^2 + 80 (16 a_2 - \\ & 160 b_2)^2 - 4096 (8 b_1 + 80 t) t + 8192 (16 a_2 - 160 b_2) b_2) - 675 (4 a_1 - 24 a_2 + \\ & 120 b_2) (-7 + 56 (8 b_1 + 80 t)^2 + 88 (16 a_2 - 160 b_2)^2 + 4096 (8 b_1 + 80 t) t - 131072 t^2 - \\ & 524288 b_2^2) + (4 a_1 - 24 a_2 + 120 b_2) (2 x - 12 b_1 - 60 t)^{10} + (-60 a_1 + 840 a_2 - \\ & 6600 b_2 + 5 (4 a_1 - 24 a_2 + 120 b_2)^3) (2 x - 12 b_1 - 60 t)^8 + (-600 a_1 - 240 a_2 + \\ & 58800 b_2 - 140 (4 a_1 - 24 a_2 + 120 b_2)^3 + 10 (4 a_1 - 24 a_2 + 120 b_2)^5 + 240 (4 a_1 - \\ & 24 a_2 + 120 b_2)^2 (16 a_2 - 160 b_2)) (2 x - 12 b_1 - 60 t)^6 + (-240 (4 a_1 - 24 a_2 + \\ & 120 b_2)^3 (8 b_1 + 80 t) - 1440 (8 b_1 + 80 t) (16 a_2 - 160 b_2) + 720 (4 a_1 - 24 a_2 + \\ & 120 b_2) (8 b_1 + 176 t)) (2 x - 12 b_1 - 60 t)^5 + (-450 (4 a_1 - 24 a_2 + 120 b_2)^3 - 210 (4 a_1 - \\ & 24 a_2 + 120 b_2)^5 + 10 (4 a_1 - 24 a_2 + 120 b_2)^7 + 300 (4 a_1 - 24 a_2 + 120 b_2)^4 (16 a_2 - \\ & 160 b_2) + 450 (4 a_1 - 24 a_2 + 120 b_2) (-3 + 12 (8 b_1 + 80 t)^2 - 4 (16 a_2 - 160 b_2)^2) - \\ & 14400 a_2 + 259200 b_2 + 1800 (4 a_1 - 24 a_2 + 120 b_2)^2 (16 a_2 - 224 b_2)) (2 x - 12 b_1 - \\ & 60 t)^4 + (-480 (4 a_1 - 24 a_2 + 120 b_2)^5 (8 b_1 + 80 t) + 14400 (4 a_1 - 24 a_2 + 120 b_2)^2 (8 b_1 + \end{aligned}$$

$$\begin{aligned}
& 80t)(16a_2 - 160b_2) + 7200(4a_1 - 24a_2 + 120b_2)(8b_1 + 48t) - 2400(4a_1 - 24a_2 + 120b_2)^3(16b_1 + 128t) - 14400(8b_1 + 80t)(16a_2 - 160b_2) + 460800(16a_2 - 160b_2)t + 921600(8b_1 + 80t)b_2(2x - 12b_1 - 60t)^3 + 5529600(8b_1 + 80t)^2b_2 + (1710(4a_1 - 24a_2 + 120b_2)^5 - 60(4a_1 - 24a_2 + 120b_2)^7 + 5(4a_1 - 24a_2 + 120b_2)^9 - 900(4a_1 - 24a_2 + 120b_2)^3(7 + 4(8b_1 + 80t)^2 - 12(16a_2 - 160b_2)^2) + 675(4a_1 - 24a_2 + 120b_2)(7 + 16(8b_1 + 80t)^2 + 16(16a_2 - 160b_2)^2) - 345600a_2 + 4492800b_2 - 21600(8b_1 + 80t)^2(16a_2 - 160b_2) - 21600(16a_2 - 160b_2)^3 + 691200(4a_1 - 24a_2 + 120b_2)^2b_2 - 1800(4a_1 - 24a_2 + 120b_2)^4(64a_2 - 448b_2))(2x - 12b_1 - 60t)^2 + (-240(4a_1 - 24a_2 + 120b_2)^7(8b_1 + 80t) - 7200(4a_1 - 24a_2 + 120b_2)^4(8b_1 + 80t)(16a_2 - 160b_2) + 10800(4a_1 - 24a_2 + 120b_2)(24b_1 + 400t + 4(8b_1 + 80t)^3 + 4(8b_1 + 80t)(16a_2 - 160b_2)^2) + 3600(4a_1 - 24a_2 + 120b_2)^3(24b_1 + 176t) + 720(4a_1 - 24a_2 + 120b_2)^5(56b_1 + 400t) + 21600(8b_1 + 80t)(16a_2 - 160b_2) - 1382400(16a_2 - 160b_2)t - 2764800(8b_1 + 80t)b_2 - 43200(4a_1 - 24a_2 + 120b_2)^2((8b_1 + 80t)(16a_2 - 160b_2) - 32(16a_2 - 160b_2)t - 64(8b_1 + 80t)b_2))(2x - 12b_1 - 60t) - 5529600(16a_2 - 160b_2)^2b_2 - 90(4a_1 - 24a_2 + 120b_2)^8(16a_2 - 160b_2) + 64800(16a_2 - 160b_2)^3 - 870(4a_1 - 24a_2 + 120b_2)^7 + 25(4a_1 - 24a_2 + 120b_2)^9 + (4a_1 - 24a_2 + 120b_2)^{11} + 5529600(8b_1 + 80t)(16a_2 - 160b_2)t - 151200a_2 + 1857600b_2) + 15(1 + (4a_1 - 24a_2 + 120b_2)^2)(2x - 12b_1 - 60t)^8 + (210 - 60(4a_1 - 24a_2 + 120b_2)^2 + 50(4a_1 - 24a_2 + 120b_2)^4 + 480(4a_1 - 24a_2 + 120b_2)(16a_2 - 160b_2))(2x - 12b_1 - 60t)^6 + (-720(4a_1 - 24a_2 + 120b_2)^2(8b_1 + 80t) - 5760b_1 + 11520t)(2x - 12b_1 - 60t)^5 + (450(4a_1 - 24a_2 + 120b_2)^2 - 150(4a_1 - 24a_2 + 120b_2)^4 + 70(4a_1 - 24a_2 + 120b_2)^6 + 1200(4a_1 - 24a_2 + 120b_2)^3(16a_2 - 160b_2) - 450 + 5400(8b_1 + 80t)^2 - 1800(16a_2 - 160b_2)^2 + 3600(4a_1 - 24a_2 + 120b_2)(16a_2 - 224b_2))(2x - 12b_1 - 60t)^4 + (-2400(4a_1 - 24a_2 + 120b_2)^4(8b_1 + 80t) + 28800(4a_1 - 24a_2 + 120b_2)(8b_1 + 80t)(16a_2 - 160b_2) + 57600b_1 + 806400t - 7200(4a_1 - 24a_2 + 120b_2)^2(16b_1 + 128t))(2x - 12b_1 - 60t)^3 + (6750(4a_1 - 24a_2 + 120b_2)^4 + 420(4a_1 - 24a_2 + 120b_2)^6 + 45(4a_1 - 24a_2 + 120b_2)^8 - 2700(4a_1 - 24a_2 + 120b_2)^2(5 + 4(8b_1 + 80t)^2 - 12(16a_2 - 160b_2)^2) - 675 - 10800(8b_1 + 80t)^2 - 10800(16a_2 - 160b_2)^2 + 21600(4a_1 - 24a_2 + 120b_2)(32a_2 - 384b_2) - 7200(4a_1 - 24a_2 + 120b_2)^3(32a_2 - 128b_2))(2x - 12b_1 - 60t)^2 + (-1680(4a_1 - 24a_2 + 120b_2)^6(8b_1 + 80t) - 28800(4a_1 - 24a_2 + 120b_2)^3(8b_1 + 80t)(16a_2 - 160b_2) - 10800(4a_1 - 24a_2 + 120b_2)^2(8b_1 + 272t) + 86400b_1 + 1209600t + 43200(8b_1 + 80t)^3 + 43200(8b_1 + 80t)(16a_2 - 160b_2)^2 + 3600(4a_1 - 24a_2 + 120b_2)^4(8b_1 - 80t) - 86400(4a_1 - 24a_2 + 120b_2)((8b_1 + 80t)(16a_2 - 160b_2) - 32(16a_2 - 160b_2)t - 64(8b_1 + 80t)b_2))(2x - 12b_1 - 60t) + 450(4a_1 - 24a_2 + 120b_2)^4(-17 + 28(8b_1 + 80t)^2 + 12(16a_2 - 160b_2)^2) + 10800(4a_1 - 24a_2 + 120b_2)(-16a_2 + 224b_2 + 4(8b_1 + 80t)^2(16a_2 - 160b_2) + 4(16a_2 - 160b_2)^3) + 675(4a_1 - 24a_2 + 120b_2)^2(-3 + 16(8b_1 + 80t)^2 + 16(16a_2 - 160b_2)^2) + 4096(8b_1 + 80t)t - 8192(16a_2 - 160b_2)b_2 - 720(4a_1 - 24a_2 + 120b_2)^7(16a_2 - 160b_2) - 3600(4a_1 - 24a_2 + 120b_2)^3(48a_2 - 1376b_2) - 720(4a_1 - 24a_2 + 120b_2)^5(272a_2 - 3168b_2) + 2764800(8b_1 + 80t)t - 11059200(16a_2 - 160b_2)b_2 + 27000(8b_1 + 80t)^2 + (2x - 12b_1 - 60t)^{10} + 91800(16a_2 - 160b_2)^2 + 353894400b_2^2 + 2190(4a_1 - 24a_2 + 120b_2)^6 + 495(4a_1 - 24a_2 + 120b_2)^8 + 11(4a_1 - 24a_2 + 120b_2)^{10}
\end{aligned}$$

and

$$\begin{aligned}
d(x, t) = & 2024 + 530841600t^2 - 120(4a_1 - 24a_2 + 120b_2)^9(16a_2 - 160b_2) - 2160(4a_1 - 24a_2 + 120b_2)^5(240a_2 - 4576b_2) - 1440(4a_1 - 24a_2 + 120b_2)^7(80a_2 -
\end{aligned}$$

$$\begin{aligned}
& 864 b_2) + (-360 (4 a_1 - 24 a_2 + 120 b_2)^8 (8 b_1 + 80 t) - 17280 (4 a_1 - 24 a_2 + 120 b_2)^5 (8 b_1 + \\
& 80 t)(16 a_2 - 160 b_2) - 1440 (4 a_1 - 24 a_2 + 120 b_2)^6 (8 b_1 + 240 t) + 32400 (4 a_1 - \\
& 24 a_2 + 120 b_2)^4 (8 b_1 - 112 t) + 64800 (4 a_1 - 24 a_2 + 120 b_2)^2 (-40 b_1 - 752 t + 4 (8 b_1 + \\
& 80 t)^3 + 4 (8 b_1 + 80 t)(16 a_2 - 160 b_2)^2) - 777600 (4 a_1 - 24 a_2 + 120 b_2)((8 b_1 + \\
& 80 t)(16 a_2 - 160 b_2) - 64 (16 a_2 - 160 b_2)t - 128 (8 b_1 + 80 t)b_2) - 172800 (4 a_1 - \\
& 24 a_2 + 120 b_2)^3 (3 (8 b_1 + 80 t)(16 a_2 - 160 b_2) - 32 (16 a_2 - 160 b_2)t - 64 (8 b_1 + \\
& 80 t)b_2) - 648000 b_1 - 8553600 t + 259200 (8 b_1 + 80 t)^3 + 1296000 (8 b_1 + 80 t)(16 a_2 - \\
& 160 b_2)^2 + 33177600 (8 b_1 + 80 t)^2 t - 33177600 (16 a_2 - 160 b_2)^2 t - 132710400 (8 b_1 + \\
& 80 t)(16 a_2 - 160 b_2)b_2(2 x - 12 b_1 - 60 t) + 80 (4 a_1 - 24 a_2 + 120 b_2)^6 (191 + \\
& 63 (8 b_1 + 80 t)^2 + 27 (16 a_2 - 160 b_2)^2) + 21600 (4 a_1 - 24 a_2 + 120 b_2)^3 (-368 a_2 + \\
& 3488 b_2 + 4 (8 b_1 + 80 t)^2 (16 a_2 - 160 b_2) + 4 (16 a_2 - 160 b_2)^3) + 240 (4 a_1 - 24 a_2 + \\
& 120 b_2)^4 (599 + 135 (8 b_1 + 80 t)^2 - 225 (16 a_2 - 160 b_2)^2 + 11520 (8 b_1 + 80 t)t - \\
& 23040 (16 a_2 - 160 b_2)b_2) - 16200 (4 a_1 - 24 a_2 + 120 b_2)(496 a_2 - 6240 b_2 + 80 (8 b_1 + \\
& 80 t)^2 (16 a_2 - 160 b_2) + 16 (16 a_2 - 160 b_2)^3 - 4096 (8 b_1 + 80 t)(16 a_2 - 160 b_2)t - \\
& 4096 (8 b_1 + 80 t)^2 b_2 + 4096 (16 a_2 - 160 b_2)^2 b_2) + 24 (4 a_1 - 24 a_2 + 120 b_2)^2 (3881 + \\
& 12150 (8 b_1 + 80 t)^2 + 28350 (16 a_2 - 160 b_2)^2 - 691200 (8 b_1 + 80 t)t + 22118400 t^2 + \\
& 88473600 b_2^2) + 1036800 (8 b_1 + 80 t)^2 (16 a_2 - 160 b_2)^2 + (-120 (4 a_1 - 24 a_2 + \\
& 120 b_2)^2 + 360 (4 a_1 - 24 a_2 + 120 b_2)(16 a_2 - 160 b_2) + 120)(2 x - 12 b_1 - 60 t)^8 + \\
& (480 (4 a_1 - 24 a_2 + 120 b_2)^2 - 240 (4 a_1 - 24 a_2 + 120 b_2)^4 + 960 (4 a_1 - 24 a_2 + \\
& 120 b_2)^3 (16 a_2 - 160 b_2) + 2320 + 2160 (8 b_1 + 80 t)^2 + 5040 (16 a_2 - 160 b_2)^2 - \\
& 1440 (4 a_1 - 24 a_2 + 120 b_2)(64 a_2 - 960 b_2))(2 x - 12 b_1 - 60 t)^6 + (-720 (4 a_1 - \\
& 24 a_2 + 120 b_2)^4 (8 b_1 + 80 t) - 17280 (4 a_1 - 24 a_2 + 120 b_2)(8 b_1 + 80 t)(16 a_2 - \\
& 160 b_2) + 4320 (4 a_1 - 24 a_2 + 120 b_2)^2 (8 b_1 + 176 t) - 51840 b_1 - 103680 t)(2 x - \\
& 12 b_1 - 60 t)^5 + (-1440 (4 a_1 - 24 a_2 + 120 b_2)^4 + 720 (4 a_1 - 24 a_2 + 120 b_2)^5 (16 a_2 - \\
& 160 b_2) + 240 (4 a_1 - 24 a_2 + 120 b_2)^2 (56 + 135 (8 b_1 + 80 t)^2 - 45 (16 a_2 - 160 b_2)^2) + \\
& 32400 (4 a_1 - 24 a_2 + 120 b_2)(16 a_2 - 288 b_2) + 7200 (4 a_1 - 24 a_2 + 120 b_2)^3 (48 a_2 - \\
& 544 b_2) + 3360 + 32400 (8 b_1 + 80 t)^2 - 54000 (16 a_2 - 160 b_2)^2 - 2764800 (8 b_1 + \\
& 80 t)t + 5529600 (16 a_2 - 160 b_2)b_2(2 x - 12 b_1 - 60 t)^4 + (-960 (4 a_1 - 24 a_2 + \\
& 120 b_2)^6 (8 b_1 + 80 t) + 57600 (4 a_1 - 24 a_2 + 120 b_2)^3 (8 b_1 + 80 t)(16 a_2 - 160 b_2) - \\
& 43200 (4 a_1 - 24 a_2 + 120 b_2)^2 (24 b_1 + 272 t) - 7200 (4 a_1 - 24 a_2 + 120 b_2)^4 (48 b_1 + \\
& 448 t) + 345600 b_1 + 5529600 t - 86400 (8 b_1 + 80 t)^3 - 86400 (8 b_1 + 80 t)(16 a_2 - \\
& 160 b_2)^2 + 172800 (4 a_1 - 24 a_2 + 120 b_2)((8 b_1 + 80 t)(16 a_2 - 160 b_2) + 32 (16 a_2 - \\
& 160 b_2)t + 64 (8 b_1 + 80 t)b_2))(2 x - 12 b_1 - 60 t)^3 + (13440 (4 a_1 - 24 a_2 + 120 b_2)^6 + \\
& 240 (4 a_1 - 24 a_2 + 120 b_2)^8 - 240 (4 a_1 - 24 a_2 + 120 b_2)^4 (-326 + 45 (8 b_1 + 80 t)^2 - \\
& 135 (16 a_2 - 160 b_2)^2) + 480 (4 a_1 - 24 a_2 + 120 b_2)^2 (-76 + 135 (8 b_1 + 80 t)^2 + \\
& 1215 (16 a_2 - 160 b_2)^2) - 129600 (4 a_1 - 24 a_2 + 120 b_2)^3 (32 a_2 - 256 b_2) - 12960 (4 a_1 - \\
& 24 a_2 + 120 b_2)^5 (32 a_2 - 256 b_2) - 64800 (4 a_1 - 24 a_2 + 120 b_2)(-96 a_2 + 1280 b_2 + \\
& 4 (8 b_1 + 80 t)^2 (16 a_2 - 160 b_2) + 4 (16 a_2 - 160 b_2)^3) + 12144 - 97200 (8 b_1 + \\
& 80 t)^2 + 32400 (16 a_2 - 160 b_2)^2 + 530841600 t^2 - 33177600 (16 a_2 - 160 b_2)b_2 + \\
& 2123366400 b_2^2)(2 x - 12 b_1 - 60 t)^2 + 24883200 (8 b_1 + 80 t)t + 120 (8 b_1 + 80 t)(2 x - \\
& 12 b_1 - 60 t)^9 - 46080 t(2 x - 12 b_1 - 60 t)^7 - 82944000 (16 a_2 - 160 b_2)b_2 + 356400 (8 b_1 + \\
& 80 t)^2 + 518400 (8 b_1 + 80 t)^4 + 874800 (16 a_2 - 160 b_2)^2 + 2123366400 b_2^2 + 3720 (4 a_1 - \\
& 24 a_2 + 120 b_2)^8 + 120 (4 a_1 - 24 a_2 + 120 b_2)^{10} + 518400 (16 a_2 - 160 b_2)^4 + (1 + \\
& (2 x - 12 b_1 - 60 t)^2 + (4 a_1 - 24 a_2 + 120 b_2)^2)^6
\end{aligned}$$

is a solution to the (NLS4) equation (1).