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Dependent default times with mixed hitting-time models

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Abstract

Mixed hitting-time models allow the analysis of competing risks through optimal stopping decisions interpreted as crossing times of latent Lévy processes with heterogeneous thresholds. In this paper, we consider a bivariate time model with dependent default, where observation times are subject to censoring and share a common latent process given by a Lévy subordinator. We establish the identifiability of the model and propose different estimators for the marginal distributions and the joint survival distribution. We establish their asymptotic properties and evaluate the finite-sample performance of our results through a simulation study on synthetic data, followed by an application using real data.

Key Words: dependent risk, mixed hitting-time model, Lévy process.

1 Introduction

Threshold-crossing models for duration analysis have recently received considerable attention in various fields, including finance, reliability engineering and survival analysis. From a practical perspective, these models are especially useful in economics, with real applications such as patent races [48, 14], technology adoption [47, 28] and smoking cessation among married couples in socio-economic settings contexts [4]. The fundamental premise of the mixed hitting-time (MHT) models is to examine the time at which a process crosses a predetermined or random threshold, leading to significant events such as defaults, failures or other critical occurrences. This approach provides a flexible description of survival mechanisms [38, 2] in the context of optimal stopping time problems, where typical solutions are usually defined by threshold-crossing rules [31, 33, 3]. To some extent, it also includes mixed proportional hazards models [39, 34], which are a well-known class of competing risk models with proportional hazard rates. In this paper, we consider a bivariate competing risks model based on the threshold-crossing duration framework, where the two durations correspond to the first-passage times of two dependent latent stochastic processes. The durations T and C are defined as the times when latent processes $(M_{1,t})_{t \geq 0}$ and $(M_{2,t})_{t \geq 0}$ exceed independent random barriers E_1 and E_2 respectively. This definition is often referred to as the *canonical construction* of default times [9, 51] and describes the standard intensity-based approach for modeling defaults. Specifically, we consider two proper cumulative hazard functions Λ_1 and Λ_2 ,

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that is, positive increasing functions on \mathbb{R}_+ such that $\Lambda_1(0) = \Lambda_2(0) = 0$. The default times are then defined by

$$T = \inf \{t \geq 0; M_{1,t} \geq E_1\}, \quad C = \inf \{t \geq 0; M_{2,t}(t) \geq E_2\} \quad (1.1)$$

and

$$M_{1,t} = \Lambda_1(t) + Y_t, \quad M_{2,t} = \Lambda_2(t) + Y_t \quad (1.2)$$

where $(Y_t)_{t \geq 0}$ is a right-continuous and increasing Lévy process, also referred to as a subordinator. The intuition behind the choice in (1.2) is that the term structure of the survival probability is modeled through an individual deterministic component, represented by the cumulative hazard functions, and a stochastic component that affects all elements simultaneously. By incorporating a mutual latent stochastic process, this model captures the possibility of both durations ending at the same time, as the two dependent processes can simultaneously jump over their respective thresholds. This property is not common to all competing risk models, but it is also shared with the bivariate distributions of the Marshall-Olkin type. Under generalized Marshall-Olkin (GMO) models, the simultaneous termination of both durations is driven by a common independent shock, as assumed in [46, 43, 27]. GMO models typically assume shocks with immediate effects; however, practical experience shows that such scenarios do not always occur, as shocks may have delayed effects. For example, during a financial crisis, weaker institutions may be affected first, while stronger institutions experience the effects later [see, e.g. 13]. Specific instances of these models originate from a particular dynamic default model proposed for valuing certain financial products [59]. In MHT models, the stochastic processes represent the accumulation of implicit shocks which occur only up to a certain threshold. This means that the observable results of these shocks, particularly when a failure occurs, are not immediate. Rather, they depend on the progression of $(Y_t)_{t \geq 0}$, which allows for a more nuanced view of risk dynamics.

In a dependent and bivariate competing risks framework, individuals experience two types of durations, but only the first to occur is observed. In survival analysis, this issue is referred to as dependent censoring and occurs when only a lower bound of the event of interest is returned. In such analyses, either the time to event T or the time to censoring C is observed for each individual in the sample. For instance, participants in clinical trials often withdraw prematurely if they find the drug to be ineffective or experience adverse effects [50]. In credit risk analysis, censoring occurs when a specific loan is being repaid at the moment of data collection [44]. Ignoring this dependence can lead to biased survival estimations, which explains why most existing statistical survival studies assume the independence of censoring. However, this assumption is challenging to verify in practice as the available data typically provide only marginal distributions. Consequently, determining whether survival is truly independent of censoring remains a challenging task, and non-parametric identifiability of the joint model is impossible without prior knowledge of the dependence structure [56]. In this context, adjusting for dependent censoring may require practitioners to have knowledge of the true dependence structure between T and C [see, e.g. 61, 10, 15, 18, 26]. For most approaches, the dependence structure is characterized by copula functions, which capture dependencies between random variables independently of their marginal distributions. However, this requirement presents significant challenges in practical applications, as specifying the correct copula is often difficult. In addition, misspecified copulas may exacerbate model bias, leading to incorrect inferences and misleading results. In survival analysis, several recent advances have addressed the limitation of assuming a known copula by exploring parametric modeling approaches in [17] and semi-parametric models in [22]. Identification results for competing risks models have also been studied with several contributions under different sets of restrictions [see, e.g. 30, 1, 41, 42, 44].

This study focuses on the analysis of dependent risk models based on the canonical construction of default models (1.1). It first addresses the issue of non-parametric identification of the dependence structure as examined in [32]. The identifiability and the estimation of the survival distribution and dependence structure of the proposed model are indeed challenging problems. Studies on structural threshold-crossing models driven by latent stochastic processes are relatively limited in the literature. In [49], the authors examined a competing risk model driven by a latent compound Poisson process, but no formal identification results exist for such a complex structural model. Without censoring, the study in [3] derived non-parametric identification results for spectral negative Lévy processes crossing a heterogeneous threshold. In [44, 27], the non-parametric identifiability of the models is addressed, although these methods require the information on whether $T \leq C$ and $T \geq C$ for each observation. In this regard, we demonstrate that our model is identified under a standard censoring model. The main advantage of our approach, compared to existing identifiability results, is that it avoids functional form restrictions on the marginal distributions, maintaining their non/semi-parametric nature, while the structure of dependence does not need to be known or assumed. We show that it is possible to apply commonly used statistical methods from survival analysis to estimate the marginal distributions of T and C as well as the joint survival distribution of (T, C) . The remainder of the paper is organized as follows. Section 2 presents the model and the notation. In Section 3, we introduce the main characteristics of the survival copula and its asymptotic properties. Section 4 contains the identification result. A non/semi-parametric estimation of the survival distributions, the joint survival distribution, the Kendall's tau and their asymptotic properties are established in Section 5. Section 6 presents simulation results that confirm the theoretical findings, first through simulations (Subsection 6.1) and then through the use of a real data set (Subsection 6.2). Some relevant concluding remarks and potential perspectives are presented in Section 7, while Section 8 includes the proofs of the theoretical results.

2 Model and notations

A Lévy process is a stochastic process with stationary and independent increments that starts at zero and has paths that are continuous from the right. Due to these properties, each such process $(Y_t)_{t \geq 0}$ is characterized by its marginal Lévy exponent κ , which is related to its Laplace-Stieltjes transformation ψ_{Y_t} by

$$\psi_{Y_t}(x) = e^{-t\kappa(x)}, \quad t \geq 0.$$

The Lévy exponent κ fully characterizes the process distribution and has the key properties of being non-decreasing, and for all $x \geq 0$, satisfying $\kappa(x) \leq x\kappa(1)$. In this work, we consider that the lifetimes of interest defined by the random times T and C are affected by the defaults of a Lévy process as defined in (1.1) where E_1 and E_2 are independent unit exponential random variables. In the sequel, we will also use the following equivalent formulation

$$T = \inf\{t \geq 0; e^{-M_{1,t}} \leq U_1\} \quad \text{and} \quad C = \inf\{t \geq 0; e^{-M_{2,t}} \leq U_2\},$$

where $U_1 = \exp(-E_1)$ and $U_2 = \exp(-E_2)$. Note that U_1 and U_2 are uniformly distributed. Based on a right-censoring approach, only the smallest default time is observed. Formally, the outcomes are limited to the couple (Z, δ) , defined by

$$Z = \min(T, C) \quad \text{and} \quad \delta = \mathbb{1}_{\{T \leq C\}}.$$

The survival functions of the random time Z , T and C are denoted by \bar{H} , \bar{F}_T and \bar{F}_C respectively. According to Theorem 2.2 in [32], \bar{F}_T and \bar{F}_C are given, for any $t \geq 0$, by

$$\bar{F}_T(t) = e^{-\Lambda_1(t) - t\kappa(1)} \quad \text{and} \quad \bar{F}_C(t) = e^{-\Lambda_2(t) - t\kappa(1)}.$$

To fully characterize the joint distribution of T and C , we apply Sklar's theorem [54], which asserts that for any joint distribution function, there exists a unique copula \mathcal{C} , defined on $F_T(\mathbb{R}_+) \times F_C(\mathbb{R}_+)$, that captures the dependence between T and C . Specifically, for any $t, s \geq 0$, the joint survival probability P is given by

$$P(t, s) = \mathbb{P}(T \leq t, C \leq s) = \mathcal{C}(F_T(t), F_C(s)).$$

In reliability studies, although this does not change the analysis, it is often more convenient to express a joint survival function in terms of its marginal survival functions, such that

$$\tilde{P}(t, s) = \mathbb{P}(T > t, C > s) = \tilde{\mathcal{C}}(\bar{F}_T(t), \bar{F}_C(s)), \quad (2.1)$$

where the function $\tilde{\mathcal{C}}$ is called the *survival copula*. In the sequel, we show that the identifiability and estimation of the distribution of T and the joint distribution P are feasible for this model. Building on this, we introduce the generalized inverses of the cumulative distribution functions \bar{F}_T and \bar{F}_C , denoted by \bar{F}_T^- and \bar{F}_C^- , respectively, which are defined as

$$\bar{F}_T^-(t) = \inf \{x \geq 0, \bar{F}_T(x) \leq t\} \quad \text{and} \quad \bar{F}_C^-(t) = \inf \{x \geq 0, \bar{F}_C(x) \leq t\}.$$

Recall that for any general distribution function F , the corresponding cumulative hazard function is defined as

$$\Lambda(t) = \int_0^t \frac{F(du)}{\bar{F}(u^-)}, \quad t \geq 0,$$

where $\bar{F} := 1 - F$ and $\bar{F}(u^-) = \lim_{s \downarrow u} \bar{F}(s)$. Conversely, the relationship between a distribution function F and its cumulative hazard function Λ can be expressed as

$$\bar{F}(t) =: \exp(-\tilde{\Lambda}(t)) = \exp\left(-\Lambda^c(t) + \sum_{s \leq t} \log(1 - \Delta\Lambda(s))\right),$$

where Λ^c and $\Delta\Lambda$ respectively denote the continuous and discontinuous parts of Λ [see p.898 in 52]. Thus, the model can be represented using the survival functions

$$\bar{F}_T(t) =: \exp(-\tilde{\Lambda}_T(t)) \quad \text{and} \quad \bar{F}_C(t) =: \exp(-\tilde{\Lambda}_C(t)),$$

where Λ_T and Λ_C define the cumulative hazard functions of T and C , respectively.

3 Copulas and dependence properties

In this section, we present several fundamental properties of the proposed model, with a particular focus on the copula function $\tilde{\mathcal{C}}$ and its relationship with the GMO copula function. In [32], a general form for the survival GMO copula is derived and used to define a large class of copulas. In line with this work, we derive the joint survival function and the survival copula associated with the proposed model in the following proposition.

Proposition 1. *The joint survival function \tilde{P} of (T, C) defined in (1.1) is given, for any $t, s \geq 0$, by*

$$\tilde{P}(t, s) = \bar{F}_C(s) \bar{F}_T(t) \min\left(\bar{F}_T(t)^{-\alpha(t)}, \bar{F}_C(s)^{-\beta(s)}\right),$$

where

$$\alpha(t) = \frac{t\xi}{\Lambda_1(t) + t\kappa(1)}, \quad \beta(s) = \frac{\xi s}{\Lambda_2(s) + s\kappa(1)},$$

and $\xi = 2\kappa(1) - \kappa(2)$. In particular, the survival copula \tilde{C} is given by

$$\tilde{C}(u, v) = uv \min \left(u^{-\alpha(\bar{F}_T^-(u))}, v^{-\beta(\bar{F}_C^-(v))} \right), \quad (u, v) \in \bar{F}_T(\mathbb{R}_+) \times \bar{F}_C(\mathbb{R}_+). \quad (3.1)$$

Note that the minimum term in (3.1) represents the deviation of the dependence structure from the independent scenario, which occurs when $\xi = 0$, and from the complete dependence case, which occurs when $\alpha \equiv \beta \equiv 1$. In a broader context, constant functions α and β are inherently connected to the MO model. The sub-density functions and the hazard functions are provided in the following proposition.

Proposition 2. *The survival function \bar{H} of Z is given by $\bar{H}(t) = e^{-\Lambda_1(t) - \Lambda_2(t) - t\kappa(2)}$ for $t \geq 0$. Moreover, the sub-densities $f_{Z,\delta=0}$ and $f_{Z,\delta=1}$ of (Z, δ) are expressed by*

$$f_{Z,\delta=0}(t) = \bar{H}(t)[\Lambda_2'(t) + \kappa(2) - \kappa(1)] \quad \text{and} \quad f_{Z,\delta=1}(t) = \bar{H}(t)[\Lambda_1'(t) + \kappa(1)]. \quad (3.2)$$

In particular, the hazard functions λ_T and λ_C respectively associated with the random variables T and C are given by $\lambda_T = f_{Z,\delta=1}/\bar{H}$ and $\lambda_C = f_{Z,\delta=0}/\bar{H} + \xi$.

Notably, the hazard function of T maintains the same form as in the case of independent censoring. This property ensures that the Nelson-Aalen estimator remains consistent even under the dependent censoring induced by the model construction. Further details on estimating the survival distribution function are provided in Section 5. To conclude this section, we present the following statement, which establishes the connection between the proposed model and the GMO model.

Proposition 3. *The model defined by (1.1)-(1.2) is a special case of the GMO model. Specifically, we have*

$$(T, C) \stackrel{d}{=} (\min(X_1, X_3), \min(X_2, X_3))$$

where $\stackrel{d}{=}$ denotes equality in distribution, X_1, X_2 and X_3 are three independent random variables such that X_3 follows an exponential distribution with parameter ξ . Moreover, the cumulative hazard functions $\Lambda_{X_1}, \Lambda_{X_2}$ of X_1 and X_2 , respectively, are given by:

$$\begin{cases} \Lambda_{X_1}(t) = \Lambda_1(t) + t\kappa(1) - \xi t \\ \Lambda_{X_2}(t) = \Lambda_2(t) + t\kappa(1) - \xi t. \end{cases}$$

It is evident that the simultaneous failure of both T and C arises from the common failure source X_3 . Although the proposed model shares the same distribution as a specific case of the GMO model, it is important to emphasize that our approach introduces key distinctions in modeling dependent censoring. MHT models offer greater flexibility by capturing more complex, time-varying dependencies through stochastic processes. This enables alternative interpretations of real-world censoring mechanisms, such as recurrent events or latent factors. In contrast, GMO models lack a dynamic temporal component and are typically limited to shock-induced dependencies that occur

instantaneously, which makes it challenging for them to effectively represent these time-varying dependencies. Additionally, GMO models are constrained by their rigid assumptions of shared shocks, which limits their ability to reflect nuanced dependency structures. MHT models also offer better adaptability in capturing latent factors and unobserved competing risks. Their flexibility also facilitates model estimation, particularly in settings with latent competing risks, where GMO models may be too rigid or face identifiability challenges. Finally, it is often more practical to summarize the complexity of the model using a universal measure of dependence, such as Kendall's tau coefficient.

Remark 1. *Based on Proposition 2 in [27], if Λ_1 and Λ_2 are strictly increasing, the Kendall's tau corresponding to the proposed model is given by*

$$\tau = 2\xi \int_0^{+\infty} \overline{H}(u)^2 du.$$

4 Identifiability

In this section, we investigate the model's identifiability. Recall that a statistical model $\mathcal{P} = (P_\gamma)_{\gamma \in \Gamma}$ is said to be identifiable if the map $\gamma \rightarrow P_\gamma$, defined on Γ , is injective. In our context, identifiability implies that the vector $(\Lambda_1, \Lambda_2, \kappa(1), \kappa(2))$ uniquely specifies the density of the observable random vector (Z, δ) . That is,

$$\text{if } f_{Z, \delta=., \Gamma_1} = f_{Z, \delta=., \Gamma_2} \text{ then } \Gamma_1 = \Gamma_2,$$

where $\Gamma_i = (\Lambda_1^i, \Lambda_2^i, \kappa(1)^i, \kappa(2)^i)$, for $i = 1, 2$. Examining the identifiability with dependent censoring is no simple task, particularly when considering the dependence structure of the survival copula $\tilde{\mathcal{C}}$. This is due to the right censoring mechanism, which prevents the simultaneous observation of the pair (T, C) , making it difficult to directly discern the relationship between T and C from the observed data. In the following theorem, we prove the identifiability of the proposed model.

Proposition 4. *Assuming that $\Lambda_1'(0) = \Lambda_2'(0) = 0$, the model defined by (1.1)–(1.2) is identifiable.*

Unlike the identifiability results in semi-parametric models discussed in [17, 21, 22] or in the MHT models in [3, 44], our approach does not require complex assumptions about the marginal distributions or the joint model to establish the model identifiability. Furthermore, compared to the model identification in [27], we do not require any additional information beyond the censoring status $T \leq C$, which is typically available in standard survival analysis. The only necessary assumption concerns the regularity of the marginal densities Λ_1 and Λ_2 , which is satisfied by a wide range of cumulative hazard functions. In particular, this preserves the semi-parametric character of the model and does not impose any functional form restrictions on the marginal distributions. Overall, the identifiability of our model suggests that we can determine the relationship between T and C , observing only their minimum through the censored data (Z, δ) .

5 Estimation

We now consider the estimation of the survival distributions \overline{F}_T and \overline{F}_C , as well as the joint survival \tilde{P} of the survival time T and the censoring time C . For this, we assume that we have an independent and identically distributed (*i.i.d.*) sample $(Z_i, \delta_i)_{1 \leq i \leq n}$ drawn from the observed model (Z, δ) .

5.1 Non-parametric estimation of \bar{F}_T

According to Equation (3.2) in Proposition 2, it turns out that the Nelson-Aalen estimator $\Lambda_{T,n}$ of the cumulative hazard function Λ_T remains consistent, unlike most dependent censoring models. The Nelson-Aalen estimator of Λ_T is then defined by

$$\Lambda_{T,n}(t) = \sum_{i:Z_{(i)} \leq t} \frac{d_i}{n-i+1} \quad \text{with} \quad d_i = \sum_{j=1}^n \mathbb{1}_{\{Z_j=Z_{(i)}, \delta_j=1\}} \quad \text{and} \quad t > 0, \quad (5.1)$$

where $Z_{(i)}$ is the i -th order statistic among the n survival time observations. According to Lemma 1 of [11], we also conclude that the Kaplan-Meier (KM) estimator $\bar{F}_{T,n}$ defined by $\bar{F}_{T,n} = \exp(-\Lambda_{T,n})$ provides a consistent estimator of the survival function \bar{F}_T . Note that its asymptotic properties can then be easily derived from established results on the KM estimator under a survival model with independent censoring (see e.g. [6, 45]).

5.2 Semi-parametric estimation of \bar{F}_C , \tilde{P} and τ

Since the survival estimation \bar{F}_T can be estimated by the KM estimator, we need to approximate the survival function \bar{F}_C and the parameter ξ in order to estimate the survival joint distribution. To this end, we consider a semi-parametric approach based on maximum likelihood (ML) with Bernstein polynomial functions. Bernstein polynomials have been used by many authors in smoothing estimators (see e.g. [58], [35] and [40]) using functions of the form

$$s \in [0, 1] \rightarrow \sum_{k=0}^m \phi_k b_{k,m}(s), \quad m \in \mathbb{N}$$

where m is the degree, $b_{k,m}(s) = C_m^k s^k (1-s)^{m-k}$ and $\bar{\phi}_m = (\phi_0, \dots, \phi_m)^T \in \mathbb{R}^{m+1}$. In this context, we assume that both random times T and C have equal supports. For compact supports $S_{F_C} = S_{F_T} = [a, b]$, we can transform the data linearly to the interval $[0, 1]$, using

$$\tilde{Z} = (Z - a)/(b - a) = \min(T - a)/(b - a), (C - a)/(b - a) = \min(\tilde{T}, \tilde{C}).$$

If $F_{\tilde{C},m}$ and $g_{\tilde{C},m}$ denote estimates of the distribution and the density functions of \tilde{C} , respectively, then estimators of the function F_C and f_C can be obtained using the following relations:

$$F_{C,m}(t) = F_{\tilde{C},m}\left(\frac{t-a}{b-a}\right) \quad \text{and} \quad f_{C,m}(t) = \frac{1}{b-a} f_{\tilde{C},m}\left(\frac{t-a}{b-a}\right),$$

for $t \in [a, b]$. Transformations such as $\tilde{Z} = Z/(1+Z)$ and $\tilde{Z} = (1/2) + (\pi/2) \tan^{-1}(Z)$ can be used also to handle the cases of random variables with support \mathbb{R}_+ and \mathbb{R} , respectively, with the proper change of variables to recover $F_{C,m}$ and $f_{C,m}$. In the following, we consider that $S_{F_C} = S_{F_T} = [0, 1]$. Based on Proposition 2, the log-likelihood is given by

$$\begin{aligned} \mathcal{L}(\xi, \bar{F}_C, f_C) &= \sum_{i=1}^n \delta_i \log\{f_{Z,\delta=1}(Z_i)\} + (1 - \delta_i) \log\{f_{Z,\delta=0}(Z_i)\} \\ &= \sum_{i=1}^n \delta_i \log\{f_Z(Z_i) - f_{Z,\delta=0}(Z_i)\} + (1 - \delta_i) \log\{f_{Z,\delta=0}(Z_i)\}, \end{aligned}$$

where $f_{Z,\delta=0}(t) = \bar{F}_T(t)e^{t\xi} [f_C(t) - \xi \bar{F}_C(t)]$ and f_Z is the density of Z . The direct maximization of this likelihood can be challenging, since it depends on an unknown functions f_Z and \bar{F}_T . The idea is now to consider an approximation of the log-likelihood \mathcal{L} through the following steps:

(i) we approximate f_Z using a non-parametric kernel method

$$f_{Z,n}(t) = \frac{1}{nh} \sum_{i=1}^n K\left(\frac{t - Z_i}{h_n}\right), \quad (5.2)$$

where K represents a kernel function and $h_n > 0$ is a bandwidth parameter.

(ii) the function \bar{F}_T is estimated using the KM estimator $\bar{F}_{T,n}$.

(iii) the functions F_C and f_C are approximated using the Bernstein polynomials. The estimators for the distribution function and the density function are then given by

$$F_{C,m}(t|\bar{\phi}_m) = \sum_{k=0}^m \phi_k b_{k,m}(t) \quad \text{and} \quad f_{C,m}(t) = \sum_{k=0}^m \phi_k b'_{k,m}(t), \quad (5.3)$$

where the coefficients $\bar{\phi}_m$ are replaced by estimates obtained from the ML approach.

Therefore, we approximate and parameterize the distribution F_C using a mixture of beta distributions, and we estimate $\bar{\phi}_m$ as parameters of the log-likelihood function \mathcal{L} . This approximation is motivated by the uniform convergence of $F_{C,m}$ as provided in Theorem 2.1 of [8], as well as by, common results on Bernstein polynomial estimation, which ensure that the best convergence rate of $F_{C,m}(t)$ to $F_C(t)$ is of order m^{-1} .

In summary, let $\mathcal{L}_m(\xi, \bar{\phi}_m)$ denote the approximate log-likelihood obtained by replacing \bar{F}_T , f_Z , \bar{F}_C and f_C with $\bar{F}_{T,n}$, $f_{Z,n}$, $\bar{F}_{C,m}$ and $f_{C,m}$ respectively. We consider the parameter vector $\theta = (\bar{\phi}_m, \xi)$ and define the ML estimator

$$\theta_n = \operatorname{argmax}_{\theta \in \mathbb{R}^{m+2}} \mathcal{L}_m(\theta).$$

Based on the estimator $\theta_n = (\bar{\phi}_{n,m}, \xi_n)$, we obtain the estimators of the survival function \bar{F}_C and the joint law \tilde{P} namely

$$\begin{aligned} \bar{F}_{C,m,n}(t) &= 1 - \sum_{k=0}^m \phi_{k,n} b_{k,m}(t), \\ \tilde{P}_{m,n}(t, s) &= \bar{F}_{C,m,n}(s) \bar{F}_{T,n}(t) \min\left(\bar{F}_{T,n}(t)^{-\alpha_n(t)}, \bar{F}_{C,m,n}(s)^{-\beta_{m,n}(s)}\right), \end{aligned}$$

where

$$\alpha_n(t) = \frac{\xi_n t}{-\ln(\bar{F}_{T,n}(t))} \quad \text{and} \quad \beta_{m,n}(t) = \frac{\xi_n t}{-\ln(\bar{F}_{C,m,n}(t))}.$$

Finally, we propose an estimator for the Kendall's tau coefficient based on the above expression. Conveniently, this coefficient depends only on the survival function of the random variable Z and the stochastic process parameter ξ . Using the ML estimate of the former parameter, we obtain the estimator

$$\tau_n = 2\xi_n \int_0^{+\infty} \bar{H}_n(u)^2 du.$$

5.3 Asymptotic properties

This section presents the consistency and asymptotic normality of the estimators defined in Subsection 5.2, along with the set of assumptions required to prove these results. To establish the asymptotic normality of θ_n , we rely on the framework provided by [60] which develops sufficient conditions for the asymptotic normality of the maximum likelihood estimators under misspecified models. To do so, we consider the optimization problem based on the minimization of the Kullback-Leibler information criterion given by $I(g, h; \theta) = \mathbb{E}(\log [g(Z, \delta; \xi)/h(Z, \delta; \theta)])$ with densities

$$g(t, b; \xi) = \left[f_Z(t) - \bar{F}_T(t)e^{t\xi} (f_C(t) - \xi \bar{F}_C(t)) \right]^b \left[\bar{F}_T(t)e^{t\xi} (f_C(t) - \xi \bar{F}_C(t)) \right]^{1-b},$$

and

$$\begin{aligned} & h(t, b; \theta) \\ &= \left[f_{Z,n}(t) - \bar{F}_{T,n}(t)e^{t\xi} (f_{C,m,n}(t|\bar{\phi}_m) - \xi \bar{F}_{C,m,n}(t|\bar{\phi}_m)) \right]^b \left[\bar{F}_{T,n}(t)e^{t\xi} (f_{C,m,n}(t|\bar{\phi}_m) - \xi \bar{F}_{C,m,n}(t|\bar{\phi}_m)) \right]^{1-b}, \end{aligned}$$

with $\bar{F}_{T,n}$, $f_{Z,n}$, $\bar{F}_{C,m,n}$ and $f_{C,m,n}$ the estimators defined in the previous section. Let us introduce the following matrices

$$A(\theta) = \left(\mathbb{E} \left[\frac{\partial^2 \log h(Z, \delta; \theta)}{\partial \theta_i \partial \theta_j} \right] \right)_{1 \leq i, j \leq m+2} \quad \text{and} \quad B(\theta) = \left(\mathbb{E} \left[\frac{\partial \log h(Z, \delta; \theta)}{\partial \theta_i} \frac{\partial \log h(Z, \delta; \theta)}{\partial \theta_j} \right] \right)_{1 \leq i, j \leq m+2}.$$

We need the following regularity conditions to establish asymptotic properties of our estimators:

(**H**₁): $\Lambda_2 e^{-\Lambda_2}$ and $\Lambda_1 e^{-\Lambda_1}$ are element of $L^2(0, \infty)$.

(**H**₂): The kernel K is bounded up by a positive constant K_{max} .

(**H**₃): I has unique minimum at θ^* .

(**H**₄): (a) θ^* is interior to \mathbb{R}_+^{m+2} .

(b) $B(\theta^*)$ is non singular.

(c) θ^* is a regular point of $A(\theta)$.

Remark 2. Assumption (**H**₁) is essential for establishing the bounded expectation result in Proposition 5. Assumption (**H**₂) is a technical requirement ensuring that condition (5) of [60] is satisfied. Assumptions (**H**₃) and (**H**₄) serve as regularity conditions commonly adopted in maximum likelihood analysis to demonstrate asymptotic properties.

One requirement in the study of the asymptotic of the estimators is to ensure that the the random time Z admit finite moments for any order. This results is stated in the following proposition.

Proposition 5. Suppose that Λ_1 and Λ_2 satisfy (**H**₁). Then, all moments of order k for Z exist, that is

$$\mathbb{E}(Z^k) < \infty.$$

This allows us, in the next proposition, to show the asymptotic normality for the parameter vector θ_n .

Proposition 6. *Under assumptions $(\mathbf{H}_1) - (\mathbf{H}_4)$, θ is identifiable and*

$$\sqrt{n}(\theta_n - \theta) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, C(\theta)),$$

where $C(\theta) = A(\theta)^{-1} B(\theta) A(\theta)^{-1}$.

The proof of Proposition 6 is done by verifying the conditions of Theorem 3.2 in [60]. In the next proposition, we finally establish the simple convergence for the joint distribution estimator and the Kendall's tau coefficient.

Proposition 7. *Under assumptions $(\mathbf{H}_1) - (\mathbf{H}_4)$,*

$$\lim_{n,m \rightarrow \infty} \widehat{P}_{n,m}(t,s) = \widetilde{P}(t,s) \text{ and } \lim_{n \rightarrow \infty} \tau_n = \tau.$$

6 Simulation and real data analysis

6.1 Simulation study

In this subsection, we evaluate the performance of the survival distribution estimators and the survival copula estimator. We considered $N = 100$ samples of sizes $n = 30, 50, 100$ under the following models:

- a) $\Lambda_1(t) = 0.01t$ and $\Lambda_2(t) = 0.02t$ defined on $[0, 100]$, and $(Y_t)_{0 < t \leq 100}$ is a Poisson process with intensity $\lambda = 2$.
- b) $\Lambda_1(t) = -\ln(1 - I_t(2.5, 6)) - 0.4t$ and $\Lambda_2(t) = -\ln(1 - I_t(3, 2)) - 0.4t$ defined on $[0, 100]$, and $(Y_t)_{0 < t \leq 100}$ is a Poisson process with intensity $\lambda = 2$, where I_t is the incomplete beta function.

The selection of the Bernstein polynomial degree m is determined by the available data based on the Akaike information criterion (AIC)

$$AIC(m) = 2(m + 2) - 2\mathcal{L}_m(\xi, \bar{\phi}_m).$$

The AIC is calculated to fit models across a grid of degree m , with the default upper bound set to n , so that the selected Bernstein degree minimizes the AIC. As a measure of performance, we evaluate the Mean Integrated Squared Error (MISE) of the mentioned estimators. Specifically, for a function F and its estimator \widehat{F} , let

$$\text{ISE}(\widehat{F}) = \int_a^b [\widehat{F}(x) - F(x)]^2 dx, \quad (6.1)$$

where $a = \min_{1 \leq i \leq n} (Z_i)$, $b = \max_{1 \leq i \leq n} (Z_i)$. Then for N sample of size n the MISE of \widehat{F} is given by the Monte Carlo approximation:

$$\text{MISE}(\widehat{F}) = \frac{1}{N} \sum_{i=1}^N \text{ISE}_i(\widehat{F}),$$

where $\text{ISE}_i(\widehat{F})$ denotes the value of the Integrated Squared Error (ISE) calculated on the i -th randomly generated sample.

Figures 1 and 2 display the survival estimators of \bar{F}_T and \bar{F}_C , respectively, under the two models a) and b). Figure 3 shows the estimation results of the functions α and β , while Figure 4 displays the

results of the survival copula estimation. Figure 1 and 2 show that the survival estimators converge toward the true marginal survival functions. This is again an attractive feature of the proposed model since the dependency between T and C does not affect the KM estimator in this scenario. Figure 3 shows that the estimators of α and β are constants under model a). This can be attributed to the fact that model a) is equivalent to a MO exponential distribution. As expected from the consistency result in Proposition 7, the discrepancy between $\tilde{P}_{n,m}$ and \tilde{P} decreases, as indicated by the plots in Figure 4. We observe that, in all cases presented in Table 1, the value of the MISE decrease along with the sample size n , for appropriate choices of the degree m .

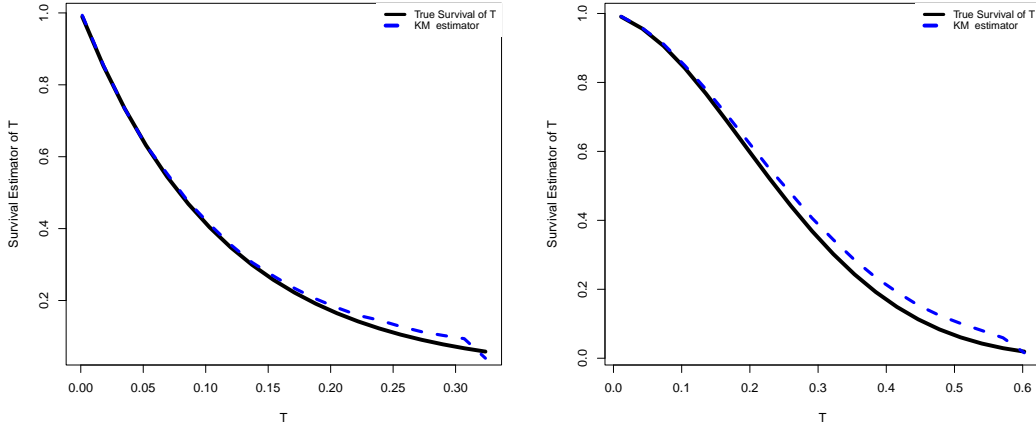


Figure 1: Kaplan Meier estimator of $\overline{F}_T(t)$ with $N = 100$ iterations, for a size $n = 100$ for the model a) (left panel) and the model b) (right panel).

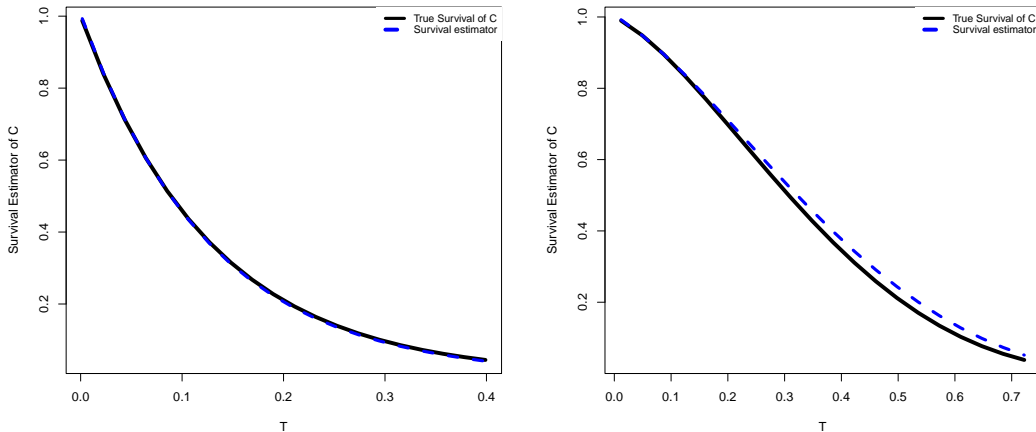


Figure 2: Bernstein estimators of $\overline{F}_C(t)$ with $N = 100$ iterations, for a size $n = 100$ for the model a) (left panel) and the model b) (right panel).

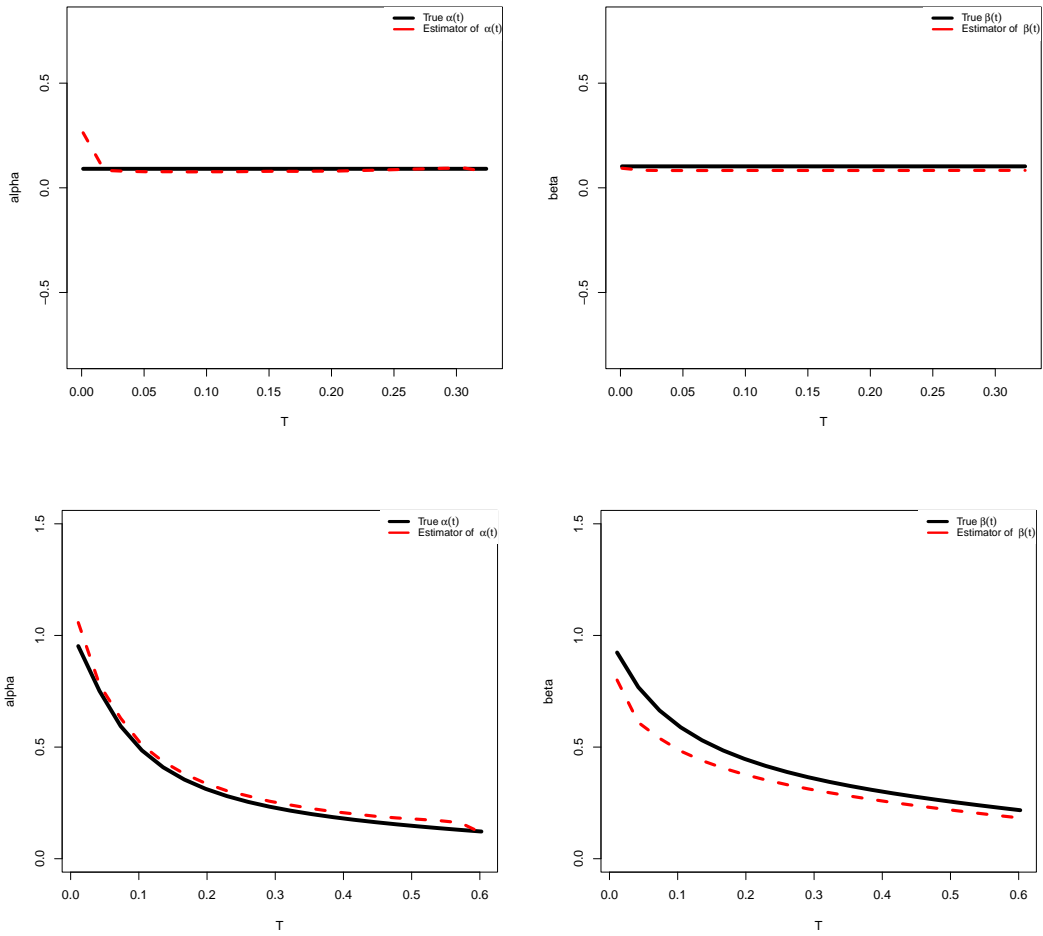


Figure 3: Estimators of $\alpha(t)$ and $\beta(t)$ with $N = 100$ iterations, for a size $n = 100$ for the model *a*) (up) and the model *b*) (down).

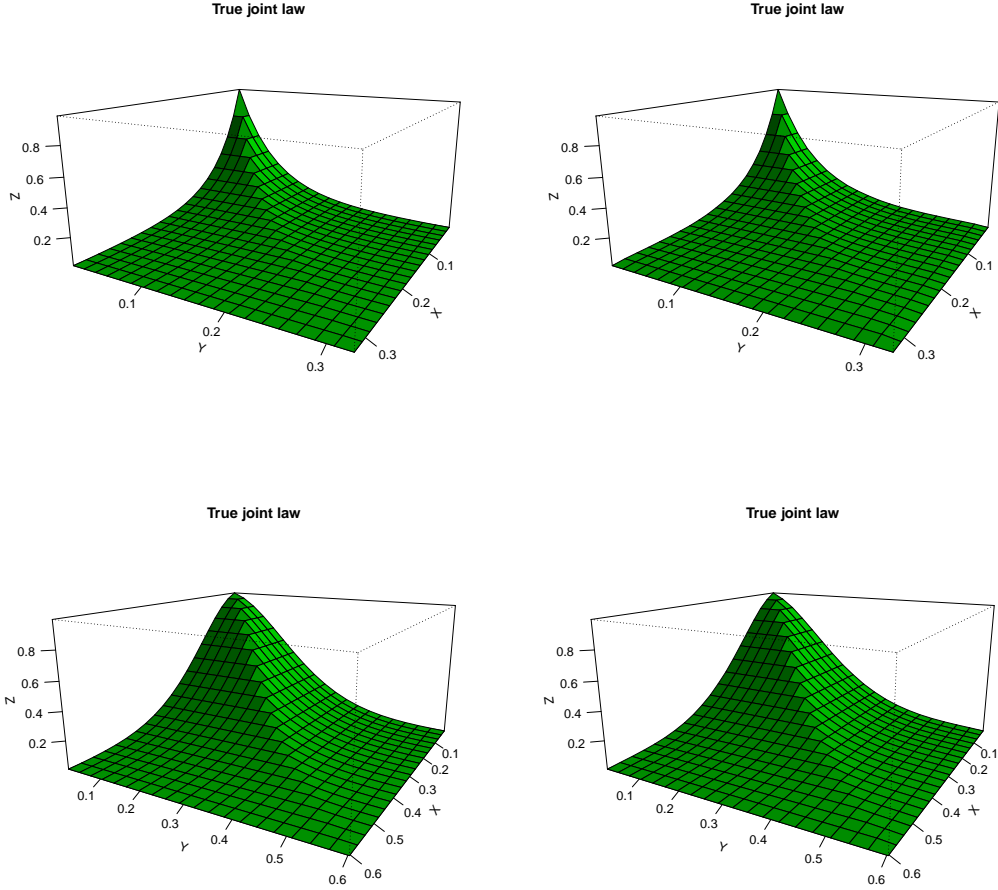


Figure 4: Survival estimator of the joint law with $N = 100$ iterations, for a size $n = 100$ for the model *a)* (up) and the model *b)* (down).

	n	$\overline{F}_{T,n}$	$\overline{F}_{C,n,m}$	$\tilde{P}_{n,m}$
Model a)	30	0.004518	0.015534	0.005218
	50	0.001618	0.001471	0.004331
	100	0.000278	0.000117	0.001298
Model b)	30	0.003696	0.003161	0.003198
	50	0.001418	0.000227	0.000573
	100	0.000180	0.000201	0.000289

Table 1: Results for MISE for $N = 100$ trials of the proposed estimators $\overline{F}_{T,n}$, $\overline{F}_{C,m,n}$ and $\tilde{P}_{n,m}$, for $n = 30$, $n = 50$ and $n = 100$.

6.2 Real data application

In this subsection, we apply the proposed estimation procedure to the joint retirement problem of married couples, using data from [34] and [44] drawn from eight waves of the Health and Retirement Study (every 2 years from 1992 to 2006). The duration variables (T, C) represent the retirement dates of the wife and the husband, respectively. The duration $Z = \min(T, C)$ represents the first entry into retirement for the corresponding member of the household and $\delta = \mathbb{1}_{\{T \leq C\}}$ indicates whether the wife retires first or the husband does. The latent stochastic processes $(\bar{Y}_t)_{t \geq 0}$ characterize the aging processes for the elderly. The sample consists of 821 households. We note that the supports are $S_T = S_C = [1, 176]$. Figure 5 depicts the KM survival estimator and the Bernstein estimator of \bar{F}_T and \bar{F}_C , respectively. Moreover, the results seem to contradict the assumption that the data come from the model proposed in [44]. In particular, with the estimation of the functions α and β , Figure 6 shows that the results are not constant and do not always reach the expected values for [44]’s model, where $\Lambda_1(t) = \Lambda_2(s) = 0$. Figure 7 display the joint survival estimator of $\tilde{P}(t, s)$.

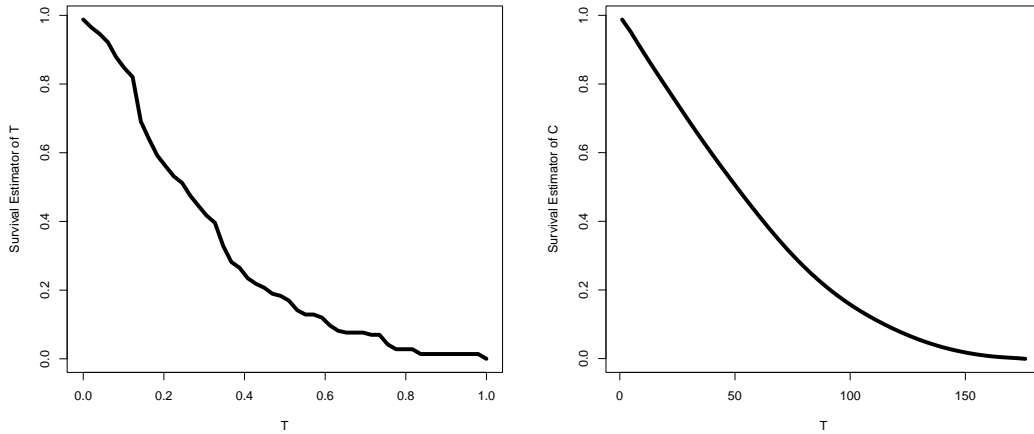


Figure 5: Results for the survival distribution estimators of $\bar{F}_T(t)$ and $\bar{F}_C(t)$ with the real data.

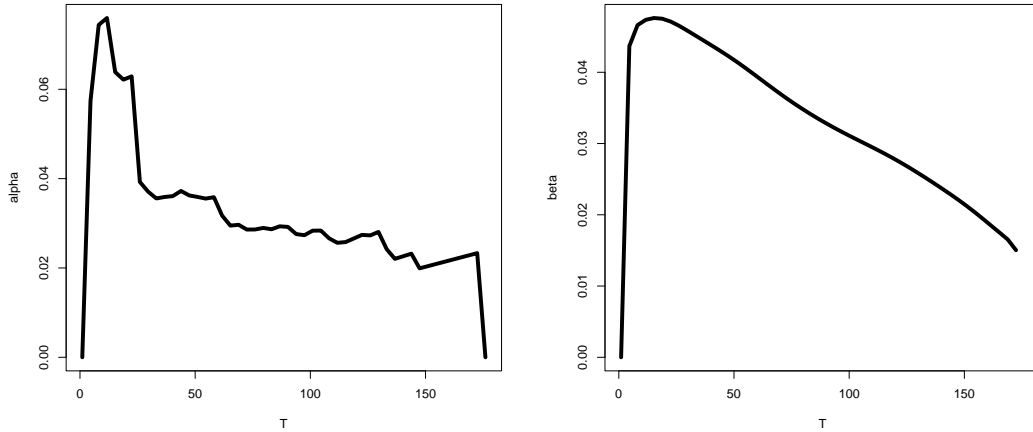


Figure 6: Results for the estimators of $\alpha(t)$ and $\beta(t)$ with the real data.

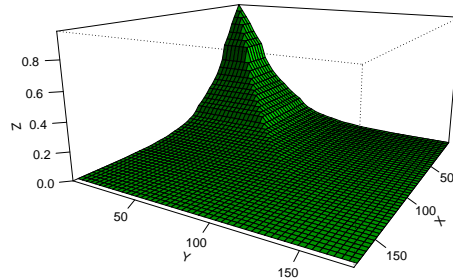


Figure 7: Results for the estimators of $\tilde{P}(t, s)$ with the real data.

7 Conclusion

This paper presents a dependent censoring model for analyzing survival times, based on the canonical default model. We establish the identifiability of this model given censored data. Several theoretical results related to the model's functions, specifically the survival copula and cumulative hazard functions, are analyzed. A flexible non-/semi-parametric estimation method is proposed and illustrated through the analysis of synthetic and real datasets. This work can be considered a preliminary study for further investigations into the survival copula based on threshold-crossing models. Indeed, our study provides a theoretical foundation for characterizing the survival copula in the context of the canonical model, based on a latent non-homogeneous process.

8 Proofs

Proof of Proposition 1:

Let $t, s \geq 0$. According to Theorem 2.2 in [32], we have

$$\tilde{P}(t, s) = \mathbb{P}(T > t, C > s) = \bar{F}_T(t)\bar{F}_C(s) \frac{\psi_{Y_t+Y_s}(1)}{\psi_{Y_t}(1)\psi_{Y_s}(1)}.$$

Suppose that $t < s$. Using the fact that $(Y_t)_{t \geq 0}$ has independent and stationary increments, and that

$$Y_t + Y_s = 2(Y_t - Y_0) + (Y_s - Y_t),$$

we obtain

$$\begin{aligned} \tilde{P}(t, s) &= \bar{F}_T(t)\bar{F}_C(s) \frac{\psi_{2(Y_t-Y_0)+(Y_s-Y_t)}(1)}{\psi_{Y_t}(1)\psi_{Y_s}(1)} \\ &= \bar{F}_T(t)\bar{F}_C(s) \frac{\psi_{Y_t}(2)\psi_{Y_s-t}(1)}{\psi_{Y_t}(1)\psi_{Y_s}(1)} \\ &= \bar{F}_T(t)\bar{F}_C(s) \frac{e^{-t\kappa(2)}e^{-(s-t)\kappa(1)}}{e^{-t\kappa(1)}e^{-s\kappa(1)}} \\ &= \bar{F}_T(t)\bar{F}_C(s)e^{-t\kappa(2)}e^{-(s-t)\kappa(1)}e^{t\kappa(1)}e^{s\kappa(1)} \\ &= \bar{F}_T(t)\bar{F}_C(s)e^{-t\kappa(2)}e^{-s\kappa(1)+t\kappa(1)}e^{t\kappa(1)}e^{s\kappa(1)} \\ &= \bar{F}_T(t)\bar{F}_C(s)e^{t\xi} \\ &= \bar{F}_T(t)\bar{F}_C(s)e^{t\xi}. \end{aligned}$$

Similarly, in the case where $s < t$, we obtain

$$\tilde{P}(t, s) = \bar{F}_T(t)\bar{F}_C(s)e^{s\xi}.$$

Therefore,

$$\tilde{P}(t, s) = \bar{F}_T(t)\bar{F}_C(s)e^{\xi \min(t, s)} = \bar{F}_T(t)\bar{F}_C(s) \min(e^{\xi t}, e^{\xi s}).$$

In other hand, according to Theorem 2.2 in [32], we have

$$\bar{F}_T(t) = e^{-\Lambda_1(t)-\kappa(1)t} \quad \text{and} \quad \bar{F}_C(t) = e^{-\Lambda_2(t)-\kappa(1)t}.$$

Hence,

$$e^{\xi t} = \bar{F}_T(t)^{-\alpha(t)} = \bar{F}_C(t)^{-\beta(t)}, \quad \forall t \geq 0,$$

where

$$\alpha(t) = \frac{\xi t}{\Lambda_1(t) + t\kappa(1)} \quad \text{and} \quad \beta(t) = \frac{\xi t}{\Lambda_2(t) + t\kappa(1)}.$$

It implies that

$$\tilde{P}(t, s) = \tilde{\mathcal{C}}(\bar{F}_T(t), \bar{F}_C(s)) = \bar{F}_C(s)\bar{F}_T(t) \min\left(\bar{F}_T(t)^{-\alpha(t)}, \bar{F}_C(s)^{-\beta(s)}\right),$$

and therefore the survival copula is given by

$$\tilde{\mathcal{C}}(u, v) = vu \min\left(u^{-\alpha(\bar{F}_T^-(u))}, v^{-\beta(\bar{F}_C^-(v))}\right).$$

Proof of Proposition 2:

Let $t \geq 0$. Then, we have,

$$\begin{aligned}
\bar{H}(t) &= \mathbb{P}(Z > t) \\
&= \mathbb{E}[\mathbb{P}(Z > t | Y_t)] \\
&= \mathbb{E}[\mathbb{P}(T > t, C > t | Y_t)] \\
&= \mathbb{E}[\mathbb{P}(\inf\{x \geq 0, e^{-M_{1,x}} \leq U_1\} > t, \inf\{x \geq 0, e^{-M_{2,x}} \leq U_2\} > t | Y_t)] \\
&= \mathbb{E}[\mathbb{P}(U_1 < e^{-M_{1,t}}, U_2 < e^{-M_{2,t}} | Y_t)] \\
&= \mathbb{E}[\mathbb{P}(U_1 < e^{-M_{1,t}} | Y_t) \mathbb{P}(U_2 < e^{-M_{2,t}} | Y_t)] \\
&= \mathbb{E}[e^{-M_{1,t}} e^{-M_{2,t}}] \\
&= e^{-\Lambda_1(t)} e^{-\Lambda_2(t)} e^{-t\kappa(2)}.
\end{aligned}$$

Let $s < t$, we have

$$\mathbb{P}(T > t, C > s) = \bar{F}_T(t) \bar{F}_C(s) e^{s\xi}. \quad (8.1)$$

Therefore, according to Theorem 1 of [7], we obtain

$$f_{Z,\delta=0}(t) = -\lim_{y \rightarrow t} \frac{\partial \mathbb{P}(T > t, C > y)}{\partial y}.$$

Using Equation (8.1), we obtain

$$\frac{\partial \mathbb{P}(T > t, C > y)}{\partial y} = \frac{\partial [\bar{F}_T(t) \bar{F}_C(y) e^{y\xi}]}{\partial y} = \bar{F}_T(t) [-f_C(y) e^{y\xi} + \xi \bar{F}_C(y) e^{y\xi}].$$

Hence,

$$\begin{aligned}
f_{Z,\delta=0}(t) &= -\lim_{y \rightarrow t} \bar{F}_T(t) [-f_C(y) e^{y\xi} + \xi \bar{F}_C(y) e^{y\xi}] \\
&= -\bar{F}_T(t) [\bar{F}'_C(t) e^{t\xi} + \xi \bar{F}_C(t) e^{t\xi}] \\
&= \bar{F}_T(t) e^{t\xi} [-\bar{F}'_C(t) - \xi \bar{F}_C(t)] \\
&= \bar{F}_T(t) e^{t\xi} [(\Lambda'_2(t) + \kappa(1)) \bar{F}_C(t) - \xi \bar{F}_C(t)] \\
&= \bar{F}_C(t) \bar{F}_T(t) e^{t\xi} [\Lambda'_2(t) + \kappa(1) - \xi] \\
&= \bar{H}(t) [\Lambda'_2(t) + \kappa(1) - \xi].
\end{aligned}$$

Moreover,

$$\begin{aligned}
f_Z(t) &= -\bar{H}'(t) \\
&= [\Lambda'_1(t) + \Lambda'_2(t) + \kappa(2)] e^{-\Lambda_1(t) - \Lambda_2(t) - t\kappa(2)} \\
&= [\Lambda'_1(t) + \Lambda'_2(t) + \kappa(2)] \bar{H}(t).
\end{aligned}$$

Consequently,

$$\begin{aligned}
f_{Z,\delta=1}(t) &= f_Z(t) - f_{Z,\delta=0}(t) \\
&= [\Lambda'_1(t) + \Lambda'_2(t) + \kappa(2)] \bar{H}(t) - \bar{H}(t) [\Lambda'_2(t) + \kappa(1) - \xi] \\
&= \bar{H}(t) [\Lambda'_1(t) + \Lambda'_2(t) + \kappa(2) - \Lambda'_2(t) - \kappa(1) + 2\kappa(1) - \kappa(2)] \\
&= \bar{H}(t) [\Lambda'_1(t) + \kappa(1)],
\end{aligned}$$

which implies that

$$\frac{f_{Z,\delta=1}(t)}{\overline{H}(t)} = \Lambda'_1(t) + \kappa(1).$$

On the other hand, we have

$$\begin{aligned} \frac{f_T(t)}{\overline{F}_T(t)} &= \frac{-\overline{F}'_T(t)}{\overline{F}_T(t)} \\ &= \frac{(\Lambda'_1(t) + \kappa(1))e^{-\Lambda_1(t)-t\kappa(1)}}{e^{-\Lambda_1(t)-t\kappa(1)}} \\ &= \Lambda'_1(t) + \kappa(1). \end{aligned}$$

Then

$$\frac{f_{Z,\delta=1}(t)}{\overline{H}(t)} = \frac{f_T(t)}{\overline{F}_T(t)}.$$

Similarly, we have

$$\frac{f_{Z,\delta=0}(t)}{\overline{H}(t)} = \frac{f_C(t)}{\overline{F}_C(t)} - \xi.$$

Proof of Proposition 3:

Consider the model defined in (1.1)-(1.2). Let X_1, X_2 , and X_3 be three independent random variables with cumulative Hazard functions $\Lambda_{X_1}, \Lambda_{X_2}$, and Λ_{X_3} , respectively. Suppose that, for $t \geq 0$

$$\begin{cases} \Lambda_{X_1}(t) = \Lambda_1(t) + t\kappa(1) - \xi t \\ \Lambda_{X_2}(t) = \Lambda_2(t) + t\kappa(1) - \xi t \\ \Lambda_{X_3}(t) = \xi t. \end{cases} \quad (8.2)$$

Define the random variables X and Y as follows

$$X = \min(X_1, X_3), \quad \text{and} \quad Y = \min(X_2, X_3).$$

Then, according to Proposition 1 in [27], the survival copula of (X, Y) is given by

$$\tilde{C}_{(X,Y)}(u, v) = uv \min \left(u^{-\alpha_1(\overline{F}_X^-(u))}, v^{-\alpha_2(\overline{F}_Y^-(v))} \right),$$

where

$$\alpha_1(t) = \frac{\Lambda_{X_3}(t)}{\Lambda_{X_3}(t) + \Lambda_{X_1}(t)} = \frac{\xi t}{\Lambda_1(t) + t\kappa(1)},$$

and

$$\alpha_2(t) = \frac{\Lambda_{X_3}(t)}{\Lambda_{X_3}(t) + \Lambda_{X_2}(t)} = \frac{\xi t}{\Lambda_2(t) + t\kappa(1)}.$$

Therefore (X, Y) has the same copula as (T, C) . In the other hand,

$$\overline{F}_X(t) = \overline{F}_{X_1}(t)\overline{F}_{X_3}(t) = e^{-\Lambda_1(t)-\Lambda_3(t)} = e^{-\Lambda_1(t)-t\kappa(1)} = \overline{F}_T(t).$$

Similarly, we have

$$\overline{F}_Y(t) = \overline{F}_{X_2}(t)\overline{F}_{X_3}(t) = e^{-\Lambda_2(t)-\Lambda_3(t)} = e^{-\Lambda_2(t)-t\kappa(1)} = \overline{F}_C(t).$$

Then, (X, Y) has the same distribution as (T, C) . Moreover,

$$T \stackrel{d}{=} \min(X_1, X_3) \quad \text{and} \quad C \stackrel{d}{=} \min(X_2, X_3).$$

Proof of Proposition 4:

Suppose that

$$f_{Z,\delta=1,\Gamma^1}(t) = f_{Z,\delta=1,\Gamma^2}(t) \quad \text{and} \quad f_{Z,\delta=0,\Gamma^1}(t) = f_{Z,\delta=1,\Gamma^2}(t),$$

where $\Gamma^i = (\Lambda_1^i, \Lambda_2^i, \kappa(1)^i, \kappa(2)^i)$ for $i = 1, 2$. Then, for $t \in [0, \infty[$, we have

$$e^{-\Lambda_1^1(t) - \Lambda_2^1(t) - t\kappa(2)^1} (\Lambda_1^1(t)' + \kappa(1)^1) = e^{-\Lambda_1^2(t) - \Lambda_2^2(t) - t\kappa(2)^2} (\Lambda_1^2(t)' + \kappa(1)^2), \quad (8.3)$$

and

$$\begin{aligned} & e^{-\Lambda_1^1(t) - \Lambda_2^1(t) - t\kappa(2)^1} (\Lambda_2^1(t)' + \kappa(2)^1 - \kappa(1)^1) \\ &= e^{-\Lambda_1^2(t) - \Lambda_2^2(t) - t\kappa(2)^2} (\Lambda_2^2(t)' + \kappa(2)^2 - \kappa(1)^2). \end{aligned} \quad (8.4)$$

Since $\Lambda_1'(0) = \Lambda_2'(0) = 0$, then for $t = 0$, Equation (8.3) implies that $\kappa(1)^1 = \kappa(1)^2$. Moreover, for $t = 0$, Equation (8.4) implies that $\kappa(2)^1 = \kappa(2)^2$. By summing Equations (8.3) and (8.4), we obtain

$$e^{-E^1(t)} (E^1(t)') = e^{-E^2(t)} (E^2(t)'),$$

where $E^i(t) = \Lambda_1^i(t) + \Lambda_2^i(t) + \kappa(2)^i$. Then $[e^{-E^1(t)}]' = [e^{-E^2(t)}]'$. It implied that, for $x \in [0, \infty[$,

$$\int_0^x [e^{-E^1(t)}]' dt = \int_0^x [e^{-E^2(t)}]' dt,$$

and therefore $e^{-E^1(x)} = e^{-E^2(x)}$, since $\Lambda_1^1(0) = \Lambda_1^2(0) = \Lambda_2^1(0) = \Lambda_2^2(0) = 0$. Substituting this result into Equation (8.3), we obtain $\Lambda_1^1(t)' = \Lambda_1^2(t)'$. Then

$$\int_0^x \Lambda_1^1(t)' dt = \int_0^x \Lambda_1^2(t)' dt,$$

and therefore $\Lambda_1^1(x) = \Lambda_1^2(x)$, since $\Lambda_1^1(0) = \Lambda_1^2(0) = 0$. Substituting this result into Equation (8.4), we obtain $\Lambda_2^1(t)' = \Lambda_2^2(t)'$. Then

$$\int_0^x \Lambda_2^1(t)' dt = \int_0^x \Lambda_2^2(t)' dt,$$

and therefore $\Lambda_2^1(x) = \Lambda_2^2(x)$, since $\Lambda_2^1(0) = \Lambda_2^2(0) = 0$. Finally, we conclude that $\Gamma^1 = \Gamma^2$.

Proof of Proposition 5:

Suppose that Λ_1 and Λ_2 satisfy (\mathbf{H}_1) . Let $k \in \mathbb{N}$. We have

$$\begin{aligned} \mathbb{E}(Z^k) &= \int_0^\infty t^k (\Lambda_1'(t) + \Lambda_2'(t) + \kappa(2)) e^{-\Lambda_1(t) - \Lambda_2(t) - t\kappa(2)} dt \\ &= \int_0^\infty t^k (\Lambda_1'(t) + \Lambda_2'(t) + \kappa(2)) e^{-\Lambda_1(t) - \Lambda_2(t) - t\frac{\kappa(2)}{2} - t\frac{\kappa(2)}{2}} dt \\ &\leq \left(\int_0^\infty t^{2k} e^{-t\kappa(2)} dt \right)^{1/2} \times \left(\int_0^\infty (\Lambda_1'(t) + \Lambda_2'(t) + \kappa(2))^2 e^{-2\Lambda_1(t) - 2\Lambda_2(t) - t\kappa(2)} dt \right)^{1/2} \\ &= \left(\frac{(2k)!}{\kappa(2)^{2k+1}} \right)^{1/2} \times I^{1/2}, \end{aligned}$$

where

$$\begin{aligned}
I &= \int_0^\infty \left(\Lambda_1'(t) + \Lambda_2'(t) + \kappa(2) \right)^2 e^{-2\Lambda_1(t)-2\Lambda_2(t)-t\kappa(2)} dt \\
&= \int_0^\infty \Lambda_1'(t)^2 e^{-2\Lambda_1(t)-2\Lambda_2(t)-t\kappa(2)} dt + \int_0^\infty \Lambda_2'(t)^2 e^{-2\Lambda_1(t)-2\Lambda_2(t)-2t\kappa(2)} dt \\
&\quad + \kappa(2)^2 \int_0^\infty e^{-2\Lambda_1(t)-2\Lambda_2(t)-t\kappa(2)} dt + 2 \int_0^\infty \Lambda_1'(t)\Lambda_2'(t) e^{-2\Lambda_1(t)-2\Lambda_2(t)-t\kappa(2)} dt \\
&\quad + 2\kappa(2) \int_0^\infty \Lambda_1'(t) e^{-2\Lambda_1(t)-2\Lambda_2(t)-t\kappa(2)} dt + 2\kappa(2) \int_0^\infty \Lambda_2'(t) e^{-2\Lambda_1(t)-2\Lambda_2(t)-t\kappa(2)} dt \\
&\leq \int_0^\infty \Lambda_1'(t)^2 e^{-2\Lambda_1(t)} dt + \int_0^\infty \Lambda_2'(t)^2 e^{-2\Lambda_2(t)} dt + \kappa(2)^2 \int_0^\infty e^{-t\kappa(2)} dt \\
&\quad + 2 \int_0^\infty \Lambda_1'(t)\Lambda_2'(t) e^{-\Lambda_1(t)-\Lambda_2(t)} dt + 2\kappa(2) \int_0^\infty \Lambda_1'(t) e^{-\Lambda_1(t)} dt + 2\kappa(2) \int_0^\infty \Lambda_2'(t) e^{-\Lambda_2(t)} dt \\
&= \int_0^\infty \Lambda_1'(t)^2 e^{-2\Lambda_1(t)} dt + \int_0^\infty \Lambda_2'(t)^2 e^{-2\Lambda_2(t)} dt + 2 \int_0^\infty \Lambda_1'(t)\Lambda_2'(t) e^{-\Lambda_1(t)-\Lambda_2(t)} dt + 5\kappa(2).
\end{aligned}$$

Using Hölder's inequality we obtain

$$\begin{aligned}
\int_0^\infty \Lambda_2'(t)\Lambda_1'(t) e^{-\Lambda_1(t)-\Lambda_2(t)} dt &\leq \left(\int_0^\infty \Lambda_1'(t)^2 e^{-2\Lambda_1(t)} dt \right)^{1/2} \times \left(\int_0^\infty \Lambda_2'(t)^2 e^{-2\Lambda_2(t)} dt \right)^{1/2} \\
&< \infty,
\end{aligned}$$

which implies that I exists and then $\mathbb{E}(Z^k) < \infty$.

Proof of Proposition 6:

In order to prove this proposition, we propose to verify the conditions (1) – (6) of [60].

The independent random vector (Z_i, δ_i) for $i = 1 \dots n$ have a common joint distribution G on $[0, 1]^2$ with measurable Radon-Nikodym density given by

$$\begin{aligned}
g(t, b; \xi) &= f_{Z, \delta=b}(t) \\
&= [f_{Z, \delta=1}(t)]^b [f_{Z, \delta=0}(t)]^{1-b} \\
&= \left[f_Z(t) - \bar{F}_T(t) e^{t\xi} (f_C(t) - \xi \bar{F}_C(t)) \right]^b \left[\bar{F}_T(t) e^{t\xi} (f_C(t) - \xi \bar{F}_C(t)) \right]^{1-b}.
\end{aligned}$$

The Radon-Nykodym density function h defined by

$$\begin{aligned}
h(t, b; \theta) &= h(t, b; \xi, \bar{\phi}_m) \\
&= \left[f_{Z,n}(t) - \bar{F}_{T,n}(t) e^{t\xi} (f_{C,m}(t|\bar{\phi}_m) - \xi \bar{F}_{C,m}(t|\bar{\phi}_m)) \right]^b \\
&\quad \times \left[\bar{F}_{T,n}(t) e^{t\xi} (f_{C,m}(t|\bar{\phi}_m) - \xi \bar{F}_{C,m}(t|\bar{\phi}_m)) \right]^{1-b},
\end{aligned}$$

is measurable on (t, b) for every $\theta \in \mathbb{R}_+^{m+2}$ and continuous on θ of every $(t, b) \in [0, 1]^2$. Then conditions (1) and (2) of [60] are verified.

In order to show condition (3)a) of [60], we need first to show that $\mathbb{E}(\log g(Z, \delta; \xi))$ exist.

★ Suppose that $\delta = 0$. Based on assumption (\mathbf{H}_1) , we have $\Lambda_2'(t)e^{-\Lambda_2(t)}$ and $\Lambda_1'(t)e^{-\Lambda_1(t)}$ are element of $L^2(0, \infty)$. Since $\bar{F}_T(Z) \leq 1$ and $f_C(Z) - \xi \bar{F}_C(Z) \leq f_C(Z)$, then

$$\begin{aligned}
\mathbb{E}(\log g(Z, \delta; \xi)) &= \mathbb{E} \left[\log \left(\bar{F}_T(Z) e^{Z\xi} (f_C(Z) - \xi \bar{F}_C(Z)) \right) \right] \\
&\leq \mathbb{E}[\log(f_C(Z))] + \mathbb{E}(\log(e^{Z\xi})) \\
&\leq \mathbb{E}(f_C(Z)) + \xi \mathbb{E}(Z) \\
&\leq \int_0^\infty f_C(t) f_Z(t) dt + \xi \mathbb{E}(Z) \\
&= \int_0^\infty \left(\Lambda_2'(t) + \kappa(1) \right) \left(\Lambda_1'(t) + \Lambda_2'(t) + \kappa(2) \right) e^{-\Lambda_1(t) - 2\Lambda_2(t) - t\kappa(2) - t\kappa(1)} dt \\
&\quad + \xi \mathbb{E}(Z) \\
&= I_1 + I_2 + (\kappa(2) + \kappa(1))I_3 + \kappa(1)I_4 + \kappa(1)\kappa(2)I_5 + \xi \mathbb{E}(Z)
\end{aligned}$$

where,

$$\begin{aligned}
I_1 &= \int_0^\infty \Lambda_2'(t) \Lambda_1'(t) e^{-\Lambda_1(t) - 2\Lambda_2(t) - t\kappa(2) - t\kappa(1)} dt \\
I_2 &= \int_0^\infty \Lambda_2'(t)^2 e^{-\Lambda_1(t) - 2\Lambda_2(t) - t\kappa(2) - t\kappa(1)} dt \\
I_3 &= \int_0^\infty \Lambda_2'(t) e^{-\Lambda_1(t) - 2\Lambda_2(t) - t\kappa(2) - t\kappa(1)} dt \\
I_4 &= \int_0^\infty \Lambda_1'(t) e^{-\Lambda_1(t) - 2\Lambda_2(t) - t\kappa(2) - t\kappa(1)} dt \\
I_5 &= \int_0^\infty e^{-\Lambda_1(t) - 2\Lambda_2(t) - t\kappa(2) - t\kappa(1)} dt.
\end{aligned}$$

In other hand, from Holder inequality we have

$$\begin{aligned}
I_1 &\leq \left[\int_0^\infty \Lambda_2'(t)^2 e^{-4\Lambda_2(t) - 2\kappa(2)t - 2\kappa(1)t} dt \right]^{1/2} \\
&\quad \times \left[\int_0^\infty \Lambda_1'(t)^2 e^{-2\Lambda_1(t)} dt \right]^{1/2} \\
&\leq \left[\int_0^\infty \Lambda_2'(t)^2 e^{-2\Lambda_2(t)} dt \right]^{1/2} \times \left[\int_0^\infty \Lambda_1'(t)^2 e^{-2\Lambda_1(t)} dt \right]^{1/2} < \infty,
\end{aligned}$$

$$I_2 \leq \int_0^\infty \Lambda_2'(t)^2 e^{-2\Lambda_2(t)} dt < \infty,$$

$$\begin{aligned}
I_3 &\leq \int_0^\infty \Lambda_2'(t) e^{-\Lambda_2(t)} dt \\
&= - \int_0^\infty \left(e^{-\Lambda_2(t)} \right)' dt \\
&= 1.
\end{aligned}$$

Similarly to I_3 , we have $I_4 < \infty$. In addition, we have

$$\begin{aligned} I_5 &\leq \int_0^\infty e^{-t(\kappa(2)+\kappa(1))} dt \\ &= \frac{1}{\kappa(2) + \kappa(1)} < \infty. \end{aligned}$$

Based on Theorem 5, $\mathbb{E}(Z) < \infty$, we conclude that $\mathbb{E}(\log g(Z, \delta; \xi)) < \infty$.

★ Now, suppose that $\delta = 1$, then

$$\begin{aligned} &\mathbb{E}(\log g(Z, \delta; \xi)) \\ &= \mathbb{E} \left[\log \left(f_Z(Z) - \bar{F}_T(Z) e^{Z\xi} (f_C(Z) - \xi \bar{F}_C(Z)) \right) \right] \\ &\leq \mathbb{E} \left[\log \left(f_Z(Z) + \bar{F}_T(Z) e^{Z\xi} \xi \bar{F}_C(Z) \right) \right] \\ &\leq \mathbb{E} \left[\log \left(f_Z(Z) + \xi e^{Z\xi} \right) \right] \\ &\leq \mathbb{E} \left(f_Z(Z) + \xi e^{Z\xi} \right) \\ &= \mathbb{E}(f_Z(Z)) + \xi \mathbb{E}(e^{Z\xi}) \\ &= \int_0^\infty \left(\Lambda_1'(t) + \Lambda_2'(t) + \kappa(2) \right)^2 e^{-2\Lambda_1(t) - 2\Lambda_2(t) - 2t\kappa(2)} dt \\ &\quad + \xi \mathbb{E}(e^{Z\xi}) \\ &= \int_0^\infty \Lambda_1'(t)^2 e^{-2\Lambda_1(t) - 2\Lambda_2(t) - 2t\kappa(2)} dt + \int_0^\infty \Lambda_2'(t)^2 e^{-2\Lambda_1(t) - 2\Lambda_2(t) - 2t\kappa(2)} dt \\ &\quad + \kappa(2)^2 \int_0^\infty e^{-2\Lambda_1(t) - 2\Lambda_2(t) - 2t\kappa(2)} dt + 2 \int_0^\infty \Lambda_1'(t) \Lambda_2'(t) e^{-2\Lambda_1(t) - 2\Lambda_2(t) - 2t\kappa(2)} dt \\ &\quad + 2\kappa(2) \int_0^\infty \Lambda_1'(t) e^{-2\Lambda_1(t) - 2\Lambda_2(t) - 2t\kappa(2)} dt + 2\kappa(2) \int_0^\infty \Lambda_2'(t) e^{-2\Lambda_1(t) - 2\Lambda_2(t) - 2t\kappa(2)} dt \\ &\quad + \xi \mathbb{E}(e^{Z\xi}) \\ &= J_1 + J_2 + \kappa(2)^2 J_3 + 2J_4 + 2\kappa(2) J_5 + 2\kappa(2) J_6 + \xi J_7. \end{aligned}$$

We have

$$\begin{aligned} J_1 &= \int_0^\infty \Lambda_1'(t)^2 e^{-2\Lambda_1(t) - 2\Lambda_2(t) - 2\kappa(2)t} dt \\ &\leq \int_0^\infty \Lambda_1'(t)^2 e^{-2\Lambda_1(t)} dt < \infty. \end{aligned}$$

Similarly to J_1 , we have $J_2 < \infty$.

$$\begin{aligned} J_3 &= \int_0^\infty e^{-2\Lambda_1(t) - 2\Lambda_2(t) - 2\kappa(2)t} dt \\ &\leq \int_0^\infty e^{-2\kappa(2)t} dt \\ &= \frac{1}{2\kappa(2)} < \infty. \end{aligned}$$

Based on Hölder's inequality, we have

$$\begin{aligned}
J_4 &= \int_0^\infty \Lambda_1'(t) \Lambda_2'(t) e^{-2\Lambda_1(t) - 2\Lambda_2(t) - 2\kappa(2)t} dt \\
&\leq \int_0^\infty \Lambda_1'(t) \Lambda_2'(t) e^{-\Lambda_1(t) - \Lambda_2(t)} dt \\
&\leq \left[\int_0^\infty \Lambda_1'(t)^2 e^{-2\Lambda_1(t)} dt \right]^{1/2} \left[\int_0^\infty \Lambda_2'(t)^2 e^{-2\Lambda_2(t)} dt \right]^{1/2} < \infty,
\end{aligned}$$

and

$$\begin{aligned}
J_5 &= \int_0^\infty \Lambda_1'(t) e^{-2\Lambda_1(t) - 2\Lambda_2(t) - 2\kappa(2)t} dt \\
&\leq \int_0^\infty \Lambda_1'(t) e^{-\Lambda_1(t)} dt \\
&= - \int_0^\infty \left(e^{-\Lambda_1(t)} \right)' dt \\
&= 1.
\end{aligned}$$

Similarly to J_5 , we have $J_6 < 1$. In addition, we have

$$\begin{aligned}
J_7 &= \mathbb{E}(e^{Z\xi}) \\
&= \int_0^\infty \left(\Lambda_1'(t) + \Lambda_2'(t) + \kappa(2) \right) e^{-\Lambda_1(t) - \Lambda_2(t) - t\kappa(2) + t\xi} dt \\
&= \int_0^\infty \left(\Lambda_1'(t) + \Lambda_2'(t) + \kappa(2) \right) e^{-\Lambda_1(t) - \Lambda_2(t) - 2t(\kappa(2) - \kappa(1))} dt \\
&\leq \int_0^\infty \Lambda_1'(t) e^{-\Lambda_1(t)} dt + \int_0^\infty \Lambda_2'(t) e^{-\Lambda_2(t)} dt + \kappa(2) \int_0^\infty e^{-2t(\kappa(2) - \kappa(1))} dt \\
&= - \int_0^\infty \left(e^{-\Lambda_1(t)} \right)' dt - \int_0^\infty \left(e^{-\Lambda_2(t)} \right)' dt + \frac{\kappa(2)}{2(\kappa(2) - \kappa(1))} \\
&= 2 + \frac{\kappa(2)}{2(\kappa(2) - \kappa(1))} < \infty.
\end{aligned}$$

We conclude that $\mathbb{E}(\log g(Z, \delta; \xi)) < \infty$.

Second, we need to show that $|\log(h(t, b; \theta))| \leq m(t, b)$, where is integrable with respect to g , for all θ . To do this, let $t \in [0, 1]$. Assume that, there exist $k > 0$, such that

$$\kappa(2) - \kappa(1) \geq k. \quad (8.5)$$

Since Λ_1 and Λ_2 are continuous differentiable functions, then there exist $\rho > 0$ such that

$$\Lambda_1(t) \leq \rho, \quad \text{and} \quad \Lambda_2(t) \leq \rho, \quad 0 \leq t \leq 1. \quad (8.6)$$

- If $b = 0$, then we have

$$|\log(h(t, b; \theta))| = \left| \log \left(\bar{F}_{T,n}(t) e^{t\xi} (f_{C,m}(t|\bar{\phi}_m) - \xi \bar{F}_{C,m}(t|\bar{\phi}_m)) \right) \right|.$$

Case 1: If $\bar{F}_{T,n}(t)e^{t\xi} (f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m)) \leq 1$, then we have

$$\begin{aligned} |\log(h(t, b; \theta))| &= \left| \log \left(\bar{F}_{T,n}(t)e^{t\xi} (f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m)) \right) \right| \\ &= -\log \left(\bar{F}_{T,n}(t)e^{t\xi} (f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m)) \right). \end{aligned}$$

In other hand, we have

$$\begin{aligned} \bar{F}_T(t)e^{t\xi} (f_C(t) - \xi\bar{F}_C(t)) &= (\Lambda_2'(t) + \kappa(2) - \kappa(1))e^{-\Lambda_1(t) - \Lambda_2(t) - \kappa(2)t} \\ &\geq (\Lambda_2'(t) + \kappa(2) - \kappa(1))e^{-2\rho - \kappa(3)} \\ &\geq (\kappa(2) - \kappa(1))e^{-2\rho - \kappa(3)} \\ &\geq ke^{-2\rho - \kappa(3)}. \end{aligned}$$

Thus,

$$-\log \left(\bar{F}_{T,n}(t)e^{t\xi} (f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m)) \right) \leq -\log(k),$$

which is integrable with respect to the density g .

Case 2: If $\bar{F}_{T,n}(t)e^{t\xi} (f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m)) \geq 1$, then we have

$$\begin{aligned} |\log(h(t, b; \theta))| &= \left| \log \left(\bar{F}_{T,n}(t)e^{t\xi} (f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m)) \right) \right| \\ &= \log \left(\bar{F}_{T,n}(t)e^{t\xi} (f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m)) \right) \\ &\leq \log \left(\bar{F}_{T,n}(t)e^{t\xi} f_{C,m}(t|\bar{\phi}_m) \right) \\ &\leq \log(f_{C,m}(t|\bar{\phi}_m)) \\ &\leq \log(\kappa(3)). \end{aligned}$$

In fact,

$$\begin{aligned} f_C(t) &= \left(\Lambda_2(t)' + \kappa(1) \right) e^{-\Lambda_2(t) - \kappa(1)t} \\ &\leq \kappa(1)e^{-\Lambda_2(t) - \kappa(1)t} \\ &\leq \kappa(3), \end{aligned}$$

which is integrable with respect to the density g .

- If $b = 1$, we obtain

$$|\log(h(t, b; \theta))| = \left| \log \left(f_{Z,n}(t) - \bar{F}_{T,n}(t)e^{t\xi} (f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m)) \right) \right|.$$

Case 1: If $f_{Z,n}(t) - \bar{F}_{T,n}(t)e^{t\xi} (f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m)) \leq 1$, then

$$\begin{aligned} |\log(h(t, b; \theta))| &= -\log \left(f_{Z,n}(t) - \bar{F}_{T,n}(t)e^{t\xi} (f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m)) \right) \\ &\leq -\log \left(f_{Z,n}(t) - \bar{F}_{T,n}(t)e^{t\xi} f_{C,m}(t|\bar{\phi}_m) \right). \end{aligned}$$

In other hand, we have

$$\begin{aligned}
f_Z(t) - \bar{F}_T(t)e^{t\xi}f_C(t) &= \left(\Lambda_1(t)' + \Lambda_2(t)' + \kappa(2)\right) e^{-\Lambda_1(t)-\Lambda_2(t)-\kappa(2)t} \\
&\quad - (\Lambda_2(t)' + \kappa(1))e^{-\Lambda_1(t)-\Lambda_2(t)-\kappa(2)t} \\
&= \left(\Lambda_1(t)' + \kappa(2) - \kappa(1)\right) e^{-\Lambda_1(t)-\Lambda_2(t)-\kappa(2)t} \\
&\geq (\kappa(2) - \kappa(1))e^{-2\rho-\kappa(3)} \\
&\geq ke^{-2\rho-\kappa(3)}.
\end{aligned} \tag{8.7}$$

Then

$$f_{Z,n}(t) - \bar{F}_{T,n}(t)e^{t\xi}f_{C,m}(t|\bar{\phi}_m) \geq ke^{-2\rho-\kappa(3)},$$

and

$$|\log(h(t, b; \theta))| \leq -\log\left(ke^{-2\rho-\kappa(3)}\right),$$

which is integrable with respect to the density g .

Case 2: If $f_{Z,n}(t) - \bar{F}_{T,n}(t)e^{t\xi}(f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m)) \geq 1$, then

$$\begin{aligned}
|\log(h(t, b; \theta))| &= \log\left(f_{Z,n}(t) - \bar{F}_{T,n}(t)e^{t\xi}(f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m))\right) \\
&\leq \log(f_{Z,n}(t)).
\end{aligned}$$

In other hand, we have

$$\begin{aligned}
f_Z(t) &= \left(\Lambda_1(t)' + \Lambda_2(t)' + \kappa(2)\right) e^{-\Lambda_1(t)-\Lambda_2(t)-\kappa(2)t} \\
&\leq \kappa(2) \leq \kappa(3).
\end{aligned}$$

Then,

$$|\log(h(t, b; \theta))| \leq \log(\kappa(3)),$$

which is integrable with respect to the density g .

Then, condition (3)a) is satisfied. Based on Assumption (**H**₃), I has unique minimum at θ^* , and then condition (3)b) is verified.

In order to prove condition (4), we note first that,

$$\begin{aligned}
\log(h(t, b; \theta)) &= b \log\left[f_{Z,n}(t) - \bar{F}_{T,n}(t)e^{t\xi}(f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m))\right] \\
&\quad \times (1-b) \log\left[\bar{F}_{T,n}(t)e^{t\xi}(f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m))\right].
\end{aligned}$$

$$\begin{aligned}
h_1(t, b; \theta) &= \frac{\partial \log(h(t, b; \theta))}{\partial \xi} \\
&= -\frac{b\bar{F}_{T,n}(t) [\xi e^{t\xi}f_{C,m}(t|\bar{\phi}_m) - \bar{F}_{C,m}(t|\bar{\phi}_m)(e^{t\xi} + \xi^2 e^{t\xi})]}{f_{Z,n}(t) - \bar{F}_{T,n}(t)e^{t\xi}(f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m))} \\
&\quad + \frac{(1-b) [\xi f_{C,m}(t|\bar{\phi}_m) - \bar{F}_{C,m}(t|\bar{\phi}_m)(1 + \xi^2)]}{(f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m))}.
\end{aligned}$$

For $k = 0, \dots, m$

$$\begin{aligned}
h_{2,k}(t, b; \theta) &= \frac{\partial \log(h(t, b; \theta))}{\partial \phi_k} \\
&= -\frac{b\bar{F}_{T,n}(t)e^{t\xi}(R_m(t) - \xi)}{f_{Z,n}(t) - \bar{F}_{T,n}(t)e^{t\xi}(f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m))} \\
&\quad + \frac{(1-b)(R_m(t) - \xi)}{(f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m))},
\end{aligned}$$

where $R_m(t) = \sum_{k=0}^m b'_{k,m}(t)$. Then, the functions h_1 and $h_{2,k}$ for $k = 0, \dots, m$ are measurable functions of (t, b) for each $\theta \in \mathbb{R}^{m+2}$ and continuously differentiable functions of θ for each $(t, b) \in \mathbb{R}_+ \times [0, 1]$. Then condition (4) is satisfied.

In order to show condition (5), we note that

- If $f_{Z,n}(t) - \bar{F}_{T,n}(t)e^{t\xi}(f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m)) \geq 1$ and $f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m) \geq 1$, then

$$\frac{1}{f_{Z,n}(t) - \bar{F}_{T,n}(t)e^{t\xi}(f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m))} \leq 1 \text{ and } \frac{1}{f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m)} \leq 1.$$

- If $f_{Z,n}(t) - \bar{F}_{T,n}(t)e^{t\xi}(f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m)) \leq 1$ and $f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m) \leq 1$, then based on Equations (8.5), (8.6) and (8.7), we have

$$\frac{1}{f_{Z,n}(t) - \bar{F}_{T,n}(t)e^{t\xi}(f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m))} \leq \frac{1}{ke^{-2\rho-\kappa(3)}}.$$

Also, we have

$$\frac{1}{f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m)} \leq \frac{1}{ke^{\rho-\kappa(3)}}.$$

In fact

$$\begin{aligned}
f_C(t) - \xi\bar{F}_C(t) &= e^{-\Lambda_2(t)-t\kappa(1)}(\Lambda'_2(t) + \kappa(1) - \xi) \\
&\geq ke^{-\rho-\kappa(3)}.
\end{aligned}$$

Then, in general, there exist constants $\mu, \nu > 0$ such that,

$$\frac{1}{f_{Z,n}(t) - \bar{F}_{T,n}(t)e^{t\xi}(f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m))} \leq \mu \text{ and } \frac{1}{f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m)} \leq \nu.$$

(i)

$$\begin{aligned}
U_1(t, b; \theta) &= \left| \frac{\partial^2 \log(h(t, b; \theta))}{\partial^2 \xi} \right| \\
&= \left| \frac{\partial h_1(t, b; \theta)}{\partial \xi} \right| \\
&\leq \mu^2 \left| e^{t\xi} f_{C,m}(t|\bar{\phi}_m)(1 + \xi^2) - e^{t\xi} \bar{F}_{C,m}(t|\bar{\phi}_m)(3\xi + \xi^3) \right| \\
&\times \left| f_{Z,n}(t) - e^{t\xi} \bar{F}_{T,n}(t) f_{C,m}(t|\bar{\phi}_m) + \xi e^{t\xi} \bar{F}_{T,n}(t) \bar{F}_{C,m}(t|\bar{\phi}_m) \right| \\
&+ \mu^2 \left| \xi e^{t\xi} f_{C,m}(t|\bar{\phi}_m) - e^{t\xi} \bar{F}_{C,m}(t|\bar{\phi}_m)(1 + \xi^2) \right| \\
&\times \left| \bar{F}_{T,n}(t) e^{t\xi} (\xi f_{C,m}(t|\bar{\phi}_m) - \bar{F}_{C,m}(t|\bar{\phi}_m)(1 + \xi^2)) \right| \\
&+ \nu^2 \left| [f_{C,m}(t|\bar{\phi}_m) - 2\xi \bar{F}_{C,m}(t|\bar{\phi}_m)] [f_{C,m}(t|\bar{\phi}_m) - \xi \bar{F}_{C,m}(t|\bar{\phi}_m)] \right| \\
&+ \nu^2 \left| \bar{F}_{C,m}(t|\bar{\phi}_m) (\xi f_{C,m}(t|\bar{\phi}_m) - \bar{F}_{C,m}(t|\bar{\phi}_m)(1 + \xi^2)) \right| \\
&\leq \mu^2 \left[e^{t\xi} R_m(t)(1 + \xi^2) + e^{t\xi} (3\xi + \xi^3) \right] \times \left[\frac{K^{max}}{h} + e^{t\xi} R_m(t) + \xi e^{t\xi} \right] \\
&+ \nu^2 \left(\left[\xi e^{t\xi} R_m(t) + e^{t\xi} (1 + \xi^2) \right]^2 + [R_m(t) + 2\xi] \times [R_m(t) + \xi] + [\xi R_m(t) + 1 + \xi] \right) := S_1(t).
\end{aligned}$$

Since the function S_1 is continuous on the compact interval $[0, 1]$, then it has a maximum ψ_1 .
Then

$$U_1(t, b; \theta) \leq \psi_1.$$

Then U_1 is dominated by an integrable function with respect to the density g for all $(t, b) \in [0, 1]^2$ and for all $\theta \in \mathbb{R}^{m+2}$.

(ii) For $k = 0, \dots, m$,

$$\begin{aligned}
U_{2,k}(t, b; \theta) &= \left| \frac{\partial^2 \log(h(t, b; \theta))}{\partial^2 \phi_k} \right| \\
&= \left| \frac{\partial h_{2,k}(t, b; \theta)}{\partial \phi_k} \right| \\
&\leq \left| \frac{b \left[\bar{F}_{T,n}(t) e^{t\xi} (R_m(t) - \xi) \right]^2}{\left[f_{Z,n}(t) - \bar{F}_{T,n}(t) e^{t\xi} (f_{C,m}(t|\bar{\phi}_m) - \xi \bar{F}_{C,m}(t|\bar{\phi}_m)) \right]^2} \right| \\
&+ \left| \frac{(1-b) [R_m(t) - \xi]^2}{(f_{C,m}(t|\bar{\phi}_m) - \xi \bar{F}_{C,m}(t|\bar{\phi}_m))^2} \right| \\
&\leq \mu^2 \left| b \left[\bar{F}_{T,n}(t) e^{t\xi} (R_m(t) - \xi) \right]^2 \right| + \nu^2 \left| (1-b) [R_m(t) - \xi]^2 \right| \\
&\leq (\mu^2 e^{t\xi} + \nu^2) (R_m(t) - \xi)^2 := S_2(t).
\end{aligned}$$

Since the function S_2 is continuous on the compact interval $[0, 1]$, then it has a maximum ψ_2 .
Then

$$U_{2,k}(t, b; \theta) \leq \psi_2.$$

Then $U_{2,k}$ is dominated by an integrable function with respect to the density g for all $(t, b) \in [0, 1]^2$ and for all $\theta \in \mathbb{R}^{m+2}$.

(iii) For $k = 0, \dots, m$,

$$\begin{aligned}
U_{3,k}(t, b; \theta) &= \left| \frac{\partial^2 \log(h(t, b; \theta))}{\partial \xi \partial \phi_k} \right| \\
&= \left| \frac{\partial h_{2,k}(t, b; \theta)}{\partial \xi} \right| \\
&\leq \mu^2 \left| \left[e^{t\xi} (\xi R_m(t) - 1 - \xi^2) \right] \left[f_{Z,n}(t) - \bar{F}_{T,n}(t) e^{t\xi} (f_{C,m}(t|\bar{\phi}_m) - \xi \bar{F}_{C,m}(t|\bar{\phi}_m)) \right] \right| \\
&+ \mu^2 \left| \left[-\bar{F}_{T,n}(t) f_{C,m}(t|\bar{\phi}_m) \xi e^{t\xi} + \bar{F}_{T,n}(t) \bar{F}_{C,m}(t|\bar{\phi}_m) (e^{t\xi} + \xi^2 e^{t\xi}) \right] \left[e^{t\xi} R_m(t) - \xi e^{t\xi} \right] \right| \\
&+ \nu^2 \left| f_{C,m}(t|\bar{\phi}_m) - \xi \bar{F}_{C,m}(t|\bar{\phi}_m) \right| + \nu^2 \left| \bar{F}_{C,m}(t|\bar{\phi}_m) (R_m(t) - \xi) \right| \\
&\leq \mu^2 \left[e^{t\xi} (\xi R_m(t) + 1 + \xi^2) \right] \times \left[\frac{K_{max}}{h} + e^{t\xi} R_m(t) + \xi e^{t\xi} \right] \\
&+ \mu^2 \left[e^{t\xi} (\xi R_m(t) + 1 + \xi^2) \right] \left[e^{t\xi} R_m(t) + \xi e^{t\xi} \right] + \nu^2 [R_m(t) + \xi]^2 := S_3(t).
\end{aligned}$$

Since the function S_3 is continuous on the compact interval $[0, 1]$, then it has a maximum ψ_3 . Then

$$U_{3,k}(t, b; \theta) \leq \psi_3.$$

Then $U_{3,k}$ is dominated by an integrable function with respect to the density g for all $(t, b) \in [0, 1]^2$ and for all $\theta \in \mathbb{R}^{m+2}$.

(iv) For $k, j = 0, \dots, m$ such that $k \neq j$,

$$\begin{aligned}
U_{4,j,k}(t, b; \theta) &= \left| \frac{\partial^2 \log(h(t, b; \theta))}{\partial \phi_j \partial \phi_k} \right| \\
&= \left| \frac{\partial h_{2,k}(t, b; \theta)}{\partial \phi_j} \right|.
\end{aligned}$$

Similarly to $U_{2,k}$, $U_{4,k}$ is dominated by an integrable function with respect to the density g for all $(t, b) \in [0, 1]^2$ and for all $\theta \in \mathbb{R}^{m+2}$.

(v) For k, \dots, m ,

$$\begin{aligned}
U_{5,k}(t, b; \theta) &= \left| \frac{\partial \log(h(t, b; \theta))}{\partial \xi} \frac{\partial \log(h(t, b; \theta))}{\partial \phi_k} \right| \\
&= \left| \frac{\partial \log(h(t, b; \theta))}{\partial \xi} \right| \left| \frac{\partial \log(h(t, b; \theta))}{\partial \phi_k} \right| \\
&= |h_1(t, b; \theta)| |h_{2,k}(t, b; \theta)|.
\end{aligned}$$

Or

$$\begin{aligned}
|h_1(t, b; \theta)| &\leq \left| \frac{b \bar{F}_{T,n}(t) [\xi e^{t\xi} f_{C,m}(t|\bar{\phi}_m) - \bar{F}_{C,m}(t|\bar{\phi}_m) (e^{t\xi} + \xi^2 e^{t\xi})]}{f_{Z,n}(t) - \bar{F}_{T,n}(t) e^{t\xi} (f_{C,m}(t|\bar{\phi}_m) - \xi \bar{F}_{C,m}(t|\bar{\phi}_m))} \right| \\
&+ \left| \frac{(1-b) [\xi f_{C,m}(t|\bar{\phi}_m) - \bar{F}_{C,m}(t|\bar{\phi}_m) (1 + \xi^2)]}{(f_{C,m}(t|\bar{\phi}_m) - \xi \bar{F}_{C,m}(t|\bar{\phi}_m))} \right|.
\end{aligned}$$

It follows that

$$\begin{aligned}
|h_1(t, b; \theta)| &\leq \mu \left| \xi e^{t\xi} f_{C,m}(t|\bar{\phi}_m) - \bar{F}_{C,m}(t|\bar{\phi}_m)(e^{t\xi} + \xi^2 e^{t\xi}) \right| \\
&\quad + \nu \left| \xi f_{C,m}(t|\bar{\phi}_m) - \bar{F}_{C,m}(t|\bar{\phi}_m)(1 + \xi^2) \right| \\
&\leq \mu \left[\xi e^{t\xi} f_{C,m}(t|\bar{\phi}_m) + (e^{t\xi} + \xi^2 e^{t\xi}) \right] + \nu \left[\xi f_{C,m}(t|\bar{\phi}_m) + (1 + \xi^2) \right] \\
&= \mu \left[\xi e^{t\xi} R_m(t) + (e^{t\xi} + \xi^2 e^{t\xi}) \right] + \nu \left[\xi R_m(t) + (1 + \xi^2) \right] := S_{5,1}(t).
\end{aligned}$$

Since the function $S_{5,1}$ is continuous on the compact interval $[0, 1]$, then it has a maximum $\psi_{5,1}$. Then

$$|h_1(t, b; \theta)| \leq \psi_{5,1} \quad (8.8)$$

On the other hand, we have

$$\begin{aligned}
|h_{2,k}(t, b; \theta)| &\leq \left| -\frac{b\bar{F}_{T,n}(t)e^{t\xi}(R_m(t) - \xi)}{f_{Z,n}(t) - \bar{F}_{T,n}(t)e^{t\xi}(f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m))} \right| \\
&\quad + \left| \frac{(1-b)(R_m(t) - \xi)}{(f_{C,m}(t|\bar{\phi}_m) - \xi\bar{F}_{C,m}(t|\bar{\phi}_m))} \right| \\
&\leq \mu e^{t\xi} |R_m(t) - \xi| + \nu |R_m(t) - \xi| := S_{5,2}(t).
\end{aligned}$$

Since the function $S_{5,2}$ is continuous on the compact interval $[0, 1]$, then it has a maximum $\psi_{5,2}$. Then

$$|h_{2,k}(t, b; \theta)| \leq \psi_{5,2}. \quad (8.9)$$

It follows from Equations (8.8) and (8.9) that

$$U_{5,k}(t, b; \theta) \leq \psi_{5,1}\psi_{5,2}.$$

Then $U_{5,k}$ is dominated by an integrable function with respect to the density g for all $(t, b) \in [0, 1]^2$ and for all $\theta \in \mathbb{R}^{m+2}$.

(vi) For $k = 0, \dots, m$,

$$\begin{aligned}
U_{6,k}(t, b; \theta) &= \left| \left[\frac{\partial \log(h(t, b; \theta))}{\partial \phi_k} \right]^2 \right| \\
&= \left| \frac{\partial \log(h(t, b; \theta))}{\partial \phi_k} \right| \left| \frac{\partial \log(h(t, b; \theta))}{\partial \phi_k} \right| \\
&= |h_{2,k}(t, b; \theta)|^2.
\end{aligned}$$

It follows from Equation (8.9) that $U_{6,k}$ is dominated by an integrable function with respect to the density g for all $(t, b) \in [0, 1]^2$ and for all $\theta \in \mathbb{R}^{m+2}$.

(vii) For $k, j = 0, \dots, m$ such that $k \neq j$,

$$\begin{aligned}
U_{7,k,j}(t, b; \theta) &= \left| \frac{\partial \log(h(t, b; \theta))}{\partial \phi_j} \frac{\partial \log(h(t, b; \theta))}{\partial \phi_k} \right| \\
&= \left| \frac{\partial \log(h(t, b; \theta))}{\partial \phi_j} \right| \left| \frac{\partial \log(h(t, b; \theta))}{\partial \phi_k} \right| \\
&= |h_{2,j}(t, b; \theta)| |h_{2,k}(t, b; \theta)| \\
&\leq \left[(e^{t\xi} + 1)R_m(t) + 2\xi \right]^2.
\end{aligned}$$

Similarly to $U_{6,k}$, $U_{7,j,k}$ is dominated by an integrable function with respect to the density g for all $(t, b) \in [0, 1]^2$ and for all $\theta \in \mathbb{R}^{m+2}$.

(viii)

$$\begin{aligned} U_8(t, b; \theta) &= \left| \left[\frac{\partial \log(h(t, b; \theta))}{\partial \xi} \right]^2 \right| \\ &= \left| \frac{\partial \log(h(t, b; \theta))}{\partial \xi} \right| \left| \frac{\partial \log(h(t, b; \theta))}{\partial \xi} \right| \\ &= |h_1(t, b; \theta)|^2. \end{aligned}$$

It follows from Equation (8.8) that U_8 is dominated by an integrable function with respect to the density g for all $(t, b) \in [0, 1]^2$ and for all $\theta \in \mathbb{R}^{m+2}$.

Then condition (5) is satisfied.

Base on assumption **(H₄)**, condition (6) of [60] is satisfied. Then the conditions (1)-(6) of Theorem 3.2 of [60] are verified, and the proof is complete.

Proof of Proposition 7:

Recall that

$$\begin{aligned} \theta &= (\theta_1, \theta_2, \dots, \theta_{m+1}, \theta_{m+2}) = (\phi_0, \dots, \phi_m, \xi), \\ \theta_n &= (\theta_{1,n}, \theta_{2,n}, \dots, \theta_{m+1,n}, \theta_{m+2,n}) = (\phi_{0,n}, \dots, \phi_{m,n}, \xi_n), \end{aligned}$$

and

$$\tilde{P}_{n,m}(t, s) = \begin{cases} \bar{F}_{C,n,m}(s) \bar{F}_{T,n}(t)^{1-\alpha_n(t)} & \text{if } \bar{F}_T(t)^{\alpha(t)} < \bar{F}_C(s)^{\beta(s)} \\ \bar{F}_{T,n}(t) \bar{F}_{C,n,m}(s)^{1-\beta_{n,m}(s)} & \text{if } \bar{F}_T(t)^{\alpha(t)} > \bar{F}_C(s)^{\beta(s)}. \end{cases}$$

Based on Theorem 2.2 of [60], $\theta_n \xrightarrow[n \rightarrow +\infty]{a.s.} \theta$, then

$$(\phi_{0,n}, \dots, \phi_{m,n}) \xrightarrow[n \rightarrow +\infty]{a.s.} (\phi_0, \dots, \phi_m) \quad (8.10)$$

and

$$\xi_n \xrightarrow[n \rightarrow +\infty]{a.s.} \xi. \quad (8.11)$$

Based on Equation (8.10), we have

$$\bar{F}_{C,n,m}(s) \xrightarrow[n \rightarrow +\infty]{a.s.} \bar{F}_{C,m}(s) \quad \text{and} \quad \beta_{n,m}(s) \xrightarrow[n \rightarrow +\infty]{a.s.} \beta_m(s). \quad (8.12)$$

On the other hand, we have

$$|\bar{F}_{C,n,m}(s) - \bar{F}(s)| \leq |\bar{F}_{C,n,m}(s) - \bar{F}_m(s)| + |\bar{F}_m(s) - \bar{F}(s)|,$$

and

$$|\beta_{n,m}(s) - \beta(s)| \leq |\beta_{n,m}(s) - \beta_m(s)| + |\beta_m(s) - \beta(s)|.$$

Next, based on Theorem 1.1 in [8] and Theorem 1 in [29] in Section VII.2, we note that for m large enough, $\lim_{m \rightarrow \infty} \bar{F}_{C,m}(s) = \bar{F}_C(s)$ and $\lim_{m \rightarrow \infty} \beta_m(s) = \beta(s)$. Then based on Equation (8.12),

$$\bar{F}_{C,n,m}(s) \xrightarrow[n, m \rightarrow +\infty]{a.s.} \bar{F}_C(s) \quad \text{and} \quad \beta_{n,m}(s) \xrightarrow[n, m \rightarrow +\infty]{a.s.} \beta(s). \quad (8.13)$$

According to Lemma 2 of [12], we have

$$\bar{F}_{T,n}(t) \xrightarrow[n \rightarrow +\infty]{a.s.} \bar{F}_T(t) \quad \text{and} \quad \alpha_n(t) \xrightarrow[n \rightarrow +\infty]{a.s.} \alpha(t). \quad (8.14)$$

Then based on Equations (8.13) and (8.14), we obtain $\lim_{n, m \rightarrow \infty} \tilde{P}_{n,m}(t, s) = \tilde{P}(t, s)$.

On the other hand, based on Glivenko-Cantelli Theorem, we have

$$\bar{H}_n(t) \xrightarrow[n \rightarrow +\infty]{unif} \bar{H}(t). \quad (8.15)$$

Then based on Equations (8.11) and (8.15), we have $\lim_{n \rightarrow \infty} \tau_n = \tau$.

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