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Bogdan Gheorghe, Florin Stoican, Ionela Prodan. A variable terminal set NMPC construction: application to multicopter stabilisation. 2023 European Control Conference (ECC), Jun 2023, Bucharest, France. pp.1-6, 10.23919/ECC57647.2023.10178227 . hal-04861083

**HAL Id: hal-04861083**

**<https://hal.science/hal-04861083v1>**

Submitted on 2 Jan 2025

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Conference Paper · June 2023

DOI: 10.23919/ECC57647.2023.10178227

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# A variable terminal set NMPC construction: application for multicopter stabilisation

Bogdan Gheorghe<sup>1</sup>, Florin Stoican<sup>1</sup> and Ionela Prodan<sup>2</sup>

**Abstract**—In this paper we relax the standard NMPC (Nonlinear Model Predictive Control) through a variable terminal set construction for stabilizing a multicopter system. This contribution allows us to increase the feasible domain and/or reduce the required prediction horizon length. Furthermore, to reduce the complexity of the computations we use zonotopic sets which prove instrumental due to their efficient representation. The theoretical results are validated in simulation and experiment for the stabilisation along a trajectory of a nano-multicopter’s translational dynamics.

**Index Terms**—Multicopter system, Trajectory tracking, Nonlinear Model Predictive Control (NMPC), Terminal constraints, Zonotopic sets.

## I. INTRODUCTION

Motion planning problems for multicopter systems are challenging in both the academic and industrial settings. Drone versatility has contributed to applications in fields like entertainment, aerial photography, precision agriculture, search and rescue, etc. [1]. Most of these applications require optimization-based control policies with constraints satisfaction, stability and robustness guarantees. Model predictive control (MPC) is one of the most enduring control techniques due to its theoretical and practical versatility [2]. Its capacity to explicitly consider constraints and a cost is essential in applications which require feasibility and stability guarantees.

Arguably, the main weakness of (nonlinear) MPC is the computation burden. Basically, MPC implementations boil down to a repeated solution of a constrained optimization problem. While there are many ways which exploit the particularities of the structure [3], the computation time may still be critical in systems with fast dynamics (and especially if they are nonlinear). On the other hand, continuous increases in computational power means that previously unattainable implementations are now within our grasp, even in embedded implementations. In particular, there is significant interest in applying MPC to robotic platforms in general and multicopters in particular [4], [5].

\*This work was partially funded by La Région, Pack Ambition Recherche 2021 - PlanMAV and the LabEx Persyval.

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For critical systems, an initial source of reluctance in using MPC was the lack of theoretical guarantees for feasibility. A number of results has provided the necessary tools but with a series of caveats. On one side is the classical terminal set approach [6] which ensures recursive feasibility. On the other, there are methods which show equivalence with the former approach for sufficiently long prediction horizons [7]. The second method is not suitable for fast dynamics due to the computational load and the first is often conservative as it constrains the set of initial states to only those which are backwards reachable from the terminal set.

One way to mitigate the restrictions is to relax the terminal set definition. E.g., [8] proposes a scaling procedure and [9] extends it to a fully dynamical formulation.

The terminal set is usually given as a maximal positive invariant set with an ancillary admissible state feedback law. In this context, our idea is two-pronged:

- i) instead of a fixed terminal set, add constraints enforcing its invariance, inclusion of the terminal state inside it and the existence of an admissible control law;
- ii) use zonotopic sets to characterize the terminal set due to their more compact representation and partial immunity to problem size [10].

Motivated by these ideas, the contributions are:

- i) provide novel set invariance conditions for zonotopic sets, similar to the polyhedral ones from [11];
- ii) propose a variable terminal set adaptation of the standard MPC, using a zonotopic description;
- iii) particularize the MPC construction for the translational dynamics of a multicopter system.

Noteworthy, zonotopic formulations have become quite popular in recent years, with results such as set inclusion [12] or even robust positive invariance [13] which are related to the theoretical novelties proposed here (Prop. 2 and its corollaries). Beyond the particularities of the usage and proving the results, the clear differentiating factor is that we consider zonotopes with independent scaling factors.

*Notation.*  $O_{m \times n} \in \mathbb{R}^{m \times n}$  is the matrix with  $m$  rows and  $n$  columns whose entries are zero. Whenever  $m = n$ , we use  $O_n$ .  $I_n \in \mathbb{R}^{n \times n}$  is the identity matrix. The column vector whose  $m$  entries have value one, is denoted by  $\mathbf{1}_m$ . For an arbitrary matrix  $G \in \mathbb{R}^{m \times n}$ ,  $G_i$  denotes its  $i$ -th column.  $|A|$ , with  $A \in \mathbb{R}^{m \times n}$ , represents the element-wise absolute value of  $A$ . The symbol ‘ $\cdot$ ’ denotes matrix multiplication. The Minkowski sum between two sets,  $X$

and  $Y$ , is defined as  $X \oplus Y = \{x + y : \forall x \in X, \forall y \in Y\}$ . ‘ $c$ ’ and ‘ $s$ ’ are shorthand for ‘ $\cos$ ’ and ‘ $\sin$ ’.

## II. PRELIMINARIES

In this section we introduce several standard notions from set-based control, see [14], [15] for further details.

A bounded and fully-dimensional polyhedron  $X \subset \mathbb{R}^d$ , has a dual representation, with both a half-space form

$$X = \{x \in \mathbb{R}^d : a_i^\top x \leq b_i, i = 1 \dots n_h\}, \quad (1)$$

as an intersection of linear inequalities and a convex sum of its extreme points (i.e., its vertices)

$$X = \{x \in \mathbb{R}^d : x = \sum_{j=1}^{n_v} \alpha_j v_j, \sum_{j=1}^{n_v} \alpha_j = 1, \alpha_j \geq 0\}. \quad (2)$$

The pairs  $(a_i, b_i) \in \mathbb{R}^d \times \mathbb{R}$  denote the inequalities and  $v_j \in \mathbb{R}^d$  denote the vertices.

Zonotopes are a subset of the polyhedra class, popular due to their robustness against problem size and numerical manipulations.

**Definition 1** ([15]). *A zonotope is a centrally symmetric polyhedron, given as a Minkowski sum of line segments. This generator representation is defined by a center  $c \in \mathbb{R}^d$  and a generator matrix  $G = [G_1 \dots G_D] \in \mathbb{R}^{d \times D}$ :*

$$Z(c, G) = \{c + \sum_{i=1}^D G_i \lambda_i : |\lambda_i| \leq 1, i = 1 \dots n_g\}. \quad (3)$$

Zonotopes have several interesting properties [15], [16]:

i) they are closed under Minkowski sum:

$$Z(c_1, G_1) \oplus Z(c_2, G_2) = Z(c_1 + c_2, [G_1 \ G_2]); \quad (4)$$

ii) they are closed under multiplication by any appropriately-sized matrix:

$$M \cdot Z(c, G) = Z(Mc, MG); \quad (5)$$

iii) have an equivalent half-space representation: to each sequence of  $d-1$  generators  $1 \leq j_1 < j_2 \dots j_{d-1} \leq D$  corresponds the pair  $(h_i, k_i) \in \mathbb{R}^d \times \mathbb{R}$ , where:

$$h_i \perp g_{j_l}, \forall j_l \in \{j_1 \dots j_{d-1}\}, k_i = \sum_{j_l \notin \{j_1 \dots j_{d-1}\}} |h_i^\top g_{j_l}|, \quad (6)$$

such that

$$Z(c, G) = \bigcap_{1 \leq j_1 < \dots j_{d-1} \leq D} \{x \in \mathbb{R}^d : |h_i(x - c)| \leq k_i\}. \quad (7)$$

iv) the inclusion  $Z(c, G) \subseteq X$ , with  $X$  defined as in (1), holds iff:

$$a_i^\top c + \sum_j |a_i^\top G_j| \leq b_i \quad \forall i. \quad (8)$$

v) the number of faces (facets, ridges, etc.) has a tight bound and is significantly less than the one associated to a random polyhedron of similar complexity.

## III. VARIABLE TERMINAL SET CONDITION

### A. Set invariance conditions for zonotopes

Consider the stable, linear and time invariant dynamic

$$x^+ = Ax, \text{ with } A \in \mathbb{R}^{n \times n}. \quad (9)$$

A set is called *positive invariant* for (9) iff implication  $x \in S \implies x^+ \in S$  holds [17]. Further, recall the original (for symmetric polyhedral sets) result of Bitsoris [11] which provides a simple condition for set invariance.

**Proposition 1.** *For dynamics such as those in (9), the symmetric polyhedral set*

$$S = \{x \in \mathbb{R}^n : |Fx| \leq \theta\}, F \in \mathbb{R}^{m \times n}, \theta \in \mathbb{R}^m, \quad (10)$$

*is positive invariant iff there exist  $H \in \mathbb{R}^{m \times m}$  such that*

$$HF = FA, \quad (|H| - I_m) \cdot \theta \leq O_{m \times 1}. \quad (11)$$

**Remark 1.** *This result, and variations inspired by it, resisted the test of time due to their simplicity. Only linear constraints appear, making it easy to integrate in larger optimization problems. Not least, relation (11) remains linear even when searching for a static gain, i.e., when  $A \mapsto A + BK$ .*

*The caveat is that we assume  $F$  fixed, otherwise the term  $HF$  becomes bilinear and difficult to solve due to the size of the problem (usually  $m$ , denoting the number of inequalities describing the set  $S$  in (10), is large). ♦*

The same reasoning should hold for zonotopic sets. They have “more” structure, hence, choosing/finding a candidate set is easier. Furthermore, the generator representation is significantly more compact than either of the polytopic ones (half-space or vertex-based).

**Proposition 2.** *For dynamics (9), the zonotopic set*

$$\mathcal{Z}(0, G) = \{x \in \mathbb{R}^n, x = G\lambda, \forall \lambda \in [-1, 1]^D\}, \quad (12)$$

*with  $G \in \mathbb{R}^{n \times D}$ , is positive invariant iff there exists matrix  $\Lambda \in \mathbb{R}^{D \times D}$  verifying*

$$AG = G\Lambda, \quad (|\Lambda| - I_D) \cdot \mathbf{1}_D \leq O_{D \times 1}. \quad (13)$$

*Proof:* The set invariance property (i.e., the implication that any state, updated via (9), is still inside the set) is equivalent with the existence of the pair of vectors  $\lambda, \lambda^+ \in [-1, 1]^D$  verifying the equivalence

$$\begin{aligned} [\forall x \in \mathcal{Z}(0, G) \implies x^+ = Ax \in \mathcal{Z}(0, G)] &\iff \\ [\forall |\lambda| \leq 1, \exists |\lambda^+| \leq 1 \text{ s.t. } AG\lambda = G\lambda^+] &. \quad (14) \end{aligned}$$

As a first step assume that there exists  $\Lambda \in \mathbb{R}^{D \times D}$  s.t.  $AG_i = G\Lambda_i, \forall i = 1 \dots D$ , a reasonable assumption since the columns  $G_i$  form an overcomplete<sup>1</sup> basis of  $\mathbb{R}^n$ .

<sup>1</sup>In the usual case when  $D \geq n$  and any selection of  $n$  columns is linearly independent.

Then, the following chain of equalities holds

$$\begin{aligned} x^+ &= AG\lambda = \sum_{i=1}^D AG_i\lambda_i = \sum_{i=1}^D G\Lambda_i\lambda_i \\ &= \sum_{i=1}^D \sum_{j=1}^D G_j\Lambda_{ji}\lambda_i = \sum_{j=1}^D \left( G_j \sum_{i=1}^D \Lambda_{ji}\lambda_i \right). \end{aligned} \quad (15)$$

Equality (15) shows that inclusion  $x^+ \in \mathcal{Z}(c, G)$  holds iff

$$\left| \sum_{i=1}^D \Lambda_{ji}\lambda_i \right| \leq 1, \forall j, \quad (16)$$

is verified (i.e., we take  $\lambda_j^+ := \sum_{i=1}^D \Lambda_{ji}\lambda_i$ ). Since (16) has to hold for any  $\lambda \in [-1, 1]^D$  we arrive at condition

$$\sum_{i=1}^D |\Lambda_{ji}| \leq 1, \forall j, \quad (17)$$

by taking  $\lambda_i \mapsto \text{sign}(\Lambda_{ji})$  in (16) and ignoring the absolute operator (since a sum of positive elements is itself positive). Gathering the column vectors  $\Lambda_i$  back into matrix  $\Lambda$  and taking for  $j = 1 \dots D$  the inequality (17) we arrive at (13), thus concluding the proof.  $\blacksquare$

**Remark 2.** *At a first glance, (13) is similar to (11). The difference lies in the dimensions. The zonotope (12), in generator form is defined by its generator matrix  $G$  having  $D$  columns. On the other hand, in half-space representation (12) (a symmetric polyhedral set, as defined in (10)) has  $m = \binom{D}{n-1}$  pairs of half-spaces [15].  $\blacklozenge$*

**Remark 3.** *The presence of the absolute operator is always worrying but  $\sum_{i=1}^D |\Lambda_{ji}| \leq 1$  describes a linear inclusion: the vector  $\Lambda_i = [\dots \Lambda_{ji} \dots]^\top$  stays inside the cross-polytope<sup>2</sup>  $C_D$ , inclusion which may be written through linear inequalities making use of either the  $2^D$  half-spaces or  $2D$  extremal vertices defining  $C_D$ .  $\blacklozenge$*

**Remark 4.** *Adding a static feedback is straightforward. By replacing  $A \mapsto A + BK$  in (13) we arrive at*

$$(A + BK)G = G\Lambda, \quad (|\Lambda| - I_D) \cdot \mathbf{1}_D \leq O_{D \times 1},$$

*which, noteworthy, remains linear.  $\blacklozenge$*

In recent papers such as [16], the case of the scaled zonotope (i.e., one where the generators directions are fixed but their scaling is given by decision variables  $\delta \in \mathbb{R}_+^D$ ) has been intensely studied:

$$\mathcal{Z}(c, G\Delta) = \{x = \sum_{i=1}^D c + G_i\lambda_i \cdot \delta_i, \forall |\lambda_i| \leq 1\}, \quad (18)$$

with the shorthand  $\Delta = \text{diag}(\delta)$ . This notation allows to expand the invariance conditions to the next corollary.

**Corollary 1.** *For dynamics (9), the scaled zonotopic set  $Z(0, G\Delta)$ , given as in (18), is invariant iff relations*

$$(A + BK)G\Delta = G\Delta\Lambda, \quad (|\Lambda| - I_D) \cdot \mathbf{1}_D \leq O_{D \times 1}, \quad (19)$$

<sup>2</sup>The dual/polar of the hypercube from  $\mathbb{R}^D$ .

hold.  $\square$

*Proof:* The result is straightforward and is obtained by replacing  $G \mapsto G\Delta$  and  $A \mapsto A + BK$  in the proof of Proposition 2.  $\blacksquare$

Neither Proposition 2 nor Corollary 1 quite cover our requirements. In both of them we considered the simpler case of a zonotopic set centered in the origin ( $c = 0$ ). This makes sense when the dynamics' equilibrium point is also the origin but it may lead to sets which are conservative wrt the region in which they have to be included.

To handle this situation, we extend the control law to the affine case and choose the zonotope's center accordingly in the next result.

**Corollary 2.** *For dynamics (9), assuming the affine control law*

$$u = Kx + \gamma, \quad (20)$$

*the scaled zonotopic set  $Z([I_n - (A + BK)]^{-1}B\gamma, G\Delta)$ , given as in (18), is invariant iff relations (19) hold.  $\square$*

*Proof:* The zonotope's center,  $[I_n - (A + BK)]^{-1}B\gamma$ , is a fixed point for dynamics (9) under control law (20). Hence, we make the change of coordinates  $x \mapsto \tilde{x} + [I_n - (A + BK)]^{-1}B\gamma$  which brings the closed-loop system to the form  $\tilde{x}^+ = (A + BK)\tilde{x}$  and the zonotopic set to the form  $Z(0, G\Delta)$ . Applying Corollary 1 concludes the proof.  $\blacksquare$

*Illustrative example*

Taking the randomly-generated generator matrix

$$G = \begin{bmatrix} -15 & -20 & 33 & -17 & -5 \\ -14 & 24 & -7 & -48 & -18 \\ 19 & 03 & 10 & -20 & -8 \\ 6 & -8 & -38 & -23 & 15 \end{bmatrix} \cdot 10^{-2}$$

and applying Proposition 2, adapted via Remark 4, we obtain the static gain  $K = [-0.52 \quad -1.55]$  which makes invariant the linear dynamics defined by:

$$A = \begin{bmatrix} 0.38 & 0.76 \\ 0.16 & 0.87 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

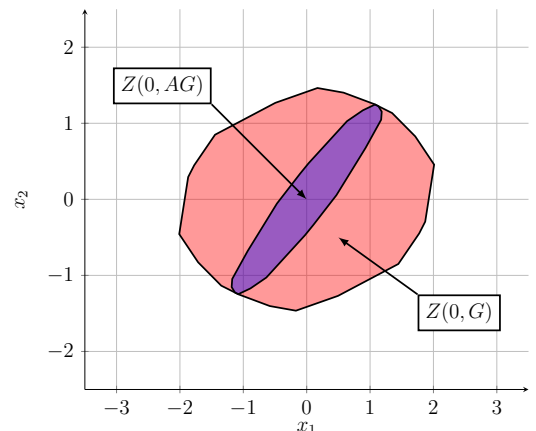


Fig. 1: Zonotope and its invariant set

Figure 1 shows the evolution of the zonotope in one step thus graphically proving its invariance under the aforementioned dynamics.

### B. Model Predictive Control

Consider the discrete-time dynamic

$$x_{i+1} = f(x_i, u_i), \quad (21)$$

where  $x_i \in \mathbb{R}^d$  denotes the state vector, and  $u_i \in \mathbb{R}^m$ , the input vector. Function  $f(\cdot, \cdot, \cdot) : \mathbb{R}^{n+m+p} \mapsto \mathbb{R}^n$  makes use of all these elements to update the state.

The control action  $u_i$  is obtained hereinafter by repeatedly solving a model predictive control (MPC) problem. MPC is one of the most popular techniques in control due to its versatility and relative ease of implementation [2]. In its standard form it reduces to solving a constrained optimization problem of the form

$$\min_{\bar{u}_0, \dots, \bar{u}_{N-1}} \sum_{k=0}^{N-1} \ell(\bar{x}_k, \bar{u}_k, r_k) + T(\bar{x}_N, r_N) \quad (22a)$$

$$\text{s.t. } \bar{x}_{k+1} = f(\bar{x}_k, \bar{u}_k), \forall k = 0, \dots, N-1, \quad (22b)$$

$$\bar{x}_0 = x_i, \quad (22c)$$

$$\bar{x}_N \in S, \quad (22d)$$

$$\bar{u}_k \in U, \bar{x}_{k+1} \in X, \forall k = 0, \dots, N-1. \quad (22e)$$

For further use, we note that  $U$  is described by the  $n_U$  pairs  $(a_{U,i}^\top, b_{U,i}) \in \mathbb{R}^m \times \mathbb{R}$  and  $X$  by the  $n_X$  pairs  $(a_{X,i}^\top, b_{X,i}) \in \mathbb{R}^n \times \mathbb{R}$ , as in (1).

The constrained optimization problem (22):

- predicts a sequence of states  $\{\bar{x}_1, \dots, \bar{x}_N\}$ , updated through dynamics (22b) by a suitable (resulted from solving (22)) sequence of inputs  $\{\bar{u}_0, \dots, \bar{u}_{N-1}\}$ ;
- such that it minimizes a stage cost  $\ell(\bar{x}_k, \bar{u}_k, r_k)$  (which usually penalizes a combination of states, inputs and references) and a terminal cost  $T(\bar{x}_N, r_N)$  (penalizing the last predicted state);
- while also verifying stage input/state constraints (22e) and a terminal state constraint (22d);
- (22) is integrated in the control scheme, by: i) initializing the prediction with the current plant state value (constraint (22c)), and, ii) applying to the plant dynamics (21) the first element of the predicted input sequence  $u_i \leftarrow \bar{u}_0$ .

While MPC often works well in practice, it still has the risk of failing to find a solution (i.e., (22) is infeasible). The standard approach, [6], is to provide a terminal cost and a terminal set inclusion condition which is guaranteed to hold under a local control law  $u = \phi(x)$ :

$$\bar{x}_{N+j} \in S \implies \bar{x}_{N+j+1} = f(\bar{x}_{N+j}, \phi(\bar{x}_{N+j})) \in S, \forall j \geq 0. \quad (23)$$

The difficulty in (23) is usually that neither the terminal set  $S$  nor the local control law  $\phi(\cdot)$  are a priori known. Usually, either one is fixed in order to obtain the other or some limiting assumptions have to be made (e.g., on the shape of the set).

In the rest of this section we provide a relaxation of the terminal condition (22d) by making use of the zonotopic invariance results from Section III-A. The idea is straightforward. Instead of pre-computing either/both of the terminal set and local control law, we replace (22d) with a collection of inequalities which, implicitly, provide the required elements, as shown in the next proposition.

**Proposition 3.** For  $\bar{x}_N$ ,  $U$  and  $X$  given as in (22), and with the linearizations  $A := \partial f(x, u)/\partial x$ ,  $B := \partial f(x, u)/\partial u$  in some suitably chosen fixed point, consider the constraints

$$\bar{x}_N - \tilde{x}_N = c, \quad (24a)$$

$$\tilde{x}_N = G\Delta\lambda, |\lambda| \leq 1, \quad (24b)$$

$$(A + BK)G\Delta = G\Delta\Lambda, (|\Lambda| - I_D) \cdot \mathbf{1}_D \leq O_{D \times 1}, \quad (24c)$$

$$\sum_j |a_{X,i}^\top G_j| \delta_j \leq b_{X,i} - a_{X,i}^\top c, \forall i = 1 : n_X, \quad (24d)$$

$$\sum_j |a_{U,i}^\top K G_j| \delta_j \leq b_{U,i} - a_{U,i}^\top c_U, \forall i = 1 : n_U, \quad (24e)$$

with the shorthand notations  $c = [I_n - (A + BK)]^{-1} B\gamma$  and  $c_U = Kc + \gamma$ .

If (24) holds, then there exist an affine law as in (20) and a zonotopic set as in (18) such that the set is made invariant for the considered linearization and the state and control action are admissible.  $\square$

*Proof:* The proposition simply gathers together the elements detailed earlier. (24a) makes a change of coordinates as in Corollary 2. (24c) ensures the set's invariance. (24d) and (24e) ensure admissibility by testing inclusion (22e) through (8), i.e., that inclusions  $Z(c, G\Delta) \subseteq X$  and  $Z(Kc + \gamma, KG\Delta) \subseteq U$  hold. Lastly, (24b) ensures the inclusion of the terminal state into the set (similar with (22d)), thus concluding the proof.  $\blacksquare$

**Remark 5.** Considering linearized dynamics in Prop. 3 sidesteps the difficult question of invariance for nonlinear dynamics. The approach is often implemented for the terminal set construction in MPC with the linearization errors as an additional source of uncertainty.  $\blacklozenge$

**Remark 6.** Equations (24) represent the most complete implementation wrt the results provided in Section III-A but at the price of nonlinear terms. The linear implementation, where the center is the origin ( $c = 0$ ) and the scaling factors are unitary ( $\Delta = \delta \cdot I$ ) is given by relations:

$$\bar{x}_N = G\lambda, |\lambda| \leq \delta^{-1} \cdot \mathbf{1}, \quad (25a)$$

$$(A + BK)G = G\Lambda, (|\Lambda| - I_D) \cdot \mathbf{1}_D \leq O_{D \times 1}, \quad (25b)$$

$$\sum_j |a_{X,i}^\top G_j| \leq \delta^{-1} b_{X,i}, \forall i = 1 : n_X, \quad (25c)$$

$$\sum_j |a_{U,i}^\top K G_j| \leq \delta^{-1} b_{U,i}, \forall i = 1 : n_U. \quad (25d)$$

The formulation is linear in  $\delta^{-1}$ .  $\blacklozenge$

#### IV. MULTICOPTER TRANSLATIONAL DYNAMICS APPLICATION

Next, we adapt the MPC construction (22) by replacing the terminal cost with the relations from either Prop. 3 or Rem. 6 for the translational dynamics of a multicopter [7].

##### A. Multicopter Dynamics

The translational dynamics of a multicopter system, given in an inertial coordinate system, are represented by the nonlinear relations

$$\dot{\xi}(t) = A\xi(t) + h_\psi(u(t)), \quad (26)$$

where:

$$A = \begin{bmatrix} O_3 & I_3 \\ O_3 & O_3 \end{bmatrix}, \quad h_\psi(t) = \begin{bmatrix} O_{3 \times 1} \\ T(c\phi s\theta c\psi + s\phi s\psi) \\ T(c\phi s\theta s\psi - s\phi c\psi) \\ -g + Tc\phi c\theta \end{bmatrix} \quad (27)$$

with the system's state (position plus velocity),  $\xi = [x \ y \ z \ \dot{x} \ \dot{y} \ \dot{z}]^\top$  and  $u(t) = [T \ \phi \ \theta]^\top$ , the input, where  $T$  is the normalized thrust,  $\phi$  is the roll angle, and  $\theta$  is the pitch angle. The yaw angle ( $\psi$ ) of the multicopter is considered a known (measured) constant.

**Remark 7.** *Translational dynamics (26) represent only the upper stage of the quadcopter's control stack. Specifically, the thrust is applied directly to the multicopter but the roll, pitch and yaw angles are given as references to be tracked at the lower level (which handles the rotational dynamics). The division is not simply about different time scales but also at the hardware implementation level: virtually any multicopter system implements the control loop (the "autopilot") of the rotational dynamics onboard, limiting (if not outright denying) access to it. ♦*

##### B. MPC implementation

We adapt the generic MPC problem from (22) to the specifics of the multicopter scheme:

- constructing a path as in [18], we track it via the stage cost  $\ell(\bar{x}_k, \bar{u}_k, r_k) = \|x_k - r_k\|_Q + \|u_k - u_{k-1}\|_R$  and the terminal cost  $T(\bar{x}_N) = \|\bar{x}_N - r_N\|_P$ ;
- the dynamics for model prediction are those from (26), discretized by a Runge-Kutta rule of order 4 and sampling time  $T_e$ ;
- the bounds are considered only on the input magnitude and the rate of variation:  $U = \{u \in \mathbb{R}^3 : \underline{u} \leq u \leq \bar{u}, |\Delta u| \leq \overline{\Delta u}\}$ ,  $X = \mathbb{R}^6$ .  $\Delta u$  denotes the input's variation between consecutive time instants.

The result is a standard output tracking scheme (the reference provides the "position" component of the state) with the main claim of novelty being the variable terminal set implementation from Proposition 3 / Remark 6.

For the subsequent analysis (both in simulation and experiment), we consider the model and MPC parameters from Table I.

To illustrate the tracking MPC approach proposed in Section III-B for the drone dynamics shown in Section IV

Parameter	Value
Crazyflie weight	$m = 0.035\text{kg}$
gravitational acc.	$g = 9.81\text{m}/\text{sec}^2$
input upper bounds	$\bar{u} = [\bar{T} \ \bar{\phi} \ \bar{\theta}] = [2g \ 10^\circ \ 10^\circ]$
input lower bounds	$\underline{u} = [\underline{T} \ \underline{\phi} \ \underline{\theta}] = [0 \ -10^\circ \ -10^\circ]$
input rate bounds	$\Delta u = [\Delta T \ \Delta \phi \ \Delta \theta] = [2g \ 10^\circ \ 10^\circ]$
cost matrices	$Q = \text{diag}(I_3, O_3)$ , $P = 10Q$ , $R = I_3$
prediction horizon	$N = 20$
sampling time	$T_e = 0.1\text{sec}$
generator matrix	$G = I$

TABLE I: Parameters for the Crazyflie model and the MPC controller.

we consider an experimental setup in which we test the performance of the control scheme (and show the same in simulation).

##### C. Experimental Setup

For ground-truth validation and to obtain an accurate estimation of the state we make use of a state-of-the-art indoor positioning system. We use an ensemble of 7 Miquis M3 cameras from Qualisys, which cover a volume of  $5 \times 5 \times 2.5$  meters and provide accurate position and Euler angles estimations for passive markers mounted on the drone (with precision error  $\leq 0.5$  mm). As aerial platform we use the Crazyflie nano-drone, an open-source system mainly used for indoor research purposes [19]. This drone is based on a STM32F405 microcontroller which has a Cortex-M4, with 196kB of RAM memory.

The Qualisys cameras emit infrared pulses which are reflected back by the asymmetrically drone-mounted markers, allowing the system to provide 6 DOF (position + Euler angles) estimation at a frequency of 50 Hz. The control action is the result of solving an optimization problem (implemented with the CasADi toolbox [20] in a Python script, running on a computer connected to the hardware installation). The command (thrust and reference pitch, roll and yaw rate) is sent forward to the drone via the CrazyRadio hardware interface, at a frequency of 10 Hz.

##### D. Results and analysis

We illustrate the NMPC-computed trajectory (reference and experimental data) in Fig. 2. The trajectory resulted in simulation closely tracks the reference, so is not depicted.

The histograms of the tracking error and computation times are depicted in Fig. 3. Fig. 3a provides the histogram of the tracking errors, i.e., the distance between the experiment trajectory and reference:  $\|\xi(1 : 3) - \bar{\xi}(1 : 3)\|_2$ . Fig. 3b illustrates the distribution of the computation times for the NMPC problem. We observe that great majority of the values are less than 0.03 sec, which is manageable for the sampling time,  $T_e = 0.1$  sec, used in the experiment to update the control action.

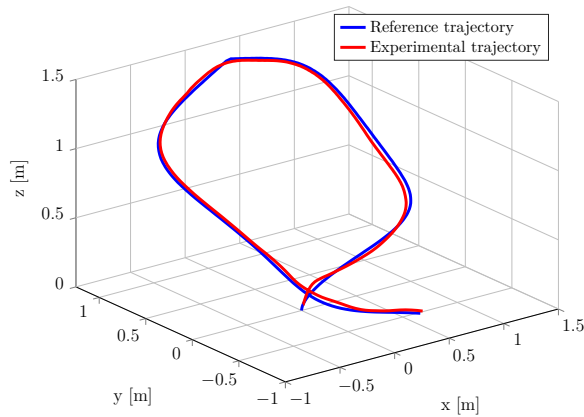
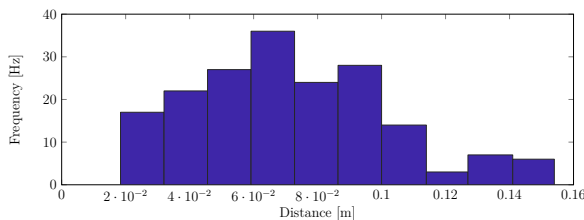
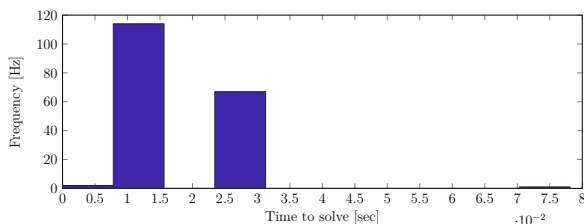


Fig. 2: Experimental results trajectory tracking

Lastly, the terminal set conditions were implemented as in Remark 6, with the nonlinear dynamics linearized at the hovering point (where the thrust cancels the weight and the roll and pitch are at zero). To the invariance condition, a slack term (penalized in the cost) was added to avoid infeasibility due to initial conditions. A suitable static feedback law was obtained, thus certifying the soundness of the approach.



(a) tracking errors



(b) computation times of MPC problem

Fig. 3: Numerical results of the experiment

## V. CONCLUSION

We provided conditions which simultaneously check zonotopic set invariance and provide an affine control law. These were integrated in a standard MPC problem to define implicitly the local admissible control law and the associated terminal set. The results have been validated in simulation and experiment for the translational dynamics of a nano-multicopter system.

with the standard, fix terminal set implementation.

Future improvements will further implement and analyze theoretical guarantees for stability and recursive feasibility, not least, to provide meaningful comparisons

Moreover, we plan to explore zonotopes' combinatorial structure to both reduce the computational load and provide a method of generator selection (not an obvious step at higher dimensions).

## REFERENCES

- [1] M. Hassanalain and A. Abdelkefi, "Classifications, applications, and design challenges of drones: A review," *Progress in Aerospace Sciences*, vol. 91, pp. 99–131, 2017.
- [2] S. V. Raković and W. S. Levine, *Handbook of model predictive control*. Springer, 2018.
- [3] H. J. Ferreau, C. Kirches, A. Potschka, H. G. Bock, and M. Diehl, "qpOASES: A parametric active-set algorithm for quadratic programming," *Mathematical Programming Computation*, vol. 6, no. 4, pp. 327–363, 2014.
- [4] S. Vargas, H. M. Becerra, and J.-B. Hayet, "MPC-based distributed formation control of multiple quadcopters with obstacle avoidance and connectivity maintenance," *Control Engineering Practice*, vol. 121, p. 105054, 2022.
- [5] N. T. Nguyen, I. Prodan, and L. Lefèvre, "Stability Guarantees for Translational Thrust-Propelled Vehicles Dynamics Through NMPC Designs," *IEEE Transactions on Control Systems Technology*, vol. 29, no. 1, pp. 207–219, 2020.
- [6] H. Chen and F. Allgower, "A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability," *Automatica*, vol. 34, no. 10, pp. 1205–1217, 1998.
- [7] N. T. Nguyen and I. Prodan, "Stabilizing a multicopter using an NMPC design with a relaxed terminal region," *IFAC-PapersOnLine*, vol. 54, no. 6, pp. 126–132, 2021.
- [8] D. Simon, J. Löfberg, and T. Glad, "Reference tracking MPC using terminal set scaling," in *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*. IEEE, 2012, pp. 4543–4548.
- [9] —, "Reference tracking MPC using dynamic terminal set transformation," *IEEE Transactions on Automatic Control*, vol. 59, no. 10, pp. 2790–2795, 2014.
- [10] M. Althoff, O. Stursberg, and M. Buss, "Computing reachable sets of hybrid systems using a combination of zonotopes and polytopes," *Nonlinear analysis: hybrid systems*, vol. 4, no. 2, pp. 233–249, 2010.
- [11] G. Bitsoris, "Positively invariant polyhedral sets of discrete-time linear systems," *International Journal of Control*, vol. 47, no. 6, pp. 1713–1726, 1988.
- [12] S. Sadraddini and R. Tedrake, "Linear encodings for polytope containment problems," in *2019 IEEE 58th Conference on Decision and Control (CDC)*. IEEE, 2019, p. 4367–4372.
- [13] V. Raghuraman and J. P. Koeln, "Set operations and order reductions for constrained zonotopes," *Automatica*, vol. 139, p. 110204, 2022.
- [14] G. M. Ziegler, *Lectures on polytopes*. Springer Science & Business Media, 2012, vol. 152.
- [15] K. Fukuda, "Polyhedral computation," 2020, publisher: Department of Mathematics, Institute of Theoretical Computer Science ETH Zurich.
- [16] D. Ioan, I. Prodan, F. Stoican, S. Oлару, and S.-I. Niculescu, "Complexity bounds for obstacle avoidance within a zonotopic framework," in *American Control Conference (ACC)*. IEEE, 2019, pp. 335–340.
- [17] F. Blanchini and S. Miani, *Set-theoretic methods in control*. Springer, 2008, vol. 78.
- [18] V. Marguet, B. Gheorghe, I. Prodan, and F. Stoican, "Reliable motion planning and coordination for a team of aerial drones," in *Journée des doctorants 2022*, 2022.
- [19] W. Giernacki, M. Skwierczyński, W. Witwicki, P. Wroński, and P. Koziński, "Crazyflie 2.0 quadrotor as a platform for research and education in robotics and control engineering," in *22nd International Conference on Methods and Models in Automation and Robotics (MMAR)*. IEEE, 2017, pp. 37–42.
- [20] J. A. Andersson, J. Gillis, G. Horn, J. B. Rawlings, and M. Diehl, "CasADi: a software framework for nonlinear optimization and optimal control," *Mathematical Programming Computation*, vol. 11, no. 1, pp. 1–36, 2019.