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THE WEIL REPRESENTATION FOR A FINITE FIELD OF CHARACTERISTIC TWO

AURÉLIE PAULL

ABSTRACT. We study the Weil representations associated to a finite field \mathbb{F} of characteristic two. Starting from a non-degenerate symplectic form on a finite-dimensional vector space W over \mathbb{F} , we consider any associated bilinear form B on W and the corresponding Heisenberg group $H(B)$. The pseudo-symplectic group $\text{Ps}(B)$ acts on $H(B)$ by automorphisms leaving its center fixed. Let $W = X \oplus Y$ be a complete polarization and let $\tilde{\chi}$ be a character of the abelian subgroup $X \times \mathbb{F}$ of $H(B)$ having non-trivial restriction χ to \mathbb{F} . From this data, we construct the projective Weil representation of $\text{Ps}(B)$. We linearize this representation and define the Weil representation of a two-fold covering $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ of $\text{Ps}(B)$. All the formulas we obtain are explicit. In particular, we exhibit explicit formulas for the projective cocycle and the character of the Weil representation. Finally, we use our results to give the complete description of the two-dimensional case $W \simeq \mathbb{F}_2^2$.

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INTRODUCTION

The Weil representation has been studied in various settings and has numerous applications in mathematics and physics. In his pioneering work, Weil [28] studies algebraic groups over local fields and adelic rings. He defines a Heisenberg group H associated to a fixed bilinear form over a finite-dimensional vector space and then the pseudo-symplectic group Ps , which is a subgroup of the automorphism group of H acting trivially on its

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center. He then obtains a unitary projective representation of Ps which he lifts to a central extension of Ps by the unit circle. In characteristic different from two and if the field is not that of complex numbers, Ps is isomorphic to the symplectic group Sp and the extension can be reduced to an extension of Sp by two elements, i.e. a two-fold covering of Sp . This is the metaplectic extension, and the linear representation obtained this way is called the oscillator, metaplectic or Weil representation.

In his article, Weil said that a possible extension of his work to the case of a finite field would have been interesting. This was done using different methods in odd characteristic for example by Gérardin [7], Teruji [26], [27] and Aubert-Przebinda [2]. In this case, the projective Weil representation appears to be a linear representation of Sp , no coverings needed. However the situation is completely different in characteristic two: Ps is not isomorphic to Sp , the symplectic group does not act on the Heisenberg group anymore and Ps does not even project onto Sp but only onto the orthogonal group O , which is a proper subgroup of Sp (with only one exception; see Proposition 1.12 below). The work of Blasco [3] shows that the metaplectic extension does not split and is a two-fold covering of Ps .

In this article, we study the Weil representations associated to a finite field \mathbb{F} of characteristic two. We consider a finite-dimensional vector space W over \mathbb{F} equipped with a non-degenerate symplectic form $\langle \cdot, \cdot \rangle$ (i.e. alternating). We fix a bilinear form B on W such that

$$\langle w_1, w_2 \rangle = B(w_1, w_2) - B(w_2, w_1)$$

for all $w_1, w_2 \in W$ and consider the quadratic form Q associated to B , i.e. $Q(w) = B(w, w)$. This leads to the definition of the symplectic group $\text{Sp}(W)$ and the orthogonal group $\text{O}(Q)$ in Definition 1.6. We point out that Weil fixes a particular form B whereas the results we obtain in the sequel (like Blasco's results) can be applied to any bilinear form B satisfying the equality above. We start by defining a Heisenberg group $\text{H}(B)$ in this setting in Definition 1.1. Notice that non-equivalent bilinear forms can lead to non-isomorphic Heisenberg groups, which is a major difference from the odd characteristic case, for which there is only one Heisenberg group to consider (cf. Proposition 1.4). If $\mathbb{F} = \mathbb{F}_2$, the situation is completely clear: $\text{H}(B)$ is an extraspecial 2-group and there are only two different Heisenberg groups up to isomorphism (see Proposition 1.5 and the appendix for the definition and structure of extraspecial groups). For arbitrary \mathbb{F} , we define the pseudo-symplectic group $\text{Ps}(B)$ associated with B in Proposition 1.8 and we describe its structure in Proposition 1.11.

The Weil representation arises from an intrinsic property of the Heisenberg group $\text{H}(B)$, namely the Stone-von Neumann theorem, of which we give a short proof in our setting in Theorem 2.9. This theorem states that, up to isomorphism, $\text{H}(B)$ has a unique unitary irreducible representation acting on its center \mathbb{F} as a fixed non-trivial character χ . For every non-trivial character χ of \mathbb{F} and extension $\tilde{\chi}$ of χ to the abelian subgroup $X \times \mathbb{F}$ of $\text{H}(B)$, we give in Lemma 2.3 a realization $\rho_{\tilde{\chi}}$ of this representation. However, we

will point out in Remark 2.10 that this theorem is in fact not necessary to obtain the intertwining operators we need to construct the Weil representation. Then, we deduce the results of Blasco but in a different manner. In fact, we were inspired at the beginning by Glasby’s article [8] for $\mathbb{F} = \mathbb{F}_2$ since it treats extraspecial 2-groups, and then we extended the formulas we obtained to every finite field \mathbb{F} of characteristic two. In particular, we give explicit formulas for the projective Weil representation of $\text{Ps}(B)$ in Proposition 3.2. (Notice that Blasco treats simultaneously the local and the finite case, and gives a formula for the finite case which does not apply to all elements of $\text{Ps}(B)$.) We also compute the associated cocycle in Corollary 3.4 and define the two-fold covering $\widetilde{\text{Ps}(B)}_\chi$ of $\text{Ps}(B)$. This definition provides possibly several non-isomorphic coverings, but we prove in Proposition 3.8 and Proposition 3.15 that there are conditions to ensure that the isomorphism class of this extension does not depend on the character χ we choose and on the choices we make to define the intertwining operators in Proposition 3.2. We obtain thus explicit formulas for the Weil representation of $\widetilde{\text{Ps}(B)}_\chi$ and its character in Corollary 3.10 and Proposition 3.11, respectively. Here we highlight that the Weil representation is unitary, faithful and irreducible.

Finally, we investigate in detail the case of the smallest dimension for W , i.e. $W \cong \mathbb{F}_2^2$. In this case, there are two different Heisenberg groups which can arise from the different possibilities for B . If $H(B) \simeq D_4$ then the two-fold covering $\widetilde{\text{Ps}(B)}_\chi$ is either isomorphic to the dihedral group D_8 or to the semi-dihedral group SD_{16} (see section 4.1). If $H(B) \simeq Q_8$ then, as mentioned in Proposition 1.12, the situation is very exceptional since $O(Q) = \text{Sp}(W)$. Here the two-fold covering $\widetilde{\text{Ps}(B)}_\chi$ is either isomorphic to the conformal special unitary group $\text{CSU}_2(\mathbb{F}_3)$ or to the general linear group $\text{GL}_2(\mathbb{F}_3)$ (see section 4.2, and [29] for more details on these finite groups). Observe that this last example is the only one which enables us to provide a Weil representation which is directly related to the entire symplectic group $\text{Sp}(W)$, and not only to $O(Q)$. The construction of the Weil representation of a metaplectic group related to the entire symplectic group over $\mathbb{F} = \mathbb{F}_2$ will be described in a forthcoming article.

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1. THE HEISENBERG GROUP $H(B)$ AND THE PSEUDO-SYMPLECTIC GROUP $\text{Ps}(B)$

Let \mathbb{F} be a finite field of characteristic two and let W be a finite dimensional \mathbb{F} -vector space of dimension $2n$. Since the Weil representation arises from the Heisenberg group, we first have to properly define the Heisenberg group in these settings. In fact, this group is in general defined from a non-degenerate symplectic form over W and this requires a division by two. Since this definition is not available here, we use a bilinear form B (related to the non-degenerate symplectic form by (5)) to define the Heisenberg group.

In particular, there are different bilinear forms B that can be used to define this group. If $\mathbb{F} = \mathbb{F}_2$, there are only two different Heisenberg groups to consider up to isomorphism (cf. Proposition 1.5).

Definition 1.1. Let B be a non-zero bilinear form defined on W with values in \mathbb{F} . We define the Heisenberg group associated to B , denoted by $H(B)$, as the set $W \times \mathbb{F}$ endowed with the following product:

$$(w_1, t_1)(w_2, t_2) = (w_1 + w_2, t_1 + t_2 + B(w_1, w_2)) \quad (1)$$

for all $w_1, w_2 \in W$ and $t_1, t_2 \in \mathbb{F}$. In particular, the identity element of $H(B)$ is $(0, 0)$ and the inverse of $(w, t) \in H(B)$ is $(-w, -t + B(w, w))$.

Example 1.2. Let X be a finite dimensional vector space over \mathbb{F} with dual space X^* . Set $W = X \times X^*$ and define a non-degenerate symplectic form on W by $\langle w_1, w_2 \rangle = x_2^*(x_1) - x_1^*(x_2)$ for all $w_1 = (x_1, x_1^*)$, $w_2 = (x_2, x_2^*) \in X \times X^*$. We denote by $H(B_{\text{Weil}})$ the Heisenberg group associated to the bilinear form B_{Weil} defined by

$$B_{\text{Weil}}(w_1, w_2) = x_2^*(x_1).$$

(We call it B_{Weil} since it is the bilinear form used by Weil in [28].)

We denote by $\mathcal{Q}(W)$ the set of quadratic forms defined on W with values in \mathbb{F} , i.e. the set of maps $f : W \rightarrow \mathbb{F}$ such that

$$(w_1, w_2) \mapsto f(w_1 + w_2) - f(w_1) - f(w_2) \text{ is a bilinear form, and} \quad (2)$$

$$f(tw) = t^2 f(w) \text{ for all } t \in \mathbb{F} \text{ and } w \in W. \quad (3)$$

We also denote by $\mathcal{Q}_a(W)$ the subset of $\mathcal{Q}(W)$ consisting of the quadratic forms which are additive.

Remark 1.3. If $\mathbb{F} = \mathbb{F}_2$ then the condition (3) is automatically satisfied by any map from W to \mathbb{F}_2 which satisfies (2).

The quadratic form associated to the bilinear form B is defined by $Q(w) = B(w, w)$ for all $w \in W$. Two quadratic forms Q and Q' are said to be equivalent if there exists an isomorphism σ of W such that $Q' = Q \circ \sigma$. Let $Q \in \mathcal{Q}(W)$ and let V be a vector subspace of W . Then V is said to be singular if $Q|_V = 0$. Following [5, §16], we denote by ν the (Witt) index of the quadratic form Q , i.e. the maximal dimension of a singular subspace V of W . We also recall the definition of the Arf invariant of a quadratic form on W (we refer to [1], [20], [5, p. 34], [4, p.197–199] and [11, Theorem 12.9] for the original definitions and more general results). If the bilinear form $(w_1, w_2) \mapsto Q(w_1 + w_2) - Q(w_1) - Q(w_2)$ is non-degenerate, then there exists a basis $\{v_i\}_{1 \leq i \leq 2n}$ of W such that for all $w = \sum_{i=1}^{2n} \lambda_i v_i$,

$$Q(w) = \lambda_1 \lambda_{n+1} + \lambda_2 \lambda_{n+2} + \cdots + \lambda_{n-1} \lambda_{2n-1} + (\xi \lambda_n^2 + \lambda_n \lambda_{2n} + \xi \lambda_{2n}^2),$$

where $\xi = 0$ or ξ is such that the polynomial $\xi X^2 + X + \xi$ is irreducible in \mathbb{F} . Then the Arf invariant of Q is defined by $\text{Arf}(Q) = \xi^2$. Two quadratic forms Q and Q' are equivalent if and only if $\text{Arf}(Q) - \text{Arf}(Q') = u^2 + u$ for some $u \in \mathbb{F}$, or equivalently, if and only if $\nu(Q) = \nu(Q')$. In particular, if $\mathbb{F} = \mathbb{F}_2$, we obtain $\text{Arf}(Q) = \xi \in \{0, 1\}$.

Proposition 1.4. Let B and B' be two non-zero bilinear forms from W to \mathbb{F} . Let Q and Q' be the quadratic forms associated to B and B' , respectively.

1. The product (1) defines a group $H(B)$ and a central extension $0 \rightarrow \mathbb{F} \rightarrow H(B) \rightarrow W \rightarrow 0$. Moreover, if the alternating form

$$(w_1, w_2) \mapsto B(w_1, w_2) - B(w_2, w_1)$$

is non-degenerate then the center of $H(B)$ is isomorphic to \mathbb{F} .

2. B is a cocycle and the cohomology class of B in $H^2(W, \mathbb{F})$ is the one associated to the isomorphism class of the central extension defined by $H(B)$.
3. The central extensions

$$0 \rightarrow \mathbb{F} \rightarrow H(B) \rightarrow W \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathbb{F} \rightarrow H(B') \rightarrow W \rightarrow 0$$

are isomorphic if and only if B and B' are cohomologous. In particular, if these extensions are isomorphic then $B - B'$ is alternating, i.e. $Q = Q'$.

4. The groups $H(B)$ and $H(B')$ are isomorphic if and only if there exists a map $f : W \rightarrow \mathbb{F}$, an automorphism of abelian groups $\sigma : W \rightarrow W$ and an automorphism of additive abelian groups $g : \mathbb{F} \mapsto \mathbb{F}$ such that for all $(w_1, w_2) \in W$:

$$f(w_1 + w_2) - f(w_1) - f(w_2) = B'(\sigma(w_1), \sigma(w_2)) - g(B(w_1, w_2)). \quad (4)$$

In this case, the isomorphism is given by

$$\begin{aligned} \Phi_{(\sigma, f, g)} : H(B) &\rightarrow H(B') \\ (w, t) &\mapsto (\sigma(w), f(w) + g(t)). \end{aligned}$$

Any isomorphism between $H(B)$ and $H(B')$ is of this form.

Proof. 1. The two group homomorphisms $\phi : \mathbb{F} \rightarrow H(B)$ and $\psi : H(B) \rightarrow W$ are such

$$t \mapsto (0, t) \quad (w, t) \mapsto w$$

that ϕ is injective, ψ is surjective and $\text{Ker}(\psi) = \text{Im}(\phi)$. Moreover, for every $t \in \mathbb{F}$, the element $(0, t)$ is in the center of $H(B)$ by (1). Let (w', t') be in the center of $H(B)$. Then, for every $(w, t) \in H(B)$,

$$\begin{aligned} (w, t) &= (w', t')(w, t)(w', t')^{-1} = (w' + w, t' + t + B(w', w))(-w', -t' + B(w', w)) \\ &= (w, t + B(w', w) + B(w', w') - B(w' + w, w')) \\ &= (w, t + B(w', w) - B(w, w')). \end{aligned}$$

This implies that $B(w', w) - B(w, w') = 0$ for all $w \in W$. Hence $w' = 0$ if $(w_1, w_2) \mapsto B(w_1, w_2) - B(w_2, w_1)$ is non-degenerate.

2. B is a cocycle because B is a bilinear form: for every $w_1, w_2, w_3 \in W$, $B(w_1, 0) = B(0, w_1) = 0$ and

$$B(w_1, w_2) + B(w_1 + w_2, w_3) = B(w_2, w_3) + B(w_1, w_2 + w_3).$$

Now, define the section $\tau : w \mapsto (w, 0)$ of the homomorphism $\psi : (w, t) \mapsto w$. Then for every $w_1, w_2 \in W$,

$$\begin{aligned} \tau(w_1)\tau(w_2)\tau(w_1 + w_2)^{-1} &= (w_1, 0)(w_2, 0)(-w_1 - w_2, B(w_1 + w_2, w_1 + w_2)) \\ &= (0, B(w_1, w_2) + B(w_1 + w_2, w_1 + w_2) - B(w_1 + w_2, w_1 + w_2)) \\ &= \phi(B(w_1, w_2)). \end{aligned}$$

This shows that B is a cocycle that can be associated to the central extension defined by $H(B)$. Thus the cohomology class of B in $H^2(W, \mathbb{F})$ is the one associated to the isomorphism class of the central extension defined by $H(B)$.

3. If the extensions defined by $H(B)$ and $H(B')$ are isomorphic then B and B' are cohomologous by the preceding point. Hence, there exists a map $\delta : W \rightarrow \mathbb{F}$ such that for every $w_1, w_2 \in \mathbb{F}$,

$$B(w_1, w_2) - B'(w_1, w_2) = \delta(w_1) + \delta(w_2) - \delta(w_1 + w_2).$$

In particular, $B(w_1, w_1) - B'(w_1, w_1) = 2\delta(w_1) - \delta(2w_1) = 0$, i.e. $B - B'$ is alternating. This implies that $Q(w_1) = B(w_1, w_1) = B'(w_1, w_1) = Q'(w_1)$, hence $Q = Q'$.

4. Suppose that there exists a map $f : W \rightarrow \mathbb{F}$, an automorphism of abelian groups $\sigma : W \rightarrow W$ and an automorphism of additive abelian groups $g : \mathbb{F} \mapsto \mathbb{F}$ such that (4) is satisfied. Then, for every $(w_1, t_1), (w_2, t_2) \in H(B)$,

$$\begin{aligned} &\Phi_{(\sigma, f, g)}(w_1, t_1)\Phi_{(\sigma, f, g)}(w_2, t_2) \\ &= (\sigma(w_1), f(w_1) + g(t_1))(\sigma(w_2), f(w_2) + g(t_2)) \\ &= (\sigma(w_1) + \sigma(w_2), f(w_1) + f(w_2) + g(t_1) + g(t_2) + B'(\sigma(w_1), \sigma(w_2))) \\ &\Phi_{(\sigma, f, g)}((w_1, t_1)(w_2, t_2)) \\ &= \Phi_{(\sigma, f, g)}(w_1 + w_2, t_1 + t_2 + B(w_1, w_2)) \\ &= (\sigma(w_1 + w_2), f(w_1 + w_2) + g(t_1 + t_2 + B(w_1, w_2))) \\ &= (\sigma(w_1) + \sigma(w_2), f(w_1) + f(w_2) + g(t_1) + g(t_2) + B'(\sigma(w_1), \sigma(w_2))) \quad \text{by (4)}. \end{aligned}$$

Hence $\Phi_{(\sigma, f, g)}((w_1, t_1)(w_2, t_2)) = \Phi_{(\sigma, f, g)}(w_1, t_1)\Phi_{(\sigma, f, g)}(w_2, t_2)$, which proves that $\Phi_{(\sigma, f, g)}$ is a group homomorphism. Moreover, for every $(w', t') \in H(B')$, since σ is bijective, there exists a unique $w \in W$ such that $\sigma(w) = w'$. Since g is bijective, there exists a unique $t \in \mathbb{F}$ such that $g(t) = f(w) + t'$. Thus, we obtain a unique element $(w, t) \in H(B)$ such that $\Phi_{(\sigma, f, g)}(w, t) = (\sigma(w), f(w) + g(t)) = (w', t')$. Therefore $\Phi_{(\sigma, f, g)}$ is an isomorphism.

Conversely, suppose that $s : H(B) \mapsto H(B')$ is a group isomorphism. Then we can decompose s as follows: $s : (w, t) \mapsto (s_1(w, t), s_2(w, t))$, where $s_1 : H(B) \mapsto W$ and $s_2 : H(B) \mapsto \mathbb{F}$. Hence, for every $(w, t) \in H(B)$,

$$\begin{aligned} s(w, t) &= s((w, 0)(0, t)) = s(w, 0)s(0, t) = (s_1(w, 0), s_2(w, 0))(0, s_2(0, t)) \\ &= (s_1(w, 0), s_2(w, 0) + s_2(0, t)), \end{aligned}$$

where we used the fact that $s_1(0, t) = 0$ because s maps the center of $H(B)$ to the center of $H(B')$. Thus $s_1(w, t) = s_1(w, 0)$, i.e. s_1 depends only on W , and $s_2(w, t) = s_2(w, 0) + s_2(0, t)$. We can set $s_1(w, t) = \sigma(w)$, $s_2(w, 0) = f(w)$ and $s_2(0, t) = g(t)$ with $\sigma : W \rightarrow W$, $f : W \rightarrow \mathbb{F}$ and $g : \mathbb{F} \rightarrow \mathbb{F}$. Then s acts on $H(B)$ by $s(w, t) = (\sigma(w), f(w) + g(t))$. Moreover,

- $f(0) = s_2(0, 0) = 0$ and $g(0) = s_2(0, 0) = 0$ because $s(0, 0) = 0$.
- σ is an abelian group homomorphism and (4) is satisfied. In fact, for every $w_1, w_2 \in W$, the fact that s is a group homomorphism implies the equality of the following two elements of $H(B')$:

$$\begin{aligned} s((w_1, 0)(w_2, 0)) &= s(w_1 + w_2, B(w_1, w_2)) = (\sigma(w_1 + w_2), f(w_1 + w_2) + g(B(w_1, w_2))) \\ s(w_1, 0)s(w_2, 0) &= (\sigma(w_1), f(w_1))(\sigma(w_2), f(w_2)) \\ &= (\sigma(w_1) + \sigma(w_2), f(w_1) + f(w_2) + B'(\sigma(w_1), \sigma(w_2))). \end{aligned}$$

- g is an abelian group homomorphism. In fact, for every $t_1, t_2 \in \mathbb{F}$, the equality $s((0, t_1)(0, t_2)) = s(0, t_1)s(0, t_2)$ implies that

$$(\sigma(0), f(0) + g(t_1 + t_2)) = (\sigma(0), f(0) + g(t_1))(\sigma(0), f(0) + g(t_2))$$

$$\text{i.e. } (0, g(t_1 + t_2)) = (0, g(t_1) + g(t_2)).$$

- σ and g are bijective. In fact, let us consider s^{-1} and write it under the form $s^{-1} : (w', t') \mapsto (\sigma'(w'), f'(w') + g'(t'))$, where $\sigma' : W \rightarrow W$ is an abelian group homomorphism and $f' : W \rightarrow \mathbb{F}$ and $g' : \mathbb{F} \rightarrow \mathbb{F}$ satisfy $f'(0) = g'(0) = 0$. Then for every $(w, t) \in H(B)$,

$$(w, t) = (ss^{-1})(w, t) = s(\sigma'(w), f'(w) + g'(t)) = (\sigma(\sigma'(w)), f(\sigma'(w)) + g(f'(w) + g'(t))).$$

Hence $w = \sigma(\sigma'(w))$, i.e. σ is bijective and $\sigma' = \sigma^{-1}$. Moreover, choosing $w = 0$, we obtain $t = f(0) + g(f'(0) + g'(t)) = g(g'(t))$, i.e. g is bijective and $g^{-1} = g'$.

Therefore, any isomorphism $s : H(B) \rightarrow H(B')$ is of the form $s = \Phi_{(\sigma, f, g)}$ given in the statement. □

From now on, we suppose that the bilinear form

$$(w_1, w_2) \mapsto \langle w_1, w_2 \rangle = B(w_1, w_2) - B(w_2, w_1) \tag{5}$$

is non-degenerate. This endows W with a symplectic vector space structure. We denote this space by $(W, \langle \cdot, \cdot \rangle)$.

Let B and B' be two non-zero bilinear forms on W . By Proposition 1.4.4, if $H(B)$ and $H(B')$ are isomorphic, then there exists f , σ and g such that (4) is satisfied. Let Q and Q' denote the quadratic forms associated to B and B' , respectively. Then $Q'(\sigma(w)) = B'(\sigma(w), \sigma(w)) = g(B(w, w)) = g(Q(w))$ for all $w \in W$, i.e. $Q' \circ \sigma = g \circ Q$. In particular, if $\mathbb{F} = \mathbb{F}_2$, then the only possibility for g is the identity. Therefore $Q' \circ \sigma = Q$, which implies that Q and Q' are equivalent. In fact, for $\mathbb{F} = \mathbb{F}_2$, we can state a more precise result in Proposition 1.5. In this case, the Heisenberg group appears to be what is called

an extraspecial 2-group. This implies that $H(B)$ and $H(B')$ are isomorphic groups if and only if Q and Q' are equivalent, and thus there are only two different Heisenberg groups. We refer to the appendix for the definition and structure of extraspecial 2-groups, the link with the Heisenberg group $H(B)$ and a proof of Proposition 1.5.

Proposition 1.5. Suppose that $\mathbb{F} = \mathbb{F}_2$. Let B be a non-zero bilinear form on W and Q be its associated quadratic form. Then $H(B)$ is an extraspecial 2-group. In particular, there are two Heisenberg groups up to isomorphism, depending whether $\nu(Q) = n$ (or equivalently $\text{Arf}(Q) = 0$) or $\nu(Q) = n - 1$ (or equivalently $\text{Arf}(Q) = 1$). This means that for two bilinear forms B and B' , the groups $H(B)$ and $H(B')$ are isomorphic if and only if Q and Q' are equivalent.

Proof. Cf. Proposition A.11. □

Now that the Heisenberg group $H(B)$ has been defined, we are able to study its automorphisms. We write $\text{Aut}(H(B))$ for the group of automorphisms of $H(B)$ and $\text{Aut}^0(H(B))$ for the subgroup of $\text{Aut}(H(B))$ consisting of the automorphisms of $H(B)$ acting trivially on the center of $H(B)$.

Definition 1.6. A vector space automorphism $\sigma : W \rightarrow W$ is said to be symplectic if it preserves the non-degenerate symplectic form $\langle \cdot, \cdot \rangle$, i.e. for all $w_1, w_2 \in W$,

$$B(\sigma(w_1), \sigma(w_2)) - B(w_1, w_2) = B(\sigma(w_2), \sigma(w_1)) - B(w_2, w_1).$$

The set of symplectic automorphisms is a group, which we denote by $\text{Sp}(W)$.

A vector space automorphism $\sigma : W \rightarrow W$ is said to be orthogonal if it preserves the quadratic form $Q : w \mapsto Q(w) = B(w, w)$, i.e. for all $w \in W$,

$$Q(\sigma(w)) = B(\sigma(w), \sigma(w)) = B(w, w) = Q(w).$$

The set of orthogonal automorphisms is a group, which we denote by $O(Q)$.

Proposition 1.7. $\text{Aut}^0(H(B))$ is the set of elements $s = (\sigma, f)$, where $\sigma \in \text{Sp}(W)$ and $f : W \rightarrow \mathbb{F}$ are such that for all $w_1, w_2 \in W$,

$$f(w_1 + w_2) - f(w_1) - f(w_2) = B(\sigma(w_1), \sigma(w_2)) - B(w_1, w_2). \quad (6)$$

The element $s = (\sigma, f)$ acts on $H(B)$ by $(\sigma, f)(w, t) = (\sigma(w), f(w) + t)$. The product of two elements of $\text{Aut}^0(H(B))$ is given by

$$(\sigma, f)(\sigma', f') = (\sigma\sigma', f' + f \circ \sigma'), \quad \text{where } (f' + f \circ \sigma')(w) = f'(w) + f(\sigma'(w)).$$

In particular, the inverse of $s = (\sigma, f)$ is $s^{-1} = (\sigma^{-1}, -f \circ \sigma^{-1})$.

Proof. Let $s \in \text{Aut}^0(H(B))$. Since s is an automorphism of $H(B)$, by Proposition 1.4.4, there exists a map $f : W \rightarrow \mathbb{F}$, an automorphism of abelian groups $\sigma : W \rightarrow W$ and an automorphism of additive abelian groups $g : \mathbb{F} \mapsto \mathbb{F}$ such that $s = \Phi_{(\sigma, f, g)}$. In particular, $s : (w, t) \mapsto (\sigma(w), f(w) + g(t))$, and for all $(w_1, w_2) \in W$,

$$f(w_1 + w_2) - f(w_1) - f(w_2) = B(\sigma(w_1), \sigma(w_2)) - g(B(w_1, w_2)).$$

Since s fixes the center of $H(B)$, $(0, t) = s(0, t) = (0, g(t))$ for all $t \in \mathbb{F}$, i.e. g is the identity and (6) is satisfied. Since the left-hand side of (6) is symmetric, σ is symplectic provided we prove that σ is \mathbb{F} -linear.

Let $\mathcal{B} = \{e_1, \dots, e_n, e_{-1}, \dots, e_{-n}\}$ be a symplectic basis of W , i.e. a basis such that $\langle e_i, e_j \rangle = \delta_{i,-j}$ for all $1 \leq i, j \leq n$. Then $\mathbb{F}e_1 = \{e_1, \dots, e_n, e_{-2}, \dots, e_{-n}\}^\perp$, where we take the orthogonal complement with respect to $\langle \cdot, \cdot \rangle$. Set $w_1 = \sigma^{-1}(e_1)$. For every $j \neq -1$, we have that $\langle w_1, \sigma^{-1}(e_j) \rangle = \langle \sigma^{-1}(e_1), \sigma^{-1}(e_j) \rangle = \langle e_1, e_j \rangle = 0$. Hence $\mathbb{F}w_1 \subseteq \{\sigma^{-1}(e_1), \dots, \sigma^{-1}(e_n), \sigma^{-1}(e_{-2}), \dots, \sigma^{-1}(e_{-n})\}^\perp$. This implies that $\sigma(\mathbb{F}w_1) \subseteq \mathbb{F}e_1$. Then, by definition of w_1 , we obtain $\sigma(\mathbb{F}w_1) \subseteq \mathbb{F}\sigma(w_1)$. In a similar way, if we set $w_{-1} = \sigma^{-1}(e_{-1})$, then $\sigma(\mathbb{F}w_{-1}) \subseteq \mathbb{F}\sigma(w_{-1})$. Thus for every $a \in \mathbb{F}$, there exist $\lambda_1(a) \in \mathbb{F}$ and $\lambda_{-1}(a) \in \mathbb{F}$ such that $\sigma(aw_1) = \lambda_1(a)\sigma(w_1)$ and $\sigma(aw_{-1}) = \lambda_{-1}(a)\sigma(w_{-1})$. Now, let $a \in \mathbb{F}$. Then

$$\langle aw_1, w_{-1} \rangle = \langle w_1, aw_{-1} \rangle = a \langle w_1, w_{-1} \rangle.$$

Hence

$$\langle \sigma(aw_1), \sigma(w_{-1}) \rangle = \langle \sigma(w_1), \sigma(aw_{-1}) \rangle = a \langle \sigma(w_1), \sigma(w_{-1}) \rangle,$$

i.e.

$$\langle \lambda_1(a)\sigma(w_1), \sigma(w_{-1}) \rangle = \langle \sigma(w_1), \lambda_{-1}(a)\sigma(w_{-1}) \rangle = a \langle \sigma(w_1), \sigma(w_{-1}) \rangle.$$

Since $\langle \sigma(w_1), \sigma(w_{-1}) \rangle = \langle e_1, e_{-1} \rangle = 1$, we conclude that

$$\lambda_1(a) = \lambda_{-1}(a) = a.$$

This proves that, for every $a \in \mathbb{F}$, $\sigma(aw_1) = a\sigma(w_1)$ and $\sigma(aw_{-1}) = a\sigma(w_{-1})$. This reasoning applies with $w_j = \sigma^{-1}(e_j)$ and $w_{-j} = \sigma^{-1}(e_{-j})$ for all $j \in \{1, \dots, n\}$. Hence $\sigma(aw_{\pm j}) = a\sigma(w_{\pm j})$. Since $\langle w_i, w_j \rangle = \delta_{i,-j}$ for all $1 \leq i, j \leq n$, the set $\{w_1, \dots, w_n, w_{-1}, \dots, w_{-n}\}$ is a basis of W . Thus $\sigma(tw) = t\sigma(w)$ for all $t \in \mathbb{F}$ and $w \in W$. \square

Definition-Proposition 1.8. Let $\text{Ps}(B)$ be the set of elements $(\sigma, f) \in \text{Sp}(W) \times \mathcal{Q}(W)$ such that for all $w_1, w_2 \in W$,

$$f(w_1 + w_2) - f(w_1) - f(w_2) = B(\sigma(w_1), \sigma(w_2)) - B(w_1, w_2), \quad \text{i.e. (6) is satisfied.}$$

Then $\text{Ps}(B)$ is a subgroup of $\text{Aut}^0(H(B))$, which we call the pseudo-symplectic group associated to B .

Remark 1.9. The only difference between $\text{Aut}^0(H(B))$ and $\text{Ps}(B)$ is that an element $s = (\sigma, f)$ in the pseudo-symplectic group is such that f satisfies the supplementary condition (3). In particular, Remark 1.3 implies that $\text{Ps}(B) = \text{Aut}^0(H(B))$ if $\mathbb{F} = \mathbb{F}_2$.

Proposition 1.10. [3, 1.3, Proposition b] The group $\text{Ps}(B)$ defines a short exact sequence:

$$0 \rightarrow \mathcal{Q}_a(W) \rightarrow \text{Ps}(B) \rightarrow \text{O}(Q) \rightarrow 1.$$

It splits if and only if either $\dim_{\mathbb{F}}(W) = 2$ or $\mathbb{F} = \mathbb{F}_2$ and $\dim_{\mathbb{F}_2}(W) = 4$.

Proposition 1.11 (The structure of $\text{Ps}(B)$). **1.** There are $|W| = |\mathbb{F}|^{2n}$ additive quadratic forms on W . More precisely, if we choose a basis $\mathcal{B} = \{v_1, \dots, v_{2n}\}$ of W , then

$$\mathcal{Q}_a(W) = \{f_z : \sum_{1 \leq j \leq 2n} \lambda_j v_j \mapsto \sum_{1 \leq j \leq 2n} \lambda_j^2 \langle z, v_j \rangle, z \in W\}.$$

If $\mathbb{F} = \mathbb{F}_2$, then $f_z = \langle z, \cdot \rangle$ for all $z \in W$. Hence $\mathcal{Q}_a(W)$ is the set of linear forms $\langle z, \cdot \rangle$, $z \in W$.

- 2.** Let χ be a character of $(\mathbb{F}, +)$. Then the maps $\chi \circ f_z$, where $z \in W$, are precisely all the characters of W .
- 3.** Let $s = (\sigma, f)$ be in $\text{Ps}(B)$. Then:
- a.** σ belongs to the orthogonal group $\text{O}(Q)$.
 - b.** For any choice of basis $\mathcal{B} = \{v_1, \dots, v_{2n}\}$ of W , we can define a quadratic form on W associated to σ by

$$f_\sigma\left(\sum_{1 \leq i \leq 2n} \lambda_i v_i\right) = \sum_{1 \leq i < j \leq 2n} \lambda_i \lambda_j \beta_\sigma(v_i, v_j), \quad (7)$$

where β_σ is the bilinear form on W defined by

$$\beta_\sigma(w_1, w_2) = B(\sigma(w_1), \sigma(w_2)) - B(w_1, w_2). \quad (8)$$

Then (σ, f_σ) is an element of $\text{Ps}(B)$.

- c.** s can be uniquely written in the form $(\sigma, f_\sigma + f_z)$, where $f_z : W \rightarrow \mathbb{F}$ is an additive quadratic form.
- 4.** Every element σ of the orthogonal group $\text{O}(Q)$ leads to exactly $|W| = |\mathbb{F}|^{2n}$ elements of $\text{Ps}(B)$. These elements can be written explicitly as $\{(\sigma, f_\sigma + f_z), z \in W\}$.
- 5.** Let $\text{Ps}_a(B)$ be the set of elements of $\text{Ps}(B)$ of the form $(1, f)$. Then $\text{Ps}_a(B)$ is an abelian subgroup of $\text{Ps}(B)$ and

$$\text{Ps}_a(B) = \{(1, f_z), z \in W\}. \quad (9)$$

Proof. **1.** If $\mathbb{F} = \mathbb{F}_2$, we already mentioned in Remark 1.3 that the only condition for a function $f : W \rightarrow \mathbb{F}_2$ to be quadratic is that $(w_1, w_2) \mapsto f(w_1 + w_2) - f(w_1) - f(w_2)$ is bilinear. Hence, if f is supposed to be quadratic and additive, it is automatically linear on \mathbb{F}_2 . Conversely, any linear form on W is also additive and thus quadratic. Moreover, the map which sends $z \in W$ to the linear form $\langle z, \cdot \rangle$ on W is an isomorphism because the symplectic form is non-degenerate. Thus $\mathcal{Q}_a(W)$ is exactly the set of linear forms $\langle z, \cdot \rangle$, where $z \in W$.

Suppose now that \mathbb{F} is an arbitrary finite field of characteristic two. Let $\mathcal{B} = \{v_1, \dots, v_{2n}\}$ be a basis of W . For every $z \in W$, define $f_z : W \rightarrow \mathbb{F}$ by

$$f_z(w) = \sum_{1 \leq j \leq 2n} \lambda_j^2 \langle z, v_j \rangle \quad \text{for all } w = \sum_{1 \leq j \leq 2n} \lambda_j v_j.$$

Then f_z is an additive quadratic form. Moreover, $f_z \neq f_{z'}$ for all $z \neq z' \in W$. In fact, if $f_z = f_{z'}$, then in particular $\langle z, v_k \rangle = \langle z', v_k \rangle$ for all $1 \leq k \leq 2n$. This implies that the linear forms $\langle z, \cdot \rangle$ and $\langle z', \cdot \rangle$ coincide on the basis \mathcal{B} of W and

thus are equal. Since $\langle \cdot, \cdot \rangle$ is non-degenerate, this implies the equality $z = z'$. In this way, we defined $|W|$ additive quadratic forms.

Conversely, every additive quadratic form f is entirely determined by its images on the basis elements: for all $w = \sum_{i=1}^{2n} \lambda_i v_i$,

$$f(w) = \sum_{i=1}^{2n} \lambda_i^2 f(v_i).$$

For all $i \in \{1, \dots, 2n\}$ there are $|\mathbb{F}|$ possible values for $f(v_i)$. This means that there are exactly $|\mathbb{F}|^{2n} = |W|$ additive quadratic forms on W . This is enough to conclude that $\mathcal{Q}_a(W) = \{f_z, z \in W\}$.

2. Let $\mathcal{B} = \{v_1, \dots, v_{2n}\}$ be a basis of W . Recall from 1 that $\mathcal{Q}_a(W) = \{f_z, z \in W\}$. For every $z \in W$, $\chi \circ f_z$ is a character of W because f_z is additive and χ is a character of \mathbb{F} . Moreover, $\chi \circ f_z = \chi \circ f_{z'}$ implies that the image of the additive quadratic form $f_z - f_{z'}$ is contained in the kernel of χ . If we set $u = z - z'$, then $f_z - f_{z'} = f_u$. If $f_u \neq 0$, there exists $w_0 \in W$ such that $f_u(w_0) = \alpha \in \mathbb{F}^*$. Fix any $\lambda \in \mathbb{F}$. Since $\mathbb{F} = \mathbb{F}^2$ (Lemma 2.1), we can set $\alpha = \beta^2$ and $\lambda = \mu^2$. Hence $f_u(\mu\beta^{-1}w_0) = \mu^2\beta^{-2}f_u(w_0) = \lambda$. This shows that the image of f_u is equal to \mathbb{F} . Since χ is non-trivial, the only possibility is $f_u = f_z - f_{z'} = 0$, i.e. $z = z'$ by 1. Thus the maps $\chi \circ f_z$ define $|W|$ distinct characters of W , i.e. they describe exactly the set of characters of W .
3. Let $s = (\sigma, f)$ be in $\text{Ps}(B)$.

- a. Recall that, by definition of $\text{Ps}(B)$, for all $w_1, w_2 \in W$,

$$f(w_1 + w_2) - f(w_1) - f(w_2) = B(\sigma(w_1), \sigma(w_2)) - B(w_1, w_2).$$

If we take $w_1 = w_2$, this gives $0 = B(\sigma(w), \sigma(w)) - B(w, w)$, i.e. σ preserves the quadratic form Q .

- b. Let $\mathcal{B} = \{v_1, \dots, v_{2n}\}$ be a basis of W over \mathbb{F} . For all $w_1 = \sum_{i=1}^{2n} \lambda_i v_i$ and

$$w_2 = \sum_{i=1}^{2n} \mu_i v_i,$$

$$\beta_\sigma(w_1, w_2) = \sum_{1 \leq i, j \leq 2n} \lambda_i \mu_j \beta_\sigma(v_i, v_j) = \sum_{1 \leq i < j \leq 2n} (\lambda_i \mu_j + \lambda_j \mu_i) \beta_\sigma(v_i, v_j),$$

where the last equality is due to the fact that β_σ is alternating (because $\sigma \in \text{O}(Q)$), and thus symmetric in characteristic two. Now, for all $w = \sum_{1 \leq i \leq 2n} \lambda_i v_i$,

and for all $t \in \mathbb{F}$,

$$f_\sigma(tw) = \sum_{1 \leq i < j \leq 2n} t \lambda_i t \lambda_j \beta_\sigma(v_i, v_j) = t^2 \sum_{1 \leq i < j \leq 2n} \lambda_i \lambda_j \beta_\sigma(v_i, v_j) = t^2 f_\sigma(w).$$

Moreover

$$\begin{aligned}
& f_\sigma(w_1 + w_2) - f_\sigma(w_1) - f_\sigma(w_2) \\
&= \sum_{1 \leq i < j \leq 2n} (\lambda_i + \mu_i)(\lambda_j + \mu_j) \beta_\sigma(v_i, v_j) - \sum_{1 \leq i < j \leq 2n} \lambda_i \lambda_j \beta_\sigma(v_i, v_j) - \sum_{1 \leq i < j \leq 2n} \mu_i \mu_j \beta_\sigma(v_i, v_j) \\
&= \sum_{1 \leq i < j \leq 2n} (\lambda_i \mu_j + \lambda_j \mu_i) \beta_\sigma(v_i, v_j) \\
&= \beta_\sigma(w_1, w_2).
\end{aligned}$$

Thus f_σ is a quadratic form on W such that (σ, f_σ) belongs to $\text{Ps}(B)$.

c. Since $s = (\sigma, f)$ and (σ, f_σ) both belong to $\text{Ps}(B)$, then for all $w_1, w_2 \in W$,

$$f(w_1 + w_2) - f(w_1) - f(w_2) = \beta_\sigma(w_1, w_2) = f_\sigma(w_1 + w_2) - f_\sigma(w_1) - f_\sigma(w_2),$$

i.e. $(f - f_\sigma)(w_1 + w_2) - (f - f_\sigma)(w_1) - (f - f_\sigma)(w_2) = 0$. This means that $f - f_\sigma$ is an additive quadratic form on W , which we can denote by $f_z = f - f_\sigma$. Hence the choice of a basis \mathcal{B} leads to a unique expression of any element of $\text{Ps}(B)$ in the form $(\sigma, f) = (\sigma, f_\sigma + f_z)$ where f_σ is defined in (7) and $f_z = f - f_\sigma \in \mathcal{Q}_a(W)$.

4. If we choose a basis $\mathcal{B} = \{v_1, \dots, v_{2n}\}$ of W then $\mathcal{Q}_a(W) = \{f_z, z \in W\}$. We can thus conclude thanks to the preceding points that any element σ in $\text{O}(Q)$ leads to exactly $|W| = |\mathbb{F}|^{2n}$ elements of $\text{Ps}(B)$, namely the elements $(\sigma, f_\sigma + f_z)$, $z \in W$.
5. Let $s_1 = (1, f_1)$ and $s_2 = (1, f_2)$ be in $\text{Ps}_a(B)$. Observe first that $f_1, f_2 \in \mathcal{Q}_a(W)$ by (6). Moreover,

$$s_1 s_2^{-1} = (1, f_1)(1, -f_2) = (1, f_1 - f_2) = (1, -f_2)(1, f_1) = s_2^{-1} s_1.$$

Since $f_1 - f_2 \in \mathcal{Q}_a(W)$, this shows that $\text{Ps}_a(B)$ is an abelian subgroup of $\text{Ps}(B)$. Then 1 yields (9). □

The next proposition describes the relation between the symplectic and the orthogonal groups in characteristic two.

Proposition 1.12 (Relation between $\text{Sp}(W)$ and $\text{O}(Q)$). We denote by q the order of the field \mathbb{F} (q is a power of the prime number 2).

1. The orders of the symplectic and orthogonal groups are given by:

$$\begin{aligned}
|\text{Sp}(W)| &= q^{n^2} \prod_{k=1}^n (q^{2k} - 1) \\
|\text{O}(Q)| &= \begin{cases} 2q^{n(n-1)}(q^n - 1) \prod_{k=1}^{n-1} (q^{2k} - 1) & \text{if } \nu = n \\ 2q^{n(n-1)}(q^n + 1) \prod_{k=1}^{n-1} (q^{2k} - 1) & \text{if } \nu = n - 1. \end{cases}
\end{aligned}$$

2. The orthogonal group is contained in the symplectic group, i.e. $O(Q) \subseteq \text{Sp}(W)$. This inclusion is strict unless $q = 2$ (i.e. $\mathbb{F} = \mathbb{F}_2$), $n = 1$ (i.e. $\dim_{\mathbb{F}_2}(W) = 2$) and $\nu = n - 1 = 0$.

Proof. 1. Cf. [4, p. 94 and p. 206].

2. Let $\sigma \in O(Q)$. Then for all $w_1, w_2 \in \text{Sp}(W)$,

$$\begin{aligned} 0 &= Q(\sigma(w_1 + w_2)) - Q(w_1 + w_2) \\ &= Q(\sigma(w_1)) + Q(\sigma(w_2)) + \langle \sigma(w_1), \sigma(w_2) \rangle - Q(w_1) - Q(w_2) - \langle w_1, w_2 \rangle \\ &= \langle \sigma(w_1), \sigma(w_2) \rangle - \langle w_1, w_2 \rangle. \end{aligned}$$

This proves that σ belongs to $\text{Sp}(W)$. Suppose now that $n = 1$. By 1,

$$|\text{Sp}(W)| = q(q^2 - 1) \quad \text{and} \quad |O(Q)| = \begin{cases} 2(q - 1) & \text{if } \nu = 1 \\ 2(q + 1) & \text{if } \nu = 0. \end{cases}$$

Hence if $q = 2$ then

$$|\text{Sp}(W)| = 6 \quad \text{and} \quad |O(Q)| = \begin{cases} 2 & \text{if } \nu = 1 \\ 6 & \text{if } \nu = 0, \end{cases}$$

i.e. $O(Q) = \text{Sp}(W)$ if $\nu = 0$. If $q > 2$ then

$$\frac{|\text{Sp}(W)|}{|O(Q)|} = \begin{cases} \frac{q(q+1)}{2} \geq 10 & \text{if } \nu = 1 \\ \frac{q(q-1)}{2} \geq 6 & \text{if } \nu = 0, \end{cases}$$

which shows in particular that $O(Q)$ is strictly contained in $\text{Sp}(W)$. In the same way, if $n \geq 2$:

$$\frac{|\text{Sp}(W)|}{|O(Q)|} = \begin{cases} \frac{q^{2n}-1}{2q^{-n}(q^n-1)} = \frac{(q^n+1)q^n}{2} \geq 10 & \text{if } \nu = n \\ \frac{q^{2n}-1}{2q^{-n}(q^n+1)} = \frac{(q^n-1)q^n}{2} \geq 6 & \text{if } \nu = n - 1, \end{cases} \quad (10)$$

i.e. $O(Q)$ is again strictly contained in $\text{Sp}(W)$. □

Remark 1.13. Comparing the orders of the groups $O(Q)$ and $\text{Sp}(W)$ in (10), we realize that, in general, the pseudo-symplectic group is really far from being sufficient to define a Weil representation related to the symplectic group, since $\text{Ps}(B)$ only surjects to the orthogonal group $O(Q)$. The construction of a metaplectic group related to the entire symplectic group over $\mathbb{F} = \mathbb{F}_2$ will be described in a forthcoming article. The only case for which the construction of the Weil representation in the present article can be related to the entire symplectic group is treated in section 4.2.

2. THE STONE-VON NEUMANN THEOREM AND THE PROJECTIVE WEIL REPRESENTATION

Recall that \mathbb{F} is a finite field of characteristic two. We denote the Pontryagin dual of $(\mathbb{F}, +)$ by $\widehat{\mathbb{F}}$. We fix a non-trivial character χ of \mathbb{F} . As before, we consider a finite

dimensional \mathbb{F} -vector space W of dimension $2n$ equipped with a non-degenerate symplectic form $\langle \cdot, \cdot \rangle$. We also fix a complete polarisation of W , i.e. two totally isotropic subspaces X and Y of W (which means that $X \subseteq X^\perp$ and $Y \subseteq Y^\perp$, where the orthogonal complement is relative to the non-degenerate symplectic form $\langle \cdot, \cdot \rangle$) of maximal dimension such that $W = X \oplus Y$. In this setup, the non-degenerate symplectic form on W is given by

$$\langle w_1, w_2 \rangle = \langle x_1 + y_1, x_2 + y_2 \rangle = \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle,$$

for all $w_1 = x_1 + y_1$ and $w_2 = x_2 + y_2$ in W , with $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Moreover, the map which sends $y \in Y$ to the linear form $\langle y, \cdot \rangle$ on X , defines an isomorphism from Y to X^* . This enables us to identify W with $X \times X^*$, fix a bilinear form B such that

$$\langle w_1, w_2 \rangle = B(w_1, w_2) - B(w_2, w_1)$$

for all $w_1, w_2 \in W$, and define the Heisenberg group $H(B)$ from section 1. For example, the expression of the bilinear form used by Weil is in this setting:

$$B_{\text{Weil}}(w_1, w_2) = \langle x_1, y_2 \rangle.$$

An important feature of the Heisenberg group is the fact that it has, up to isomorphism, a unique irreducible unitary representation acting as χ on its center. In our setting (finite dimensional vector spaces over a finite field of characteristic two), we can give an easy proof of this result, which is known as the Stone-von Neumann theorem. This proof is based on counting the irreducible representations of $H(B)$. We begin by associating to each non-trivial character of \mathbb{F} a unitary irreducible representation of $H(B)$.

- Lemma 2.1.**
1. Every element of \mathbb{F} is a square in \mathbb{F} .
 2. Every character of \mathbb{F} takes its values in $\{\pm 1\}$ and can uniquely be written in the form $\chi_a : t \mapsto \chi(at)$, where $a \in \mathbb{F}$. In particular there are $|\mathbb{F}| - 1$ non-trivial characters of \mathbb{F} .
 3. χ can be extended to the commutative subgroup $X \times \mathbb{F}$ of $H(B)$ in $|\mathbb{F}|^n$ different ways. Let $\tilde{\chi}$ be such an extension. Then $\tilde{\chi}$ takes its values in $\{\pm 1, \pm i\}$. More precisely, $\tilde{\chi}$ takes the values $\pm i$ if and only if X is not a singular space, i.e. $Q|_X \neq 0$.

Proof. Suppose that the order of \mathbb{F} is equal to $q = 2^m$ with $m \geq 1$, i.e. $\mathbb{F} = \mathbb{F}_{2^m}$.

1. Let $a, b \in \mathbb{F}$ be such that $a^2 = b^2$. Then $(a+b)^2 = a^2 + b^2 = 0$. Hence $a = b$, which implies that the map $a \mapsto a^2$ is injective. Since \mathbb{F} is finite, this map is bijective.
2. Let ψ be a character of \mathbb{F} . Then ψ takes its values in $\{\pm 1\}$ because for all $t \in \mathbb{F}$,

$$1 = \psi(0) = \psi(2t) = \psi(t)^2.$$

The fact that $\mathbb{F} \rightarrow \widehat{\mathbb{F}}$ (where $\chi_a : t \mapsto \chi(at)$) is an isomorphism follows from the

$$a \mapsto \chi_a$$

fundamental theorem for finitely generated abelian groups: $(\mathbb{F}_{2^m}, +)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$ and $\widehat{\mathbb{Z}/2\mathbb{Z}} \simeq \mathbb{Z}/2\mathbb{Z}$. Hence $\widehat{\mathbb{F}} \simeq (\widehat{\mathbb{Z}/2\mathbb{Z}})^m \simeq (\mathbb{Z}/2\mathbb{Z})^m \simeq (\mathbb{Z}/2\mathbb{Z})^m \simeq \mathbb{F}$.

3. Since $\mathbb{F} = \mathbb{F}_{2^m}$, \mathbb{F} is a \mathbb{F}_2 -vector space of dimension m . Let $\{t_1, \dots, t_m\}$ be a basis of \mathbb{F} over \mathbb{F}_2 and let $\{x_1, \dots, x_n\}$ be a basis of X over \mathbb{F} . Suppose that

$\tilde{\chi}$ is an extension of χ to $X \times \mathbb{F}$. Observe that for every integer $N \geq 2$ and $w_1, \dots, w_N \in W$, we have

$$\left(\sum_{k=1}^N w_k, 0 \right) = \left(0, - \sum_{j=1}^{N-1} \sum_{l=j+1}^N B(w_j, w_l) \right) \prod_{k=1}^N (w_k, 0) \quad (11)$$

in $H(B)$.

Now let (x, t) be an element of the commutative subgroup $X \times \mathbb{F}$ of $H(B)$. Then there exists $\mu_{k,l} \in \mathbb{F}_2$ and $\lambda_k = \sum_{l=1}^m \mu_{k,l} t_l \in \mathbb{F}$ such that $(x, t) = \left(\sum_{k=1}^n \lambda_k x_k, t \right)$. Applying (11) to $w_k = \sum_{1 \leq l \leq m} \mu_{k,l} t_l x_k$, where $1 \leq k \leq n$, we obtain:

$$\left(\sum_{k=1}^n \sum_{l=1}^m \mu_{k,l} t_l x_k, 0 \right) = (0, T) \prod_{k=1}^n \left(\sum_{l=1}^m \mu_{k,l} t_l x_k, 0 \right) \quad (12)$$

where

$$T = - \sum_{j=1}^{n-1} \sum_{l=j+1}^n B \left(\sum_{r=1}^m \mu_{j,r} t_r x_j, \sum_{s=1}^m \mu_{l,s} t_s x_l \right) = - \sum_{j=1}^{n-1} \sum_{l=j+1}^n \left(\sum_{r=1}^m \mu_{j,r} t_r \right) \left(\sum_{s=1}^m \mu_{l,s} t_s \right) \beta_{j,l}$$

and

$$\beta_{j,l} = B(x_j, x_l).$$

Applying (11) to $v_l = \mu_{k,l} t_l x_k$, where $1 \leq l \leq m$ and k is fixed, we obtain

$$\left(\sum_{l=1}^m \mu_{k,l} t_l x_k, 0 \right) = (0, T_k) \prod_{l=1}^m (\mu_{k,l} t_l x_k, 0), \quad (13)$$

where

$$T_k = - \sum_{r=1}^{m-1} \sum_{s=r+1}^m B(\mu_{k,r} t_r x_k, \mu_{k,s} t_s x_k) = - \left(\sum_{r=1}^{m-1} \mu_{k,r} t_r \right) \left(\sum_{s=r+1}^m \mu_{k,s} t_s \right) \beta_{k,k}.$$

Thus:

$$\begin{aligned} \tilde{\chi}(x, t) &= \tilde{\chi}((0, t)(x, 0)) \\ &= \chi(t) \tilde{\chi} \left(\sum_{1 \leq k \leq n, 1 \leq l \leq m} \mu_{k,l} t_l x_k, 0 \right) \\ &\stackrel{(12)}{=} \chi(t) \chi(T) \tilde{\chi} \left(\prod_{k=1}^n \left(\sum_{l=1}^m \mu_{k,l} t_l x_k, 0 \right) \right) \\ &= \chi(t) \chi(T) \prod_{k=1}^n \tilde{\chi} \left(\sum_{l=1}^m \mu_{k,l} t_l x_k, 0 \right) \\ &\stackrel{(13)}{=} \chi(t) \chi(T) \prod_{k=1}^n \chi(T_k) \prod_{l=1}^m \tilde{\chi}(\mu_{k,l} t_l x_k, 0). \end{aligned}$$

Since $\mu_{k,l} \in \mathbb{F}_2$, the character $\tilde{\chi}$ is entirely determined by the images $\tilde{\chi}(t_l x_k)$ for all $1 \leq k \leq n$ and $1 \leq l \leq m$, via the formula

$$\tilde{\chi}(x, t) = \chi(t)\chi(T) \prod_{k=1}^n \chi(T_k) \prod_{l=1}^m \tilde{\chi}(t_l x_k, 0)^{\mu_{k,l}}. \quad (14)$$

Now let $1 \leq k \leq n$ and $1 \leq l \leq m$. Since $(t_l x_k, 0)(t_l x_k, 0) = (0, t_l^2 \beta_{k,k})$, the value of $\tilde{\chi}(t_l x_k)$ depends entirely on $\beta_{k,k} = Q(x_k)$. In fact, if $\beta_{k,k} = 0$ then the order of $(t_l x_k, 0)$ in $X \times \mathbb{F}$ is 2, and thus $\tilde{\chi}(t_l x_k) \in \{\pm 1\}$, whereas if $\beta_{k,k} \neq 0$ then the order of $(t_l x_k, 0)$ in $X \times \mathbb{F}$ is 4, and thus $\tilde{\chi}(t_l x_k) \in \{\pm i\}$. In any case, there are two possibilities for $\tilde{\chi}(t_l x_k)$. This leads to at most 2^{nm} possibilities for the extension $\tilde{\chi}$. Conversely, any choice of $\tilde{\chi}(t_l x_k)$ in $\{\pm 1\}$ if $\beta_{k,k} = 0$, or in $\{\pm i\}$ if $\beta_{k,k} \neq 0$, for all $1 \leq k \leq n$ and $1 \leq l \leq m$, defines by (14) a character of $X \times \mathbb{F}$ which extends χ . Therefore, there are exactly $2^{nm} = q^n = |\mathbb{F}|^n$ extensions of χ to $X \times \mathbb{F}$, which can be explicitly computed thanks to the formula (14) above. We have also shown that any such extension $\tilde{\chi}$ takes its values in $\{\pm 1, \pm i\}$, and more precisely the values $\pm i$ are taken if and only if there exists $x \in X$ such that $Q(x) = B(x, x) \neq 0$, i.e. if and only if X is not Q -singular. \square

Now we fix the counting measures of Y and \hat{Y} , respectively, by

$$\mu_Y = \frac{1}{\sqrt{|Y|}} \sum_{y \in Y} \delta_y \quad \text{and} \quad \mu_{\hat{Y}} = \frac{1}{\sqrt{|Y|}} \sum_{\xi \in \hat{Y}} \delta_\xi.$$

Since Y is finite, the set $L^2(Y)$ of square-integrable functions $\phi : Y \rightarrow \mathbb{C}$ coincides with the set of all functions $\phi : Y \rightarrow \mathbb{C}$. We denote by $\mathcal{B}_Y = \{\delta_y, y \in Y\}$ the natural basis of $L^2(Y)$. Hence $\dim_{\mathbb{C}}(L^2(Y)) = |Y| = |\mathbb{F}|^n$. Moreover, we can define a scalar product $\langle \cdot, \cdot \rangle_2$ on $L^2(Y)$ by:

$$\langle \phi, \psi \rangle_2 = \int_Y \phi(y) \overline{\psi(y)} d\mu_Y(y),$$

for all $\phi, \psi \in L^2(Y)$. We also define the Fourier transform \mathcal{F} over $L^2(Y)$ and its inverse:

$$\begin{aligned} \mathcal{F}(\phi)(\xi) &= \int_Y \phi(y) \xi(y) d\mu_Y(y) \\ \phi(y) &= \int_{\hat{Y}} \mathcal{F}(\phi)(\xi) \xi(-y) d\mu_{\hat{Y}}(\xi), \end{aligned}$$

for all $\phi \in L^2(Y)$, $y \in Y$ and $\xi \in \hat{Y}$. In this way, the Plancherel theorem states that $\|\mathcal{F}(\phi)\|_2 = \|\phi\|_2$ for all $\phi \in L^2(Y)$. Finally we define the adjoint of an operator $U \in \text{End}_{\mathbb{C}}(L^2(Y))$ as the operator U^* which satisfies for all $\phi, \psi \in L^2(Y)$:

$$\langle U^*(\phi), \psi \rangle_2 = \langle \phi, U(\psi) \rangle_2.$$

Now we define the Schrödinger model of the Heisenberg representation of the group $H(B)$. Then, in Lemma 2.3, we prove that it is a unitary irreducible representation of $H(B)$ and present its realization on $L^2(Y)$.

Definition 2.2 (Schrödinger model of the Heisenberg representation). Let $\tilde{\chi}$ be an extension of χ to the abelian subgroup $X \times \mathbb{F}$ of $H(B)$. We denote by $\mathcal{H}(X, \tilde{\chi})$ the \mathbb{C} -vector space of functions $\phi : H(B) \rightarrow \mathbb{C}$ such that for all $(x, t) \in X \times \mathbb{F}$ and for all $h \in H(B)$,

$$\phi(h(x, t)) = \tilde{\chi}(x, t)^{-1} \phi(h).$$

Let $\rho_{\tilde{\chi}} : H(B) \rightarrow \text{End}_{\mathbb{C}}(\mathcal{H}(X, \tilde{\chi}))$ be the map defined by $\rho_{\tilde{\chi}}(g)\phi(h) = \phi(g^{-1}h)$.

Lemma 2.3. $\mathcal{H}(X, \tilde{\chi})$ is isomorphic to $L^2(Y)$ by restriction to Y . As a representation of $H(B)$ on the Hilbert space $L^2(Y)$, $\rho_{\tilde{\chi}}$ is unitary, irreducible and acts as χ on the center. It is given by the following formula: for all $g = (x + y, t) \in H(B)$, for all $\varphi \in L^2(Y)$ and for all $y' \in Y$,

$$\rho_{\tilde{\chi}}(x + y, t)\varphi(y') = \tilde{\chi}(x, t - Q(y) + \langle x, y' \rangle - B(x + y', y))\varphi(y' - y). \quad (15)$$

The character of $\rho_{\tilde{\chi}}$ is given by

$$\text{Trace}(\rho_{\tilde{\chi}}(x + y, t)) = \begin{cases} \chi(t)|Y| & \text{if } x = y = 0 \\ 0 & \text{otherwise,} \end{cases} \quad (16)$$

and does not depend on the choice of extension $\tilde{\chi}$. In particular, another choice of extension χ^{\sharp} gives a representation $\rho_{\chi^{\sharp}}$ which is equivalent to $\rho_{\tilde{\chi}}$.

Remark 2.4. The fact that $\rho_{\tilde{\chi}}$ is a representation of $H(B)$ implies in particular that for all $w, w' \in W$,

$$\rho_{\tilde{\chi}}(w)\rho_{\tilde{\chi}}(w') = \rho_{\tilde{\chi}}(w + w', B(w, w')) = \chi(B(w, w'))\rho_{\tilde{\chi}}(w + w'). \quad (17)$$

Remark 2.5. We can wonder why we need to extend the character χ of \mathbb{F} to the abelian subgroup $X \times \mathbb{F}$ of $H(B)$. In fact, we could want to extend trivially χ to $X \times \mathbb{F}$, and define the space $\mathcal{H}(X, \chi)$ as the set of functions $\phi : H(B) \rightarrow \mathbb{C}$ such that for all $(x, t) \in X \times \mathbb{F}$ and for all $h \in H(B)$,

$$\phi(h(x, t)) = \chi(t)^{-1} \phi(h).$$

However, this is only possible if $B|_{X \times X} = 0$. In fact, suppose that $\tilde{\chi}$ extends χ trivially. Then for all $(x_1, t_1), (x_2, t_2) \in X \times \mathbb{F}$,

$$\begin{cases} \tilde{\chi}(x_1, t_1)\tilde{\chi}(x_2, t_2) = \chi(t_1)\chi(t_2) = \chi(t_1 + t_2) \\ \tilde{\chi}((x_1, t_1)(x_2, t_2)) = \tilde{\chi}(x_1 + x_2, t_1 + t_2 + B(x_1, x_2)) = \chi(t_1 + t_2 + B(x_1, x_2)). \end{cases} \quad (18)$$

Since $\tilde{\chi}$ is a character, (18) implies that $\chi(B(x_1, x_2)) = 1$ for all $x_1, x_2 \in X$. If $B|_{X \times X}$ is non-zero then its image is \mathbb{F} . This means that $\mathbb{F} \subseteq \text{Ker}(\chi)$, which is impossible since χ is non-trivial by hypothesis. Hence $B|_{X \times X} = 0$. This happens for example if B is B_{Weil} . Consequently, if $B|_{X \times X}$ is non-zero then the trivial extension of χ is not a character of $X \times \mathbb{F}$. This is why, following [3], we consider arbitrary characters $\tilde{\chi}$ of $X \times \mathbb{F}$ that restrict to χ on \mathbb{F} in Definition 2.2, in order to get a realization of the Heisenberg representation which is valid for any bilinear form B .

Proof of Lemma 2.3. First we show that $\rho_{\tilde{\chi}}$ is a representation of $H(B)$ on $\mathcal{H}(X, \tilde{\chi})$ acting as χ on the center. For all $g, g', h \in H(B)$, for all $(x, t) \in X \times \mathbb{F}$ and for all $\phi \in \mathcal{H}(X, \tilde{\chi})$,

- $\rho_{\tilde{\chi}}(g)\phi(h(x, t)) = \phi(g^{-1}h(x, t)) = \tilde{\chi}(x, t)^{-1}\phi(g^{-1}h) = \tilde{\chi}(x, t)^{-1}\rho_{\tilde{\chi}}(g)\phi(h)$. Hence $\rho_{\tilde{\chi}}(g)\phi$ is in the space $\mathcal{H}(X, \tilde{\chi})$.
 - $\rho_{\tilde{\chi}}(0, 0)\phi(h) = \phi((0, 0)^{-1}h) = \phi(h)$ and
- $$\rho_{\tilde{\chi}}(gg')\phi(h) = \phi((gg')^{-1}h) = \phi(g'^{-1}g^{-1}h) = \rho_{\tilde{\chi}}(g')\phi(g^{-1}h) = (\rho_{\tilde{\chi}}(g)\rho_{\tilde{\chi}}(g'))\phi(h).$$
- $\rho_{\tilde{\chi}}(0, t)\phi(h) = \phi((0, t)^{-1}h) = \phi((0, -t)h) = \phi(h(0, -t)) = \chi(-t)^{-1}\phi(h) = \chi(t)\phi(h)$, so $\rho_{\tilde{\chi}}(0, t) = \chi(t)\text{Id}_{\mathcal{H}(X, \tilde{\chi})}$.

To show that $\rho_{\tilde{\chi}}$ is unitary and irreducible, we first prove the vector space isomorphism between $\mathcal{H}(X, \tilde{\chi})$ and $L^2(Y)$, and then we use the realization of $\rho_{\tilde{\chi}}$ on $L^2(Y)$.

Consider the linear map $\mathcal{H}(X, \tilde{\chi}) \rightarrow L^2(Y)$, where $\phi|_Y(y) = \phi(y, 0)$ for all $y \in Y$.

$$\phi \mapsto \phi|_Y$$

- If $\phi \in \mathcal{H}(X, \tilde{\chi})$ satisfies $\phi|_Y = 0$, then for all $h = (x + y, t) \in \text{H}(B)$,

$$\phi(h) = \phi((y, 0)(x, t - B(y, x))) = \tilde{\chi}(x, t - B(y, x))^{-1}\phi(y, 0) = 0.$$

Hence the map $\phi \mapsto \phi|_Y$ is injective.

- Let $\varphi \in L^2(Y)$. For all $(x + y, t) \in \text{H}(B)$, set $\phi_\varphi(x + y, t) = \tilde{\chi}(x, t - B(y, x))^{-1}\varphi(y)$. Then

$$\phi_\varphi(y, 0) = \chi(B(y, 0))\varphi(y) = \varphi(y),$$

i.e. $\phi_\varphi|_Y = \varphi$, and for all $h = (x' + y', t') \in \text{H}(B)$,

$$\begin{aligned} \phi_\varphi(h(x, t)) &= \phi_\varphi(x' + y' + x, t' + t + B(x' + y', x)) \\ &= \tilde{\chi}(x + x', t + t' + B(x' + y', x) - B(y', x + x'))^{-1}\varphi(y') \\ &= \tilde{\chi}((x', t' - B(y', x'))(x, t))^{-1}\varphi(y') \\ &= \tilde{\chi}(x, t)^{-1}\tilde{\chi}(x', t' - B(y', x'))^{-1}\varphi(y') \\ &= \tilde{\chi}(x, t)^{-1}\phi_\varphi(h). \end{aligned}$$

Hence $\phi_\varphi \in \mathcal{H}(X, \tilde{\chi})$ and the map $\phi \mapsto \phi|_Y$ is surjective.

Thus the map $\phi \mapsto \phi|_Y$ defines an isomorphism between $\mathcal{H}(X, \tilde{\chi})$ and $L^2(Y)$, allowing us to realize $\rho_{\tilde{\chi}}$ on $L^2(Y)$. For every $\varphi \in L^2(Y)$, for every $(x + y, t) \in \text{H}(B)$ and for every $y' \in Y$,

$$\begin{aligned} \rho_{\tilde{\chi}}(x + y, t)\varphi(y') &= \rho_{\tilde{\chi}}(x + y, t)\phi_\varphi(y', 0) \\ &= \phi_\varphi((x + y, t)^{-1}(y', 0)) \\ &= \phi_\varphi((-x - y, -t + B(x + y, x + y))(y', 0)) \\ &= \phi_\varphi(-x - y + y', -t + B(x + y, x + y) - B(x + y, y')) \\ &= \tilde{\chi}(-x, -t + B(x + y, x + y - y') - B(y - y', x))^{-1}\varphi(y' - y) \\ &= \tilde{\chi}(x, t - Q(x) - \langle x, y - y' \rangle - B(y, x - y') - Q(y) + Q(x))\varphi(y' - y) \\ &= \tilde{\chi}(x, t - Q(y) + \langle x, y' \rangle - B(x, y) - B(y, y'))\varphi(y' - y) \\ &= \tilde{\chi}(x, t - Q(y) + \langle x, y' \rangle - B(x + y', y))\varphi(y' - y). \end{aligned}$$

Suppose now that $\mathcal{V} \subseteq L^2(Y)$ is a $\rho_{\tilde{\chi}}$ -invariant non-zero vector subspace, i.e. $\mathcal{V} \neq 0$ and $\rho_{\tilde{\chi}}(h)\mathcal{V} \subseteq \mathcal{V}$ for all $h \in H(B)$. If $\mathcal{V} \neq L^2(Y)$ then there exists a non-zero $\varphi_0 \in \mathcal{V}$ such that $\rho_{\tilde{\chi}}(H(B))\varphi_0 \neq L^2(Y)$. Thus the subspace orthogonal to $\rho_{\tilde{\chi}}(H(B))\varphi_0$ is different from $\{0\}$, i.e. there exists a non-zero $\psi_0 \in L^2(Y)$ such that

$$\langle \rho_{\tilde{\chi}}(h)\varphi_0, \psi_0 \rangle_2 = 0 \quad \text{for all } h \in H(B). \quad (19)$$

For all functions $\varphi, \psi \in L^2(Y)$ and for all $w \in W$ we define:

$$\Lambda_{\varphi, \psi}(w) = \langle \rho_{\tilde{\chi}}(w, 0)\varphi, \psi \rangle_2,$$

i.e. for $w = x + y$, where $x \in X$ and $y \in Y$:

$$\begin{aligned} \Lambda_{\varphi, \psi}(x + y) &= \int_Y \rho_{\tilde{\chi}}(x + y)\varphi(y')\overline{\psi(y')} dy' \\ &\stackrel{(15)}{=} \int_Y \tilde{\chi}(x, -Q(y) + \langle x, y' \rangle - B(x + y', y))\varphi(y' - y)\overline{\psi(y')} dy' \\ &= \tilde{\chi}(x, -Q(y) - B(x, y)) \int_Y \chi(\langle x, y' \rangle - B(y', y))\varphi(y' - y)\overline{\psi(y')} dy' \\ &= \tilde{\chi}(x, -Q(y) - B(x, y)) \int_Y \xi_x(y')f_{\varphi, \psi, y}(y') dy', \end{aligned}$$

where $f_{\varphi, \psi, y} : y' \mapsto \chi(-B(y', y))\varphi(y' - y)\overline{\psi(y')}$ is an element of $L^2(Y)$ and $\xi_x : y' \mapsto \chi(\langle x, y' \rangle)$ is a character of Y . Thus:

$$\Lambda_{\varphi, \psi}(x + y) = \tilde{\chi}(x, -Q(y) - B(x, y)) \mathcal{F}(f_{\varphi, \psi, y})(\xi_x).$$

Hence

$$\begin{aligned} \|\Lambda_{\varphi, \psi}\|_2^2 &= \int_X \int_Y |\Lambda_{\varphi, \psi}(x + y)|^2 dy dx \\ &= \int_X \int_Y |\tilde{\chi}(x, -Q(y) - B(x, y))|^2 |\mathcal{F}(f_{\varphi, \psi, y})(\xi_x)|^2 dy dx \\ &= \int_X \int_Y |\mathcal{F}(f_{\varphi, \psi, y})(\xi_x)|^2 dy dx \\ &= \int_{\hat{Y}} \int_Y |\mathcal{F}(f_{\varphi, \psi, y})(\xi)|^2 dy d\xi \quad \text{because } x \mapsto \xi_x \text{ is a group isomorphism between} \\ &\quad X \text{ and } \hat{Y} \\ &= \int_Y \|\mathcal{F}(f_{\varphi, \psi, y})\|_2^2 dy \end{aligned}$$

Applying Plancherel's formula, we obtain:

$$\begin{aligned}
\|\Lambda_{\varphi,\psi}\|_2^2 &= \int_Y \|f_{\varphi,\psi,y}\|_2^2 dy \\
&= \int_{Y \times Y} |\chi(-B(y',y))\varphi(y+y')\overline{\psi(y')}|^2 dy' dy \\
&= \int_{Y \times Y} |\varphi(y+y')|^2 |\overline{\psi(y')}|^2 dy dy' \\
&= \int_Y |\varphi(y'')| dy'' \int_Y |\overline{\psi(y')}|^2 dy' \\
&= \|\varphi\|_2^2 \|\psi\|_2^2.
\end{aligned}$$

Thus

$$\|\varphi_0\|_2^2 \|\psi_0\|_2^2 = \|\Lambda_{\varphi_0,\psi_0}\|_2^2 = \int_X \int_Y |\Lambda_{\varphi_0,\psi_0}(x+y)|^2 dy dx = \int_X \int_Y |\langle \rho_{\tilde{\chi}}(x+y)\varphi_0, \psi_0 \rangle_2|^2 dy dx \stackrel{(19)}{=} 0.$$

This contradicts the fact that φ_0 and ψ_0 are non-zero. Hence $\mathcal{V} = L^2(Y)$, and thus $\rho_{\tilde{\chi}}$ is irreducible.

Now, we prove that $\rho_{\tilde{\chi}}$ is unitary. Let $h = (x+y, t) \in H(B)$ and $\varphi \in L^2(Y)$, then

$$\begin{aligned}
\|\rho_{\tilde{\chi}}(h)\varphi\|_2^2 &= \int_Y |\rho_{\tilde{\chi}}(x+y, t)\varphi(y')|^2 dy' \\
&\stackrel{(15)}{=} \int_Y |\tilde{\chi}(x, t - Q(y) + \langle x, y' \rangle - B(x+y', y))\varphi(y' - y)|^2 dy' \\
&= \int_Y |\varphi(y' - y)|^2 dy' \\
&= \int_Y |\varphi(y'')|^2 dy'' \\
&= \|\varphi\|_2^2.
\end{aligned}$$

This shows that the operator $\rho_{\tilde{\chi}}(h)$ preserves the norm, and thus is unitary.

The last point to prove is that if $\tilde{\chi}$ and χ^\sharp are two different extensions of χ to $X \times \mathbb{F}$ then the representations $\rho_{\tilde{\chi}}$ and ρ_{χ^\sharp} are equivalent. To this extent, we first determine the character of the representation $\rho_{\tilde{\chi}}$ in the basis $\mathcal{B}_Y = \{\delta_z, z \in Y\}$ of $L^2(Y)$. By (15), for every $x \in X$, $y, y', z \in Y$ and $t \in \mathbb{F}$,

$$\begin{aligned}
\rho_{\tilde{\chi}}(x+y, t)\delta_z(y') &= \tilde{\chi}(x, t - Q(y) + \langle x, y' \rangle - B(x+y', y))\delta_z(y' - y) \\
&= \tilde{\chi}(x, t - Q(y) + \langle x, z+y \rangle - B(x+z+y, y))\delta_{z+y}(y') \\
&= \tilde{\chi}(x, t - Q(y) + \langle x, z \rangle - B(z+y, y) - B(y, x))\delta_{z+y}(y') \\
&= \tilde{\chi}(x, t - 2Q(y) + \langle x, z \rangle - B(y, x+z))\delta_{z+y}(y')
\end{aligned}$$

i.e.

$$\rho_{\tilde{\chi}}(x+y, t)\delta_z = \tilde{\chi}(x, t + \langle x, z \rangle - B(y, x+z))\delta_{z+y}. \quad (20)$$

Hence by Schur's orthogonality relations, $\rho_{\tilde{\chi}}(x + y, t)\delta_z$ contributes to the trace of $\rho_{\tilde{\chi}}(x + y, t)$ with value $\tilde{\chi}(x, t + \langle x, z \rangle)$ if $y = 0$, and 0 otherwise. Thus:

$$\text{Trace}(\rho_{\tilde{\chi}}(x + y, t)) = \begin{cases} \sum_{z \in Y} \tilde{\chi}(x, t + \langle x, z \rangle) = \tilde{\chi}(x, t) \sum_{z \in Y} \chi(\langle x, z \rangle) & \text{if } y = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The character $\chi(\langle x, \cdot \rangle)$ is trivial on Y if and only if $x = 0$. Hence (16) follows. \square

Remark 2.6. In the proof of Lemma 2.3, the irreducibility of $\rho_{\tilde{\chi}}$ can also be proven using Lemma 2.8 below. In fact, if $\mathcal{V} \subseteq L^2(Y)$ is a $\rho_{\tilde{\chi}}$ -invariant vector subspace, then \mathcal{V} is invariant under the operator $\rho_{\tilde{\chi}}(w)$ for every $w \in W$, hence \mathcal{V} is invariant under all the operators of $\text{End}_{\mathbb{C}}(L^2(Y))$. Then $\mathcal{V} = L^2(Y)$. A third proof can be given using Schur's orthogonality relations. We compute (16) first and then write:

$$\begin{aligned} (\rho_{\tilde{\chi}}, \rho_{\tilde{\chi}}) &= \frac{1}{|\mathbb{H}(B)|} \sum_{h \in \mathbb{H}(B)} |\text{Trace}(\rho_{\tilde{\chi}}(h))|^2 = \frac{1}{|\mathbb{H}(B)|} \sum_{(x+y, t) \in \mathbb{H}(B)} |\chi(t)|^2 |Y|^2 \delta_{(0,0)}(x, y) \\ &= \frac{1}{|\mathbb{H}(B)|} \sum_{(0, t) \in \mathbb{H}(B)} |Y|^2 \\ &= \frac{|\mathbb{F}||Y|^2}{|\mathbb{H}(B)|} \\ &= 1. \end{aligned}$$

Corollary 2.7. The character of $\rho_{\tilde{\chi}}$ is a function supported on the center of $\mathbb{H}(B)$. For $h = (w, t) \in \mathbb{H}(B)$, it is given by

$$\text{Trace}(\rho_{\tilde{\chi}}(w, t)) = \chi(t)|Y|\delta_0(w).$$

Proof. This is exactly the proof of (16) in Lemma 2.3 above. \square

Lemma 2.8. $\{\rho_{\tilde{\chi}}(w), w \in W\}$ is a basis of $\text{End}_{\mathbb{C}}(L^2(Y))$.

Proof. Suppose that for every $w = x + y \in W$, there exists $\lambda(w) = \lambda(x, y) \in \mathbb{C}$ such that

$$\sum_{w \in W} \lambda(w) \rho_{\tilde{\chi}}(w) = 0.$$

Then for every $y' \in Y$, by (20):

$$\begin{aligned} 0 &= \sum_{w \in W} \lambda(w) \rho_{\tilde{\chi}}(w) \delta_{y'} = \sum_{w \in W} \lambda(w) \tilde{\chi}(x, \langle x, y' \rangle - B(y, x + y')) \delta_{y'+y} \\ &= \sum_{y \in Y} \chi(-B(y, y')) \sum_{x \in X} \lambda(x, y) \tilde{\chi}(x, \langle x, y' \rangle - B(y, x)) \delta_{y'+y}. \end{aligned}$$

Since $\{\delta_{y'+y}, y \in Y\}$ is a basis of $L^2(Y)$,

$$\sum_{x \in X} \lambda(x, y) \tilde{\chi}(x, \langle x, y' \rangle - B(y, x)) = 0 \tag{21}$$

for all $y, y' \in Y$. Since X and Y are finite and have the same cardinality $N = |Y| = |\mathbb{F}|^n$, we can define the following matrices:

$$\begin{cases} A = (a_{y',x})_{y' \in Y, x \in X} \in M_N(\mathbb{C}) & \text{by } a_{y',x} = \chi(\langle x, y' \rangle) \\ B = (b_{x,y})_{x \in X, y \in Y} \in M_N(\mathbb{C}) & \text{by } b_{x,y} = \lambda(x, y) \tilde{\chi}(x, -B(y, x)). \end{cases}$$

Then, writing (21) for every $y, y' \in Y$ is equivalent to writing $AB = 0$. Now, for all $(y, y') \in Y^2$,

$$(AA^T)_{y',y} = \sum_{x \in X} \chi(\langle x, y' \rangle) \chi(\langle y, x \rangle) = \sum_{x \in X} \chi(\langle x, y' - y \rangle) = |X| \delta_y(y'),$$

where the last equality is justified by Schur's orthogonality relations. This proves that A is invertible, its inverse being $|X|^{-1}A^T$. In particular, $AB = 0$ implies that $B = 0$. Hence, for every $x \in X$ and $y \in Y$, $\lambda(x, y) \tilde{\chi}(x, -B(y, x)) = 0$. Since $\tilde{\chi}$ is a character, it takes its values in $U(1)$. Thus $\lambda(w) = 0$ for all $w \in W$. This proves that $\{\rho_{\tilde{\chi}}(w), w \in W\}$ is a family of $|W| = |Y|^2 = N^2$ linearly independent operators of $\text{End}_{\mathbb{C}}(\text{L}^2(Y))$, which is of dimension $\dim_{\mathbb{C}}(\text{L}^2(Y))^2 = |Y|^2 = N^2$. This is enough to conclude that $\{\rho_{\tilde{\chi}}(w), w \in W\}$ is a basis of $\text{End}_{\mathbb{C}}(\text{L}^2(Y))$. \square

The usual approach for constructing the metaplectic representation of a covering of the pseudo-symplectic group is based on the Stone-von Neumann Theorem. As we shall see below, we shall not use it in our construction. However, for the sake of completeness, we now state and give a simple proof of this theorem.

Theorem 2.9 (Stone-von Neumann). The Heisenberg group $H(B)$ has, up to isomorphism, a unique unitary irreducible representation acting as χ on its center, which will be called the Heisenberg representation. For every non-trivial character χ of \mathbb{F} and extension $\tilde{\chi}$ of χ to $X \times \mathbb{F}$, the map $\rho_{\tilde{\chi}}$ defined by (15) is a realization of the Heisenberg representation on $\text{L}^2(Y)$, which is therefore of dimension $|Y| = |\mathbb{F}|^n$.

Proof. We begin by listing the irreducible representations of $H(B)$. Notice first that for all $w, w' \in W$ and $t, t' \in \mathbb{F}$,

$$\begin{aligned} & (w, t)(w', t')(w, t)^{-1}(w', t')^{-1} \\ &= (w + w', t + t' + B(w, w'))(-w - w', -t - t' + Q(w) + Q(w') + B(w, w')) \\ &= (0, Q(w) + Q(w') + 2B(w, w') - B(w + w', w + w')) \\ &= (0, B(w, w') - B(w', w)) \\ &= (0, \langle w, w' \rangle). \end{aligned}$$

Hence the commutator subgroup $[H(B), H(B)]$ of $H(B)$ is exactly the center of $H(B)$, and is isomorphic to \mathbb{F} . The quotient $H(B)/[H(B), H(B)]$ is thus isomorphic to W , which is an abelian group and has $|W| = |Y|^2$ one-dimensional representations. This leads to $|Y|^2$ one-dimensional representations of $H(B)$. Moreover, for every non-trivial character χ , extending χ to $X \times \mathbb{F}$ and using Lemma 2.3, we provide a unitary irreducible representation of $H(B)$ acting as χ on its center. This implies in particular that if χ and ψ are two distinct

characters of $H(B)$, then $\rho_{\tilde{\chi}}$ and $\rho_{\tilde{\psi}}$ are not equivalent. This leads to $|\mathbb{F}| - 1$ additional inequivalent irreducible representations of $H(B)$ of dimension $|Y|$ (cf. Lemma 2.1). If we sum their dimensions squared, we obtain

$$|Y|^2 + (|\mathbb{F}| - 1)|Y|^2 = |Y|^2|\mathbb{F}| = |H(B)|,$$

which shows (as a consequence of Schur's orthogonality relations) that we found all the pairwise inequivalent irreducible representations of $H(B)$. In particular, for every non-trivial character χ of \mathbb{F} , $\rho_{\tilde{\chi}}$ is up to isomorphism the unique unitary irreducible representation of $H(B)$ acting as χ on its center. \square

The Stone-von Neumann theorem enables us to define a projective representation of $\text{Ps}(B)$. In fact, for every $s \in \text{Ps}(B)$, we define the map $\rho_{\tilde{\chi}}^s : H(B) \rightarrow \text{End}_{\mathbb{C}}(L^2(Y))$ by $\rho_{\tilde{\chi}}^s(g) = \rho_{\tilde{\chi}}(s(g))$. Then $\rho_{\tilde{\chi}}^s$ is a unitary irreducible representation of $H(B)$ which acts as χ on the center. Therefore, the Stone-von Neumann theorem implies that $\rho_{\tilde{\chi}}$ and $\rho_{\tilde{\chi}}^s$ are equivalent, i.e. there exists $\omega_{\tilde{\chi}}(s) \in U(L^2(Y))$ such that

$$\rho_{\tilde{\chi}}^s(h) = \omega_{\tilde{\chi}}(s)\rho_{\tilde{\chi}}(h)\omega_{\tilde{\chi}}(s)^{-1} \quad \text{for all } h \in H(B). \tag{22}$$

Schur's lemma also implies that $\omega_{\tilde{\chi}}(s)$ is unique up to multiplication by an element of $U(1)$.

Remark 2.10. Notice that proving the existence of the intertwining operators in (22) does not require the Stone-von Neumann theorem. Indeed, since the representations we consider are all finite-dimensional, it is enough to use Corollary 2.7: for every $s \in \text{Ps}(B)$, $\rho_{\tilde{\chi}}$ and $\rho_{\tilde{\chi}}^s$ have the same character. Since these representations are irreducible, they are equivalent. Thus the existence of the intertwining operator $\omega_{\tilde{\chi}}(s) \in U(L^2(Y))$ satisfying (22) follows.

Lemma 2.11. The map $s \mapsto \omega_{\tilde{\chi}}(s)$ defines a projective representation of $\text{Ps}(B)$ on $L^2(Y)$.

Proof. By definition of $\omega_{\tilde{\chi}}(s)$ in (22), if $s = (1, 0)$ then for every $h \in H(B)$,

$$\rho_{\tilde{\chi}}(h) = \omega_{\tilde{\chi}}(1, 0)\rho_{\tilde{\chi}}(h)\omega_{\tilde{\chi}}(1, 0)^{-1}.$$

Hence $\omega_{\tilde{\chi}}(1, 0)$ intertwines $\rho_{\tilde{\chi}}$ with itself. By Schur's lemma, there exists $z_0 \in U(1)$ such that $\omega_{\tilde{\chi}}(1, 0) = z_0 \text{Id}_{L^2(Y)}$. In particular, $\omega_{\tilde{\chi}}(1, 0)$ belongs to the center of $\text{GL}(L^2(Y))$ and we can choose $z_0 = 1$.

Now let $s_1, s_2 \in \text{Ps}(B)$. Then for every $h \in H(B)$, by (22),

$$\begin{aligned} \rho_{\tilde{\chi}}^{s_1 s_2}(h)\omega_{\tilde{\chi}}(s_1)\omega_{\tilde{\chi}}(s_2) &= \rho_{\tilde{\chi}}^{s_1}(s_2(h))\omega_{\tilde{\chi}}(s_1)\omega_{\tilde{\chi}}(s_2) \\ &= \omega_{\tilde{\chi}}(s_1)\rho_{\tilde{\chi}}(s_2(h))\omega_{\tilde{\chi}}(s_2) \\ &= \omega_{\tilde{\chi}}(s_1)\rho_{\tilde{\chi}}^{s_2}(h)\omega_{\tilde{\chi}}(s_2) \\ &= \omega_{\tilde{\chi}}(s_1)\omega_{\tilde{\chi}}(s_2)\rho_{\tilde{\chi}}(h). \end{aligned}$$

Hence $\omega_{\tilde{\chi}}(s_1 s_2)$ and $\omega_{\tilde{\chi}}(s_1)\omega_{\tilde{\chi}}(s_2)$ both intertwine $\rho_{\tilde{\chi}}$ and $\rho_{\tilde{\chi}}^{s_1 s_2}$. Thus there exists $c_{\tilde{\chi}}(s_1, s_2) \in U(1)$ such that

$$\omega_{\tilde{\chi}}(s_1)\omega_{\tilde{\chi}}(s_2) = c_{\tilde{\chi}}(s_1, s_2)\omega_{\tilde{\chi}}(s_1 s_2). \tag{23}$$

This proves that $s \mapsto \omega_{\tilde{\chi}}(s)$ is a projective representation of $\text{Ps}(B)$. \square

Remark 2.12. The fact that $s \mapsto \omega_{\tilde{\chi}}(s)$ is a projective representation implies that the map $c_{\tilde{\chi}} : \text{Ps}(B) \times \text{Ps}(B) \rightarrow U(1)$ defined in (23) is a cocycle.

3. LIFTING THE PROJECTIVE WEIL REPRESENTATION TO A TWO-FOLD COVERING $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ OF $\text{Ps}(B)$

The next step is to see if the projective representation $\omega_{\tilde{\chi}}$ defined in Lemma 2.11 can be lifted to a linear representation, either of $\text{Ps}(B)$ or of a finite central extension of this group. To answer this question, we first construct an explicit section of $\omega_{\tilde{\chi}}$. This allows us to give an explicit formula for the cocycle $c_{\tilde{\chi}}$. Then we prove that the best result we can obtain is to lift $\omega_{\tilde{\chi}}$ to a non-split extension $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ of $\text{Ps}(B)$ by $\mu_2(\mathbb{C}) = \{\pm 1\}$, i.e. a two-fold covering of $\text{Ps}(B)$.

Lemma 3.1. Let $s = (\sigma, f)$ be in $\text{Ps}(B)$. Set $K(\sigma) = \text{Ker}(1 - \sigma)$ and $I(\sigma) = \text{Im}(1 - \sigma)$.

1. $K(\sigma)$ and $I(\sigma)$ are orthogonal subspaces of W with respect to the non-degenerate symplectic form $\langle \cdot, \cdot \rangle$.
2. The restriction $f|_{K(\sigma)}$ of f to $K(\sigma)$ is additive and there exists $w_s \in W$ such that

$$\chi \circ f|_{K(\sigma)} = \chi \circ \langle w_s, \cdot \rangle|_{K(\sigma)}. \quad (24)$$

In particular, if $\mathbb{F} = \mathbb{F}_2$ then $f|_{K(\sigma)} = \langle w_s, \cdot \rangle|_{K(\sigma)}$.

3. $w'_s \in W$ satisfies (24) if and only if $w'_s - w_s$ belongs to $I(\sigma)$.
4. If $s \in \text{Ps}_a(B)$ then the element w_s satisfying (24) is unique. More precisely,
 - $w_{(1,0)} = 0$.
 - If $\mathbb{F} = \mathbb{F}_2$ then w_s is the unique element v such that $s = (1, \langle v, \cdot \rangle)$.
 - If \mathbb{F} is arbitrary and $s = (1, f_z)$, then w_s is the unique element v such that $\chi \circ f_z = \chi \circ (\langle v, \cdot \rangle)$.
5. For every $s_1 = (\sigma_1, f_1)$ and $s_2 = (\sigma_2, f_2)$ in $\text{Ps}(B)$, and for all choices of w_{s_1}, w_{s_2} and $w_{s_1 s_2}$ satisfying (24) for s_1, s_2 and $s_1 s_2$, respectively:
 - $K(\sigma_1) \cap K(\sigma_2) \subseteq K(\sigma_1 \sigma_2)$ and $I(\sigma_1 \sigma_2) \subseteq I(\sigma_1) + I(\sigma_2)$.
 - There exists $w_{s_1, s_2} \in I(\sigma_1) + I(\sigma_2)$ such that $w_{s_1 s_2} - w_{s_1} - w_{s_2} = w_{s_1, s_2}$.

Proof. 1. For every $u = (1 - \sigma)(w) \in I(\sigma)$ and $w' \in K(\sigma)$,

$$\begin{aligned} \langle u, w' \rangle &= \langle (1 - \sigma)(w), w' \rangle = \langle w, w' \rangle - \langle \sigma(w), w' \rangle \\ &= \langle w, w' \rangle - \langle w, \sigma^{-1}(w') \rangle \\ &= \langle w, w' \rangle - \langle w, w' \rangle \\ &= 0. \end{aligned}$$

This shows that $I(\sigma) \subseteq K(\sigma)^\perp$. Since $\langle \cdot, \cdot \rangle$ is non-degenerate,

$$\dim(K(\sigma)^\perp) = \dim W - \dim(K(\sigma)) = \dim(I(\sigma)).$$

Thus $K(\sigma)^\perp = I(\sigma)$.

2. Recall from (8) that we defined $\beta_\sigma : (w_1, w_2) \mapsto B(\sigma(w_1), \sigma(w_2)) - B(w_1, w_2)$. If w_1 and w_2 belong to $K(\sigma)$, then $f(w_1 + w_2) - f(w_1) - f(w_2) = \beta_\sigma(w_1, w_2) = 0$. Hence f is additive on $K(\sigma)$. We can extend $f|_{K(\sigma)}$ to W by taking any vector subspace V_σ such that $W = K(\sigma) \oplus V_\sigma$ and setting $F(w) = F(z + v) = f(z)$ for all $w = z + v \in K(\sigma) \oplus V_\sigma$. We obtain in this way an additive quadratic form on W . If $\mathbb{F} = \mathbb{F}_2$, we can apply Proposition 1.11.1 and deduce that there exists $w_s \in W$ such that $F = \langle w_s, \cdot \rangle$, which implies that $f|_{K(\sigma)} = \langle w_s, \cdot \rangle|_{K(\sigma)}$. Now if \mathbb{F} is any finite field of characteristic two, then the fact that $f|_{K(\sigma)}$ is additive implies that $\chi \circ F$ is a character of W . By Proposition 1.11.2, there exists $w_s \in W$ such that $\chi \circ F = \chi \circ \langle w_s, \cdot \rangle$. In particular, $\chi(f(w)) = \chi(\langle w_s, w \rangle)$ for all $w \in K(\sigma)$.
3. Suppose that $w'_s \in W$ satisfies the same property (24) as w_s . Then for all $w \in K(\sigma)$,

$$\chi(\langle w'_s - w_s, w \rangle) = \chi(f(w) - f(w)) = 1.$$

Hence the image of the linear form $\langle w'_s - w_s, \cdot \rangle$ defined on $K(\sigma)$ is contained in the kernel of χ . If this linear form were not identically zero then we would have $\mathbb{F} \subseteq \text{Ker}(\chi)$, which is impossible since χ is non-trivial by hypothesis. Hence $\langle w'_s - w_s, w \rangle = 0$ for all $w \in K(\sigma)$, which is equivalent to the fact that $w'_s - w_s \in K(\sigma)^\perp = I(\sigma)$.

Conversely, if $w'_s - w_s \in I(\sigma) = K(\sigma)^\perp$, then $\langle w'_s - w_s, w \rangle = 0$ for all $w \in K(\sigma)$. Hence $\chi(\langle w'_s, w \rangle) = \chi(\langle w_s, w \rangle) = \chi(f(w))$ for all $w \in K(\sigma)$, i.e. w'_s satisfies (24).

4. Suppose that $s = (1, f) \in \text{Ps}_a(B)$. Then $K(\sigma) = W$, $I(\sigma) = \{0\}$ and $f \in \mathcal{Q}_a(W)$ by (6). By 3, w'_s satisfies (24) if and only if $w'_s - w_s \in I(\sigma)$, i.e. $w'_s = w_s$. This shows that the element w_s satisfying (24) is unique. If $s = (1, 0)$ then (24) is written

$$1 = \chi(\langle w_s, w \rangle) \quad \text{for all } w \in W.$$

Since $\langle w_s, \cdot \rangle$ is a linear form, its image is either $\{0\}$ or \mathbb{F} . Since χ is non-trivial, this implies that $\langle w_s, \cdot \rangle = 0$, i.e. $w_s = 0$ because $\langle \cdot, \cdot \rangle$ is non-degenerate. Suppose now that $\mathbb{F} = \mathbb{F}_2$. By Propositions 1.11.1 and 1.11.5, there exists a unique element $z \in W$ such that $f = \langle z, \cdot \rangle$. Hence (24) is written

$$\chi(\langle z, w \rangle) = \chi(\langle w_s, w \rangle) \quad \text{for all } w \in W,$$

and we can conclude in the same way that $w_s - z = 0$, i.e. $w_s = z$. Finally, suppose that \mathbb{F} is arbitrary. Then there exists a unique $z \in W$ such that $f = f_z$ by Propositions 1.11.1 and 1.11.5. Since the maps $\chi \circ f_w$, $w \in W$, describe completely the set of characters of W by Proposition 1.11.2, there exists a unique $v \in W$ such that $\chi \circ f_z = \chi \circ \langle v, \cdot \rangle$. Then (24) is written:

$$\chi(\langle w_s, w \rangle) = \chi(f_z(w)) = \chi(\langle v, w \rangle) \quad \text{for all } w \in W.$$

Hence $w_s = v$, which concludes the proof of 4.

5. Let $s_1 = (\sigma_1, f_1)$ and $s_2 = (\sigma_2, f_2)$ be in $\text{Ps}(B)$.
- Let $w \in W$ such that $\sigma_1(w) = \sigma_2(w) = w$, then $(\sigma_1\sigma_2)(w) = \sigma_1(w) = w$. Hence $K(\sigma_1) \cap K(\sigma_2) \subseteq K(\sigma_1\sigma_2)$. Taking the orthogonal complement with

respect to $\langle \cdot, \cdot \rangle$, we obtain $K(\sigma_1\sigma_2)^\perp \subseteq (K(\sigma_1) \cap K(\sigma_2))^\perp$, i.e.

$$I(\sigma_1\sigma_2) \subseteq K(\sigma_1)^\perp + K(\sigma_2)^\perp = I(\sigma_1) + I(\sigma_2).$$

- By definition of w_{s_1} , w_{s_2} and $w_{s_1s_2}$, for every $w \in K(\sigma_1) \cap K(\sigma_2) \subseteq K(\sigma_1\sigma_2)$,

$$\begin{aligned} \chi((f_2 + f_1 \circ \sigma_2)(w)) &= \chi(\langle w_{s_1s_2}, w \rangle) \\ \chi(f_2(w))\chi(f_1(w)) &= \chi(\langle w_{s_1s_2}, w \rangle) \\ \chi(\langle w_{s_1}, w \rangle)\chi(\langle w_{s_2}, w \rangle) &= \chi(\langle w_{s_1s_2}, w \rangle) \\ \chi(\langle w_{s_1s_2} - w_{s_1} - w_{s_2}, w \rangle) &= 1. \end{aligned}$$

This implies that $\langle w_{s_1s_2} - w_{s_1} - w_{s_2}, w \rangle = 0$ for all $w \in K(\sigma_1) \cap K(\sigma_2)$, since otherwise χ would be trivial. Thus $w_{s_1s_2} - w_{s_1} - w_{s_2} \in (K(\sigma_1) \cap K(\sigma_2))^\perp = I(\sigma_1) + I(\sigma_2)$, and we can set $w_{s_1s_2} = w_{s_1s_2} - w_{s_1} - w_{s_2} \in I(\sigma_1) + I(\sigma_2)$. \square

Definition-Proposition 3.2 (Projective Weil representation of $\text{Ps}(B)$). For every $s = (\sigma, f) \in \text{Ps}(B)$ and for every $w_s \in W$ satisfying (24), we define:

$$\omega_{\tilde{\chi}}(s, w_s) = |K(\sigma)|^{-1} \gamma(\sigma) \sum_{w \in W} \rho_{\tilde{\chi}}(s(w))^{-1} \rho_{\tilde{\chi}}(w_s) \rho_{\tilde{\chi}}(w) \quad (25)$$

where $\gamma(\sigma)$ is a complex number such that $|\gamma(\sigma)|^2 = 2^{-\dim I(\sigma)}$ and $\gamma(1) = 1$. Then $\omega_{\tilde{\chi}}(s, w_s)$ satisfies the following properties:

1. For every $h \in H(B)$,

$$\rho_{\tilde{\chi}}(s(h)) = \omega_{\tilde{\chi}}(s, w_s) \rho_{\tilde{\chi}}(h) \omega_{\tilde{\chi}}(s, w_s)^{-1}. \quad (26)$$

2. For every $u \in I(\sigma)$, choose any $w \in W$ such that $u = (1 - \sigma)(w)$. Then

$$\omega_{\tilde{\chi}}(s, w_s) = \gamma(\sigma) \sum_{u \in I(\sigma)} \chi(f(w) + \langle w_s, w \rangle + B(u, w + w_s)) \rho_{\tilde{\chi}}(u + w_s). \quad (27)$$

3. $\omega_{\tilde{\chi}}(s, w_s) \in U(L^2(Y))$.

4. If we make another choice of element w'_s satisfying (24) then $\omega_{\tilde{\chi}}(s, w_s)$ and $\omega_{\tilde{\chi}}(s, w'_s)$ can only differ by a sign. More precisely:

$$\omega_{\tilde{\chi}}(s, w'_s) = \chi(-f(v) + \langle v, w_s \rangle + B(w'_s - w_s, v - w_s)) \omega_{\tilde{\chi}}(s, w_s) \quad (28)$$

where v is any element such that $w'_s - w_s = (1 - \sigma)(v)$.

Thus for each choice $\{w_s \text{ satisfying (24), } s \in \text{Ps}(B)\}$, the map $s \mapsto \omega_{\tilde{\chi}}(s, w_s)$ is an explicit section of the projective Weil representation on $L^2(Y)$.

Proof. **1.** Let $s = (\sigma, f) \in \text{Ps}(B)$, $w_s \in W$ satisfying (24) and $h = (v, t) \in H(B)$. Then:

$$\begin{aligned} \rho_{\tilde{\chi}}(s(h)) \omega_{\tilde{\chi}}(s, w_s) &= |K(\sigma)|^{-1} \gamma(\sigma) \sum_{w \in W} \rho_{\tilde{\chi}}(s(h)) \rho_{\tilde{\chi}}(s(w))^{-1} \rho_{\tilde{\chi}}(w_s) \rho_{\tilde{\chi}}(w) \\ &= |K(\sigma)|^{-1} \gamma(\sigma) \sum_{w \in W} \rho_{\tilde{\chi}}(s(h)) s(w)^{-1} w_s w. \end{aligned}$$

Setting $w' = w - v$, we see that

$$\begin{aligned}
 & s(h)s(w)^{-1}w_s w \\
 &= (\sigma(v), f(v) + t)(-\sigma(w), -f(w) + Q(\sigma(w)))w_s w \\
 &= (\sigma(v - w), f(v) - f(w) + t + Q(\sigma(w)) - B(\sigma(v), \sigma(w)))w_s w \\
 &= (\sigma(v - w), -f(w - v) - B(\sigma(w - v), \sigma(v)) + B(w - v, v) + t + Q(\sigma(w)) - B(\sigma(v), \sigma(w)))w_s w \\
 &= (-\sigma(w'), -f(w') - B(\sigma(w'), \sigma(v)) + B(w', v) + t + Q(\sigma(w')) + Q(\sigma(v)) + \langle \sigma(w'), \sigma(v) \rangle \\
 &\quad - B(\sigma(v), \sigma(w' + v)))w_s w \\
 &= (-\sigma(w'), -f(w') + B(w', v) + t + Q(\sigma(w')) + \langle \sigma(w'), \sigma(v) \rangle + \langle \sigma(v), \sigma(w') \rangle - 2B(\sigma(v), \sigma(w'))) \\
 &\quad \cdot w_s w \\
 &= (-\sigma(w'), -f(w') + B(w', v) + t + Q(\sigma(w')))w_s(w' + v) \\
 &= (-\sigma(w'), -f(w') + Q(\sigma(w')))(0, B(w', v) + t)w_s w'(v, -B(w', v)) \\
 &= s(w')^{-1}w_s w'(v, t) \\
 &= s(w')^{-1}w_s w' h.
 \end{aligned}$$

Hence:

$$\begin{aligned}
 \rho_{\tilde{\chi}}(s(h))\omega_{\tilde{\chi}}(s, w_s) &= |\mathbf{K}(\sigma)|^{-1}\gamma(\sigma) \sum_{w \in W} \rho_{\tilde{\chi}}(s(h)s(w)^{-1}w_s w) \\
 &= |\mathbf{K}(\sigma)|^{-1}\gamma(\sigma) \sum_{w' \in W} \rho_{\tilde{\chi}}(s(w')^{-1}w_s w' h) \\
 &= \omega_{\tilde{\chi}}(s, w_s)\rho_{\tilde{\chi}}(h).
 \end{aligned}$$

2. Let $s = (\sigma, f) \in \text{Ps}(B)$ and $w_s \in W$ satisfying (24). Then:

$$\omega_{\tilde{\chi}}(s, w_s) = |\mathbf{K}(\sigma)|^{-1}\gamma(\sigma) \sum_{w \in W} \rho_{\tilde{\chi}}(s(w)^{-1}w_s w),$$

where

$$\begin{aligned}
 s(w)^{-1}w_s w &= (-\sigma(w), -f(w) + Q(\sigma(w)))(w_s + w, B(w_s, w)) \\
 &= ((1 - \sigma)(w) + w_s, -f(w) + Q(\sigma(w)) + B(w_s, w) - B(\sigma(w), w_s + w)) \\
 &= ((1 - \sigma)(w) + w_s, -f(w) + Q(\sigma(w)) - Q(w) + \langle w_s, w \rangle + B((1 - \sigma)(w), w_s + w)) \\
 &= ((1 - \sigma)(w) + w_s, f(w) + \langle w_s, w \rangle + B((1 - \sigma)(w), w_s + w))
 \end{aligned}$$

since $0 = f(2w) = 2f(w) + Q(\sigma(w)) - Q(w) = Q(\sigma(w)) - Q(w)$ by (6). Hence

$$\rho_{\tilde{\chi}}(s(w)^{-1}w_s w) = \chi(f(w) + \langle w_s, w \rangle + B((1 - \sigma)(w), w_s + w))\rho_{\tilde{\chi}}((1 - \sigma)(w) + w_s). \quad (29)$$

Now for every $v \in K(\sigma)$, by (29):

$$\begin{aligned}
& \rho_{\tilde{\chi}}(s(w+v)^{-1}w_s(w+v)) \\
&= \chi(f(w+v) + \langle w_s, w+v \rangle + B((1-\sigma)(w+v), w_s+w+v))\rho_{\tilde{\chi}}((1-\sigma)(w+v) + w_s) \\
&= \chi(f(w) + f(v) + B(\sigma(w), \sigma(v)) - B(w, v) + \langle w_s, w \rangle + \langle w_s, v \rangle + B((1-\sigma)(w), w_s+w) \\
&\quad + B((1-\sigma)(w), v))\rho_{\tilde{\chi}}((1-\sigma)(w) + w_s) \\
&= \chi(f(w) + f(v) + \langle w_s, w \rangle + \langle w_s, v \rangle + B((1-\sigma)(w), w_s+w))\rho_{\tilde{\chi}}((1-\sigma)(w) + w_s) \text{ since } 2B = 0 \\
&= \chi(f(v) + \langle w_s, v \rangle)\chi(f(w) + \langle w_s, w \rangle + B((1-\sigma)(w), w_s+w))\rho_{\tilde{\chi}}((1-\sigma)(w) + w_s) \\
&= \chi(f(v) + \langle w_s, v \rangle)\rho_{\tilde{\chi}}(s(w)^{-1}w_s w),
\end{aligned}$$

i.e. by (24):

$$\rho_{\tilde{\chi}}(s(w+v)^{-1}w_s(w+v)) = \rho_{\tilde{\chi}}(s(w)^{-1}w_s w).$$

This shows that $\rho_{\tilde{\chi}}(s(w)^{-1}w_s w)$ does not depend on the left coset of w modulo $K(\sigma)$. This together with (29) enables us to write:

$$\begin{aligned}
\omega_{\tilde{\chi}}(s, w_s) &= |K(\sigma)|^{-1}\gamma(\sigma) \sum_{w \in W} \rho_{\tilde{\chi}}(s(w)^{-1}w_s w) \\
&= |K(\sigma)|^{-1}\gamma(\sigma) \sum_{w+K(\sigma) \in W/K(\sigma)} |K(\sigma)|\rho_{\tilde{\chi}}(s(w)^{-1}w_s w) \\
&= \gamma(\sigma) \sum_{w+K(\sigma) \in W/K(\sigma)} \chi(f(w) + \langle w_s, w \rangle + B((1-\sigma)(w), w_s+w))\rho_{\tilde{\chi}}((1-\sigma)(w) + w_s) \\
&= \gamma(\sigma) \sum_{u \in I(\sigma)} \chi(f(w) + \langle w_s, w \rangle + B(u, w_s+w))\rho_{\tilde{\chi}}(u + w_s),
\end{aligned}$$

where $w \in W$ is any element such that $u = (1-\sigma)(w)$, and we use the isomorphism between $W/K(\sigma)$ and $I(\sigma)$.

- 3.** Because of the intertwining property (26) and Schur's lemma, in order to verify that $\omega_{\tilde{\chi}}(s, w_s)$ is invertible, it is enough to show that it is not the zero operator. The expression (27) of $\omega_{\tilde{\chi}}(s, w_s)$ implies that $\omega_{\tilde{\chi}}(s, w_s)$ has coefficients equal to ± 1 in the basis $\{\rho_{\tilde{\chi}}(w), w \in W\}$ of $\text{End}_{\mathbb{C}}(L^2(Y))$ (cf. Lemmas 2.1.2 and 2.8). Thus $\omega_{\tilde{\chi}}(s, w_s)$ cannot be the zero operator. Furthermore, using the unitarity of $\rho_{\tilde{\chi}}$, we can write

$$\begin{aligned}
\omega_{\tilde{\chi}}(s, w_s)\omega_{\tilde{\chi}}(s, w_s)^* &= |K(\sigma)|^{-2}\gamma(\sigma)^2 \sum_{w \in W} \rho_{\tilde{\chi}}(s(w)^{-1}w_s w) \sum_{w' \in W} \rho_{\tilde{\chi}}(s(w')^{-1}w_s w')^* \\
&= |K(\sigma)|^{-2}|\gamma(\sigma)|^2 \sum_{w, w' \in W} \rho_{\tilde{\chi}}(s(w)^{-1}w_s w)\rho_{\tilde{\chi}}(w'^{-1}w_s^{-1}s(w')).
\end{aligned}$$

To compute the expression on the right-hand side, observe first that by (29):

$$\begin{aligned}
 & s(w)^{-1}w_s w w'^{-1}w_s^{-1}s(w') \\
 &= ((1-\sigma)(w) + w_s, f(w) + \langle w_s, w \rangle + B((1-\sigma)(w), w_s + w)) \\
 &\quad \cdot (-(1-\sigma)(w') - w_s, -f(w') - \langle w_s, w' \rangle - B((1-\sigma)(w'), w_s + w') + Q((1-\sigma)(w') + w_s)) \\
 &= ((1-\sigma)(w - w'), f(w) - f(w') + \langle w_s, w - w' \rangle + B((1-\sigma)(w), w_s + w) \\
 &\quad - B((1-\sigma)(w'), w_s + w') + Q((1-\sigma)(w') + w_s) - B((1-\sigma)(w) + w_s, (1-\sigma)(w') + w_s)) \\
 &= ((1-\sigma)(w - w'), T(w, w', w_s)), \\
 &\quad \text{where, if we set } z = w - w',
 \end{aligned}$$

$$\begin{aligned}
 & T(w, w', w_s) \\
 &= f(w - w') + B(\sigma(w - w'), \sigma(w')) - B(w - w', w') + \langle w_s, w - w' \rangle + B((1-\sigma)(w), w_s + w) \\
 &\quad - B((1-\sigma)(w'), w_s + w') + Q((1-\sigma)(w') + w_s) - B((1-\sigma)(w) + w_s, (1-\sigma)(w') + w_s) \\
 &= f(z) + B(\sigma(z), \sigma(w')) - B(z, w') + \langle w_s, z \rangle + B((1-\sigma)(z + w'), w_s + z + w') \\
 &\quad - B((1-\sigma)(w'), w_s + w') + Q((1-\sigma)(w') + w_s) - B((1-\sigma)(z + w') + w_s, (1-\sigma)(w') + w_s) \\
 &= f(z) + Q(z) + \langle w_s, z \rangle - B(\sigma(z), z) + \langle z, (1-\sigma)(w') \rangle.
 \end{aligned}$$

Thus, by (29),

$$\begin{aligned}
 \omega_{\tilde{\chi}}(s, w_s)\omega_{\tilde{\chi}}(s, w_s)^* &= |\mathbf{K}(\sigma)|^{-2}|\gamma(\sigma)|^2 \sum_{w, w' \in W} \rho_{\tilde{\chi}}(s(w)^{-1}w_s w)\rho_{\tilde{\chi}}(w'^{-1}w_s^{-1}s(w')) \\
 &= |\mathbf{K}(\sigma)|^{-2}|\gamma(\sigma)|^2 \sum_{z \in W} \chi(f(z) + Q(z) + \langle w_s, z \rangle - B(\sigma(z), z)) \\
 &\quad \sum_{w' \in W} \chi(\langle z, (1-\sigma)(w') \rangle)\rho_{\tilde{\chi}}((1-\sigma)(z)) \\
 &= |\mathbf{K}(\sigma)|^{-2}|\gamma(\sigma)|^2 \sum_{z \in W} \chi(f(z) + Q(z) + \langle w_s, z \rangle - B(\sigma(z), z)) \\
 &\quad \sum_{w' \in W} \chi(\langle (1-\sigma^{-1})(z), w' \rangle)\rho_{\tilde{\chi}}((1-\sigma)(z)) \\
 &= |\mathbf{K}(\sigma)|^{-2}|\gamma(\sigma)|^2 \sum_{z \in \mathbf{K}(\sigma)} |W|\rho_{\tilde{\chi}}(0) \\
 &= |\mathbf{K}(\sigma)|^{-1}|\gamma(\sigma)|^2 |W| \text{Id}_{L^2(Y)} \\
 &= 2^{-\dim \mathbf{K}(\sigma) - \dim \mathbf{I}(\sigma)} 2^{\dim \mathbf{K}(\sigma) + \dim \mathbf{I}(\sigma)} \text{Id}_{L^2(Y)} \\
 &= \text{Id}_{L^2(Y)},
 \end{aligned}$$

where the three last equalities are justified by the fact that $w' \mapsto \chi(\langle (1-\sigma^{-1})(z), w' \rangle)$ is a character of W which is trivial if and only if $z \in \mathbf{K}(\sigma)$, Schur's orthogonality relations, and the isomorphism between $W/\mathbf{K}(\sigma)$ and $\mathbf{I}(\sigma)$. Therefore $\omega_{\tilde{\chi}}(s, w_s)$ is a unitary operator of $L^2(Y)$.

4. Let $w'_s \in W$ be such that (24) is satisfied, i.e. $\chi \circ f|_{\mathbf{K}(\sigma)} = \chi \circ \langle w'_s, \cdot \rangle|_{\mathbf{K}(\sigma)}$. Then Lemma 3.1.3 implies that $w'_s - w_s \in \mathbf{I}(\sigma)$. Hence we can set $w'_s - w_s = (1 - \sigma)(v)$ for some $v \in W$. Then

$$\omega_{\tilde{\chi}}(s, w'_s) = |\mathbf{K}(\sigma)|^{-1} \gamma(\sigma) \sum_{w \in W} \rho_{\tilde{\chi}}(s(w)^{-1} w'_s w),$$

where, by (29) and if we set $z = w + v$,

$$\begin{aligned} s(w)^{-1} w'_s w &= ((1 - \sigma)(w) + w'_s, f(w) + \langle w'_s, w \rangle + B((1 - \sigma)(w), w'_s + w)) \\ &= ((1 - \sigma)(w) + w_s + (1 - \sigma)(v), f(w) + \langle w_s + (1 - \sigma)(v), w \rangle \\ &\quad + B((1 - \sigma)(w), w_s + (1 - \sigma)(v) + w)) \\ &= ((1 - \sigma)(z) + w_s, f(z - v) + \langle w_s + (1 - \sigma)(v), z - v \rangle \\ &\quad + B((1 - \sigma)(z - v), w_s + (1 - \sigma)(v) + z - v)) \\ &= ((1 - \sigma)(z) + w_s, f(z) + \langle w_s, z \rangle + B((1 - \sigma)(z), w_s + z) - f(v) - \langle w_s, v \rangle \\ &\quad - B((1 - \sigma)(v), w_s - v)) \\ &= ((1 - \sigma)(z) + w_s, f(z) + \langle w_s, z \rangle + B((1 - \sigma)(z), w_s + z)) \\ &\quad \cdot (0, -f(v) - \langle w_s, v \rangle + B(w'_s - w_s, v - w_s)) \\ &\stackrel{(29)}{=} s(z)^{-1} w_s z (0, -f(v) - \langle w_s, v \rangle + B(w'_s - w_s, v - w_s)) \end{aligned}$$

Thus if we apply the change of variables $z = w + v$ in the sum, this leads to

$$\begin{aligned} \omega_{\tilde{\chi}}(s, w'_s) &= |\mathbf{K}(\sigma)|^{-1} \gamma(\sigma) \sum_{z \in W} \chi(-f(v) + \langle v, w_s \rangle + B(w'_s - w_s, v - w_s)) \rho_{\tilde{\chi}}(s(z)^{-1} w_s z) \\ &= \chi(-f(v) + \langle v, w_s \rangle + B(w'_s - w_s, v - w_s)) \omega_{\tilde{\chi}}(s, w_s). \end{aligned}$$

□

Notation 3.3. In Proposition 3.2, we defined an operator $\omega_{\tilde{\chi}}(s, w_s)$ for all $s = (\sigma, f) \in \text{Ps}(B)$ and for all w_s satisfying (24). From now on, to simplify the notation, if it is clear that a choice of w_s has been made, then we will write indifferently $\omega_{\tilde{\chi}}(s, w_s)$ or $\omega_{\tilde{\chi}}(s)$.

Corollary 3.4 (Computation of the metaplectic cocycle). We keep the same notations as in Lemma 3.1 and Proposition 3.2, and we assume that a choice of $w_s \in W$ satisfying (24) has been made for all $s \in \text{Ps}(B)$. The cocycle $c_{\tilde{\chi}}$ associated to the projective Weil representation $\omega_{\tilde{\chi}}$ will be called the metaplectic cocycle. It is given by the following formula. For every $s_1 = (\sigma_1, f_1)$ and $s_2 = (\sigma_2, f_2)$ in $\text{Ps}(B)$,

$$c_{\tilde{\chi}}(s_1, s_2) = C_{1,2} \sum_{\substack{u_1 \in \mathbf{I}(\sigma_1), u_2 \in \mathbf{I}(\sigma_2) \\ u_1 + u_2 = w_{s_1, s_2}}} \chi(f_1(w_1) + \langle w_{s_1}, w_1 \rangle + B(u_1, w_1 + w_{s_1}) + f_2(w_2) + \langle w_{s_2}, w_2 \rangle) \chi(B(u_2, w_2 + w_{s_2}) + B(u_1 + w_{s_1}, u_2 + w_{s_2})) \quad (30)$$

where $C_{1,2} = \gamma(\sigma_1) \gamma(\sigma_2) \gamma(\sigma_1 \sigma_2)^{-1}$, $w_{s_1, s_2} = w_{s_1 s_2} - w_{s_1} - w_{s_2}$ and w_i is any element such that $(1 - \sigma)(w_i) = u_i$ for $i \in \{1, 2\}$. In particular, the cocycle $c_{\tilde{\chi}}$ does not depend on the choice of the extension $\tilde{\chi}$ of χ : we can denote it by c_{χ} instead of $c_{\tilde{\chi}}$.

Remark 3.5. The fact that c_χ is a cocycle implies that for every $s = (\sigma, f) \in \text{Ps}(B)$,

$$c_\chi(s, (1, 0)) = c_\chi((1, 0), s) = 1.$$

This can also be recovered directly from (30) since $\gamma(1) = 1$.

Proof of Corollary 3.4. Let $s_1 = (\sigma_1, f_1)$ and $s_2 = (\sigma_2, f_2)$ be in $\text{Ps}(B)$. The fact that $\omega_{\tilde{\chi}}$ is a projective representation implies that $c_{\tilde{\chi}}$ satisfies

$$\omega_{\tilde{\chi}}(s_1)\omega_{\tilde{\chi}}(s_2) = c_{\tilde{\chi}}(s_1, s_2)\omega_{\tilde{\chi}}(s_1s_2), \quad (31)$$

where we recall from Remark 3.3 that $\omega_{\tilde{\chi}}(s_1) = \omega_{\tilde{\chi}}(s_1, w_{s_1})$ and $\omega_{\tilde{\chi}}(s_2) = \omega_{\tilde{\chi}}(s_2, w_{s_2})$. Now we compute separately $\omega_{\tilde{\chi}}(s_1)\omega_{\tilde{\chi}}(s_2)$ and $\omega_{\tilde{\chi}}(s_1s_2)$, using the expression (27) found in Proposition 3.2 and (17):

$$\begin{aligned} & \omega_{\tilde{\chi}}(s_1)\omega_{\tilde{\chi}}(s_2) \\ &= \gamma(\sigma_1)\gamma(\sigma_2) \sum_{\substack{u_1 \in \text{I}(\sigma_1) \\ u_2 \in \text{I}(\sigma_2)}} \chi(f_1(w_1) + \langle w_{s_1}, w_1 \rangle + B(u_1, w_1 + w_{s_1}) + f_2(w_2) + \langle w_{s_2}, w_2 \rangle) \\ & \quad \chi(B(u_2, w_2 + w_{s_2}))\rho_{\tilde{\chi}}(u_1 + w_{s_1})\rho_{\tilde{\chi}}(u_2 + w_{s_2}) \\ &= \gamma(\sigma_1)\gamma(\sigma_2) \sum_{\substack{u_1 \in \text{I}(\sigma_1) \\ u_2 \in \text{I}(\sigma_2)}} \chi(f_1(w_1) + \langle w_{s_1}, w_1 \rangle + B(u_1, w_1 + w_{s_1}) + f_2(w_2) + \langle w_{s_2}, w_2 \rangle) \\ & \quad \chi(B(u_2, w_2 + w_{s_2}) + B(u_1 + w_{s_1}, u_2 + w_{s_2}))\rho_{\tilde{\chi}}(u_1 + w_{s_1} + u_2 + w_{s_2}) \end{aligned}$$

and

$$\omega_{\tilde{\chi}}(s_1s_2) = \gamma(\sigma_1\sigma_2) \sum_{u \in \text{I}(\sigma_1\sigma_2)} \chi((f_2 + f_1 \circ \sigma_2)(w) + \langle w_{s_1s_2}, w \rangle + B(u, w + w_{s_1s_2}))\rho_{\tilde{\chi}}(u + w_{s_1s_2}).$$

Now, using the fact that the operators $\rho_{\tilde{\chi}}(w)$, $w \in W$, form a basis of $\text{End}_{\mathbb{C}}(\mathbb{L}^2(Y))$ (cf. Lemma 2.8) and identifying the coefficients of $\omega_{\tilde{\chi}}(s_1)\omega_{\tilde{\chi}}(s_2)$ and $\omega_{\tilde{\chi}}(s_1s_2)$ in this basis, we obtain for all $u \in \text{I}(\sigma_1\sigma_2)$:

$$\begin{aligned} & c_{\tilde{\chi}}(s_1, s_2)\gamma(\sigma_1\sigma_2)\chi((f_2 + f_1 \circ \sigma_2)(w) + \langle w_{s_1s_2}, w \rangle + B(u, w + w_{s_1s_2})) \\ &= \gamma(\sigma_1)\gamma(\sigma_2) \\ & \quad \cdot \sum_{\substack{u_1 \in \text{I}(\sigma_1), u_2 \in \text{I}(\sigma_2) \\ u_1 + u_2 + w_{s_1} + w_{s_2} = u + w_{s_1s_2}}} \chi(f_1(w_1) + \langle w_{s_1}, w_1 \rangle + B(u_1, w_1 + w_{s_1}) + f_2(w_2) + \langle w_{s_2}, w_2 \rangle) \\ & \quad \chi(B(u_2, w_2 + w_{s_2}) + B(u_1 + w_{s_1}, u_2 + w_{s_2})). \end{aligned}$$

Taking $u = 0$ and replacing $w_{s_1s_2} - w_{s_1} - w_{s_2}$ by w_{s_1, s_2} (defined in Lemma 3.1.5) in the previous equation, we finally obtain:

$$\begin{aligned} c_{\tilde{\chi}}(s_1, s_2) &= \gamma(\sigma_1)\gamma(\sigma_2)\gamma(\sigma_1\sigma_2)^{-1} \\ & \quad \cdot \sum_{\substack{u_1 \in \text{I}(\sigma_1), u_2 \in \text{I}(\sigma_2) \\ u_1 + u_2 = w_{s_1, s_2}}} \chi(f_1(w_1) + \langle w_{s_1}, w_1 \rangle + B(u_1, w_1 + w_{s_1}) + f_2(w_2) + \langle w_{s_2}, w_2 \rangle) \\ & \quad \chi(B(u_2, w_2 + w_{s_2}) + B(u_1 + w_{s_1}, u_2 + w_{s_2})). \end{aligned}$$

In particular, this expression does not depend on the choice of the extension $\tilde{\chi}$ of χ , so we can write c_χ instead of $c_{\tilde{\chi}}$. \square

Now that we have determined an explicit formula for the metaplectic cocycle, the next goal is to know its range in order to construct the smallest possible extension of $\text{Ps}(B)$. In fact, the range of c_χ will depend on the choice made for $\gamma(\sigma)$ in the definition (25) of the operator $\omega_{\tilde{\chi}}(s, w_s)$, for every $s = (\sigma, f) \in \text{Ps}(B)$. Recall from Proposition 3.2 that the only conditions on $\gamma(\sigma)$ are that $|\gamma(\sigma)|^2 = 2^{-\dim \mathbb{I}(\sigma)}$, to ensure that $\omega_{\tilde{\chi}}(s, w_s)$ is a unitary operator, and $\gamma(1) = 1$, to have $\omega_{\tilde{\chi}}((1, 0), 0) = \text{Id}_{L^2(Y)}$. This means that we can set

$$\gamma(\sigma) = 2^{-\frac{\dim \mathbb{I}(\sigma)}{2}} e^{i\theta(\sigma)}, \quad \text{where } \theta(\sigma) \in \mathbb{R} \quad \text{and} \quad \theta(1) \equiv 0 \pmod{2\pi}. \quad (32)$$

Corollary 3.6 (Range of the metaplectic cocycle). The cocycle c_χ takes its values in $U(1)$. Furthermore, c_χ takes its values in $\mu_2(\mathbb{C}) = \{\pm 1\}$ if and only if

$$\theta(\sigma_1) + \theta(\sigma_2) - \theta(\sigma_1\sigma_2) \equiv 0 \pmod{\pi} \quad \text{for all } \sigma_1, \sigma_2 \in \text{O}(Q), \quad (33)$$

where the map $\sigma \mapsto \theta(\sigma)$ is defined by (32). In particular, this is the case if $\gamma(\sigma) \in \mathbb{R}$ for all $\sigma \in \text{O}(Q)$, for example if $\gamma(\sigma) = 2^{-\frac{\dim \mathbb{I}(\sigma)}{2}}$.

Proof. Equation (31), together with the unitarity of the operators $\omega_{\tilde{\chi}}(s)$ for all $s \in \text{Ps}(B)$, implies that c_χ takes its values in $U(1)$. Since χ takes its values in $\mu_2(\mathbb{C})$, formula (30) implies that c_χ takes its values in \mathbb{R} if and only if $C_{1,2} = \gamma(\sigma_1)\gamma(\sigma_2)\gamma(\sigma_1\sigma_2)^{-1}$ is real for all $\sigma_1, \sigma_2 \in \text{O}(Q)$. By (32),

$$\begin{aligned} C_{1,2} \in \mathbb{R} &\iff e^{i(\theta(\sigma_1) + \theta(\sigma_2) - \theta(\sigma_1\sigma_2))} \in \mathbb{R} \\ &\iff \theta(\sigma_1) + \theta(\sigma_2) - \theta(\sigma_1\sigma_2) \equiv 0 \pmod{\pi}. \end{aligned}$$

□

Remark 3.7. Recall that we want to extend the projective Weil representation of $\text{Ps}(B)$ defined in Proposition 3.2 to a linear representation of either $\text{Ps}(B)$ or a smallest possible extension of $\text{Ps}(B)$. The range of c_χ will therefore play a crucial role, since it determines the size of the extension we construct. If the cocycle were a coboundary, then it would be symmetric on $\text{Ps}_a(B)$ (defined in Proposition 1.11.5), which we are going to prove to be false in Proposition 3.8.2. Since the cocycle c_χ is not a coboundary, the projective Weil representation is not a linear representation of $\text{Ps}(B)$. Thus the best we can obtain is to lift the projective representation $s \mapsto \omega_{\tilde{\chi}}(s)$ to an extension of $\text{Ps}(B)$ by two elements. By Corollary 3.6, this is possible if and only if (33) is satisfied.

Now that we know that the best we can do is to lift the projective Weil representation to a two-fold covering of $\text{Ps}(B)$, we define such a covering in the following result.

Definition-Proposition 3.8 (Two-fold covering of $\text{Ps}(B)$). We keep the notations and assumptions from Lemma 3.1 and Proposition 3.2. In particular, we suppose that a choice of w_s satisfying (24) has been made for all $s \in \text{Ps}(B)$. We fix also $\gamma(\sigma)$ defined in Proposition 3.2 such that (32) and (33) are satisfied. Set

$$\widetilde{\text{Ps}(B)}_{\tilde{\chi}} = \{(s, (-1)^k \omega_{\tilde{\chi}}(s)), s \in \text{Ps}(B), k \in \mathbb{Z}\}. \quad (34)$$

Then:

1. $\widetilde{\text{Ps}}(B)_{\tilde{\chi}}$ is a subgroup of $\text{Ps}(B) \times \text{U}(\text{L}^2(Y))$.
2. $\widetilde{\text{Ps}}(B)_{\tilde{\chi}}$ is a two-fold covering of $\text{Ps}(B)$, i.e. it defines a non-split central exact sequence

$$1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\tilde{\Phi}} \widetilde{\text{Ps}}(B)_{\tilde{\chi}} \xrightarrow{\tilde{\Psi}} \text{Ps}(B) \longrightarrow 1.$$

Moreover, $\widetilde{\text{Ps}}(B)_{\tilde{\chi}}$ is a stem extension of $\text{Ps}(B)$ and $c_{\tilde{\chi}}$ is a cocycle which can be associated to this extension.

3. Different choices of $\{w_s, s \in \text{Ps}(B)\}$ lead to isomorphic extensions of $\text{Ps}(B)$.
4. Different choices of extensions $\tilde{\chi}$ of χ lead to isomorphic extensions of $\text{Ps}(B)$.
5. Let $\{\gamma'(\sigma), \sigma \in \text{O}(Q)\}$ be another set of elements satisfying (32) and (33). If

$$\theta'(\sigma) \equiv \theta(\sigma) \pmod{\pi} \quad \text{for all } \sigma \in \text{O}(Q) \tag{35}$$

then the extensions $\widetilde{\text{Ps}}(B)_{\tilde{\chi}}$ and $\widetilde{\text{Ps}}(B)_{\tilde{\chi}'}$ (associated to the choices $\gamma(\sigma)$ and $\gamma'(\sigma)$, respectively) are isomorphic. In particular, this is the case if $\gamma(\sigma)$ and $\gamma'(\sigma)$ are chosen to be real for all $\sigma \in \text{O}(Q)$.

6. Let $\widetilde{\text{Ps}}(B)_{\tilde{\chi}}$ be the two-fold covering of $\text{Ps}(B)$ defined with the extension $\tilde{\chi}$ of χ and the choices $\{w_s, s \in \text{Ps}(B)\}$ and $\{\gamma(\sigma), \sigma \in \text{O}(Q)\}$ such that (24), (32) and (33) are satisfied. Define similarly $\widetilde{\text{Ps}}(B)_{\tilde{\chi}'}$ with the extension $\tilde{\chi}'$ of χ and the choices $\{w'_s, s \in \text{Ps}(B)\}$ and $\{\gamma'(\sigma), \sigma \in \text{O}(Q)\}$ such that (24), (32) and (33) are satisfied. If (35) is satisfied then

$$\begin{aligned} \mathcal{I} : \widetilde{\text{Ps}}(B)_{\tilde{\chi}} &\longrightarrow \widetilde{\text{Ps}}(B)_{\tilde{\chi}'} & (36) \\ (s, (-1)^k \omega_{\tilde{\chi}}(s, w_s, \gamma(\sigma))) &\mapsto (s, (-1)^k \nu(s)^{-1} \gamma(\sigma) \gamma'(\sigma)^{-1} \omega_{\tilde{\chi}'}(s, w'_s, \gamma'(\sigma))) \end{aligned}$$

is an isomorphism of central extensions, where ν is defined by (38). This means that the following diagram is commutative:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\tilde{\Phi}} & \widetilde{\text{Ps}}(B)_{\tilde{\chi}} & \xrightarrow{\tilde{\Psi}} & \text{Ps}(B) \longrightarrow 1 \\ & & \downarrow \text{id} & & \downarrow \mathcal{I} & & \downarrow \text{id} \\ 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\tilde{\Phi}'} & \widetilde{\text{Ps}}(B)_{\tilde{\chi}'} & \xrightarrow{\tilde{\Psi}'} & \text{Ps}(B) \longrightarrow 1 \end{array} \tag{D_1}$$

Proof. **1.** Let $\tilde{s}_1 = (s_1, (-1)^{k_1} \omega_{\tilde{\chi}}(s_1))$ and $\tilde{s}_2 = (s_2, (-1)^{k_2} \omega_{\tilde{\chi}}(s_2))$ be in $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$. Then, by definition, \tilde{s}_1 and \tilde{s}_2 belong to $\text{Ps}(B) \times \text{U}(\text{L}^2(Y))$. Moreover,

$$\begin{aligned} \tilde{s}_1 \tilde{s}_2^{-1} &= (s_1, (-1)^{k_1} \omega_{\tilde{\chi}}(s_1))(s_2^{-1}, (-1)^{-k_2} \omega_{\tilde{\chi}}(s_2)^{-1}) \\ &= (s_1 s_2^{-1}, (-1)^{k_1 - k_2} \omega_{\tilde{\chi}}(s_1) \omega_{\tilde{\chi}}(s_2)^{-1}) \\ &= (s_1 s_2^{-1}, (-1)^{k_1 - k_2} c_{\chi}(s_2, s_2^{-1})^{-1} \omega_{\tilde{\chi}}(s_1) \omega_{\tilde{\chi}}(s_2^{-1})) \quad \text{by (31)} \\ &= (s_1 s_2^{-1}, (-1)^{k_1 - k_2} c_{\chi}(s_2, s_2^{-1})^{-1} c_{\chi}(s_1, s_2^{-1}) \omega_{\tilde{\chi}}(s_1 s_2^{-1})) \quad \text{by (31)}. \end{aligned}$$

Since (33) is satisfied, Corollary 3.6 implies that $(-1)^{k_1 - k_2} c_{\chi}(s_2, s_2^{-1})^{-1} c_{\chi}(s_1, s_2^{-1})$ belongs to $\mu_2(\mathbb{C}) = \{\pm 1\}$. Hence $\tilde{s}_1 \tilde{s}_2^{-1} \in \widetilde{\text{Ps}(B)}_{\tilde{\chi}}$, and thus $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ is a subgroup of $\text{Ps}(B) \times \text{U}(\text{L}^2(Y))$.

2. Define $\tilde{\Phi} : \mu_2(\mathbb{C}) \rightarrow \widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ and $\tilde{\Psi} : \widetilde{\text{Ps}(B)}_{\tilde{\chi}} \rightarrow \text{Ps}(B)$. Then $\tilde{\Phi}$ and $\tilde{\Psi}$ are group homomorphisms such that:

- if $\tilde{\Phi}(z) = ((1, 0), \text{Id}_{\text{L}^2(Y)})$ then $z = 1$ by definition of $\tilde{\Phi}$, and thus $\tilde{\Phi}$ is injective.
- For every $s \in \text{Ps}(B)$, we proved in Proposition 3.2 that $\omega_{\tilde{\chi}}(s)$ is a unitary operator on $\text{L}^2(Y)$. Hence $(s, \omega_{\tilde{\chi}}(s)) \in \widetilde{\text{Ps}(B)}_{\tilde{\chi}}$. Moreover, $\tilde{\Psi}(s, \omega_{\tilde{\chi}}(s)) = s$, which implies that $\tilde{\Psi}$ is surjective.
- If $\tilde{\Psi}(s, (-1)^k \omega_{\tilde{\chi}}(s)) = (1, 0)$ then $s = (1, 0)$. Thus $(s, (-1)^k \omega_{\tilde{\chi}}(s)) \in \text{Im}(\tilde{\Phi})$ by definition of $\tilde{\Phi}$, $\tilde{\Psi}$ and $\omega_{\tilde{\chi}}$. Conversely, it is clear that $\text{Im}(\tilde{\Phi}) \subseteq \text{Ker}(\tilde{\Psi})$. Therefore $\text{Im}(\tilde{\Phi}) = \text{Ker}(\tilde{\Psi})$.

This shows that $1 \rightarrow \mu_2(\mathbb{C}) \xrightarrow{\tilde{\Phi}} \widetilde{\text{Ps}(B)}_{\tilde{\chi}} \xrightarrow{\tilde{\Psi}} \text{Ps}(B) \rightarrow 1$ is a central extension. Now we want to show that this extension does not split. First we remark that c_{χ} is a cocycle associated to this extension. In fact, define $\tau(s) = (s, \omega_{\tilde{\chi}}(s))$ for all $s \in \text{Ps}(B)$. Then τ is a section of $\tilde{\Psi}$, which satisfies for every $s_1, s_2 \in \text{Ps}(B)$:

$$\begin{aligned} \tau(s_1) \tau(s_2) &= (s_1, \omega_{\tilde{\chi}}(s_1))(s_2, \omega_{\tilde{\chi}}(s_2)) = (s_1 s_2, \omega_{\tilde{\chi}}(s_1) \omega_{\tilde{\chi}}(s_2)) = (s_1 s_2, c_{\chi}(s_1, s_2) \omega_{\tilde{\chi}}(s_1 s_2)) \\ &= \tilde{\Phi}(c_{\chi}(s_1, s_2)) \tau(s_1 s_2). \end{aligned}$$

Suppose that this extension splits. Then c_{χ} is a coboundary, i.e. there exists $\nu : \text{Ps}(B) \rightarrow \mu_2(\mathbb{C})$ such that for all $s_1, s_2 \in \text{Ps}(B)$,

$$c_{\chi}(s_1, s_2) = \nu(s_1) \nu(s_2) \nu(s_1 s_2)^{-1}. \quad (37)$$

Recall from Proposition 1.11.5 that $\text{Ps}_a(B)$ denotes the abelian subgroup of $\text{Ps}(B)$ consisting of the elements of the form $(1, f)$. In particular, (37) has to be satisfied for all elements of $\text{Ps}_a(B)$. We therefore have for every $s_1 = (1, f_1)$ and $s_2 = (1, f_2)$ in $\text{Ps}_a(B)$:

$$c_{\chi}(s_2, s_1) = \nu(s_2) \nu(s_1) \nu(s_2 s_1)^{-1} = \nu(s_1) \nu(s_2) \nu(s_1 s_2)^{-1} = c_{\chi}(s_1, s_2).$$

This means that the cocycle c_χ is symmetric on $\text{Ps}_a(B)$. Now let w_1 and w_2 in W be such that $\chi(\langle w_1, w_2 \rangle) \neq 1$. Such a choice is possible. For example, if we pick an arbitrary w_1 in $W \setminus \{0\}$, then $\langle w_1, \cdot \rangle$ is a linear form on W with image \mathbb{F} . Hence there exists w_2 such that $\chi(\langle w_1, w_2 \rangle) \neq 1$ because χ is non-trivial. In turn, $\chi(\langle w_1, w_2 \rangle) \neq 1$ is equivalent to have $\chi(B(w_1, w_2)) \neq \chi(B(w_2, w_1))$. Moreover, $\chi(\langle w_1, \cdot \rangle)$ and $\chi(\langle w_2, \cdot \rangle)$ are two distinct characters of W . By Proposition 1.11.2, there exists f_{z_1} and f_{z_2} in $\mathcal{Q}_a(W)$ such that $\chi(\langle w_1, \cdot \rangle) = \chi \circ f_{z_1}$, $\chi(\langle w_2, \cdot \rangle) = \chi \circ f_{z_2}$ and $z_1 \neq z_2$. Define now $s_1 = (1, f_{z_1})$ and $s_2 = (1, f_{z_2})$, which are in $\text{Ps}_a(B)$ by definition. Then by Lemma 3.1.4, the two elements w_{s_1} and w_{s_2} satisfying (24) for s_1 and s_2 , respectively, are such that $w_{s_1} = w_1$ and $w_{s_2} = w_2$. Thus if we compute the cocycle with the formula (30) found in Corollary 3.4, we obtain

$$c_\chi(s_1, s_2) = \chi(B(w_{s_1}, w_{s_2})) = \chi(B(w_1, w_2))$$

and

$$c_\chi(s_2, s_1) = \chi(B(w_{s_2}, w_{s_1})) = \chi(B(w_2, w_1)).$$

In particular, $c_\chi(s_1, s_2) \neq c_\chi(s_2, s_1)$ since we supposed that $\chi(\langle w_1, w_2 \rangle) \neq 1$. Therefore, c_χ is not symmetric on $\text{Ps}_a(B)$ and the extension $\text{Ps}(B)_{\tilde{\chi}}$ does not split.

Finally, it remains to prove that $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ is a stem extension, i.e. that $\tilde{\Phi}(-1)$ is in the commutator subgroup $[\widetilde{\text{Ps}(B)}_{\tilde{\chi}}, \widetilde{\text{Ps}(B)}_{\tilde{\chi}}]$. As in the last paragraph above, let $w_1, w_2 \in W$ be such that $\chi(\langle w_1, w_2 \rangle) \neq 1$. Consider again $s_1 = (1, f_{z_1})$ and $s_2 = (1, f_{z_2})$ in $\text{Ps}_a(B)$. Set $\tilde{s}_1 = (s_1, \omega_{\tilde{\chi}}(s_1, w_1))$ and $\tilde{s}_2 = (s_2, \omega_{\tilde{\chi}}(s_2, w_2))$. Then by (27), \tilde{s}_1 and \tilde{s}_2 are two elements of $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ such that

$$\begin{aligned} \tilde{s}_1 \tilde{s}_2 \tilde{s}_1^{-1} \tilde{s}_2^{-1} &= (s_1 s_2 s_1^{-1} s_2^{-1}, \omega_{\tilde{\chi}}(s_1, w_1) \omega_{\tilde{\chi}}(s_2, w_2) \omega_{\tilde{\chi}}(s_1, w_1)^{-1} \omega_{\tilde{\chi}}(s_2, w_2)^{-1}) \\ &= ((1, 0), \rho_{\tilde{\chi}}(w_1) \rho_{\tilde{\chi}}(w_2) \rho_{\tilde{\chi}}(w_1)^{-1} \rho_{\tilde{\chi}}(w_2)^{-1}) \quad \text{since } \text{Ps}_a(B) \text{ is commutative} \\ &= ((1, 0), \rho_{\tilde{\chi}}(w_1) \rho_{\tilde{\chi}}(w_2) \rho_{\tilde{\chi}}(-w_1, Q(w_1)) \rho_{\tilde{\chi}}(-w_2, Q(w_2))) \\ &= ((1, 0), \rho_{\tilde{\chi}}(0, B(w_1, w_2) + Q(w_1) + Q(w_2) + B(w_1, w_2) - Q(w_1 + w_2))) \\ &= ((1, 0), \rho_{\tilde{\chi}}(0, -\langle w_1, w_2 \rangle)) \\ &= ((1, 0), \chi(-\langle w_1, w_2 \rangle) \text{Id}_{L^2(Y)}) \end{aligned}$$

Since $\chi(\langle w_1, w_2 \rangle) \neq 1$ and χ takes its values in $\{\pm 1\}$ (see Lemma 2.1.2), we obtain that $\tilde{s}_1 \tilde{s}_2 \tilde{s}_1^{-1} \tilde{s}_2^{-1} = ((1, 0), -\text{Id}_{L^2(Y)}) = \tilde{\Phi}(-1)$.

3. Suppose that for every $s \in \text{Ps}(B)$, w'_s also satisfies (24). Denote by $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}'$ the set of elements $\{(s, (-1)^k \omega_{\tilde{\chi}}(s, w'_s), s \in \text{Ps}(B), k \in \mathbb{Z}\}$. Then by (28) in Proposition 3.2, for every $s \in \text{Ps}(B)$,

$$\omega_{\tilde{\chi}}(s, w'_s) = \chi(-f(v) + \langle v, w_s \rangle + B(w'_s - w_s, v - w_s)) \omega_{\tilde{\chi}}(s, w_s),$$

where v is any element such that $w'_s - w_s = (1 - \sigma)(v)$. Since χ takes its values in $\mu_2(\mathbb{C})$, the set of operators $(-1)^k \omega_{\tilde{\chi}}(s, w'_s)$ is exactly the same as the set of

operators $(-1)^k \omega_{\tilde{\chi}}(s, w_s)$, where $k \in \mathbb{Z}$. Hence $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ and $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}'$ are equal as sets.

Let c'_χ denote the cocycle associated to the projective representation $s \mapsto \omega_{\tilde{\chi}}(s, w'_s)$ (and then also associated to the extension $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}'$). Then we can prove that c_χ and c'_χ are cohomologous. In fact, $\omega_{\tilde{\chi}}(s, w_s)$ and $\omega_{\tilde{\chi}}(s, w'_s)$ satisfy the same intertwining property: for every $s \in \text{Ps}(B)$ and $h \in \text{H}(B)$,

$$\rho_{\tilde{\chi}}(s(h))\omega_{\tilde{\chi}}(s) = \omega_{\tilde{\chi}}(s)\rho_{\tilde{\chi}}(h).$$

Schur's lemma then implies the existence of $\nu : \text{Ps}(B) \rightarrow U(1)$ such that $\omega_{\tilde{\chi}}(s, w'_s) = \nu(s)\omega_{\tilde{\chi}}(s, w_s)$. We computed such a map ν in Proposition 3.2 (28), since we proved that

$$\omega_{\tilde{\chi}}(s, w'_s) = \chi(-f(v) + \langle v, w_s \rangle + B(w'_s - w_s, v - w_s))\omega_{\tilde{\chi}}(s, w_s),$$

where v is any element such that $w'_s - w_s = (1 - \sigma)(v)$. Hence we obtained

$$\nu(s) = \chi(-f(v) + \langle v, w_s \rangle + B(w'_s - w_s, v - w_s)) \in \{\pm 1\}. \quad (38)$$

Thus for all $s_1, s_2 \in \text{Ps}(B)$,

$$\begin{aligned} \omega_{\tilde{\chi}}(s_1, w'_{s_1})\omega_{\tilde{\chi}}(s_2, w'_{s_2}) &= \nu(s_1)\nu(s_2)\omega_{\tilde{\chi}}(s_1, w_{s_1})\omega_{\tilde{\chi}}(s_2, w_{s_2}) \\ c'_\chi(s_1, s_2)\omega_{\tilde{\chi}}(s_1 s_2, w'_{s_1 s_2}) &= \nu(s_1)\nu(s_2)c_\chi(s_1, s_2)\omega_{\tilde{\chi}}(s_1 s_2, w_{s_1 s_2}) \\ c'_\chi(s_1, s_2)\nu(s_1 s_2)\omega_{\tilde{\chi}}(s_1 s_2, w_{s_1 s_2}) &= \nu(s_1)\nu(s_2)c_\chi(s_1, s_2)\omega_{\tilde{\chi}}(s_1 s_2, w_{s_1 s_2}). \end{aligned}$$

Hence

$$c'_\chi(s_1, s_2)c_\chi(s_1, s_2)^{-1} = \nu(s_1)\nu(s_2)\nu(s_1 s_2)^{-1}. \quad (39)$$

Since ν takes its values in $\{\pm 1\}$, this shows that c_χ and c'_χ are cohomologous. Therefore, $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ and $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}'$ are extensions whose associated cocycles are cohomologous: they are isomorphic extensions. In particular, the groups $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ and $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}'$ are isomorphic.

4. Formula (30) found for the metaplectic cocycle in Corollary 3.4 does not depend on the choice of extension $\tilde{\chi}$ of χ . In particular, two different extensions $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ and $\widetilde{\text{Ps}(B)}_{\chi^\sharp}$ corresponding to the extensions $\tilde{\chi}$ and χ^\sharp of χ , respectively, can be associated to the same cocycle c_χ . This implies that the extensions (hence also the groups) $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ and $\widetilde{\text{Ps}(B)}_{\chi^\sharp}$ are isomorphic.
5. Let $\{\gamma'(\sigma), \sigma \in \text{O}(Q)\}$ be another set of elements satisfying (32) and (33). We denote by $\omega'_{\tilde{\chi}}(s)$ and c'_χ the operators and the cocycle associated to this choice. Then formula (30) implies that for all $s_1, s_2 \in \text{Ps}(B)$,

$$c'_\chi(s_1, s_2) = \gamma'(\sigma_1)\gamma'(\sigma_2)\gamma'(\sigma_1\sigma_2)^{-1}\gamma(\sigma_1)^{-1}\gamma(\sigma_2)^{-1}\gamma(\sigma_1\sigma_2)c_\chi(s_1, s_2),$$

i.e.

$$c'_\chi(s_1, s_2)c_\chi(s_1, s_2)^{-1} = \gamma'(\sigma_1)\gamma(\sigma_1)^{-1}\gamma'(\sigma_2)\gamma(\sigma_2)^{-1}(\gamma'(\sigma_1\sigma_2)\gamma(\sigma_1\sigma_2)^{-1})^{-1}. \quad (40)$$

If $\theta'(\sigma) \equiv \theta(\sigma) \pmod{\pi}$ for all $\sigma \in \mathcal{O}(Q)$, then

$$\gamma'(\sigma)\gamma(\sigma)^{-1} = e^{i(\theta'(\sigma)-\theta(\sigma))} \in \{\pm 1\},$$

which shows that the map $\sigma \mapsto \gamma'(\sigma)\gamma(\sigma)^{-1}$ takes its values in $\{\pm 1\}$. Thus $c'_\chi c_\chi^{-1}$ is a coboundary by (40). This implies that the extensions $\widetilde{\text{Ps}}(B)_{\widetilde{\chi}}$ and $\widetilde{\text{Ps}}(B)_{\widetilde{\chi}'}$ are isomorphic.

6. Denote by c_χ and c'_χ the cocycles associated to the extensions $\widetilde{\text{Ps}}(B)_{\widetilde{\chi}}$ and $\widetilde{\text{Ps}}(B)_{\chi^\#}$ defined in the statement, respectively. (Recall that $\chi^\#$ does not appear in the notation of these cocycles since the metaplectic cocycle does not depend on the choice of extension of χ by Corollary 3.4.) Let $\widetilde{s}_1 = (s_1, (-1)^{k_1}\omega_{\widetilde{\chi}}(s_1, w_{s_1}, \gamma(\sigma_1)))$ and $\widetilde{s}_2 = (s_2, (-1)^{k_2}\omega_{\widetilde{\chi}}(s_2, w_{s_2}, \gamma(\sigma_2)))$ in $\widetilde{\text{Ps}}(B)_{\widetilde{\chi}}$. Then

$$\begin{aligned} \mathcal{I}(\widetilde{s}_1)\mathcal{I}(\widetilde{s}_2) &= (s_1, (-1)^{k_1}\nu(s_1)^{-1}\gamma(\sigma_1)\gamma'(\sigma_1)^{-1}\omega_{\chi^\#}(s_1, w'_{s_1}, \gamma'(\sigma_1))) \\ &\quad \cdot (s_2, (-1)^{k_2}\nu(s_2)^{-1}\gamma(\sigma_2)\gamma'(\sigma_2)^{-1}\omega_{\chi^\#}(s_2, w'_{s_2}, \gamma'(\sigma_2))) \\ &= (s_1s_2, (-1)^{k_1+k_2}\nu(s_1)^{-1}\nu(s_2)^{-1}\gamma(\sigma_1)\gamma'(\sigma_1)^{-1}\gamma(\sigma_2)\gamma'(\sigma_2)^{-1}\omega_{\chi^\#}(s_1, w'_{s_1}, \gamma'(\sigma_1)) \\ &\quad \omega_{\chi^\#}(s_2, w'_{s_2}, \gamma'(\sigma_2))) \\ &= (s_1s_2, (-1)^{k_1+k_2}\nu(s_1)^{-1}\nu(s_2)^{-1}\gamma(\sigma_1)\gamma'(\sigma_1)^{-1}\gamma(\sigma_2)\gamma'(\sigma_2)^{-1}c'_\chi(s_1, s_2) \\ &\quad \omega_{\chi^\#}(s_1s_2, w'_{s_1s_2}, \gamma'(\sigma_1\sigma_2))) \\ \mathcal{I}(\widetilde{s}_1\widetilde{s}_2) &= \mathcal{I}(s_1s_2, (-1)^{k_1+k_2}\omega_{\widetilde{\chi}}(s_1, w_{s_1}, \gamma(\sigma_1))\omega_{\widetilde{\chi}}(s_2, w_{s_2}, \gamma(\sigma_2))) \\ &= \mathcal{I}(s_1s_2, (-1)^{k_1+k_2}c_\chi(s_1, s_2)\omega_{\widetilde{\chi}}(s_1s_2, w_{s_1s_2}, \gamma(\sigma_1\sigma_2))) \\ &= (s_1s_2, (-1)^{k_1+k_2}c_\chi(s_1, s_2)\nu(s_1s_2)^{-1}\gamma(\sigma_1\sigma_2)\gamma'(\sigma_1\sigma_2)^{-1}\omega_{\chi^\#}(s_1s_2, w'_{s_1s_2}, \gamma'(\sigma_1\sigma_2))). \end{aligned} \tag{41}$$

Hence \mathcal{I} is a group homomorphism if and only if

$$\nu(s_1)^{-1}\nu(s_2)^{-1}\gamma(\sigma_1)\gamma'(\sigma_1)^{-1}\gamma(\sigma_2)\gamma'(\sigma_2)^{-1}c'_\chi(s_1, s_2) = c_\chi(s_1, s_2)\nu(s_1s_2)^{-1}\gamma(\sigma_1\sigma_2)\gamma'(\sigma_1\sigma_2)^{-1},$$

i.e.

$$c'_\chi(s_1, s_2)c_\chi(s_1, s_2)^{-1} = \nu(s_1)\nu(s_2)\nu(s_1s_2)^{-1}\gamma(\sigma_1\sigma_2)\gamma'(\sigma_1\sigma_2)^{-1}\gamma(\sigma_1)^{-1}\gamma'(\sigma_1)\gamma(\sigma_2)^{-1}\gamma'(\sigma_2). \tag{42}$$

Denote by c''_χ the cocycle associated to the two-fold covering of $\text{Ps}(B)$ which is defined with the extension $\widetilde{\chi}$ of χ and the choices $\{w'_s, s \in \text{Ps}(B)\}$ and $\{\gamma(\sigma), \sigma \in \mathcal{O}(Q)\}$ such that (24), (32) and (33) are satisfied. By (39),

$$c''_\chi(s_1, s_2)c_\chi(s_1, s_2)^{-1} = \nu(s_1)\nu(s_2)\nu(s_1s_2)^{-1}. \tag{43}$$

By (40), since (35) is satisfied,

$$c'_\chi(s_1, s_2)c''_\chi(s_1, s_2)^{-1} = \gamma'(\sigma_1)\gamma(\sigma_1)^{-1}\gamma'(\sigma_2)\gamma(\sigma_2)^{-1}(\gamma'(\sigma_1\sigma_2)\gamma(\sigma_1\sigma_2)^{-1})^{-1}. \tag{44}$$

Since $c'_\chi(s_1, s_2)c_\chi(s_1, s_2)^{-1} = c'_\chi(s_1, s_2)c''_\chi(s_1, s_2)^{-1}c''_\chi(s_1, s_2)c_\chi(s_1, s_2)^{-1}$, (43) and (44) imply (42). Let $\tilde{s} = (s, (-1)^k \omega_{\tilde{\chi}}(s, w_s, \gamma(\sigma))) \in \text{Ker}(\mathcal{I})$. Then $s = (1, 0)$ and

$$\begin{aligned} \text{Id}_{L^2(Y)} &= (-1)^k \nu(s)^{-1} \gamma(\sigma) \gamma'(\sigma)^{-1} \omega_{\chi^\sharp}(s, w'_s, \gamma'(\sigma)) \\ &= (-1)^k \nu(1, 0)^{-1} \gamma(1) \gamma'(1)^{-1} \omega_{\chi^\sharp}((1, 0), 0, \gamma'(1)) \\ &= (-1)^k \text{Id}_{L^2(Y)}. \end{aligned}$$

Hence $k \in 2\mathbb{Z}$. This enables us to conclude that $(s, (-1)^k \omega_{\tilde{\chi}}(s, w_s, \gamma(\sigma))) = ((1, 0), \omega_{\tilde{\chi}}((1, 0), 0, \gamma(1))) = ((1, 0), \text{Id}_{L^2(Y)})$, i.e. \mathcal{I} is injective. Since $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ and $\widetilde{\text{Ps}(B)}_{\chi^\sharp}$ have the same order, \mathcal{I} is an isomorphism.

Now, it only remains to prove that the diagram (D_1) commutes. First we have that

$$\begin{aligned} (\mathcal{I} \circ \tilde{\Phi})(-1) &= \mathcal{I}((1, 0), -\text{Id}_{L^2(Y)}) = ((1, 0), -\omega_{\chi^\sharp}((1, 0), 0, \gamma(1))) = ((1, 0), -\text{Id}_{L^2(Y)}) \\ &= \tilde{\Phi}'(-1). \end{aligned}$$

In addition, let $\tilde{s} = (s, (-1)^k \omega_{\tilde{\chi}}(s, w_s, \gamma(\sigma))) \in \widetilde{\text{Ps}(B)}_{\tilde{\chi}}$. Then

$$(\tilde{\Psi}' \circ \mathcal{I})(\tilde{s}) = \tilde{\Psi}'(s, (-1)^k \nu(s)^{-1} \gamma(\sigma) \gamma'(\sigma)^{-1} \omega_{\chi^\sharp}(s, w'_s, \gamma'(\sigma))) = s = \tilde{\Psi}(s).$$

This proves that the diagram (D_1) is commutative. \square

Remark 3.9. • We do not know if the converse of the statement of Proposition 3.8.5 holds in general, i.e. if the isomorphism of extensions implies that $\theta'(\sigma) \equiv \theta(\sigma) \pmod{\pi}$ for all $\sigma \in O(Q)$. There is a necessary condition for the extensions to be isomorphic since we can construct non-isomorphic extensions (cf. [8], Theorem 5, or the two-dimensional case detailed in section 4). Proposition 3.8.5 shows in particular that if we choose $\gamma(\sigma)$ and $\gamma'(\sigma)$ to be real (consistent with the fact that χ is real) then the extensions obtained with these choices are isomorphic. If W is two-dimensional, then we prove in Propositions 4.2 and 4.5 that the converse of Proposition 3.8.5 holds.

- We can wonder whether we can obtain all the possible non-split stem extensions of $\text{Ps}(B)$ by two elements with all possible choices of $\{\gamma(\sigma), \sigma \in \text{Ps}(B)\}$ satisfying (33). This is false in general, as we shall see for the two-dimensional case (cf. Remark 4.3). Nevertheless, we prove in this example that we obtain all the possible non-split stem extensions of $\text{Ps}(B)$ by two elements which contain a copy of the Heisenberg group $H(B)$.

Now we have everything we need to lift the projective Weil representation of $\text{Ps}(B)$ to a linear representation of $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$.

Corollary 3.10 (The Weil representation). 1. The projective Weil representation $\omega_{\tilde{\chi}}$ of $\text{Ps}(B)$ can be lifted to a linear representation $\tilde{\omega}_{\tilde{\chi}}$ of $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$, which will

be called the Weil representation. $\tilde{\omega}_{\tilde{\chi}}$ is defined by

$$\tilde{\omega}_{\tilde{\chi}}(\tilde{s}) = (-1)^k \omega_{\tilde{\chi}}(s) \quad \text{for all } \tilde{s} = (s, (-1)^k \omega_{\tilde{\chi}}(s)) \in \widetilde{\text{Ps}(B)}_{\tilde{\chi}}. \quad (45)$$

2. $\tilde{\omega}_{\tilde{\chi}}$ is unitary, faithful and irreducible.
3. The image of $\tilde{\omega}_{\tilde{\chi}}$ contains all the operators $\rho_{\tilde{\chi}}(w)$, $w \in W$ (which form a basis of $\text{End}_{\mathbb{C}}(\mathbb{L}^2(Y))$). More precisely, for every $w \in W$,

$$\rho_{\tilde{\chi}}(w) = \omega_{\tilde{\chi}}((1, f_z), w), \quad (46)$$

where $f_z \in \mathcal{Q}_a(W)$ is such that $\chi \circ f_z = \chi(\langle w, \cdot \rangle)$.

Proof. Define the map $\tilde{\omega}_{\tilde{\chi}}$ by (45).

1. Observe first that for $s = (1, 0)$, since $w_s = 0$ by Lemma 3.1.4, we obtain by (25) that

$$\omega_{\tilde{\chi}}((1, 0), 0) = |\mathbb{K}(1)|^{-1} \gamma(1) \sum_{w \in W} \rho_{\tilde{\chi}}(w)^{-1} \rho_{\tilde{\chi}}(0) \rho_{\tilde{\chi}}(w) = \text{Id}_{\mathbb{L}^2(Y)}.$$

Then $\tilde{\omega}_{\tilde{\chi}}((1, 0), \omega_{\tilde{\chi}}((1, 0), 0)) = \text{Id}_{\mathbb{L}^2(Y)}$. Moreover, for every $\tilde{s}_1 = (s_1, (-1)^{k_1} \omega_{\tilde{\chi}}(s_1))$ and $\tilde{s}_2 = (s_2, (-1)^{k_2} \omega_{\tilde{\chi}}(s_2))$ in $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$,

$$\tilde{\omega}_{\tilde{\chi}}(\tilde{s}_1) \tilde{\omega}_{\tilde{\chi}}(\tilde{s}_2) = (-1)^{k_1} \omega_{\tilde{\chi}}(s_1) (-1)^{k_2} \omega_{\tilde{\chi}}(s_2) = c_{\chi}(s_1, s_2) (-1)^{k_1+k_2} \omega_{\tilde{\chi}}(s_1 s_2)$$

and

$$\begin{aligned} \tilde{\omega}_{\tilde{\chi}}(\tilde{s}_1 \tilde{s}_2) &= \tilde{\omega}_{\tilde{\chi}}(s_1 s_2, (-1)^{k_1} \omega_{\tilde{\chi}}(s_1) (-1)^{k_2} \omega_{\tilde{\chi}}(s_2)) = \tilde{\omega}_{\tilde{\chi}}(s_1 s_2, (-1)^{k_1+k_2} c_{\chi}(s_1, s_2) \omega_{\tilde{\chi}}(s_1 s_2)) \\ &= c_{\chi}(s_1, s_2) (-1)^{k_1+k_2} \omega_{\tilde{\chi}}(s_1 s_2). \end{aligned}$$

Thus $\tilde{\omega}_{\tilde{\chi}}$ is a linear representation of $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$, which lifts the projective Weil representation of $\text{Ps}(B)$.

2. Let $s = (\sigma, f) \in \text{Ps}(B)$ and $\tilde{s} = (s, (-1)^k \omega_{\tilde{\chi}}(s)) \in \widetilde{\text{Ps}(B)}_{\tilde{\chi}}$. Suppose that $\tilde{\omega}_{\tilde{\chi}}(\tilde{s}) = \text{Id}_{\mathbb{L}^2(Y)}$, then $(-1)^k \omega_{\tilde{\chi}}(s) = \text{Id}_{\mathbb{L}^2(Y)}$. Then by (27) in Proposition 3.2,

$$(-1)^k \gamma(\sigma) \sum_{u \in \text{I}(\sigma)} \chi(f(w) + \langle w_s, w \rangle + B(u, w + w_s)) \rho_{\tilde{\chi}}(u + w_s) = \text{Id}_{\mathbb{L}^2(Y)} = \rho_{\tilde{\chi}}(0),$$

where $w \in W$ is any element such that $u = (1 - \sigma)(w)$. The operators $\rho_{\tilde{\chi}}(w)$ form a basis of $\text{End}_{\mathbb{C}}(\mathbb{L}^2(Y))$ according to Lemma 2.8, and χ takes its values in $\{\pm 1\}$. Hence only one term $\rho_{\tilde{\chi}}(u + w_s)$ can appear in the sum, the one for which $u + w_s = 0$. Moreover, since $u \in \text{I}(\sigma)$, this means that $\text{I}(\sigma)$ is a vector space which contains only one element in this case. Hence $\text{I}(\sigma) = \{0\}$ and $w_s = u = 0$. This implies that $\mathbb{K}(\sigma) = W$, i.e. $\sigma = 1$. In addition, $w_s = 0$ implies by (24) that $\chi \circ f|_{\mathbb{K}(\sigma)} = \chi \circ f = \chi \circ \langle w_s, \cdot \rangle = 1$, i.e. $\chi \circ f = 1$. If $f \neq 0$ then $f(W) = \mathbb{F}$, which implies that $\mathbb{F} \subseteq \text{Ker}(\chi)$. This is impossible because χ is non-trivial by hypothesis. Hence $f = 0$ and $s = (1, 0)$. In particular, $\omega_{\tilde{\chi}}(s) = \text{Id}_{\mathbb{L}^2(Y)}$. Since $(-1)^k \omega_{\tilde{\chi}}(s) = \text{Id}_{\mathbb{L}^2(Y)}$ by hypothesis, this implies that $k \in 2\mathbb{Z}$, i.e. $\tilde{s} = 1$. Finally, this proves that $\tilde{\omega}_{\tilde{\chi}}$ is faithful. We prove the irreducibility of $\tilde{\omega}_{\tilde{\chi}}$ in the next point.

3. For every $w \in W$, we proved in Proposition 3.8.2 that there exists $z \in W$ and $s = (1, f_z) \in \text{Ps}_a(B)$ such that $w_s = w$. Then, by (27) in Proposition 3.2,

$$\omega_{\tilde{\chi}}(s, w) = \sum_{u \in I(1)} \chi(f_z(v) + \langle w, v \rangle + B(u, v + w)) \rho_{\tilde{\chi}}(u + w) = \rho_{\tilde{\chi}}(w)$$

where v is any element such that $u = (1 - 1)(v) = 0$. This shows that $\widetilde{\omega_{\tilde{\chi}}(\text{Ps}(B)_{\tilde{\chi}})}$ contains the operators $\rho_{\tilde{\chi}}(w)$ for all $w \in W$. Now, if $\mathcal{V} \subseteq L^2(Y)$ is a vector subspace invariant by $\widetilde{\omega_{\tilde{\chi}}}$ then, in particular, \mathcal{V} is invariant under all the operators $\rho_{\tilde{\chi}}(w)$, $w \in W$. Since these operators form a basis of $\text{End}_{\mathbb{C}}(L^2(Y))$ by Lemma 2.8, then either $\mathcal{V} = \{0\}$ or $\mathcal{V} = L^2(Y)$, i.e. $\widetilde{\omega_{\tilde{\chi}}}$ is irreducible. \square

Proposition 3.11 (Character of the Weil representation). Let $\Theta_{\tilde{\chi}}$ be the character of $\widetilde{\omega_{\tilde{\chi}}}$. We suppose as before that a choice of $\{\gamma(\sigma), \sigma \in O(Q)\}$ and $\{w_s, s \in \text{Ps}(B)\}$ has been made. Then, for every $\tilde{s} = (s, (-1)^k \omega_{\tilde{\chi}}(s, w_s))$ in $\widetilde{\text{Ps}(B)_{\tilde{\chi}}}$,

$$\Theta_{\tilde{\chi}}(\tilde{s}) = (-1)^k \gamma(\sigma) |Y| \chi(f(w) - B(w_s + w, w_s)) \mathbb{1}_{I(\sigma)}(w_s), \quad (47)$$

where $w \in W$ is any element such that $(\sigma - 1)(w) = w_s$. In particular, $\Theta_{\tilde{\chi}}$ does not depend on the choice of extension $\tilde{\chi}$ of χ and we can denote it by Θ_{χ} . Moreover, Θ_{χ} satisfies the following properties:

1. The support of Θ_{χ} is

$$\text{supp}(\Theta_{\chi}) = \{\tilde{s} = (s, (-1)^k \omega_{\tilde{\chi}}(s, w_s)) \in \widetilde{\text{Ps}(B)_{\tilde{\chi}}}, s = (\sigma, f) \text{ with } \chi \circ f|_{K(\sigma)} = 1\}.$$

Hence it does not depend on the choices of $\{\gamma(\sigma), \sigma \in O(Q)\}$ and $\{w_s, s \in \text{Ps}(B)\}$.

2. For every $\sigma \in O(Q)$, there exists $\tilde{s} = (s, (-1)^k \omega_{\tilde{\chi}}(s, w_s)) \in \widetilde{\text{Ps}(B)_{\tilde{\chi}}}$ such that the projection of s on $O(Q)$ is σ and $\Theta_{\chi}(\tilde{s}) \neq 0$.
3. Let $\sigma \in O(Q)$. Then the following assertions are equivalent:
 - $\Theta_{\chi}(\tilde{s}) \neq 0$ for all $\tilde{s} = (s, (-1)^k \omega_{\tilde{\chi}}(s, w_s)) \in \widetilde{\text{Ps}(B)_{\tilde{\chi}}}$ such that the projection of s on $O(Q)$ is σ .
 - $1 - \sigma$ is invertible.

Proof. 1. Let $\tilde{s} = (s, (-1)^k \omega_{\tilde{\chi}}(s)) \in \widetilde{\text{Ps}(B)_{\tilde{\chi}}}$, where $s = (\sigma, f) \in \text{Ps}(B)$. By definition of $\widetilde{\omega_{\tilde{\chi}}}$, $\Theta_{\tilde{\chi}}(\tilde{s}) = \text{Trace}(\widetilde{\omega_{\tilde{\chi}}}(\tilde{s})) = (-1)^k \text{Trace}(\omega_{\tilde{\chi}}(s))$. Recall the expression (27) proved in Proposition 3.2:

$$\omega_{\tilde{\chi}}(s) = \gamma(\sigma) \sum_{u \in I(\sigma)} \chi(f(w) + \langle w_s, w \rangle + B(u, w + w_s)) \rho_{\tilde{\chi}}(u + w_s),$$

where $w \in W$ is any element such that $u = (1 - \sigma)(w)$. We can first compute the trace of every operator of the form $\rho_{\tilde{\chi}}(w)$, $w \in W$. In fact, this computation has already been done in Lemma 2.3 in the basis $\mathcal{B}_Y = \{\delta_y, y \in Y\}$ of $L^2(Y)$, and the result is given by (16): for all $w_0 \in W$,

$$\text{Trace}(\rho_{\tilde{\chi}}(w_0)) = |Y| \delta_0(w_0).$$

Then we obtain

$$\begin{aligned}
 \text{Trace}(\omega_{\tilde{\chi}}(s)) &= \gamma(\sigma) \sum_{u \in \mathbb{I}(\sigma)} \chi(f(w) + \langle w_s, w \rangle + B(u, w + w_s)) \text{Trace}(\rho_{\tilde{\chi}}(u + w_s)) \\
 &= \gamma(\sigma) \sum_{u \in \mathbb{I}(\sigma)} \chi(f(w) + \langle w_s, w \rangle + B(u, w + w_s)) |Y| \delta_0(u + w_s) \\
 &= \gamma(\sigma) |Y| \chi(f(w) + \langle w_s, w \rangle - B(w_s, w + w_s)) \mathbb{1}_{\mathbb{I}(\sigma)}(w_s) \\
 &= \gamma(\sigma) |Y| \chi(f(w) - B(w_s + w, w_s)) \mathbb{1}_{\mathbb{I}(\sigma)}(w_s),
 \end{aligned}$$

where $w \in W$ is any element such that $w_s = (\sigma - 1)(w)$. This proves (47). Now, $\Theta_{\chi}(\tilde{s})$ is non-zero if and only if $w_s \in \mathbb{I}(\sigma)$, which does not depend on the choice made for $\gamma(\sigma)$. By Lemma 3.1.3, the fact that w_s belongs to $\mathbb{I}(\sigma)$ is independent of the choice of w_s . Moreover, since $\mathbb{K}(\sigma)$ and $\mathbb{I}(\sigma)$ are orthogonal with respect to the non-degenerate symplectic form $\langle \cdot, \cdot \rangle$ (Lemma 3.1), $w_s \in \mathbb{I}(\sigma)$ is equivalent to have $\chi \circ f|_{\mathbb{K}(\sigma)} = 1$ by (24).

2. Let σ be an element of $\mathbb{O}(Q)$. Let $\{v_1, \dots, v_l\}$ be a basis of $\mathbb{K}(\sigma)$ and complete it into a basis $\mathcal{B} = \{v_1, \dots, v_{2n}\}$ of W . Then we can associate to σ an element $s_{\sigma} = (\sigma, f_{\sigma}) \in \text{Ps}(B)$ by Proposition 1.11, where f_{σ} and β_{σ} are given by (7) and (8):

$$f_{\sigma}(w) = \sum_{1 \leq i \leq 2n} \lambda_i v_i = \sum_{1 \leq i < j \leq 2n} \lambda_i \lambda_j \beta_{\sigma}(v_i, v_j),$$

and for every $w_1, w_2 \in W$,

$$\beta_{\sigma}(w_1, w_2) = B(\sigma(w_1), \sigma(w_2)) - B(w_1, w_2).$$

In particular, if $w = \sum_{1 \leq i \leq l} \lambda_i v_i$ belongs to $\mathbb{K}(\sigma)$, then $\beta_{\sigma}(v_i, v_j) = 0$ for all $1 \leq i < j \leq l$, and thus $f_{\sigma}(w) = 0$. Then (24) is written

$$1 = \chi(f_{\sigma}(w)) = \chi(\langle w_{s_{\sigma}}, w \rangle)$$

for all $w \in \mathbb{K}(\sigma)$. This implies that $w_{s_{\sigma}}$ belongs to $\mathbb{I}(\sigma)$. In fact, this is automatically true if $\mathbb{I}(\sigma) = W$, and otherwise $\mathbb{K}(\sigma) \neq \{0\}$ and the image \mathbb{F} of the linear map $\langle w_{s_{\sigma}}, \cdot \rangle|_{\mathbb{K}(\sigma)}$ would be contained in the kernel of χ . If we set $\tilde{s}_{\sigma} = (s_{\sigma}, \omega_{\tilde{\chi}}(s_{\sigma}, w_{s_{\sigma}})) \in \text{Ps}(B)_{\tilde{\chi}}$, then we obtain that $\Theta_{\chi}(\tilde{s}) \neq 0$ by (47).

Remark 3.12. This proof shows that, for every $\sigma \in \mathbb{O}(Q)$, there always exists a basis of W such that $f_{\sigma}|_{\mathbb{K}(\sigma)} = 0$. However, it requires to fix a basis of $\mathbb{K}(\sigma)$ and then complete it into a basis of W . There is in fact an alternative solution if we do not want to fix these bases. We can define f_{σ} from any basis of W with (7). Then, since $(\sigma, f_{\sigma}) \in \text{Ps}(B)$, we know that f_{σ} is additive on $\mathbb{K}(\sigma)$ by (6). Hence there exists $v \in W$ such that $\chi \circ f_{\sigma} = \chi \circ \langle v, \cdot \rangle$ on $\mathbb{K}(\sigma)$ by Proposition 1.11.1. Set $s = (\sigma, f = f_{\sigma} - \langle v, \cdot \rangle)$. Then $s \in \text{Ps}(B)$ and $\chi \circ f = 1$ on $\mathbb{K}(\sigma)$. This enables to conclude by (24) that $w_s \in \mathbb{I}(\sigma)$, and we also obtain 2 in this way.

3. Let $\sigma \in O(Q)$. If $1 - \sigma$ is invertible, then $I(\sigma) = W$. Then every w_s satisfying (24) belongs to $I(\sigma)$. By (47), this implies that $\Theta_\chi(\tilde{s}) \neq 0$ for all $\tilde{s} = (s, (-1)^k \omega_{\tilde{\chi}}(s)) \in \widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ such that the projection of s on $O(Q)$ is σ . Conversely, suppose that $1 - \sigma$ has a non-trivial kernel $K(\sigma)$. Define first f_σ as in the proof of 2 above, i.e. with a basis $\mathcal{B} = \{v_1, \dots, v_{2n}\}$ of W such that $\{v_1, \dots, v_l\}$ is a basis of $K(\sigma)$, with $l \geq 1$. In particular, f_σ satisfies $f_{\sigma|_{K(\sigma)}} = 0$. Since $K(\sigma)^\perp = I(\sigma)$, $1 - \sigma$ is not invertible and $\langle \cdot, \cdot \rangle$ is non-degenerate, then $K(\sigma)^\perp \neq W$. Hence there exists an element $w_0 \notin K(\sigma)^\perp = I(\sigma)$. In particular, the map $\langle w_0, \cdot \rangle|_{K(\sigma)}$ is not identically zero. Then the image of the linear form $\langle w_0, \cdot \rangle|_{K(\sigma)}$ is equal to \mathbb{F} . It follows that $\chi \circ \langle w_0, \cdot \rangle|_{K(\sigma)} \neq 1$ since χ is non-trivial by hypothesis. Now, by Proposition 1.11.2, there exists $z \in W$ such that $\chi \circ f_z = \chi \circ \langle w_0, \cdot \rangle$, where $f_z \in \mathcal{Q}_a(W)$. Set $s = (\sigma, f_\sigma + f_z) \in \text{Ps}(B)$. Then for all $w \in K(\sigma)$:

$$\chi(\langle w_0, w \rangle) = \chi(f_z(w)) = \chi((f_\sigma + f_z)(w)) = \chi(f(w)),$$

i.e. w_0 satisfies (24) for $s = (\sigma, f)$ with $f = f_\sigma + f_z$. Thus, by Lemma 3.1.3, $w_s - w_0 \in I(\sigma)$, which implies that w_s does not belong to $I(\sigma)$. This shows that $\Theta_\chi(\tilde{s}) = 0$ for $\tilde{s} = (s, \omega_{\tilde{\chi}}(s, w_s))$. Since the projection of s on $O(Q)$ is σ , this completes the proof by contraposition. \square

Corollary 3.13. For every $\sigma \in O(Q)$, choose $\gamma(\sigma)$ satisfying (32) such that $\gamma(\sigma)$ is real. Then the Weil representation $\tilde{\omega}_{\tilde{\chi}}$ defined by (45) is self-dual.

Proof. Let $\tilde{\omega}_{\tilde{\chi}}^*$ be the dual representation of $\tilde{\omega}_{\tilde{\chi}}$ and denote by Θ_χ^* its character. Since $\tilde{\omega}_{\tilde{\chi}}$ is unitary, for every $\tilde{s} = (s, (-1)^k \omega_{\tilde{\chi}}(s, w_s)) \in \widetilde{\text{Ps}(B)}_{\tilde{\chi}}$,

$$\Theta_\chi^*(\tilde{s}) = \text{Trace}(\tilde{\omega}_{\tilde{\chi}}(\tilde{s}^{-1})^T) = \text{Trace}((\tilde{\omega}_{\tilde{\chi}}(\tilde{s})^*)^T) = \overline{\text{Trace}(\tilde{\omega}_{\tilde{\chi}}(\tilde{s}))} = \overline{\Theta_\chi(\tilde{s})}.$$

If $\gamma(\sigma)$ is real for all $\sigma \in O(Q)$ then the character Θ_χ of the Weil representation $\tilde{\omega}_{\tilde{\chi}}$ is real by (47). In particular, the character Θ_χ^* of the dual representation $\tilde{\omega}_{\tilde{\chi}}^*$ is exactly equal to the character Θ_χ of the Weil representation $\tilde{\omega}_{\tilde{\chi}}$. \square

Corollary 3.14. Let $\tilde{\chi}$ and χ^\sharp be two extensions of χ to $X \times \mathbb{F}$. For every $\sigma \in O(Q)$, choose $\gamma(\sigma)$ and $\gamma'(\sigma)$ satisfying (32). For every $s \in \text{Ps}(B)$, choose w_s and w'_s satisfying (24). Suppose that (33) and (35) are satisfied. Then the Weil representation $\tilde{\omega}_{\tilde{\chi}}$, obtained with $\tilde{\chi}$, $\{w_s \text{ satisfying (24), } s \in \text{Ps}(B)\}$ and $\{\gamma(\sigma) \text{ satisfying (32), } \sigma \in O(Q)\}$, is isomorphic to the Weil representation $\tilde{\omega}_{\chi^\sharp} \circ \mathcal{I}$, obtained with χ^\sharp , $\{w'_s \text{ satisfying (24), } s \in \text{Ps}(B)\}$ and $\{\gamma'(\sigma) \text{ satisfying (32), } \sigma \in O(Q)\}$. In other terms, there exists an isomorphism $T \in U(L^2(Y))$ such that the following diagram commutes:

$$\tilde{\omega}_{\chi^\sharp}(\mathcal{I}(\tilde{s})) = T \circ \tilde{\omega}_{\tilde{\chi}}(\tilde{s}) \circ T^{-1} \quad \text{for all } \tilde{s} \in \widetilde{\text{Ps}(B)}_{\tilde{\chi}}, \quad \text{i.e.} \quad \begin{array}{ccc} L^2(Y) & \xrightarrow{\tilde{\omega}_{\tilde{\chi}}(\tilde{s})} & L^2(Y) \\ \downarrow T & & \downarrow T \\ L^2(Y) & \xrightarrow{\tilde{\omega}_{\chi^\sharp}(\mathcal{I}(\tilde{s}))} & L^2(Y) \end{array} \quad (D_2)$$

where \mathcal{I} is defined by (36).

Proof. Since (35) is satisfied, by Proposition 3.8.6, the extensions $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ and $\widetilde{\text{Ps}(B)}_{\chi^\sharp}$ associated to the choices $(\tilde{\chi}, \gamma(\sigma), w_s)$ and $(\chi^\sharp, \gamma'(\sigma), w'_s)$, respectively, are isomorphic. An explicit isomorphism \mathcal{I} is given by (36). Hence, \mathcal{I} is also a group isomorphism between $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ and $\widetilde{\text{Ps}(B)}_{\chi^\sharp}$. Now, we denote by Θ_χ and Θ'_χ the characters of $\tilde{\omega}_{\tilde{\chi}}$ and $\tilde{\omega}_{\chi^\sharp}$, respectively. Then by definition of \mathcal{I} , Θ_χ and Θ'_χ , for every $\tilde{s} = (s, (-1)^k \omega_{\tilde{\chi}}(s, w_s, \gamma(\sigma))) \in \widetilde{\text{Ps}(B)}_{\tilde{\chi}}$,

$$\begin{aligned}
 & \text{Trace}(\tilde{\omega}_{\chi^\sharp}(\mathcal{I}(\tilde{s}))) \\
 &= \text{Trace}((-1)^k \nu(s)^{-1} \gamma(\sigma) \gamma'(\sigma)^{-1} \omega_{\chi^\sharp}(s, w'_s, \gamma'(\sigma))) \\
 &= (-1)^k \nu(s)^{-1} \gamma(\sigma) \gamma'(\sigma)^{-1} \Theta'_\chi(s, \omega_{\chi^\sharp}(s, w'_s, \gamma'(\sigma))) \\
 &\stackrel{(38),(47)}{=} (-1)^k \gamma(\sigma) \gamma'(\sigma)^{-1} \chi(f(v) - \langle v, w_s \rangle - B(w'_s - w_s, v - w_s)) \gamma'(\sigma) |Y| \\
 &\quad \chi(f(w') - B(w'_s + w', w'_s)) \mathbb{1}_{\text{I}(\sigma)}(w'_s) \\
 &= (-1)^k \gamma(\sigma) |Y| \chi(f(v) - \langle v, w_s \rangle - B(w'_s - w_s, v - w_s) + f(w') - B(w'_s + w', w'_s)) \\
 &\quad \mathbb{1}_{\text{I}(\sigma)}(w'_s),
 \end{aligned}$$

where v is any element such that $w'_s - w_s = (1 - \sigma)(v)$ and $w' \in W$ is any element such that $(\sigma - 1)(w') = w'_s$. By Lemma 3.1.3, $\mathbb{1}_{\text{I}(\sigma)}(w'_s) = \mathbb{1}_{\text{I}(\sigma)}(w_s)$. Moreover,

$$\begin{aligned}
 & f(v) - \langle v, w_s \rangle - B(w'_s - w_s, v - w_s) + f(w') - B(w'_s + w', w'_s) \\
 &= f(v + w') - B(\sigma(v), \sigma(w')) + B(v, w') - B(v, w_s) + B(w_s, v) - B(w'_s, v - w_s) - B(w_s, w_s) \\
 &\quad + B(w_s, v) - B(w'_s, w'_s) - B(w', (1 - \sigma)(v) + w_s) \\
 &= f(v + w') - B(w_s + v + w', w_s) - B(\sigma(v), \sigma(w')) + B(v, w') - B(w'_s, v - w_s + w'_s) \\
 &\quad - B(w', (1 - \sigma)(v)) \\
 &= f(v + w') - B(w_s + v + w', w_s) - B(\sigma(v), \sigma(w')) + B(v, w') + B((\sigma - 1)(w'), \sigma(v)) \\
 &\quad - B(w', (1 - \sigma)(v)) \\
 &= f(v + w') - B(w_s + v + w', w_s) - \langle \sigma(v), \sigma(w') \rangle + \langle v, w' \rangle \\
 &= f(v + w') - B(w_s + v + w', w_s) \quad \text{since } \sigma \in \text{Sp}(W).
 \end{aligned}$$

Hence

$$\text{Trace}(\tilde{\omega}_{\chi^\sharp}(\mathcal{I}(\tilde{s}))) = (-1)^k \gamma(\sigma) |Y| \chi(f(v + w') - B(w_s + v + w', w_s)) \mathbb{1}_{\text{I}(\sigma)}(w_s),$$

where $(\sigma - 1)(v + w') = w_s$. Therefore, by Proposition 3.11, we obtain $\text{Trace}(\tilde{\omega}_{\chi^\sharp}(\mathcal{I}(\tilde{s}))) = \Theta_\chi(\tilde{s})$. This proves that $\tilde{\omega}_{\tilde{\chi}}$ and $\tilde{\omega}_{\chi^\sharp} \circ \mathcal{I}$ are two irreducible representations with the same character, hence they are isomorphic. \square

We proved in Proposition 3.8.6 that the two-fold covering $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ of $\text{Ps}(B)$ we construct does not depend on the choices made for the extension $\tilde{\chi}$ of χ , $\{w_s$ satisfying (24), $s \in \text{Ps}(B)\}$ and $\{\gamma(\sigma)$ satisfying (32) and (33), $\sigma \in \text{O}(Q)\}$, provided (35) is satisfied. However, this

two-fold covering of $\text{Ps}(B)$ depends a priori on the choice of non-trivial character χ of \mathbb{F} that we fixed in the beginning of section 1. In fact, we can prove that different choices of characters lead to isomorphic extensions of $\text{Ps}(B)$, provided (35) is satisfied.

Proposition 3.15. Let ψ be a non-trivial character of \mathbb{F} . We suppose that a choice of w_s satisfying (24) for χ has been made for all $s \in \text{Ps}(B)$. We fix also $\gamma(\sigma)$ satisfying (32) for all $\sigma \in O(Q)$ and (33). Then:

1. There exists $a = b^2 \in \mathbb{F} \setminus \{0\}$ such that $\psi(t) = \chi(at)$ for all $t \in \mathbb{F}$.
2. For all $s = (\sigma, f) \in \text{Ps}(B)$, the element $b^{-1}w_s$ satisfies (24) for ψ , i.e.

$$\psi \circ f|_{\mathbb{K}(\sigma)} = \psi \circ \langle b^{-1}w_s, \cdot \rangle|_{\mathbb{K}(\sigma)}.$$

3. Let $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ be the two-fold covering of $\text{Ps}(B)$ obtained from the extension $\tilde{\chi}$ of χ to $X \times \mathbb{F}$ and the choices $\{w_s, s \in \text{Ps}(B)\}$ and $\{\gamma(\sigma), \sigma \in O(Q)\}$. Let $\widetilde{\text{Ps}(B)}'_{\tilde{\psi}}$ be the two-fold covering of $\text{Ps}(B)$ obtained from the extension $\tilde{\psi}$ of ψ to $X \times \mathbb{F}$ and the choices $\{b^{-1}w_s, s \in \text{Ps}(B)\}$ mentioned in 2, and $\{\gamma'(\sigma), \sigma \in O(Q)\}$. If (35) is satisfied then the extensions $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ and $\widetilde{\text{Ps}(B)}'_{\tilde{\psi}}$ are isomorphic.
4. Let $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ be the extension defined in 3. Let $\widetilde{\text{Ps}(B)}_{\tilde{\psi}}$ be the two-fold covering of $\text{Ps}(B)$ obtained from the extension $\tilde{\psi}$ of ψ to $X \times \mathbb{F}$ and the choices $\{v_s, s \in \text{Ps}(B)\}$ and $\{\delta(\sigma), \sigma \in O(Q)\}$ such that (24), (32) and (33) are satisfied. If (35) is satisfied then the extensions $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ and $\widetilde{\text{Ps}(B)}_{\tilde{\psi}}$ are isomorphic. An explicit isomorphism of central extensions is

$$\mathcal{J} : \widetilde{\text{Ps}(B)}_{\tilde{\chi}} \rightarrow \widetilde{\text{Ps}(B)}_{\tilde{\psi}} \quad (48)$$

$$(s, (-1)^k \omega_{\tilde{\chi}}(s, w_s, \gamma(\sigma))) \mapsto (s, (-1)^k \nu(s)^{-1} \gamma(\sigma) \delta(\sigma)^{-1} \omega_{\tilde{\psi}}(s, v_s, \delta(\sigma)))$$

where ν relates the choices $\{b^{-1}w_s, s \in \text{Ps}(B)\}$ and $\{v_s, s \in \text{Ps}(B)\}$ by (38), i.e.

$$\nu(s) = \psi(-f(v) + \langle v, b^{-1}w_s \rangle + B(v_s - b^{-1}w_s, v - b^{-1}w_s)), \quad (49)$$

where v is any element such that $v_s - b^{-1}w_s = (1 - \sigma)(v)$. This means that the following diagram is commutative:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\tilde{\Phi}} & \widetilde{\text{Ps}(B)}_{\tilde{\chi}} & \xrightarrow{\tilde{\Psi}} & \text{Ps}(B) \longrightarrow 1 \\ & & \downarrow \text{id} & & \downarrow \mathcal{J} & & \downarrow \text{id} \\ 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\hat{\Phi}} & \widetilde{\text{Ps}(B)}_{\tilde{\psi}} & \xrightarrow{\hat{\Psi}} & \text{Ps}(B) \longrightarrow 1 \end{array} \quad (D_3)$$

Proof. 1. This was proven in Lemma 2.1.

2. For every $w \in \mathbb{K}(\sigma)$, by 1 and the fact that f is quadratic,

$$\psi(f(w)) = \chi(af(w)) = \chi(f(bw)) = \chi(\langle w_s, bw \rangle) = \chi(a \langle b^{-1}w_s, w \rangle) = \psi(\langle b^{-1}w_s, w \rangle),$$

where the third equality comes from the fact that w_s satisfies (24) for χ . This proves that $b^{-1}w_s$ satisfies (24) for ψ .

- 3.** To show that the extensions $\widetilde{\text{Ps}}(B)_{\widetilde{\chi}}$ (associated to $\{w_s, s \in \text{Ps}(B)\}$ and $\{\gamma(\sigma), \sigma \in \text{O}(Q)\}$) and $\widetilde{\text{Ps}}(B)_{\widetilde{\psi}}$ (associated to $\{b^{-1}w_s, s \in \text{Ps}(B)\}$ and $\{\gamma'(\sigma), \sigma \in \text{O}(Q)\}$) are isomorphic, it is enough to show that the cocycles c_χ and c'_ψ associated to these extensions are cohomologous. Let $s_1 = (\sigma_1, f_1)$ and $s_2 = (\sigma_2, f_2)$ be in $\text{Ps}(B)$. Then, by (30) in Corollary 3.4,

$$c_\chi(s_1, s_2) = C_{1,2} \sum_{\substack{u_1 \in \text{I}(\sigma_1), u_2 \in \text{I}(\sigma_2) \\ u_1 + u_2 = w_{s_1, s_2}}} \chi(f_1(w_1) + \langle w_{s_1}, w_1 \rangle + B(u_1, w_1 + w_{s_1}) + f_2(w_2)) \\ \chi(\langle w_{s_2}, w_2 \rangle + B(u_2, w_2 + w_{s_2}) + B(u_1 + w_{s_1}, u_2 + w_{s_2}))$$

where $C_{1,2} = \gamma(\sigma_1)\gamma(\sigma_2)\gamma(\sigma_1\sigma_2)^{-1}$, $(1 - \sigma)(w_i) = u_i$ for $i \in \{1, 2\}$, and $w_{s_1, s_2} = w_{s_1 s_2} - w_{s_1} - w_{s_2}$. This implies that $b^{-1}w_{s_1, s_2} = b^{-1}w_{s_1 s_2} - b^{-1}w_{s_1} - b^{-1}w_{s_2}$. If we set $u'_i = b^{-1}u_i$ and $w'_i = b^{-1}w_i$, then $(1 - \sigma)(w'_i) = u'_i$ and

$$c_\chi(s_1, s_2) = C_{1,2} \sum_{\substack{u'_1 \in \text{I}(\sigma_1), u'_2 \in \text{I}(\sigma_2) \\ u'_1 + u'_2 = b^{-1}w_{s_1, s_2}}} \chi(f_1(bw'_1) + \langle w_{s_1}, bw'_1 \rangle + B(bu'_1, bw'_1 + w_{s_1}) + f_2(bw'_2)) \\ \chi(\langle w_{s_2}, bw'_2 \rangle + B(bu'_2, bw'_2 + w_{s_2}) + B(bu'_1 + w_{s_1}, bu'_2 + w_{s_2})) \\ = C_{1,2} \sum_{\substack{u'_1 \in \text{I}(\sigma_1), u'_2 \in \text{I}(\sigma_2) \\ u'_1 + u'_2 = b^{-1}w_{s_1, s_2}}} \chi(af_1(w'_1) + a\langle b^{-1}w_{s_1}, w'_1 \rangle + aB(u'_1, w'_1 + b^{-1}w_{s_1}) + af_2(w'_2)) \\ \chi(a\langle b^{-1}w_{s_2}, w'_2 \rangle + aB(u'_2, w'_2 + b^{-1}w_{s_2}) + aB(u'_1 + b^{-1}w_{s_1}, u'_2 + b^{-1}w_{s_2})) \\ = C_{1,2} \sum_{\substack{u'_1 \in \text{I}(\sigma_1), u'_2 \in \text{I}(\sigma_2) \\ u'_1 + u'_2 = b^{-1}w_{s_1, s_2}}} \psi(f_1(w'_1) + \langle b^{-1}w_{s_1}, w'_1 \rangle + B(u'_1, w'_1 + b^{-1}w_{s_1}) + f_2(w'_2)) \\ \psi(\langle b^{-1}w_{s_2}, w'_2 \rangle + B(u'_2, w'_2 + b^{-1}w_{s_2}) + B(u'_1 + b^{-1}w_{s_1}, u'_2 + b^{-1}w_{s_2})) \\ = \gamma(\sigma_1)\gamma'(\sigma_1)^{-1}\gamma(\sigma_2)\gamma'(\sigma_2)^{-1}\gamma(\sigma_1\sigma_2)^{-1}\gamma'(\sigma_1\sigma_2)c'_\psi(s_1, s_2),$$

i.e.

$$c_\chi(s_1, s_2)c'_\psi(s_1, s_2)^{-1} = \gamma(\sigma_1)\gamma'(\sigma_1)^{-1}\gamma(\sigma_2)\gamma'(\sigma_2)^{-1}\gamma(\sigma_1\sigma_2)^{-1}\gamma'(\sigma_1\sigma_2). \quad (50)$$

Since $\sigma \mapsto \gamma(\sigma)\gamma'(\sigma)^{-1}$ takes its values in $\{\pm 1\}$ by (35), $c_\chi c'_\psi^{-1}$ is a coboundary.

- 4.** Consider again the extension $\widetilde{\text{Ps}}(B)_{\widetilde{\chi}}$ defined in 3. Let $\{\delta(\sigma), \sigma \in \text{O}(Q)\}$ such that (32) and (33) are satisfied. Suppose also that (35) is satisfied. Let $\widetilde{\text{Ps}}(B)_{\widetilde{\psi}}''$ be the extension defined from $\widetilde{\psi}$, $\{b^{-1}w_s, s \in \text{Ps}(B)\}$ and $\{\delta(\sigma), \sigma \in \text{O}(Q)\}$. By 3, the extensions $\widetilde{\text{Ps}}(B)_{\widetilde{\chi}}$ and $\widetilde{\text{Ps}}(B)_{\widetilde{\psi}}''$ are isomorphic. Define now the extension $\widetilde{\text{Ps}}(B)_{\widetilde{\psi}}$ from the extension $\widetilde{\psi}$ of ψ and the choices $\{w_s, s \in \text{Ps}(B)\}$ and $\{\delta(\sigma), \sigma \in \text{O}(Q)\}$. Then the extensions $\widetilde{\text{Ps}}(B)_{\widetilde{\psi}}''$ and $\widetilde{\text{Ps}}(B)_{\widetilde{\psi}}$ are isomorphic by Proposition 3.8.3. This is enough to conclude that the extensions $\widetilde{\text{Ps}}(B)_{\widetilde{\chi}}$ and $\widetilde{\text{Ps}}(B)_{\widetilde{\psi}}$ are isomorphic. More precisely, define \mathcal{J} as in (48). We denote by c''_ψ and c_ψ the cocycles associated to $\widetilde{\text{Ps}}(B)_{\widetilde{\psi}}''$ and $\widetilde{\text{Ps}}(B)_{\widetilde{\psi}}$, respectively. Let $\widetilde{s}_1 = (s_1, (-1)^{k_1}\omega_{\widetilde{\chi}}(s_1, w_{s_1}, \gamma(\sigma_1)))$, $\widetilde{s}_2 =$

$$\begin{aligned}
& (s_2, (-1)^{k_2} \omega_{\tilde{\chi}}(s_2, w_{s_2}, \gamma(\sigma_2))) \in \widetilde{\text{Ps}(B)}_{\tilde{\chi}}. \text{ As in Proposition 3.8.6, we prove that} \\
\mathcal{J}(\tilde{s}_1) \mathcal{J}(\tilde{s}_2) &= (s_1 s_2, (-1)^{k_1+k_2} \nu(s_1)^{-1} \nu(s_2)^{-1} \gamma(\sigma_1) \delta(\sigma_1)^{-1} \gamma(\sigma_2) \delta(\sigma_2)^{-1} c_\psi(s_1, s_2) \\
&\quad \omega_{\tilde{\psi}}(s_1 s_2, v_{s_1 s_2}, \delta(\sigma_1 \sigma_2))) \\
\mathcal{J}(\tilde{s}_1 \tilde{s}_2) &= (s_1 s_2, (-1)^{k_1+k_2} c_\chi(s_1, s_2) \nu(s_1 s_2)^{-1} \gamma(\sigma_1 \sigma_2) \delta(\sigma_1 \sigma_2)^{-1} \omega_{\tilde{\psi}}(s_1 s_2, v_{s_1 s_2}, \delta(\sigma_1 \sigma_2))). \tag{51}
\end{aligned}$$

Hence \mathcal{J} is a group homomorphism if and only if

$$c_\psi(s_1, s_2) c_\chi(s_1, s_2)^{-1} = \nu(s_1) \nu(s_2) \nu(s_1 s_2)^{-1} \gamma(\sigma_1 \sigma_2) \delta(\sigma_1 \sigma_2)^{-1} \gamma(\sigma_1)^{-1} \delta(\sigma_1) \gamma(\sigma_2)^{-1} \delta(\sigma_2). \tag{52}$$

By (39),

$$c_\psi(s_1, s_2) c_\psi''(s_1, s_2)^{-1} = \nu(s_1) \nu(s_2) \nu(s_1 s_2)^{-1}. \tag{53}$$

By (50),

$$c_\psi''(s_1, s_2) c_\chi(s_1, s_2)^{-1} = \delta(\sigma_1) \gamma(\sigma_1)^{-1} \delta(\sigma_2) \gamma(\sigma_2)^{-1} (\delta(\sigma_1 \sigma_2) \gamma(\sigma_1 \sigma_2)^{-1})^{-1}. \tag{54}$$

Since $c_\psi(s_1, s_2) c_\chi(s_1, s_2)^{-1} = c_\psi(s_1, s_2) c_\psi''(s_1, s_2)^{-1} c_\psi''(s_1, s_2) c_\chi(s_1, s_2)^{-1}$, (53) and (54) imply (52). The fact that \mathcal{J} is an isomorphism such that the diagram (D_3) commutes can be proven exactly in the same way as for \mathcal{I} in Proposition 3.8.6. \square

Corollary 3.16. Let χ and ψ be two non-trivial characters of \mathbb{F} (such that $\psi(t) = \chi(at)$ for all $t \in \mathbb{F}$, where $a = b^2$, cf. Lemma 2.1). Let $\tilde{\chi}$ and $\tilde{\psi}$ be two extensions of χ and ψ to $X \times \mathbb{F}$, respectively. For every $\sigma \in \text{O}(Q)$, choose $\gamma(\sigma)$ and $\delta(\sigma)$ satisfying (32) with χ and ψ , respectively. For every $s \in \text{Ps}(B)$, choose w_s and v_s satisfying (24) with χ and ψ , respectively. Suppose that (33) and (35) are satisfied. Then the Weil representation $\tilde{\omega}_{\tilde{\chi}}$, obtained with $\tilde{\chi}$, $\{\gamma(\sigma) \text{ satisfying (32), } \sigma \in \text{O}(Q)\}$ and $\{w_s \text{ satisfying (24), } s \in \text{Ps}(B)\}$ is isomorphic to the Weil representation $\tilde{\omega}_{\tilde{\psi}} \circ \mathcal{J}$, obtained with $\tilde{\psi}$, $\{v_s \text{ satisfying (24), } s \in \text{Ps}(B)\}$ and $\{\delta(\sigma) \text{ satisfying (32), } \sigma \in \text{O}(Q)\}$. In other terms, there exists an isomorphism $U \in \text{U}(L^2(Y))$ such that the following diagram commutes:

$$\begin{array}{ccc}
\tilde{\omega}_{\tilde{\chi}}(\mathcal{J}(\tilde{s})) = U \circ \tilde{\omega}_{\tilde{\chi}}(\tilde{s}) \circ U^{-1} & \text{for all } \tilde{s} \in \widetilde{\text{Ps}(B)}_{\tilde{\chi}}, & \text{i.e.} \\
& & \begin{array}{ccc}
L^2(Y) & \xrightarrow{\tilde{\omega}_{\tilde{\chi}}(\tilde{s})} & L^2(Y) \\
\downarrow U & & \downarrow U \\
L^2(Y) & \xrightarrow{\tilde{\omega}_{\tilde{\psi}}(\mathcal{J}(\tilde{s}))} & L^2(Y)
\end{array} \tag{D_4}
\end{array}$$

where \mathcal{J} is defined by (48).

Proof. Since (35) is satisfied, by Proposition 3.15.4, the extensions $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ and $\widetilde{\text{Ps}(B)}_{\tilde{\psi}}$ associated to the choices $(\tilde{\chi}, \gamma(\sigma), w_s)$ and $(\tilde{\psi}, \delta(\sigma), v_s)$, respectively, are isomorphic. An explicit isomorphism \mathcal{J} is given by (48). Hence, \mathcal{J} is also a group isomorphism between $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ and $\widetilde{\text{Ps}(B)}_{\tilde{\psi}}$. Now, we denote by Θ_χ and Θ_ψ the characters of $\tilde{\omega}_{\tilde{\chi}}$ and $\tilde{\omega}_{\tilde{\psi}}$, respectively. Then by definition of \mathcal{J} , Θ_χ and Θ_ψ , for every $\tilde{s} = (s, (-1)^k \omega_{\tilde{\chi}}(s, w_s, \gamma(\sigma))) \in$

$\widetilde{\text{Ps}}(B)_{\widetilde{\chi}}$, we have that

$$\begin{aligned}
& \text{Trace}(\widetilde{\omega}_{\widetilde{\psi}}(\mathcal{J}(\widetilde{s}))) \\
&= \text{Trace}((-1)^k \nu(s)^{-1} \gamma(\sigma) \delta(\sigma)^{-1} \omega_{\widetilde{\psi}}(s, v_s, \delta(\sigma))) \\
&= (-1)^k \nu(s)^{-1} \gamma(\sigma) \delta(\sigma)^{-1} \Theta_{\psi}(s, \omega_{\widetilde{\psi}}(s, v_s, \delta(\sigma))) \\
&\stackrel{(47),(49)}{=} (-1)^k \gamma(\sigma) \delta(\sigma)^{-1} \psi(f(v) - \langle v, b^{-1}w_s \rangle - B(v_s - b^{-1}w_s, v - b^{-1}w_s)) \delta(\sigma) |Y| \\
&\quad \psi(f(w) - B(v_s + w, v_s)) \mathbb{1}_{I(\sigma)}(v_s) \\
&= (-1)^k \gamma(\sigma) |Y| \psi(f(v) - \langle v, b^{-1}w_s \rangle - B(v_s - b^{-1}w_s, v - b^{-1}w_s) + f(w) - B(v_s + w, v_s)) \\
&\quad \mathbb{1}_{I(\sigma)}(v_s),
\end{aligned}$$

where v is any element such that $v_s - b^{-1}w_s = (1 - \sigma)(v)$ and $w \in W$ is any element such that $(\sigma - 1)(w) = v_s$. As in Corollary 3.14, we can prove that

$$\begin{aligned}
& f(v) - \langle v, b^{-1}w_s \rangle - B(v_s - b^{-1}w_s, v - b^{-1}w_s) + f(w) - B(v_s + w, v_s) \\
&= f(v + w) - B(b^{-1}w_s + v + w, b^{-1}w_s).
\end{aligned}$$

Hence

$$\begin{aligned}
\text{Trace}(\widetilde{\omega}_{\widetilde{\psi}}(\mathcal{J}(\widetilde{s}))) &= (-1)^k \gamma(\sigma) |Y| \psi(f(v + w) - B(b^{-1}w_s + v + w, b^{-1}w_s)) \mathbb{1}_{I(\sigma)}(w_s) \\
&= (-1)^k \gamma(\sigma) |Y| \chi(af(v + w) - aB(b^{-1}w_s + v + w, b^{-1}w_s)) \mathbb{1}_{I(\sigma)}(w_s) \\
&= (-1)^k \gamma(\sigma) |Y| \chi(f(bv + bw) - B(w_s + bv + bw, w_s)) \mathbb{1}_{I(\sigma)}(w_s),
\end{aligned}$$

where $(\sigma - 1)(bv + bw) = w_s - bv_s + bv_s = w_s$. Therefore, by Proposition 3.11, we obtain $\text{Trace}(\widetilde{\omega}_{\widetilde{\psi}}(\mathcal{J}(\widetilde{s}))) = \Theta_{\chi}(\widetilde{s})$. This proves that $\widetilde{\omega}_{\widetilde{\chi}}$ and $\widetilde{\omega}_{\widetilde{\psi}} \circ \mathcal{J}$ are two irreducible representations with the same character, hence they are isomorphic. \square

4. THE TWO-DIMENSIONAL CASE FOR $\mathbb{F} = \mathbb{F}_2$, I.E. $W \simeq \mathbb{F}_2^2$

In this section, we apply the results of the previous three sections in order to describe explicitly the groups involved and the Weil representation when W is a symplectic \mathbb{F}_2 -vector space of dimension two. Most of the groups are found explicitly thanks to the database ‘‘GroupNames’’ [29]. We keep the notations used in the preceding sections and we use them with $n = 1$. In particular, W is a \mathbb{F}_2 -vector space of dimension 2, endowed with a non-degenerate and alternating form $\langle \cdot, \cdot \rangle$. Then W is isomorphic to \mathbb{F}_2^2 , and we can set $W = X \oplus Y$ with $X = \text{Vect}\{(1, 0)\}$ and $Y = \text{Vect}\{(0, 1)\}$. This means that $W = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$, where $0, 1 \in \mathbb{F}_2$. We will write an element $w \in W$ either as $w = x + y$ with $x \in X$ and $y \in Y$, or as $w = (x, y)$. With respect to the fixed complete polarization $W = X \oplus Y$, the non-degenerate symplectic form is given by

$$\langle x_1 + y_1, x_2 + y_2 \rangle = x_1 y_2 - y_1 x_2$$

for all $w_i = x_i + y_i \in W$, i.e. its matrix is $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since W is a two-dimensional vector space over \mathbb{F}_2 , the space Y is one-dimensional and hence $\dim_{\mathbb{C}}(L^2(Y)) = 2$. Thus

all the operators we obtain in this example are two-by-two matrices.

The first step is to determine the Heisenberg groups $H(B)$ in this example, which amounts to determine all the possible bilinear forms B such that for every $w_i = x_i + y_i \in W$, $i \in \{1, 2\}$,

$$\langle x_1 + y_1, x_2 + y_2 \rangle = B(w_1, w_2) - B(w_2, w_1).$$

In addition, since $H(B)$ is an extraspecial 2-group (see Proposition 1.5 and appendix), it is enough by Proposition A.11 to find two bilinear forms B whose associated quadratic forms Q have different Witt index/Arf invariant.

In terms of matrices, we are looking for $B = \begin{pmatrix} \beta_{1,1} & \beta_{1,2} \\ \beta_{2,1} & \beta_{2,2} \end{pmatrix}$ such that for every $x_1 + y_1 \in W$ and $x_2 + y_2 \in W$,

$$(x_1 \ y_1) B \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - (x_2 \ y_2) B \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = (x_1 \ y_1) \Omega \begin{pmatrix} x_2 \\ y_2 \end{pmatrix},$$

i.e. $\beta_{1,2}(x_1 y_2 - y_1 x_2) - \beta_{2,1}(x_1 y_2 - y_1 x_2) = x_1 y_2 - y_1 x_2$. Thus $\beta_{1,2} - \beta_{2,1} = 1$. This leads to eight possibilities for B :

$$B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, B_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$B_5 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B_6 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, B_7 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, B_8 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

These bilinear forms are given in coordinates by:

- $B_1(w_1, w_2) = x_1 y_2$, i.e. $B_1 = B_{\text{Weil}}$
- $B_2(w_1, w_2) = x_1 y_2 + x_1 x_2$
- $B_3(w_1, w_2) = x_1 y_2 + y_1 y_2$
- $B_4(w_1, w_2) = x_1 y_2 + x_1 x_2 + y_1 y_2$
- $B_5(w_1, w_2) = y_1 x_2$
- $B_6(w_1, w_2) = y_1 x_2 + x_1 x_2$
- $B_7(w_1, w_2) = y_1 x_2 + y_1 y_2$
- $B_8(w_1, w_2) = y_1 x_2 + x_1 x_2 + y_1 y_2$.

Thus the corresponding quadratic forms are written as follows:

- $Q_1(w) = xy$
- $Q_2(w) = xy + x^2$
- $Q_3(w) = xy + y^2$
- $Q_4(w) = xy + x^2 + y^2$
- $Q_5(w) = xy$
- $Q_6(w) = xy + x^2$
- $Q_7(w) = xy + y^2$
- $Q_8(w) = xy + x^2 + y^2$.

In particular, the quadratic forms of Witt index 1 (Arf invariant 0) are $Q_1 = Q_5$, $Q_2 = Q_6$ and $Q_3 = Q_7$, whereas the Witt index of $Q_4 = Q_8$ is 0 (Arf invariant 1). Therefore, to study all the possibilities for the Heisenberg groups, it is enough to consider $B_1 = B_{\text{Weil}}$ and B_4 . Before separating the two cases depending on the index of Q , we can give some details about the additive maps and the symplectic group involved in this example. We will identify any linear map $\sigma : W \rightarrow W$ with its matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in the

basis $\mathcal{B} = \{(1, 0), (0, 1)\}$ of W and, similarly, any linear map $T : L^2(Y) \rightarrow L^2(Y)$ with its matrix in the basis $\mathcal{B}_Y = \{\delta_0, \delta_1\}$ of $L^2(Y)$.

Proposition 4.1. **1.** The set $\mathcal{Q}_a(W)$ of additive quadratic forms on W is given by the set of linear forms $(x, y) \mapsto 0$, $(x, y) \mapsto x$, $(x, y) \mapsto y$ and $(x, y) \mapsto x + y$.
2. The symplectic group of W is

$$\mathrm{Sp}(W) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

i.e. $\mathrm{Sp}(W) = \mathrm{GL}_2(\mathbb{F}_2) = \mathrm{SL}_2(\mathbb{F}_2)$, and in particular

$$\mathrm{Sp}(W) \simeq S_3 = \langle a, b \mid a^3 = b^2 = 1, bab = a^{-1} \rangle,$$

where for example $a = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Proof. **1.** The additive quadratic forms on W with values in \mathbb{F}_2 are just the linear maps $f : W \rightarrow \mathbb{F}_2$. Such a map f is entirely determined by $f(1, 0)$ and $f(0, 1)$. There are two possibilities for each of them, so this gives four possibilities for f : $f(x, y) = xf(1, 0) + yf(0, 1)$.

2. $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is symplectic if and only if $\sigma^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which is equivalent to $\alpha\delta - \gamma\beta = 1$, i.e. $\sigma \in \mathrm{GL}_2(\mathbb{F}_2) = \mathrm{SL}_2(\mathbb{F}_2)$. □

4.1. The case $\mathrm{Arf}(Q) = 0$. In this part, we suppose that the bilinear form B is such that Q has Arf invariant 0, or equivalently (Witt) index 1. In particular, we can consider the bilinear form $B = B_{\mathrm{Weil}}$ which appears in Weil's article, i.e. $B(w_1, w_2) = x_1y_2$ for all $w_i = x_i + y_i \in W$, $i \in \{1, 2\}$. Hence the matrix of B in the basis $\{(1, 0), (0, 1)\}$ of W is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $Q(w) = xy$ for all $w = x + y \in W$. In this case, $\mathrm{H}(B)$ is the set of elements $(x, y, t) \in \mathbb{F}_2^3$ with product

$$(x_1, y_1, t_1)(x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + x_1y_2).$$

In particular, $\mathrm{H}(B)$ is isomorphic to the dihedral group $D_4 = \langle a, b \mid a^4 = b^2 = 1, bab = a^{-1} \rangle$, where for example $a = (1, 1, 0)$ and $b = (1, 0, 0)$.

We need to fix an extension $\tilde{\chi}$ of the unique non-trivial character χ of \mathbb{F}_2 to $X \times \mathbb{F}_2$. This extension is entirely determined by $\tilde{\chi}(1, 0, 0)$, which can take the values ± 1 because X is Q -singular (cf. Lemma 2.1). Set

$$\tilde{\varepsilon} = \tilde{\chi}(1, 0, 0) \in \{\pm 1\}. \tag{55}$$

The first groups we compute explicitly are the orthogonal group $\mathrm{O}(Q)$ and the pseudo-symplectic group $\mathrm{Ps}(B)$. Then, in order to obtain all the possible two-fold coverings defined in Proposition 3.8.2

$$1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\tilde{\Phi}} \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \xrightarrow{\tilde{\Psi}} \mathrm{Ps}(B) \longrightarrow 1,$$

we consider all the possible choices of $\{\gamma(\sigma)$ satisfying (32) and (33), $\sigma \in O(Q)\}$. We will see in (56) that, up to isomorphism, there are only two different coverings.

Proposition 4.2. **1.** The orthogonal group is $O(Q) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$. If we

set $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then $O(Q) = \{1, \tau\} = \langle \tau \rangle$.

2. The pseudo-symplectic group is

$$\text{Ps}(B) = \left\{ \begin{array}{l} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right), \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, x \right), \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, y \right), \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, x+y \right), \\ \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, xy \right), \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, xy+x \right), \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, xy+y \right), \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, xy+x+y \right) \end{array} \right\}.$$

We will denote these elements in the same order by $\{s_1, s_2, s_3, s_4, r_1, r_2, r_3, r_4\}$. In particular, $\text{Ps}(B)$ is isomorphic to the dihedral group of order eight $D_4 = \langle a, b \mid a^4 = b^2 = 1, bab = a^{-1} \rangle$, an isomorphism being for example given by $a = r_2$ and $b = s_2$.

3. The different choices of $\{\gamma(\sigma) = 2^{-\frac{\dim I(\sigma)}{2}} e^{i\theta(\sigma)}, \sigma \in O(Q)\}$ satisfying (32) and (33) lead to two non-isomorphic pseudo-symplectic extensions

$$1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\tilde{\Phi}} \widetilde{\text{Ps}(B)}_{\tilde{\chi}} \xrightarrow{\tilde{\Psi}} \text{Ps}(B) \longrightarrow 1.$$

More precisely,

- If $\theta(\tau) \equiv 0 \pmod{\pi}$ then the pseudo-symplectic covering

$$\widetilde{\text{Ps}(B)}_{\tilde{\chi}} = \{(s, (-1)^l \omega_{\tilde{\chi}}(s)), s \in \text{Ps}(B), l \in \mathbb{Z}\}$$

is given by

$$\widetilde{\text{Ps}(B)}_{\tilde{\chi}} = \left\{ \begin{array}{l} \left(s_1, (-1)^l \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \left(s_2, (-1)^l \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), \left(s_3, (-1)^l \tilde{\varepsilon} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right), \\ \left(s_4, (-1)^l \tilde{\varepsilon} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right), \left(r_1, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} \tilde{\varepsilon} & 1 \\ 1 & -\tilde{\varepsilon} \end{pmatrix} \right), \left(r_2, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 1 & -\tilde{\varepsilon} \\ \tilde{\varepsilon} & 1 \end{pmatrix} \right), \\ \left(r_3, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 1 & \tilde{\varepsilon} \\ -\tilde{\varepsilon} & 1 \end{pmatrix} \right), \left(r_4, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} \tilde{\varepsilon} & -1 \\ -1 & -\tilde{\varepsilon} \end{pmatrix} \right) \end{array} \right\}$$

where $\tilde{\varepsilon} = \tilde{\chi}(1, 0, 0)$ is as in (55). In particular, $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ is isomorphic to the dihedral group $D_8 = \langle a, b \mid a^8 = b^2 = 1, bab = a^{-1} \rangle$, an isomorphism being given for example by $a = \left(r_2, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\tilde{\varepsilon} \\ \tilde{\varepsilon} & 1 \end{pmatrix} \right)$ and $b = \left(s_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$.

- If $\theta(\tau) \equiv \frac{\pi}{2} \pmod{\pi}$ then the pseudo-symplectic covering

$$\widetilde{\text{Ps}(B)}_{\tilde{\chi}} = \{(s, (-1)^l \omega_{\tilde{\chi}}(s)), s \in \text{Ps}(B), l \in \mathbb{Z}\}$$

is given by

$$\widetilde{\text{Ps}}(B)_{\tilde{\chi}} = \left\{ \begin{array}{l} \left(s_1, (-1)^l \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \left(s_2, (-1)^l \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), \left(s_3, (-1)^l \tilde{\varepsilon} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right), \\ \left(s_4, (-1)^l \tilde{\varepsilon} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right), \left(r_1, \frac{(-1)^l i}{\sqrt{2}} \begin{pmatrix} \tilde{\varepsilon} & 1 \\ 1 & -\tilde{\varepsilon} \end{pmatrix} \right), \left(r_2, \frac{(-1)^l i}{\sqrt{2}} \begin{pmatrix} 1 & -\tilde{\varepsilon} \\ \tilde{\varepsilon} & 1 \end{pmatrix} \right), \\ \left(r_3, \frac{(-1)^l i}{\sqrt{2}} \begin{pmatrix} 1 & \tilde{\varepsilon} \\ -\tilde{\varepsilon} & 1 \end{pmatrix} \right), \left(r_4, \frac{(-1)^l i}{\sqrt{2}} \begin{pmatrix} \tilde{\varepsilon} & -1 \\ -1 & -\tilde{\varepsilon} \end{pmatrix} \right) \end{array} \right\}$$

where $\tilde{\varepsilon} = \tilde{\chi}(1, 0, 0)$ is as in (55). In particular, $\widetilde{\text{Ps}}(B)_{\tilde{\chi}}$ is isomorphic to the semi-dihedral group $\text{SD}_{16} = \langle a, b \mid a^8 = b^2 = 1, bab = a^3 \rangle$, an isomorphism being given for example by $a = \left(r_2, \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & -\tilde{\varepsilon} \\ \tilde{\varepsilon} & 1 \end{pmatrix} \right)$ and $b = \left(s_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$.

Proof. 1. σ is orthogonal if and only if $(x \ y) \sigma^T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \sigma \begin{pmatrix} x \\ y \end{pmatrix} = (x \ y) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

for all $x, y \in \mathbb{F}_2$.

2. According to Proposition 1.11, the elements of $\text{Ps}(B)$ are given by $(\sigma, f_\sigma + f)$, where $\sigma \in O(Q)$ and $f \in \mathcal{Q}_a(W)$. This gives $f_1 = 0$ and $f_\tau(x, y) = xy$. Proposition 4.1 gives all the eight elements of $\text{Ps}(B)$ listed above.
3. First we make a choice of w_s satisfying (24) for all $s = (\sigma, f) \in \text{Ps}(B)$. If $s = s_i$ then $\sigma = 1$ and $K(\sigma) = W$. Hence there is only one choice of w_s satisfying $f|_{K(\sigma)} = f = \langle w_s, \cdot \rangle$, namely $w_{s_1} = 0$, $w_{s_2} = (0, 1)$, $w_{s_3} = (1, 0)$ and $w_{s_4} = (1, 1)$. If $s = r_i$ then $\sigma = \tau$ and $K(\sigma) = \{0, (1, 1)\}$. Hence there are two choices of w_s for each s , namely $w_{r_1} \in \{(1, 0), (0, 1)\}$, $w_{r_2} \in \{0, (1, 1)\}$, $w_{r_3} \in \{0, (1, 1)\}$ and $w_{r_4} \in \{(1, 0), (0, 1)\}$. Since $\text{Ps}(B)_{\tilde{\chi}}$ does not depend of the choice of $\{w_s, s \in \text{Ps}(B)\}$, we can choose for example $w_{r_1} = w_{r_4} = (1, 0)$ and $w_{r_2} = w_{r_3} = 0$. This enables us to compute the operators $\omega_{\tilde{\chi}}(s)$ for all $s \in \text{Ps}(B)$ thanks to Proposition 3.2:

$$\omega_{\tilde{\chi}}(s) = \gamma(\sigma) \sum_{u \in I(\sigma)} \chi(f(w) + \langle w_s, w \rangle + B(u, w + w_s)) \rho_{\tilde{\chi}}(u + w_s),$$

where $w \in W$ is any element such that $u = (1 - \sigma)(w)$ and $\gamma(\sigma)$ satisfies (32) and (33). This formula together with (15) with $t = 0$ gives:

- $\omega_{\tilde{\chi}}(s_1) = \rho_{\tilde{\chi}}(w_{s_1}) = \rho_{\tilde{\chi}}(0) = I_2$
- $\omega_{\tilde{\chi}}(s_2) = \rho_{\tilde{\chi}}(w_{s_2}) = \rho_{\tilde{\chi}}(0, 1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $\omega_{\tilde{\chi}}(s_3) = \rho_{\tilde{\chi}}(w_{s_3}) = \rho_{\tilde{\chi}}(1, 0) = \begin{pmatrix} \tilde{\varepsilon} & 0 \\ 0 & -\tilde{\varepsilon} \end{pmatrix}$
- $\omega_{\tilde{\chi}}(s_4) = \rho_{\tilde{\chi}}(w_{s_4}) = \rho_{\tilde{\chi}}(1, 1) = \begin{pmatrix} 0 & -\tilde{\varepsilon} \\ \tilde{\varepsilon} & 0 \end{pmatrix}$
- $\omega_{\tilde{\chi}}(r_1) = \frac{e^{i\theta(\tau)}}{\sqrt{2}} (\rho_{\tilde{\chi}}(1, 0) + \rho_{\tilde{\chi}}(0, 1)) = \frac{e^{i\theta(\tau)}}{\sqrt{2}} \begin{pmatrix} \tilde{\varepsilon} & 1 \\ 1 & -\tilde{\varepsilon} \end{pmatrix}$

$$\begin{aligned}
\bullet \omega_{\tilde{\chi}}(r_2) &= \frac{e^{i\theta(\tau)}}{\sqrt{2}}(\rho_{\tilde{\chi}}(0) + \rho_{\tilde{\chi}}(1, 1)) = \frac{e^{i\theta(\tau)}}{\sqrt{2}} \begin{pmatrix} 1 & -\tilde{\varepsilon} \\ \tilde{\varepsilon} & 1 \end{pmatrix} \\
\bullet \omega_{\tilde{\chi}}(r_3) &= \frac{e^{i\theta(\tau)}}{\sqrt{2}}(\rho_{\tilde{\chi}}(0) - \rho_{\tilde{\chi}}(1, 1)) = \frac{e^{i\theta(\tau)}}{\sqrt{2}} \begin{pmatrix} 1 & \tilde{\varepsilon} \\ -\tilde{\varepsilon} & 1 \end{pmatrix} \\
\bullet \omega_{\tilde{\chi}}(r_4) &= \frac{e^{i\theta(\tau)}}{\sqrt{2}}(\rho_{\tilde{\chi}}(1, 0) - \rho_{\tilde{\chi}}(0, 1)) = \frac{e^{i\theta(\tau)}}{\sqrt{2}} \begin{pmatrix} \tilde{\varepsilon} & -1 \\ -1 & -\tilde{\varepsilon} \end{pmatrix}.
\end{aligned}$$

Moreover, since $\tau^2 = 1$, (33) implies that

$$\theta(\tau) + \theta(\tau) - \theta(1) \equiv 0 \pmod{\pi}.$$

Since $\theta(1) \equiv 0 \pmod{2\pi}$ by (32), we obtain that $\theta(\tau) \equiv 0 \pmod{\frac{\pi}{2}}$. Hence there are four possibilities for $e^{i\theta(\tau)}$, but these choices lead to at most two non-isomorphic extensions by Proposition 3.8.5, corresponding to the choices $\theta(\tau) \equiv 0 \pmod{\pi}$ and $\theta(\tau) \equiv \frac{\pi}{2} \pmod{\pi}$, respectively. Then Proposition 3.8 leads to the list of all elements of $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ given in the statement above. Finally, we want to identify the two groups $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ defined in this way in the database [29]. We know by 2 that $\text{Ps}(B)$ is isomorphic to D_4 and we know by Proposition 3.8.2 that $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ is a non-split stem extension of $\text{Ps}(B)$ by two elements. According to [29], there are only three possibilities for $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$, namely the groups D_8 , SD_{16} and Q_{16} . These groups have different numbers of elements of order two. Since we have already computed all the elements of the groups $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$, we can easily obtain the orders of these elements and then deduce the following group isomorphisms:

$$\widetilde{\text{Ps}(B)}_{\tilde{\chi}} \simeq \begin{cases} D_8 & \text{if } \theta(\tau) \equiv 0 \pmod{\pi} \\ SD_{16} & \text{if } \theta(\tau) \equiv \frac{\pi}{2} \pmod{\pi}. \end{cases} \quad (56)$$

□

Remark 4.3 (related to Remark 3.9). The result obtained in (56) proves that in general, we cannot obtain all the non-split stem extensions of $\text{Ps}(B)$ by two elements with all possible choices of $\{\gamma(\sigma), \sigma \in O(Q)\}$ satisfying (32) and (33). In fact, the number of central extensions of $\text{Ps}(B) \simeq D_4$ by two elements can be found with the universal coefficient theorem, which provides the following short split exact sequence:

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_1(D_4, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \rightarrow H^2(D_4, \mathbb{Z}/2\mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(H_2(D_4, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \rightarrow 0.$$

The Schur multiplier of D_4 is $H_2(D_4, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ (see [17, p. 278]). Moreover, $H_1(D_4, \mathbb{Z}) \simeq D_4/[D_4, D_4] \simeq (\mathbb{Z}/2\mathbb{Z})^2$, and thus $\text{Ext}_{\mathbb{Z}}^1(H_1(D_4, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = \text{Ext}_{\mathbb{Z}}^1((\mathbb{Z}/2\mathbb{Z})^2, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^2$. Hence $|H^2(D_4, \mathbb{Z}/2\mathbb{Z})| = 8$, which means that there are eight non-isomorphic extensions of $\text{Ps}(B) \simeq D_4$ by two elements. The database [29] gives six possible groups associated to these extensions, three of them being associated to non-split stem extensions: D_8 , SD_{16} and Q_{16} . However, by (56), there is no choice of $\{\gamma(\sigma), \sigma \in O(Q)\}$ which enables us to obtain a group $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ which is isomorphic to the generalised quaternionic

group Q_{16} . Nevertheless, $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ contains a copy of the Heisenberg group $H(B)$, which is isomorphic to D_4 in this example, and Q_{16} does not contain D_4 . Therefore, we obtain in fact all the possible groups $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ associated to non-split stem extensions of $\text{Ps}(B)$ by two elements which contain the Heisenberg group $H(B)$. Moreover, (56) shows also that 3.8.5 is an equivalence in this case.

Remark 4.4. We give a short proof of the fact that the extension defined by the pseudo-symplectic group splits when $\dim(W) = 2$ and $\mathbb{F} = \mathbb{F}_2$; see Proposition 1.10. Recall that $\text{Ps}(B)$ defines a short exact sequence:

$$0 \rightarrow \mathcal{Q}_a(W) \xrightarrow{\Phi} \text{Ps}(B) \xrightarrow{\Psi} \text{O}(Q) \rightarrow 1$$

where $\Phi : f_z = \langle z, \cdot \rangle \mapsto (1, f_z)$ and $\Psi : (\sigma, f) \mapsto \sigma$. Recall also that we defined a section $\sigma \mapsto f_\sigma$ of $\tilde{\Psi}$ in Proposition 1.11. If we consider the basis $\mathcal{B} = \{(1, 0), (0, 1)\}$ of W , then for every $\sigma \in \text{O}(Q)$ and $(x, y) \in W$,

$$f_\sigma(x, y) = \begin{cases} 0 & \text{if } \sigma = 1 \\ xy & \text{if } \sigma = \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{cases}$$

Hence

$$\begin{aligned} (f_1 + f_1 \circ 1)(x, y) &= 0 = f_{1,1} \\ (f_1 + f_\tau \circ 1)(x, y) &= xy = f_{\tau,1} \\ (f_\tau + f_1 \circ \sigma_1)(x, y) &= xy = f_{1,\tau} \\ (f_\tau + f_\tau \circ \tau)(x, y) &= xy + yx = 0 = f_{\tau,2} = f_1. \end{aligned}$$

This shows that $\sigma \mapsto f_\sigma$ is a group homomorphism, which implies that the extension splits. This proof can also be easily adapted to the case of a \mathbb{F} -vector space of dimension 2, where \mathbb{F} is an arbitrary finite field of characteristic two.

4.2. The case $\text{Arf}(Q) = 1$. In this part, we suppose that the bilinear form B is such that Q has Arf invariant 1, or equivalently (Witt) index 0. In particular, we can consider the bilinear form B given by $B(w_1, w_2) = x_1y_2 + x_1x_2 + y_1y_2$ for all $w_i = x_i + y_i \in W, i \in \{1, 2\}$.

Thus the matrix of B in the basis $\{(1, 0), (0, 1)\}$ of W is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $Q(w) = xy + x^2 + y^2$ for all $w = x + y \in W$. In this case, $H(B)$ is the set of elements $(x, y, t) \in \mathbb{F}_2^3$ with product

$$(x_1, y_1, t_1)(x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + x_1y_2 + x_1x_2 + y_1y_2).$$

In particular, $H(B)$ is isomorphic to the quaternionic group

$$Q_8 = \langle a, b \mid a^4 = 1, b^2 = a^2, bab^{-1} = a^{-1} \rangle,$$

where for example $a = (1, 0, 0)$ and $b = (0, 1, 0)$.

We need to fix an extension $\tilde{\chi}$ of the unique non-trivial character χ of \mathbb{F}_2 to $X \times \mathbb{F}_2$. This extension is entirely determined by $\tilde{\chi}(1, 0, 0)$, which can take the values $\pm i$ since

$B((1, 0), (1, 0)) = 1$ (cf. Lemma 2.1). Set

$$\tilde{\xi} = \tilde{\chi}(1, 0, 0) \in \{\pm i\}. \quad (57)$$

As in the previous subsection, we compute first the groups $O(Q)$ and $\text{Ps}(B)$. Then we consider all the possible choices of $\{\gamma(\sigma)$ satisfying (32) and (33), $\sigma \in O(Q)\}$ in order to obtain all the possible two-fold coverings

$$1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\tilde{\Phi}} \widetilde{\text{Ps}(B)}_{\tilde{\chi}} \xrightarrow{\tilde{\Psi}} \text{Ps}(B) \longrightarrow 1$$

defined in Proposition 3.8.2. As in Proposition 4.2, we will find in (62) two different coverings up to isomorphism.

Let $v \in W$. We denote by σ_v the transvection in $\text{Sp}(W)$ defined by $\sigma_v(w) = w + \langle w, v \rangle v$, for every $w \in W$.

Proposition 4.5. **1.** The orthogonal group is $O(Q) = \text{GL}_2(\mathbb{F}_2) = \text{SL}_2(\mathbb{F}_2)$, i.e.

$$O(Q) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

In particular, this is the only case for which $O(Q) = \text{Sp}(W)$ (cf Proposition 1.12).

2. The pseudo-symplectic group is

$$\text{Ps}(B) = \left\{ \begin{array}{l} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right), \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, x \right), \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, y \right), \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, x + y \right), \\ \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, xy \right), \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, xy + x \right), \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, xy + y \right), \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, xy + x + y \right), \\ \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, 0 \right), \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, x \right), \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, y \right), \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, x + y \right), \\ \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, xy \right), \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, xy + x \right), \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, xy + y \right), \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, xy + x + y \right), \\ \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, xy \right), \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, xy + x \right), \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, xy + y \right), \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, xy + x + y \right), \\ \left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, 0 \right), \left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, x \right), \left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, y \right), \left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, x + y \right) \end{array} \right\}.$$

In particular, $\text{Ps}(B)$ is isomorphic to the symmetric group

$$S_4 = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^2 = 1, cac^{-1} = dad = ab = ba, abc^{-1} = a, bd = db, dcd = c^{-1} \rangle,$$

an isomorphism being for example given by $a = (1, x)$, $b = (1, y)$, $c = \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, 0 \right)$

and $d = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, xy + x + y \right)$.

3. The different choices of $\{\gamma(\sigma) = 2^{-\frac{\dim I(\sigma)}{2}} e^{i\theta(\sigma)}, \sigma \in O(Q)\}$ satisfying (32) and (33) lead to two non-isomorphic pseudo-symplectic extensions

$$1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\tilde{\Psi}} \widetilde{\text{Ps}(B)}_{\tilde{\chi}} \xrightarrow{\tilde{\Psi}} \text{Ps}(B) \longrightarrow 1.$$

More precisely,

- If $\theta(\sigma_{(1,0)}) \equiv 0 \pmod{\pi}$ then the pseudo-symplectic covering is

$$\widetilde{\text{Ps}(B)}_{\tilde{\chi}} = \{(s, (-1)^l \omega_{\tilde{\chi}}(s)), s \in \text{Ps}(B), l \in \mathbb{Z}\},$$

where the set of operators $(-1)^l \omega_{\tilde{\chi}}(s)$ is given by (in the same order as the elements of $\text{Ps}(B)$ above, and where $\tilde{\xi}$ is as in (57)):

$$\left\{ \begin{array}{l} (-1)^l \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (-1)^l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, (-1)^l \begin{pmatrix} \tilde{\xi} & 0 \\ 0 & -\tilde{\xi} \end{pmatrix}, (-1)^l \begin{pmatrix} 0 & \tilde{\xi} \\ \tilde{\xi} & 0 \end{pmatrix}, \\ \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} \tilde{\xi} & -1 \\ 1 & -\tilde{\xi} \end{pmatrix}, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 1 & \tilde{\xi} \\ \tilde{\xi} & 1 \end{pmatrix}, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 1 & -\tilde{\xi} \\ -\tilde{\xi} & 1 \end{pmatrix}, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} \tilde{\xi} & 1 \\ -1 & -\tilde{\xi} \end{pmatrix}, \\ \frac{(-1)^l}{2} \begin{pmatrix} 1 - \tilde{\xi} & 1 - \tilde{\xi} \\ -1 - \tilde{\xi} & 1 + \tilde{\xi} \end{pmatrix}, \frac{(-1)^l}{2} \begin{pmatrix} 1 - \tilde{\xi} & -1 + \tilde{\xi} \\ 1 + \tilde{\xi} & 1 + \tilde{\xi} \end{pmatrix}, \frac{(-1)^l}{2} \begin{pmatrix} 1 + \tilde{\xi} & -1 - \tilde{\xi} \\ 1 - \tilde{\xi} & 1 - \tilde{\xi} \end{pmatrix}, \frac{(-1)^l}{2} \begin{pmatrix} 1 + \tilde{\xi} & 1 + \tilde{\xi} \\ -1 + \tilde{\xi} & 1 - \tilde{\xi} \end{pmatrix}, \\ \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \frac{(-1)^l \tilde{\xi}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \frac{(-1)^l \tilde{\xi}}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \\ \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 1 - \tilde{\xi} & 0 \\ 0 & 1 + \tilde{\xi} \end{pmatrix}, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 0 & -1 - \tilde{\xi} \\ 1 - \tilde{\xi} & 0 \end{pmatrix}, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 1 + \tilde{\xi} & 0 \\ 0 & 1 - \tilde{\xi} \end{pmatrix}, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 0 & -1 + \tilde{\xi} \\ 1 + \tilde{\xi} & 0 \end{pmatrix}, \\ \frac{(-1)^l}{2} \begin{pmatrix} 1 + \tilde{\xi} & -1 + \tilde{\xi} \\ 1 + \tilde{\xi} & 1 - \tilde{\xi} \end{pmatrix}, \frac{(-1)^l}{2} \begin{pmatrix} 1 - \tilde{\xi} & 1 + \tilde{\xi} \\ -1 + \tilde{\xi} & 1 + \tilde{\xi} \end{pmatrix}, \frac{(-1)^l}{2} \begin{pmatrix} 1 - \tilde{\xi} & -1 - \tilde{\xi} \\ 1 - \tilde{\xi} & 1 + \tilde{\xi} \end{pmatrix}, \frac{(-1)^l}{2} \begin{pmatrix} 1 + \tilde{\xi} & 1 - \tilde{\xi} \\ -1 - \tilde{\xi} & 1 - \tilde{\xi} \end{pmatrix} \end{array} \right\}.$$

In particular, $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ is isomorphic to the conformal special unitary group

$$\text{CSU}_2(\mathbb{F}_3) = \langle a, b, c, d \mid a^4 = c^3 = 1, b^2 = d^2 = a^2, bab^{-1} = dbd^{-1} = a^{-1}, cac^{-1} = ab, \\ dad^{-1} = a^2b, cbc^{-1} = a, dcd^{-1} = c^{-1} \rangle,$$

an isomorphism being given for example by

$$a = \left((1, x), \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right), \quad b = \left((1, y), \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right), \\ c = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, (x+y)\delta_{-i}(\tilde{\xi}) \right), -\frac{1}{2} \begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix} \right) \text{ and} \\ d = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, xy + (x+y)\delta_i(\tilde{\xi}) \right), \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix} \right).$$

- If $\theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \pmod{\pi}$ then the pseudo-symplectic covering is

$$\widetilde{\text{Ps}(B)}_{\tilde{\chi}} = \{(s, (-1)^l \omega_{\tilde{\chi}}(s)), s \in \text{Ps}(B), l \in \mathbb{Z}\},$$

where the set of operators $(-1)^l \omega_{\tilde{\chi}}(s)$ is exactly as the previous one, only replacing every factor $\frac{(-1)^l}{\sqrt{2}}$ by $\frac{(-1)^{l_i}}{\sqrt{2}}$. In particular, $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ is isomorphic to the general linear group

$$\text{GL}_2(\mathbb{F}_3) = \langle a, b, c, d \mid a^4 = c^3 = d^2 = 1, b^2 = a^2, bab^{-1} = dbd = a^{-1}, cac^{-1} = ab, \\ dad = a^2b, cbc^{-1} = a, dcd = c^{-1} \rangle,$$

an isomorphism being given for example by

$$a = \left((1, x), \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right), \quad b = \left((1, y), \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right), \\ c = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, (x+y)\delta_{-i}(\tilde{\xi}) \right), -\frac{1}{2} \begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix} \right) \text{ and} \\ d = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, xy + (x+y)\delta_i(\tilde{\xi}) \right), \frac{i}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix} \right).$$

Proof. 1. σ is orthogonal if and only if $(x \ y) \sigma^T \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} x \\ y \end{pmatrix} = (x \ y) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

for all $x, y \in \mathbb{F}_2$.

2. According to Proposition 1.11, the elements of $\text{Ps}(B)$ are given by $(\sigma, f_\sigma + f)$, where $\sigma \in \text{O}(Q)$ and $f \in \mathcal{Q}_a(W)$. This gives $f_\sigma = 0$ if $\sigma \in \left\{ I_2, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$

and $f_\sigma(x, y) = xy$ if $\sigma \in \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$. Proposition 4.1 gives all the 24 elements of $\text{Ps}(B)$ listed in the statement above.

3. Before computing the operators $\omega_{\tilde{\chi}}(s, w_s)$ for every $s \in \text{Ps}(B)$ with the formula (25), we are going to study the different possibilities for the choices of $\{\gamma(\sigma) = 2^{-\frac{\dim 1(\sigma)}{2}} e^{i\theta(\sigma)}, \sigma \in \text{O}(Q)\}$ satisfying (32) and (33). Let $v \in W$ and let σ_v be the transvection in $\text{Sp}(W)$ defined by $\sigma_v(w) = w + \langle w, v \rangle v$, for every $w \in W$. Then the group $\text{O}(Q) = \text{Sp}(W)$ is given by

$$\text{Sp}(W) = \left\{ \sigma_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_{(1,1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_{(1,0)}\sigma_{(0,1)} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \sigma_{(0,1)} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \right. \\ \left. \sigma_{(1,0)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \sigma_{(0,1)}\sigma_{(1,0)} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

By (32), $\theta(\sigma_{(0,0)}) \equiv 0 \pmod{2\pi}$. Moreover, $\sigma_{(1,0)}^2 = \sigma_{(0,1)}^2 = \sigma_{(1,1)}^2 = 1$, so by (33),

$$\theta(\sigma_v) \equiv 0 \pmod{\frac{\pi}{2}} \quad \text{for every } v \neq 0. \quad (58)$$

Observe now that $\sigma_{(1,0)}\sigma_{(0,1)} = \sigma_{(0,1)}\sigma_{(1,1)} = \sigma_{(1,1)}\sigma_{(1,0)}$. By (33), this implies that

$$\theta(\sigma_{(1,0)}\sigma_{(0,1)}) \equiv \begin{cases} \theta(\sigma_{(1,0)}) + \theta(\sigma_{(0,1)}) & \text{mod } \pi \\ \theta(\sigma_{(0,1)}) + \theta(\sigma_{(1,1)}) & \text{mod } \pi \\ \theta(\sigma_{(1,1)}) + \theta(\sigma_{(1,0)}) & \text{mod } \pi. \end{cases}$$

Hence

$$\theta(\sigma_{(1,0)}) \equiv \theta(\sigma_{(0,1)}) \equiv \theta(\sigma_{(1,1)}) \pmod{\pi}. \quad (59)$$

Now by (58) and (59), there are two possibilities:

- If $\theta(\sigma_{(1,0)}) \equiv 0 \pmod{\pi}$ then by (59), $\theta(\sigma_{(0,1)}) \equiv \theta(\sigma_{(1,1)}) \equiv 0 \pmod{\pi}$. By (33), we also have $\theta(\sigma_{(1,0)}\sigma_{(0,1)}) \equiv 0 \pmod{\pi}$ and $\theta(\sigma_{(0,1)}\sigma_{(1,0)}) \equiv 0 \pmod{\pi}$. Finally, this proves that

$$\theta(\sigma) \equiv 0 \pmod{\pi} \quad \text{for every } \sigma \in \text{Sp}(W). \quad (60)$$

In particular, by Proposition 3.8.5, any other choice $\{\gamma'(\sigma) = 2^{-\frac{\dim \mathbb{I}(\sigma)}{2}} e^{i\theta'(\sigma)}, \sigma \in \text{Sp}(W)\}$ satisfying (32),(33) and such that $\theta'(\sigma_{(1,0)}) \equiv 0 \pmod{\pi}$, will lead to an extension which is isomorphic to this one.

- If $\theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \pmod{\pi}$ then by (59), $\theta(\sigma_{(0,1)}) \equiv \theta(\sigma_{(1,1)}) \equiv \frac{\pi}{2} \pmod{\pi}$. By (33), we have $\theta(\sigma_{(1,0)}\sigma_{(0,1)}) \equiv 0 \pmod{\pi}$ and $\theta(\sigma_{(0,1)}\sigma_{(1,0)}) \equiv 0 \pmod{\pi}$. In particular, by Proposition 3.8.5, any other choice $\{\gamma'(\sigma) = 2^{-\frac{\dim \mathbb{I}(\sigma)}{2}} e^{i\theta'(\sigma)}, \sigma \in \text{Sp}(W)\}$ satisfying (32),(33) and such that $\theta'(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \pmod{\pi}$, will lead to an extension which is isomorphic to this one.

This shows that, up to isomorphism, the different choices $\{\gamma(\sigma) = 2^{-\frac{\dim \mathbb{I}(\sigma)}{2}} e^{i\theta(\sigma)}, \sigma \in \text{Sp}(W)\}$ satisfying (32) and (33) lead to at most two pseudo-symplectic extensions. Now, we can compute explicitly the operator $\omega_{\tilde{\chi}}(s, w_s)$ for every $s \in \text{Ps}(B)$ with Proposition 3.2:

$$\omega_{\tilde{\chi}}(s, w_s) = \gamma(\sigma) \sum_{u \in \mathbb{I}(\sigma)} \chi(f(w) + \langle w_s, w \rangle + B(u, w + w_s)) \rho_{\tilde{\chi}}(u + w_s), \quad (61)$$

where $w \in W$ is any element such that $u = (1 - \sigma)(w)$, and $\gamma(\sigma)$ satisfies (32) and (33). We give the details of the computations only for $s = \left(\sigma_{(1,0)}\sigma_{(0,1)} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, 0 \right)$, the other ones being obtained in a similar way. First we make a choice of w_s satisfying (24) (the isomorphism class of the extension defined by $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ does not depend on this choice according to Proposition 3.8). Here we have $\text{K}(\sigma_{(1,0)}\sigma_{(0,1)}) = \text{Ker} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \{0\}$, so any $w_s \in W$ satisfies (24), and we can choose for example

$w_s = 0$. By (61),

$$\begin{aligned}
\omega_{\tilde{\chi}}(s, 0) &= \gamma(\sigma_{(1,0)}\sigma_{(0,1)}) \sum_{u \in \mathbb{I}(\sigma)} \chi(B(u, w)) \rho_{\tilde{\chi}}(u) \\
&= \frac{e^{i\theta(\sigma_{(1,0)}\sigma_{(0,1)})}}{2} (\rho_{\tilde{\chi}}(0) + \chi(B((1, 0), (0, 1))) \rho_{\tilde{\chi}}(1, 0) + \chi(B((0, 1), (1, 1))) \rho_{\tilde{\chi}}(0, 1) \\
&\quad + \chi(B((1, 1), (1, 0))) \rho_{\tilde{\chi}}(1, 1)) \\
&= \frac{e^{i\theta(\sigma_{(1,0)}\sigma_{(0,1)})}}{2} (\rho_{\tilde{\chi}}(0) - \rho_{\tilde{\chi}}(1, 0) - \rho_{\tilde{\chi}}(0, 1) - \rho_{\tilde{\chi}}(1, 1)) \\
&= \frac{e^{i\theta(\sigma_{(1,0)}\sigma_{(0,1)})}}{2} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \tilde{\xi} & 0 \\ 0 & -\tilde{\xi} \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \tilde{\xi} \\ \tilde{\xi} & 0 \end{pmatrix} \right),
\end{aligned}$$

i.e.

$$\omega_{\tilde{\chi}}(s, 0) = \frac{e^{i\theta(\sigma_{(1,0)}\sigma_{(0,1)})}}{2} \begin{pmatrix} 1 - \tilde{\xi} & 1 - \tilde{\xi} \\ -1 - \tilde{\xi} & 1 + \tilde{\xi} \end{pmatrix},$$

where $e^{i\theta(\sigma_{(1,0)}\sigma_{(0,1)})} = \pm 1$ since $\theta(\sigma_{(1,0)}\sigma_{(0,1)}) \equiv 0 \pmod{\pi}$ for any choice of $\theta(\sigma_{(1,0)})$.

This enables us to establish the two lists of elements of the groups $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ given in the above statement. Now, it only remains to identify the groups and extensions defined by the two possible choices of $\theta(\sigma_{(1,0)})$. We know that $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ is a non-split stem central extension of $\text{Ps}(B) \simeq S_4$ by two elements. Hence, by [29], $\widetilde{\text{Ps}(B)}_{\tilde{\chi}}$ is isomorphic to one of the following groups: $\text{CSU}_2(\mathbb{F}_3)$ or $\text{GL}_2(\mathbb{F}_3)$. These groups having different numbers of elements of order two, we can immediately deduce from the lists of elements given above the following group isomorphisms:

$$\widetilde{\text{Ps}(B)}_{\tilde{\chi}} \simeq \begin{cases} \text{CSU}_2(\mathbb{F}_3) & \text{if } \theta(\sigma_{(1,0)}) \equiv 0 \pmod{\pi} \\ \text{GL}_2(\mathbb{F}_3) & \text{if } \theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \pmod{\pi}. \end{cases} \quad (62)$$

□

Remark 4.6 (related to Remarks 3.9 and 4.3). The result obtained in (62) proves that in this case, we obtain all the non-split stem extensions of $\text{Ps}(B)$ by two elements with all possible choices of $\{\gamma(\sigma), \sigma \in O(Q)\}$. In fact, the number of central extensions of $\text{Ps}(B) \simeq S_4$ by two elements can be found with the universal coefficient theorem, which provides the following short split exact sequence:

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(\text{H}_1(S_4, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \rightarrow \text{H}^2(S_4, \mathbb{Z}/2\mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(\text{H}_2(S_4, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \rightarrow 0.$$

The Schur multiplier of S_4 is $\text{H}_2(S_4, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ (see [17, p. 279]). Moreover, $\text{H}_1(S_4, \mathbb{Z}) \simeq S_4/[S_4, S_4] = S_4/A_4 \simeq \mathbb{Z}/2\mathbb{Z}$, and thus $\text{Ext}_{\mathbb{Z}}^1(\text{H}_1(S_4, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Hence $|\text{H}^2(S_4, \mathbb{Z}/2\mathbb{Z})| = 4$, which means that there are four non-isomorphic extensions of $\text{Ps}(B) \simeq S_4$ by two elements. The database [29] gives exactly four possible associated groups, only two of them being associated to non-split stem extensions: $\text{CSU}_2(\mathbb{F}_3)$

and $\mathrm{GL}_2(\mathbb{F}_3)$. Moreover, $\widetilde{\mathrm{Ps}(B)}_{\bar{\chi}}$ contains a copy of the Heisenberg group $\mathrm{H}(B)$, which is isomorphic to Q_8 in this example, and the groups $\mathrm{CSU}_2(\mathbb{F}_3)$ and $\mathrm{GL}_2(\mathbb{F}_3)$ are also the only ones containing Q_8 among the central extensions of S_4 by two elements. Therefore, in this case, we obtain all the possible groups $\widetilde{\mathrm{Ps}(B)}_{\bar{\chi}}$ associated to non-split stem extensions of $\mathrm{Ps}(B)$ by two elements, and this set of extensions coincides with the extensions of $\mathrm{Ps}(B)$ by two elements which contain the Heisenberg group $\mathrm{H}(B)$. Moreover, (62) shows also that Proposition 3.8.5 is an equivalence in this case.

Recall from Proposition 1.12 that $\mathrm{O}(Q) \subseteq \mathrm{Sp}(W)$. The case $\dim_{\mathbb{F}_2}(W) = 2$ and $\mathrm{Arf}(Q) = 1$ is the only one when $\mathrm{O}(Q) = \mathrm{Sp}(W)$. The fact that $\mathrm{O}(Q) = \mathrm{Sp}(W)$ implies that the pseudo-symplectic covering, hence the Weil representation, is here directly related to the entire symplectic group.

APPENDIX A. EXTRASPECIAL 2-GROUPS

Let p be a prime number. Unless otherwise stated, all the groups in the following are supposed to be finite. We denote by $\Phi(G)$ the Frattini subgroup of a finite group G , which is the intersection of all (proper) maximal subgroups of G . The aim of this section is to describe the structure of extraspecial 2-groups in order to prove Proposition 1.5.

Definition A.1. Let G be a p -group. Then G is said to be extraspecial if $\Phi(G)$ has order p and $\Phi(G) = [G, G] = Z(G)$.

Remark A.2. This definition of extraspecial group is given for instance in [9], [14] and [19]. Glasby uses in [8] another definition: G is extraspecial if it is a p -group such that $[G, G] = Z(G)$ has order p and $G/Z(G)$ is elementary abelian (i.e. all elements different from the trivial one have the same order). These definitions are equivalent. In fact, let G be a p -group. Suppose that $[G, G] = Z(G)$ has order p and $G/Z(G)$ is elementary abelian. Then $Z(G) = [G, G] \cap Z(G) \subseteq \Phi(G)$ by [14, III.3.12]. Since $\Phi(G)$ is the smallest normal subgroup N of G such that G/N is elementary abelian (cf. [14, III.3.14.a]), one has $\Phi(G) \subseteq Z(G)$, and thus $\Phi(G) = Z(G)$. Conversely, if G satisfies Definition A.1 then $G/Z(G) = G/\Phi(G)$ and $G/\Phi(G)$ is elementary abelian, again by [14, III.3.14.a].

In order to understand the structure of extraspecial groups, we first need to recall the central product of two groups.

Definition A.3. (cf. [19, 2.2.6])

- A group G is the internal central product of two subgroups H and K if $[H, K] = 1$, $G = HK$ and $H \cap K = Z(G)$.
- A group G is the external central product of two groups H and K if there exists an isomorphism $\theta : Z(H) \rightarrow Z(K)$ such that $G = (H \times K)/N$, where

$$N = \{(h, k) \in Z(H) \times Z(K) \mid \theta(h) = k^{-1}\}.$$

We shall denote G by $H \circ_{\theta} K$.

Notation A.4. The structure of a central product $H \circ_{\theta} K$ defined in Definition A.3 depends on the isomorphism θ , i.e. on the way to identify the elements of the center of H with the elements of the center of K . If θ is implicit for the groups H and K which are studied, or if θ is unique, then we simply denote the central product of H and K by $G = H \circ K$. We point out that constructing the central product of a finite number of groups is a commutative and associative operation.

From now on, unless otherwise stated, since we are only interested in 2-groups, we suppose that $p = 2$. However, the reader interested in p odd can find the results and proofs for general p in [14, III], [9, Chapter 5], [25, Chapter 4, 4] and [19, 2.2]. Now, we include the proof of a crucial result for the study of extraspecial 2-groups, which shows in particular that the decomposition of a central product of two groups is not unique. We recall the presentations of the dihedral and quaternionic groups of order eight:

$$\begin{aligned} D_4 &= \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle \\ Q_8 &= \langle x, y \mid x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} \rangle. \end{aligned}$$

Lemma A.5. The central products $D_4 \circ D_4$ and $Q_8 \circ Q_8$ are isomorphic.

Proof. Since the centers of D_4 and Q_8 contain only two elements, there is only one way to identify them and Notation A.4 applies. In particular there is no need to mention θ . Define two copies of Q_8 as $G_1 = \langle x_1, y_1 \mid x_1^4 = 1, y_1^2 = x_1^2, y_1x_1y_1^{-1} = x_1^{-1} \rangle$ and $G_2 = \langle x_2, y_2 \mid x_2^4 = 1, y_2^2 = x_2^2, y_2x_2y_2^{-1} = x_2^{-1} \rangle$. The central product of these groups is written

$$\begin{aligned} G_1 \circ G_2 &= \langle x_1, y_1, x_2, y_2 \mid y_1^2 = x_1^2 = y_2^2 = x_2^2 = z, z^2 = 1, y_1x_1y_1^{-1} = x_1^{-1}, y_2x_2y_2^{-1} = x_2^{-1}, \\ & x_1x_2x_1^{-1} = x_2, x_1y_2x_1^{-1} = y_2, y_1x_2y_1^{-1} = x_2, y_1y_2y_1^{-1} = y_2 \rangle, \end{aligned}$$

where we identify the centers $Z(G_i) = \langle x_i^2 = y_i^2 \rangle$ and the last four equalities come from the fact that G_1 centralizes G_2 in the central product. In a similar way, if we set $A_1 = \langle a_1, b_1 \mid a_1^4 = b_1^2 = 1, b_1a_1b_1^{-1} = a_1^{-1} \rangle$ and $A_2 = \langle a_2, b_2 \mid a_2^4 = b_2^2 = 1, b_2a_2b_2^{-1} = a_2^{-1} \rangle$, then A_1 and A_2 are two copies of D_4 and their central product is written

$$\begin{aligned} A_1 \circ A_2 &= \langle a_1, b_1, a_2, b_2 \mid a_1^2 = a_2^2 = u, u^2 = b_1^2 = b_2^2 = 1, b_1a_1b_1^{-1} = a_1^{-1}, b_2a_2b_2^{-1} = a_2^{-1}, \\ & a_1a_2a_1^{-1} = a_2, a_1b_2a_1^{-1} = b_2, b_1a_2b_1^{-1} = a_2, b_1b_2b_1^{-1} = b_2 \rangle. \end{aligned}$$

Now set $H_1 = \langle x_1, y_1x_2 \rangle \subseteq G_1 \circ G_2$ and $H_2 = \langle x_2, y_2x_1 \rangle \subseteq G_1 \circ G_2$. Then $(y_1x_2)^2 = y_1^2x_2^2 = y_1^4 = 1$ and $(y_1x_2)x_1(y_1x_2)^{-1} = y_1x_2x_1y_1x_2 = y_1x_1y_1x_2^2 = x_1^{-1}y_1^2x_2^2 = x_1^{-1}$. This shows that $H_1 \simeq D_4$, and similar computations show that $H_2 \simeq D_4$. Moreover, H_1 centralizes H_2 because G_1 centralizes G_2 . Let $\phi : A_1 \circ A_2 \rightarrow G_1 \circ G_2$ be defined by $\phi(a_1) = x_1$, $\phi(b_1) = y_1x_2$, $\phi(a_2) = x_2$ and $\phi(b_2) = y_2x_1$. Then ϕ can be extended to a group homomorphism in the natural way. Moreover, ϕ is surjective because $x_1 = \phi(a_1)$, $y_1 = \phi(b_1a_2^{-1})$, $x_2 = \phi(a_2)$ and $y_2 = \phi(b_2a_1^{-1})$. Since $A_1 \circ A_2$ and $G_1 \circ G_2$ have the same number of elements, this is enough to conclude that these two groups are isomorphic. \square

The next technical result will be needed to give a symplectic structure to extraspecial groups and relate them to the Heisenberg groups defined in section 1.

Lemma A.6. Let G be an extraspecial p -group, with p prime. Then:

1. G is nilpotent of step 2.
2. The exponent of G is either p or p^2 , and more precisely it equals 4 if $p = 2$.
3. $G/Z(G)$ can be identified with a finite-dimensional \mathbb{F}_p -vector space. More precisely, $G/Z(G)$ can be endowed with a symplectic vector space structure by defining the non-degenerate symplectic form $\langle xZ(G), yZ(G) \rangle = l$, where $Z(G) = \langle z \rangle$ and $[x, y] = z^l$, $0 \leq l \leq p - 1$.
4. Let $n \geq 1$ such that $|G| = p^{2n+1}$. Then every maximal abelian normal subgroup of G has order p^{n+1} .

Proof. 1. The lower central series of G is $G \triangleright [G, G] \triangleright 1$ because G is non-abelian and $[[G, G], G] = [Z(G), G] = 1$.

2. For all x and y in G , $1 = [x, y]^p = [x^p, y]$ by [14, III.1.3] and the fact that $[G, G] = Z(G)$ has order p . Hence $x^p \in Z(G)$ and thus $x^{p^2} = 1$ because $Z(G)$ has order p . This shows that the exponent of G is either p or p^2 . If we suppose that $p = 2$ and G has exponent 2 then G should be abelian since every element has order two. This is impossible by definition of an extraspecial group, hence G has exponent $p^2 = 4$.

3. Remark A.2 implies that $G/Z(G)$ is elementary abelian. Since $G/Z(G)$ is also a p -group, it follows from the fundamental theorem of abelian groups that there exists $m \geq 1$ such that $G/Z(G) \simeq (\mathbb{Z}/p\mathbb{Z})^m$, if we use an additive notation for $G/Z(G)$. Thus there exists a generating set $\{x_1, \dots, x_m\}$ of $G/Z(G)$, i.e. for all $x \in G/Z(G)$ there exists $\lambda_i \in \{0, \dots, p - 1\}$ such that $x = \sum_{1 \leq i \leq m} \lambda_i x_i$. Since every element of

$G/Z(G)$ is of order p , we can consider the λ_i as elements of $\mathbb{F}_p = \{0, \dots, p - 1\}$. For $\lambda \in \mathbb{F}_p$, define $\lambda \cdot x = \sum_{1 \leq i \leq m} \lambda \lambda_i x_i$. In this way, $G/Z(G)$ becomes an \mathbb{F}_p -vector space of dimension m with basis $\{x_1, \dots, x_m\}$.

Now let x and y be in G . Since $[G, G] = Z(G)$ is cyclic of order p , there exists $z \in G$ such that $[G, G] = Z(G) = \langle z \rangle$. Hence there exists a unique element $0 \leq l \leq p - 1$ such that $[x, y] = z^l$, i.e. $\langle \cdot, \cdot \rangle$ is well defined. Now let $0 \leq k \leq p - 1$ and x, y, u be in G such that $[x, y] = z^{l_1}$ and $[u, y] = z^{l_2}$. Then:

- $[x, x] = 1$ and $[y, x] = [x, y]^{-1}$. Hence $\langle xZ(G), xZ(G) \rangle = 0$ and $\langle yZ(G), xZ(G) \rangle = -l_1 = -\langle xZ(G), yZ(G) \rangle$.
- $[xu, y] = xuyx^{-1}y^{-1} = x[u, y]yx^{-1}y^{-1} = [x, y][u, y]$ because $[G, G] \subseteq Z(G)$. This implies that $[xu^k, y] = [x, y][u, y]^k$. Hence $\langle xZ(G)(uZ(G))^k, yZ(G) \rangle = \langle xu^k Z(G), yZ(G) \rangle = l_1 + kl_2 = \langle xZ(G), yZ(G) \rangle + k \langle uZ(G), yZ(G) \rangle$.
- Suppose that $\langle xZ(G), vZ(G) \rangle = 0$ for all $v \in G$, then $[x, v] = 1$ for all $v \in G$, i.e. $x \in Z(G)$.

This shows that $\langle \cdot, \cdot \rangle$ is an anti-symmetric, alternating and non-degenerate bilinear form on $G/Z(G)$, i.e. $(G/Z(G), \langle \cdot, \cdot \rangle)$ is a symplectic vector space. In particular, the dimension of $G/Z(G)$ is even.

4. The existence of $n \geq 1$ such that $|G| = p^{2n+1}$ has been established in 3. Let A be a maximal abelian normal subgroup of G . Then A contains the center of

G . There is a bijective correspondence between subgroups of G containing $Z(G)$ and subgroups of $G/Z(G)$, and $G/Z(G)$ is identified with a symplectic vector space. Therefore $A/Z(G)$ is a vector subspace of $G/Z(G)$ of maximal dimension, which satisfies $\langle a_1Z(G), a_2Z(G) \rangle = 0$ for all $a_1, a_2 \in A$. In other terms, $A/Z(G)$ is a maximal isotropic subspace of $G/Z(G)$ and hence of dimension n . Then $|A| = |A/Z(G)||Z(G)| = p^n p = p^{n+1}$. \square

Lemma A.7. (cf. [9, Chapter 5, Lemma 4.6]) Let G be a p -group and H be an extraspecial p -group, with p prime.

- (1) Every automorphism of H acting trivially on $H/Z(H)$ is an inner automorphism of H .
- (2) If H is a subgroup of G such that $[G, H] \subseteq Z(H)$ then $G = H \circ C_G(H)$.

Lemma A.8. (cf. [9, Chapter 5, Theorem 5.1]) Let G be a non-abelian group of order 2^3 . Then G is extraspecial and more precisely G is isomorphic to either D_4 or Q_8 .

Now we come to the central result of this section: up to isomorphism, there are exactly two distinct extraspecial 2-groups of the same order and they can be described as central products of extraspecial groups of order 2^3 .

Proposition A.9. (cf. [9, Chapter 5, Theorem 5.2]) Let G be an extraspecial 2-group. Then there exists $n \geq 1$ such that $|G| = 2^{2n+1}$ and G is isomorphic to either the central product of n copies of the dihedral group D_4 (which we denote by $D_4^{\circ n}$ or 2_+^{1+2n}) or the central product of $n - 1$ copies of D_4 and one copy of the quaternionic group Q_8 (which we denote by $Q_8 \circ D_4^{\circ(n-1)}$ or 2_-^{1+2n}).

The last step before proving Proposition 1.5 is to exhibit the link between extraspecial 2-groups and the Heisenberg group $H(B)$ introduced in Definition 1.1, where the form

$$\langle \cdot, \cdot \rangle : (w_1, w_2) \mapsto B(w_1, w_2) - B(w_2, w_1)$$

is non-degenerate and $\mathbb{F} = \mathbb{F}_2$. Recall that this endows W with a symplectic vector space structure $(W, \langle \cdot, \cdot \rangle)$ of dimension $2n$ over \mathbb{F}_2 . A priori the group $H(B)$ depends on the bilinear form chosen to define it, and there are many such bilinear forms. However, as we are going to prove in Proposition A.11, this group is very particular: $H(B)$ is in fact an extraspecial group of order 2^{2n+1} . This implies that, in spite of the multiple possibilities for B , there are up to isomorphism only two different groups $H(B)$, corresponding to the two non-equivalent quadratic forms from W to \mathbb{F}_2 . Recall from the paragraph after Remark 1.3 the index $\nu(Q)$ and the Arf invariant $\text{Arf}(Q)$ of a quadratic form Q on W . Then $\nu(Q) = n$ corresponds to $\text{Arf}(Q) = 0$, and $\nu(Q) = n - 1$ corresponds to $\text{Arf}(Q) = 1$.

Lemma A.10. To every extraspecial group G , we can associate a symplectic structure over $G/Z(G)$. More precisely, if $Z(G) = \langle z \rangle$ then we can define a non-degenerate symplectic form $\langle \cdot, \cdot \rangle$ and a quadratic form Q associated to $\langle \cdot, \cdot \rangle$ by:

$$\langle xZ(G), yZ(G) \rangle = k, \quad \text{where } [x, y] = z^k, \quad \text{and} \quad Q(xZ(G)) = l, \quad \text{where } x^2 = z^l.$$

Moreover, if $G \simeq 2_+^{1+2n}$ then $\nu(Q) = n$, whereas if $G \simeq 2_-^{1+2n}$ then $\nu(Q) = n - 1$.

Proof. The fact that $(G/Z(G), \langle \cdot, \cdot \rangle)$ is a symplectic vector space has already been proven in Lemma A.6.3. Let $x, y \in G$ be such that $[x, y] = z^k$. By Lemma A.6.2, we can also write $x^2 = z^{l_1}$ and $y^2 = z^{l_2}$ since $x^2, y^2 \in Z(G)$. Hence Q is well defined. In addition:

$$(xy)^2 = xyxy = [x, y]yx^2y = x^2y^2[x, y] = z^{l_1+l_2+k},$$

which shows that $Q(xZ(G)yZ(G)) - Q(xZ(G)) - Q(yZ(G)) = k = \langle xZ(G), yZ(G) \rangle$, i.e. $(xZ(G), yZ(G)) \mapsto Q(xZ(G)yZ(G)) - Q(xZ(G)) - Q(yZ(G))$ is bilinear. Thus Q is quadratic.

Suppose now that $G \simeq 2_+^{1+2n} = D_4^{\circ n}$, with presentation

$$G = \langle a_1, \dots, a_n, b_1, \dots, b_n \mid a_i^4 = b_i^2 = 1, b_i a_i b_i^{-1} = a_i^{-1}, a_i^2 = a_j^2, \\ [a_i, a_j] = [a_i, b_j] = [b_i, b_j] = 1 \text{ for } i \neq j \rangle.$$

Then every element x of G can be written as a product $x = a_1^{k_1} b_1^{l_1} \dots a_n^{k_n} b_n^{l_n}$, where $0 \leq k_i \leq 3$ and $0 \leq l_i \leq 1$. This enables us to compute the expression of $Q(xZ(G))$. First, let $1 \leq i \leq n$. If we do not distinguish between $0 \leq k_i \leq 3$ and its projection in $\mathbb{Z}/2\mathbb{Z}$, and if we use that $[a_i^{k_i}, b_i^{l_i}] = [a_i, b_i]^{k_i l_i}$ (proven in Lemma A.6.3), then:

$$Q(a_i^{k_i} Z(G) b_i^{l_i} Z(G)) = Q(a_i^{k_i} Z(G)) + Q(b_i^{l_i} Z(G)) + \langle a_i^{k_i} Z(G), b_i^{l_i} Z(G) \rangle = k_i + k_i l_i.$$

Hence, since two copies of D_4 commute in the central product, we have by induction that

$$Q(xZ(G)) = \sum_{i=1}^n Q(a_i^{k_i} Z(G) b_i^{l_i} Z(G)) = \sum_{i=1}^n (k_i + k_i l_i).$$

Applying the transformation $(k_i, l_i) \mapsto (k_i, k_i + l_i)$, we modify $Q(xZ(G))$ into $\sum_{i=1}^n k_i l_i$, which implies that Q has index n .

Finally we suppose that $G \simeq 2_-^{1+2n} = Q_8 \circ D_4^{\circ(n-1)}$, with presentation

$$G = \langle x_1, y_1, a_2, \dots, a_n, b_2, \dots, b_n \mid x_1^4 = 1, y_1 x_1 y_1^{-1} = x_1^{-1}, a_i^4 = b_i^2 = 1, b_i a_i b_i^{-1} = a_i^{-1}, \\ x_1^2 = y_1^2 = a_i^2 = a_j^2, [a_i, a_j] = [a_i, b_j] = [b_i, b_j] = 1 \text{ for } i \neq j \rangle.$$

Then every element x of G can be written as a product $x = x_1^{k_1} y_1^{l_1} a_2^{k_2} b_2^{l_2} \dots a_n^{k_n} b_n^{l_n}$, where $0 \leq k_i \leq 3$ for all i , $0 \leq l_1 \leq 3$, and $0 \leq l_i \leq 1$ for $i \geq 2$. This enables us to compute the expression of $Q(xZ(G))$. First, as above, for every $2 \leq i \leq n$:

$$Q(a_i^{k_i} Z(G) b_i^{l_i} Z(G)) = Q(a_i^{k_i} Z(G)) + Q(b_i^{l_i} Z(G)) + \langle a_i^{k_i} Z(G), b_i^{l_i} Z(G) \rangle = k_i + k_i l_i,$$

and similarly,

$$Q(x_1^{k_1} Z(G) y_1^{l_1} Z(G)) = Q(x_1^{k_1} Z(G)) + Q(y_1^{l_1} Z(G)) + \langle x_1^{k_1} Z(G), y_1^{l_1} Z(G) \rangle = k_1 + l_1 + k_1 l_1.$$

Hence

$$Q(xZ(G)) = Q(x_1^{k_1} Z(G) y_1^{l_1} Z(G)) + \sum_{i=2}^n Q(a_i^{k_i} Z(G) b_i^{l_i} Z(G)) = k_1 + l_1 + k_1 l_1 + \sum_{i=2}^n (k_i + k_i l_i).$$

Applying the transformation $(k_i, l_i) \mapsto (k_i, k_i + l_i)$, we modify $Q(xZ(G))$ into $k_1 + l_1 + \sum_{i=1}^n k_i l_i$, which implies that Q has index $n - 1$. \square

Proposition A.11. The Heisenberg group $H(B)$ is extraspecial. In particular, $H(B)$ is isomorphic to 2_+^{1+2n} if $\nu(Q) = n$, or to 2_-^{1+2n} if $\nu(Q) = n - 1$.

Proof. Let $h_1 = (w_1, t_1)$ and $h_2 = (w_2, t_2)$ be in $H(B)$. Then:

$$\begin{aligned} [h_1, h_2] &= (w_1 + w_2, t_1 + t_2 + B(w_1, w_2))(-w_1 - w_2, -t_1 - t_2 + Q(w_1) + Q(w_2) + B(w_1, w_2)) \\ &= (0, -Q(w_1 + w_2) + Q(w_1) + Q(w_2)) = (0, -\langle w_1, w_2 \rangle). \end{aligned}$$

This shows that $[H(B), H(B)] \simeq \mathbb{F}_2$, i.e. $[H(B), H(B)]$ has order two and $[H(B), H(B)] = Z(H(B))$. Moreover $H(B)/Z(H(B)) \simeq W$ is elementary abelian, since it is isomorphic to \mathbb{F}_2^{2n} . Therefore we can conclude that $H(B)$ is extraspecial by Remark A.2. In addition, $h^2 = (0, Q(w))$ for every $h = (w, t) \in H(B)$. This implies that the quadratic form associated to $H(B)$ as an extraspecial group by Lemma A.10 is exactly the quadratic form Q associated to B . In this way, Lemma A.10 implies that $H(B)$ is isomorphic to 2_+^{1+2n} if $\nu(Q) = n$, or to 2_-^{1+2n} if $\nu(Q) = n - 1$. \square

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