Compactness of the set of solutions to a Liouville type equation with boundary singularity

Samy Skander Bahoura

Compactness of the set of solutions to a Liouville type equation with boundary singularity.

Samy Skander Bahoura*
Equipe d'Analyse Complexe et Géométrie.
Université Pierre et Marie Curie, 75005 Paris, France.

Abstract

We give a blow-up behavior for solutions to a Liouville type equation with boundary singularity and Dirichlet boundary condition. An application, we have a compactness result for solutions to this Problem under a Lipschitz condition.

Keywords: Blow-up, boundary, singularity, a priori estimate, compactness, Lipschitz condition, Dirichlet boundary condition, singular weight.
MSC: 35J60, 35B44, 35B45

1 Introduction and Main Results

We set $\Delta = \partial_{11} + \partial_{22}$ on an analytic domain $\Omega \subset \mathbb{R}^2$.

We consider the following boundary value problem:

$$(P)\begin{cases} -\Delta u = -|x|^{-2\alpha}\log\frac{|x|}{2d}Ve^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

Here:

$$d = diam(\Omega),\ \alpha \in (0, 1/2),\ 0 \in \partial\Omega,$$

and,

$$0 \leq V \leq b < +\infty,\ -|x|^{-2\alpha}\log\frac{|x|}{2d}e^u \in L^1(\Omega),\ u \in W_0^{1,1}(\Omega).$$

The previous equation of the previous boundary value problem, was studied by many authors, with or without the boundary condition, also for Riemannian surfaces, see [1-14], we can find some existence and compactness results. The solutions are in the sense of the distributions.

*e-mails: samybahoura@gmail.com

Remark that: the weight $-|x|^{-2\alpha}\log\frac{|x|}{2d} >> |x|^{-2\alpha}$, but the solutions with the previous conditions on u and V, stay again in $C^{1,\epsilon}(\bar{\Omega}), \epsilon > 0$. Here we look to the case of a weight with singularity, bigger than the usual weight, $|x|^{-2\alpha}, \alpha \in (0, 1/2)$.

Among other results, we can see in [7] the following important Theorem,

Theorem. *(Brezis-Merle [7]).If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the problem (P) with, $0 < a \leq V_i \leq b < +\infty$, then, for all compact set K of Ω,*

$$\sup_K u_i \leq c = c(a, b, \alpha, K, \Omega).$$

For a potential V with more regularity, there are other type of estimates, for example, $\sup + \inf$ type inequalities. It was proved by Shafrir see [14], that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C\left(\frac{a}{b}\right)\sup_K u_i + \inf_\Omega u_i \leq c = c(a, b, K, \Omega).$$

Now, if we suppose $(V_i)_i$ uniformly Lipschitz with A the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$, see [5].

Here we give the behavior of solutions which blow-up on the boundary and we give a compactness result for a boundary value Problem, with Lipschitz condition on the prescribed scalar curvature.

Here, we have the following problem as the Brezis-Merle Problem (see [7]):

Problem. Suppose that $V_i \to V$ in $C^0(\bar{\Omega})$, with, $0 \leq V_i$. Also, we consider a sequence of solutions (u_i) of (P) relatively to (V_i) such that,

$$\int_\Omega -|x|^{-2\alpha}\log\frac{|x|}{2d}e^{u_i}dx \leq C,$$

is it possible to have:

$$||u_i||_{L^\infty} \leq C?$$

Here, we give a characterization of the behavior of the blow-up points on the boundary and also a proof of the compactness theorem when the prescribed scalar curvature are uniformly Lipschitz. For the behavior blow-up solutions on the boundary, the following condition is enough,

$$0 \leq V_i \leq b,$$

The condition $V_i \to V$ in $C^0(\bar{\Omega})$ is not necessary.

But for the proof of the compactness result for the previous problem we assume that:

$$||\nabla V_i||_{L^\infty} \leq A.$$

Our main results are:

We have the following characterization of the behavior of the blow-up points on the boundary.

Theorem 1.1 *Assume that* $\max_\Omega u_i \to +\infty$*, Where* (u_i) *are solutions of the probleme* (P) *with:*

$$\alpha \in (0,1/2),\ 0 \le V_i \le b,\ \text{ and } \int_\Omega -|x|^{-2\alpha} \log \frac{|x|}{2d} e^{u_i} dx \le C,\ \ \forall\ i,$$

then; after passing to a subsequence, there is a function u*, there is a number* $N \in \mathbb{N}^*$ *and there are* N *points:*
$x_1 = 0, x_2, \dots, x_N \in \partial\Omega - \{0\}$*, such that,*

$$\partial_\nu u_i \to \partial_\nu u + \sum_{j=1}^{N} \alpha_j \delta_{x_j},\ \ \alpha_1 \ge 4\pi(1-\alpha), \alpha_j \ge 4\pi \text{ weakly in the sense of measures.}$$

and,

$$u_i \to u \text{ in } C^1_{loc}(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

or,
$x_1, \dots, x_N \in \partial\Omega - \{0\}$*,*

$$\partial_\nu u_i \to \partial_\nu u + \sum_{j=1}^{N} \alpha_j \delta_{x_j},\ \ \alpha_1 \ge 4\pi, \alpha_j \ge 4\pi \text{ weakly in the sense of measures.}$$

and,

$$u_i \to u \text{ in } C^1_{loc}(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

In the following theorem, we have a compactness result which concern the boundary value problem (P).

Theorem 1.2 *Assume that* (u_i) *are solutions of* (P) *relatively to* (V_i) *with the following conditions:*

$$d = diam(\Omega),\ \alpha \in (0,1/2),\ 0 \in \partial\Omega,$$

$$0 \le V_i \le b,\ ||\nabla V_i||_{L^\infty} \le A,\ \text{ and } \int_\Omega -|x|^{-2\alpha} \log \frac{|x|}{2d} e^{u_i} \le C,$$

We have,

$$||u_i||_{L^\infty} \le c(b, \alpha, A, C, \Omega),$$

2 Proof of the theorems

Proof of theorem 1.1:

We have,

$$u_i \in W_0^{1,1}(\Omega).$$

By [7], $e^{u_i} \in L^k,\ \forall k > 1$ and by the elliptic estimates:

$$u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}),\ \epsilon > 0.$$

By the Stokes formula, we have,

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C,$$

Without loss of generality, we can assume that $\partial_\nu u_i \geq 0$. Thus, (using the weak convergence in the space of Radon measures), we have the existence of a non-negative Radon measure μ such that,

$$\int_{\partial\Omega} \partial_\nu u_i \phi d\sigma \to \mu(\phi),\ \forall\ \phi \in C^0(\partial\Omega).$$

Let us consider a point $x_0 \in \partial\Omega$. We say that x_0 is regular if, $x_0 \neq 0$ and $\mu(\{x_0\}) < 4\pi$, or $x_0 = 0$ and $\mu(\{0\}) < 4\pi(1-\alpha)$. A point $x_0 \in \partial\Omega$ is a nonregular point, if the previous conditions are not satisfied.

For a regular point $x_0 \in \partial\Omega$, we may assume that the following curve, $B(x_0, \epsilon) \cap \partial\Omega := I_\epsilon$ is an interval. (In this case, it is simpler to construct the following test function η_ϵ).

Case 1: $\mu(\{0\}) \geq 4\pi(1-\alpha)$.

This means that 0 is a nonregular point for the measure μ.

$$\begin{cases} \eta_\epsilon \equiv 1, \text{ on } I_\epsilon,\ 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, \text{ outside } I_{2\epsilon}, \\ 0 \leq \eta_\epsilon \leq 1, \\ ||\nabla \eta_\epsilon||_{L^\infty(I_{2\epsilon})} \leq \dfrac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

We take a $\tilde{\eta}_\epsilon$ such that,

$$\begin{cases} -\Delta\tilde{\eta}_\epsilon = 0 & \text{in } \Omega \\ \tilde{\eta}_\epsilon = \eta_\epsilon & \text{on } \partial\Omega. \end{cases}$$

We use the following estimate, see [8],

$$||\nabla u_i||_{L^q} \leq C_q,\ \forall\ i \text{ and } 1 < q < 2.$$

We deduce from the last estimate that, (u_i) converge weakly in $W_0^{1,q}(\Omega)$, almost everywhere to a function $u \geq 0$ and $\int_\Omega -|x|^{-2\alpha} \log \dfrac{|x|}{2d} e^u < +\infty$ (by

Fatou lemma). Also, V_i weakly converge to a nonnegative function V in L^∞. The function u is in $W_0^{1,q}(\Omega)$ solution of :

$$\begin{cases} -\Delta u = -|x|^{-2\alpha} \log \frac{|x|}{2d} V e^u \in L^1(\Omega) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

As in the corollary 1 of Brezis-Merle result, see [7], we have $e^{ku} \in L^1(\Omega), k > 1$. By the elliptic estimates, we have $u \in C^1(\bar{\Omega})$.

We can write,

$$-\Delta((u_i - u)\tilde{\eta}_\epsilon) = -|x|^{-2\alpha} \log \frac{|x|}{2d}(V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon + 2 < \nabla(u_i - u)|\nabla\tilde{\eta}_\epsilon > . \quad (1)$$

We use the interior esimate of Brezis-Merle, see [7],

Step 1: Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between $\tilde{\eta}_\epsilon$ and u, we obtain,

$$\int_\Omega -|x|^{-2\alpha} \log \frac{|x|}{2d} V e^u \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u \eta_\epsilon \leq 4\epsilon ||\partial_\nu u||_{L^\infty} = C\epsilon \quad (2)$$

We have,

$$\begin{cases} -\Delta u_i = -\log \frac{|x|}{2d} V_i e^{u_i} & \text{in } \Omega \\ \quad u_i = 0 & \text{on } \partial\Omega, \end{cases}$$

We use the Green formula between u_i and $\tilde{\eta}_\epsilon$ to have:

$$\int_\Omega -|x|^{-2\alpha} \log \frac{|x|}{2d} V_i e^{u_i} \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon d\sigma \to \mu(\eta_\epsilon) \leq \mu(I_{2\epsilon}) \leq 4\pi - \epsilon_0, \ \ \epsilon_0 > 0 \quad (3)$$

From (2) and (3) we have for all $\epsilon > 0$ there is $i_0 = i_0(\epsilon)$ such that, for $i \geq i_0$,

$$\int_\Omega -|x|^{-2\alpha} \log \frac{|x|}{2d} |(V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon| dx \leq 4\pi - \epsilon_0 + C\epsilon \quad (4)$$

Step 2: Estimate of integral of the second term of the right hand side of (1).

Let $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^3\}$ and $\Omega_{\epsilon^3} = \{x \in \Omega, d(x, \partial\Omega) \geq \epsilon^3\}$, $\epsilon > 0$. Then, for ϵ small enough, Σ_ϵ is hypersurface.

The measure of $\Omega - \Omega_{\epsilon^3}$ is $k_2\epsilon^3 \leq \mu_L(\Omega - \Omega_{\epsilon^3}) \leq k_1\epsilon^3$.

Remark: for the unit ball $\bar{B}(0,1)$, our new manifold is $\bar{B}(0, 1 - \epsilon^3)$.

We write,

$$\int_{\Omega} | < \nabla(u_i - u)|\nabla\tilde{\eta}_\epsilon > |dx = \int_{\Omega_{\epsilon^3}} | < \nabla(u_i - u)|\nabla\tilde{\eta}_\epsilon > |dx +$$

$$+ \int_{\Omega - \Omega_{\epsilon^3}} < \nabla(u_i - u)|\nabla\tilde{\eta}_\epsilon > |dx. \tag{5}$$

Step 2.1: Estimate of $\int_{\Omega-\Omega_{\epsilon^2}} | < \nabla(u_i - u)|\nabla\tilde{\eta}_\epsilon > |dx$.

First, we know from the elliptic estimates that $||\nabla\tilde{\eta}_\epsilon||_{L^\infty} \leq C_1/\epsilon^2$, C_1 depends on Ω

We know that $(|\nabla u_i|)_i$ is bounded in $L^q, 1 < q < 2$, we can extract from this sequence a subsequence which converge weakly to $h \in L^q$. But, we know that we have locally the uniform convergence to $|\nabla u|$ (by Brezis-Merle theorem), then, $h = |\nabla u|$ a.e. Let q' be the conjugate of q.

We have, $\forall f \in L^{q'}(\Omega)$

$$\int_{\Omega} |\nabla u_i| f dx \to \int_{\Omega} |\nabla u| f dx$$

If we take $f = 1_{\Omega - \Omega_{\epsilon^3}}$, we have:

$$\text{for } \epsilon > 0 \ \exists\ i_1 = i_1(\epsilon) \in \mathbb{N},\ \ i \geq i_1,\ \int_{\Omega-\Omega_{\epsilon^3}} |\nabla u_i| \leq \int_{\Omega-\Omega_{\epsilon^3}} |\nabla u| + \epsilon^3.$$

Then, for $i \geq i_1(\epsilon)$,

$$\int_{\Omega-\Omega_{\epsilon^3}} |\nabla u_i| \leq mes(\Omega - \Omega_{\epsilon^3})||\nabla u||_{L^\infty} + \epsilon^2 = C\epsilon.$$

Thus, we obtain,

$$\int_{\Omega-\Omega_{\epsilon^3}} | < \nabla(u_i - u)|\nabla\tilde{\eta}_\epsilon > |dx \leq \epsilon C_1(2k_1||\nabla u||_{L^\infty} + 1) \tag{6}$$

The constant C_1 does not depend on ϵ but on Ω.

Step 2.2: Estimate of $\int_{\Omega_{\epsilon^2}} | < \nabla(u_i - u)|\nabla\tilde{\eta}_\epsilon > |dx$.

We know that, $\Omega_\epsilon \subset\subset \Omega$, and (because of Brezis-Merle's interior estimates) $u_i \to u$ in $C^1(\Omega_{\epsilon^3})$. We have,

$$||\nabla(u_i - u)||_{L^\infty(\Omega_{\epsilon^3})} \leq \epsilon^2, \text{ for } i \geq i_3 = i_3(\epsilon).$$

We write,

$$\int_{\Omega_{\epsilon_3}} | < \nabla(u_i - u)|\nabla\tilde{\eta}_\epsilon > |dx \leq ||\nabla(u_i - u)||_{L^\infty(\Omega_{\epsilon^3})}||\nabla\tilde{\eta}_\epsilon||_{L^\infty} \leq C_1\epsilon \text{ for } i \geq i_3,$$

For $\epsilon > 0$, we have for $i \in \mathbb{N}$, $i \geq \max\{i_1, i_2, i_3\}$,

$$\int_\Omega | < \nabla(u_i - u)|\nabla\tilde{\eta}_\epsilon > |dx \leq \epsilon C_1(2k_1||\nabla u||_{L^\infty} + 2) \tag{7}$$

From (4) and (7), we have, for $\epsilon > 0$, there is $i_3 = i_3(\epsilon) \in \mathbb{N}, i_3 = \max\{i_0, i_1, i_2\}$ such that,

$$\int_\Omega |\Delta[(u_i - u)\tilde{\eta}_\epsilon]|dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1(2k_1||\nabla u||_{L^\infty} + 2 + C) \tag{8}$$

We choose $\epsilon > 0$ small enough to have a good estimate of (1).

Indeed, we have:

$$\begin{cases} -\Delta[(u_i - u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega, \\ (u_i - u)\tilde{\eta}_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

with $||g_{i,\epsilon}||_{L^1(\Omega)} \leq 4\pi - \epsilon_0$.

We can use Theorem 1 of [7] to conclude that there is $q \geq \tilde{q} > 1$ such that:

$$\int_{V_\epsilon(x_0)} e^{\tilde{q}|u_i - u|}dx \leq \int_\Omega e^{q|u_i - u|\tilde{\eta}_\epsilon}dx \leq C(\epsilon, \Omega).$$

where, $V_\epsilon(x_0)$ is a neighberhood of x_0 in $\bar{\Omega}$.

Thus, for each $x_0 \in \partial\Omega - \{\bar{x}_1, \ldots, \bar{x}_m\}$ there is $\epsilon_{x_0} > 0, q_{x_0} > 1$ such that:

$$\int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0}u_i}dx \leq C, \ \forall \ i.$$

Now, we consider a cutoff function $\eta \in C^\infty(\mathbb{R}^2)$ such that:

$$\eta \equiv 1 \text{ on } B(x_0, \epsilon_{x_0}/2) \text{ and } \eta \equiv 0 \text{ on } \mathbb{R}^2 - B(x_0, 2\epsilon_{x_0}/3).$$

We write,

$$-\Delta(u_i\eta) = -|x|^{-2\alpha}\log\frac{|x|}{2d}V_i e^{u_i}\eta - 2 < \nabla u_i|\nabla\eta > -u_i\Delta\eta.$$

By the elliptic estimates, $(u_i\eta)_i$ is uniformly bounded in $W^{2,q_1}(\Omega)$ and also, in $C^1(\bar{\Omega})$.

Finaly, we have, for some $\epsilon > 0$ small enough,

$$||u_i||_{C^{1,\theta}[B(x_0,\epsilon)]} \leq c_3 \ \forall \ i. \tag{9}$$

We have proved that, there is a finite number of points $\bar{x}_1, \ldots, \bar{x}_m$ such that the squence $(u_i)_i$ is locally uniformly bounded in $\bar{\Omega} - \{\bar{x}_1, \ldots, \bar{x}_m\}$.

And, finaly, we have:

$$\partial_\nu u_i \to \partial_\nu u + \sum_{j=1}^N \alpha_j \delta_{x_j}, \ \alpha_1 \geq 4\pi(1-\alpha) \ \alpha_j \geq 4\pi \text{ weakly in the sens of measures.} \tag{10}$$

Case 2: $\mu(\{0\}) < 4\pi(1-\alpha)$.

Then, 0 is a regular point for the measure μ.

Let us consider $B_\epsilon(0)$, a ball of center 0 and radius $\epsilon > 0$. As in the previous case, we use the uniform estimate in $W_0^{1,q}(\Omega), (1 \leq q < 2)$ and Brezis-Merle's method, see [7], to have

$$e^{u_i} \in L^{(1-\epsilon')/(1-\alpha-\epsilon')}(B_\epsilon(0)).$$

with a uniform bound.
Thus, by the Hölder inequality we have

$$u_i \in L^\infty(B_\epsilon(0)).$$

with uniform bound. If we take $\mu(\{0\}) < 4\pi$, by the Brezis-Merle estimate we have $e^{u_i} \in L^r(B_\epsilon(0))$ with $r > 1$, but this r may not be large enough to ensure $u_i \in L^\infty(B_\epsilon(0))$, because we have the term $-|x|^{-2\alpha} \log \frac{|x|}{2d}$ in the equation.
Then, by the elliptic estimates, for $\alpha \in (0,1)$

$$u_i \in W^{2,1+\epsilon'}(B_\epsilon(0)) \cap C^{0,\epsilon'}(B_\epsilon(0)) \cap C^{2,\epsilon'}_{loc}(\Omega - \{0, x_1, x_2, \ldots, x_N\}), \qquad (11)$$

and, for $\alpha \in (0, 1/2)$, we have

$$u_i \in W^{2,1+\epsilon'}(B_\epsilon(0)) \cap C^{1,\epsilon'}(B_\epsilon(0)) \cap C^{2,\epsilon'}_{loc}(\Omega - \{0, x_1, x_2, \ldots, x_N\}). \qquad (12)$$

And thus, we have

$$\partial_\nu u_i \to \partial_\nu u + \sum_{j=1}^{N} \alpha_j \delta_{x_j},\ \alpha_j \geq 4\pi \text{ weakly in the sense of measures on } \partial\Omega. \quad (13)$$

If we consider a neighborhood of a regular point $x_0 \neq 0$, we are far from the singularity and the scheme of the first step work in this case; we have the uniform convergence of $\partial_\nu u_i$ around x_0. In the case $\alpha \in (0, 1/2)$, the elliptic estimates gives us the C^1 convergence of u_i.

Proof of theorem 1.2:

Without loss of generality, we can assume that 0 is a blow-up point (either, we are in the regular case). Also, by a conformal transformation, we can assume that $\Omega = B_1^+$, the half ball, and $\partial^+ B_1^+$ is the exterior part, a part which not contain 0 and on which u_i converge in the C^1 norm to u. Let us consider B_ϵ^+, the half ball with radius $\epsilon > 0$.

The Pohozaev identity gives :

$$\int_{B_\epsilon^+} -|x|^{-2\alpha} \log \frac{|x|}{2d} V_i e^{u_i}(1+o(1))dx + \int_{B_\epsilon^+} < x|\nabla V_i > -|x|^{-2\alpha} \log \frac{|x|}{2d} V_i e^{u_i} dx =$$

$$= \int_{\partial^+ B_\epsilon^+} g(\nabla u_i) d\sigma + o(1), \tag{14}$$

$$\int_{B_\epsilon^+} -|x|^{-2\alpha} \log \frac{|x|}{2d} V e^u (1 + o(1)) dx + \int_{B_\epsilon^+} < x|\nabla V > -|x|^{-2\alpha} \log \frac{|x|}{2d} V e^u dx =$$

$$= \int_{\partial^+ B_\epsilon^+} g(\nabla u) d\sigma + o(1), \tag{15}$$

Thus,

$$\int_{B_\epsilon^+} -|x|^{-2\alpha} \log \frac{|x|}{2d} V_i e^{u_i} dx - \int_{B_\epsilon^+} -|x|^{-2\alpha} \log \frac{|x|}{2d} V e^u dx +$$

$$+ \int_{B_\epsilon^+} < x|\nabla V_i > -|x|^{-2\alpha} \log \frac{|x|}{2d} V_i e^{u_i} dx - \int_{B_\epsilon^+} < x|\nabla V > -|x|^{-2\alpha} \log \frac{|x|}{2d} V e^u dx =$$

$$= \int_{\partial^+ B_\epsilon^+} g(\nabla u_i) - g(\nabla u) d\sigma + o(1) = o(1),$$

First, we tend i to infinity after ϵ to 0, we obtain:

$$\lim_{\epsilon \to 0} \lim_{i \to +\infty} \int_{B_\epsilon^+} -|x|^{-2\alpha} \log \frac{|x|}{2d} V_i e^{u_i} dx = 0, \tag{16}$$

But,

$$\int_{B_\epsilon^+} -|x|^{-2\alpha} \log \frac{|x|}{2d} V_i e^{u_i} dx = \int_{\partial B_\epsilon^+} \partial_\nu u_i + o(\epsilon) + o(1) \to \alpha_1 > 0.$$

A contradiction.

References

[1] T. Aubin. Some Nonlinear Problems in Riemannian Geometry. Springer-Verlag, 1998.

[2] C. Bandle. Isoperimetric Inequalities and Applications. Pitman, 1980.

[3] Bartolucci, D. A "sup+Cinf" inequality for Liouville-type equations with singular potentials. Math. Nachr. 284 (2011), no. 13, 1639-1651.

[4] Bartolucci, D. A 'sup+Cinf' inequality for the equation $-\Delta u = V e^u / |x|^{2\alpha}$. Proc. Roy. Soc. Edinburgh Sect. A 140 (2010), no. 6, 1119-1139

[5] H. Brezis, YY. Li and I. Shafrir. A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities. J.Funct.Anal.115 (1993) 344-358.

[6] Brezis. H, Marcus. M, Ponce. A. C. Nonlinear elliptic equations with measures revisited. Mathematical aspects of nonlinear dispersive equations, 55-109, Ann. of Math. Stud., 163, Princeton Univ. Press, Princeton, NJ, 2007.

[7] H. Brezis, F. Merle. Uniform estimates and Blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimension. Commun. in Partial Differential Equations, 16 (8 and 9), 1223-1253(1991).

[8] H. Brezis, W. A. Strauss. Semi-linear second-order elliptic equations in L1. J. Math. Soc. Japan 25 (1973), 565-590.

[9] W. Chen, C. Li. A priori estimates for solutions to nonlinear elliptic equations. Arch. Rational. Mech. Anal. 122 (1993) 145-157.

[10] C-C. Chen, C-S. Lin. A sharp sup+inf inequality for a nonlinear elliptic equation in $\mathbb{R}^2$. Commun. Anal. Geom. 6, No.1, 1-19 (1998).

[11] YY. Li, I. Shafrir. Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two. Indiana. Math. J. Vol 3, no 4. (1994). 1255-1270.

[12] YY. Li. Harnack Type Inequality: the method of moving planes. Commun. Math. Phys. 200,421-444 (1999).

[13] L. Ma, J-C. Wei. Convergence for a Liouville equation. Comment. Math. Helv. 76 (2001) 506-514.

[14] I. Shafrir. A sup+inf inequality for the equation $-\Delta u = Ve^u$. C. R. Acad.Sci. Paris Sér. I Math. 315 (1992), no. 2, 159-164.