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A NOTE ON CONTOUR INTEGRALS REPRESENTING JACKSON INTEGRALS AND THEIR APPLICATIONS

CHANGGUI ZHANG

ABSTRACT. In our paper with M. E. H. Ismail [6], we are interested to study the zeros of a family of entire functions introduced by Ramanujan, using contour integrals and connection formulas for solutions of linear q -difference equations. Many years ago, M. E. H. Ismail provided a simple proof of Ramanujan's ${}_1\psi_1$ summation formula [5]. In this note, we aim to present an analytic proof of this formula using similar approaches to those in [6]. Furthermore, building on these methods, we derive a decomposition formula for the sum-function of a q -analog of the Euler series, which has the potential to be generalized to a broader class of q -difference equations.

INTRODUCTION

Throughout the whole paper, q denotes a complex number such that $0 < |q| < 1$. Given a non-zero complex number λ and a function defined in a neighborhood of the q -spiral $\lambda q^{\mathbb{Z}}$, the Jackson integral $\int_0^{\lambda\infty} f(x) d_q x$ is defined as follows :

$$(0.1) \quad \int_0^{\lambda\infty} f(z) d_q z = (1-q)\lambda \sum_{n \in \mathbb{Z}} f(\lambda q^n) q^n,$$

provided that the right-hand side converges; see [2, Remark A.5.]. The following notation will be used:

- (1) First of all, the generalized q -Pochhammer symbol is defined for positive n or infinite limits as ([3, §1.2]):

$$(a; q)_n = (1-a)(1-aq)\dots(1-aq^{n-1}), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1-aq^k).$$

It follows that

$$(0.2) \quad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}$$

for any non-negative integer n , provided $aq^n \notin q^{\mathbb{Z}_{\leq 0}}$. Furthermore, this expression is extended to $n \in \mathbb{Z}_{< 0}$, assuming $aq^n \notin q^{\mathbb{Z}_{\leq 0}}$. Consequently, for $n \in \mathbb{Z}_{\geq 1}$ and $a \notin (q^{n+\mathbb{Z}_{\leq 0}} \cup \{0\})$, substituting n with $-n$ into (0.2) yields :

$$(0.3) \quad (a; q)_{-n} = \frac{(-1)^n}{a^n (q/a; q)_n} q^{n(n+1)/2}.$$

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We dedicate this note to our dear friend, Professor Mourad E. H. Ismail, in celebration of his 80th anniversary. This project is supported by the Labex CEMPI (ANR-11-LABX-0007).

Additionally, when a_1, \dots, a_k are k complex numbers, we write simply:

$$(a_1, \dots, a_k; q)_n = (a_1; q)_n \dots (a_k; q)_n; \quad (a_1, \dots, a_k; q)_\infty = (a_1; q)_\infty \dots (a_k; q)_\infty.$$

(2) Define the Jacobi theta function as:

$$(0.4) \quad \theta(z) = \theta(q; z) = \sum_{n \in \mathbb{Z}} q^{n(n-1)/2} z^n, \quad z \in \mathbb{C}^*.$$

The classic Jacobi triple-product formula states:

$$(0.5) \quad \theta(z) = \left(q, -z, -\frac{q}{z}; q \right)_\infty.$$

Hence, $\theta(z) = 0$ if and only if $z \in (-q^{\mathbb{Z}})$. Additionally, the theta function satisfies:

$$(0.6) \quad \theta(q^n z) = x^{-n} q^{-n(n-1)/2} \theta(z), \quad n \in \mathbb{Z}.$$

(3) As in [2, (1.9)], the q -logarithm $\ell(q; z)$ is defined as follows:

$$(0.7) \quad \ell(z) = \ell(q; z) = -z \frac{\theta'(-z)}{\theta(-z)}.$$

Differentiating both sides of Equation (0.6) with respect to z , it follows that $\ell(z)$ satisfies the linear q -difference equations:

$$(0.8) \quad \ell(q^n z) = \ell(z) - n, \quad n \in \mathbb{Z},$$

and has simple poles at $q^{\mathbb{Z}}$.

(4) Given $\lambda \in \mathbb{C}^*$, let $[\lambda] = [\lambda; q] = \lambda q^{\mathbb{Z}}$ denote the q -spiral passing through λ . Define

$$\varepsilon(q) = \min \left(\frac{|1 - q^n|}{1 + |q|^n} : n \in \mathbb{Z}_{>0} \right),$$

and let $D(\alpha; r)$ denote the open disc $|z - \alpha| < r$. For any $\epsilon \in (0, \varepsilon(q))$ and $\lambda \in \mathbb{C}^*$, notice that:

$$(q^m \bar{D}(\lambda; \epsilon|\lambda)) \cap (q^n \bar{D}(\lambda; \epsilon|\lambda)) = \emptyset, \quad (m, n) \in \mathbb{Z}^2, \quad n \neq m.$$

Define

$$(0.9) \quad \bar{D}_\epsilon^{[\lambda]} = \cup_{n \in \mathbb{Z}} (q^n \bar{D}(\lambda; \epsilon|\lambda)), \quad \mathcal{C}_\epsilon^{[\lambda]} = \cup_{n \in \mathbb{Z}} (q^n \partial^+ D(\lambda; \epsilon|\lambda)).$$

From now on, we are ready to represent the Jackson integral (0.1) as follows.

Lemma 0.1. *Let $\lambda \in \mathbb{C}^*$, $\epsilon \in (0, \varepsilon(q))$, and let U be any domain in \mathbb{C}^* with $\bar{D}_\epsilon^{[\lambda]} \subset U$. The following equality holds for any analytic function f in U :*

$$(0.10) \quad \int_0^{\lambda\infty} f(z) d_q z = \frac{1-q}{2\pi i} \int_{\mathcal{C}_\epsilon^{[\lambda]}} \ell\left(\frac{x}{\lambda}\right) f(z) dz,$$

provided that the left-hand side converges.

Proof. This result follows immediately from Cauchy's residue theorem; see [9, Chap. VI]. \square

The remainder of this paper is divided into two parts. In the first part, we use Lemma 0.1 to present an analytic proof of Ramanujan's ${}_1\psi_1$ summation formula, employing approaches similar to those in [6]. In the second part, building on these methods, we derive a decomposition formula for the sum-function of a q -analog of the Euler series.

1. A PROOF OF RAMANUJAN'S ${}_1\psi_1$ SUM

The following bilateral summation formula, known as Ramanujan's ${}_1\psi_1$ sum:

$$(1.1) \quad \sum_{n \in \mathbb{Z}} \frac{(a; q)_n}{(b; q)_n} z^n = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \quad |b/a| < |z| < 1,$$

where the left-hand side is by definition ${}_1\psi_1(a; b; q, z)$, was first given by Ramanujan [4, p. 222, (12.12.2)]. Several mathematicians, including M. E. H. Ismail [5], have published different proofs of (1.1). For further details, see [3, p.126]. This section is to outline a proof based upon the analytic theory of q -difference equations with the help of Lemma 0.1.

Indeed, by applying (0.2), the bilateral basic hypergeometric series ${}_1\psi_1(a; b; q, z)$ can be rewritten as:

$$(1.2) \quad {}_1\psi_1(a; b; q, z) = \frac{(a; q)_\infty}{(b; q)_\infty} \sum_{n \in \mathbb{Z}} \frac{(bq^n; q)_\infty}{(aq^n; q)_\infty} z^n.$$

Using (0.2) and (0.3), one obtains that the Laurent series ${}_1\psi_1(a; b; q, z)$ converges to an analytic function in the annulus $|b/a| < |z| < 1$. Moreover, this function can be continued as a meromorphic solution in \mathbb{C}^* to the q -difference equation:

$$(1.3) \quad \left(\frac{b}{q} - az \right) y(qz) = (1 - z)y(z)$$

Since $\frac{(az, q/az; q)_\infty}{(z, b/az; q)_\infty}$ is also a solution of Equation (1.3), there exists a meromorphic function $C(a, b; q, z)$ in \mathbb{C}^* such that

$$(1.4) \quad {}_1\psi_1(a; b; q, z) = C(a, b; q, z) \frac{(az, q/az; q)_\infty}{(z, b/az; q)_\infty},$$

where $C(a, b; q, qz) = C(a, b; q, z)$.

Theorem 1.1. *The q -periodic function $C(a, b; q, z)$ defined in the connection formula (1.4) is constant and can be expressed as:*

$$(1.5) \quad C(a, b; q, z) = \frac{(q, b/a; q)_\infty}{(b, q/a; q)_\infty}.$$

Proof. For the rest of this section, we assume that $|b| < |qa|$. Thus, the annulus $|b/a| < |z| < 1$ covers the fundamental domain $|q| \leq |z| < 1$ with respect to the action $z \mapsto qz$. It follows that the q -periodic function $C(a, b; q, z)$ defined in (1.4) is constant. Setting $z = q$ in both (1.2) and (1.4), we obtain:

$$C(a, b; q, q) = \frac{(q, b/qa, a; q)_\infty}{(qa, 1/a, b; q)_\infty} \sum_{n \in \mathbb{Z}} \frac{(bq^n; q)_\infty}{(aq^n; q)_\infty} q^n,$$

or, equivalently:

$$(1.6) \quad C(a, b; q, q) = -a \frac{(q, b/qa; q)_\infty}{(b, q/a; q)_\infty} \sum_{n \in \mathbb{Z}} \frac{(bq^n; q)_\infty}{(aq^n; q)_\infty} q^n.$$

By (0.1), it follows that

$$\sum_{n \in \mathbb{Z}} \frac{(bq^n; q)_\infty}{(aq^n; q)_\infty} q^n = \frac{1}{1 - q} \int_0^\infty \frac{(bz; q)_\infty}{(az; q)_\infty} d_q z.$$

Thus, using (0.10) for $\lambda = 1$ yields:

$$(1.7) \quad \sum_{n \in \mathbb{Z}} \frac{(bq^n; q)_\infty}{(aq^n; q)_\infty} q^n = \frac{1}{2\pi i} \int_{\mathcal{C}_\epsilon^{[1]}} \ell(z) \frac{(bz; q)_\infty}{(az; q)_\infty} dz,$$

where $\epsilon \in (0, \varepsilon(q))$. Let

$$\mathcal{C}_{\epsilon, \infty}^{[1/a]} = \cup_{n \in \mathbb{Z}_{\leq 0}} (q^n \partial^+ D(1/a; \epsilon/|a|)).$$

Assume that $\epsilon \in (0, \varepsilon(q))$ is chosen such that $\mathcal{C}_{\epsilon, \infty}^{[1/a]} \cap \mathcal{C}_\epsilon^{[1]} = \emptyset$. By applying the residue theorem, we have:

$$\left(\int_{\mathcal{C}_\epsilon^{[1]}} + \int_{\mathcal{C}_{\epsilon, \infty}^{[1/a]}} \right) \ell(z) \frac{(bz; q)_\infty}{(az; q)_\infty} dz = 0.$$

Finally, we find that

$$(1.8) \quad \begin{aligned} \sum_{n \in \mathbb{Z}} \frac{(bq^n; q)_\infty}{(aq^n; q)_\infty} q^n &= -\frac{1}{2\pi i} \int_{\mathcal{C}_{\epsilon, \infty}^{[1/a]}} \ell(z) \frac{(bz; q)_\infty}{(az; q)_\infty} dz \\ &= -\sum_{n=0}^{\infty} \ell\left(\frac{q^{-n}}{a}\right) \text{Res} \left(\frac{(bz; q)_\infty}{(az; q)_\infty} : z = \frac{q^{-n}}{a} \right). \end{aligned}$$

By considering (0.8), we have:

$$(1.9) \quad \ell\left(\frac{q^{-n}}{a}\right) = \ell\left(\frac{1}{a}\right) + n.$$

A direct computation shows:

$$\text{Res} \left(\frac{(bz; q)_\infty}{(az; q)_\infty} : z = \frac{q^{-n}}{a} \right) = -\frac{(qa/b; q)_n (b/a; q)_\infty}{a(q; q)_n (q; q)_\infty} \left(\frac{b}{qa}\right)^n$$

Then, it follows from (1.8) that

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \frac{(bq^n; q)_\infty}{(aq^n; q)_\infty} q^n &= \frac{(b/a; q)_\infty}{a(q; q)_\infty} \sum_{n=0}^{\infty} \left(\ell\left(\frac{1}{a}\right) + n \right) \frac{(qa/b; q)_n}{(q; q)_n} \left(\frac{b}{qa}\right)^n \\ &= \frac{(b/a; q)_\infty}{a(q; q)_\infty} \left(\ell\left(\frac{1}{a}\right) + \delta \right) {}_1\phi_0\left(\frac{qa}{b}; q, z\right) \Big|_{z=b/qa}, \end{aligned}$$

where $\delta = z \frac{d}{dz}$ denotes the Euler derivative. Using the q -binomial theorem [3, p.7, (1.3.2)] yields:

$${}_1\phi_0\left(\frac{qa}{b}; q, z\right) = \frac{(qaz/b; q)_\infty}{(z; q)_\infty},$$

which implies:

$${}_1\phi_0\left(\frac{qa}{b}; q, z\right) \Big|_{z=b/qa} = 0, \quad \delta {}_1\phi_0\left(\frac{qa}{b}; q, z\right) \Big|_{z=b/qa} = -\frac{(q; q)_\infty}{(b/qa; q)_\infty}.$$

Thus, it follows from the above that

$$\sum_{n \in \mathbb{Z}} \frac{(bq^n; q)_\infty}{(aq^n; q)_\infty} q^n = -\frac{1}{a(1-b/qa)}.$$

Using (1.6) gives immediately (1.5). □

Combining (1.4) with (1.5) results in (1.1) under the conditions $|b/a| < |z| < 1$ and $|b/a| < |q|$. It is worth noting that the latter condition, $|b/a| < |q|$, can be removed by employing the standard argument of analytic continuation with respect to the parameter a or b .

2. q -LAPLACE TRANSFORMS AND APPELL-LERCH SERIES

As a simple example of an irregular singular q -difference equation, consider the following q -analogue of the Euler differential equation:

$$(2.1) \quad zy\left(\frac{z}{q}\right) + y(z) = 1.$$

Define the power series $\hat{E}_q(z)$ as:

$$(2.2) \quad \hat{E}_q(z) = \sum_{n \geq 0} q^{-n(n-1)/2} (-z)^n$$

It is straightforward to verify that \hat{E}_q is divergent anywhere in \mathbb{C}^* , but it formally satisfies Eq. (2.1). To construct an analytic or meromorphic solution from this formal solution, one can employ a q -analogue of the Borel-Laplace summation method. For more details, see [1], [7] or [8].

In this section, we utilize the q -Laplace transform operator $\mathcal{L}_{q;1}^{[\lambda]}$, introduced in [10]. It is defined as:

$$(2.3) \quad \mathcal{L}_{q;1}^{[\lambda]} \phi(z) = \sum_{\xi \in [\lambda]} \frac{\phi(\xi)}{\theta(\xi/z)},$$

where $[\lambda] = \lambda q^{\mathbb{Z}}$ denotes a q -spiral, and ϕ is a germ of an analytic function at $\xi = 0$ in \mathbb{C} that can be continued into a domain containing $[\lambda]$ and exhibits suitable growth at infinity. Substituting $\xi = \lambda q^n$ into (2.3), the summation can be rewritten as a Jackson integral:

$$\mathcal{L}_{q;1}^{[\lambda]} \phi(z) = \frac{1}{1-q} \int_0^{\lambda\infty} \frac{\phi(\xi)}{\theta(\xi/z)} \frac{d_q \xi}{\xi}.$$

By applying Lemma 0.1, the transform can also be expressed as:

$$(2.4) \quad \mathcal{L}_{q;1}^{[\lambda]} \phi(z) = \frac{1}{2\pi i} \int_{\mathcal{C}_\epsilon^{[\lambda]}} \ell\left(\frac{\xi}{\lambda}\right) \frac{\phi(\xi)}{\theta(\xi/z)} \frac{d_q \xi}{\xi},$$

where $\epsilon \in (0, \varepsilon(q))$, and $\mathcal{C}_\epsilon^{[\lambda]}$ represents a small deformation of the contour near the q -spiral $[\lambda]$ (see (0.9)).

Concerning Eq. (2.1) and its formal solution $\hat{E}_q(z)$, introduced in (2.2), the q -Borel transform $\hat{\mathcal{B}}_{q;1}$ is defined as follows ([2]):

$$\hat{E}_q(z) \mapsto \phi(\xi) := \hat{\mathcal{B}}_{q;1} \hat{E}_q(\xi) = \frac{1}{1+\xi}.$$

Letting this into (2.4) yields the sum-function $\mathcal{S}_{q;1}^{[\lambda]} \hat{E}_q(z)$ of $\hat{E}_q(z)$ associated with the q -spiral $[\lambda]$. It is defined as:

$$(2.5) \quad \mathcal{S}_{q;1}^{[\lambda]} \hat{E}_q(z) = \frac{1}{2\pi i} \int_{\mathcal{C}_\epsilon^{[\lambda]}} \ell\left(\frac{\xi}{\lambda}\right) \frac{1}{(1+\xi)\theta(\xi/z)} \frac{d\xi}{\xi},$$

where $\lambda \in \mathbb{C}^* \setminus [-1]$ and $z \in \mathbb{C}^* \setminus [-\lambda]$. Notice that $\mathcal{S}_{q;1}^{[\lambda]} \hat{E}_q(z)$ is related with Appell-Lerch series and Mordell's integrals (see [11])

Theorem 2.1. For $\lambda \in \mathbb{C}^* \setminus [-1]$ and $z \in \mathbb{C}^* \setminus [-\lambda]$, the following identity holds:

$$(2.6) \quad \mathcal{S}_{q;1}^{[\lambda]} \hat{E}_q(z) = \frac{\ell(-1/\lambda) - \ell(-z/\lambda)}{\theta(-1/z)} - \frac{1}{(q; q)_\infty^3} \sum_{n \in \mathbb{Z}} \frac{(-1)^n n q^{n(n-1)/2}}{1 - q^n z},$$

where $\ell(z) = \ell(q; z)$ is the q -logarithm as defined in (0.7).

Proof. Let $\lambda \in \mathbb{C}^* \setminus [-1]$, and let K be any non-empty compact of $\mathbb{C}^* \setminus [-\lambda]$. We only need to prove (2.6) for $z \in K$. To do this, fix $\epsilon \in (0, \varepsilon(q))$ such that the three integral contours $\mathcal{C}_\epsilon^{[-z]}$, $\mathcal{C}_\epsilon^{[\lambda]}$ and $\mathcal{C}(-1; \epsilon) = \partial^+ D(-1; \epsilon)$ are mutually disjoint for all $z \in K$. Differentiating both sides of Equation (0.6) with respect to z and letting $z = -1$ yields:

$$(2.7) \quad \theta'(-q^n) = (-1)^n q^{-n(n+1)/2} \theta'(-1),$$

from what we deduce the expression:

$$(2.8) \quad \frac{1}{\theta(-1/z)} = -\frac{1}{\theta'(-1)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n(n-1)/2}}{1 - q^n z}.$$

Using the residue theorem and using equation (2.7), it follows from (2.5) that:

$$(2.9) \quad \begin{aligned} \mathcal{S}_{q;1}^{[\lambda]} \hat{E}_q(z) &= -\frac{1}{2\pi i} \left(\int_{\mathcal{C}_\epsilon^{[-z]}} + \int_{\mathcal{C}(-1; \epsilon)} \right) \ell\left(\frac{\xi}{\lambda}\right) \frac{1}{(1+\xi)\theta(\xi/z)} \frac{d\xi}{\xi} \\ &= \frac{\ell(-1/\lambda)}{\theta(-1/z)} + \frac{1}{\theta'(-1)} \sum_{n \in \mathbb{Z}} \ell\left(-\frac{q^n z}{\lambda}\right) \frac{(-1)^n q^{n(n-1)/2}}{1 - q^n z}. \end{aligned}$$

Substituting n with $-n$ into (1.9) gives:

$$\ell\left(-\frac{q^n z}{\lambda}\right) = \ell\left(-\frac{z}{\lambda}\right) - n.$$

With the help of (2.8), we deduce:

$$(2.10) \quad \sum_{n \in \mathbb{Z}} \ell\left(-\frac{q^n z}{\lambda}\right) \frac{(-1)^n q^{n(n-1)/2}}{1 - q^n z} = -\theta'(-1) \frac{\ell(-z/\lambda)}{\theta(-1/z)} - \sum_{n \in \mathbb{Z}} \frac{(-1)^n n q^{n(n-1)/2}}{1 - q^n z}.$$

Gathering (2.9) with (2.10) and using $\theta'(-1) = (q; q)_\infty^3$, we arrive at the desired identity (2.6), completing the proof. \square

Using (2.6), we derive the Stokes multiplier as follows:

$$(2.11) \quad \mathcal{S}_{q;1}^{[\lambda]} \hat{E}_q(z) - \mathcal{S}_{q;1}^{[\mu]} \hat{E}_q(z) = \frac{1}{\theta(-1/z)} \left(\ell\left(-\frac{1}{\lambda}\right) - \ell\left(-\frac{z}{\lambda}\right) - \ell\left(-\frac{1}{\mu}\right) + \ell\left(-\frac{z}{\mu}\right) \right),$$

where both λ and μ belong to $\mathbb{C}^* \setminus [-1]$. It is worth noting that Eq. (2.11) provides a new expression of the q -periodic factor K as defined in [2, Theorem 3.13].

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