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ADDITIVE AND MULTIPLICATIVE COINVARIANT SPACES OF WEYL GROUPS IN THE LIGHT OF HARMONICS AND GRADED TRANSFER

SEBASTIAN DEBUS AND TOBIAS METZLAFF

ABSTRACT. A finite group with an integral representation has two induced canonical actions, one on polynomials and one on Laurent polynomials. Knowledge about the invariants is in either case applied in many computations by means of symmetry reduction techniques, for example in algebraic systems solving or optimization. In this article, we realize the two actions as the additive action on the symmetric algebra and the multiplicative action on the group algebra of a lattice with Weyl group symmetry. By constructing explicit equivariant isomorphisms, we draw algorithmic relations between the two, which allow the transfer and preservation of representation- and invariant-theoretic properties. Our focus lies on the multiplicative coinvariant space, which is identified with the regular representation and harmonic polynomials.

INTRODUCTION

The Weyl group \mathcal{W} of a crystallographic root system with weight lattice Ω has an additive action on the symmetric algebra $\text{Sym}(\Omega)$ and a multiplicative action on the group algebra $\mathbb{C}[\Omega]$. In this article, we relate the coinvariant spaces of both actions to one another.

Weight lattices provide optimal configurations for a variety of problems in geometry and information theory, with incidence in physics and chemistry. The A_2 lattice (the hexagonal lattice) is classically known to be optimal for sampling, packing, covering, and quantization in the plane [CS99, KAH05], but also proved, or conjectured, to be optimal for energy minimization problems [PS20, BF23]. More recently, the E_8 lattice was proven to give an optimal solution for the sphere packing problem and a large class of energy minimization problems in dimension 8 [Via17, CKM⁺22]. From an approximation point of view, weight lattices of root systems describe Gaussian cubature [LX10, MP11], a rare occurrence on multidimensional domains. In a different direction, the triangulations associated with infinite families of root systems are relevant in graphics and computational geometry, see for instance [CKW20] and the references therein.

The distinguishing feature of these lattices is their symmetry with respect to the respective Weyl group \mathcal{W} . Much is known about the coinvariant space $\text{Sym}(\Omega)_{\mathcal{W}}$ of the additive action, which is the quotient of $\text{Sym}(\Omega)$ modulo the ideal generated by the homogeneous invariants of positive degree, known as the Hilbert ideal. The motivation to study this quotient is to eliminate redundant copies of irreducible \mathcal{W} -modules inside $\text{Sym}(\Omega)$. In particular, a symmetry adapted basis of $\text{Sym}(\Omega)_{\mathcal{W}}$ is used to perform symmetry reductions in several areas of computational mathematics, such as polynomial optimization [GP04], algebraic system solving [FLS⁺23] and sphere packing [DLM21].

Finding a symmetry adapted basis for $\text{Sym}(\Omega)_{\mathcal{W}}$, which respects the isotypic decomposition of the associated group representation, has been subject to many works in algebraic combinatorics. For symmetric groups (Weyl groups of type A_{n-1}), a basis can be constructed from Young tableaux and higher Specht polynomials [FH91, Yam93, ATY97]. Bitableaux combinatorics can furthermore be used to describe coinvariants of hyperoctahedral groups (type B_n and C_n) and certain subgroups of index 2 (type D_n) [ATY97, MY98]. Recently, there has been an interest in finding symmetry adapted bases or so-called higher Specht bases for the coinvariant spaces of diagonal actions of the symmetric group and Garsia-Procesi modules [Gil24, GR21].

This extensive study for additive coinvariants raises questions concerning the second action, that is, the multiplicative action of the Weyl group \mathcal{W} on the group algebra $\mathbb{C}[\Omega]$. The invariant ring is fully described by a theorem of Bourbaki, but although certain notions of multiplicative coinvariant spaces appear in the literature, see [Dem74, Ste75, GS84, Gri06, Hua14], it has, to the knowledge of the authors, by far not received the same amount of attention from algebraic combinatorics as the additive one.

We therefore argue that the transfer of representation- and invariant-theoretic properties from the additive to the multiplicative coinvariant space is more enlightening and fruitful than focusing exclusively on a combinatorial study of the latter.

First, we prove that also the multiplicative coinvariant space affords the regular representation of \mathcal{W} using only the fact that the group algebra $\mathbb{C}[\Omega]$ is a free module of rank $|\mathcal{W}|$ over the invariant ring $\mathbb{C}[\Omega]^{\mathcal{W}}$ [Ste75]. Second, we introduce a notion of multiplicative \mathcal{W} -harmonics via Euler derivations and prove that the space of multiplicative harmonics also affords the regular representation which mirrors the additive situation. We then proceed to the associated graded algebras of $\text{Sym}(\Omega)$ and $\mathbb{C}[\Omega]$ to construct an equivariant isomorphism, based on [BNZ12, CPZ13], and show how to algorithmically compute a symmetry adapted coinvariant basis for $\mathbb{C}[\Omega]_{\mathcal{W}}$ from one for $\text{Sym}(\Omega)_{\mathcal{W}}$. Finally, we conclude with case studies for the Weyl groups of types A_{n-1} and C_n .

All vector spaces and algebras are defined over the complex numbers \mathbb{C} unless explicitly stated otherwise. The nonnegative integers $\{0, 1, 2, \dots\}$ are denoted by \mathbb{N} .

1. ADDITIVE AND MULTIPLICATIVE ACTIONS

Let Ω be a free \mathbb{Z} -module of finite rank n and \mathcal{W} be a finite subgroup of $\text{GL}(\Omega)$.

Denote by $V = \text{span}_{\mathbb{C}} \Omega$ the vector space generated by Ω . The symmetric algebra $\text{Sym}(\Omega)$ is the quotient of the tensor algebra $\mathbb{C} \oplus V \oplus (V \otimes V) \oplus \dots$ modulo the relations $\mu_1 \otimes \dots \otimes \mu_\ell \equiv \mu_{\sigma(1)} \otimes \dots \otimes \mu_{\sigma(\ell)}$ for all permutations $\sigma \in \mathfrak{S}_\ell$. The **additive action** of \mathcal{W} on $\text{Sym}(\Omega)$ is the linear action defined by

$$(s, \mu) \mapsto s \cdot \mu := s(\mu).$$

When a \mathbb{Z} -basis $\Omega = \mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_n$ is fixed, then $\text{Sym}(\Omega) = \mathbb{C}[X_1, \dots, X_n]$ is a polynomial ring with indeterminates $X_i := \omega_i$. In particular, we have $s \cdot X_i = s_{1,i} X_1 + \dots + s_{n,i} X_n$, where $s_{\ell,i} \in \mathbb{Z}$ are the matrix entries of $s \in \mathcal{W}$.

The elements of $\text{Sym}(\Omega)$, which are invariant under this action, form the **additive invariant ring** $\text{Sym}(\Omega)^{\mathcal{W}}$. By Hilbert's finiteness theorem, $\text{Sym}(\Omega)^{\mathcal{W}}$ is a finitely generated algebra, that is, $\text{Sym}(\Omega)^{\mathcal{W}} = \mathbb{C}[\mathcal{S}_a]$ for a minimal set of fundamental invariants \mathcal{S}_a .

Denote by $\mathcal{H}_a := (\mathcal{S}_a)$ the **additive Hilbert ideal**, that is, the ideal in $\text{Sym}(\Omega)$ generated by \mathcal{S}_a . The **additive coinvariant space** is the \mathcal{W} -module

$$\text{Sym}(\Omega)_{\mathcal{W}} := \text{Sym}(\Omega)/\mathcal{H}_a.$$

The above objects are well understood for reflection groups. A **pseudoreflection** s is an automorphism on V , such that $\text{codim}(\ker(\text{id}_V - s)) = 1$, that is, the fixed point space has codimension 1. In particular, s has eigenvalue 1 with multiplicity $n - 1$ and another eigenvalue in $\mathbb{C} \setminus \{0\}$, which is a root of unity. If this last eigenvalue is -1 , then s is called a (real) **reflection**. If the finite group $\mathcal{W} \subseteq \text{GL}(\Omega)$ is generated by reflections, then it is the Weyl group of some crystallographic root system with weight lattice Ω , see [Kan01, Ch. 9].

This article focuses on Weyl groups, but it should be noted that there are also groups generated by pseudoreflections, which leave complex lattices stable, see [Neb99], and the following famous result holds in this more general setting.

Theorem 1.1 (Shephard-Todd [ST54], Chevalley [Che55]). Let \mathcal{W} be the Weyl group of a crystallographic root system with weight lattice Ω . Then the following statements hold.

- (1) $\text{Sym}(\Omega)^{\mathcal{W}}$ is a polynomial algebra with Krull-dimension n .
- (2) $\text{Sym}(\Omega)$ is a free $\text{Sym}(\Omega)^{\mathcal{W}}$ -module of rank $|\mathcal{W}|$.
- (3) $\text{Sym}(\Omega)_{\mathcal{W}}$ affords the regular representation of \mathcal{W} .

Statement (3) means that there is a \mathcal{W} -module isomorphism between $\text{Sym}(\Omega)_{\mathcal{W}}$ and the (noncommutative) group algebra $\mathbb{C}[\mathcal{W}]$ of the Weyl group.

Furthermore, with statement (1) we can consider the Jacobian determinant D of the map $\mathbb{C}^n \rightarrow \mathbb{C}^n$ defined by a set of (homogeneous) fundamental invariants \mathcal{S}_a with cardinality n . The vector space \mathcal{L}_a generated by $\{\partial_{i_1} \dots \partial_{i_\ell} D \mid \ell \in \mathbb{N}, 1 \leq i_1 \leq \dots \leq i_\ell \leq n\}$ is called the space of **additive \mathcal{W} -harmonics**.

Theorem 1.2 (Steinberg [Ste64]). Let \mathcal{W} be the Weyl group of a crystallographic root system with weight lattice Ω . Then $\text{Sym}(\Omega)_{\mathcal{W}}$ is isomorphic as a \mathcal{W} -module to \mathcal{L}_a .

On the other hand, the group algebra $\mathbb{C}[\Omega]$ is the commutative algebra with vector space basis $\mathbf{e}^\Omega := \{\mathbf{e}^\mu \mid \mu \in \Omega\}$ and multiplication $\mathbf{e}^\mu \mathbf{e}^\nu = \mathbf{e}^{\mu+\nu}$. The **multiplicative action** of \mathcal{W} on $\mathbb{C}[\Omega]$ is the linear action defined by

$$(s, \mathbf{e}^\mu) \mapsto s \cdot \mathbf{e}^\mu := \mathbf{e}^{s(\mu)}.$$

When a \mathbb{Z} -basis $\Omega = \mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_n$ is fixed, then $\mathbb{C}[\Omega] = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is a Laurent polynomial ring with indeterminates $x_i := \mathbf{e}^{\omega_i}$. In particular, we have $s \cdot x_i = x_1^{s_{1,i}} \dots x_n^{s_{n,i}}$.

The elements of $\mathbb{C}[\Omega]$, which are invariant under this action, form the **multiplicative invariant ring** $\mathbb{C}[\Omega]^{\mathcal{W}}$. This is a finitely generated algebra, that is, $\mathbb{C}[\Omega]^{\mathcal{W}} = \mathbb{C}[\mathcal{S}_m]$ for a minimal set of fundamental invariants \mathcal{S}_m .

Theorem 1.3 (Bourbaki [Bou68], Steinberg [Ste75], Farkas [Far84]). Let \mathcal{W} be the Weyl group of a crystallographic root system with weight lattice Ω . Then the following statements hold.

- (1) $\mathbb{C}[\Omega]^{\mathcal{W}}$ is a polynomial algebra with Krull dimension n .
- (2) $\mathbb{C}[\Omega]$ is a free $\mathbb{C}[\Omega]^{\mathcal{W}}$ -module of rank $|\mathcal{W}|$.

Note that statements (1) and (2) correspond to those in Theorem 1.1.

The notion ‘‘multiplicative’’ refers to the fact that \mathfrak{e}^Ω is a multiplicative group in $\mathbb{C}[\Omega]$, where the inverse of \mathfrak{e}^μ is $\mathfrak{e}^{-\mu}$. On the other hand, $V = \text{span}_{\mathbb{C}} \Omega$ is an additive group in $\text{Sym}(\Omega)$.

Example 1.4. The lattice $\Omega = \mathbb{Z}^2$ is the weight lattice of the root system C_2 in $\mathfrak{h} = \mathbb{R}^2$ with fundamental weights $\omega_1 = e_1, \omega_2 = e_1 + e_2$ and Weyl group $\mathcal{W} \cong \mathfrak{S}_2 \wr \{\pm 1\}$ of order 8, see [Bou68, Pl. III]. We choose between two \mathbb{Z} -bases for the weight lattice, namely $\Omega = \mathbb{Z} e_1 \oplus \mathbb{Z} e_2 = \mathbb{Z} \omega_1 \oplus \mathbb{Z} \omega_2$, to emphasize that the choice matters for computations.

In the standard basis e_1, e_2 , the Weyl group is generated by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with an additive action on $\text{Sym}(\Omega) = \mathbb{C}[Y_1, Y_2]$ and a multiplicative action on $\mathbb{C}[\Omega] = \mathbb{C}[y_1^{\pm 1}, y_2^{\pm 1}]$. The fundamental invariants are $Y_1^2 + Y_2^2, Y_1^2 Y_2^2$ and $y_1 + y_1^{-1} + y_2 + y_2^{-1}, (y_1 + y_1^{-1})(y_2 + y_2^{-1})$ respectively.

On the other hand, in the basis of fundamental weights ω_1, ω_2 , the Weyl group is generated by

$$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

with an additive action on $\text{Sym}(\Omega) = \mathbb{C}[X_1, X_2]$ and a multiplicative action on $\mathbb{C}[\Omega] = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$. We have $X_1 = Y_1, X_2 = Y_1 + Y_2$ and $x_1 = y_1, x_2 = y_1 y_2$. In particular, the fundamental invariants are $2X_1^2 - 2X_1X_2 + X_2^2, X_1^4 - 2X_1^3X_2 + X_1^2X_2^2$ and $x_1 + x_1^{-1} + x_1^{-1}x_2 + x_1x_2^{-1}, x_2 + x_2^{-1} + x_1^2x_2^{-1} + x_1^{-2}x_2$ respectively.

As a convention, we always use the variables X_i for $\text{Sym}(\Omega)$ and $x_i^{\pm 1}$ for $\mathbb{C}[\Omega]$, if the basis for Ω is that of fundamental weights of a crystallographic root system, while we use Y_i and $y_i^{\pm 1}$ for other bases such as the standard one in Example 1.4.

2. MULTIPLICATIVE COINVARIANT SPACES

Let \mathcal{W} be the Weyl group of a crystallographic root system with weight lattice Ω and $\mathcal{S}_m = \{\theta_1, \dots, \theta_n\}$ be a set of fundamental invariants for the multiplicative invariant ring $\mathbb{C}[\Omega]^{\mathcal{W}}$. Denote by $\mathcal{H}_m := (\mathcal{S}_m)$ the **multiplicative Hilbert ideal**, that is, the ideal in $\mathbb{C}[\Omega]$ generated by \mathcal{S}_m . The **multiplicative coinvariant space** is the \mathcal{W} -module

$$\mathbb{C}[\Omega]_{\mathcal{W}} := \mathbb{C}[\Omega] / \mathcal{H}_m.$$

In this section, we show that statement (3) in Theorem 1.1 also holds for multiplicative invariants, that is, $\mathbb{C}[\Omega]_{\mathcal{W}}$ affords the regular representation.

Lemma 2.1. For $c = (c_1, \dots, c_n) \in \mathbb{C}^n$, denote $\mathcal{S}_c := \{\theta_1 - c_1, \dots, \theta_n - c_n\}$. Then \mathcal{S}_c is also a set of fundamental invariants for $\mathbb{C}[\Omega]^{\mathcal{W}}$.

Proof. We have $\theta_i - c_i \in \mathbb{C}[\mathcal{S}_0] = \mathbb{C}[\mathcal{S}_m] = \mathbb{C}[\Omega]^{\mathcal{W}}$ and conversely $\theta_i = (\theta_i - c_i) + c_i \in \mathbb{C}[\mathcal{S}_c]$. \square

Fix a global monomial ordering on \mathfrak{e}^Ω and let I be any ideal in $\mathbb{C}[\Omega]$ with strong Gröbner basis G . A normal set for $\mathbb{C}[\Omega]/I$ is a finite set of normal forms in $\mathbb{C}[\Omega]$ with respect to G , such that their equivalence classes form a vector space basis of $\mathbb{C}[\Omega]/I$. We do not assume that a normal set consists only of monomials.

Lemma 2.2. For $c \in \mathbb{C}^n$, denote $\mathcal{H}_c := (\mathcal{S}_c)$. The vector space dimension of $\mathbb{C}[\Omega]/\mathcal{H}_c$ is $|\mathcal{W}|$ and, if $\{h_1, \dots, h_{|\mathcal{W}|}\} \in \mathbb{C}[\Omega]$ is a normal set for $\mathbb{C}[\Omega]_{\mathcal{W}} = \mathbb{C}[\Omega]/\mathcal{H}_0$, then it is also a normal set for $\mathbb{C}[\Omega]/\mathcal{H}_c$ and vice versa.

Proof. By Theorem 1.3 and Lemma 2.1, there are $h_1, \dots, h_{|\mathcal{W}|} \in \mathbb{C}[\Omega]$, such that

$$\mathbb{C}[\Omega] = \bigoplus_{\ell=1}^{|\mathcal{W}|} h_\ell \cdot \mathbb{C}[\Omega]^{\mathcal{W}} = \bigoplus_{\ell=1}^{|\mathcal{W}|} h_\ell \cdot \mathbb{C}[\mathcal{S}_c].$$

Hence, for any $f = \sum_{\ell} h_\ell g_\ell(\theta_1 - c_1, \dots, \theta_n - c_n) \in \mathbb{C}[\Omega]$ with $g_\ell \in \mathbb{C}[z_1, \dots, z_n]$, we have $f - \sum_{\ell} h_\ell g_\ell(0) \in \mathcal{H}_c$. Since the h_ℓ are linearly independent in the quotient, the statement follows. \square

Proposition 2.3. For $c \in \mathbb{C}^n$ generic, the ideal \mathcal{H}_c in $\mathbb{C}[\Omega]$ is radical and $\mathbb{C}[\Omega]/\mathcal{H}_c$ affords the regular representation of \mathcal{W} .

Proof. The group elements $s \in \mathcal{W}$ act nonlinearly on elements $x = (x_1, \dots, x_n) \in (\mathbb{C} \setminus \{0\})^n$ of the torus by $(s, x) \mapsto s \star x$, where

$$(s \star x)_i := x_1^{\alpha_{i,1}} \dots x_n^{\alpha_{i,n}}, \alpha_{i,j} \in \mathbb{Z}, s^{-1}(\omega_i) = \alpha_{i,1} \omega_1 + \dots + \alpha_{i,n} \omega_n.$$

This induces the multiplicative action on $\mathbb{C}[\Omega]$, that is, $s \cdot f(x) = f(s^{-1} \star x)$. Denote by V_c the set of all points in $(\mathbb{C} \setminus \{0\})^n$, where all Laurent polynomials $f \in \mathcal{H}_c$ vanish. Since the c_i are generic, this set is nonempty. By [HMR24, Prop. 3.1], V_c is the orbit $\mathcal{W}(x)$ of a point $x \in (\mathbb{C} \setminus \{0\})^n$. In particular, the orbit has full length, that is, $|\mathcal{W}| = |\mathcal{W}(x)|$, and \mathcal{H}_c is zero-dimensional. We have

$$\dim_{\mathbb{C}}(\mathbb{C}[\Omega]/\mathcal{H}_c) \geq |V_c| = |\mathcal{W}(x)| = |\mathcal{W}|$$

and equality holds if and only if \mathcal{H}_c is radical. With Lemma 2.2, we have $\dim_{\mathbb{C}}(\mathbb{C}[\Omega]/\mathcal{H}_c) = |\mathcal{W}|$, which shows that \mathcal{H}_c is radical.

In particular, the coordinate ring $\mathbb{C}[\Omega]/\mathcal{H}_c$ of V_c corresponds to the algebra $\mathbb{C}[\mathcal{W}(x)]$ of regular functions on $\mathcal{W}(x)$. We claim that $\text{Hom}_{\mathcal{W}}(\mathcal{W}(x), \mathbb{C})$ and $\mathbb{C}[\mathcal{W}(x)]$ are isomorphic as \mathcal{W} -modules. For $x' \in \mathcal{W}(x)$, let $\delta_{x'} \in \mathbb{C}[\mathcal{W}(x)]$ denote the indicator function of x' , that is, $\delta_{x'}(x') = 1$ and $\delta_{x'}(\tilde{x}) = 0$ for all $\tilde{x} \in \mathcal{W}(x) \setminus \{x'\}$. For any $s \in \mathcal{W}$, we have $s \cdot \delta_{x'}(\tilde{x}) = \delta_{x'}(s^{-1} \star \tilde{x}) = \delta_{s \star x'}(\tilde{x})$. This shows that

$$\text{Hom}_{\mathcal{W}}(\mathcal{W}(x), \mathbb{C}) \rightarrow \mathbb{C}[\mathcal{W}(x)], \delta_{x'} \mapsto x'$$

does indeed define a \mathcal{W} -module homomorphism and it is bijective. Hence, the claim is true and the statement follows from the observation that $\mathbb{C}[\mathcal{W}]$ and $\mathbb{C}[\mathcal{W}(x)]$ are isomorphic as \mathcal{W} -modules. \square

Theorem 2.4. The multiplicative coinvariant space $\mathbb{C}[\Omega]_{\mathcal{W}}$ affords the regular representation of \mathcal{W} .

Proof. Let $c \in \mathbb{C}^n$ be a generic point. We prove that the vector spaces $\mathbb{C}[\Omega]_{\mathcal{W}}$ and $\mathbb{C}[\Omega]/\mathcal{H}_c$ are isomorphic as \mathcal{W} -modules. Since Proposition 2.3 implies $\mathbb{C}[\Omega]/\mathcal{H}_c \cong \mathbb{C}[\mathcal{W}]$, this will show the claim.

By Lemma 2.2, there exists a normal set $\{h_1, \dots, h_{|\mathcal{W}|}\} \subset \mathbb{C}[\Omega]$ of \mathcal{H}_0 and \mathcal{H}_c , such that

$$\mathcal{R}_c := \mathbb{C}[\Omega]/\mathcal{H}_c = \text{span}\{[h_1]_{\mathcal{R}_c}, \dots, [h_{|\mathcal{W}|}]_{\mathcal{R}_c}\}$$

with $\mathcal{R}_0 = \mathbb{C}[\Omega]_{\mathcal{W}}$ and thus

$$\Phi_c : \mathcal{R}_c \rightarrow \mathbb{C}[\Omega]_{\mathcal{W}}, \quad \sum_{\ell=1}^{|\mathcal{W}|} \lambda_{\ell} [h_{\ell}]_{\mathcal{R}_c} \mapsto \sum_{\ell=1}^{|\mathcal{W}|} \lambda_{\ell} [h_{\ell}]_{\mathcal{R}_0}$$

is a well-defined vector space isomorphism. Moreover, the ideals \mathcal{H}_c and \mathcal{H}_0 are \mathcal{W} -stable and, since $\mathbb{C}[\Omega]$ is a free $\mathbb{C}[\Omega]^{\mathcal{W}}$ -module, we have $s \cdot h_{\ell} \in \text{span}\{h_1, \dots, h_{|\mathcal{W}|}\}$ for all $s \in \mathcal{W}$. Thus, Φ_c is a \mathcal{W} -module isomorphism. \square

A multiplicative coinvariant basis can for example be computed as follows.

Algorithm 1: Multiplicative Coinvariant Bases

Data: $\mathcal{S}_m \subseteq \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

a set of fundamental invariants for $\text{Sym}(\Omega)^{\mathcal{W}}$

Result: $\mathcal{B}_m \subseteq \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

a normal set for $\mathbb{C}[\Omega]_{\mathcal{W}}$

- 1 $\mathcal{H}_m = (\mathcal{S}_m)$;
 - 2 $G = \text{GB}(\mathcal{H}_m)$ ▷ strong Gröbner basis;
 - 3 $N = \text{NS}(G)$ ▷ normal set;
 - 4 **return** $\mathcal{B}_m = N$;
-

Example 2.5. Recall from Example 1.4 that the multiplicative fundamental invariants for C_2 in the basis of fundamental weights are the orbit polynomials $\theta_1 = x_1 + x_1^{-1} + x_1^{-1}x_2 + x_1x_2^{-1}$ and $\theta_2 = x_2 + x_2^{-1} + x_1^{-2}x_2 + x_1^2x_2^{-1}$. A normal set of the Hilbert ideal $\mathcal{H}_m = (\theta_1, \theta_2)$ with respect to the graded reverse lexicographical ordering $x_1 \succ x_2 \succ x_1^{-1} \succ x_2^{-1}$ is

$$\{1, x_1, x_1x_2^{-1}, x_2, x_1^{-1}, x_1^{-2}, x_1^{-1}x_2^{-1}, x_2^{-1}\}.$$

Note that this normal set consists of monomials but does not tell us anything about the \mathcal{W} -module structure of $\mathbb{C}[\Omega]_{\mathcal{W}}$.

3. MULTIPLICATIVE \mathcal{W} -HARMONICS

Let \mathcal{W} be a Weyl group with Weight lattice $\Omega = \mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_n$, where ω_i are the fundamental weights of the corresponding root system. In analogy to Theorem 1.2, we show in the present section that the multiplicative coinvariant space can also be understood as a space of harmonics.

For this, we define the **Euler derivation** ∂_i as the linear operator on $\mathbb{C}[\Omega]$ with $\partial_i x^{\alpha} = \alpha_i x^{\alpha}$ for $\alpha \in \mathbb{Z}^n$. The associated nabla operator is denoted by $\nabla := [\partial_1, \dots, \partial_n]^{\ell}$. This definition is motivated by the derivative of the exponential function, see also [HW88, MNR13].

Recall from [Bou68, Ch. VI, §3, Thm. 1] that the multiplicative invariant ring $\mathbb{C}[\Omega]^{\mathcal{W}}$ of a Weyl group is generated as an algebra by the orbit polynomials $\theta_i := \frac{1}{|\mathcal{W}|} \sum_{s \in \mathcal{W}} \mathbf{e}^{s(\omega_i)}$, where ω_i are the fundamental weights. Denote the corresponding Jacobian matrix by $J := [\nabla \theta_1 | \dots | \nabla \theta_n] \in (\mathbb{C}[\Omega])^{n \times n}$.

Proposition 3.1. Let $\delta := \omega_1 + \dots + \omega_n \in \Omega$. The Jacobian determinant $\text{Det}(J)$ is, up to a nonzero scalar, equal to

$$\Upsilon_\delta := \frac{1}{|\mathcal{W}|} \sum_{s \in \mathcal{W}} \text{Det}(s) \mathbf{e}^{s(\delta)} \in \mathbb{C}[\Omega].$$

A Laurent polynomial $f \in \mathbb{C}[\Omega]$ with $s \cdot f = \text{Det}(s)f$ for all $s \in \mathcal{W}$ is called **anti-invariant**. By Weyl's character formula, the space of all anti-invariants is a free module of rank 1 over $\mathbb{C}[\Omega]^{\mathcal{W}}$ generated by Υ_δ , which is therefore also called the **Weyl denominator** [Bou75, Ch. VIII, §9, Thm. 1].

Proof. For $s \in \mathcal{W}$ and $x \in (\mathbb{C} \setminus \{0\})^n$, one observes that $J(s^{-1} \star x) = (s_{i,j})_{1 \leq i,j \leq n} J(x)$, where the right hand side is given by matrix multiplication. Hence, $\text{Det}(J)$ is anti-invariant and thus there exists a unique $f \in \mathbb{C}[\Omega]^{\mathcal{W}}$, such that $\text{Det}(J) = f \Upsilon_\delta$. We have

$$J_{ij} = \partial_i \theta_j = \frac{\partial_i}{|\mathcal{W}|} \sum_{s \in \mathcal{W}} \mathbf{e}^{s(\omega_j)} = \frac{\partial_i}{|\mathcal{W}(\omega_j)|} \left(\mathbf{e}^{\omega_j} + \sum_{\mu \prec \omega_j} \tilde{c}_\mu \mathbf{e}^\mu \right),$$

where $\tilde{c}_\mu \in \{0, 1\}$ and $\mu \prec \omega_j$ if and only if $0 \neq \omega_j - \mu$ is a sum of positive roots of the corresponding root system. The diagonal elements of the matrix J have highest monomial \mathbf{e}^{ω_i} in this ordering, while the non-diagonal entries have highest monomial \mathbf{e}^μ for some $\mu \prec \omega_j$. Hence, there exist $c_\mu \in \mathbb{Z}$ such that

$$f \Upsilon_\delta = \text{Det}(J) = \frac{1}{\prod_{i=1}^n |\mathcal{W}(\omega_j)|} \left(\mathbf{e}^\delta + \sum_{\mu \prec \delta} c_\mu \mathbf{e}^\mu \right).$$

By comparison of leading monomials, f must be a scalar. \square

We call the vector space \mathcal{L}_m generated by $\{\partial_{i_1} \dots \partial_{i_\ell} \text{Det}(J) \mid \ell \in \mathbb{N}, 1 \leq i_1 \leq \dots \leq i_\ell \leq n\}$ the space of **multiplicative \mathcal{W} -harmonics**.

Theorem 3.2. The space of multiplicative \mathcal{W} -harmonics affords the regular representation of \mathcal{W} .

Proof. By Proposition 3.1 the Jacobian determinant of \mathcal{W} is, up to a scalar, equal to the Weyl denominator Υ_δ . Since $\delta = \omega_1 + \dots + \omega_n$ is the sum of all the fundamental weights, it is contained in the interior of the fundamental Weyl chamber. But \mathcal{W} acts transitively on the set of all chambers and thus $|\mathcal{W}(\delta)| = |\mathcal{W}|$ and Υ_δ is a Laurent polynomial with $|\mathcal{W}|$ -many terms $c_\mu x^\mu$, where $\mu \in \mathcal{W}(e)$ and $c_\mu = \pm 1/|\mathcal{W}|$. From the definition of ∂_i , it follows that the vector space generated by all Euler derivations is a subspace of $L := \text{span}_{\mathbb{C}}\{\mathbf{e}^\mu \mid \mu \in \mathcal{W}(\delta)\}$. Since $|\mathcal{W}(\delta)| = |\mathcal{W}|$, L affords the regular representation of \mathcal{W} .

We show that $\mathcal{L}_m = L$ with a recursive argument. Let $W_0 := \{\mu = \alpha_1 \omega_1 + \dots + \alpha_n \omega_n \in \mathcal{W}(\delta) \mid \forall 1 \leq i \leq n : \alpha_i \neq 0\}$ be the set of all monomials in Υ_δ with full support. First,

we want to show that any monomial occurring in Υ_δ is contained in \mathcal{L}_m . We have $f := \partial_1 \cdots \partial_n \Upsilon_\delta = \sum_{\mu \in W_0} c_\mu \mathbf{e}^\mu \in \mathcal{L}_m$ for some $c_\mu \in \mathbb{C} \setminus \{0\}$. We order all the elements in W_0 totally $\mu^{(1)} \prec \mu^{(2)} \prec \cdots \prec \mu^{(n_0)}$ and show that $\mathbf{e}^{\mu^{(1)}} \in \mathcal{L}_m$. Let $\mu^{(\ell)} = \alpha_1^\ell \omega_1 + \cdots + \alpha_n^\ell \omega_n$ and $1 \leq j \leq n$ with $\mu_j^{(1)} \neq \mu_j^{(2)}$. Consider $\partial_j f - \alpha_j^2 f = \sum_{i=1}^{n_0} c_{\mu^{(i)}} (\alpha_j^{(i)} - \alpha_j^{(2)}) \mathbf{e}^{\mu^{(i)}} \in \mathcal{L}_m$. By iterating this procedure we find $\mathbf{e}^{\mu^{(1)}} \in \mathcal{L}_m$ and eventually that $\mathbf{e}^{\mu^{(i)}} \in \mathcal{L}_m$ for any $1 \leq i \leq n_0$. We can repeat the same strategy by considering $f_1 := \sum_{\mu \in \mathcal{W}(\delta) \setminus W_0} c_\mu \mathbf{e}^\mu$ and an inclusion-wise maximal support set W_1 in f_1 of cardinality n_1 . After finitely many steps we obtain $\mathcal{L}_m = L$. \square

Concluding with Theorem 2.4, the multiplicative coinvariant space $\mathbb{C}[\Omega]_{\mathcal{W}}$ is, as a \mathcal{W} -module, isomorphic to the space of multiplicative \mathcal{W} -harmonics.

Remark 3.3. For the action of a pseudoreflection group on polynomials and, in particular, for the additive action of a Weyl group on $\text{Sym}(\Omega) = \mathbb{C}[X_1, \dots, X_n]$, the additive \mathcal{W} -harmonic polynomials can equivalently be defined as the orthogonal complement of \mathcal{H}_a with respect to the inner product $\langle \cdot, \cdot \rangle : \text{Sym}(\Omega) \times \text{Sym}(\Omega) \rightarrow \mathbb{C}$, $\langle f, g \rangle := (\partial_f(g))(0)$, where $\partial_f := \sum_\alpha c_\alpha \frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial X_1^{\alpha_1} \cdots \partial X_n^{\alpha_n}}$ if $f = \sum_\alpha c_\alpha X^\alpha$, see for example [SW23]. The name ‘‘harmonics’’ originates from this construction. As a consequence we have that the tensor product over \mathbb{C} of the additive coinvariant space $\text{Sym}(\Omega)/\mathcal{H}_a$ with the space of additive \mathcal{W} -harmonics is isomorphic to $\text{Sym}(\Omega)$.

Unfortunately, this does not hold in the multiplicative setting. For instance, for \mathcal{W} the Weyl group of the root system A_2 , we found that

$$\mathbb{C}[\Omega]^{\mathcal{W}} \otimes_{\mathbb{C}} \text{span}_{\mathbb{C}} \{ \partial_{i_1} \cdots \partial_{i_\ell} \Upsilon_\delta \mid \ell \in \mathbb{N}, 1 \leq i_1 \leq \dots \leq i_\ell \leq n \}$$

is a proper subset of $\mathbb{C}[\Omega]$, where the Weyl denominator Υ_δ is equal to the Laurent polynomial g in Table 2 up to a scalar.

4. ASSOCIATED GRADED EQUIVARIANTS

In the previous two sections, we have described the multiplicative coinvariant space in terms of representation-theoretic properties. From a computational point of view however, the problem with Algorithm 1 is that the obtained basis for $\mathbb{C}[\Omega]_{\mathcal{W}}$ does not reflect these properties. Our next goal in this and the following section is therefore to provide a algorithmic relation between additive and multiplicative coinvariants, that allows to preserve these properties in computations.

To commence, we follow [BNZ12] and consider the algebra homomorphisms

$$\text{Sym}(\Omega) \rightarrow \mathbb{C}, X_i \mapsto 0 \quad \text{and} \quad \mathbb{C}[\Omega] \rightarrow \mathbb{C}, x_i \mapsto 1$$

with kernels $\mathcal{I}_a = (X_i \mid 1 \leq i \leq n)$ and $\mathcal{I}_m = (1 - x_i^{-1} \mid 1 \leq i \leq n)$.

Lemma 4.1. For a nonnegative integer $d \in \mathbb{N}$,

$$\phi_d : \text{Sym}(\Omega)/\mathcal{I}_a^{d+1} \rightarrow \mathbb{C}[\Omega]/\mathcal{I}_m^{d+1}, [X_i] \mapsto [1 - x_i^{-1}]$$

is an algebra isomorphism.

Proof. The inverse homomorphism is defined by $\phi_d^{-1}([x_i^{-1}]) = [1 - X_i]$ and $\phi_d^{-1}([x_i]) = [1 + X_i + X_i^2 + \dots + X_i^d]$ by geometric series expansion modulo \mathcal{I}_a^{d+1} . \square

Note that ϕ_d can be lifted to an algebra homomorphism $\phi : \text{Sym}(\Omega) \rightarrow \mathbb{C}[\Omega]$. Conversely, any $\psi : \mathbb{C}[\Omega] \rightarrow \text{Sym}(\Omega)$ with $1 = \psi(1) = \psi(x_i x_i^{-1}) = \psi(x_i) \psi(x_i^{-1})$ forces $\psi(x_i)$ to be a unit in $\text{Sym}(\Omega)$, that is, $\psi(x_i) \in \mathbb{C} \setminus \{0\}$. Hence, ϕ is not invertible.

Instead, we consider the associated graded algebras [BH98]

$$\mathcal{G}_a^*(\Omega) := \bigoplus_{d \in \mathbb{N}} \mathcal{I}_a^d / \mathcal{I}_a^{d+1} \quad \text{and} \quad \mathcal{G}_m^*(\Omega) := \bigoplus_{d \in \mathbb{N}} \mathcal{I}_m^d / \mathcal{I}_m^{d+1}.$$

The quotient $\mathcal{I}_a^d / \mathcal{I}_a^{d+1}$ is isomorphic to the vector space $\text{Sym}(\Omega)_d$ of homogeneous polynomials of degree d . In particular, we have $\mathcal{G}_a^*(\Omega) \cong \text{Sym}(\Omega)$ as graded algebras.

For $f \in \mathbb{C}[\Omega]$ on the other hand, we denote by $f^* := [f_0] + [f_1] + [f_2] + \dots$ the associated graded element in $\mathcal{G}_m^*(\Omega)$, where f_0 is the normal form of f with respect to \mathcal{I}_m and iteratively f_d is the normal form of $f - f_{d-1} - f_{d-2} - \dots - f_0$ with respect to \mathcal{I}_m^{d+1} so that $[f_d] \in \mathcal{I}_m^d / \mathcal{I}_m^{d+1}$. We have $f = f_0 + f_1 + f_2 + \dots$ and, since $\mathcal{G}_m^*(\Omega)$ is Noetherian, this iteration stops at some point.

Example 4.2. (1) Consider the homogeneous polynomial

$$F = X_1 X_2 - X_1^2 \in \mathbb{C}[X_1, X_2]_2 \cong \mathcal{I}_a^2 / \mathcal{I}_a^3.$$

Its image under $X_i \mapsto 1 - x_i^{-1}$ decomposes into

$$\begin{aligned} f &= \frac{1}{x_1} - \frac{1}{x_1^2} - \frac{1}{x_2} + \frac{1}{x_1 x_2} \\ &= \underbrace{\left(\frac{1}{x_1 x_2} - x_1 - \frac{2}{x_1} - \frac{1}{x_2} + 3 \right)}_{=f_2} + \underbrace{\left(-\frac{1}{x_1^2} + x_1 + \frac{3}{x_1} - 3 \right)}_{=f_3} \end{aligned}$$

with $[f_2] \in \mathcal{I}_m^2 / \mathcal{I}_m^3$ and $[f_3] \in \mathcal{I}_m^3 / \mathcal{I}_m^4$.

(2) Now, let \mathcal{W} be a Weyl group of the root system C_2 as in Example 2.5. Since $\mathcal{W}(\omega_1) = \{\omega_1, \omega_1 - \omega_2, \omega_2 - \omega_1, -\omega_1\}$, we have

$$\begin{aligned} \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]^{\mathcal{W}} &\ni x_1 + \frac{x_1}{x_2} + \frac{x_2}{x_1} + \frac{1}{x_1} - 4 \\ &= \underbrace{\left(\frac{-2}{x_1 x_2} + 2x_1 + x_2 + \frac{4}{x_1} + \frac{3}{x_2} - 8 \right)}_{=f_2} \\ &\quad + \underbrace{\left(\frac{x_1}{x_2} + \frac{x_2}{x_1} + \frac{2}{x_1 x_2} - x_1 - x_2 - \frac{3}{x_1} - \frac{3}{x_2} + 4 \right)}_{=f_3}. \end{aligned}$$

Neither of the two summands is stable under the \mathcal{W} -transformation $s_1 : (x_1, x_2) \mapsto (x_1^{-1} x_2, x_2)$.

We conclude from the example that ϕ_d preserves neither homogeneity nor equivariance. We explain now how working over the associated graded algebras instead solves this circumstance.

Proposition 4.3 ([BNZ12, CPZ13]). The quotients $\text{Sym}(\Omega)_d \cong \mathcal{I}_a^d / \mathcal{I}_a^{d+1}$ and $\mathcal{I}_m^d / \mathcal{I}_m^{d+1}$ are \mathcal{W} -stable under the induced additive and multiplicative action respectively. Furthermore,

the map

$$\begin{aligned} \phi^{(d)} : \text{Sym}(\Omega)_d &\rightarrow \mathcal{I}_m^d / \mathcal{I}_m^{d+1}, \\ \prod_{j=1}^d X_{i_j} &\mapsto \left[\prod_{j=1}^d (1 - x_{i_j}^{-1}) \right] \end{aligned}$$

is a \mathcal{W} -module isomorphism.

Proof. Let $s \in \mathcal{W}$ with matrix entries $s_{\ell,i} \in \mathbb{Z}$. Then

$$\phi^{(d)} \left(s \cdot \prod_{j=1}^d X_{i_j} \right) = \phi^{(d)} \left(\prod_{j=1}^d \sum_{\ell=1}^n s_{\ell,i_j} X_{\ell} \right) = \left[\prod_{j=1}^d \sum_{\ell=1}^n s_{\ell,i_j} (1 - x_{\ell}^{-1}) \right].$$

Since $(1 - x^{-\alpha})(1 - x^{-\beta}) = (1 - x^{-\alpha}) + (1 - x^{-\beta}) - (1 - x^{-\alpha-\beta})$ for any $\alpha, \beta \in \mathbb{Z}^n$ and $[1 - x_{\ell}^{-1}] = [-(1 - x_{\ell})]$ in $\mathcal{I}_m / \mathcal{I}_m^2$, we have $[s_{\ell,i_j}(1 - x_{\ell}^{-1})] = [1 - x_{\ell}^{-s_{\ell,i_j}}]$ in $\mathcal{I}_m / \mathcal{I}_m^2$ and consequently

$$\left[\prod_{j=1}^d \sum_{\ell=1}^n s_{\ell,i_j} (1 - x_{\ell}^{-1}) \right] = \left[\prod_{j=1}^d \sum_{\ell=1}^n (1 - x_{\ell}^{-s_{\ell,i_j}}) \right]$$

in $\mathcal{I}_m^d / \mathcal{I}_m^{d+1}$. Thus,

$$\phi^{(d)} \left(s \cdot \prod_{j=1}^d X_{i_j} \right) = \left[\prod_{j=1}^d \sum_{\ell=1}^n (1 - x_{\ell}^{-s_{\ell,i_j}}) \right] = s \cdot \phi^{(d)} \left(\prod_{j=1}^d X_{i_j} \right).$$

We conclude that $\phi^{(d)}$ is well-defined and \mathcal{W} -equivariant. For bijectiveness, note that $\{(1 - x_{i_1}^{-1}) \cdots (1 - x_{i_d}^{-1}) \mid 1 \leq i_1 \leq \dots \leq i_d \leq n\}$ is a vector space basis for $\mathcal{I}_m^d / \mathcal{I}_m^{d+1}$. \square

In particular,

$$\phi^* = \bigoplus_{d \in \mathbb{N}} \phi^{(d)} : \text{Sym}(\Omega) \rightarrow \mathcal{G}_m^*(\Omega).$$

is a \mathcal{W} -equivariant graded algebra isomorphism.

5. GRADED TRANSFER OF COINVARIANTS

In this section, we show how to preserve group symmetry between additive and multiplicative coinvariants. The kinds of basis we are specifically looking for are called symmetry-adapted.

Let \mathcal{W} be the Weyl group of a crystallographic root system with Weight lattice Ω and fundamental weights ω_i . As fundamental invariants, we consider the orbit polynomials $\theta_i := \frac{1}{|\mathcal{W}|} \sum_{s \in \mathcal{W}} \mathbf{e}^{s(\omega_i)}$. By Lemma 2.1, the set $\{\theta_1 - 1, \dots, \theta_n - 1\} \subseteq \mathcal{I}_m \cap \mathbb{C}[\Omega]^{\mathcal{W}}$ also generates $\mathbb{C}[\Omega]^{\mathcal{W}}$ as an algebra and thus we may fix $\mathcal{H}_m = (\mathcal{I}_m \cap \mathbb{C}[\Omega]^{\mathcal{W}})$ as the multiplicative Hilbert ideal in $\mathbb{C}[\Omega]$.

Proposition 5.1. The ideal $\mathcal{H}_m^* := (h^* \mid h \in \mathcal{H}_m)$ in $\mathcal{G}_m^*(\Omega)$ is generated by the set $\{(\theta_1 - 1)^*, \dots, (\theta_n - 1)^*\}$. Equip $\mathcal{G}_m^*(\Omega) / \mathcal{H}_m^*$ with the induced multiplicative action. Then the map

$$\mathbb{C}[\Omega]_{\mathcal{W}} \rightarrow \mathcal{G}_m^*(\Omega) / \mathcal{H}_m^*, f + \mathcal{H}_m \mapsto f^* + \mathcal{H}_m^*$$

is a \mathcal{W} -module isomorphism.

Proof. To see that the map is well-defined, let $f - g \in \mathcal{H}_m$, that is, $f - g = \sum_i k_i (\theta_i - 1)$ for some $k_i \in \mathbb{C}[\Omega]$. Then $(f - g)^* = \sum_i k_i^* (\theta_i - 1)^* \in \mathcal{H}_m^*$. Furthermore, since every element $\tilde{f} \in \mathcal{G}_m^*(\Omega)$ is of the form $\tilde{f} = [f_0] + [f_1] + [f_2] + \dots$ for some $f_d \in \mathcal{I}_m^d$, the inverse is given by $\tilde{f} + \mathcal{H}_m^* \mapsto f_0 + f_1 + f_2 + \dots + \mathcal{H}_m$ and thus the map is bijective. Finally, equivariance follows from Proposition 4.3. \square

Theorem 5.2. Let $\{H_\ell \mid 1 \leq \ell \leq |\mathcal{W}|\} \subseteq \text{Sym}(\Omega)$ be a normal set for $\text{Sym}(\Omega)_{\mathcal{W}}$ with H_ℓ homogeneous of degree d_ℓ . Then

$$\{\phi^{(d_\ell)}(H_\ell) \mid 1 \leq \ell \leq |\mathcal{W}|\}$$

is a normal set for $\mathcal{G}_m^*(\Omega)/\mathcal{H}_m^*$.

Proof. Thanks to Theorem 2.4, we only need to show that the $\phi^{(d_\ell)}(H_\ell) \in \mathcal{G}_m^*(\Omega)$ are linearly independent modulo \mathcal{H}_m^* .

We may assume without loss of generality that

$$d_1 = \dots = d_{k_1} \prec d_{k_1+1} = \dots = d_{k_2} \prec d_{k_2+1} = \dots = d_{k_3} \prec \dots$$

with $k_0 = 0$ and $k_r = |\mathcal{W}|$. Let $\lambda_\ell \in \mathbb{C}$ be scalars such that

$$\begin{aligned} \mathcal{H}_m^* \ni \sum_{\ell=1}^{|\mathcal{W}|} \lambda_\ell \phi^{(d_\ell)}(H_\ell) &= \sum_{j=1}^r \sum_{\ell=k_{j-1}+1}^{k_j} \lambda_\ell \phi^{(d_{k_j})}(H_\ell) \\ &= \sum_{j=1}^r \phi^{(d_{k_j})} \left(\sum_{\ell=k_{j-1}+1}^{k_j} \lambda_\ell H_\ell \right) \in \bigoplus_{j=1}^r \mathcal{I}_m^{d_{k_j}} / \mathcal{I}_m^{d_{k_j}+1}. \end{aligned}$$

This decomposition implies

$$\phi^{(d_{k_j})} \left(\sum_{\ell=k_{j-1}+1}^{k_j} \lambda_\ell H_\ell \right) \in \mathcal{H}_m^* \cap \mathcal{I}_m^{d_{k_j}} / \mathcal{I}_m^{d_{k_j}+1}$$

and thus, using the \mathcal{W} -equivariance from Proposition 4.3,

$$\sum_{\ell=k_{j-1}+1}^{k_j} \lambda_\ell H_\ell \in \mathcal{H}_a \cap \text{Sym}(\Omega)_{d_{k_j}}.$$

In particular, since the H_ℓ form a normal set for $\text{Sym}(\Omega)/\mathcal{H}_a$, their only linear combination in \mathcal{H}_a is 0, which finally yields $\lambda_\ell = 0$. \square

Since the finite dimensional vector spaces $\text{Sym}(\Omega)_d$ and $\mathcal{I}_m^d / \mathcal{I}_m^{d+1}$ are isomorphic as \mathcal{W} -modules by the map $\phi^{(d)}$ from Proposition 4.3, they have coinciding isotypic decompositions

$$\text{Sym}(\Omega)_d \cong \mathcal{I}_m^d / \mathcal{I}_m^{d+1} \cong \bigoplus_{i=1}^{\#\text{irrep.}} W_i^{m_i^{(d)}},$$

where $\#\text{irrep.}$ is the number of irreducible \mathcal{W} -modules W_i , which appear with multiplicity $m_i^{(d)}$ [Ser77]. A vector space basis for (a subspace of)

$$\text{Sym}(\Omega) = \bigoplus_{d \in \mathbb{N}} \text{Sym}(\Omega)_d \quad \text{or} \quad \mathcal{G}_m^*(\Omega) = \bigoplus_{d \in \mathbb{N}} \mathcal{I}_m^d / \mathcal{I}_m^{d+1}$$

respecting the isotypic decomposition of the homogeneous components is called **symmetry adapted** (see [DK02, GP04] for general finite groups).

Corollary 5.3. Assume that the normal set $\{H_\ell \mid 1 \leq \ell \leq |\mathcal{W}|\}$ for $\text{Sym}(\Omega)_{\mathcal{W}}$ in Theorem 5.2 is symmetry-adapted. Then the same holds for the normal set $\{\phi^{(d_\ell)}(H_\ell) \mid 1 \leq \ell \leq |\mathcal{W}|\}$ for $\mathcal{G}_m^*(\Omega)/\mathcal{H}_m^*$.

With Proposition 5.1 we have

$$\mathbb{C}[\Omega] \cong \mathbb{C}[\Omega]^{\mathcal{W}} \otimes_{\mathbb{C}} \mathbb{C}[\Omega]_{\mathcal{W}} \cong \mathbb{C}[\Omega]^{\mathcal{W}} \otimes_{\mathbb{C}} \mathcal{G}_m^*(\Omega)/\mathcal{H}_m^*,$$

and can thus derive a symmetry adapted basis for $\mathbb{C}[\Omega]$ by multiplication with invariants in \mathcal{H}_m , see also [DR23] for additive invariants.

Algorithm 2: Symmetry Adapted Coinvariant Transfer

Data: $\mathcal{B}_a \subseteq \mathbb{C}[X_1, \dots, X_n]$

a symmetry adapted homogeneous normal set for $\text{Sym}(\Omega)_{\mathcal{W}}$

Result: $\mathcal{B}_m \subseteq \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

a symmetry adapted homogeneous normal set for $\mathbb{C}[\Omega]_{\mathcal{W}}$

1 $K = \emptyset$;

2 $L = \mathcal{B}_a$;

3 $\mathcal{I}_m = (1 - x_i^{-1} \mid 1 \leq i \leq n)$;

4 **while** $L \neq \emptyset$ **do**

5 **choose** $H \in L$;

6 $d = \deg(H)$;

7 $h = H|_{X_i \mapsto 1 - x_i^{-1}}$;

8 $h = \text{NF}(h, \mathcal{I}_m^{d+1})$ ▷ normal form;

9 $L = L \setminus \{H\}$;

10 $K = K \cup \{h\}$;

11 **end**

12 **return** $\mathcal{B}_m = K$;

Example 5.4. Recall from Example 1.4, that the weight lattice $\Omega = \mathbb{Z}^2$ is generated by the fundamental weights ω_1, ω_2 defining variables X_i, x_i and by the standard basis e_1, e_2 defining variables Y_i, y_i for $\text{Sym}(\Omega)$ and $\mathbb{C}[\Omega]$ respectively.

A symmetry adapted basis of the additive coinvariant space of \mathbb{C}_2 with coordinates Y_1, Y_2 can be determined using higher Specht polynomials [MY98]. From this construction, we display in Table 1 symmetry adapted bases for the additive and multiplicative coinvariant spaces for both bases. Although they are not a monomial basis as the normal set in Example 2.5, we can deduce the isotypic decomposition.

6. HOMOMORPHISMS FOR TYPES A_{n-1} AND C_n

Although $\mathbb{C}[\Omega]$ and $\text{Sym}(\Omega)$ are not isomorphic as rings, one can find homomorphisms of \mathcal{W} -modules for types A_{n-1} and C_n .

Irrep	Dim	Basis elements
$\mathbb{S}^{(2),(0)}$	1	1
$\mathbb{S}^{(1),(1)}$	2	Y_1, Y_2 $1 - 1/y_1, 1/y_1 - 1/(y_1 y_2)$ $X_1, -X_1 + 2X_2$ $1 - 1/x_1, 1/x_1 - 1/x_2$ $Y_2^2 \cdot Y_1, Y_1^2 \cdot Y_2$ $3/y_2 - 1/y_1^2 + 1/(y_1^2 y_2) + y_1 y_2 + 4/y_1 - 3/(y_1 y_2) - 1, -1/y_2 + 1/y_1^2 - 1/(y_1^2 y_2) - 2/y_1 + 2/(y_1 y_2) + 1$ $X_1^3 - 4X_1^2 X_2 + 4X_1 X_2^2, -X_1^3 + 2X_1^2 X_2$ $3x_1/x_2 - 1/x_1^2 + 1/(x_2 x_1) + x_2 + 4/x_1 - 3/x_2 - 1, -x_1/x_2 + 1/x_1^2 - 1/(x_2 x_1) - 2/x_1 + 2/x_2 + 1$
$\mathbb{S}^{(0),(2)}$	1	$Y_1 \cdot Y_2$ $1/(y_1^2 y_2) - y_1 - 2/y_1 - 1/(y_1 y_2) + 3$ $-X_1^2 + 2X_1 X_2$ $1/(x_2 x_1) - x_1 - 2/x_1 - 1/x_2 + 3$
$\mathbb{S}^{(1,1),(0)}$	1	$Y_1^2 - Y_2^2$ $2/(y_1^2 y_2) - y_1 y_2 - 2/y_1 - 3/(y_1 y_2) + 4$ $4X_1 X_2 - 4X_2^2$ $2/(x_2 x_1) - x_2 - 2/x_1 - 3/x_2 + 4$
$\mathbb{S}^{(0),(1,1)}$	1	$(Y_1^2 - Y_2^2) \cdot Y_1 \cdot Y_2$ $6/y_2 + 4/y_1^2 + 14/(y_1^2 y_2) - 10y_1 - 10y_1 y_2 - 18/y_1 - 10/(y_1 y_2) + 24$ $-4X_1^3 X_2 + 12X_1^2 X_2^2 - 8X_1 X_2^3$ $6x_1/x_2 + 4/x_1^2 + 14/(x_2 x_1) - 10x_1 - 10x_2 - 18/x_1 - 10/x_2 + 24$

TABLE 1. Additive and Multiplicative coinvariants for the group C_2 in standard and weight bases

The reason why we find the linear maps only for these types is that the change of basis from fundamental weights x_i to standard y_i over the Laurent polynomials is bijective only for types A_{n-1} and C_n among the infinite families of Weyl groups, see [HMR24, Lem. 5.2, 6.2].

6.1. **Type A_{n-1} .** The symmetric group \mathfrak{S}_n acts on the affine algebraic varieties

$$V := \{Y = (Y_1, \dots, Y_n) \in \mathbb{C}^n \mid Y_1 + \dots + Y_n = 0\}$$

and $W := \{y = (y_1, \dots, y_n) \in \mathbb{C}^n \mid y_1 \cdots y_n - 1 = 0\}$

by permutation of coordinates. This defines a representation $\rho : \mathfrak{S}_n \rightarrow \mathrm{GL}_n(\mathbb{C})$ with induced actions on the coordinate rings $\mathbb{C}[V] \cong \mathbb{C}[Y_1, \dots, Y_n]/(Y_1 + \dots + Y_n)$ and $\mathbb{C}[W] \cong \mathbb{C}[y_1, \dots, y_n]/(y_1 \cdots y_n - 1)$.

Recall from [Bou68, Pl. I] that $\mathcal{W} = \mathfrak{S}_n$ is the Weyl group of the root system A_{n-1} . Denote by $\omega_1, \dots, \omega_{n-1}$ the fundamental weights of the weight lattice Ω . This defines a representation $\pi : \mathfrak{S}_n \rightarrow \mathrm{GL}_{n-1}(\mathbb{Z})$ with an additive action on $\mathrm{Sym}(\Omega) \cong \mathbb{C}[X_1, \dots, X_{n-1}]$ and a multiplicative action on $\mathbb{C}[\Omega] \cong \mathbb{C}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$. Furthermore, we have

$$\mathcal{W}(\omega_1) = \{\omega_1, \omega_2 - \omega_1, \dots, \omega_{n-1} - \omega_{n-2}, -\omega_{n-1}\},$$

which leads to the following statement (see also [HMR24, §5]).

Lemma 6.1. The maps $\mathbb{C}[V] \rightarrow \text{Sym}(\Omega)$ with

$$Y_1 \mapsto X_1, Y_2 \mapsto X_2 - X_1, \dots, Y_{n-1} \mapsto X_{n-1} - X_{n-2}, Y_n \mapsto -X_{n-1}$$

and $\mathbb{C}[W] \rightarrow \mathbb{C}[\Omega]$ with

$$y_1 \mapsto x_1, y_2 \mapsto x_2 x_1^{-1}, \dots, y_{n-1} \mapsto x_{n-1} x_{n-2}^{-1}, y_n \mapsto x_{n-1}^{-1}$$

are well-defined ρ - π -equivariant algebra isomorphisms.

We note that the isomorphism $\mathbb{C}[W] \rightarrow \mathbb{C}[\bar{\Omega}]$ was already observed in [GS84, §9].

Proof. The choice of coordinates $X_i = \omega_i, Y_i = \omega_i - \omega_{i-1}, x_i = \mathbf{e}^{\omega_i}, y_i = \mathbf{e}^{\omega_i - \omega_{i-1}}$, by which the Y_i and y_i form \mathcal{W} -orbits of length n , implies that the maps are well-defined and ρ - π -equivariant. For bijectiveness, note that the inverse maps are given by $X_i \mapsto Y_1 + \dots + Y_i$ and $x_i \mapsto y_1 \cdots y_i$. \square

Hence, the induced actions on $\mathbb{C}[V]$ and $\mathbb{C}[W]$ can be identified with the additive and multiplicative action respectively.

Example 6.2. The Weyl group \mathcal{W} of A_2 is generated by s_1, s_2 with

$$\rho(s_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \rho(s_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$\pi(s_1) = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \pi(s_2) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

We now compute a coinvariant basis for $\mathbb{C}[\Omega]_{\mathcal{W}}$. The fundamental theorem of symmetric polynomials states that the invariant ring $\mathbb{C}[y_1, \dots, y_n]^{\rho(\mathfrak{S}_n)}$ is, as an algebra, generated by the elementary symmetric polynomials

$$\sigma_i(y) = \sum_{\substack{J \subseteq \{1, \dots, n\} \\ \text{with } |J|=i}} \prod_{j \in J} y_j.$$

In particular, $\mathbb{C}[W]^{\mathcal{W}} \cong \mathbb{C}[\sigma_1(y), \dots, \sigma_{n-1}(y)] / (\sigma_n(y) - 1)$.

On the other hand, the multiplicative invariant ring $\mathbb{C}[\Omega]^{\mathcal{W}} \cong \mathbb{C}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]^{\pi(\mathfrak{S}_n)}$ is generated by orbit polynomials $\theta_i(x)$. Denote by $\Psi : \mathbb{C}[W] \rightarrow \mathbb{C}[\Omega]$ the second map from Lemma 6.1.

Proposition 6.3. The map

$$\mathbb{C}[y] / (\sigma_i(y) \mid 1 \leq i \leq n) \rightarrow \mathbb{C}[x^{\pm}] / (\theta_i \mid 1 \leq i \leq n-1)$$

that takes $h(y)$ to $\Psi(h(y))$ is a \mathcal{W} -module isomorphism.

Proof. By [HMR24, Prop. 5.1], $\Psi(\sigma_i(y))$ is equal to $\theta_i(x)$ up to a scalar and thus

$$\begin{aligned} \mathbb{C}[W]_{\mathcal{W}} &\cong \mathbb{C}[y] / (\sigma_1(y), \dots, \sigma_{n-1}(y), \sigma_n(y) - 1) \\ &\cong \mathbb{C}[x^{\pm}] / (\theta_1(x), \dots, \theta_{n-1}(x)) \cong \mathbb{C}[\Omega]_{\mathcal{W}}. \end{aligned}$$

Then the statement follows as in Lemma 2.2. \square

The irreducible representations of \mathfrak{S}_n are called **Specht modules** and are denoted by \mathbb{S}^λ where λ is a partition of n (we refer to [Sag01] for details).

Example 6.4. The Weyl group $\mathcal{W} = \mathfrak{S}_3$ of A_2 has three irreducible representations \mathbb{S}^λ , where λ is a partition of 3. Moreover the group algebra $\mathbb{C}[\mathcal{W}]$ has isotypic decomposition $1 \cdot \mathbb{S}^{(3)} \oplus 2 \cdot \mathbb{S}^{(2,1)} \oplus 1 \cdot \mathbb{S}^{(1,1,1)}$. In Table 2, we start with a symmetry adapted basis of the additive coinvariant space. The additive basis consists of higher Specht polynomials and was calculated with the M2 package SpechtModule [NC].

	$\mathbb{S}^{(3)}$	$\mathbb{S}^{(2,1)}$	$\mathbb{S}^{(1,1,1)}$
a	1	$\{Y_1 - Y_2, Y_1 - Y_3\}$ $\{(Y_1 - Y_2)Y_3, (Y_1 - Y_3)Y_2\}$	$(Y_1 - Y_2)(Y_1 - Y_3)(Y_2 - Y_3)$
m	1	$\{f_{1,1} := x_1 - x_2x_1^{-1}, f_{1,2} := x_1 - x_2^{-1}\}$ $\{f_{2,1} := x_1x_2^{-1} - x_1^{-1}, f_{2,2} := x_2 - x_1^{-1}\}$	$g := x_1x_2 - x_1^{-1}x_2^2 + x_1^{-2}x_2$ $-x_1^2x_2^{-1} + x_1x_2^{-2} - x_1^{-1}x_2^{-1}$

TABLE 2. Additive and multiplicative coinvariant bases for A_2 .

A filtration of $\mathbb{C}[\Omega]$ based on the Voronoï cell of the dual lattice was introduced in [Met25] with the purpose of optimization of trigonometric polynomials with crystallographic symmetry. In said article, a multiplicity pattern of the irreducible representations was computationally observed for A_2 . The symmetry adapted basis from Example 6.4 allows us to prove this observation.

We equip the real vector space $\mathfrak{h} = \text{span}_{\mathbb{R}} \Omega$ with an inner product $\langle \cdot, \cdot \rangle : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{R}$, so that \mathcal{W} is orthogonal, that is, $\langle s(X), Y \rangle = \langle X, s^{-1}(Y) \rangle$ for all $X, Y \in \mathfrak{h}$. The **coroot lattice** Λ is the dual lattice of Ω , that is, $\Lambda = \{\lambda \in \mathfrak{h} \mid \forall \mu \in \Omega : \langle \lambda, \mu \rangle \in \mathbb{Z}\}$.

Let $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$ be the induced norm. Then the **Voronoï cell** of Λ is $\text{Vor}(\Lambda) := \{X \in \mathfrak{h} \mid \forall \lambda \in \Lambda : \|X\| \leq \|\lambda - X\|\}$.

Since Ω is full-dimensional, so is Λ and in particular $\text{Vor}(\Lambda)$ is compact. Thus, for $d \in \mathbb{N}$, the subset $\Omega_d := \Omega \cap d \text{Vor}(\Lambda)$ of the lattice is finite, where $d \text{Vor}(\Lambda)$ denotes the dilation of Λ by the integer d , see Figure 1. We define $\mathbb{C}[\Omega]_d$ as the finite-dimensional vector subspace of $\mathbb{C}[\Omega]$ containing all Laurent polynomials with support in Ω_d .

By [Bou68, Ch. VI, §2, Prop. 4] and [CS99, Ch. 21, §3, Thm. 5], we have $\text{Vor}(\Lambda) = \mathcal{W}\Delta$, where Δ is the closure of an alcove of the affine Weyl group $\mathcal{W} \ltimes \Lambda$ so that $0 \in \Delta$. In particular, the set $\mathcal{W}(d\Delta \cap \Omega) = d \text{Vor}(\Lambda) \cap \Omega = \Omega_d$ is \mathcal{W} -stable and thus \mathcal{W} acts multiplicatively on $\mathbb{C}[\Omega]_d$.

Theorem 6.5. For A_2 , the isotypic decomposition of $\mathbb{C}[\Omega]_d$ is

$$\frac{(d+1)(d+2)}{2} \cdot \mathbb{S}^{(3)} \oplus d(d+1) \cdot \mathbb{S}^{(2,1)} \oplus \frac{(d-1)d}{2} \cdot \mathbb{S}^{(1,1,1)}.$$

Proof. Note that for any $s \in \mathcal{W}$ and $\mu \in d\Omega \setminus (d-1)\Omega$, we have $s(\mu) \in d\Omega \setminus (d-1)\Omega$, because \mathcal{W} is an isometry group.

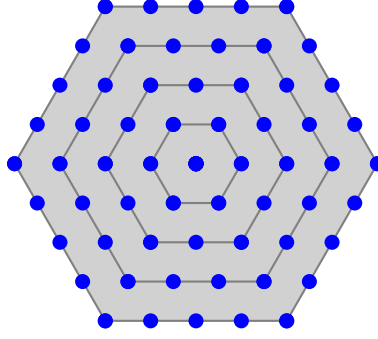


FIGURE 1. The subsets $\Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \Omega_4$ of the hexagonal lattice in the plane, the weight lattice Ω of A_2 .

Using the symmetry adapted basis of the coinvariant space $\mathbb{C}[x^\pm]_{\mathcal{W}}$ from Example 6.4, we find

$$\mathbb{C}[\Omega]_0 = \langle 1 \rangle_{\mathbb{C}} = \mathbb{S}^{(3)},$$

$$\mathbb{C}[\Omega]_1 = \mathbb{C}[\Omega]_0 \oplus \langle \theta_1, \theta_2, f_{1,1}, f_{1,2}, f_{2,1}, f_{2,2} \rangle_{\mathbb{C}} = 3 \cdot \mathbb{S}^{(3)} \oplus 2 \cdot \mathbb{S}^{(2,1)}$$

$$\mathbb{C}[\Omega]_2 = \mathbb{C}[\Omega]_1 \oplus \langle \theta_i^2, \theta_1 \theta_2, \theta_i f_{j,k}, g \mid 1 \leq i, j, k \leq 2 \rangle_{\mathbb{C}} = 6 \cdot \mathbb{S}^{(3)} \oplus 6 \cdot \mathbb{S}^{(2,1)} \oplus 1 \cdot \mathbb{S}^{(1,1,1)}.$$

For integers $a, b \in \mathbb{N}$, we have

$$\begin{aligned} \theta_1^a \theta_2^b &\in \mathbb{C}[\Omega]_{a+b} \setminus \mathbb{C}[\Omega]_{a+b-1}, \\ \theta_1^a \theta_2^b f_{j,k} &\in \mathbb{C}[\Omega]_{a+b+1} \setminus \mathbb{C}[\Omega]_{a+b}, \\ \theta_1^a \theta_2^b g &\in \mathbb{C}[\Omega]_{a+b+2} \setminus \mathbb{C}[\Omega]_{a+b+1}, \end{aligned}$$

which can be seen from $\Omega_d + \Omega_{d'} \subset \Omega_{d+d'}$, see [Met25, Lem. 2.1 (2)], and by observing that 1 is the coefficient of the four monomials

$$x_1^a x_2^b, x_1^{a+1} x_2^b, x_1^a x_2^{b+1}, x_1^{a+1} x_2^{b+1}$$

in the Laurent polynomials

$$\theta_1^a \theta_2^b, \theta_1^a \theta_2^b f_{1,1}, \theta_1^a \theta_2^b f_{2,2}, \theta_1^a \theta_2^b g.$$

Using the decomposition $\mathbb{C}[\Omega] = \mathbb{C}[\theta_1, \theta_2] \otimes_{\mathbb{C}} \mathbb{C}[\Omega]/(\theta_1, \theta_2)$, we derive the claimed multiplicities of \mathbb{S}^λ in $\mathbb{C}[\Omega]_d$, because

$$|\{(a, b) \in \mathbb{N}^2 \mid a + b \leq d\}| = \sum_{i=0}^d (d+1-i) = \frac{(d+1)(d+2)}{2}.$$

□

Finding the analogous isotypic decompositions for A_{n-1} seems very challenging. We have $\dim_{\mathbb{C}} \mathbb{C}[\Omega]_1 = 2^n - 1$ and, for $n = 4$, we found

$$\mathbb{C}[\Omega]_1 = 4 \cdot \mathbb{S}^{(4)} \oplus 3 \cdot \mathbb{S}^{(3,1)} \oplus 1 \cdot \mathbb{S}^{(2,2)}.$$

We do not foresee a pattern for the multiplicities of the isotypic decomposition of $\mathbb{C}[\Omega]_1$ for larger n .

6.2. **Type C_n .** The hyperoctahedral group $\mathfrak{S}_n \wr \{\pm 1\}$ acts on the affine algebraic varieties

$$V := \{Y = (Y_1, \dots, Y_n) \in \mathbb{C}^n\}$$

and $W := \{y = (y_1, \dots, y_n) \in (\mathbb{C} \setminus \{0\})^n\}$

by permutation as well as sign-change and inversion of coordinates respectively. This defines a representation $\rho : \mathfrak{S}_n \wr \{\pm 1\} \rightarrow \mathrm{GL}_n(\mathbb{C})$ with induced actions on the coordinate rings $\mathbb{C}[V] \cong \mathbb{C}[Y_1, \dots, Y_n]$ and $\mathbb{C}[W] \cong \mathbb{C}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]$.

Recall from [Bou68, Pl. I] that $\mathcal{W} = \mathfrak{S}_n \wr \{\pm 1\}$ is the Weyl group of the root system C_n . Denote by $\omega_1, \dots, \omega_n$ the fundamental weights of the weight lattice Ω . This defines a representation $\pi : \mathfrak{S}_n \wr \{\pm 1\} \rightarrow \mathrm{GL}_n(\mathbb{Z})$ with an additive action on the symmetric algebra $\mathrm{Sym}(\Omega) \cong \mathbb{C}[X_1, \dots, X_n]$ and a multiplicative action on the group algebra $\mathbb{C}[\Omega] \cong \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Furthermore, we have

$$\mathcal{W}(\omega_1) = \{\omega_1, \omega_2 - \omega_1, \dots, \omega_n - \omega_{n-1}\},$$

which leads to the following statement analogous to type A_{n-1} (see also [HMR24, §6]).

Lemma 6.6. The maps $\mathbb{C}[V] \rightarrow \mathrm{Sym}(\Omega)$ with

$$Y_1 \mapsto X_1, Y_2 \mapsto X_2 - X_1, \dots, Y_n \mapsto X_n - X_{n-1}$$

and $\mathbb{C}[W] \rightarrow \mathbb{C}[\Omega]$ with

$$y_1 \mapsto x_1, y_2 \mapsto x_2 x_1^{-1}, \dots, y_n \mapsto x_n x_{n-1}^{-1}$$

are well-defined ρ - π -equivariant algebra isomorphisms.

Hence, the induced actions on $\mathbb{C}[V]$ and $\mathbb{C}[W]$ can be identified with the additive and multiplicative action respectively.

Example 6.7. The Weyl group \mathcal{W} of C_2 is generated by s_1, s_2 with

$$\rho(s_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \rho(s_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\pi(s_1) = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \pi(s_2) = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}.$$

As in Lemma 6.1 we find a \mathcal{W} -module isomorphism between $\mathbb{C}[V]$ and $\mathbb{C}[W]$.

Proposition 6.8. The map $\Psi : \mathbb{C}[V] \rightarrow \mathbb{C}[W]$ with

$$Y_i^{2k} \mapsto y_i^k + y_i^{-k}, Y_i^{2k-1} \mapsto y_i^k - y_i^{-k}, \text{ for all } i \in \{1, \dots, n\}, k \in \mathbb{N},$$

and $\prod_{i=1}^n Y_i^{\alpha_i} \mapsto \prod_{i=1}^n \Psi(Y_i^{\alpha_i})$ is an $\rho - \pi$ -equivariant isomorphism. Moreover, the image of \mathcal{H}_a is not a proper ideal in $\mathbb{C}[W]$.

Proof. By construction the map is well-defined, injective and $\rho - \pi$ -equivariant. The map is also surjective since

$$y_i^{\pm k} = \frac{y_i^k + y_i^{-k}}{2} \pm \frac{y_i^k - y_i^{-k}}{2} = \frac{\Psi(Y_i^{2k}) \pm \Psi(Y_i^{2k-1})}{2} \in \mathrm{Im}(\Psi).$$

Assume that \mathcal{H}_a is mapped to an ideal $I \subseteq \mathbb{C}[W]$, that is,

$$\Psi \left(\sum_{i=1}^n Y_i^2 \right), \Psi \left(\sum_{i=1}^n Y_i^4 \right) \text{ and } \Psi \left(\sum_{1 \leq i < j \leq n} Y_i^2 Y_j^2 \right) \in I.$$

Since I is closed under multiplication and addition, this would imply

$$2n = \left(\Psi \left(\sum_{i=1}^n Y_i^2 \right) \right)^2 - \Psi \left(\sum_{i=1}^n Y_i^4 \right) - 2 \Psi \left(\sum_{1 \leq i < j \leq n} Y_i^2 Y_j^2 \right) \in I,$$

that is, $I = \mathbb{C}[W]$. □

In particular, the image of the additive Hilbert ideal is not the multiplicative Hilbert ideal. Thus the map Ψ does not extend to an \mathcal{W} -morphism of the coinvariant spaces.

Example 6.9. The additive coinvariants in Table 1 are mapped to linearly independent equivariants in $\mathbb{C}[W]$ by Proposition 6.8. The irreducible representations of $\mathfrak{S}_n \wr \{\pm 1\}$ are also called Specht modules and are indexed by pairs of partitions (λ, μ) where the sizes of λ and μ sum to n (see for example [Mus93]).

Irrep	Multiplicative Equivariants
$\mathfrak{S}^{((2),(0))}$	1
$\mathfrak{S}^{((1),(1))}$	$y_1 - y_1^{-1}, y_2 - y_2^{-1}$ $(y_2 + y_2^{-1}) \cdot (y_1 - y_1^{-1}), (y_1 + y_1^{-1}) \cdot (y_2 - y_2^{-1})$
$\mathfrak{S}^{((0),(2))}$	$(y_1 - y_1^{-1}) \cdot (y_2 - y_2^{-1})$
$\mathfrak{S}^{((1,1),(0))}$	$(y_1 + y_1^{-1}) - (y_2 + y_2^{-1})$
$\mathfrak{S}^{((0),(1,1))}$	$(y_1^2 - y_1^{-2})(y_2 - y_2^{-1}) - (y_1 - y_1^{-1})(y_2^2 - y_2^{-2})$

TABLE 3. The image of a symmetry adapted basis of $\text{Sym}(\Omega)_{\mathcal{W}}$ under the map Ψ from Proposition 6.8.

Since the action of \mathcal{W} in the standard basis y is more intuitive than in the action in the basis x of fundamental weights, it might be simpler to relate the additive coinvariant basis consisting of higher Specht polynomials to a symmetry adapted basis of $\mathbb{C}[y^{\pm}]/(\sigma_n(y+y^{-1}))$, where $y + y^{-1} := (y_1 + y_1^{-1}, \dots, y_n + y_n^{-1})$. The isomorphisms in Lemma 6.6 allow us to consider the multiplicative coinvariant basis in a more intuitive way. The invariant ring $\mathbb{C}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]^{\rho(\mathfrak{S}_n \wr \{\pm 1\})}$ is, as an algebra, generated by

$$\sigma_i(y + y^{-1}) = \sum_{\substack{J \subseteq \{1, \dots, n\} \\ \text{with } |J|=i}} \prod_{j \in J} y_j + y_j^{-1},$$

that is, $\mathbb{C}[W]^{\mathcal{W}} \cong \mathbb{C}[\sigma_1(y + y^{-1}), \dots, \sigma_n(y + y^{-1})]$, see [Lor05, Ex. 3.5.4].

Again, the multiplicative invariant ring $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{\pi(\mathfrak{S}_n \wr \{\pm 1\})}$ is generated by orbit polynomials θ_i . Denote by $\Gamma : \mathbb{C}[W] \rightarrow \mathbb{C}[\Omega]$ the second map from Lemma 6.6.

Proposition 6.10. The map

$$\mathbb{C}[y^\pm]/(\sigma_i(y + y^{-1}) \mid 1 \leq i \leq n) \rightarrow \mathbb{C}[x^\pm]/(\theta_i(x) \mid 1 \leq i \leq n)$$

that takes $h(y)$ to $\Gamma(h(y))$ is a \mathcal{W} -module isomorphism.

Proof. Since $\Gamma(\sigma_i(y + y^{-1})) = 2^i \binom{n}{i} \theta_i(x)$ by [HMR24, Prop. 6.1], this follows analogously to Proposition 6.3. \square

CONCLUSION

We drew the relation between the additive and multiplicative coinvariant spaces for Weyl groups \mathcal{W} . As \mathcal{W} -modules, each affords the regular representation of \mathcal{W} and is isomorphic to the space of \mathcal{W} -harmonics respectively. Therefore, the two coinvariant spaces and in particular the algebras $\mathbb{C}[\Omega]$ and $\text{Sym}(\Omega)$ are isomorphic \mathcal{W} -modules. However, the construction of \mathcal{W} -homomorphisms is challenging, as we showed, and can be overcome by working with the associated graded algebras. Only in the cases of A_{n-1} and partly C_n , intuitive \mathcal{W} -homomorphisms were found.

We presented in Proposition 6.8 an intuitive equivariant isomorphism of $\text{Sym}(\Omega)$ and $\mathbb{C}[\Omega]$ for the root system of type C_n which does not induce an isomorphism of the coinvariant spaces. Nonetheless, in Example 6.9 and Table 1, the symmetry adapted basis of $\text{Sym}(\Omega)_{\mathcal{W}}$ consisting of higher Specht polynomials is mapped to the Laurent polynomials in Table 3, which form a vector space basis of $\mathbb{C}[\Omega]_{\mathcal{W}}$. A natural question is whether this also holds for any $n \in \mathbb{N}$. Moreover, by Proposition 6.10 the multiplicative coinvariant space can be expressed as $\mathbb{C}[y^\pm]/(\sigma_i(y + y^{-1}))$ on which the group $\mathfrak{S}_n \wr \{\pm 1\}$ acts more intuitively. We could not determine a Gröbner basis of the ideal $\mathcal{H}_m = (\sigma_i(y + y^{-1}))$ for arbitrary $n \in \mathbb{N}$, although the reduced Gröbner basis with respect to the lexicographical ordering of the ideal $\mathcal{H}_a = (\sigma_i(Y))$ is known [MS03]. It seems interesting to ask whether there is a natural identification of the modules $\mathbb{C}[y^\pm]_{\mathcal{W}}$ and $\mathbb{C}[Y]_{\mathcal{W}}$ despite that Ψ does not map \mathcal{H}_a to \mathcal{H}_m ?

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