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# Note on the inverse limit of localizations

Daniel Ferrand

**Abstract.** The inclusion of prime ideals of a commutative ring  $A$  provides its spectrum with an order relation; for any  $A$ -module  $M$ , this order relation induces an (inverse) limit, written  $\varprojlim_{\mathfrak{p} \in \text{Spec}(A)} M_{\mathfrak{p}}$ , between the localizations  $M_{\mathfrak{p}}$  of  $M$ . This note concerns the canonical application  $\lambda_M : M \rightarrow \varprojlim M_{\mathfrak{p}}$ : we identify conditions for this map to be bijective; the basic condition is that if the ring  $A$  is semi-local then  $\lambda_M$  is bijective for any module  $M$ , but for the other rings, finiteness conditions on the modules are necessary.

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## 1. Introduction

Let  $A$  be a commutative ring. The *opposite* of the inclusion of prime ideals defines an order relation on the spectrum  $X = \text{Spec}(A)$ , and for any  $A$ -module  $M$ , the inclusion  $\mathfrak{q} \subset \mathfrak{p}$  gives rise to a localization map  $\rho_{\mathfrak{q}, \mathfrak{p}} : M_{\mathfrak{p}} \rightarrow M_{\mathfrak{q}}$ ; we are interested in the inverse limit<sup>1</sup> of this system, more precisely in conditions under which the canonical map

$$(1.1) \quad \lambda_M : M \rightarrow \varprojlim M_{\mathfrak{p}}$$

is an isomorphism. The case of a torsionless module  $M$  over an integral domain  $A$  is the mother result: let  $K$  be the field of fractions of  $A$ ; the torsionless hypothesis means that the map  $M \rightarrow K \otimes_A M = M_{(0)}$  is injective; that implies that all the maps  $M_{\mathfrak{p}} \rightarrow K \otimes_A M$  are also injective, thus we may identify the modules  $M$  and  $M_{\mathfrak{p}}$  with their image in  $K \otimes_A M$ ; then the limit may be seen to be isomorphic to  $\bigcap M_{\mathfrak{m}}$ , where the  $\mathfrak{m}$  run through the set of maximal ideals, and in that case, the equality

$$(1.2) \quad M = \bigcap M_{\mathfrak{m}}$$

is well known (see [CA, II 3.3 Cor. 4], or the discussion in [GOD, p.124-127], and also the proof of (3.1)).

On the other hand there are plenty of modules which are not isomorphic to the limit of their localizations. For example, consider a ring  $A$  whose set  $\Omega$  of maximal ideals is infinite, and consider the  $A$ -module  $M = \bigoplus_{\mathfrak{m} \in \Omega} A/\mathfrak{m}$ ; then for a prime ideal  $\mathfrak{p}$ , one has  $M_{\mathfrak{p}} = 0$  if  $\mathfrak{p}$  is not maximal, and  $M_{\mathfrak{m}} = A/\mathfrak{m}$  for  $\mathfrak{m} \in \Omega$ ; thus

$$(1.3) \quad M = \bigoplus_{\mathfrak{m} \in \Omega} A/\mathfrak{m} \subsetneq \varprojlim M_{\mathfrak{p}} = \prod_{\mathfrak{m} \in \Omega} A/\mathfrak{m}$$

Note that if the set  $\Omega$  is finite then not only is the inclusion above an equality, but for *any*  $A$ -module  $M$  the map  $\lambda_M$  is an isomorphism; this is quite obvious if  $A$  is a local ring, and, in the semi-local case, that comes easily from faithfully flat descent (4.3), after identifying the limit as a kind of gluing of localizations at maximal ideals (4.1.3). It is the unique situation where a mere hypothesis on the base ring is enough to imply the result for any  $A$ -module

In the situations below the hypotheses often involve the set  $S$  of  $M$ -regular elements of  $A$ ; for example, if the set  $A - S$  of zero-divisors in  $M$  is contained in any maximal ideal of  $A$ , then  $\lambda_M$  is an isomorphism (3.1).

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<sup>1</sup>In the following, the word *limit* alone will always mean *inverse, or projective limit*.

The following criterium motivates many proofs : the map  $\lambda_M$  is bijective if and only if there exists a multiplicative subset  $T \subset A$  and an injective map  $\theta : \varprojlim_X M_{\mathfrak{p}} \rightarrow T^{-1}M$  (5.1). In particular, together with the above semi-local case, it gives a proof of the general result:

(1.4) *The map  $\lambda_M$  is an isomorphism if there exists a finite set of prime ideals,  $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n$  such that the canonical map*

$$M \longrightarrow M_{\mathfrak{p}_1} \times \dots \times M_{\mathfrak{p}_n}$$

*is injective*

This hypothesis is satisfied in two extreme situations : when the ring  $A$  is semi-local, the  $\mathfrak{p}_i$  being the maximal ideals, and when the ring  $S^{-1}A$  is semi-local, where  $S$  the set of all the  $M$ -regular elements, (the primes  $\mathfrak{p}_i$  corresponding then to the maximal ideals of  $S^{-1}A$ ), and also if  $\text{Ass}(M)$  is finite.

(1.5) For proving that an element  $\xi \in \varprojlim M_{\mathfrak{p}}$  comes from  $M$ , a geometric point of view should lead to look for a covering of  $\text{Spec}(A)$  by affine open sets  $U$ , each endowed with a section  $x$  whose image  $\lambda(x) \in \varprojlim_U M_{\mathfrak{p}}$  be equal to  $\xi|U$ ; these conditions together imply that the gluing of these sections is possible and that it gives a global solution. This gluing method has been successfully used long ago assuming  $\text{Ass}(M)$  to be finite (see §6 and §7).

We also indicate a method Grothendieck sketched in a hand written manuscript [GRO] (published after his death) where he uses constructible methods to define the open sets alluded to above, although this way demand noetherian hypotheses, which are not required elsewhere.

## 2. Preliminaries

*In this Note all the rings are tacitly assumed to be commutative.*

**2.1.** Let  $A$  be a ring and let  $M$  be an  $A$ -module ; the spectrum  $X = \text{Spec}(A)$  is viewed as a set ordered by the opposite of the inclusion of primes ideals : an inclusion  $\mathfrak{q} \subset \mathfrak{p}$  induces the *localization map*  $\rho_{\mathfrak{q},\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow M_{\mathfrak{q}}$ .

For a subset  $Y \subset X$ , with the order induced by the one of  $X$ , we define the limit  $\varprojlim_Y M_{\mathfrak{p}}$  as the set of families  $(x_{\mathfrak{p}})_{\mathfrak{p} \in Y} \in \prod_Y M_{\mathfrak{p}}$  such that for all inclusion  $\mathfrak{q} \subset \mathfrak{p}$  of prime ideals in  $Y$ , one has

$$(2.1.1) \quad \rho_{\mathfrak{q},\mathfrak{p}}(x_{\mathfrak{p}}) = x_{\mathfrak{q}},$$

These relations imply that if prime ideals  $\mathfrak{p}$  and  $\mathfrak{p}'$  in  $Y$  contain a prime ideal  $\mathfrak{q} \in Y$  in common, then  $\rho_{\mathfrak{q},\mathfrak{p}}(x_{\mathfrak{p}}) = \rho_{\mathfrak{q},\mathfrak{p}'}(x_{\mathfrak{p}'})$ .

We note  $\lambda_{M,Y} : M \rightarrow \varprojlim_Y M_{\mathfrak{p}}$ , (or simply  $\lambda : M \rightarrow \varprojlim M_{\mathfrak{p}}$  if the module and the range of the prime ideals are clear) the map which associates to an element  $x \in M$  the family of its images in each  $M_{\mathfrak{p}}$ .

By extension of the case of the map  $\prod_X \rightarrow \prod_Y$  we call *projection* the map  $\varprojlim_X M_{\mathfrak{p}} \rightarrow \varprojlim_Y M_{\mathfrak{p}}$  defined by  $(x_{\mathfrak{p}})_{\mathfrak{p} \in X} \mapsto (x_{\mathfrak{p}})_{\mathfrak{p} \in Y}$

**2.2. Regular elements.** Let  $T$  be a multiplicative set in  $A$ , and let  $M$  be an  $A$ -module; the kernel of the map  $\iota_M^T : M \rightarrow T^{-1}M$  is the set of  $x \in M$  such that there exists  $t \in T$  with  $tx = 0$ ; an element  $s \in A$  is said to be  *$M$ -regular* if the map  $M \rightarrow M, x \mapsto sx$  is injective; the set of these elements, usually denoted by  $S$ , is multiplicative; the map  $\iota_M^S : M \rightarrow S^{-1}M$  is injective.

(2.2.1) We introduce the notations  $\text{Spec}(A - S)$  (resp.  $\text{Max}(A - S)$ ) to denote the set of prime ideals of  $A$  which do not meet  $S$  (resp. the set of those ideals which are maximal among those who do not meet  $S$ ); the canonical injection  $\text{Spec}(S^{-1}A) \rightarrow \text{Spec}(A)$  induces bijections  $\text{Spec}(S^{-1}A) \simeq \text{Spec}(A - S)$  and  $\text{Max}(S^{-1}A) \simeq \text{Max}(A - S)$  [CA, II, §2, Prop.11, ii)].

This set  $\text{Max}(A - S)$  of prime ideals of  $A$  (attached to  $M$  via  $S$ ) will play a prominent role in the sequel, in place of the set of associated primes (2.4) often used.

**2.3. Proposition.** *Let  $A$  be a ring,  $X = \text{Spec}(A)$  its spectrum and denote by  $\Omega$  the set of the maximal ideals of  $A$ . Then the following canonical maps are injective :*

- i)  $M \rightarrow \prod_{\Omega} M_{\mathfrak{m}}$ ;*
- ii)  $\lambda_M : M \rightarrow \varprojlim_X M_{\mathfrak{p}}$ ;*
- iii)  $\omega_M : \varprojlim_X M_{\mathfrak{p}} \rightarrow \varprojlim_{\Omega} M_{\mathfrak{m}} = \prod_{\Omega} M_{\mathfrak{m}}$ ;*
- iv)  $M \rightarrow \prod_{\text{Max}(A-S)} M_{\mathfrak{p}}$ , where  $\text{Max}(A-S)$  is the set of prime ideals defined above.*

**Proof:** *i)* One can invoke the “passage from the local to the global”, namely the fact that the  $A$ -module  $\bigoplus_{\Omega} A_{\mathfrak{m}}$  is faithfully flat, as in [CA, II, §3, Prop.10, and Cor.2], but it is more direct to consider the annihilator ideal  $\text{Ann}(x)$  of an element  $x \in M$  of the kernel : the ideal  $\text{Ann}(x)$  is contained in no maximal ideal, so it is equal to  $A$ , thus  $x = 0$ .

*ii)* The map from *i)* factors as  $M \xrightarrow{\lambda_M} \varprojlim_X M_{\mathfrak{p}} \rightarrow \prod_{\Omega} M_{\mathfrak{m}}$ .

*iii)* Any prime ideal  $\mathfrak{p}$  is contained in a maximal one, say  $\mathfrak{m}$ , and for an element  $\xi \in \varprojlim M_{\mathfrak{p}}$ , its  $\mathfrak{p}$ -component  $\xi(\mathfrak{p})$  is, by the definition of the limit (2.1), the image of  $\xi(\mathfrak{m})$  by the localization map  $\rho_{\mathfrak{p},\mathfrak{m}} : M_{\mathfrak{m}} \rightarrow M_{\mathfrak{p}}$  ; thus, if all the maximal components are zero, the same is true for all components.

*iv)* Let  $S$  be the set of  $M$ -regular elements of  $M$ ; the map  $M \rightarrow S^{-1}M$  is injective, and  $S^{-1}M$  is endowed with the structure of an  $S^{-1}A$ -module. The set  $\text{Max}(A-S)$  may be seen as the set of maximal ideals of  $S^{-1}A$ , so the first part shows that the map  $S^{-1}M \rightarrow \prod_{\text{Max}(A-S)} M_{\mathfrak{p}}$  is injective.  $\square$

**2.4. Associated primes.** For an  $A$ -module  $M$ , a prime ideal is said to be *associated* to  $M$  if it is minimal containing the annihilator of an element  $x \in M$ , that is if it is a generic point of the support of the sub-module  $xA \subset M$ ; the set of the prime ideals associated to  $M$  is denoted by  $\text{Ass}(M)$ . This notion is explained in [ACL 6.5], or, under the older name of *weakly* associated prime ideal, in [CA IV, §1, exerc.17 and 19].

The set of  $M$ -regular elements is the complementary subset to the union of the prime ideals associated to  $M$  [ACL, 6.5.8]; it is also the complementary subset to the union of the prime ideals in  $\text{Max}(A-S)$ ; therefore, if  $\text{Ass}(M)$  is finite, then the set  $\text{Max}(A-S)$  is contained in  $\text{Ass}(M)$  and thus it is also finite; but the converse is not true : if  $\text{Max}(A-S)$  is finite any associated prime ideal is contained in at least one element of  $\text{Max}(A-S)$ , but many of them may be contained in the same element of  $\text{Max}(A-S)$ , so we can't conclude that  $\text{Ass}(M)$  is finite. Finally, the formation of  $\text{Ass}(M)$  commutes with localizations [ACL, 6.5.10], whereas the formation of  $\text{Max}(A-S)$  doesn't.

The set  $\text{Ass}(M)$  is finite if the  $A$ -module  $M$  is noetherian [ACL, 6.5.16], for example if  $A$  is noetherian and  $M$  is an  $A$ -module of finite type.

By analogy with (2.3), we find that the following maps are injective :

$$(2.4.1) \quad M \rightarrow S^{-1}M \rightarrow \prod_{\mathfrak{p} \in \text{Ass}(M)} M_{\mathfrak{p}}, \text{ ; and also : } M \xrightarrow{\lambda_M} \varprojlim_X M_{\mathfrak{p}} \rightarrow \varprojlim_{\text{Ass}(M)} M_{\mathfrak{p}} \rightarrow \prod_{\text{Ass}(M)} M_{\mathfrak{p}}$$

### 3. The limit as an intersection of modules

**3.1. Proposition.** [NOR, Th.3.15 ] *Let  $A$  be a ring,  $X = \text{Spec}(A)$  its spectrum and let  $\Omega$  be the set of its maximal ideals. Let  $M$  be an  $A$ -module whose set  $S$  of regular elements contains  $\bigcup_{\mathfrak{m} \in \Omega} (A - \mathfrak{m})$ , in other words each maximal ideal contains all the zero divisors for  $M$ .*

*Then  $S^{-1}M$  contains, for each maximal ideal  $\mathfrak{m}$ , a submodule  $M_{\mathfrak{m}}^*$ , isomorphic to  $M_{\mathfrak{m}}$ , and the maps*

$$M \longrightarrow \varprojlim_{\text{Spec}(A)} M_{\mathfrak{p}} \longrightarrow \bigcap_{\Omega} M_{\mathfrak{m}}^*$$

*are both isomorphism.*

This statement, which generalizes (1.2), is due to Fredrik NORDSTRÖM; his proof is hard to follow; the proof I give below is more in the spirit of (1.2).

**Proof:** For  $\mathfrak{m} \in \Omega$ , the set  $A - \mathfrak{m}$  is assumed to be contained in  $S$ , thus the injective map  $j : M \rightarrow S^{-1}M$  factors through  $M_{\mathfrak{m}}$  giving the maps  $M \xrightarrow{\iota_{\mathfrak{m}}} M_{\mathfrak{m}} \xrightarrow{j_{\mathfrak{m}}} (S^{-1}M)_{\mathfrak{m}} = S^{-1}M$ , and  $j_{\mathfrak{m}}$  is injective as  $j$  is. We denote  $M_{\mathfrak{m}}^*$  the submodule  $j_{\mathfrak{m}}(M_{\mathfrak{m}}) \subset S^{-1}M$ .

The maximal ideals of the ring  $S^{-1}A$  correspond to the prime ideals in  $\text{Max}(A - S)$  (2.2.1). As  $S^{-1}M$  is an  $S^{-1}A$ -module, the map  $v : S^{-1}M \rightarrow \prod_{\text{Max}(A-S)} M_{\mathfrak{p}}$  is injective (2.3 i))

Now let  $\xi \in \varprojlim_X M_{\mathfrak{p}}$ , and let  $\mathfrak{m}$  and  $\mathfrak{n}$  be in  $\Omega$ ; we will show that the  $\mathfrak{m}$  and  $\mathfrak{n}$  components of  $\xi$  have the same image:  $j_{\mathfrak{m}}(\xi(\mathfrak{m})) = j_{\mathfrak{n}}(\xi(\mathfrak{n}))$ .

$$\begin{array}{ccccc}
 & & M_{\mathfrak{m}} & & \\
 & \nearrow \iota_{\mathfrak{m}} & & \searrow j_{\mathfrak{m}} & \\
 M & \xrightarrow{j} & S^{-1}M & \xrightarrow{v} & \prod_{\text{Max}(A-S)} M_{\mathfrak{p}} \\
 & \searrow & & \nearrow j_{\mathfrak{n}} & \\
 & & M_{\mathfrak{n}} & & 
 \end{array}$$

For  $\mathfrak{p} \in \text{Max}(A - S)$  let  $v_{\mathfrak{p}} : S^{-1}M \rightarrow M_{\mathfrak{p}}$  the localization map. Every prime ideal  $\mathfrak{p} \in \text{Max}(A - S)$  is disjoint from  $S$  and thus, by assumption, it is contained in any maximal ideal of  $A$ , and in particular in  $\mathfrak{m}$  and in  $\mathfrak{n}$ , thus the compound maps  $v_{\mathfrak{p}} \circ j_{\mathfrak{m}}$  and  $v_{\mathfrak{p}} \circ j_{\mathfrak{n}}$  are the localization maps  $\rho_{\mathfrak{p},\mathfrak{m}}$  and  $\rho_{\mathfrak{p},\mathfrak{n}}$ ; but (2.1)  $\rho_{\mathfrak{p},\mathfrak{m}}(\xi(\mathfrak{m})) = \rho_{\mathfrak{p},\mathfrak{n}}(\xi(\mathfrak{n}))$ ; since this relation is true for all the ideals  $\mathfrak{p} \in \text{Max}(A - S)$ , and since  $v$  is injective, we conclude that all the images  $j_{\mathfrak{m}}(\xi(\mathfrak{m})) \in S^{-1}M$  are equal and thus they define one unique element  $\xi^* \in \bigcap_{\Omega} M_{\mathfrak{m}}^* \subset S^{-1}M$ . The map  $\xi \mapsto \xi^*$  thus defined is injective since the relation  $\xi^* = 0$  means that for all  $\mathfrak{m} \in \Omega$ ,  $j_{\mathfrak{m}}(\xi(\mathfrak{m})) = 0$ , but the maps  $j_{\mathfrak{m}}$  are injective. So we have defined an injective<sup>2</sup> map  $\varprojlim_X M_{\mathfrak{p}} \rightarrow \bigcap_{\Omega} M_{\mathfrak{m}}^*$  which factors  $j$ .

Let us show now that the map  $M \rightarrow \bigcap_{\Omega} M_{\mathfrak{m}}^*$  is surjective. Let  $\zeta$  be any element in  $\bigcap_{\Omega} M_{\mathfrak{m}}^*$ . Let  $I \subset A$  be the ideal of the  $t \in A$  such that  $t\zeta \in j(M)$ ; we have to check that  $I$  is not contained in any maximal ideal and so  $1 \in I$ . For each  $\mathfrak{m} \in \Omega$ , there exists  $\zeta_{\mathfrak{m}} \in M_{\mathfrak{m}}$  such that  $j_{\mathfrak{m}}(\zeta_{\mathfrak{m}}) = \zeta$ , and thus, for any  $t \in I$ ,  $j_{\mathfrak{m}}(t\zeta_{\mathfrak{m}}) \in j(M) = j_{\mathfrak{m}} \circ \iota_{\mathfrak{m}}(M)$ ; since  $j_{\mathfrak{m}}$  is injective, this relation is equivalent to  $t\zeta_{\mathfrak{m}} \in \iota_{\mathfrak{m}}(M)$ ; but  $\zeta_{\mathfrak{m}}$  being in  $M_{\mathfrak{m}}$ , there exists such a  $t$  which is not in  $\mathfrak{m}$ ; being true for all  $\mathfrak{m} \in \Omega$ , this property shows that  $I = A$  and thus  $\zeta \in j(M)$ . The map  $\varprojlim_X M_{\mathfrak{p}} \rightarrow \bigcap_{\Omega} M_{\mathfrak{m}}^*$  being injective and the compound map  $M \rightarrow \varprojlim_X M_{\mathfrak{p}} \rightarrow \bigcap_{\Omega} M_{\mathfrak{m}}^*$  being an isomorphism, we can conclude that the two following arrows

$$M \longrightarrow \varprojlim_X M_{\mathfrak{p}} \longrightarrow \bigcap_{\Omega} M_{\mathfrak{m}}^*$$

are both isomorphisms. □

#### 4. The limit as a gluing of maximal localizations

**4.1.** The description of limits via products and equalizers, i.e. kernels of parallel arrows, is classical in the context of general categories (see for example [MCL, p.108]), but here we concentrate on the limits of localizations of a module, and the remarks below are directly inspired by the very first steps of descent theory of Grothendieck; they lead to define those limits as equalizers between products of maximal local rings, or products of modules over such rings; this is intuitively clear as soon as the projection map  $\varprojlim_{\mathfrak{p} \subset \mathfrak{m}} A_{\mathfrak{p}} \rightarrow A_{\mathfrak{m}}$  is seen as an isomorphism.

<sup>2</sup>Due to lemma (5.1), this property alone implies that  $\lambda$  is bijective; but the next paragraph is more precise.

Let  $M$  be an  $A$ -module. Let  $X = \text{Spec}(A)$ , and let  $\Omega$  be the set of its maximal ideals. For an inclusion of prime ideals  $\mathfrak{q} \subset \mathfrak{p}$ , and when the context is clear, we denote by the same symbol  $\rho_{\mathfrak{q},\mathfrak{p}}$  the localisation maps  $A_{\mathfrak{p}} \rightarrow A_{\mathfrak{q}}$  and  $M_{\mathfrak{p}} \rightarrow M_{\mathfrak{q}}$ .

**4.1.1.** *The image of the projection map  $\omega_M : \varprojlim_X M_{\mathfrak{p}} \rightarrow \prod_{\Omega} M_{\mathfrak{m}}$  is the set of the families  $(x_{\mathfrak{m}})$  characterised by the following property : if two maximal ideals  $\mathfrak{m}$  and  $\mathfrak{n}$  both contain a prime ideal  $\mathfrak{p}$ , then  $\rho_{\mathfrak{p},\mathfrak{m}}(x_{\mathfrak{m}}) = \rho_{\mathfrak{p},\mathfrak{n}}(x_{\mathfrak{n}})$ . If this property is satisfied for  $(x_{\mathfrak{m}})_{\mathfrak{m} \in \Omega}$  then this element is the projection of the family  $(x_{\mathfrak{p}}) \in \varprojlim_X M_{\mathfrak{p}}$  defined by  $x_{\mathfrak{p}} = \rho_{\mathfrak{p},\mathfrak{m}}(x_{\mathfrak{m}})$  where  $\mathfrak{m}$  is any maximal ideal containing  $\mathfrak{p}$ .*

The indicated property says that the given definition of  $x_{\mathfrak{p}}$  is in fact independent from the choice of the maximal ideal containing  $\mathfrak{p}$ . Now consider an inclusion  $\mathfrak{q} \subset \mathfrak{p}$  of prime ideals, in order to check the relation (2.1.1) one can choose any maximal ideal  $\mathfrak{m}$  containing  $\mathfrak{p}$ , and thus  $x_{\mathfrak{p}} = \rho_{\mathfrak{p},\mathfrak{m}}(x_{\mathfrak{m}})$ ; the transitivity of the localization maps implies that  $\rho_{\mathfrak{q},\mathfrak{p}}(x_{\mathfrak{p}}) = \rho_{\mathfrak{q},\mathfrak{p}}(\rho_{\mathfrak{p},\mathfrak{m}}(x_{\mathfrak{m}})) = \rho_{\mathfrak{q},\mathfrak{m}}(x_{\mathfrak{m}}) = x_{\mathfrak{q}}$ .  $\square$

Two maximal ideals  $\mathfrak{m}, \mathfrak{n} \in \Omega$ , give rise to the two maps  $A_{\mathfrak{m}} \rightarrow A_{\mathfrak{m}} \otimes_A A_{\mathfrak{n}}$  and  $A_{\mathfrak{n}} \rightarrow A_{\mathfrak{m}} \otimes_A A_{\mathfrak{n}}$ ; taking the product of these maps, we get two morphisms of commutative rings

$$\prod_{\Omega} A_{\mathfrak{m}} \begin{array}{c} \xrightarrow{v} \\ \xrightarrow{w} \end{array} \prod_{\Omega \times \Omega} A_{\mathfrak{m}} \otimes_A A_{\mathfrak{n}},$$

namely the maps

$$v : (a_{\mathfrak{m}})_{\mathfrak{m} \in \Omega} \mapsto (a_{\mathfrak{m}} \otimes 1_{\mathfrak{n}})_{(\mathfrak{m},\mathfrak{n}) \in \Omega \times \Omega}, \quad \text{and} \quad w : (a_{\mathfrak{m}})_{\mathfrak{m} \in \Omega} \mapsto (1_{\mathfrak{m}} \otimes a_{\mathfrak{n}})_{(\mathfrak{m},\mathfrak{n}) \in \Omega \times \Omega}.$$

**4.1.2. Proposition.** *The following diagram is exact*

$$\varprojlim A_{\mathfrak{p}} \xrightarrow{\omega} \prod_{\Omega} A_{\mathfrak{m}} \begin{array}{c} \xrightarrow{v} \\ \xrightarrow{w} \end{array} \prod_{\Omega \times \Omega} A_{\mathfrak{m}} \otimes_A A_{\mathfrak{n}}.$$

**Proof:** The exactness means that the map  $\omega$  is injective, a fact already seen in (2.3, *iii*), and that its image is equal to the equalizer of the two maps  $v$  and  $w$ . From (4.1.1) we know that the image of the map  $\omega$  is the set of elements  $\alpha = (\alpha_{\mathfrak{m}}) \in \prod_{\Omega} A_{\mathfrak{m}}$  such that

$$(\star) \quad \rho_{\mathfrak{p},\mathfrak{m}}(\alpha_{\mathfrak{m}}) = \rho_{\mathfrak{p},\mathfrak{n}}(\alpha_{\mathfrak{n}}) \quad \text{for all prime ideals } \mathfrak{p} \subset \mathfrak{m} \cap \mathfrak{n},$$

a condition we have to check to be equivalent to  $v(\alpha) = w(\alpha)$ , that is for all  $\mathfrak{m}$  and  $\mathfrak{n}$  in  $\Omega$ ,

$$(\star\star) \quad \alpha_{\mathfrak{m}} \otimes 1_{\mathfrak{n}} = 1_{\mathfrak{m}} \otimes \alpha_{\mathfrak{n}} \quad \text{in } A_{\mathfrak{m}} \otimes_A A_{\mathfrak{n}}$$

Since  $A_{\mathfrak{m}} \otimes_A A_{\mathfrak{n}} = (A - \mathfrak{m})^{-1}(A - \mathfrak{n})^{-1}A$  the set  $\text{Spec}(A_{\mathfrak{m}} \otimes_A A_{\mathfrak{n}})$  will be seen as the subset  $\text{Spec}(A_{\mathfrak{m}}) \cap \text{Spec}(A_{\mathfrak{n}})$  of  $\text{Spec}(A)$ , that is the set of the prime ideals contained in both  $\mathfrak{m}$  and  $\mathfrak{n}$ ; for such a prime  $\mathfrak{p} \subset \mathfrak{m} \cap \mathfrak{n}$ , the map  $A_{\mathfrak{m}} \otimes_A A_{\mathfrak{n}} \rightarrow A_{\mathfrak{p}}$  is the product, in the ring  $A_{\mathfrak{p}}$ , of the localization maps on each factors :

$$A_{\mathfrak{m}} \otimes_A A_{\mathfrak{n}} \xrightarrow{\rho_{\mathfrak{p},\mathfrak{m}} \otimes \rho_{\mathfrak{p},\mathfrak{n}}} A_{\mathfrak{p}} \otimes_A A_{\mathfrak{p}} = A_{\mathfrak{p}}, \quad a_{\mathfrak{m}} \otimes a_{\mathfrak{n}} \mapsto \rho_{\mathfrak{p},\mathfrak{m}}(a_{\mathfrak{m}})\rho_{\mathfrak{p},\mathfrak{n}}(a_{\mathfrak{n}}).$$

But the map  $A_{\mathfrak{m}} \otimes_A A_{\mathfrak{n}} \rightarrow \varprojlim_{\mathfrak{p} \subset \mathfrak{m} \cap \mathfrak{n}} A_{\mathfrak{p}}$  is injective (2.3, *ii*), so the previous two conditions are indeed equivalent.  $\square$

The above description of the limit of localizations of a ring  $A$  extends formally to limits of localizations of an  $A$ -module  $M$ ; it is enough to replace the rings  $A_{\mathfrak{p}}$ ,  $A_{\mathfrak{m}}$  and  $A_{\mathfrak{m}} \otimes_A A_{\mathfrak{n}}$ , by, respectively,  $M \otimes_A A_{\mathfrak{p}}$ ,  $M \otimes_A A_{\mathfrak{m}}$  and  $M \otimes_A A_{\mathfrak{m}} \otimes_A A_{\mathfrak{n}}$ , and to replace each map between the above indicated rings by tensor on the left with  $\text{Id}_M$ . The main point is the injectivity of the map  $M \otimes_A A_{\mathfrak{m}} \otimes_A A_{\mathfrak{n}} \rightarrow \varprojlim_{\mathfrak{p} \subset \mathfrak{m} \cap \mathfrak{n}} M \otimes_A A_{\mathfrak{p}}$ , which is nothing but (2.3 *ii*) with  $A_{\mathfrak{m}} \otimes_A A_{\mathfrak{n}}$  as base ring. Then we get :

**4.1.3. Proposition.** *Let  $\Omega$  be the set of maximal ideals of the ring  $A$ , and let  $M$  be an  $A$ -module. By considering the above defined maps  $\omega' = \text{Id}_M \otimes_A \omega$ , and the same for  $v'$  and  $w'$ , we get the diagram*

$$\varprojlim M_{\mathfrak{p}} \xrightarrow{\omega'} \prod_{\Omega} M \otimes_A A_{\mathfrak{m}} \xrightleftharpoons[w']{v'} \prod_{\Omega \times \Omega} M \otimes_A A_{\mathfrak{m}} \otimes_A A_{\mathfrak{n}}.$$

*This diagram is exact.* □

**4.2. Corollary.** *Let  $A$  be a semi-local ring and let  $M$  be an  $A$ -module. Then the canonical map  $\lambda_M : M \rightarrow \varprojlim M_{\mathfrak{p}}$  is an isomorphism.*

**Proof:** Let  $\mathfrak{m}_1, \dots, \mathfrak{m}_n$  be the maximal ideals of  $A$ . The morphism  $A \rightarrow A' = \prod_i A_{\mathfrak{m}_i}$  is faithfully flat; consider the diagram  $A \rightarrow A' \rightrightarrows A' \otimes_A A'$  where the two parallel maps are  $a' \mapsto a' \otimes 1$  and  $a' \mapsto 1 \otimes a'$ ; this diagram is exact and also its companion diagram [ACL, (8.7.10)]

$$(4.2.1) \quad M \rightarrow M \otimes_A A' \rightrightarrows M \otimes_A A' \otimes_A A'$$

Since the set of maximal ideals of  $A$  is finite, the canonical morphisms

$$M \otimes_A \prod_{i=1}^n A_{\mathfrak{m}_i} \rightarrow \prod_{i=1}^n M \otimes_A A_{\mathfrak{m}_i}$$

are isomorphisms, and now the exact diagram (4.2.1) may be read as

$$M \longrightarrow \prod_{i=1}^n M \otimes_A A_{\mathfrak{m}_i} \rightrightarrows \prod_{(i,j)} M \otimes_A A_{\mathfrak{m}_i} \otimes_A A_{\mathfrak{m}_j}$$

The comparison with (4.1.3) shows that  $M \rightarrow \varprojlim M_{\mathfrak{p}}$  is an isomorphism. □

## 5. A general sufficient condition

**5.1. Lemma.** *Let  $A$  be a ring and  $X = \text{Spec}(A)$  be its spectrum. For an  $A$ -module  $M$  the map  $M \xrightarrow{\lambda_M} \varprojlim_X M_{\mathfrak{p}}$  is an isomorphism if and only if, there is a multiplicative subset  $S \subset A$  consisting of  $M$ -regular elements, such that the canonical map  $M \rightarrow S^{-1}M$  factors through  $\varprojlim_X M_{\mathfrak{p}}$  as*

$$M \xrightarrow{\lambda_M} \varprojlim_X M_{\mathfrak{p}} \xrightarrow{\theta} S^{-1}M$$

*where the map  $\theta$  is injective.*

**Proof:** The condition is necessary: with the set  $S$  of all the  $M$ -regular elements, the map  $M \simeq \varprojlim_X M_{\mathfrak{p}} \rightarrow S^{-1}M$  is injective.

Conversely, let  $M \xrightarrow{\lambda_M} \varprojlim_X M_{\mathfrak{p}} \xrightarrow{\theta} S^{-1}M$  be a factorisation of the map  $M \rightarrow S^{-1}M$  with an injective map  $\theta$ ; due to (2.3,  $i$ ), it remains to check that  $\lambda = \lambda_M$  is surjective.

Let  $\xi \in \varprojlim_X M_{\mathfrak{p}}$ ; its image  $\theta(\xi) \in S^{-1}M$  may be written as  $\theta(\xi) = (x/s)$  with  $x \in M$  and  $s \in S$ , thus  $\theta(\lambda(x) - s\xi) = 0$ . Since  $\theta$  is injective, we have, in  $\varprojlim M_{\mathfrak{p}}$ , the equality  $\lambda(x) = s\xi$ ; an equality we read, in  $M_{\mathfrak{p}}$  as  $x/1 = s\xi(\mathfrak{p})$ ; thus, for all  $\mathfrak{p}$ , one has  $xA_{\mathfrak{p}} \subset sM_{\mathfrak{p}}$ , that is  $Ax \subset sM$  [CA, II, Cor.1 of Thm.1], or  $x = sz$  with  $z \in M$ . But then  $\theta(\lambda(z)) = (x/s) = \theta(\xi)$  and the injectivity of  $\theta$  gives the conclusion  $\xi = \lambda(z)$ . □

**5.2. Theorem.** *Let  $A$  be a ring and let  $M$  be an  $A$ -module. If there exists a finite set of prime ideals,  $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n$  such that the canonical map*

$$(5.2.1) \quad M \longrightarrow M_{\mathfrak{p}_1} \times \dots \times M_{\mathfrak{p}_n}$$

*is injective, then the map*

$$M \longrightarrow \varprojlim_{\mathfrak{p} \in \text{Spec}(A)} M_{\mathfrak{p}}$$

*is an isomorphism.*

**Proof:** Let  $S$  be the complementary subset in  $A$  of the union of the  $\mathfrak{p}_i$ ; since the map  $S^{-1}A \rightarrow \prod_{i=1}^n A_{\mathfrak{p}_i}$  is faithfully flat, it remains injective by tensoring by  $M$  [CA, I §3.1] so the map  $S^{-1}M \rightarrow \prod_{i=1}^n M_{\mathfrak{p}_i}$  is also injective; the injectivity of the map (5.2.1) shows that

$$M \longrightarrow S^{-1}M$$

is injective, thus all the elements in  $S$  are regular for  $M$ .

Let  $X = \text{Spec}(A)$ , and  $Y = \text{Spec}(S^{-1}A)$ . The subset  $Y \subset X$  induces the projection morphism

$$\varprojlim_X M_{\mathfrak{p}} \longrightarrow \varprojlim_Y M_{\mathfrak{p}};$$

this morphism is injective: in fact, let  $\xi \in \varprojlim_X M_{\mathfrak{p}}$  whose image in  $\varprojlim_Y M_{\mathfrak{p}}$  is zero; in particular  $\xi(\mathfrak{p}_i) = 0$  for all  $i$ ; let  $\mathfrak{p}$  be any prime ideal in  $X$ ; we have to check that  $\xi(\mathfrak{p}) \in M_{\mathfrak{p}}$  is zero. Consider the map

$$M_{\mathfrak{p}} \longrightarrow \prod (M_{\mathfrak{p}_i})_{\mathfrak{p}}.$$

Since  $\text{Spec}((A_{\mathfrak{p}_i})_{\mathfrak{p}}) = \text{Spec}(A_{\mathfrak{p}_i}) \cap \text{Spec}(A_{\mathfrak{p}})$ , we dispose, as in (4.1.2), of an injective map

$$(M_{\mathfrak{p}_i})_{\mathfrak{p}} \longrightarrow \varprojlim_{\mathfrak{q} \in \mathfrak{p}_i \cap \mathfrak{p}} M_{\mathfrak{q}};$$

but  $\xi(\mathfrak{q}) = \rho_{\mathfrak{q}, \mathfrak{p}_i}(\xi(\mathfrak{p}_i)) = 0$ , so the image of  $\xi(\mathfrak{p})$  in  $(M_{\mathfrak{p}_i})_{\mathfrak{p}}$  is zero for all  $\mathfrak{p}_i$ , and the map  $M_{\mathfrak{p}} \rightarrow \prod (M_{\mathfrak{p}_i})_{\mathfrak{p}}$  is injective by assumption; so  $\xi(\mathfrak{p}) = 0$ .

Since  $S^{-1}A$  is semilocal, the canonical map

$$\lambda' : S^{-1}M \longrightarrow \varprojlim_Y M_{\mathfrak{p}}$$

is an isomorphism (4.3), and, using its inverse, we get an injective map  $\theta : \varprojlim_{\mathfrak{p} \in X} M_{\mathfrak{p}} \longrightarrow S^{-1}M$ . The conclusion now follows from the lemma (5.1).  $\square$

**5.3. Corollary.** *Let  $M$  be an  $A$ -module and  $S$  be its set of regular elements. If the ring  $S^{-1}A$  is semilocal, then the map*

$$\lambda_M : M \rightarrow \varprojlim M_{\mathfrak{p}}$$

*is bijective when  $\mathfrak{p}$  runs through  $\text{Spec}(A)$ .*

**Proof:** The map

$$M \rightarrow \prod_{\text{Max}(A-S)} M_{\mathfrak{p}}$$

is injective (2.3, *iv*).  $\square$

## 6. Gluing methods

For proving that an element  $\xi \in \varprojlim M_{\mathfrak{p}}$  comes from  $M$ , a natural idea - i.e. a geometric one - is to find a covering of  $X = \text{Spec}(A)$  by affine open sets  $U$ , each endowed with a section  $x \in \Gamma(U, \widetilde{M})$  whose image via the map  $\Gamma(U, \widetilde{M}) \rightarrow \varprojlim_U M_{\mathfrak{p}}$  be equal to the restriction  $\xi|_U$ ; given two such open sets  $U$  and  $U'$  with sections  $x$  and  $x'$ , their restrictions on  $U \cap U'$  will both induce the same element  $\xi|_{U \cap U'}$  in  $\varprojlim_{U \cap U'} M_{\mathfrak{p}}$ , and thus, by Lemma (2.3, *ii*)), these sections  $x$  and  $x'$  are equal on  $U \cap U'$ ; so, by gluing, they come from a global section of  $M$  [GOD, p. 109].

Unfortunately the use of the set  $\text{Max}(A - S)$  (2.2) of the maximal ideal from  $S^{-1}A$  is not possible with gluing because this set doesn't localize, contrary to  $\text{Ass}(M)$  [ACL, 6.5.10]. So here is a proof of the surjectivity of  $\lambda_M$  by gluing but under the assumption that  $\text{Ass}(M)$  is a finite set<sup>3</sup>.

Giving an element  $\xi \in \varprojlim_X M_{\mathfrak{p}}$ , for each maximal ideal  $\mathfrak{m}$  we choose  $m \in M$  and  $s \in A - \mathfrak{m}$  such that  $\xi(\mathfrak{m}) = (m/s)$ ; for each prime ideal  $\mathfrak{p} \in D(s)$  we consider the canonical localization map  $j_{\mathfrak{p}} : M_s \rightarrow M_{\mathfrak{p}}$  and we define the set

$$U = \{\mathfrak{p} \in D(s), j_{\mathfrak{p}}(m/s) = \xi(\mathfrak{p})\}.$$

If  $U$  is open, or, if need be, an affine open subset, containing  $\mathfrak{m}$ , then the section  $m/s \in \Gamma(U, \widetilde{M})$  will induce  $\xi|_U \in \varprojlim_U M_{\mathfrak{p}}$ . For proving that  $U$  is open, the point is :

*Let  $\mathfrak{p}$  be a prime ideal in  $D(s)$ . Then  $\mathfrak{p} \in U$  if and only if  $\text{Ass}(M_{\mathfrak{p}}) \subset U$ .*

(Proof: If  $\mathfrak{p}$  is in  $U$  then each  $\mathfrak{q} \in \text{Ass}(M_{\mathfrak{p}})$  (which is a subset of  $\text{Spec}(A_{\mathfrak{p}})$ ) is also in  $U$  since  $j_{\mathfrak{q}} = \rho_{\mathfrak{q}, \mathfrak{p}} \circ j_{\mathfrak{p}}$ . Conversely, if  $\text{Ass}(M_{\mathfrak{p}}) \subset U$  then for any  $\mathfrak{q} \in \text{Ass}(M_{\mathfrak{p}})$  one has  $\rho_{\mathfrak{q}, \mathfrak{p}}(j_{\mathfrak{p}}(m/s) - \xi(\mathfrak{p})) = j_{\mathfrak{q}}(m/s) - \xi_{\mathfrak{q}} = 0$ ; but the map  $M_{\mathfrak{p}} \rightarrow \prod_{\mathfrak{q} \in \text{Ass}(M_{\mathfrak{p}})} M_{\mathfrak{q}}$  is injective (2.4.1) thus  $j_{\mathfrak{p}}(m/s) - \xi(\mathfrak{p}) = 0$  and  $\mathfrak{p} \in U$ .)

Now let  $T$  be the union of the closure  $V(\mathfrak{q})$  of the  $\mathfrak{q} \in \text{Ass}(M_s)$  which are *not* in  $U$ ; it is a closed set in  $D(s)$  since  $\text{Ass}(M_s) \subset \text{Ass}(M)$  is finite. Then  $U = D(s) - (D(s) \cap T)$ .  $\square$

## 7. The Grothendieck's point of view

Here is what I understood from a rough draft from Grothendieck [GRO], for the lecture of which L. Malgoire helped me a lot. The first part of the proof given below is a mere translation of [EGA, 0<sub>I</sub>, 2.4.7]

**7.1. Theorem.** *Let  $A$  be a noetherian ring and let  $M$  be an  $A$ -module of finite type. Then the canonical map*

$$\lambda_M : M \rightarrow \varprojlim_{\mathfrak{p} \in \text{Spec}(A)} M_{\mathfrak{p}}$$

*is an isomorphism.*

**Proof:** Let  $\xi = (\xi(\mathfrak{p})) \in \varprojlim_X M_{\mathfrak{p}}$ , and let  $\mathfrak{m}$  be a prime ideal; on a suitable open set  $D(s)$  containing  $\mathfrak{m}$ , we may extend  $\xi(\mathfrak{m})$  as an element  $(x/s) \in M_s$ . For each prime ideal  $\mathfrak{p} \in D(s)$  we consider the canonical localization map  $j_{\mathfrak{p}} : M_s \rightarrow M_{\mathfrak{p}}$  and we define the set (noted  $U$  above but here we follow the notation in EGA)

$$E = \{\mathfrak{p} \in D(s), j_{\mathfrak{p}}(x/s) = \xi(\mathfrak{p})\}.$$

The point is again to check that  $E$  is an open neighborhood of  $\mathfrak{m}$ . By construction,  $\mathfrak{m}$  is in  $E$ , and  $E$  is stable by generization: if  $\mathfrak{q} \subset \mathfrak{p}$  with  $\mathfrak{p} \in E$ , then we have also  $\mathfrak{q} \in E$ , since the localization map  $\rho_{\mathfrak{q}, \mathfrak{p}} : M_{\mathfrak{p}} \rightarrow M_{\mathfrak{q}}$  sends  $\xi(\mathfrak{p})$  to  $\xi(\mathfrak{q})$ . Let  $U$  be the greatest open set of  $X = \text{Spec}(A)$  contained in  $E$  (i.e. the "interior" of  $E$ ). Since  $A$  is noetherian the closed set  $X - U$  is the finite union of its irreducible components. Arguing by contradiction we suppose that  $U \neq E$ ; thus one of these components, say  $\{\bar{\mathfrak{q}}\}$  meets  $E$ , therefore its *generic* prime  $\mathfrak{q}$  is in  $E$ ; let  $T$  be the (finite) union of the other components; the

<sup>3</sup>This is a simplification of the proof I found in 1966

set  $V = X - T$  is open in  $X$ . Since  $X - U = \{\bar{\mathfrak{q}}\} \cup T = (\{\bar{\mathfrak{q}}\} \cap V) \sqcup T$ ,  $V$  is the disjoint union of  $U$  and of the set  $Y = \{\bar{\mathfrak{q}}\} \cap V$ .

Let us show that  $E \cap Y$  contains a non empty subset  $W$  which is open in  $Y$ :

The kernel  $N$  of the map  $j_{\mathfrak{q}} : M_s \rightarrow M_{\mathfrak{q}}$  is of finite type and  $N_{\mathfrak{q}} = 0$ ; thus there exists  $t \in A - \mathfrak{q}$  such that  $N_t = 0$ . The set  $W = \{\bar{\mathfrak{q}}\} \cap V \cap D(st)$  is open in  $Y = \{\bar{\mathfrak{q}}\} \cap V$ . Let us show the inclusion  $W \subset E$ : let  $\mathfrak{p} \in W$ ; since  $t \notin \mathfrak{p}$ , the localisation morphism  $\rho_{\mathfrak{q}, \mathfrak{p}} : M_{\mathfrak{p}} \rightarrow M_{\mathfrak{q}}$  is injective; but the elements  $j_{\mathfrak{p}}(x/s)$  and  $\xi(\mathfrak{p})$  have the same image  $\xi(\mathfrak{q})$  in  $M_{\mathfrak{q}}$ , so one has  $j_{\mathfrak{p}}(x/s) = \xi(\mathfrak{p})$  and thus  $\mathfrak{p} \in E$ .

From that we shall conclude that  $U \cup W$  is open in  $V = U \sqcup Y$ , thus it is open in  $X$ , and this is a contradiction since  $U$  is the interior of  $E$ ; so  $U = E$ , and  $E$  is indeed open.  $\square$

## 8. Chronology

In the middle of the sixties the theme of this Note arises between members of the Samuel seminar ; I gave a proof for the isomorphism of a module with the limit of its localizations, by an heavy gluing method, and under noetherian hypotheses ; I owe to Daniel LAZARD the observation that the use of “weakly” associated prime ideals is enough for the proof to work; then I wrote a detailed manuscript which did not circulate away from a little circle of colleagues, and I only published an abstract, without any proof [FER].

Some years later D. LAZARUS asked me for explanations about the case of semi-local ring, giving me an occasion for finding out the simplification provided by the faithfully flat descent (4.3)

On 2003 I had the opportunity to meet Fredrik NORDSTRÖM at the K.T.H. in Stockholm; his master thesis [NOR], with Roy SKJELNES as academic supervisor, contains most of the results of the present Note, and also the statement (3.1), which, as far as I know, was new. Unfortunately his proofs are often hard to follow.

At the end of 2009, the question : “*When is a commutative ring the limit of its local rings ?*” appears on Mathoverflow, without giving rise to very much work, essentially just a counter-example.

Recently, by chance, I have found in the GROTHENDIECK archives the rough draft entitled : *Réseaux d’anneaux locaux*; it is hand-written and I needed the help of Jean MALGOIRE to decipher it. The method in this draft seems to require noetherian assumptions inevitably.

Now I feel the times has come to write the proofs down. All the proofs in this Note, except for (7.1), are new and often simpler than those from the 1966 manuscript, but today not all the statements in [FER] seem to deserve to be proved here.

## References

- [CA] N. BOURBAKI (1989) *Elements of Mathematics, Commutative Algebra, Chapters 1-7*, Springer
- [ACL] A. CHAMBERT-LOIR (2021), *(Mostly) Commutative Algebra*, Springer.
- [FER] D. FERRAND, (1966), Sur les modules qui sont limite projective de leurs localisés, *C. R. Acad. Sc. Paris*, **t.262**, p.609-611 (14 mars 1966).
- [GOD] R. GODEMENT, (1958), *Théorie des faisceaux*, Paris, Hermann.
- [EGA] A. GROTHENDIECK, J.A. DIEUDONNÉ, (1971), *Éléments de Géométrie Algébrique*, Chapitre I, Springer.
- [GRO] A. GROTHENDIECK, (?), Réseaux d’anneaux locaux, *Institut Montpellierain Alexander Grothendieck*, [8 pages], <https://grothendieck.umontpellier.fr/web/viewer.html?file=../42.pdf>

[MCL] S. MAC LANE,(1971), *Categories for the Working Mathematicien*, Springer-Verlag

[NOR] F. NORDSTRÖM,(2003), Recovering a module from its local structure, Master's Thesis - MAT-2003-06 Dept. Math., KTH, Stockholm.

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