



HAL
open science

New probabilistic methods for physics

Guy Cirier

► **To cite this version:**

| Guy Cirier. New probabilistic methods for physics. 2024. hal-04859700

HAL Id: hal-04859700

<https://hal.science/hal-04859700v1>

Preprint submitted on 30 Dec 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

New probabilistic methods for physics

Guy Cirier

LSTA, University Pierre et Marie Curie Sorbonne, France

Email: guy.cirier@gmail.com

Résumé

On présente une méthode probabiliste pour étudier les itérations. Cette méthode est utilisée pour des EDO et est appliquée pour analyser les comportements asymptotiques en mécanique ou en physique.

Abstract

We present how a probabilistic model can describe the asymptotic behavior of the iterations, with applications for ODE and approach of some problems in mechanics in \mathbb{R}^d .

Keywords

Perron-Frobenius's measure, ODE, Poincaré-Bendixon problem, Lorenz's equations, self-similarity, Hamiltonian

Introduction

Since a long time, probabilistic methods are used in physics. They concern generally a great quantity of objects having the same behavior (particles, atoms, stars...). Then statistical methods can be applied as the number of objects is very great. These results are well known. So, Boltzmann has developed the concept of ergodism for study the kinetic theory of the gaz. Roughly speaking, in mathematics, the ergodic hypothesis consists to write that the time average equals spatial mean. Then, a probability of presence can be defined in the space. Many theoretical results exist, but the hypothesis is often open to debates.

But the notion of probability of presence can be defined intrinsically, without ergodism. We consider an invariant measure P under a measurable function f :

P is invariant under f if, for all borelian set B , P verifies the Perron-Frobenius's equation PF (See Lasota & Mc Kay):

$$P_f(B) = P \circ f^{-1}(B) = P(B)$$

This equation doesn't suppose that the borelians are bounded. However, with this choice, many demonstrations are simplified and it assumes easily the convergence of the studied series. Effectively, if f is iterated indefinitely in a bounded or compact set, the probability of presence get a sense with the more or less important density of points in some places. If such a P can be defined explicitly, we get important information about the objects verifying the f iteration. But this functional equation is not easy to study. However, this is the subject of the present paper.

So, f is supposed be a function applying a set $C \subset \mathbb{R}^d$ bounded in itself. This condition is verified for all phenomena of the infinitely small or for many phenomena on the earth. Let $\Delta \subset C$ the domain of this invariant measure associated. Naturally, iterations of f means $f^{(k)} = f \circ f \circ \dots \circ f$ for every k . Let the measure P invariant: $P_f(B) = P \circ f^{-1}(B) = P_{f^{(k)}}(B)$.

But, if Δ_k is the domain of $P_{f^{(k)}}$: $\Delta \subset \Delta_{k-1} \subset \Delta_k \subset C$, because $B \subset f^{-1} \circ f(B)$. All these bounded domains have points of accumulation in $C \subset \mathbb{R}^d$. Implicitly, we suppose that the iteration starts from a point in C . A function f may have many invariant domains.

Main results concerning physics:

This method brings new explications for many bounded phenomena:

- Under general conditions, this measure exists, in particular if f is C^∞ and applies a bounded set $C \subset \mathbb{R}^d$ in itself. It implies self-similarity in some cases.
- We consider a differential ordinary equation $da/dt = F(a)$ as an iteration $f: f(a) = a + \delta F(a)$ where $\delta = t/n$ is the path. if f is C^∞ and applies a bounded set $C \subset \mathbb{R}^d$ in itself, the final behavior is almost asymptotically cyclical and periodic.
- But, there are some levels hard to cross continuously due to the Fredholm's alternatives. At that time, breakings of the behavior can appear. The Hamilton's equation is reviewed. The n-bodies problem has new perspective.

The conditions of these results will be clarified in the following paper.

So, many results for physics may be obtained by this method and seemed ignored by scientists.

The paper is presented in two parts with chapters:

Part I: Invariant measure of Perron-Frobenius

Part II: Differential ordinary equations as iterations

Part I: Invariant measure of Perron-Frobenius

A- Laplace-Fourier's transform

1- Laplace-Fourier's transform of the invariant measure P

Hypothesis

We suppose that $C \subset \mathbb{R}^d$ is bounded. f applies C in C and f at least C^∞ .

Here, we seek an analytic approach of the f -invariant measure P with the Fourier-Laplace's transform. We use the known property of this measure: for all positive P -measurable function g , we have the formula (see Lasota & Mc Kay):

$$\int g \circ f(x) dP(x) = \int g(x) dP(x).$$

For $g(x) = e^{yx}$, we write the Fourier-Laplace's transform $\phi(y) = \mathcal{L}(e^{yX}) = E(e^{yX})$. As $x \in C \subset \mathbb{R}^d$ is bounded, $|x| < D = \text{diam}(C)$, we develop $\phi(y)$ with series $\phi(y) = \sum_n b_n y^n$.

It is convergent because $|b_n| < E(\frac{|x|^n}{n!}) < D^n/n!$. Let $\phi_f(y) = \mathcal{L}(e^{yf(X)})$. The second series is also convergent because $|f(x)| < D$. If the measure P is invariant:

$$\phi(y) = \phi_f(y)$$

Here, $y \in \mathbb{R}^d$ ou \mathbb{C}^d . If $y = it$, $\phi(y)$ is the characteristic function of the measure P . In this case, we recall that equality or convergence of the characteristic functions implies equality or convergence law of probability. This will be often implied in this paper.

Translation of the distribution with a small fixed vector $a \in \mathbb{R}^d$, $X \mapsto X + a$.

We translate X : $\phi(y, a) = E(e^{y(X+a)})$.

And by f -transformation: $\phi(y, a) \mapsto \phi_f(y, a) = E(e^{yf(X+a)})$.

As the measure is invariant: $\phi(y, a) = \phi_f(y, a)$.

And: $\theta_f(y, a) = \phi(y, a) - \phi_f(y, a) = 0$

Proposition :

$\theta_f(y, a) \equiv 0$ is an identity and $\partial^p \partial^q \theta_f(y, a) / \partial y^p \partial a^q \equiv 0$ for all p and q . For $\forall a \in C$ and $\forall y$:

$$\theta_f(y, a) = \Sigma_n b_n \partial^n (e^{ya} - e^{yf(a)}) / \partial a^n \equiv 0.$$

■ For $a = 0$, $\theta_f(y, 0) \equiv 0$. For $\forall a \neq 0$, $e^{-ya} \theta_f(y, a) = \emptyset(y, 0) - E(e^{yf(X+a)-ya}) = 0$. So $\theta_f(y, a) \equiv 0$ is an identity.

If the random vector $X \in C \subset \mathbb{R}^d$ has a measure P with density $p(x)$, the translated random variable $X + a$ has the same density for every small translation $a \in C \subset \mathbb{R}^d$ of the random vector X . Using the convergent series $\emptyset(y) = \Sigma_n b_n y^n$, we have the translated density:

$$p(x - a) = \mathcal{L}^{-1}(e^{ta} \emptyset(t)) = \mathcal{L}^{-1}(\Sigma_n b_n t^n e^{ta}).$$

So, we can write $p(x - a)$ as a distribution in the sense of Schwartz with the Dirac's δ :

$$p(x - a) = \Sigma_n b_n \partial^n \delta(x - a) / \partial a^n.$$

As: $E(e^{y(X+a)}) = \Sigma_n b_n y^n e^{ya} = \Sigma_n b_n \partial^n e^{ya} / \partial a^n$.

And: $E(e^{yf(X+a)}) = \int e^{yf(x+a)} dP(x) = \int e^{yf(x)} p(x - a) dx$,

$$E(e^{yf(X+a)}) = \Sigma_n b_n \partial^n (\int e^{yf(x)} \delta(x - a) dx) / \partial a^n,$$

$$E(e^{yf(X+a)}) = \Sigma_n b_n \partial^n e^{yf(a)} / \partial a^n.$$

By difference, we get: $\theta_f(y, a) = \Sigma_n b_n \partial^n (e^{ya} - e^{yf(a)}) / \partial a^n \equiv 0$. ■

Notations

We call $\theta_f(y, a) \equiv 0$ resolving equation R_a of PF and $e^n(y, a) = \partial^n (e^{ya} - e^{yf(a)}) / \partial a^n$ gap of order n :

$$\theta_f(y, a) = \Sigma_n b_n e^n(y, a) \equiv 0$$

If $a = 0$: $\theta_f(y) = \theta_f(y, 0) = \Sigma_n b_n e^n(y, 0) \equiv 0$.

Remarks

- The solution of the Perron-Frobenius's equation is a particular case of $\theta_f(y) = 0$.

- We observe that $\partial^n (e^{yf(a)}) / \partial a^n = H_n(y, a) e^{yf(a)}$ where $H_n(y, a)$ is a Bell-polynomial in y with degree n even f is C^∞ or analytic. We can note the gap:

$$e^n(y, a) = \partial^n (e^{ya} - e^{yf(a)}) / \partial a^n = y^n e^{ya} - H_n(y, a) e^{yf(a)}$$

And, for $a = 0$: $e^n(y, 0) = y^n - H_n(y)$ is a polynomial in y with degree n .

- We obtain $\emptyset_f(y)$ by putting $H_n(y)$ instead of y^n in the series of $\emptyset(y) = \Sigma_n b_n y^n$.

- $\emptyset_f(y)$ is obtained with taking $H_n(y)$ instead of y^n in the series $\emptyset(y) = \Sigma_n b_n y^n$.

- If x and $x + a \in C \subset \mathbb{R}^d$, the series are convergent.

- If $a = 0$, then: $\emptyset(0) = 1$, $\emptyset_f(0) = 1 = e^{yf(0)}$, $\forall y$ and $f(0) = 0$. The point of reference is a fixed point. If we take $f^{(k)}$ instead of f , the point of reference, yet denoted 0, is a point of a cycle. These points, and the eigen values λ of the linear part of f at these points, are most significant for the study of the convergence of the process. They are well known for the linearization of f . That means: find a function φ such $\varphi \circ f = \lambda \varphi$. A big problem is linked to the eigen values λ in the resonance's case $\lambda^n = 1$, not studied here.

- If $y = 0$, then: $\emptyset_f(0) = \emptyset(0) = 1$ and $b_0 = 1$. But, the other b_n are unknown.

2- The lattice distribution of probability

Proposition

The general solution of the linear equation $\theta_f(y) = 0$ is $\emptyset(y) = 1 + b\varphi(y)$ with an arbitrary constant b . It means that $\emptyset(y) = 1$, for all $\varphi(y) = 0$. We have a lattice distribution of probability for $\varphi(y) = 0$.

■ $\emptyset(y)$ can be written $\emptyset(y) = 1 + b\varphi(y)$ with $\varphi(0) = 0$. The linearity of $\theta_f(y) \equiv 0$ implies:
 $b\varphi(y) - b\varphi_f(y) = b(\varphi(y) - \varphi_f(y)) \equiv 0, \text{ for } \forall b$. So: $\emptyset(y) = 1 + b\varphi(y), \forall b$. ■

Iteration induces derivation on $\theta_f(y, 0) = 0$

Proposition

If $\lambda_\ell \neq 1, \ell = 1, \dots, d$, the iteration $f_\ell(a)$ acts as a derivation on $\theta_f(y, 0) = 0$ in the sense:

$$a_\ell \mapsto f_\ell(a) \Rightarrow \theta_f(y, 0) \mapsto \partial\theta_f(y, a)/\partial a_\ell|_{a=0}$$

Denoted:

$$a \mapsto f(a) \Rightarrow \theta_f(y, 0) \mapsto \partial\theta_f(y, a)/\partial a|_{a=0}$$

By induction, all the coordinates of $n \in N^d$ in $e^n(y, 0)$ are: $n_1 = \dots = n_\ell \dots = n_d$ if there is no resonance $\lambda^n = 1$:

$$n = \dots = n_\ell \dots = n_d.$$

■ For all derivable function $g(a)$ in \mathbb{R} , such as $g(0) = 0$, we write $g(a) \sim a\partial g(a)/\partial a$ near 0.

Let a function $f(a)$ such as $f(0) = 0$ and $\frac{\partial f(a)}{\partial a}|_{a=0} = \lambda \neq 1$ at 0. Then, we have:

$$gf(a) \sim a\lambda \partial g(a)/\partial a|_{a=0}.$$

We apply this result to the impact of $a_\ell \mapsto f_\ell(a)$ on $D = \theta_f(y, f_\ell(a)) - \theta_f(y, a_\ell) = 0$, as all the other coordinates $\bar{a}_\ell \neq a_\ell$ remain fixed:

As:

$$\theta_f(y, a_\ell) = 0 :$$

when $a_\ell \rightarrow 0$

$$f_\ell(a) - a_\ell \sim a_\ell(\lambda_\ell - 1)$$

$$D \sim a_\ell(\lambda_\ell - 1)(\partial\theta_f(y, 0)/\partial a_\ell)$$

As an iteration of f is the iteration of each coordinate:

$$D \sim \prod_{\ell=1}^d a_\ell(\lambda_\ell - 1)(\partial\theta_f(y, 0)/\partial a). \blacksquare$$

B- Solution of the Perron-Frobenius's equation

Now: $n \in N^d$, with $n = n_1 = \dots = n_\ell \dots = n_d$. Let $\theta_{nf}(y) = \sum_{m \leq n} b_m e^m(y)$. We chose n sufficiently large to get uniformly the inequality for $\forall y \in \mathbb{R}^d$ bounded:

$$|\theta_{nf}(y) - \theta_f(y)| < \epsilon.$$

As $\theta_f(y) = 0$, we must search an estimator $\theta_{*n}(y)$ of the polynomial $|\theta_{nf}(y)| < \epsilon$. More, we have seen that $\theta_f(y) = 0$ defines a function $\emptyset(y) = 1 + b\varphi(y)$ with $\varphi(y) = 0$. $\varphi(y)$ is defined by the convergent series $\varphi(y) = \sum_n b_n y^n = 0$. So, we have also:

$$|\varphi_n(y) - \varphi(y)| < \epsilon,$$

with $\varphi_n(y) = \sum_{m \leq n} b_m y^m$. Similarly, the polynomial $|\varphi_n(y)| < \epsilon$ has non-null coefficients.

We have to find a solution φ_n of the inequalities $|\varphi_n(y)| < \epsilon$ and $|\varphi_n(y) - \varphi_{nf}(y)| < \epsilon$.

Under the condition that y is any root of $|\varphi_n(y)| < \epsilon$, we must have $|\theta_{*nf}(y)| < \epsilon$.

Lemma

We suppose that a solution of $\theta_{*nf}(y) = \sum_{m \leq n} b_{*m} e^m(y) = 0$ exists. Under non-resonance conditions, for all $b_n = b \neq 0$ fixed, the zeros of $e^n(y)$ are zeros of $\theta_{*n}(y)$.

■ The solution of this equation is obtained as the following:

Let:

$$\theta_{*n}(y) = \sum_{m \leq n} b_{*m} e^m(y) = 0.$$

- For $y = 0$: $b_{*0} = 1$. As $\theta_{*n}(y)$ is a polynomial, the condition $\theta_{*n}(y) = 0$ implies either all the coefficients of $\theta_{*n}(y)$ are null or the solution is valid only for the y verifying $\theta_{*n}(y) = 0$. But, as the term of highest degree of $\theta_{*n}(y)$ is: $(1 - \lambda^n) b_n y^n$, we must have, under non-resonance conditions and for all $b = b_n \neq 0$, $e^n(y) = 0$. (Because all the other gaps $e^m(y)$ have a lower total degree for all $m < n$). Then, zeros of $e^n(y)$ are zeros of $\theta_{*n}(y)$. ■

In order to verify that $e^n(y) = 0$ gives the solution, we have to calculate all the b^*_m , $m < n$, under the condition $e^n(y) = 0$. We put the arbitrary constant $b = b^*_n$.

Theorem

Under the non-resonance condition, we find a unique convergent solution of $\theta^*_n(y) = 0$, up to an arbitrary constant b :

$$\Phi_n(y) = 1 - be^n(y).$$

We obtain a lattice distribution defined by the zeros of $e^n(y)$.

In the repellent case where $\lambda^n \gg 1$, we have:

$$\Phi_n(y) \sim 1 - bH_n(y).$$

Then, the distribution of the real zeros of the polynomials $H_n(y)$ gives the distribution of the Perron-Frobenius's measure when $n \rightarrow \infty$.

We obtain a lattice distribution defined by the zeros of $H_n(y)$.

■ By definition: $\theta^*_n(y) = \sum_{m \leq n} b^*_m e^m(y) = 0$.

1 - Let the polynomials $\Phi^*_n(y) = 1 + \sum_{0 < m \leq n} b^*_m y^m$,

and $\Phi_f^*_n(y) = 1 + \sum_{0 < m \leq n} b^*_m H_m(y)$,

Where the coefficients of the Bell's polynomial $H_m(y) = \sum_{0 < k \leq m} h_{mk} y^k$ are known.

As $\theta^*_n(y) = \sum_{m \leq n} b^*_m e^m(y) = 0$, we note $A^*_{n-1}(y)$:

$$A^*_{n-1}(y) = \sum_{m < n} b^*_m e^m(y) = \theta^*_n(y) - b^*_n e^n(y).$$

So

$$A^*_{n-1}(y) = \theta^*_{n-1}(y) \text{ has a degree less than } n.$$

As $\theta^*_n(y) = 0$:

$$A^*_{n-1}(y) = -be^n(y) = -b \sum_{0 < k \leq n} h_{nk} y^k$$

But, $A^*_{n-1}(y) = \sum_{m < n} b^*_m e^m(y) = \sum_{m < n} b^*_m (y^m - \sum_{0 < k \leq m} h_{mk} y^k)$.

So, we develop $A^*_{n-1}(y)$ in terms of y^k with degree $k \leq m < n$:

$$A^*_{n-1}(y) = \sum_{m < n} b^*_m (y^m - \sum_{0 < k \leq m} h_{mk} y^k) = -b \sum_{0 < k < n} h_{nk} y^k$$

for a fixed arbitrarily $b^*_n = b \neq 0$.

2- If $e^n(y) = 0$, $A^*_{n-1}(y) = 0$ and we obtain a finite triangular system of linear equations: it can be solved step by step and all the unknown coefficients b^*_m can be defined in a unique way in function of b and the known coefficients h_{mk} of $H_m(y)$ with $m \leq n \in \mathbb{N}^d$.

3 - This solution is unique for all $b^*_n = b \neq 0$ arbitrarily fixed, near to the solution of $\theta_f(y) = 0$, as the b^*_m converge to the b_m . So, we have constructed the polynomials $\Phi^*_n(y) - 1$ and $\Phi_f^*_n(y) - 1$ and we can write $\Phi^*_n(y) = 1 + b\varphi_n(y)$ where b is arbitrary. Taking $\varphi_n(y) = e^n(y) = 0$, we have $|\theta^*_{nf}(y)| < \epsilon$.

Different cases can happen according to $\lambda^n \gg 1$ or $\lambda^n \ll 1$.

If all the coordinates of $|\lambda|$ are less than 1, the process converges to the fixed point.

If some of them are less than 1, but others are greater than 1, we have a hyperbolic situation under no resonance conditions.

When $y^n \lambda^n \gg y^n$, we can write for large n :

$$\Phi_n(y) \sim 1 - bH_n(y). \blacksquare$$

And now we have to study the zeros of $H_n(y)$.

C- Asymptotic density of the real zeros of $H_n(y)$

1- Study of the asymptotic density of the real zeros of $H_n(y)$

We study now the real zeros of $H_n(y)$ in order to get their asymptotic distribution. f is real, C^∞ , without resonance, and applies $C \subset \mathbb{R}^d$ bounded in itself. As each iteration induces a derivation on $H_n(y) = 0$, we take in general a common index $n = n_1 = n_\ell = \dots = n_d$.

We recall that the polynomial:

$$H_n(y) = e^{-yf(a)} \partial^n e^{yf(a)} / \partial a^n |_{a=0} = \partial^n e^{yf(a)} / \partial a^n |_{a=0}$$

can be represented by the Cauchy's integral:

$$H_{n-1}(y) = c \oint_{\Gamma} \frac{e^{yf(a)}}{a^n} da = c \oint_{\Gamma} e^{yf(a)-n \ln a} da$$

where Γ is a closed polydisk around the fixed point 0 of f , $a \in \mathbb{C}^d$, c is a finite non-null real function, without importance in the context. We take $n = n_1 = n_\ell = n_d$.

The integrand is: $n\gamma(a) = yf(a) - n \ln a$

If we note $y = ns$ ($s_\ell = y_\ell / n$): $\gamma(a) = sf(a) - \ln a$.

$\gamma(a)$ is called here the Plancherel-Rotach's function because they are the firsts to use this method to approximate the Hermite's polynomials. The method consists to search the critical point α of $\gamma(a)$. Under the numerous conditions of the general position, the critical point α maximizing $e^{n\gamma(a)}$ gives a first approximation of the integral. The critical point α is defined by the equation:

$$\partial\gamma(\alpha)/\partial a = s\partial f(\alpha)/\partial a - 1/\alpha = 0.$$

The nullify of the approximation gives us the real zeros of $H_n(y)$. Some coordinates of α can be real, the others are complex. We note $H_\infty(s) = 0$ the asymptotic distribution if it exists.

Main conditions to use this method

The critical point must be isolated from the other critical points, at a finite distance and far from the edge ∂C . A sufficient condition to get this maximum is that the hessian matrix of $\gamma(a)$, which is hermitian, is definite negative at α . that means $x[\partial^2\gamma(a)/\partial a^2] x < 0$. But, as the Morse's theory says that the hessian matrix has the form $\sum x'_\ell{}^2 - \sum x_\ell{}^2$ and the "positive" coordinates are not suitable. Moreover, when the hessian is not definite, we have many problems with the Stokes's phenomena or with the catastrophe theory.

At last, α must have complex coordinates.

2- The asymptotic density for f real

Theorem

Under the conditions of the general position, the critical point α of the PR-function must be complex to annul $H_n(y)$. Among the multiple critical points, first, we must choose α such as the real part of the Cauchy's integral is maximum.

- Then, only for all complex coordinates (denoted yet α) of α , we have:

$$\partial\gamma(\alpha)/\partial a = s\partial f(\alpha)/\partial a - 1/\alpha = 0.$$

$$n\Im(\gamma) = \Im(ns f(\alpha) - n \ln \alpha) = k\pi.$$

When $n \rightarrow \infty$, $k/n = \chi_n \rightarrow \chi$, with χ uniform on $[0,1]$ for each r complex coordinates.

The asymptotic distribution is:

$$\Im(sf(\alpha) - \ln \alpha) = \chi\pi. \quad \text{With } \kappa \in [0,1]^r$$

And: $\Im(\gamma_\ell(\alpha)) - \chi_\ell\pi = 0$.

The κ_ℓ are identically independent uniform on $[0,1]$.

- In the unidimensional case, the distribution of the zeros is:

$$q(s) ds = \Im(f(\alpha)) ds / \pi \text{ with } s\partial f(\alpha)/\partial a - 1/\alpha = 0$$

■ For all real coordinates of α , the exponential defining $H_{n-1}(y)$ cannot be cancelled. Only a complex coordinate can annul $H_{n-1}(y)$. Suppose there are r complex coordinates. So, as f has real coefficients, if $\partial\gamma(\alpha)/\partial a = 0$, then $\partial\gamma(\bar{\alpha})/\partial a = 0$ where $\bar{\alpha}$ is the conjugate of α . As in Plancherel- Rotach, the contribution to estimate $H_{n-1}(y)$ is:

$$H_{n-1}(y) = c \oint_{\Gamma} (\exp(n\gamma(a)) - \exp(n\gamma(\bar{a}))) da.$$

- As in the steepest descent, $\Im(n\gamma)$ is constant:

$$H_{n-1}(y) = 2c \sin(\Im(n\gamma)) \Re(\oint_{\Gamma} e^{n\gamma(a)} da)$$

$c\Re(\oint_{\Gamma} e^{n\gamma(a)} da)$ is real and cannot annul $H_{n-1}(y)$; but, among the solutions given by the critical equation, we must choose the maximum. Then, only the imaginary part $\Im(\gamma)$ of $\gamma(a)$ can nullify $H_{n-1}(y)$. That means:

$$\sin(\Im(n\gamma)) = 0 \text{ and } \Im(n\gamma) = k\pi$$

$$\Im(sf(\alpha)) - \theta = \pi k/n = \pi\chi_n$$

Writing: $\gamma_{\ell}(\alpha) = s_{\ell}f_{\ell}(\alpha) - \ln \alpha_{\ell}$, $\gamma(\alpha) = \sum_{\ell} \gamma_{\ell}(\alpha)$ and with: $\chi = \sum_{\ell} \chi_{\ell}$

$$\sum_{\ell} (n\Im(\gamma_{\ell}(\alpha)) - \chi_{\ell}\pi) = 0.$$

But, at time $n + 1_{\ell}$, where all the n except $n_{\ell} = n$ which becomes $n_{\ell} + 1_{\ell}$, the PR function becomes

$$\sum_{\ell} (n\Im(\gamma_{\ell}(\alpha)) - \chi_{\ell}\pi) + \gamma_{\ell}(\alpha) - \chi_{\ell}\pi = 0$$

Because $H_{n-1}(y) = 0$ becomes $H_{n-1+1_{\ell}}(y) = 0$.

then: $\Im(\gamma_{\ell}(\alpha)) - \chi_{\ell}\pi/n = 0$

When $n \rightarrow \infty$, $k_{\ell}/n \rightarrow \chi_{\ell} \in [0,1]$,

with κ_{ℓ} uniform on $[0,1]$ for each complex coordinate and we have $H_{\infty}(s)=0$.

- In the case of unidimensional function, the repartition of the zeros verifies for $\chi = k/n$:

$$\Im(\gamma(\alpha)) = \chi\pi \quad \text{with} \quad \partial\gamma/\partial\alpha = 0$$

So: $q(s) ds = \text{Prob}\{1 \text{ zero} \in (s, s + ds)\} = dq$

$$q(s) ds = \Im(d\gamma/ds) ds/\pi = \Im(f(\alpha)) ds/\pi.$$

Because: $d\gamma/ds = \partial\gamma/\partial s + \partial\gamma/\partial\alpha \cdot \partial\alpha/\partial s = f(\alpha)$. ■

Remarks

- As some coordinates of α can be real, then random manifolds can appear.

- These distributions of the PF-equation are linked to each fixed (or cycle) point $f(0) = 0$. Then, the solutions are local. They preexist independently of all convergence, as the fixed points, cycles or invariant varieties. The convergence to one of the asymptotic situations depends on the initial point.

- All these distributions can be masked in various situations. The principle of the maximum of the real part $\Re(\gamma)$ of $\gamma(a)$ provides a method to define the fuzzy frontiers of the different domains of attraction.

- In the case where $H_{n-1}(ns) = 0$, an iteration induces a derivation as we have seen for $\theta_f(y, 0) = 0$. The relation $k/n < \frac{(k+1)}{n+1} < (k+1)/n$ can explain that the zeros of H_n are interleaved when the reciprocal image s of $\Im(sf(\alpha)) - \theta = k\pi/n$ are monotonic.

- If f is complex, we can identify \mathbb{C} as \mathbb{R}^2 and the calculus are quite the same but more difficult.

For instance, if f has real coefficients, with s real, α and κ fix: $\gamma(a) = sf(a) - \ln a$. The distribution of zeros is defined by:

$$\Im(\gamma(\alpha)) = k\pi/n ; \partial\gamma(\alpha)/\partial\alpha = s\partial f(\alpha)/\partial\alpha - 1/\alpha = 0.$$

Let $\xi = e + id$ a complex number, we want to study the iterations of ξf . The equations of the distribution are now defined for the PR function at the critical point α' :

$$\gamma'(\alpha') = s'\xi f(\alpha') - \ln \alpha'$$

$$\Im(\gamma') = k'\pi/n \text{ and } \partial\gamma(\alpha')/\partial\alpha' = s'\xi\partial f(\alpha')/\partial\alpha' - 1/\alpha' = 0.$$

The steepest descent induces: $H_{n-1}(y) = 2c\sin(\Im(n\gamma')) \Re(\oint_{\Gamma} e^{n\gamma'(a)} da)$.

- Most important is the fact that iterations are asymptotically commutative as derivations.

- As everybody knows, the steepest descent's method is difficult to use, but it shows a very large variety of behaviors.

3- Relation between the distribution of the zeros of $H_n(y)$ and the invariant density

We calculate the distribution P with the asymptotic distribution of zeros of $H_n(y)$ in a

particular case. We simplify in supposing that we have n^d points $y_n \in \mathbb{R}^d$ verifying $H_n(y) = 0$. The Laplace's transform of the y_n is:

$$\Phi_n(y) = (1/n^d) \sum_n e^{-y y_n}$$

At each point y_n , correspond a point x in C with non-null density. But, we must normalize the coordinates of the zeros y , to get a distribution at finite distance: $n x_{n\ell} = y_{n\ell} = n s_\ell$. At the volume dy correspond a normalized volume dx : $dy = n^d dx$. So, we get $n^d q(x) dx$ points.

. When $n \rightarrow \infty$, we have:

$$\begin{aligned} \Phi_n(ns) &= (1/n^d) \sum_n e^{-s x_n} n^d q(x_n) dx_n \\ \Phi_n(ns) &\rightarrow \int e^{-s x} q(x) dx = \Phi(s) \end{aligned}$$

$\Phi_n(s)$ is the Laplace's transform of the density of zeros of $H_n(y)$.

$\Phi(s)$ is asymptotically the Laplace's transform of the invariant distribution $q(x)$ the density of zeros solution of $H_n(y) = 0$.

In fact, there are few phenomena where we have n^d points $y \in \mathbb{R}^d$ verifying $H_n(y) = 0$, except the case unidimensional because all the zeros of $H_n(y)$ are not real. Let $p(x)$ be the density of the measure invariant with regard to the Lebesgue's measure of \mathbb{R}^d ($dP(x) = p(x) dx$). What is the relation of $p(x)$ with the asymptotic density of zeros $q(x)$?

Proposition

Suppose that C is rectangular with null density $p(x)$ on ∂C . Then $p(x)$ is:

$$p(x) = (-x \partial q(x) / \partial x).$$

■ For all solution of $\theta(y, a) = 0$, the measure P must verify the identity $\theta(y, a) \equiv 0$, for $\forall y$ and

small $\forall a$. If $\varphi(y)$ is solution of $\theta(y, a) = 0$, then, the Laplace's transform must verify:

$$\varphi(y) = \partial^2 \Phi(y, a) / \partial a \partial y |_{a=0} = \partial(y \Phi(y)) / \partial y$$

The inverse transformation gets the result. ■

Remark: the method of the kernel for the invariant measure

This method seems complicated, but more productive than others more elegant but difficult to use as the following method of the kernel:

Let $\Phi(y)$ the Fourier-Laplace's transform of the invariant measure P corresponding to f .

$$\Phi(y) = \Phi_f(y)$$

$$\Phi_f(y) = E(e^{y f(x)}) = \mathcal{L}(e^{y f(x)}) = \int e^{y f(x)} dP(x).$$

But:

$$p(x) = \mathcal{L}^{-1}(\Phi(t)) = \frac{1}{(2\pi i)^d} \int_{c-i\infty}^{c+i\infty} e^{-tx} \Phi(t) dt.$$

So:

$$\Phi_f(y) = \int e^{y f(x)} dP(x) = \int e^{y f(x)} \frac{1}{(2\pi i)^d} \int_{c-i\infty}^{c+i\infty} e^{-tx} \Phi(t) dt dx$$

As $\Phi(y) = \Phi_f(y)$:

$$\Phi(y) = \frac{1}{(2\pi i)^d} \int \int_{c-i\infty}^{c+i\infty} e^{y f(x) - tx} \Phi(t) dt dx$$

If:

$$C(y, t) = \frac{1}{(2\pi i)^d} \int_{c-i\infty}^{c+i\infty} e^{y f(x) - tx} dx$$

We obtain the functional: $\Phi(y) = \int C(y, t) \Phi(t) dt$

Where $C(y, t)$ is the kernel for the eigen value 1. The steepest descent can be used to approximate $C(y, t)$, but it doesn't seem productive.

4- Analysis of the domination in the case of have many fixed points

When we have many fixed points, there is a problem to know the domain of attraction of each point. Let α and β two points of $Fix(f)$ in a bounded set $C \subset \mathbb{R}^d$. f applies $C \subset \mathbb{R}^d$ bounded in itself. Let $a \in C$:

$$a = \alpha + u = \beta + v$$

Changing the origin:

$$f_\alpha(u) = f(\alpha + u) - \alpha \text{ and } f_\beta(v) = f(\beta + v) - \beta$$

with $f(\alpha) - \alpha = f(\beta) - \beta = 0$. Let: $\gamma_\alpha(u) = sf_\alpha(u) - \ln u$ and: $\gamma_\beta(v) = sf_\beta(v) - \ln v$
 The critical point is defined by: $s\partial f(\alpha + u)/\partial u - 1/u = 0$ et $s\partial f(\beta + v)/\partial v - 1/v = 0$
 As $a = \alpha + u = \beta + v$: $v = u$

The steepest descent method that we must take the greatest real part of the integrand at the critical point. By difference, the sign of the real part of $s(f_\alpha(u) - f_\beta(u))$ at the critical point get the criterium. But this frontier is random.

5- Examples

- Let the logistic map [4]: $f(a) = \lambda a - a^2/2$; and $\gamma(a) = s(\lambda a - a^2/2) - \ln a$;
 $\partial\gamma/\partial a = s(\lambda a - a^2) - 1 = 0$

If the discriminant $\Delta = (s\lambda)^2 - 4s < 0$ and $s > 0$, we put $\lambda\sqrt{s} = 2\cos\vartheta$, we have:
 $2\cos\vartheta a\sqrt{s} - sa^2 - 1 = 0$ with roots: $a\sqrt{s} = e^{\pm i\vartheta}$

and: $\Im m(f(a)) = \Im m(\lambda a - a^2/2) = \sin 2\vartheta/s$.

$q(s) ds = \Im m(f(a)) ds/\pi = (1 - \cos 2\vartheta) d\vartheta/\pi$

So: $q(s) = (\lambda/2\pi)\sqrt{1/s - \lambda^2/4}$.

If we put $t = \cos\vartheta = \frac{\lambda\sqrt{s}}{2}$

Then t follows: $W(t) dt = (2/\pi)\sqrt{1 - t^2} dt$.

We recover directly a well-known result: Let $H_n(y, a) = \partial^n(e^{y(\lambda a - a^2/2)})/\partial a^n$

where $e^{y(\lambda a - a^2/2)}$ is (with easy transformations) like the generatrix function $e^{(2ta - a^2)}$ of the Hermite polynomials $H_n(t)$. The law of the zeros of $H_n(x)$ is known as the semi-circular Wigner's law: $W(t) dt = (2/\pi)\sqrt{1 - t^2} dt$.

- Then, the density of the logistic corresponding to $q(s)$ is $p(a)$:

$$p(a) = -adq/da = -a(2/\pi)d\left(\frac{\lambda}{4\sqrt{a}}\sqrt{1 - \frac{a\lambda^2}{4}}\right)/da$$

$$p(a) = \lambda/(2\pi\sqrt{4a - a^2\lambda^2}).$$

The density of the logistic map follows a Beta(1/2, 1/2) law in a more general situation than in the Ulam-Von Neumann's case.

- *m-Hermitian case*: $f(a) = \lambda a - a^m/m$.

The Plancherel-Rotach's function is: $\gamma(a) = s(\lambda a - a^m/m) - \ln a$.

With the critical point a defined by the trinomial equation:

$$d\gamma(a)/da = s(\lambda a - a^m) - 1 = 0$$

studied by H. Fell. We note that the discriminant $\Delta < 0$ may establish the existence of a complex root of this equation. We recall: $\Delta = m^m/s^{m-1} - (m-1)^{m-1}\lambda^m < 0$

- *Quadratic function*: We take now a quadratic function f in \mathbb{R}^d with $f(0) = 0$. We write the PR function $\gamma(a)$ for every fixed point 0 of $f(a)$:

$$f(a) = \lambda a + Qa^2/2$$

the hessian of sQ is symmetric. For all s such as sQ is non-degenerate, it exists an orthogonal transformation $T: a = Tu$, with: $T'sQT = D$, the diagonal matrix of eigenvalues of sQ and:

$$\ln a = \sum_{\ell=1}^{\ell=d} \ln a_\ell = \ln \prod_{\ell=1}^{\ell=d} a_\ell = \ln \text{Vol}(a) = \ln \text{Vol}(u) = \ln \prod_{\ell=1}^{\ell=d} u_\ell = \sum_{\ell=1}^{\ell=d} \ln u_\ell.$$

Because the volume $\text{Vol}(a) = \text{Vol}(u)$ is invariant under an orthogonal transformation.

Then, the P.R. function $\gamma(a)$ becomes:

$$\gamma(u) = sf(Tu) - \ln Tu = s\lambda Tu + Du^2/2 - \ln u.$$

We note $D = D^+$ if $\ell = 1, \dots, p$ and $D = D^-$ if $\ell = p + 1, \dots, d$.

$$\begin{aligned} &= \sum_{\ell=1}^{\ell=d} \Lambda_\ell u_\ell + \sum_{\ell=1}^{\ell=p} D^+_\ell u_\ell^2/2 - \sum_{\ell=p+1}^{\ell=d} D^-_\ell u_\ell^2/2 - \sum_{\ell=1}^{\ell=d} \ln u_\ell \\ &= \sum_{\ell=1}^{\ell=p} (\Lambda_\ell u_\ell + D^+_\ell u_\ell^2/2 - \ln u_\ell) + \sum_{\ell=p+1}^{\ell=d} (\Lambda_\ell u_\ell + D^-_\ell/2 u_\ell^2 - \ln u_\ell) \end{aligned}$$

where:

$$\Lambda u = s\lambda T u.$$

If we note:

$$\gamma_+(u_\rho) = \Lambda_\rho u_\rho + D^+_\rho u_\rho^2 / 2 - \ln u_\rho$$

and

$$\gamma_-(u_\rho) = \Lambda_\rho u_\rho + D^-_\rho u_\rho^2 / 2 - \ln u_\rho$$

So:

$$\gamma(u) = \sum_{\ell=1}^{\ell=p} \gamma_+(u_\rho) + \sum_{\ell=p+1}^{\ell=d} \gamma_-(u_\rho).$$

Applying the logistic calculus to each $\gamma_+(u_\rho)$ and $\gamma_-(u_\rho)$, we obtain p conditions $\Lambda_\rho u_\rho = 0$ half the time and $d-p$ random independent variables following a Beta (1/2,1/2) low. But, we may have other fixed points: $a(1 - \lambda) = Qa^2$.

These results can be extended to a C^∞ function f with the Morse-Palais Lemma as if the hessian is definite. We have studied f , we examine now $h = g \circ f$ and $f^{(p)}$.

D- Self similarity

1- Analysis of $h = g \circ f$ and applications to $f^{(p)}$

Let f and g two functions C^∞ which apply a bounded set $C \subset \mathbb{R}^d$ in itself. We search invariant measures of $h = g \circ f$ from the study of the invariant measures of f .

- We know $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. If the measure P is invariant under f , $P \circ f^{-1}(B) = P(B)$,

So :

$$P \circ (g \circ f)^{-1}(B) = P \circ f^{-1} \circ g^{-1}(B) = P \circ g^{-1}(B).$$

If g is only continuous, g is borelian, then, as g applies $C \subset \mathbb{R}^d$ in itself, $g^{-1}(B) = A$ is a borelian and:

$$P \circ (g \circ f)^{-1}(B) = P \circ f^{-1} \circ g^{-1}(B) = P \circ f^{-1}(A) = P(A).$$

- We have to search zeros of $H_{n-1,h}(y)$ for $h = g \circ f$ in relation whit those of $H_{n-1,f}(y)$ for f . We suppose that the fixed point 0 of f is fixed point of g : $g(0) = 0$.

For f , let α, s_f solution of $H_{n-1,h}(y) = 0$ completely defined for fixed χ by:

$$\mathfrak{I}m(\gamma_f(\alpha)) = \chi \pi \text{ et } \partial \gamma_f / \partial a = 0.$$

We have to search solution $(s_{g \circ f}$ et $\alpha_{g \circ f})$ for $h = g \circ f$.

Lemma

Under the previous hypothesis, if we choose $s_{g \circ f}$ verifying $\partial \frac{g(a)}{\partial a} |_{a=f(\alpha)} = s_f$, the critical complex point α of γ_f : $\alpha_f = \alpha$ remains the critical point $\alpha_{g \circ f} = \alpha$ for the iteration $g \circ f$.

Then, the distribution of $h = g \circ f$ provided by the reciprocal image of the distribution of f :

$$\mathfrak{I}m(\gamma_{g \circ f}(\alpha)) = \chi_{g \circ f} \pi$$

■ Let:

$$\gamma_f(a) = s_f f(a) - \ln a \text{ and } \gamma_{g \circ f}(a) = s_{g \circ f} g \circ f(a) - \ln a$$

The critical point α for f is defined by:

$$\partial \gamma_f / \partial a = 0,$$

And:

$$s_f \partial f(\alpha) / \partial a = 1 / \alpha.$$

The critical point $\alpha_{g \circ f}$ for $g \circ f$ is defined by:

$$\partial \gamma_{g \circ f} / \partial a = 0,$$

That means:

$$s_{g \circ f} \partial g \circ f(\alpha_{g \circ f}) / \partial a = 1 / \alpha_{g \circ f}.$$

But:

$$\partial g \circ f(\alpha_{g \circ f}) / \partial a = \partial g(f) / \partial f \cdot \partial f(\alpha_{g \circ f}) / \partial a.$$

If we take:

$$s_{g \circ f} \partial g(f) / \partial f = s_f,$$

Then:

$$s_f \partial f(\alpha_{g \circ f}) / \partial a = 1 / \alpha_{g \circ f}.$$

The critical point $\alpha_{g \circ f}$ verifies the critical equation of f , so $\alpha_{g \circ f} = \alpha$. ■

Remarks

Even if s_f is real, $s_{g \circ f}$ is not real because $\partial g(f) / \partial f$ depends on α . It is not a big problem because the steepest descent is valid for a complex function. But the complex part of $s_{g \circ f}$ can also be nullify. In order to have $\partial g(f) / \partial f$ independent from a , $\partial^2 g(f) / \partial a^2 \cdot \partial f(a) / \partial a = 0$.

The hessian of g must be degenerated at the critical point α . However, g may induce other solutions and linearities.

Corollary

$g = f^{(p)}$, then the critical point α of γ_f is critical point of $\gamma_{f^{(p)}}$, $\forall p$.

2- Analysis of $f^{(p)}$, $\forall p$

Some distributions can be linked to cycles of order p . By translation the origin at a point $0 \in \mathbb{R}^d$ of the cycle of $f^{(p)}$: $f^{(p)}(0) = 0$, the iteration is written:

$$h(a) = f^{(p)}(a) - f^{(p)}(0) = f^{(p)}(a) = f^{(p-1)} \circ f(a).$$

But, before all, we have to study the behavior of these distributions near a fixed realization of the asymptotic random vector χ uniform on $[0,1]^r$. This means that $\partial\gamma(\alpha)/\partial a = 0$ has complex solutions.

Lemma

For $\forall \mu \in \mathbb{Z}$, the asymptotic distribution of zeros of $H_n(\mu y)$ is the same than this of $H_n(y)$ and asymptotically, if $\mu \in \mathbb{R}$, we have the same result.

■ Let κ and α such as $H_\infty(s) = 0 : \partial\gamma(\alpha)/\partial a = s\partial f(\alpha)/\partial a - 1/\alpha = 0$

If μ is rational, $\mu = p/q$ with $p, q \in \mathbb{N}$, then: $n\partial\gamma(\alpha)/\partial a = y\partial f(\alpha)/\partial a - n/\alpha = 0$.

Now $n\partial\gamma_\mu(\alpha)/\partial a = \mu y\partial f(\alpha)/\partial a - n/\alpha = 0$

$$n\partial\gamma_\mu(\alpha)/\partial a = y\partial f(\alpha)/\partial a - nq/p\alpha = 0$$

If we take $n = pm$: $n\partial\gamma_\mu(\alpha)/\partial a = y\partial f(\alpha)/\partial a - qm/\alpha = 0$.

We obtain the same equation with qm instead of n and the same asymptotic distribution of zeros with $y = qms$.

Let $H_\infty(s) = 0$ with fixed α , s and κ fixed. As $H_{n-1}(s_n)$ is polynomial in s_n , the solution s_n of the equations $\partial\gamma_n(\alpha)/\partial a = 0$ and $\Im m(\gamma_n) = \pi \kappa_n$ is uniformly continuous function of κ_n near to κ for $\forall n > N$. With a multidimensional Dirichlet's approximation, for $\forall n > N$, $\exists k$ and $\chi_n = k/n$:

$$|\chi - \chi_n| < 1/n^2 < 1/N^2. \blacksquare$$

As in the previous lemma, for $\forall n > N$ we have s_n near to s for a known critical point α .

Proposition

For $\forall n > N$, s_n verify the recurrence: $s_{n+1}\partial f(f(\alpha))/\partial a = s_n$. But asymptotically, we can have solutions of the equation: $s_{n+1}\partial f(f(\alpha))/\partial a = \mu s_n$ with $\mu \in \mathbb{R}$.

If the matrix $\partial f(f(\alpha))/\partial a$ is diagonalizable, we search the conditions on s_n to be an eigen vector of this matrix. If we note eigen vector $\mu = \rho e^{i\vartheta}$, it induces:

$$\vartheta = k\pi/n,$$

and asymptotically $\vartheta = \omega\pi$ with uniform $\omega \in [0,1]$ under the condition that the vector κ is uniform on $[0,1]^r$.

■ At time n , we have: $\partial\gamma(\alpha)/\partial a = s_n\partial f(\alpha)/\partial a - 1/\alpha = 0$ and $H_{n-1}(s_n) = 0$. We iterate

$$f : s_{n+1}\partial f \circ f(\alpha)/\partial a - 1/\alpha = 0.$$

$$\text{As: } s_n\partial f(\alpha)/\partial a - 1/\alpha = 0,$$

$$\text{we obtain: } s_{n+1}\partial f(f)/\partial f|_{f=f(\alpha)=s_n}$$

The complex matrix $\partial f(f(\alpha))/\partial f$ doesn't depend on n . We can search the asymptotic behavior of s_n . That means:

$$s\partial f(f(\alpha))/\partial a = s.$$

There is no solution, except if an eigen value is 1.

But we have seen in the lemma that we can modify asymptotically the equation for $\forall \mu \in \mathbb{R}$:

$$s \partial f(f(\alpha)) / \partial a = \mu s.$$

If we write $\mu = \rho e^{i\vartheta}$, can approximate the real part of μ , but the imaginary part must be null: $\Im(\lambda) = \rho \sin \vartheta = 0$. If such s exist, for all s_n which converges to s , the asymptotic relation between s_n and s_{n+1} : $s_{n+1}T = \mu s_n T = \mu^n s_1 T$ is verified. All the μ^n must be real and $\sin n\vartheta = 0$. So: $\vartheta = k\pi/n$ and asymptotically $\vartheta = \omega\pi$ with uniform $\omega \in [0,1]$ under the condition that the vector κ is uniform on $[0,1]^r$. This may explain the self-similarity phenomenon because at each iteration, the curve is reciprocal image of :

$$k/n < \frac{(k+1)}{n+1} < (k+1)/n. \blacksquare$$

3- Example: the Henon's curves

Let the iteration: $f(a, b) = (a_1, b_1) = (g(a) + b, va)$

with $g(a) = \delta a - \sigma a^2$,

Explicitly: $a_1 = g(a) + b, \quad b_1 = va$

We can put: $\delta = \lambda + \lambda' \quad \text{and} \quad v = -\lambda\lambda'$

Where λ and λ' are the eigen value of the linear part of f at 0.

We observe: $H_n(x, y) = \partial^n e^{xg(a)+yva} / \partial a^n |_{a=0} \partial^n e^{xb} / \partial b^n |_{b=0}$

$$H_n(x, y) = x^n \partial^n e^{xg(a)+yva} / \partial a^n |_{a=0}$$

The hessian of f is degenerated and we have only to study $\partial^n e^{xg(a)-yva} / \partial a^n |_{a=0}$.

After normalization (x, y divided by n), we put: $x\sigma a^2 = u^2/2 \quad \text{and} \quad (x\mu + yv)a = \beta u$.

Then: $\beta = (x\mu + yv) / \sqrt{2x\sigma}$

We recognize in $\gamma(a) = xg(a) + yva - \ln a$, the PR function of the logistic map: $f(u) = \beta u - u^2/2$. We have the density of probability of presence under the conditions if the discriminant of the PR function: $\Delta = (s\lambda)^2 - 4s < 0$ and $s > 0$. This density is:

$$p(a) = \beta / (2\pi \sqrt{4a - a^2\beta^2}).$$

With: $\beta = (x\delta + yv) / \sqrt{2x\sigma}$

We have a piece of random parabola.

But now, we consider $f \circ f = f(a_1, b_1) = (g(a_1) + b_1, va_1)$

$$f \circ f(a, b) = (g(g(a) + b) + va, v(g(a) + b)).$$

The PR function becomes:

$$\gamma_1(\alpha_1) = x(g(g(a) + b) + va) + yv(g(a) + b) - \ln a - \ln b$$

Its critical point is: $\partial \gamma_1(\alpha_1) / \partial a = \{x g'(g(a) + b) + yv\} g'(a) - 1/a = 0$.

$$\partial \gamma_1(\alpha_1) / \partial b = \{x g'(g(a) + b) + yv\} - 1/b = 0.$$

Then, fixing $x_1 = x g'(g(a) + b) + yv$, we have yet: $x_1 g'(a) - 1/a = 0$ and $x_1 - 1/b = 0$; which determines the critical point $\alpha = (a, b)$ of $\gamma(a)$ for f where b is unknown and a is invariant. But, with $\partial \gamma_1(\alpha_1) / \partial b = 0$, we have a new equation of b function of a :

$$bx g'(g(a) + b) + yvb + 1 = 0$$

As: $g'(a) = \delta - 2\sigma a, g'(g(a) + b) = \delta - 2\sigma(g(a) + b)$; the equation in b is:

$$x b(g'(g(a)) - 2\sigma b) + yvb + 1 = 0.$$

$$-2\sigma x b^2 + (x\delta + yv - 2\sigma x(\Re(g(a)) + i\Im(g(a))))b + 1 = 0.$$

But: $\partial f(f(\alpha)) / \partial a = \begin{bmatrix} \delta - 2\sigma g(a) & 1 \\ v & 0 \end{bmatrix}$

And the eigen vector $s = (\mu, v)$ verifies: $\mu(\mu - \delta - 2\sigma g(a)) - v = 0$.

The sequence $s_n = (\mu^n, v \mu^{n-1})$ If we write $\mu = \rho e^{-i\theta}$, the imaginary part of μ will be null.

That means: $n\theta = k\pi/n \rightarrow \omega$ uniform on $(0,1)$. But, before the uniformity, the self-similarity dominates. \blacksquare

E- The complex iterations

1- The trinomial iterations

We study here only trinomial iterations as the following.

Let the application of a set $C \subset \mathbb{C}$ in C :

$$z_1 = f(z) = \alpha z + z^m/m; m > 1$$

$z, \alpha \in \mathbb{C}$, and f is trinomial. All the following calculus are formal.

The fixed points are: $z_0 = 0$ et $z_{m0} = ((1 - \alpha)m)^{1/(m-1)}$. As $\alpha \in \mathbb{C}$, we note $\alpha = re^{i\sigma}$ and $z = \rho e^{i\vartheta}$ and $\vartheta + \sigma = \omega$. We have $m - 1$ fixed points z_{m0} :

$$z_{m0k} = z_{m0} e^{i2k\pi/(m-1)}.$$

The derivatives of f are $\lambda_0 = \alpha$ at point 0 and $\lambda_{m0} = \alpha + m(1 - \alpha)$ at points z_{m0} .

The steepest descent method is supposed yet valid. We observe a great variety of behaviors.

- Study at 0

The Plancherel-Rotach's function is at 0 : $\gamma(z) = sf(z) - \ln(z) = s(\alpha z + z^m/m) - \ln(z)$, the critical points are given by : $\partial\gamma(z)/\partial z = s\partial f/\partial z - 1/z = s(\alpha + z^{m-1}) - 1/z = 0$.

$$\partial\gamma(z)/\partial z = z^m + \alpha z - 1/s = 0$$

The discriminant of this equation is: $\Delta = (-1)^{m-1} (m-1)^{m-1} (\alpha)^m + (m)^m (1/s)^{m-1}$

If α is real and Δ is negative, or if α is complex, we have a complex root. In polar coordinates:

$$\partial\gamma(z)/\partial z = s\rho^m e^{im\vartheta} + sr\rho e^{i\omega} - 1 = 0.$$

And:

$$\Im(\partial\gamma/\partial z) = \Im(z^m + \alpha z) = 0,$$

$$\rho^{m-1} \sin m\vartheta + r \sin \omega = 0.$$

We have ρ function of ϑ :

$$\rho^{m-1} = -r \sin \omega / \sin m\vartheta > 0 \text{ with condition: } |\rho^{m-1} \sin m\vartheta| \leq r$$

But we have also:

$$\Re(s\rho^m e^{im\vartheta} + sr\rho e^{i\omega}) = 1$$

$$sr \sin(m\vartheta - \omega) = \sin m\vartheta$$

We can write:

$$s = \sin m\vartheta / r \sin(m\vartheta - \omega) = \varphi(\theta)/r$$

Let χ uniform on (0,1). If s is real, the distribution of probability of s is:

$$\pi d\chi = \Im(f(z)) ds;$$

As:

$$\Im(\partial\gamma/\partial z) = \Im(z^m + \alpha z) = 0,$$

$$\pi d\chi = (m-1)/m \Im(\alpha z) ds.$$

Then we have:

$$\pi d\chi = (m-1)/m r \rho \sin \omega ds$$

And we can express χ in function of ϑ and get the law of ϑ :

$$\pi d\chi = (m-1)/m (-r \sin \omega / \sin m\vartheta)^{1/(m-1)} \sin \omega \varphi'(\theta) d\theta$$

If, now we iterate k times f , we have to study the asymptotical equation:

$$s \Re(\partial f(f)/\partial f|_{f=f(\alpha)}) = s \Re((\alpha z + z^m/m)^{m-1} + \alpha)$$

Where z is critical point of γ . Let $\mu = \Re((\alpha z + z^m/m)^{m-1} + \alpha)$

The separation between the bounded iteration and the divergence is given by:

$$|\mu| = 1$$

But, we must have also the similarity's condition: $\Im(\mu^k) = 0$

- Study at the points z_{m0}

We have $m - 1$ fixed points z_{m0} : $z_{m0k} = z_{m0} e^{i2k\pi/(m-1)}$

It suffice to translate the calculus at these points. The only novelty is the domain of attraction of each fixed point: $(0, z_{m0k})$. The relation of domination is given by comparison of the real part of the Plancherel-Rotach's functions.

2- The Fatou-Julia's sets

- $m=2$, if $0 > \alpha \in \mathbb{R}$, we recognize the logistic map when $4s + (\alpha s)^2 < 0$. If $\alpha \in \mathbb{C}$, we note: $\alpha = re^{i\sigma}$ and $z = \rho e^{i\vartheta}$. The critical points are solution of : $\partial\gamma(z)/\partial z = s(\alpha z + z^2) - 1 = 0$.

The imaginary part is: $\rho \sin(2\vartheta) = -r \sin(\sigma + \vartheta)$

The real part is:

$$s \rho(r \cos(\sigma + \vartheta) + \rho \cos(2\vartheta)) = 1.$$

And:

$$s \rho r (\sin(2\vartheta) \cos(\sigma + \vartheta) - \sin(\sigma + \vartheta) \cos(2\vartheta)) = \sin(2\vartheta)$$

$$s = -\sin(2\vartheta)^2 / r^2 \sin(\vartheta + \sigma) \sin(\vartheta - \sigma)$$

We write

$$s = (-1/r^2) \varphi(\vartheta, \sigma)$$

the distribution of probability of s is:

$$\pi d\chi = \Im(\alpha z/2) ds = -r^2 \sin(\sigma + \vartheta)^2 / 2 \sin(2\vartheta) ds.$$

$$\pi d\chi = \sin(\sigma + \vartheta)^2 \varphi'(\vartheta, \sigma) d\vartheta / 2 \sin(2\vartheta)$$

It is more difficult than the real case where $\alpha \in \mathbb{R}$.

If, now we iterate k times f , we have to study the asymptotical equation:

$$s_n = s_{n+1} \operatorname{Re}(\alpha + \alpha z + z^2/2)$$

Where z is critical point of γ . Let $\mu = \operatorname{Re}(\alpha + \alpha z + z^2/2)$

The separation between the bounded iteration and the divergence is given by:

$$|\mu| = 1$$

But, we must have also the similarity's condition: $\Im(\mu^k) = 0$

$$\Im(\mu) = \Im(\alpha + \alpha z/2) = r(\sin(\sigma) - r \sin(\sigma + \vartheta)/2 \sin(2\vartheta)).$$

- $m = 3$,

In this case, we have analogues of the derivatives of the Airy's function or the Hopf's bifurcation the differential iteration: $f(z) = z(1 + \delta(\alpha + b|z|^2))$, $z \in \mathbb{C}$ where δ is the path.

We study: $f(z) = \alpha z + z^3/3$. The critical point of $\gamma(z) = s(\alpha z + z^3/3) - \ln(z)$, is solution of: $\partial\gamma/\partial z = z^3 + \alpha z - 1/s = 0$. If $0 > \alpha \in \mathbb{R}$;

the two imaginary solutions are obtained with the negative discriminant: $\Delta = 4\alpha^3 + 27/s^2 < 0$.

Part II: Differential ordinary equations as iterations

A- The probabilistic approach of the Bendixon- Poincaré problem

In this part, we apply the results on the bounded iteration especially to the ordinary differential equation. In this study, we can say, under good conditions, that a bounded ODE is deterministic near the origin of the process, but may have random or fixed cycles after a very long time.

With our probabilistic method, we obtain some new results in mechanic or in physics, but we meet also many new difficulties due to the particular steepest descent's method used to study the Plancherel-Rotach's function.

We consider ordinary differential equation *ODE*:

$$da/dt = F(a)$$

where $a \in \mathbb{C} \subset \mathbb{R}^d$ or \mathbb{C}^d , $t \in \mathbb{R}^+$, $F(a)$ is a C^∞ -application of $a \in \mathbb{C}$ in \mathbb{C} . The domain \mathbb{C} is supposed bounded. The problem is to find a function $a(t)$ verifying this equation with an initial condition:

$$a(t_0) = a_0.$$

1- The differential iteration

We associate to the ODE, the differential iteration $f(a)$ belonging in a bounded domain \mathbb{C} :

$$f(a) = a + \delta F(a)$$

$$a_1 = a + \delta F(a)$$

where $\delta = t/n$ is the path. We recall that the fixed points of a differential iteration are the zeros α of F : $F(\alpha) = 0$. We suppose that the theoretical Caratheodory's solution exists $a(t)$ for some $t > t_0$ with $a_0 = a(t_0)$ and a_0 belongs to a known fixed point.

$$a(t) = a_0 + \int_{t_0}^t F(a(u)) du = a_0 + S(a_0, a(t)).$$

When we iterate n times f , we have the iteration:

$$a_n = f^{(n)}(a(t_0)).$$

That we can write:
$$a_n = f^{(n)}(a_0) = a_0 + \delta \left(\sum_{p=0}^{n-1} F(a_p) \right)$$

Then, when $n \rightarrow \infty$, this solution $a_n \rightarrow a(t)$ converges to the Caratheodory's solution:

$$\lim_{n \rightarrow \infty} f^{(n)}(a_0) = a(t) = a_0 + \int_{t_0}^t F(a(u)) du$$

A priori, this deterministic solution depends on a_0 . But, the asymptotic probability of presence doesn't depend completely on the initial conditions. It depends essentially on the basin of attraction of the fixed points. Subject to this condition, we submit a probabilistic version of the Poincaré-Bendixon's problem in \mathbb{R}^d .

Proposition

Under the existence of the Caratheodory's solution, we have asymptotic random cycles around each fixed point for the random coordinates. The times of return in each very small borelian set around a point of a cycle are constant in probability. Along each cycle, the conditional probability has a constant density.

We have:
$$\int_{t_0}^{t_0+T} F(a(u)) du = 0$$

■ With the path $\delta = t/n$, we have: $a_n = f^{(n)}(a_0) = a_0 + \delta \left(\sum_{p=0}^{n-1} F(a_p) \right)$

When $n \rightarrow \infty$, the convergence to the Caratheodory's solution gives:

$$a(t_0 + t) = a(t_0) + \int_0^t F(a(v)) dv$$

As we search a probabilistic behavior, after an infinite time, we start at an asymptotic time $t = t_0$, with a realization of a random vector $X(t_0) = a_0$. But; for every n , we have the Perron-Frobenius equation for the iteration $f^{(n)}$ independently of a_0 :

$$\theta_{n,f}(y) = E \left(e^{yX} - e^{y f^{(n)}(X)} \right) = 0$$

We choose n for the path $\delta = t/n$ especially for $f^{(n)}$. As the PF equation is verified, when $n \rightarrow \infty$, the convergence to the Caratheodory's solution gives: $f^{(n)}(X) \rightarrow X(t)$ with convergence in law. In other words, $X(t) = X(t_0)$ in law for the invariant non-null measure and some t_0 . It means that exist t_0 and T such as:

$$X(t_0 + T) = X(t_0) + \int_0^T F(a(v)) dv.$$

The asymptotic behavior is random periodic cycles with an unknown almost period $T = t - t_0$. The conditional probability of presence on each periodic cycle is uniform. As $X(t_0 + T) = X(t_0)$ in law, we have: $\int_0^T F(a(v)) dv = 0$. ■

Corollary

All the coordinates of the asymptotic periodic solution are multiple of a minimal period τ except if they are independent.

■ Suppose the coordinates not functionally independent. Suppose the common period is T . Suppose the minimal period of the coordinate a_ρ is $\tau \neq T$ and $\tau < T$. As the common period is T : $\int_0^T F_\rho(a(v)) dv = 0$. As the period of a_ρ is τ : $\int_0^\tau F_\rho(a(v)) dv = 0$. So, $\int_\tau^T F_\rho(a(v)) dv = 0$. Then: $T - \tau$ is again a period of a_ρ . For the same reason: $T - 2\tau$ is a period of a_ρ . And $T - n\tau$ is a period of a_ρ . T is a multiple of τ . ■

Remarks

- When we have many fixed points, the complete solution is more difficult to apprehend because we meet some problems with the transitions from a domain of a fixed point to another when all the domains are repulsive (as in the Lorenz's equations).

- We can try to extend these results to PDE equations and obtain other new results, as in the following. Let the PDE: $\partial a / \partial u = F(a)$

Where $a \in C \subset \mathbb{R}^d$ or \mathbb{C}^d , $u \in \mathbb{R}^p$ with $d > p$. As we write the iteration δ

$$a_1 = a + \delta_u F(a)$$

There is no great difference with the ODE in changing a scalar t in a vector u and the limits conditions. After transformation of the PDE into iterations, suppose that one can use the Caratheodory's solution for PDE:

$$a(u) = a(u_0) + \int F(a(v)) dv.$$

And only the asymptotic solutions for a non-null measure are periodic cycles with the unknown almost period $T = u - u_0$. And we have often to consider the limits conditions.

But overall, we meet two very difficult problems:

- the more simple is the Fredholm's alternative,
- the second very complicate is the interaction between the asymptotic periodic cycles.

2- The Fredholm's alternative

Asymptotically, we have some random cycles. Let $a(t)$ be a point on a such asymptotic cycle and a very small invariant borelian around this point. So, we have many large times to return in this borelian. In the differential iteration, we have many and large $\tau = (t + kT)/n$ which give the same $a(t)$ where T is a random quasi-period. We write now the PR integrand and the equations of the critical point for a very large $n(kT + t)$, $\delta = t$:

$$\gamma(a) = sa + tF(a) - \ln(a)$$

And:
$$\partial \gamma(a) / \partial a = s + t \partial F(a) / \partial a - 1/a = 0$$

Proposition

When the number of iterations $n \rightarrow \infty$, with $s = y/n$ the critical point is defined by:

$$s + ts \partial F(a) / \partial a - 1/a = 0$$

If s_a is a particular solution and if \vec{s} is an eigenvector of $-\partial F(a) / \partial a$ for the eigenvalue $1/t$, the general solution is disjunctive:

$$s = s_a \text{ if } t \neq -1/\lambda_a \text{ or } s = \vec{s} \text{ if } t = -1/\lambda_a.$$

The eigenvalue $1/t$ can be interpreted as a critical asymptotic frequency.

■ Contrary to the previous §, we don't write the critical point a as a function of s , but s as a function of a . For fixed a , we recognize the linear affine equation of s depending on the parameter t . We have to find a particular solution s_a :

$$s_a + ts_a \partial F(a) / \partial a - 1/a = 0.$$

Formally:
$$s_a = (Id + t \partial F(a) / \partial a)^{-1} (1/a).$$

This solution s_a is valid for all $t \neq -1/\lambda_a$ where λ_a is eigenvalue of $\partial F(a) / \partial a$ at the critical point a . The general solution will be $s = s_a + \vec{s}$, with: $\vec{s} = -t \vec{s} \partial F(a) / \partial a$. As $t \neq -1/\lambda_a$, $\vec{s} = 0$. But, as t is increasing from 0 to ∞ , the general solution is disjunctive and shows discontinuities at the eigenvalues λ_a . ■

Remark: calculation of s_a

s_a is obtained with $(Id + t \partial F(a) / \partial a)^{-1}$ for all $t \neq -1/\lambda_a$ which doesn't belong to the spectrum of $-\partial F(a) / \partial a$ with the series development of τ .

Remark: the Fredholm's alternative

Here, we have the Fredholm's alternative: either we have s_a for all $t \neq -1/\lambda_a$ or \vec{s} for $t = -1/\lambda_a > 0$. Suppose we start with $0 < t < -1/\lambda_a$, but t is increasing:

What happens when $t \rightarrow -1/\lambda_a$? What is the physical interpretation? Can we connect this phenomenon to some physical constants or boundaries?

Under general conditions, we have said that a physical phenomenon may have a periodic behavior, but the period is unknown. We can say now that if this movement is periodical, the period must be included inside two real consecutive eigenvalues λ_a . It depends on the initial conditions.

3- The interaction between the asymptotic periodic cycles

This study corresponds to the asymptotic analysis of $f^{(p)}, \forall p$.

We have seen the recurrence of s_n at the critical point α : $s_{n+1} \partial f(f(\alpha))/\partial a = \mu s_n$ with $\mu \in \mathbb{R}$.

Here, the differential iteration is: $f(a) = a + \delta F(a)$

We obtain: $s_{n+1}(I + \delta \partial F(f(\alpha))/\partial a) = \mu s_n$

As $\delta \rightarrow 0$: $s_{n+1}(I + \delta \partial F(\alpha)/\partial a) = \mu s_n$

And we have to search an asymptotic s eigen vector of :

$$\begin{aligned} s(I + \delta \partial F(\alpha)/\partial a) &= \mu s \\ s \partial F(\alpha)/\partial a &= ((\mu - 1)/\delta) s = \lambda s \end{aligned}$$

Application to the mechanics

Example 1: The Hamilton's equations in \mathbb{R}^d for an autonomous system.

Let the equations: $\begin{aligned} dp/dt &= -\partial H(p, q)/\partial q, \\ dq/dt &= \partial H(p, q)/\partial p, \end{aligned} \quad a = (p, q) \in \mathbb{R}^{2d}$

we note: $F(a) = (-\partial H(a)/\partial q, \partial H(a)/\partial p)$

then: $da/dt = F(a)$,

the corresponding iteration: $f(a) = a + \delta F(a)$, with $\delta = t/n$.

where $a \in C \subset \mathbb{R}^{2d}$, $t \in \mathbb{R}^+$, $F(a)$ is a C^∞ -application of $a \in C$ in C . The domain C is supposed bounded. We have to search the fixed points defined by:

$$\begin{aligned} \partial H(p, q)/\partial q &= 0, \\ \partial H(p, q)/\partial p &= 0. \end{aligned}$$

We study now the asymptotic probabilistic solution of the differential iteration with the PR integrand:

$$\gamma(a) = xp + yq - x\delta \partial H(p, q)/\partial q + y\delta \partial H(p, q)/\partial p - \ln(p) - \ln(q)$$

The critical point is for $\delta = t$:

$$\begin{aligned} \partial \gamma(a)/\partial p &= x - xt \partial^2 H/\partial q \partial p + yt \partial^2 H/\partial p^2 - 1/p = 0, \\ \partial \gamma(a)/\partial q &= y - xt \partial^2 H/\partial q^2 + yt \partial^2 H/\partial p \partial q - 1/q = 0. \end{aligned}$$

Let the Hamiltonian matrix:

$$\partial^2 \tilde{H} = \begin{pmatrix} -\partial^2 H/\partial q \partial p & \partial^2 H/\partial p^2 \\ -\partial^2 H/\partial q^2 & \partial^2 H/\partial p \partial q \end{pmatrix}$$

With the antisymmetric matrix: $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\partial^2 \tilde{H} J$ is symmetric.

If $1/a = (1/p, 1/q)$ and $s = (x, y)$:

$$st \partial^2 \tilde{H} + s = 1/a$$

If we have a particular solution resolving this equation, it remains a linear asymptotic system:

$$st \partial^2 \tilde{H} + s = 0.$$

The eigen vector corresponding to the smallest eigen value when $t \rightarrow \infty$, gives $s \partial^2 \tilde{H} = 0$.

Asymptotically, the projection of the Laplacian on this eigen vector is null:

$$\alpha \partial^2 H/\partial p^2 + \beta \partial^2 H/\partial q \partial p + \gamma \partial^2 H/\partial q^2 = 0$$

Example 2

In the case of a potential $V(q)$ in \mathbb{R}^d with derivatives decreasing at ∞ :

$$H(p; q) = V(q) + p^2/2m$$

The Hamilton's equations are: $dp/dt = -\partial V(q)/\partial q$,

$$dq/dt = p/m.$$

The asymptotic Plancherel-Rotach's function is:

$$\gamma(p, q) = xp + yq - x\delta\partial V(q)/\partial q + \delta yp/m - \ln(p) - \ln(q)$$

We have two independent equations $\gamma(p, q) = \gamma_0(p) + \gamma(q)$

with: $\gamma_0(p) = xp + \delta yp/m - \ln(p)$,

and: $\gamma(q) = yq - y\delta\partial V(q)/\partial q - \ln(q)$.

the critical point is: $x + \delta y/m - 1/p = 0$,

and: $y - x\delta\partial^2 V(q)/\partial q^2 - 1/q = 0$.

The fixed point is defined by $p = 0$ and $\partial V(q)/\partial q = 0$.

The probabilistic periodic solution depends only on $V(q)$ because the independency of $\gamma(q)$ and $\gamma_0(p)$.

If we have a particular solution resolving this equation, it remains a linear system:

$$x + \delta y/m = 0 \quad \text{and} \quad y - x\delta\partial^2 V(q)/\partial q^2 = 0.$$

So: $y - m\delta^2 y\partial^2 V(q)/\partial q^2 = 0$.

It means: y is eigen vector of the real symmetric matrix $\partial^2 V(q)/\partial q^2$. As $t \rightarrow \infty$, if the phenomenon is bounded and cyclical $\delta \rightarrow \infty$: $y\partial^2 V(q)/\partial q^2 = my/\delta^2 \rightarrow 0$. So, we must take in priority the eigen vectors corresponding to the smallest eigen values:

$$ydp/dt = -y\partial V(q)/\partial q = C.$$

In \mathbb{R}^d , the projection of the forces on the eigen vectors y is constant; taking $q \rightarrow \infty$, $C=0$.

Along the vector y , the behavior of the bodies is linear as the barycenter. Asymptotically the cyclical behavior exists only in few dimensions corresponding to the smallest eigen values.

We can imagine for very small bodies a unique vibration along only each eigen vector.

Example 3

The n-bodies problem: the solar system

We have a sun with a mass M and $n-1$ planets with a total mass $\mu = M/1500$. The heaviest planet of the system is Jupiter with a mass $M/1000$. So, we call ε_ℓ the mass of the planets. Let a be a point of \mathbb{R}^d ; a_1 and a_ℓ are the positions of the sun and the planets; $a(i, \ell)$ is the distance between two bodies; the potential function is:

$$V(q) = M\sum_{\ell=2}^{\ell=n} \varepsilon_\ell g(a(1, \ell)) + \sum_{\ell>1}^{\ell=n} \sum_{i>1}^{i=n} \varepsilon_\ell \varepsilon_i g(a(i, \ell)),$$

with $|a_\ell| = \|a_\ell - a_1\| = a(1, \ell)$; $g(|a_\ell|) = 1/|a_\ell|$ and $q = a$. (But g can be more general).

The main difficulty is the discontinuity at $|a_\ell| = 0$. We suppose the validity of the calculus and weak inter actions: then, all the second member $\sum_{\ell>1}^{\ell=n} \sum_{i>1}^{i=n} \varepsilon_\ell \varepsilon_i g(a(i, \ell))$ is negligible and:

$$V(q) = M\sum_{\ell=2}^{\ell=n} \varepsilon_\ell g(|a_\ell|)$$

The fixed points of the iteration are defined by: $\partial V(a)/\partial a = 0$, $p = 0$ where the forces and the celerity are null. They are the Lagrange's points.

Let: $\partial V(a)/\partial a = M\sum_{\ell=2}^{\ell=n} \varepsilon_\ell \partial g(|a_\ell|)/\partial |a_\ell| \partial |a_\ell|/\partial a$

That means: $\partial |a_\ell|/\partial a_1 = 2(a_1 - a_\ell)$ and $\partial |a_\ell|/\partial a_\ell = 2(a_\ell - a_1) = -\partial a/\partial a_1$

$$\partial^2 |a_\ell|/\partial a_1^2 = \partial^2 |a_\ell|/\partial a_\ell^2 = 2 \quad \text{and} \quad \partial^2 |a_\ell|/\partial a_\ell \partial a_1 = -2$$

$$\begin{aligned} \partial V(a)/\partial a_1 &= M\sum_{\ell=2}^{\ell=n} \varepsilon_\ell g'(|a_\ell|) \partial |a_\ell|/\partial a_1 \\ &= 2M\sum_{\ell=2}^{\ell=n} \varepsilon_\ell g'(|a_\ell|)(a_1 - a_\ell) \end{aligned}$$

$$\partial V(a)/\partial a_\ell = M\varepsilon_\ell g'(|a_\ell|) \partial |a_\ell|/\partial a_\ell.$$

$$\begin{aligned} \partial^2 V(q)/\partial a_1^2 &= M\partial(\sum_{\ell=2}^{\ell=n} \varepsilon_\ell g'(|a_\ell|) \partial |a_\ell|/\partial a_1)/\partial a_1 \\ &= M\sum_{\ell=2}^{\ell=n} \varepsilon_\ell (g''(|a_\ell|)(\partial a/\partial a_1)^2 + 2g'(|a_\ell|)) \end{aligned}$$

$$\begin{aligned} \partial^2 V(q)/\partial a_\ell^2 &= -\partial^2 V(a)/\partial a_1 \partial a_\ell \\ &= M\varepsilon_\ell (g''(|a_\ell|)(\partial a/\partial a_1)^2 + 2g'(|a_\ell|)). \end{aligned}$$

We note: $c_\ell = \partial^2 V(q)/\partial a_\ell^2$ then $c_1 = \sum_{\ell=2}^{\ell=n} c_\ell$

And: $\partial^2 V(a)/\partial a_1 \partial a_\ell = -c_\ell$

$\partial^2 V(a)/\partial a_l \partial a_\ell = 0$ for $l \neq 1, \ell$.
 We observe: $\partial^2 V(a)/\partial a_1 \partial a_\ell + \partial^2 V(a)/\partial a_\ell^2 = 0$
 By integration: $\partial V(q)/\partial q_1 + \partial V(q)/\partial q_\ell = C, \forall q_1 q_\ell$.
 Then $C = 0$ and the problem is reduced to the two-body problem for the $n-1$ planets.
 We have the asymptotic critical equation: $y - \tau y \partial^2 V(q)/\partial q^2 = 0$.
 The determinant Δ of $\partial^2 V(q)/\partial q^2$ is null. What are the eigen values other than $\tau = 0$?
 Let the determinant: $\Delta(\lambda) = |\partial^2 V(q)/\partial q^2 - \lambda I|$
 Let $\Pi_n(\lambda) = \prod_{\ell=2}^n (\lambda - c_\ell)$, $\tau = 1/\lambda$ and $\lambda^{-n} \Pi_n(\lambda) = \tau^n \Pi_n(1/\tau)$ is a polynomial in τ with degree n .

Proposition

$$\Delta(\lambda) = (-1)^{n+1} \lambda^{2+n} d\lambda^{-n} \Pi_n(\lambda) / d\lambda$$

So, the eigen values λ_ℓ of $\Delta(\lambda)$ give the extrema of the polynomials $\tau^n \Pi_n(1/\tau)$ and c_ℓ are the roots of $\Pi_n(\lambda)$. Each eigen value is between two consecutive roots: $c_\ell < \lambda_\ell < c_{\ell+1}$.
 Asymptotically the cyclical behavior exists only in few dimensions corresponding to the smallest eigen values. All the forces and the orbits become flatter on a plane.

■ As $\partial^2 V(q)/\partial q^2$ is symmetric real, the eigen values are real and the question is if the eigen value is more or less than 1.

We have:

$$\Delta(\lambda) = \begin{vmatrix} c_1 - \lambda & -c_2 & -c_n \\ -c_2 & c_2 - \lambda & 0 \\ \vdots & \vdots & \vdots \\ -c_n & 0 & c_n - \lambda \end{vmatrix}$$

With $c_1 = \sum_{\ell=2}^n c_\ell$

Adding all columns in the first column, then getting $-\lambda$ in factor, we obtain by summation on the first row:

$$= (-\lambda) \begin{vmatrix} n & -\lambda & -\lambda \\ 1 & c_2 - \lambda & 0 \\ 1 & 0 & c_n - \lambda \end{vmatrix} = (-1)^n \lambda \begin{vmatrix} n & \lambda & \lambda \\ 1 & \lambda - c_2 & 0 \\ 1 & 0 & \lambda - c_n \end{vmatrix}$$

Then:

$$\Delta(\lambda) = (-1)^n \lambda (n \Pi_n(\lambda) - \lambda B_n(\lambda))$$

With:

$$B_n(\lambda) = \begin{vmatrix} 0 & \lambda & \lambda \\ 1 & \lambda - c_2 & 0 \\ 1 & 0 & \lambda - c_n \end{vmatrix}$$

We have the relation:

$$\begin{aligned}
 B_n(\lambda) &= (\lambda - c_n) B_{n-1}(\lambda) + \Pi_{n-1}(\lambda) \\
 &= (\lambda - c_n) B_{n-1}(\lambda) + \Pi_n(\lambda) / (\lambda - c_n) \\
 &= (\lambda - c_n) (c_{n-1} - \lambda) B_{n-2}(\lambda) + \Pi_n(\lambda) (1/(\lambda - c_n) + 1/(\lambda - c_{n-1}))
 \end{aligned}$$

So, by induction:

$$B_n(\lambda) = \Pi_n(\lambda) (\sum_{\ell=2}^n 1/(\lambda - c_\ell))$$

And:

$$\Delta(\lambda) = (-1)^n \lambda^2 \Pi_n(\lambda) (n/\lambda - (\sum_{\ell=2}^n 1/(\lambda - c_\ell)))$$

But we recognize:

$$d\lambda^{-n} \Pi_n(\lambda) / d\lambda = -\lambda^{-n} \Pi_n(\lambda) (n/\lambda - (\sum_{\ell=2}^n 1/(\lambda - c_\ell))).$$

$$\Delta(\lambda) = (-1)^{n+1} \lambda^{2+n} d\lambda^{-n} \Pi_n(\lambda) / d\lambda$$

If σ_k is symmetric function of order of the $n - 1$ roots c_ℓ of $\Pi_n(\lambda)$ with $\sigma_0 = 1$:

$$\Pi_n(\lambda) = \sum_{k=1}^{n-1} (-1)^k \sigma_k \lambda^{n-1-k}$$

$$d\lambda^{-n} \Pi_n(\lambda) / d\lambda = -\sum_{k=1}^{n-1} (-1)^k (k+1) \sigma_k \lambda^{-2-k}$$

$$\Delta(\lambda) = (-1)^n \sum_{k=1}^{n-1} (-1)^k (k+1) \sigma_k \lambda^{n-k}$$

We note:

$$\Delta = \lambda(\lambda - \lambda_2) \dots (\lambda - \lambda_\ell) \dots (\lambda - \lambda_n)$$

The eigen values λ_ℓ give the extrema of the polynomial $\tau^n \Pi_n(1/\tau)$ where the $1/c_\ell$ are the roots of $\tau^n \Pi_n(1/\tau)$; then, we have: $c_\ell < \lambda_\ell < c_{\ell+1}$.

As we have seen, in \mathbb{R}^d , the projection of the forces on the eigen vectors y is constant. Except strong correlations between the small bodies (as moon with earth). In this case, we must rewrite the problem with: $V(q) = M \sum_{\ell=2}^{\ell=n} \varepsilon_{\ell} g(a(1, \ell)) + \sum_{\ell>1}^{\ell=n} \sum_{i>1}^{i=n} \varepsilon_{\ell} \varepsilon_i g(a(i, \ell))$
 We don't must forget that the c_{ℓ} are randomized. We can use also the tools of the stellar statistics when we have many bodies. ■

Example 4 Case where the hessian is degenerated: the Lorenz's equation

Generally, the hessian is not definite negative. The Lorenz's equation is a particularly important example because the differential iteration can be broken down into three independent iterations which have a remarkable feature: a partial linearity; an iteration with a negative hessian which induces a probabilistic solution and another with a positive hessian. It is an ideal example to clarify the previous results. However, as there is an interpenetration of the distributions related to each fixed point, the connection between the various results remains delicate. The probabilistic presentation seems to be the best: it gives the probability of presence except at the places where the domination changes; in this case, we go from a basin to an another as a ping-pong ball.

The iteration at its repellent fixed points

The vectors of this equation are written in bold notations: $d\mathbf{a}/dt = F(\mathbf{a})$ where $\mathbf{a} = (a, b, c)$:

$$\begin{aligned} da/dt &= \sigma(b-a) \\ db/dt &= \rho a - b - ac \\ dc/dt &= -\beta c + ab. \end{aligned}$$

The differential equation applies a bounded set C in itself for $\delta > t > 0$ (the phenomenon is occurring between a cold sphere at -50° and hot sphere, the earth, at $+15^\circ$ as the terrestrial atmosphere is modelled by Lorenz).

The differential iteration $\mathbf{a}_1 = f(\mathbf{a})$ associated with a given path $\delta = t/n$ is:

$$\begin{aligned} a_1 &= a + \delta\sigma(b-a) \\ b_1 &= b + \delta(\rho a - b - ac) \\ c_1 &= c + \delta(-\beta c + ab). \end{aligned}$$

This iteration is quadratic, but has a linearity in a .

We recall the known results concerning the fixed points:

The fixed points are zeros of $F(\mathbf{a})=0$. If $\rho > 1$ and $\alpha = \sqrt{\beta(\rho - 1)}$, it exists three fixed points: the point $\mathbf{0} = (0,0,0)$, and two others symmetric with respect to the axis of c :

$$\alpha_+ = (\alpha, \alpha, \alpha^2/\beta) \text{ and } \alpha_- = (-\alpha, -\alpha, \alpha^2/\beta).$$

At $\mathbf{0}$, the eigenvalue's equation λ of the linear part is:

$$(\beta + \lambda) [(\sigma + \lambda)(1 + \lambda) - \sigma\rho] = 0,$$

But, at α_+ or at α_- :

$$\lambda(\beta + \lambda)(1 + \sigma + \lambda) - \alpha^2(2\sigma + \lambda) = 0$$

Coefficients β, σ, ρ are such as these three fixed points are repellent; that means we have to study the distributions around each fixed point. We don't speak here about attractive cycles, resonances, and some particular values of the parameters, etc. It remains many points to clarify.

Analysis of the hessian

Projecting $f(\mathbf{a})$ on an axis $\mathbf{y} = (x, y, z)$, we write:

$$\mathbf{y} f(\mathbf{a}) = L(\mathbf{a}) + \delta Q(\mathbf{a})$$

where $L(\mathbf{a})$ is linear for \mathbf{a} : $L(\mathbf{a}) = x(a + \delta\sigma(b-a)) + y(b + \delta(\rho a - b)) + zc(1 - \delta\beta)$

with: $L(\mathbf{a}) = aL_1 + bL_2 + cL_3$

$$L_1 = x(1 - \delta\sigma) + \delta\rho y$$

$$L_2 = \delta\sigma x + y(1 - \delta)$$

$$L_3 = z(1 - \delta\beta)$$

and $Q(\mathbf{a})$ is quadratic: $Q(\mathbf{a}) = (zb - yc)a$.

The hessian $Q(\mathbf{a})$ is degenerated and not definite negative. But, Q doesn't change when we translate the origin from a fixed point to another.

First, we examine the matrix of $Q(\mathbf{a})$:

$$Q = \begin{bmatrix} 0 & z & -y \\ z & 0 & 0 \\ -y & 0 & 0 \end{bmatrix}$$

Let $\mu = \sqrt{y^2 + z^2}$ the positive eigenvalue of the characteristic equation of Q :

$$\mu(\mu^2 - y^2 - z^2) = 0$$

The matrix of the eigenvectors T is orthogonal and constant for all \mathbf{a} .

$$T = \frac{1}{\mu\sqrt{2}} \begin{bmatrix} 0 & \mu & \mu \\ y\sqrt{2} & -z & z \\ -z\sqrt{2} & y & -y \end{bmatrix}$$

Corresponding to the diagonal matrix of the eigenvectors: $\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$.

Change of basis near 0

- We calculate in the basis of eigenvectors directly with the Hermite's polynomials. As T is orthogonal, the transposed T' is also its inverse: $T' = T^{-1}$.

Then, the application $\mathbf{u} = T\mathbf{a}$ with $\mathbf{u} = (u, v, w)$ transforms:

$$\begin{aligned} \mathbf{y}f(\mathbf{a}) &\mapsto G(\mathbf{u}) = \mathbf{y}f(T'\mathbf{u}) \\ Q(\mathbf{a}) &\mapsto Q(T'\mathbf{u}) = \delta\mu(w^2 - v^2) \\ L(\mathbf{a}) &\mapsto LT'\mathbf{u}. \end{aligned}$$

Now, in the basis \mathbf{u} , the function $\mathbf{y}f(T'\mathbf{u}) = G(\mathbf{u})$ is factorized into three independent functions:

$$G(\mathbf{u}) = g_1(u) + g_2(v) + g_3(w)$$

with: $g_1(u) = l_1 u$; $g_2(v) = l_2 v - \delta\mu v^2$; $g_3(w) = l_3 w + \delta\mu w^2$.

Where: $l_1 = (\delta\sigma x + y(1 - \delta) + z(1 - \delta\beta)) / \sqrt{2}$

$$l_2 = (x - \delta\sigma x + \delta\rho y)y / \mu - (\delta\sigma x + y(1 - \delta)) - z(1 - \delta\beta)z / \mu \sqrt{2}$$

$$l_3 = (x - \delta\sigma x + \delta\rho y)z / \mu + (\delta\sigma x + y(1 - \delta)) - z(1 - \delta\beta)y / \mu \sqrt{2}$$

- We get 3 independent iterations:

- . the first iteration g_1 is linear;
- . the second g_2 is a random iteration;
- . the third g_3 remains positive, except if $l_3 = 0$ half the time.

- To calculate l_1, l_2 et l_3 , we form:

$$L(\mathbf{a}) = a(x - \delta\sigma x + \delta\rho y) + b(\delta\sigma x + y(1 - \delta)) + zc(1 - \delta\beta)$$

With: $L_1 = x(1 - \delta\sigma) + \delta\rho y$; $L_2 = \delta\sigma x + y(1 - \delta)$; $L_3 = z(1 - \delta\beta)$

Then: $l\mathbf{u} = (l_1, l_2, l_3)\mathbf{u} = LT'\mathbf{u} = (L_1, L_2, L_3) \frac{1}{\mu\sqrt{2}} \begin{bmatrix} 0 & y\sqrt{2} & z\sqrt{2} \\ \mu & -z & y \\ \mu & z & -y \end{bmatrix} \mathbf{u}$.

- Let the resolving gap $e^n(\mathbf{y}) = \partial(\partial^n(e^{\mathbf{y}f(\mathbf{a})}) / \partial \mathbf{a}^n) \partial \delta|_{\mathbf{a}=0} = 0$

For $\forall t \leq \delta$, putting $\mathbf{a} = T'\mathbf{u}$, we have three separate coordinates:

$$e^n(\mathbf{u}) = T^n \partial(\partial^n(e^{\mathbf{y}f(T'\mathbf{u})}) / \partial \mathbf{u}^n) \partial \delta|_{\mathbf{u}=0} = 0$$

$$\partial^n(e^{\mathbf{y}f(T'\mathbf{u})}) / \partial \mathbf{u}^n = \partial^n(e^{g_1(u)}) / \partial u^n \cdot \partial^n(e^{g_2(v)}) / \partial v^n \cdot \partial^n(e^{g_3(w)}) / \partial w^n.$$

This gives:

$$\partial^n(e^{g_1(u)}) / \partial u^n = l_1^n e^{g_1(u)},$$

$$\partial^n(e^{g_2(v)}) / \partial v^n = H_n(g_2(v)) e^{g_2(v)},$$

$$\partial^n(e^{g_3(w)})/\partial w^n = H_n(g_3(w))e^{g_3(w)},$$

And:

$$e^n(\mathbf{u}) = \partial l_1^n H_n(g_2(v)) H_n(g_3(w)) (e^{y f(T^u)}) \partial \delta|_{u=0} = 0.$$

Proposition

The solution around the fixed point $\mathbf{0}$ consists of the intersection of the family of random surfaces defined by: $l_2/2\sqrt{\mu} \mapsto \text{low } \beta(1/2, 1/2)$

with the surfaces $\sigma x - y - z\beta = 0$ and $(-\sigma x + \rho y)z + (\sigma x - y + z\beta) y/\sqrt{2} = 0$.

■ With the same calculations of encodings and interchanging the derivations, we have:

$$\partial l_1^n / \partial \delta = 0; \partial H_n(g_2(v)) / \partial \delta = 0; \partial H_n(g_3(w)) / \partial \delta = 0$$

We study separately the three expressions:

- First: $\partial l_1^n / \partial \delta = n (\partial l_1 / \partial \delta) l_1^{n-1} = 0.$

Either $\partial l_1 / \partial \delta = \sigma x - y - z\beta = 0$, or: $l_1 \sim (y + z)/\sqrt{2} = 0$

- Second: the polynomial $H_n(g_3(w))$ when $w = 0$ is a Hermite's polynomial $H_n(x)$ where x is $x = il_3/(\sqrt{2\delta\mu})$. this polynomial $i^n H_n(il_3/(\sqrt{2\delta\mu}))$ is always positive half the time. In a general way:

$$\partial H_n(x) / \partial \delta = n H_{n-1}(x) \partial x / \partial \delta = 0. \text{ So: } d(l_3/\sqrt{2\delta\mu})/d\delta = 0,$$

And

$$l_3 \sim (xz\sqrt{2} + (y-z)y)/\mu\sqrt{2} = 0 \text{ half the time.}$$

- Third: in the case of $H_n(g_2(w))$, in addition to the solution $l_2 = 0$, we have to find the possible invariant distribution of $H_n(l_2/\sqrt{2\delta\mu}) = 0$.

Let the integrand of $n\gamma(w) = g_2(w) - n \ln w$.

When $\delta \rightarrow 0$, $l_2 \sim (x y \sqrt{2} + (y-z)z) / \sqrt{2}\mu$ with $\mu = \sqrt{y^2 + z^2}$.

By normalization of the coordinates $\mathbf{x} = (x, y, z) = \delta n \mathbf{s} = (\delta nr, \delta ns, \delta nt)$, we obtain:

$$l_2 \sim n\delta(rs\sqrt{2} + (s-t)t) / 2(s^2 + t^2)^{1/2} = n\delta l_2(\mathbf{s})$$

$$\delta\mu = n\delta^2(s^2 + t^2)^{1/2} = n\delta^2 \mu(\mathbf{s})$$

$$n\gamma(v) = n(\delta l_2(\mathbf{s})v - \mu(\mathbf{s})(\delta v)^2 - \ln \delta v + \ln \delta)$$

Putting $\delta v = v$, we have: $n\gamma(v) = n(l_2(\mathbf{s})v - \mu(\mathbf{s})v^2 - \ln v)$.

We search the critical point: $d\gamma(v)/dv = l_2(\mathbf{s}) - 2\mu(\mathbf{s})v - 1/v = 0$

The imaginary roots are: $v(\mathbf{s}) = l_2(\mathbf{s})/4\mu(\mathbf{s}) \pm i\sqrt{1/2\mu(\mathbf{s}) - l_2(\mathbf{s})^2/16\mu(\mathbf{s})^2}$.

Under the condition: $l_2(\mathbf{s})^2 < 8\mu(\mathbf{s})$:

$$l_3 \sim (rt\sqrt{2} + (s-t)s)/\mu(\mathbf{s})\sqrt{2} = 0,$$

Implies: $l_2(\mathbf{s}) = -(s-t)^2 / \sqrt{2}(s^2 + t^2)^{1/2}$

The condition becomes: $(s-t)^4 / (s^2 + t^2)^{3/2} < 16$

$l_1 = 0$ implies $s + t = 0$, then: $s < 8$.

In any case, we observe that the conditions $l_3 = l_1 = 0$ allow us to express r et t depending on s and we can write that the density of zeros of s is now:

$$q(s)ds = \text{Prob}(1 \text{ zero between } s, s+ds) = |\Im m f(v(s))| ds / \pi$$

$$q(s)ds = l_2(\mathbf{s})\sqrt{8\mu(\mathbf{s}) - l_2(\mathbf{s})^2} / 8\pi\mu(\mathbf{s}) ds = d\kappa.$$

Then, κ follows a uniform law on $(0, 1)$ with: $s + t = 0$ (or: $\sigma x - y - z\beta = 0$) and: $x y \sqrt{2} + (y-z)z = 0$.

We also remark that the normalization doesn't affect the coefficients of the orthogonal matrix:

$$T(x, y, z) = T(\delta nr, n\delta s, n\delta t) = T(r, s, t). \blacksquare$$

Analysis near α_+ and α_-

We now verify similar results the two other fixed points α_+ and α_- .

We search the distributions around the two other fixed points. To pass from the fixed point $\mathbf{0}$ to the fixed point α_+ or α_- , we have just to put in the iteration instead of $\mathbf{a} = (a, b, c)$:

$\mathbf{a}' + \alpha_+ = (a' + \alpha, b' + \alpha, c' + \alpha^2/\beta)$ or $\mathbf{a}'' + \alpha_- = (a'' - \alpha, b'' - \alpha, c'' + \alpha^2/\beta)$.

- Calculation for α_+

So, for $\mathbf{a}' + \alpha_+ = \mathbf{a}$; $\mathbf{a}'_1 = \mathbf{a}' + \alpha_+$ and $\mathbf{a}_1 = f(\mathbf{a})$ where $\mathbf{a}_1 = (a_1, b_1, c_1)$ becomes

$$\mathbf{a}_1 = \mathbf{a}'_1 + \alpha_+ = f(\mathbf{a}) = f(\mathbf{a}' + \alpha_+);$$

then:

$$\mathbf{a}'_1 = \mathbf{a}' + \delta F(\mathbf{a}' + \alpha_+)$$

As:

$$F(\mathbf{a}) = (\sigma(b-a), (\rho a - b - ac), (-\beta c + ab))$$

$\mathbf{a}_1 = f(\mathbf{a})$ becomes for $\mathbf{a} + \alpha_+$:

$$a'_1 = a + \delta\sigma(b-a) = a_1$$

$$b'_1 = b + \delta(\rho a - b - ac) + \delta(-\alpha c - \alpha a^2/\beta) = b_1 + \delta(-\alpha c - \alpha a^2/\beta)$$

$$c'_1 = c + \delta(-\beta c + ab) + \delta\alpha(a+b) = c_1 + \delta\alpha(a+b)$$

The projection of $f(\mathbf{a})$ on an axis $\mathbf{y} = (x, y, z)$ can be written:

$$\mathbf{y} f(\mathbf{a}') = x a_1 + y b_1 + \delta y(-\alpha c - \alpha a^2/\beta) + z c_1 + z \delta \alpha(a+b)$$

$$\mathbf{y} f(\mathbf{a}') = \mathbf{y} f(\mathbf{a}) + \delta(a(z\alpha - y\alpha^2/\beta) + z\alpha b - y\alpha c)$$

and $Q(\mathbf{a})$ is invariant: $\mathbf{y} f(\mathbf{a}') = L'(\mathbf{a}) + \delta Q(\mathbf{a})$

$L(\mathbf{a})$ is linear for \mathbf{a} : $L'(\mathbf{a}) = L(\mathbf{a}) + \delta(a(z\alpha - y\alpha^2/\beta) + z\alpha b - y\alpha c)$

$$L'(\mathbf{a}) = aL'_1 + bL'_2 + cL'_3$$

with:

$$L'_1 = L_1 + \delta(z\alpha - y\alpha^2/\beta); L'_2 = L_2 + \delta z\alpha; L'_3 = L_3 - \delta y\alpha.$$

Then T and Λ remain invariant. The following is only a calculus.

We calculate l'_1, l'_2 et l'_3 , with

$$L(\mathbf{a}) = a(x - \delta\sigma x + \delta\rho y) + b(\delta\sigma x + y(1-\delta)) + zc(1-\delta\beta) :$$

Where

$$L_1 = x(1-\delta\sigma) + \delta\rho y; L_2 = \delta\sigma x + y(1-\delta); L_3 = z(1-\delta\beta)$$

And:

$$l'\mathbf{u} = (l'_1, l'_2, l'_3)\mathbf{u} = L T^p \mathbf{u}$$

$$= (L_1 + \delta(z\alpha - y\alpha^2/\beta), L_2 + \delta z\alpha, L_3 - \delta y\alpha) \frac{1}{\mu\sqrt{2}} \begin{bmatrix} 0 & y\sqrt{2} & z\sqrt{2} \\ \mu & -z & y \\ \mu & z & -y \end{bmatrix}.$$

The results are modified; if $l = (l_1, l_2, l_3)$ is related to 0 and $l' = (l'_1, l'_2, l'_3)$ to α_+

$$l'_1 = l_1 + \delta\alpha(z-y)/\sqrt{2}$$

$$l'_2 = l_2 + \delta\alpha((z-y\alpha/\beta)y\sqrt{2} - z(z+y))/\mu\sqrt{2}$$

$$l'_3 = l_3 + \delta\alpha((z-y\alpha/\beta)z\sqrt{2} + y(z+y))/\mu\sqrt{2}$$

The following calculations remain the same with these modifications.

- Calculation for α_- -

When \mathbf{a} becomes $\mathbf{a}'' + \alpha_-$ the calculation is the same with the coordinates:

$$a''_1 = a + \delta\sigma(b-a) = a_1$$

$$b''_1 = b + \delta(\rho a - b - ac) + \delta(\alpha c - \alpha a^2/\beta) = b_1 + \delta(\alpha c - \alpha a^2/\beta)$$

$$c''_1 = c + \delta(-\beta c + ab) - \delta\alpha(a+b) = c_1 - \delta\alpha(a+b).$$

It remains the problems of domination and frontiers between the various distributions attached at each fixed point.

Remark

We have to go back to the original coordinates. And the solution gives only the probabilities of presence...

The Rössler's attractor (particular case of the Lorenz attractor)

Vectors are written in bold characters $\mathbf{a} = (a, b, c)$:

$$da/dt = -(b+c),$$

$$db/dt = a + \sigma b,$$

$$dc/dt = \beta + c(a - \rho).$$

The differential iteration for a fixed path $\delta = t/n$ is:

$$\mathbf{a}_1 = f(\mathbf{a})$$

$$a_1 = a - \delta(b+c) \quad b_1 = b + \delta(a + \sigma b) \quad c_1 = c + \delta(\beta + c(a - \rho)).$$

The iteration is quadratic in c , but has two linearities in a and b .

The fixed points are zeros of $F(\mathbf{a})=0$.

If $c = \tau$ is the coordinate of the fixed point: $b = -\tau$, $a = \sigma\tau$, we have the equation:

$$\beta + \tau(\sigma\tau - \rho) = 0;$$

solutions are:

$$\tau = \frac{-\rho \pm \sqrt{\rho^2 - 4\sigma\beta}}{2\sigma}$$

If $\sigma \leq \beta \ll \rho$, it exists two fixed points fixed: $\boldsymbol{\tau} = (\sigma\tau, -\tau, \tau)$ and $\boldsymbol{\tau}' = (\sigma\tau', -\tau', \tau')$.

At fixed point $\boldsymbol{\tau}$, the iteration becomes with $\alpha = \sigma\tau + \rho$:

$$\begin{aligned} \mathbf{a}_1 &= \mathbf{a} + \delta F(\mathbf{a}): \\ a_1 &= a - \delta(b + c) \\ b_1 &= b + \delta(a + \sigma b) \\ c_1 &= c + \delta(\tau a + c(a - \alpha)). \end{aligned}$$

Let the PR function

$$\gamma(\mathbf{a}) = \mathbf{y}f(\mathbf{a}) - n \ln(\mathbf{a}) \text{ with } \mathbf{x} = (x, y, z)$$

$$\gamma(\mathbf{a}) = x(a - \delta(b + c)) + y(b + \delta(a + \sigma b)) + z(c + \delta(\tau a + c(a - \alpha))) - n \ln(\mathbf{a})$$

$$\gamma(\mathbf{a}) = L(\mathbf{a}) + \delta Q(\mathbf{a})/2 - n \ln(\mathbf{a})$$

With $Q(\mathbf{a}) = 2zca$ and $L(\mathbf{a}) = x(a - \delta(b + c)) + y(b + \delta(a + \sigma b)) + z(c + \delta(\tau a - c\alpha))$

$L(\mathbf{a})$ is linear in \mathbf{a} : $(x + \delta y + \delta \tau z, -x\delta + y(1 + \delta\sigma), -x\delta + z - z\delta\alpha)$

$Q(\mathbf{a}) = 2zca$ is quadratic symmetric form with matrix:

$$Q = \begin{bmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ z & 0 & 0 \end{bmatrix}$$

The eigen vectors $\lambda = z$, $\lambda = -z$ with the orthogonal matrix of eigen vectors.

Example 5: Non autonomous Hamiltonian system

It is easy to verify if a Caratheodory's solution exists. If it exists, the non autonomous Hamiltonian system $H(p, q, t)$ is depending on the time and we have a new equation:

$$dH(p, q, t)/dt = \partial H(p, q, t)/\partial t$$

The Hamilton-Jacobi's equation with the method of the characteristics leads to the known Lie equations:

$$\begin{aligned} dx/dt &= \partial H / \partial p \\ dp/dt &= -\partial H / \partial x - p \partial H / \partial z \\ dz/dt &= -H + p \partial H / \partial p \end{aligned}$$

The PR function is:

$$\gamma = u(x + \delta \frac{\partial H}{\partial p}) + v(p - \delta (\frac{\partial H}{\partial x} + p \frac{\partial H}{\partial z})) + w(z - \delta (H - p \frac{\partial H}{\partial p})) - \ln(p) - \ln(x) - \ln(z).$$

The critical point is now defined by:

$$\begin{aligned} \frac{\partial \gamma}{\partial x} &= u \left(1 + \delta \frac{\partial^2 H}{\partial p \partial x} \right) - v \delta \left(\frac{\partial^2 H}{\partial x^2} + p \frac{\partial^2 H}{\partial z \partial x} \right) - w \left(\delta \left(\frac{\partial H}{\partial x} - p \frac{\partial^2 H}{\partial p \partial x} \right) - \frac{1}{x} \right) = 0 \\ \frac{\partial \gamma}{\partial p} &= u \delta \frac{\partial^2 H}{\partial p^2} + v - v \delta \left(\frac{\partial^2 H}{\partial p \partial x} + p \frac{\partial^2 H}{\partial z \partial p} + \frac{\partial H}{\partial z} \right) + w \delta p \frac{\partial^2 H}{\partial p^2} - \frac{1}{p} = 0 \\ \frac{\partial \gamma}{\partial z} &= u \delta \frac{\partial^2 H}{\partial p \partial z} - v \delta \left(\frac{\partial^2 H}{\partial z \partial x} + p \frac{\partial^2 H}{\partial z^2} \right) + w \left(1 - \delta \left(\frac{\partial H}{\partial z} - p \frac{\partial^2 H}{\partial p \partial z} \right) - \frac{1}{z} \right) = 0 \end{aligned}$$

$$\partial^2 \hat{H} = \begin{vmatrix} \frac{\partial^2 H}{\partial p \partial x} & -\frac{\partial^2 H}{\partial x^2} - p \frac{\partial^2 H}{\partial z \partial x} & -\frac{\partial H}{\partial x} + p \frac{\partial^2 H}{\partial p \partial x} \\ \frac{\partial^2 H}{\partial p^2} & -\frac{\partial^2 H}{\partial p \partial x} - p \frac{\partial^2 H}{\partial z \partial p} - \frac{\partial H}{\partial z} & +p \frac{\partial^2 H}{\partial p^2} \\ \vdots & \vdots & \vdots \end{vmatrix}$$

$$\left| \begin{array}{cc} \frac{\partial^2 H}{\partial p \partial z} & -\frac{\partial^2 H}{\partial z \partial x} - p \frac{\partial^2 H}{\partial z^2} \\ -\frac{\partial H}{\partial z} + p \frac{\partial^2 H}{\partial p \partial z} \end{array} \right|$$

Let $U = (u, v, w)$ and $\frac{1}{P} = (\frac{1}{x}, \frac{1}{p}, \frac{1}{z})$, the critical point is defined by:

$$U + \delta \partial^2 \hat{H} U - \frac{1}{P} = 0$$

And we apply the asymptotic probabilistic method. Various problems in physics concerning objects in movement obey more or less directly the Hamilton's equations $H(p, q, t)$. They must obey asymptotically differential partial equations of second order. When the movements are autonomous, the equations are the Laplacians.

Conclusion

After this study, we can say, under good conditions, that an EDO is deterministic near the origin of the process, but may have random or fixed cycles after a very long time.

With this probabilistic method, we obtain some new results, but we meet also many new difficulties due to the particular steepest descent's method used to study the Plancherel-Rotach's function.

Reference

- Cirier, G. (2017), Les itérations polynomiales bornées dans \mathbb{R}^d . *Ed. Univ. Europ.*
 (some demonstrations must be improved). - *About the measure of Perron-Frobenius, its density, ...*
 Lasota, A. and Mackey, M.C. (1991): Chaos, Fractals and Noise. *Second Ed., Springer-Verlag.*
 Delabaere E. and Howls C. J, 2002. *Global asymptotics for multiple integrals with boundaries*, Duke Math. J. 112, n° 2 p199-264.
 Plancherel, M. and Rotach, W. (1929), Sur les valeurs asymptotiques des polynômes d'Hermite.... *Commentarii Mathematici Helvetici n 1, p 227 - 254.*
 Ulam, S.M. and von Neumann, J. (1947): On combination of stochastic and deterministic processes. *Bull. 53. AMS, 1120.*
 Cirier, G. (2021), Asymptotic Frequencies of bounded differential equations; *J. of Math. And Stat. Sc.; vol. 7, 11.*
 - *About the equation of Lorenz*
 Lorenz, E. N. (1963), Deterministic Nonperiodic Flow. *J. Atmos. Sci., vol. 20, p. 130-141.*