



**HAL**  
open science

## QUANTIZATION ON FILTERED MANIFOLDS

Clotilde Fermanian Kammerer, Véronique Fischer, Steven Flynn

► **To cite this version:**

Clotilde Fermanian Kammerer, Véronique Fischer, Steven Flynn. QUANTIZATION ON FILTERED MANIFOLDS. 2026. <hal-04852868v2>

**HAL Id: hal-04852868**

**<https://hal.science/hal-04852868v2>**

Preprint submitted on 15 Apr 2026

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons CC BY 4.0 - Attribution - International License

# QUANTIZATION ON FILTERED MANIFOLDS

CLOTILDE FERMANIAN-KAMMERER, VÉRONIQUE FISCHER AND STEVEN FLYNN

ABSTRACT. In this article, we develop a pseudodifferential calculus on a general filtered manifold  $M$ . The symbols are fields of operators  $\sigma(x, \pi)$  parametrised by  $x \in M$  and the unitary dual  $\widehat{G}_x M$  of the osculating Lie group  $G_x M$ . We define classes of symbols and a local quantization formula associated to a local frame adapted to the filtration. We prove that the collection of operators on  $M$  coinciding locally with the quantization of symbols enjoys the essential properties of a pseudodifferential calculus: composition, adjoint, parametrices, continuity on adapted Sobolev spaces. Moreover, we show that the polyhomogeneous subcalculus coincides with the calculus constructed by van Erp and Yuncken via groupoids.

## CONTENTS

1. Introduction	2
1.1. Symbolic pseudodifferential calculus and applications	2
1.2. Strategy of the construction	5
1.3. Important technical ingredients of the proofs	6
1.4. Organisation of the paper	7
2. Symbols classes on graded Lie groups	8
2.1. Definitions for graded Lie groups and algebras	8
2.2. Group Fourier transform on a nilpotent Lie group	13
2.3. Rockland operators and Sobolev spaces	16
2.4. Invariant symbol classes	19
2.5. Smoothing symbols	24
2.6. Homogeneous and polyhomogeneous symbol classes	26
2.7. Difference operators and Leibniz property	28
2.8. Transformation of symbols by morphisms	30
3. Filtered manifolds	31
3.1. Filtered manifolds and their osculating bundles	31
3.2. Local frames adapted to the filtration	33
3.3. Geometric exponentiation	35
3.4. Homogeneous order and principal part of a differential operator	37
3.5. Group exponentiation and BCH formula	39
4. Action of geometric maps on the osculating group	41
4.1. Higher order terms on graded spaces	41

---

1991 *Mathematics Subject Classification.* 47G30, 43A85, 53C30, 58J40, 35H20, 35A27.

*Key words and phrases.* Pseudodifferential theory, filtered manifolds and subelliptic operators, microlocal analysis.

4.2.	Higher order remainders on filtered manifolds	44
4.3.	Composition of geometric exponential mappings	48
5.	Symbols classes on filtered manifolds	52
5.1.	The symbols and their convolution kernels	52
5.2.	The classes of symbols $S^m(\widehat{GM})$	59
5.3.	Operations on symbols	61
5.4.	Spectral multipliers of Rockland symbols	66
5.5.	Homogeneous and polyhomogeneous symbols	71
6.	Kernel estimates and related results	76
6.1.	Kernel estimates	77
6.2.	Kernel estimates and higher order terms	78
6.3.	Kernel estimates and perturbed convolution	85
7.	The local quantization	86
7.1.	Local quantization	86
7.2.	$L^2$ boundedness for the local quantization	88
7.3.	Independence of the local quantization in the cut-off and the frame	91
8.	Symbolic calculus for the local quantization	94
8.1.	Amplitudes	94
8.2.	Local quantization of amplitudes	101
8.3.	The adjoints of locally quantized operators	104
8.4.	The composition of locally quantized operators	107
9.	Pseudodifferential calculus on $M$	109
9.1.	Pseudodifferential operators on $M$	109
9.2.	Properly supported pseudodifferential operators and composition	114
9.3.	Principal Symbol	118
9.4.	Parametrices	123
9.5.	Local Sobolev spaces	126
9.6.	The classical $H$ -calculus	132
9.7.	$H$ -calculus and tangent groupoid	135
Appendix A.	Combinatoric	142
Appendix B.	Functional spaces on a vector bundle	144
B.1.	Vertical densities on $E$	144
B.2.	Schwartz functions and vertical tempered densities on $E$	145
B.3.	Vertical Schwartz densities	146
References		146

## 1. INTRODUCTION

1.1. **Symbolic pseudodifferential calculus and applications.** Starting from about 1960, the seminal works of Hörmander [47, 48] provided a flexible framework for studying elliptic operators through pseudodifferential calculi, along with remarkable characterisations and

properties for hypoellipticity. For the former, many other contemporaneous contributions also played a significant role in this development, see e.g. [51, 56]. The latter was the motivation behind the research programme led by Folland and Stein and their collaborators in the 70's and 80's (see eg [39, 40, 60]) on subelliptic operators modelled via nilpotent Lie groups. Nowadays, the geometric contexts for these ideas are called subRiemannian. The main method revolved around the construction of parametrices of these operators, and more generally of algebras of operators adapted to these settings. Usually, the pseudodifferential calculi constructed then were in effect much closer to algebras of singular integral operators described via their integral kernels (in a similar spirit, see Melin's unpublished preprint [53] and Street's monographs [62, 63]); indeed, these calculi didn't use notions of symbols as in the Riemannian or Euclidean settings for elliptic operators. Furthermore, they were set on contact manifolds and their generalisations (e.g. Heisenberg manifolds) [7, 28, 53] or on nilpotent Lie groups [15]. These ideas and methods were further deployed in the 90's in the studies of spectral or functional problems of operators such as subLaplacians on stratified nilpotent Lie groups in harmonic analysis (see the seminal paper [54]).

In the past twenty years, a pseudodifferential theory based on Alain Connes idea of tangent groupoid [20] emerged from works by Ponge [58] and by van Erp [66] on Heisenberg and contact manifolds. It was then realised that the context to generalise these considerations were manifolds equipped with filtrations of the tangent bundle, i.e. filtered manifolds, see the works of Choi and Ponge [12, 13, 14] and of van Erp and Yuncken [67, 68]. Van Erp and Yuncken's results can be considered as the development in the context of filtered manifolds of Debord and Skandalis' works [24, 25] which provided the first abstract characterisation of the classical pseudodifferential operators in terms of actions on Connes' tangent groupoid. While the class of filtered manifolds contains equiregular subRiemannian manifolds, recent progress suggests that these ideas can be further generalised to manifolds whose space of vector fields admits a filtration [3], encompassing also singular subRiemannian manifolds and quasi-subRiemannian settings. It is worth noting that the latter calculi constructed within the tangent groupoid framework are included in the more general construction developed by Street [62, 63].

The pseudodifferential theory via groupoids was devised by Alain Connes around 1990 as a framework to show index theorems. Since then, it has become an efficient tool to study hypoellipticity and Fredholmness in many contexts. However, at the time of writing, it does not show the same breadth and depth of applications to PDE's as Hörmander's calculus. One reason is that Hörmander's pseudodifferential operators enjoy an explicit symbolic calculus related to the symplectic structure of the cotangent bundle with an explicit quantization procedure; this make it flexible to deal with solutions of PDEs and quantum dynamical problems via classical dynamics. Moreover, Hörmander's calculus allows for  $(\rho, \delta)$ -generalisations and semiclassical calculus including symbolic smoothing operators [35] which aren't yet developed with the groupoid approach.

One motivation for this work is to construct a noncommutative counterpart of Hörmanders symbolic pseudodifferential calculus on filtered manifolds, formulated in terms of the natural

noncommutative cotangent bundle. We show in Section 9.7 that, upon restriction to the polyhomogeneous subcalculus, it agrees with the construction of van Erp and Yuncken [67, 68]. Moreover, we believe that this approach is as flexible as Hörmander's calculus, allowing for the incorporation of  $(\rho, \delta)$ -type generalizations and the development of a corresponding semiclassical pseudodifferential calculus.

In the mean time, the last decades have seen a rising interest of the PDE community for subelliptic operators [4, 8, 11, 17, 19, 52, 59]. The methods underlying these works have involved mainly Riemannian tools based on Hörmander's calculus. These tools are very powerful and are the basis for microlocal and semiclassical analysis with various domains of applications, see for instance the seminal papers in control theory [6], spectral geometry [16, 69], homogenisation [42, 64]. However, recent works [4, 17, 59] suggest that they can be fruitful for subelliptic or subRiemannian problems only in the context of very low-dimensional contact manifolds. Beyond this, the non-commutativity inherent to subRiemannian and filtered settings seems to make any semiclassical or microlocal question rapidly intractable when using the traditional (Riemannian) Hörmander calculus. However, other approaches for instance via groupoids have recently given impressive results such as the proof in full generality of the Helffer-Nourrigat conjecture [45, 3]. This reinforces the motivation for the construction of an adapted *symbolic* pseudodifferential calculus, with the goal of providing the tools for dealing with questions from PDEs in subRiemannian and subelliptic contexts.

A notable exception to the literature of the non-symbolic subelliptic pseudodifferential calculi of the late twentieth century is Taylor's monograph [65]: he proposed to define pseudodifferential operators symbolically using the operator-valued group Fourier transform, focusing mainly on the Heisenberg group (see also [5]). Using also this group Fourier transform together with many tools and methods from harmonic analysis (singular integral theory and spectral properties of subLaplacians), the recent monograph [37] laid the foundations of a symbolic and global pseudodifferential calculus on any graded nilpotent Lie group. We call these pseudodifferential calculi *noncommutative* because, unlike Hörmander's, their symbols do not commute.

The pseudodifferential calculus in [37] has allowed for results similar to the problems mentioned above in the Euclidean or Riemannian settings. For instance, the article [30] proves a div-curl Lemma in the context of nilpotent graded Lie groups, in the same spirit as the Euclidean work [42]. The article [32] emphasises invariance properties of quantum limits while [33] deals with spectral geometry for subLaplacians on 2-step nilmanifolds. The article [34] is a control result. These applications underscore the relevance of the non-commutative analogue of the cotangent bundle arising from the group Fourier transform. It also highlights the necessity of incorporating smoothing symbols when high-frequency phenomena appear and semiclassical analysis becomes essential. Motivated by these considerations, we follow the framework developed in [37] to construct a *noncommutative symbolic pseudodifferential calculus* in the more geometric context of filtered manifolds. Our study further establishes the construction of parametrices for symbols which are invertible at high

frequency (Rockland-type conditions). We also introduce corresponding *Sobolev spaces* of arbitrary real order.

**1.2. Strategy of the construction.** Let us sketch now our strategy - the rigorous definitions of the objects involved and their subsequent analysis will be done later in the core of the paper. The readers will find in Section 1.4 information about where to find definitions and statements about each of the items mentioned here.

To the filtered manifold  $M$ , we associate its osculating Lie group  $GM$  and osculating Lie algebra bundles  $\mathfrak{g}M$ . The symbols we introduce are fields of operators

$$\sigma = \{\sigma(x, \pi) : x \in M, \pi \in \widehat{G}_x M\}$$

with  $\sigma(x, \pi)$  densely defined on the space of the representation  $\pi$ . The noncommutative analogue to the cotangent bundle

$$\widehat{GM} := \{(x, \pi) : x \in M, \pi \in \widehat{G}_x M\},$$

parametrises our symbols. We define symbol classes, as well as a local quantization associating an operator to one of these symbols. This local quantization uses the geometric exponentiation of vector fields and allows for the identification of a point  $y$  near  $x \in M$  as  $y = \exp_x^{\mathbb{X}} v$ ,  $v \in \mathbb{R}^n$ , once a convenient frame  $\mathbb{X}$  has been chosen and  $T_x M$  identified with  $\mathbb{R}^n$ . This exponential map has to be considered in contrast with the exponential map  $\text{Exp}_x^{\mathbb{X}}$  of the group  $G_x M$ . The upper index  $\mathbb{X}$  recalls that all these objects depend on a chosen frame, and the identifications it induces.

More technically, we associate to a symbol  $\sigma$  an operator defined via the following local quantization. Given a local frame  $\mathbb{X}$  on an open set  $U$  adapted to the filtration of  $M$  and a cut-off function  $\chi$ , the operator acts on  $f \in C_c^\infty(M)$  by

$$(1.1) \quad \text{Op}^{\mathbb{X}, \chi}(\sigma)f(x) := \int_{v \in \mathbb{R}^n} \kappa_x^{\mathbb{X}}(v) (\chi_x f)(\exp_x^{\mathbb{X}}(-v)) dv, \quad x \in U.$$

In this definition, the distribution  $\kappa^{\mathbb{X}}$  is called the *convolution kernel* associated with  $\sigma$  in the  $\mathbb{X}$ -coordinates and corresponds to the group Fourier transform inverse of  $\sigma$  in the following sense:

$$\sigma(x, \pi) = \int_{\mathbb{R}^n} \kappa_x^{\mathbb{X}}(v) \pi(\text{Exp}_x^{\mathbb{X}} v)^* dv.$$

We define symbol classes  $S^m(\widehat{GM})$ ,  $m \in \mathbb{R}$ , on which the local quantization process (1.1) makes sense. Moreover, the resulting operators enjoy symbolic properties for composition and adjoint amongst others. Ensuring that these objects are rigorously and intrinsically defined poses many regularity questions. In particular, the convolution kernels  $\kappa_x^{\mathbb{X}}(v)$  display singularity in  $v = 0$  and the operator in (1.1) is of Calderon-Zygmund's type. This is the main difficulty of the work and it highly relies on previous works on nilpotent Lie groups of the second author [37].

The passage from the local quantization in (1.1) to a global calculus follows classical lines [46, 70]. We define pseudodifferential operators as being locally described by the local quantization defined above. We show that they are described equivalently by a global quantization

on  $M$  of our symbols via adapted atlases up to smoothing operators. To sum-up, we obtain classes of operators

$$\Psi^\infty(M) := \cup_m \Psi^m(M),$$

which forms a *pseudodifferential calculus* in the sense that it satisfies:

- (1) The continuous inclusions  $\Psi^m(M) \subset \Psi^{m'}(M)$  hold for any  $m \leq m'$ .
- (2)  $\Psi^\infty(M)$  is an algebra of operators. Furthermore if  $T_1 \in \Psi^{m_1}(M)$ ,  $T_2 \in \Psi^{m_2}(M)$ , then  $T_1 T_2 \in \Psi^{m_1+m_2}(M)$ , and the composition is continuous as a map  $\Psi^{m_1}(M) \times \Psi^{m_2}(M) \rightarrow \Psi^{m_1+m_2}(M)$ .
- (3)  $\Psi^\infty(M)$  is stable under taking the adjoint. Furthermore if  $T \in \Psi^m(M)$  then  $T^* \in \Psi^m(M)$ , and taking the adjoint is continuous as a map  $\Psi^m(M) \rightarrow \Psi^m(M)$ .

The calculus also makes it possible to construct parametrices and to introduce local Sobolev spaces  $L_{s,loc}^2(M)$ ,  $s \in \mathbb{R}$ , on which the pseudodifferential operators act in the following way: if  $T \in \Psi^m(M)$ , then  $T \in \mathcal{L}(L_{s+m,loc}^2(M), L_{s,loc}^2(M))$ . We will also show that its polyhomogeneous subcalculus coincides with the calculus developed by van Erp and Yuncken [67, 68].

The construction of this pseudodifferential calculus paves the way for new approaches to the study of PDEs on filtered manifolds. These include the analysis of the propagation of singularities and the investigation of compactness defects for square-integrable families on  $M$ , among other applications. A development in this direction is presented in [10], where the calculus introduced therein provides an alternative proof of the quantum ergodicity result established on three-dimensional contact manifolds in [17], while also extending it to higher-dimensional contact manifolds endowed with a compatible complex structure. The overarching motivation of our work lies in *spectral geometry on filtered manifolds*, including contact, quasi-contact, and Engel manifolds, with particular emphasis on conditions ensuring the quantum ergodicity of sub-Laplacians or more general subelliptic operators, and on the geometric interpretation of the coefficients appearing in the Weyl trace formula.

**1.3. Important technical ingredients of the proofs.** The strategies behind the proofs of properties such as  $L^2$ -boundedness, composition and adjoint follow established lines in pseudodifferential theory, see e.g. [2, 37, 61]. However, their implementations in the case of filtered manifolds are technically highly challenging.

In the Riemannian case, pseudodifferential operators on manifolds are defined as being described through coordinates patches as pseudodifferential operators on  $\mathbb{R}^n$ . This cannot be the case on a filtered manifold, as natural coordinates do not exist - although privileged ones have been studied [1, 9, 13, 14] since the inception of subRiemannian geometry. This is a fundamental obstacle that we overcome with the method of moving frames: the time-1 flow of vector fields allows us to define the exponential map  $\exp^{\mathbb{X}}$  with respect to a given local frame  $\mathbb{X}$ . This map can be compared with the bisubmersions of [3].

A crucial milestone in this paper is the analysis of the composition of the exponential map  $\exp^{\mathbb{X}}$  with respect to a local frame  $\mathbb{X}$  adapted to the filtration. As expected from ideas dating back from the work of Nagel, Stein and Wainger in [55], we show that this composition is

connected to the osculating group structure. However, in order to describe precisely this connection, we introduce a new technical concept of algebraic nature, namely terms that we call *higher order*. This concept is also useful in other situations, for instance to compare the product laws of two osculating groups  $G_xM$  and  $G_yM$  above two points  $x, y \in M$  close enough, or to describe the effect of changing adapted frames in exponential maps.

As can be anticipated from the group case [37], our proofs of the calculus properties rely fundamentally on the analysis of the integral kernels of the operators through the convolution kernels  $\kappa^{\mathbb{X}}$  of their symbols. Introducing the notion of amplitudes corresponding to our symbols separates steps in the proofs for composition and adjoint. It is remarkable that our analysis of the impact of the geometric setting then boils down to understanding the two following situations:

- (1) perturbing the variable of a convolution kernel  $\kappa^{\mathbb{X}}$  with a term of higher order,
- (2) perturbing the group convolution of two convolution kernels with another function.

**1.4. Organisation of the paper.** The paper is structured linearly. In Section 2, we recall results about symbol classes on graded Lie groups that will be useful for constructing the symbol classes  $S^m(\widehat{GM})$ ,  $m \in \mathbb{R}$ . In particular, we revisit standard estimates in this setting with a special concern with the constants. Indeed, the symbol  $\sigma(x, \cdot)$  will be related with the nilpotent graded group  $\widehat{G}_xM$ , which depends on  $x \in M$ . It is thus important to analyse how constants depend on the gradation of the group, the filtration of the Lie algebra  $\mathfrak{g}_xM$  above each point  $x \in M$ , and on its Lie structure, i.e. the norm of the Lie bracket  $[\cdot, \cdot]_{\mathfrak{g}_xM}$ , viewed as a bilinear form on  $\mathfrak{g}_xM$ .

Section 3 recalls definitions about filtered manifolds and Section 4 focuses on the second crucial tool of our analysis, namely the geometric exponential map  $\exp^{\mathbb{X}}$ . It is here that we use the notion of higher order to study the composition of exponential map.

With all these tools in hands, we define the symbol classes  $S^m(\widehat{GM})$ ,  $m \in \mathbb{R}$ , in Section 5, we study their properties and give examples. A special attention is devoted to kernel estimates in Section 6 and to the analysis of convolution kernels obtained by the two perturbations mentioned at the end of Section 1.3. The results in Section 6 are used when quantizing the symbols for studying their actions on smooth functions on  $M$ , or on elements of  $L^2(M)$  (for symbols of non-positive order), and for proving properties of symbolic calculus.

The local quantization is presented in Section 7, where we also study the boundedness on  $L^2(M)$  of local pseudodifferential operators of order 0 and the dependence of the local quantization with respect to the frame  $\mathbb{X}$  and the cut-off function  $\chi$ . Theorems establishing the symbolic calculus are proved in Section 8: determination of the symbol of the adjoint of a pseudodifferential operator, and of the composition of two pseudodifferential operators. These results are obtained by an approach based on amplitudes, in a classical manner for pseudodifferential calculus, adapted to filtered manifolds.

The concept of local quantization allows us to define a pseudodifferential calculus on the manifold  $M$  in Section 9. It comprises operators that are locally given as the local quantization of a symbol, or that are smoothing (i.e. have a smooth integral kernel). In this

section, we also analyse the characterisation of these pseudodifferential operators in terms of atlases, and introduce the adapted hierarchy of local Sobolev spaces. Finally, we conclude our study with the analysis of the links between the calculus we have constructed and van Erp and Yuncken's [67, 68].

**Acknowledgements.** The authors thank Robert Yuncken for several insightful discussions that played an important role in shaping their ideas, and Lino Benedetto for his helpful remarks and comments during the preparation of this text. They are also grateful to the anonymous referees for their thoughtful suggestions, which greatly contributed to improving the introduction.

The three authors acknowledge the support of The Leverhulme Trust via Research Project Grant RPG 2020-037. The first author also benefits from the support of the Région Pays de la Loire via the Connect Talent Project HiFrAn 2022 07750, the grant ANR-23-CE40-0016 (OpART), and the France 2030 program, Centre Henri Lebesgue ANR-11-LABX-0020-01.

## 2. SYMBOLS CLASSES ON GRADED LIE GROUPS

In this section, we recall the definitions for the objects appearing in the analysis on graded Lie groups and the natural symbol classes in this context. This gives us the opportunities to introduce the notation used throughout the paper. In addition, we present their main properties, paying a special attention to how estimates depend on the group structure and the gradation. The material essentially follows [37], see also [39, 41] and [29].

**2.1. Definitions for graded Lie groups and algebras.** Here we recall some definitions and set some notation for the rest of the paper. References for this material include [37] and [41].

### 2.1.1. Graded Lie algebra.

**Definition 2.1.** The (finite dimensional real) vector space  $\mathcal{V}$  admits the gradation by the sequence of vector subspaces  $\mathcal{V}_j$ ,  $j \in \mathbb{N}$ , when  $\mathcal{V} = \mathcal{V}_1 \oplus \dots \oplus \mathcal{V}_j \oplus \dots$

In the above definition, as  $\mathcal{V}$  is finite dimensional, all but a finite number of the  $\mathcal{V}_j$ 's are  $\{0\}$ . Continuing with the setting of this definition, we define the linear map

$$A : \mathcal{V} \rightarrow \mathcal{V} \quad \text{via} \quad A(v) = jv, \quad v \in \mathcal{V}_j \text{ and } j \in \mathbb{N},$$

and the linear maps

$$\delta_r := \text{Exp}((\ln r)A) = \sum_{k=0}^{\infty} \frac{1}{k!} ((\ln r)A)^k, \quad r > 0,$$

or, in other words, the linear maps given via  $\delta_r v = r^j v$ ,  $v \in \mathcal{V}_j$  and  $r > 0$ .

**Definition 2.2.** Let  $\mathcal{V} = \bigoplus_{j=1}^{\infty} \mathcal{V}_j$  be a graded finite dimensional space.

- (1) The maps  $\delta_r$ ,  $r > 0$ , defined above are called the *dilations* of the graded vector space  $\mathcal{V}$ .

- (2) The eigenvalues of the map  $A$  defined above are called the *weights* of the dilations; they are the integers  $j \in \mathbb{N}$  such that  $\mathcal{V}_j \neq \{0\}$ . A basis of  $\mathcal{V}$  made up of eigenvectors for  $A$  is called an *adapted basis*.

By convention, if  $\mathbb{V}$  is an adapted basis, we will always assume that they are indexed so that the weights are increasing, that is,  $\mathbb{V} = (V_1, \dots, V_n)$  satisfies

$$AV_i = v_i V_i, \quad i = 1, \dots, n = \dim \mathcal{V}, \quad \text{with} \quad 0 < v_1 \leq \dots \leq v_n.$$

- (3) The number of distinct  $v_j$ ,  $j \geq 1$ , is called the step of the gradation.  
(4) The *homogeneous dimension* of the gradation is given by

$$Q := \sum_{j=1}^{\infty} j \dim \mathcal{V}_j = v_1 + \dots + v_n.$$

- (5) A function  $f : \mathcal{V} \rightarrow \mathbb{C}$  is *m-homogeneous* with respect to the  $\delta_r$ -dilations when  $f(\delta_r v) = r^m f(v)$  for all  $v \in \mathcal{V}$  and  $r > 0$ . This extends to measurable functions and distributions on  $\mathcal{V}$ . A differential operator  $T$  on  $\mathcal{V}$  is *m-homogeneous* with respect to the  $\delta_r$ -dilations when  $T(f \circ \delta_r) = r^m (Tf) \circ \delta_r$  for all  $f \in C_c^\infty(\mathcal{V})$  and  $r > 0$ . This also extends to more general linear operators, e.g.  $T : \mathcal{S}(\mathcal{V}) \rightarrow \mathcal{S}'(\mathcal{V})$  etc.  
(6) A *quasinorm* on  $\mathcal{V}$  is a continuous map  $|\cdot| : \mathcal{V} \rightarrow [0, +\infty)$  that is symmetric, definite and 1-homogeneous with respect to the  $\delta_r$ -dilations, i.e. we have for any  $v \in \mathcal{V}$  and  $r > 0$ :

$$|-v| = |v| \geq 0, \quad |v| = 0 \iff v = 0, \quad |\delta_r v| = r|v|.$$

*Example 2.3.* If  $\mathbb{V} = (V_1, \dots, V_n)$  is an adapted basis of the graded vector space  $\mathcal{V}$ , then fixing  $\alpha > 0$ , the following defines a quasinorm.

$$|v|_{\alpha, \mathbb{V}} = \left( \sum_{i=1}^n |v_i|^{\alpha/v_i} \right)^{1/\alpha}, \quad v = \sum_{i=1}^n v_i V_i.$$

*Remark 2.4.* Any two quasinorms  $|\cdot|$  and  $|\cdot|'$  on the same graded space  $\mathcal{V}$  are equivalent in the sense that

$$\exists C > 0, \quad \forall v \in \mathcal{V}, \quad C^{-1}|v| \leq |v|' \leq C|v|.$$

**Definition 2.5.** A (real and finite dimensional) Lie algebra  $\mathfrak{g}$  is *graded* when the vector space  $\mathfrak{g}$  admits a gradation  $\mathfrak{g} = \bigoplus_{j=1}^{\infty} \mathfrak{g}_j$  satisfying

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \quad \text{for all } i, j \in \mathbb{N}_0,$$

with the convention that  $\mathfrak{g}_0 = \{0\}$  is trivial.

We readily check that a Lie algebra admitting a gradation is graded if and only if the dilations  $\delta_r$ ,  $r > 0$ , are morphisms of the Lie algebra  $\mathfrak{g}$ .

*Remark 2.6.* If a Lie algebra is graded, then it is nilpotent and the step of the Lie algebra, i.e. its degree of nilpotency for the Lie bracket, is less or equal to the step of the gradation, (the number of distinct  $v_j$ ).

The dilations extend naturally to its universal enveloping Lie algebra  $\mathcal{U}(\mathfrak{g})$ . Keeping the same notation for the resulting maps

$$\delta_r : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g}), \quad r > 0,$$

these maps are morphisms of the algebra  $\mathcal{U}(\mathfrak{g})$ . They allow us to define a notion of homogeneity in the universal enveloping Lie algebra: an element  $T \in \mathcal{U}(\mathfrak{g})$  is homogeneous of degree  $\nu$  when  $\delta_r T = r^\nu T$  for any  $r > 0$ .

*Notation 2.7.* The subspace of  $\mathcal{U}(\mathfrak{g})$  comprised of elements of homogeneous degree equal to  $\nu$  is denoted by  $\mathcal{U}_\nu(\mathfrak{g})$ .

If  $\mathbb{V} = (V_1, \dots, V_n)$  is an adapted basis of  $\mathfrak{g}$ , then  $\mathcal{U}_N$  consists of all the linear combination of

$$\mathbb{V}^\alpha := V_1^{\alpha_1} \dots V_n^{\alpha_n}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n,$$

with  $[\alpha] = N$ , where

$$[\alpha] := \alpha_1 v_1 + \dots + \alpha_n v_n.$$

Above, we have adopted the convention  $\mathbb{V}^0 = 1$  for  $\alpha = 0$ .

2.1.2. *Graded Lie groups.* Recall that there is a one-to-one correspondence between nilpotent Lie algebras and connected simply connected nilpotent Lie groups. This leads to the following definition:

**Definition 2.8.** A connected simply connected Lie group is *graded* when its Lie algebra is graded.

Any graded Lie group is therefore nilpotent. By convention, in this paper, all the nilpotent Lie groups are assumed to be connected and simply connected. The exponential mapping

$$\text{Exp}_G : \mathfrak{g} \rightarrow G$$

of a nilpotent Lie group  $G$  with Lie algebra  $\mathfrak{g}$  is a bijection and furthermore a global diffeomorphism. This allows us to define the dilations on a graded Lie group:

**Definition 2.9.** The *dilations* of a graded Lie group  $G$  are the mappings defined via

$$\delta_r : G \rightarrow G, \quad \delta_r \text{Exp}_G V = \text{Exp}_G(\delta_r V), \quad V \in \mathfrak{g}, \quad r > 0.$$

The dilations of a graded Lie group are morphisms of the Lie group. With the exponential mapping  $\text{Exp}_G$  and a choice of adapted basis on the graded Lie algebra  $\mathfrak{g}$ , it is possible to identify the spaces of functions on  $G$  with spaces of functions on the vector space underlying  $\mathfrak{g}$ . The dilations then lead naturally to the notion of homogeneity of a function on  $G$  and operators acting on functional spaces on  $G$  as in Definition 2.2.

*Example 2.10.* If  $\mathbb{V} = (V_1, \dots, V_n)$  is an adapted basis of the graded Lie algebra  $\mathfrak{g}$ , then identifying elements of  $\mathfrak{g}$  with left-invariant vector fields, the differential operators  $V_j$ ,  $j = 1, \dots, n$ , and more generally the differential operators  $\mathbb{V}^\alpha = V_1^{\alpha_1} \dots V_n^{\alpha_n}$  are homogeneous for the dilations  $\delta_r$ ,  $r > 0$ , with respective degrees  $v_j$  and  $[\alpha]$ . We can also consider the

right-invariant vector fields  $\tilde{V}_j$ ,  $j = 1, \dots, n$ , associated with  $\mathbb{V}$ , and more generally the corresponding right-invariant differential operator  $\tilde{\mathbb{V}}^\alpha$ , which will also be  $[\alpha]$ -homogeneous.

*Notation 2.11.* If  $T \in \mathcal{U}(\mathfrak{g})$ , we may use the same notation for the corresponding left-invariant differential operator, and we denote by  $\tilde{T}$  the corresponding right-invariant differential operator. We will use this notation throughout this article.

In all this section, we are concerned with the dependence of constants with respect to the structure of the graded group, in particular its gradation and its Lie structure. We consider the *structural constant*  $\|[\cdot, \cdot]\|_{\mathbb{V}}$  defined as follows: If  $T : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$  is a map and  $\mathbb{V}$  a basis of  $\mathcal{V}$ , we set

$$\|T\|_{\mathbb{V}} := \max_{1 \leq i, j, k \leq n} |T_{i, j, k}| \quad \text{where} \quad T(V_i, V_j) = \sum_{k=1}^n T_{i, j, k} V_k.$$

This defines a norm on the space of bilinear maps  $\mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ . And  $\|[\cdot, \cdot]_{\mathfrak{g}}\|_{\mathbb{V}}$  denotes taking this norm for the Lie bracket  $[\cdot, \cdot]_{\mathfrak{g}}$  of  $\mathfrak{g}$  which is a bilinear map  $\mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ . Most of the constants we will see below depends on the Lie structure only via this structural constant  $\|[\cdot, \cdot]\|_{\mathbb{V}}$ .

**2.1.3. Baker-Campbel-Hausdorff formula.** We will describe the law of the group in coordinates via the *Baker-Campbel-Hausdorff formula*, abbreviated here to BCH, the exponential mapping  $\text{Exp}_G$  and a choice of a basis  $\mathbb{V} = (V_1, \dots, V_n)$  of  $\mathfrak{g}$  adapted to the gradation.

The BCH formal writes: if  $a$  and  $b$  are two non-commutative indeterminates, we have in the sense of formal power series of  $a$  and  $b$  that  $e^a e^b = e^c$  where

$$(2.1) \quad c = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sum_{\alpha_i + \beta_i \neq 0} \frac{1}{\alpha! \beta! |\alpha + \beta|} (\text{ad } a)^{\alpha_1} (\text{ad } b)^{\beta_1} \dots (\text{ad } a)^{\alpha_k} (\text{ad } b)^{\beta_k - 1} b.$$

When  $\beta_k = 0$ , the term should be modified so that the last factor is  $(\text{ad } a)^{\alpha_k - 1} a$ . Above,  $\text{ad } a$  denotes the commutator with  $a$ , i.e.  $\text{ad } a(f) = af - fa$ , and similarly for  $b$ . In particular, BCH describes, on the one hand, the group product of a nilpotent Lie group (aka Dynkin formula, via the group exponential after a choice of a basis of the Lie algebra) and, on the other hand, the composition of flows of vector fields (see the appendix of [55]). We will use the BCH formula in various ways later on for Theorem 4.13.

*Notation 2.12.* The truncated BCH formula with a sum with  $|\alpha| + |\beta| \leq N$  is denoted by:

$$\begin{aligned} & \text{BCH}_N(a, b) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sum_{\alpha_i + \beta_i \neq 0, |\alpha| + |\beta| \leq N} \frac{1}{\alpha! \beta! |\alpha + \beta|} (\text{ad } a)^{\alpha_1} (\text{ad } b)^{\beta_1} \dots (\text{ad } a)^{\alpha_k} (\text{ad } b)^{\beta_k - 1} b. \end{aligned}$$

Various noncommutative setting may be used for the BCH formula, and we may write  $\text{BCH}_N(a, b; \text{ad})$  to specify which commutator structure  $\text{ad}$  is used.

2.1.4. *Quasinorms and Taylor formula.* Quasinorms were defined on graded vector space in Definition 2.2, and in particular on graded Lie algebra. Using the exponential mapping  $\text{Exp}_G$ , we may identify the quasinorms on a graded group  $G$  and on its Lie algebra  $\mathfrak{g}$ . Then a quasinorm  $|\cdot|$  on  $G$  satisfies the triangle inequality up to a constant:

$$(2.2) \quad \exists C > 0 \quad \forall x, y \in G \quad |xy| \leq C(|x| + |y|).$$

*Example 2.13.* We define  $|x|_{\mathbb{V}} = |\text{Exp}_G^{-1} \cdot |_{\alpha, \mathbb{V}}$  with  $\mathbb{V}$  an adapted basis,  $|\cdot|_{\alpha, \mathbb{V}}$  defined in Example 2.3 and  $\alpha = 2M_0$  where  $M_0$  is a common multiple of the weights  $v_1, \dots, v_n$ , the least one to fix the idea. Then  $|\cdot|_{\mathbb{V}}$  is a quasinorm defined on  $G$ .

*Notation 2.14.* We call the quasinorm  $|\cdot|_{\mathbb{V}}$  of Example 2.13 the *standard quasinorm associated to  $\mathbb{V}$* . It is smooth away from 0 and  $|\cdot|_{\mathbb{V}}^{2M_0}$  is smooth, even at 0. We will use this notation all along the article.

*Remark 2.15.* In the triangle inequality (2.2) for the quasinorm  $|\cdot|_{\mathbb{V}}$  associated to  $\mathbb{V}$  of Example 2.13, we can choose the constant  $C$  so that it is an increasing function of  $\|[\cdot, \cdot]_{\mathfrak{g}}\|_{\mathbb{V}}$  depending on the gradation of  $\mathfrak{g}$  and on the choice of adapted basis  $\mathbb{V}$ , but otherwise not on the Lie structure, i.e. the full description of the Lie bracket  $[\cdot, \cdot]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  (up to isomorphism of Lie algebras).

Using the exponential mapping  $\text{Exp}_G$  and a fixed basis  $\mathbb{V} = (V_1, \dots, V_n)$  of  $\mathfrak{g}$  adapted to the gradation, we will often identify a point  $x$  in  $G$  with the coordinates of  $\text{Exp}_G^{-1}x$  in  $\mathfrak{g}$  with respect to  $\mathbb{V}$ . As already mentioned, having fixed an adapted basis  $\mathbb{V}$ , we set

$$x = \text{Exp}_G(x_1V_1 + \dots + x_nV_n) \in G.$$

A function on  $G$  can then be realized as a function of the variables  $(x_1, \dots, x_n) \in \mathbb{R}^n$ . For example, the monomials with respect to  $\mathbb{V}$  are defined as follows:

*Example 2.16.* Having fixed an adapted basis  $\mathbb{V}$ , we set

$$x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}, \quad x = \text{Exp}_G(x_1V_1 + \dots + x_nV_n) \in G, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n.$$

The functions  $x^\alpha$ ,  $\alpha \in \mathbb{N}_0^n$  are called the monomials with respect to  $\mathbb{V}$ . The monomial  $x^\alpha$  is  $[\alpha]$ -homogeneous for the dilations  $\delta_r$ ,  $r > 0$ .

Our analysis will require Taylor estimates adapted to graded groups and due to Folland and Stein [41], see also Theorem 3.1.51 in [37].

**Theorem 2.17.** *Let  $G$  be a graded Lie group. We fix a basis  $\mathbb{V}$  of  $\mathfrak{g}$  adapted to the gradation and a quasinorm  $|\cdot|$  on  $G$ .*

(1) Mean value theorem. *There exists  $C_0 > 0$  and  $\eta > 1$  such that for any  $f \in C^1(G)$  we have*

$$\forall x, y \in G, \quad |f(xy) - f(x)| \leq C_0 \sum_{j=1}^n |y|^{v_j} \sup_{|z| \leq \eta|y|} |(V_j f)(xz)|.$$

(2) Taylor estimate. More generally, with the constant  $\eta$  of point (1), for any  $N \in \mathbb{N}_0$ , there exists  $C_N > 0$  such that for any  $f \in C^{\lceil N \rceil}(G)$  we have

$$\forall x, y \in G, \quad |f(xy) - \mathbb{P}_{G,f,x,N}(y)| \leq C_N \sum_{\substack{|\alpha| \leq \lceil N \rceil + 1 \\ [\alpha] > N}} |y|^{[\alpha]} \sup_{|z| \leq \eta^{\lceil N \rceil + 1} |y|} |(\mathbb{V}^\alpha f)(xz)|.$$

Above,  $\lceil N \rceil$  denotes  $\max\{|\alpha| : \alpha \in \mathbb{N}_0^n \text{ with } [\alpha] \leq N\}$  and  $\mathbb{P}_{G,f,x,N}$  denotes the Taylor polynomial of  $f$  at  $x$  of order  $N$  for the graded group  $G$ , i.e. the unique linear combination of monomials of homogeneous degree  $\leq N$  satisfying  $\mathbb{V}^\beta \mathbb{P}_{G,f,x,N}(0) = \mathbb{V}^\beta f(x)$  for any  $\beta \in \mathbb{N}_0^n$  with  $[\beta] \leq N$ .

*Remark 2.18.* If we choose the quasinorm  $|\cdot| = |\cdot|_{\mathbb{V}}$  associated with  $\mathbb{V}$  of Example 2.13, the constants  $\eta$  and  $C_N$  can be chosen so that they are increasing functions of  $\|[\cdot]_{\mathfrak{g}}\|_{\mathbb{V}}$  depending on the gradation of  $\mathfrak{g}$  and on the choice of adapted basis  $\mathbb{V}$ , but otherwise not on the Lie structure.

In the sequel it will be useful to describe the differentiation of the composition of functions. We denote by  $*_G$  the law of the group  $G$  and, once given a basis  $\mathbb{V}$  of  $\mathfrak{g}$  adapted to the gradation, we identify  $G$  with  $\mathbb{R}^n$ .

**Lemma 2.19.** *Let  $G, G'$  be two graded Lie groups. We fix a basis  $\mathbb{V}$  (resp.  $\mathbb{V}'$ ) of  $\mathfrak{g}$  (resp. of  $\mathfrak{g}'$ ) adapted to the gradation. If  $f : G' \rightarrow \mathbb{R}$  and  $g : G \rightarrow G'$  are smooth, then with the convention above,*

$$V_j(f \circ g)(v) = \sum_{\ell=1}^{\dim G'} h_\ell(v) V'_\ell f(g(v)), \quad \text{where } h_\ell(v) := \partial_{t=0} [(-g(v)) *_G g(v *_G te_j)]_\ell.$$

Above  $(e_1, \dots, e_{\dim G})$  denotes the canonical basis of  $\mathbb{R}^{\dim G}$ , and  $[v']_\ell$  the coefficient of  $v' \in \mathbb{R}^{\dim G'}$ .

*Proof of Lemma 2.19.* This follows readily from writing

$$\begin{aligned} V_j(f \circ g)(v) &= \partial_{t=0} (f(g(v *_G te_j))) \\ &= \partial_{t=0} f(g(v) *_G' ((-g(v)) *_G' g(v *_G te_j))) \\ &= \sum_{\ell=1}^{\dim G'} \partial_{t=0} [(-g(v)) *_G' g(v *_G te_j)]_\ell V'_\ell f(g(v)), \end{aligned}$$

□

**2.2. Group Fourier transform on a nilpotent Lie group.** Let  $G$  be a nilpotent Lie group. Any representation  $(\pi, \mathcal{H}_\pi)$  of  $G$  we consider in this paper is assumed to be unitary, strongly continuous and its Hilbert space  $\mathcal{H}_\pi$  separable. We follow [26, 37].

2.2.1. *Definition on  $L^1(G)$ .* We fix a Haar measure  $dx$  on  $G$ . The *group Fourier transform* of a function  $\kappa \in L^1(G)$  at a representation  $\pi$  of  $G$  is the operator

$$\int_G \kappa(x)\pi(x)^* dx \in \mathcal{L}(\mathcal{H}_\pi),$$

denoted by

$$\mathcal{F}_G \kappa(\pi) = \widehat{\kappa}(\pi) = \pi(\kappa)$$

Note that if  $\pi_1$  and  $\pi_2$  are equivalent, i.e.  $U\pi_1 = \pi_2 U$  for some unitary map  $U$  between their Hilbert spaces, then  $U\mathcal{F}_G \kappa(\pi_1) = \mathcal{F}_G \kappa(\pi_2)U$ . Therefore, we may consider the group Fourier transform of  $\kappa \in L^1(G)$  at the equivalence class of the representation  $\pi$  of  $G$ .

The *unitary dual*  $\widehat{G}$  is the set of irreducible representations of  $G$  modulo equivalence equipped with the hull kernel topology. It is a standard Borel set and the group Fourier transform of  $\kappa \in L^1(G)$  may be viewed as the measurable field of equivalence classes of bounded operators

$$\mathcal{F}_G \kappa = \{\pi(\kappa) \in \mathcal{L}(\mathcal{H}_\pi), \pi \in \widehat{G}\}.$$

We denote by  $1_{\widehat{G}}$  (the class of) the trivial representation of  $G$ .

Note that

$$(2.3) \quad \mathcal{F}_G(f) \circ \mathcal{F}_G(g) = \mathcal{F}(g \star_G f), \quad g \star_G f = \int_G g(y)f(y^{-1}x)dy = \int_G g(xy^{-1})f(y)dy.$$

2.2.2. *Extension to  $L^2(G)$ .* There exists a measure  $\mu_{\widehat{G}}$  on  $\widehat{G}$  such that

$$\forall f \in L^1(G) \cap L^2(G), \quad \int_G |f(x)|^2 dx = \int_{\widehat{G}} \|\pi(f)\|_{HS(\mathcal{H}_\pi)}^2 d\mu_{\widehat{G}}(\pi),$$

where  $\|\cdot\|_{HS(\mathcal{H}_\pi)}$  denotes the Hilbert-Schmidt norm on  $\mathcal{H}_\pi$ ; the measure  $\mu_{\widehat{G}}$  is unique once the Haar measure  $dx = \mu_G$  has been fixed. The measure  $\mu_{\widehat{G}}$  and the above formula are called the *Plancherel measure* and the *Plancherel formula* (see [21, 50] or [26, 27] for abstract version of Plancherel theorem in a general context).

We denote by  $L^2(\widehat{G})$  the Hilbert space of measurable fields of Hilbert-Schmidt operators  $\{\sigma(\pi) : \mathcal{H}_\pi \rightarrow \mathcal{H}_\pi \text{ Hilbert-Schmidt}\}$  (up to  $\mu_{\widehat{G}}$ -a.e. equivalence and unitary equivalence) such that the following quantity is finite

$$\|\sigma\|_{L^2(\widehat{G})} := \sqrt{\int_{\widehat{G}} \|\sigma\|_{HS(\mathcal{H}_\pi)}^2 d\mu_{\widehat{G}}(\pi)}.$$

This allows for a unitary extension of the group Fourier transform  $\mathcal{F}_G : L^2(G) \rightarrow L^2(\widehat{G})$ .

2.2.3. *Extension to distributions, the set of bounded invariant symbols.* We start with defining the set of bounded invariant symbols  $L^\infty(\widehat{G})$ :

**Definition 2.20.** A *bounded invariant symbol* of  $G$  is a measurable field of equivalence classes of bounded operators  $\sigma = \{\sigma(\pi) \in \mathcal{L}(\mathcal{H}_\pi) : \pi \in \widehat{G}\}$  on  $\widehat{G}$  such that the following quantity is finite

$$\|\sigma\|_{L^\infty(\widehat{G})} := \sup_{\pi \in \widehat{G}} \|\sigma(\pi)\|_{\mathcal{L}(\mathcal{H}_\pi)}.$$

The supremum above is understood as the essential supremum with respect to the Plancherel measure  $\mu_{\widehat{G}}$  and the notion of measurable fields of bounded operators is up to  $\mu_{\widehat{G}}$ -a.e. equivalence and unitary equivalence. The space of bounded invariant symbols of  $G$  is denoted by  $L^\infty(\widehat{G})$ .

Note that differing from [37] and related papers, the symbols  $\sigma$  in this section are invariant in the sense that they do not depend on  $x \in G$ .

We check readily

$$\forall \kappa \in L^1(G), \quad \|\pi(\kappa)\|_{L^\infty(\widehat{G})} \leq \|\kappa\|_{L^1(G)}, \quad \text{so} \quad \mathcal{F}_G(L^1(G)) \subset L^\infty(\widehat{G}).$$

However, there are many other bounded invariant symbols. In fact, it follows from Dixmier's full Plancherel theorem [26, §18] that  $L^\infty(\widehat{G})$  is a von Neumann algebra (sometimes called the von Neumann algebra of the group) isomorphic to the subspace

$$\mathcal{L}(L^2(G))^G$$

of bounded operators on  $L^2(G)$  that commutes with left-translation of the group. Indeed, if  $\sigma \in L^\infty(\widehat{G})$ , then it is straightforward to check that the Fourier multiplier defined via  $f \mapsto \mathcal{F}_G^{-1}(\sigma \widehat{f})$  is in  $\mathcal{L}(L^2(G))^G$  and its symbol is  $\sigma$ . Conversely, if  $T \in \mathcal{L}(L^2(G))^G$ , there exists a unique bounded invariant symbol  $\widehat{T} \in L^\infty(\widehat{G})$  such that  $\mathcal{F}_G(Tf) = \widehat{T} \widehat{f}$  for any  $f \in L^2(G)$ . Moreover  $\|\widehat{T}\|_{L^\infty(\widehat{G})} = \|T\|_{\mathcal{L}(L^2(G))}$ .

From the Schwartz kernel theorem, it follows that if  $T \in \mathcal{L}(L^2(G))^G$ , there exists a unique distribution  $\kappa \in \mathcal{S}'(G)$  such that  $Tf = f \star_G \kappa$  for any  $f \in \mathcal{S}(G)$ . The distribution  $\kappa$  is called the (right) convolution kernel of  $T$ . We set  $\mathcal{F}_G \kappa := \widehat{T}$  and we check easily that this extends the group Fourier transform to

$$\mathcal{K}(G) := \{\kappa \in \mathcal{S}'(G) : (f \mapsto f \star_G \kappa) \in \mathcal{L}(L^2(G))\},$$

and that  $\mathcal{F}_G : \mathcal{K} \rightarrow L^\infty(\widehat{G})$  is injective, hence bijective. We readily check the following property:

**Lemma 2.21.** *Let  $\theta$  be an automorphism of the nilpotent Lie group  $G$ . We keep the same notation for the corresponding automorphism of its Lie algebra  $\mathfrak{g}$ . The Jacobian determinant of  $\theta$  on  $G$  is then constant and equal to  $\det \theta$ . For any  $T \in \mathcal{L}(L^2(G))^G$ , the operator  $T_\theta$  defined by*

$$T_\theta f = (T(f \circ \theta^{-1})) \circ \theta$$

is in  $\mathcal{L}(L^2(G))^G$  with

$$\|T_\theta\|_{\mathcal{L}(L^2(G))} = \|T\|_{\mathcal{L}(L^2(G))}.$$

Moreover, denoting by  $\kappa$  and  $\sigma$  the convolution kernel and symbol of  $T$ , the convolution kernel and symbol of  $T_\theta$  are given by

$$\kappa_\theta := (\det \theta) \kappa \circ \theta \quad \text{and} \quad \theta_* \sigma(\pi) = \sigma(\pi \circ \theta^{-1}), \quad \pi \in \widehat{G},$$

with  $\|\theta_* \sigma\|_{L^\infty(\widehat{G})} = \|\sigma\|_{L^\infty(\widehat{G})}$ .

#### 2.2.4. Invariant symbols.

**Definition 2.22.** An *invariant symbol*  $\sigma$  on  $G$  is a measurable field of operators  $\sigma = \{\sigma(\pi) : \mathcal{H}_\pi^{+\infty} \rightarrow \mathcal{H}_\pi^{-\infty} : \pi \in \widehat{G}\}$  over  $\widehat{G}$ . Here,  $\mathcal{H}_\pi^{+\infty}$  and  $\mathcal{H}_\pi^{-\infty}$  denote the spaces of smooth and distributional vectors in  $\mathcal{H}_\pi$ .

We identify the Lie algebra  $\mathfrak{g}$  of  $G$  with the space of real left-invariant vector fields on  $G$  and the universal enveloping Lie algebra  $\mathcal{U}(\mathfrak{g})$  with the space of left-invariant differential operators on  $G$ . If  $\pi$  is a representation of  $G$ , we keep the same notation for the corresponding representations of  $\mathfrak{g}$  and  $\mathcal{U}(\mathfrak{g})$ .

*Example 2.23.* For any  $T \in \mathcal{U}(\mathfrak{g})$ ,  $\widehat{T} = \{\pi(T), \pi \in \widehat{G}\}$  is an invariant symbol. In fact, each  $\pi(T)$  acts on  $\mathcal{H}_\pi^{+\infty}$  and, by duality, on  $\mathcal{H}_\pi^{-\infty}$ .

*Remark 2.24.* With this general definition, two invariant symbols may not be composable, although an invariant symbol can always be composed (on the right and on the left) with  $\widehat{T}$ ,  $T \in \mathcal{U}(\mathfrak{g})$ .

In Section 2.4 below, we will restrict ourselves to smaller subspaces of invariant symbols which will contain  $\{\pi(T), \pi \in \widehat{G}\}$ ,  $T \in \mathcal{U}(\mathfrak{g})$ . To define these subspaces, we will consider the Sobolev spaces on  $G$  whose definition is recalled in Section 2.3.1 below. This relies on the notion of Rockland operator recalled in Section 2.3 and on the graded Lie structure of the group  $G$  while the theory recalled above in Sections 2.2.1, 2.2.2 and 2.2.3 only requires  $G$  to be a locally compact unimodular group  $G$  of type 1.

**2.3. Rockland operators and Sobolev spaces.** As in [37], we adopt the following conventions:

**Definition 2.25.** A left-invariant differential operator  $\mathcal{R} \in \mathcal{U}(\mathfrak{g})$  on  $G$  is *Rockland* when it is homogeneous and for any  $\pi \in \widehat{G} \setminus \{1_{\widehat{G}}\}$ ,  $\pi(\mathcal{R})$  is injective on  $\mathcal{H}_\pi^\infty$ .

It is positive when

$$\forall f \in \mathcal{S}(G), \quad \int_G \mathcal{R}f(x) \overline{f(x)} dx \geq 0.$$

*Example 2.26.* A positive Rockland operator always exists, for instance

$$(2.4) \quad \mathcal{R} = \sum_j (-1)^{\frac{M_0}{v_j}} V_j^{\frac{2M_0}{v_j}},$$

having fixed a basis  $\mathbb{V} = (V_1, \dots, V_n)$  adapted to the gradation and a multiple  $M_0$  of the resulting weights  $v_1, \dots, v_n$ . Its homogeneous degree is  $2M_0$ . For  $M_0$  the least common multiple of the weights  $v_1, \dots, v_n$ , we denote by  $\mathcal{R}_{\mathbb{V}}$  the resulting positive Rockland operator.

The Helffer-Nourrigat theorem [44] states that a Rockland operator is hypoelliptic (this result valid on graded groups was further investigated on manifolds by the same authors in [45] in the 80's, leading to the Helffer-Nourrigat conjecture proved recently in [3]). Moreover, if  $\mathcal{R}$  is symmetric, then  $\mathcal{R}$  and  $\pi(\mathcal{R})$  for any  $\pi \in \widehat{G}$  are essentially self-adjoint on  $\mathcal{S}(G) \subset$

$L^2(G)$  and  $\mathcal{H}_\pi^\infty \subset \mathcal{H}_\pi$  respectively; we keep the same notation for their self-adjoint extensions. Denoting by  $E_{\mathcal{R}}$  and  $E_{\pi(\mathcal{R})}$  the spectral resolutions of  $\mathcal{R}$  and  $\pi(\mathcal{R})$ ,

$$(2.5) \quad \text{i.e. } \mathcal{R} = \int_{\mathbb{R}} \lambda dE_{\mathcal{R}}(\lambda) \quad \text{and} \quad \pi(\mathcal{R}) = \int_{\mathbb{R}} \lambda dE_{\pi(\mathcal{R})}(\lambda),$$

then for any interval  $I \subset \mathbb{R}$ , we have

$$\mathcal{F}_G(E_{\mathcal{R}}(I)) = \{E_{\pi(\mathcal{R})}(I), \pi \in \widehat{G}\},$$

or in other words

$$\forall f \in \mathcal{S}(G), \quad \mathcal{F}_G(E_{\mathcal{R}}(I)f) = E_{\pi(\mathcal{R})}(I)\mathcal{F}_G f.$$

The spectral resolutions of  $\mathcal{R}$  and  $\pi(\mathcal{R})$  allows us to consider spectral multipliers in these operators.

If in addition of being Rockland,  $\mathcal{R}$  is positive then  $\pi(\mathcal{R})$  is also positive. In fact, there is an equivalence between  $\mathcal{R}$  being positive and  $\pi(\mathcal{R})$  positive for any  $\pi \in \widehat{G}$ . In this case, for each  $\pi \in \widehat{G} \setminus \{1_{\widehat{G}}\}$ , the operator  $\pi(\mathcal{R})$  has a discrete spectrum in  $(0, +\infty)$  and each eigenvalue has finite multiplicity [49].

**2.3.1. Inhomogeneous Sobolev spaces  $L_s^2(G)$  on  $G$ .** For any  $s \in \mathbb{R}$  and  $\mathcal{R}$  a positive Rockland operator of homogeneous degree  $\nu$ , the domain of the spectral multiplier  $(I + \mathcal{R})^s$  contains  $\mathcal{S}(G)$ , and  $\pi(I + \mathcal{R})^s$ ,  $\pi \in \widehat{G}$ , acts on  $\mathcal{H}_\pi^\infty$  and on  $\mathcal{H}_\pi^{-\infty}$ . We denote by  $L_{s,\mathcal{R}}^2(G)$  on  $G$  the closure of  $\mathcal{S}(G)$  in  $\mathcal{S}'(G)$  for the norm

$$\|f\|_{L_{s,\mathcal{R}}^2(G)} := \|(I + \mathcal{R})^{s/\nu} f\|_{L^2(G)}.$$

**Proposition 2.27.** *The spaces  $L_{s,\mathcal{R}}^2(G)$  enjoy the following properties:*

- (1) *Each  $L_{s,\mathcal{R}}^2(G)$  is a Hilbert space and for  $s = 0$ ,  $L_{0,\mathcal{R}}^2(G) = L^2(G)$ .*
- (2) *For any  $s_1 \geq s_2$ , we have the continuous inclusions  $L_{s_1,\mathcal{R}}^2(G) \subseteq L_{s_2,\mathcal{R}}^2(G)$ .*
- (3) *The following interpolation property holds*

$$\|f\|_{L_{t,\mathcal{R}}^2(G)} \leq \|f\|_{L_{s,\mathcal{R}}^2(G)}^\theta \|f\|_{L_{u,\mathcal{R}}^2(G)}^{1-\theta}$$

for  $t \in (s, u)$  with  $s < u$  and  $t = \theta s + (1 - \theta)u$ ,  $\theta \in (0, 1)$ .

- (4) *The spaces  $L_{s,\mathcal{R}}^2(G)$  and  $L_{-s,\mathcal{R}}^2(G)$  are dual to each other.*
- (5) *If  $\mathbb{V}$  is a basis of  $\mathfrak{g}$  adapted to the gradation and  $s \in \mathbb{N}$  is a common multiple of the weights of the gradation, then  $L_{s,\mathcal{R}}^2(G)$  is the space of  $f \in L^2(G)$  such that  $\mathbb{V}^\alpha f \in L^2(G)$  for any multi-index  $\alpha$  with  $[\alpha] = s$ . Furthermore, a norm equivalent to the norm  $\|\cdot\|_{L_{s,\mathcal{R}}^2(G)}$  is given by*

$$f \mapsto \|f\|_{L^2(G)} + \sum_{[\alpha]=s} \|\mathbb{V}^\alpha f\|_{L^2(G)}.$$

- (6) *If  $\mathcal{R}_1$  and  $\mathcal{R}_2$  are two positive Rockland operators, then for any  $s \in \mathbb{R}$ ,*

$$\exists C = C_{s,G,\mathcal{R}_1,\mathcal{R}_2} > 0, \quad \forall f \in \mathcal{S}(G), \quad \|f\|_{L_{s,\mathcal{R}_1}^2(G)} \leq C \|f\|_{L_{s,\mathcal{R}_2}^2(G)}.$$

- (7) Sobolev embedding. For  $s > Q/2$  the inclusion  $L_{s,\mathcal{R}}^2 \subset C_b(G)$  is continuous; here  $C_b(G)$  is the Banach space of continuous bounded function on  $G$  equipped with the norm  $\|\cdot\|_{L^\infty(G)}$ .

It follows from Proposition 2.27 that for each  $s \in \mathbb{R}$ , the space  $L_{s,\mathcal{R}}^2(G)$  is independent of  $\mathcal{R}$  or  $\mathbb{V}$ , and all the norms  $\|f\|_{L_{s,\mathcal{R}}^2(G)}$  for various  $\mathcal{R}$  are equivalent. Moreover,  $L_{s,\mathcal{R}}^2(G)$  depends only on  $s \in \mathbb{R}$  and on the connected simply connected nilpotent Lie group  $G$ . We then omit the index  $\mathcal{R}$  to denote the space by  $L_s^2(G)$ . This space is called the Sobolev space on  $G$ .

We can be more precise about the constant in Proposition 2.27 (6):

*Remark 2.28.* Assume that the gradation of the vector space underlying  $\mathfrak{g}$  as well as an adapted basis  $\mathbb{V}$  are fixed. Then, the constant  $C$  may be chosen as

$$C = \tilde{C}(\max(s, \|\cdot\|_{\mathfrak{g}}\|_{\mathbb{V}}, C_{\mathbb{V}}(\mathcal{R}_1), C_{\mathbb{V}}(\mathcal{R}_2))),$$

where  $\tilde{C} : [0, \infty) \rightarrow (0, \infty)$  is an increasing function, and the constants  $C_{\mathbb{V}}(\mathcal{R}_i)$ ,  $i = 1, 2$ , are defined as follows: let  $\mathcal{R}$  be a positive Rockland operator on  $G$ , then it can be written as  $\mathcal{R} = \sum_{[\alpha]=\nu} c_\alpha \mathbb{V}^\alpha$  (here  $\nu$  is the homogeneous degree of  $\mathcal{R}$ ), and we set  $C_{\mathbb{V}}(\mathcal{R}) := \max_{[\alpha]=\nu} |c_\alpha|$ .

2.3.2. *Further extensions of the group Fourier transform.* We set

$$\mathcal{K}_{a,b}(G) := \{\kappa \in \mathcal{S}'(G), (f \mapsto f \star_G \kappa) \in \mathcal{L}(L_a^2(G), L_b^2(G))\}.$$

Fixing a positive Rockland operator  $\mathcal{R}$  of homogeneous degree  $\nu$ , if  $\sigma$  is an invariant symbol, then so is the symbol

$$(I + \widehat{\mathcal{R}})^{b/\nu} \sigma (I + \widehat{\mathcal{R}})^{-a/\nu} = \{(I + \pi(\mathcal{R}))^{b/\nu} \sigma(\pi)(I + \pi(\mathcal{R}))^{-a/\nu} : \pi \in \widehat{G}\}.$$

We then define the following subspace of invariant symbols

$$\sigma \in L_{a,b}^\infty(\widehat{G}) \text{ if and only if } (I + \widehat{\mathcal{R}})^{b/\nu} \sigma (I + \widehat{\mathcal{R}})^{-a/\nu} \in L^\infty(\widehat{G}).$$

Equipped with the norm

$$(2.6) \quad \|\sigma\|_{L_{a,b}^\infty(\widehat{G}), \mathcal{R}} := \|(I + \widehat{\mathcal{R}})^{b/\nu} \sigma (I + \widehat{\mathcal{R}})^{-a/\nu}\|_{L^\infty(\widehat{G})},$$

it is a Banach space, isomorphic and isometric to the subspace of left-invariant operators in  $\mathcal{L}(L_a^2(G), L_b^2(G))$ . The properties of Rockland operators (see Proposition 2.27) imply that  $L_{a,b}^\infty(\widehat{G})$  is independent of the choice of a positive Rockland operator  $\mathcal{R}$ . Moreover, the Schwartz kernel theorem allows us to extend the group Fourier transform into a bijection  $\mathcal{F}_G : \mathcal{K}_{a,b}(G) \rightarrow L_{a,b}^\infty(G)$  (see Proposition 5.1.29 in [37]).

2.3.3. *Homogeneous Sobolev spaces.* For any  $s \in \mathbb{R}$  and any positive Rockland operator  $\mathcal{R}$  of homogeneous degree  $\nu$ , the map

$$f \mapsto \|\mathcal{R}^{s/\nu} f\|_{L^2(G)} := \|f\|_{L_{s,\mathcal{R}}^2(G)}$$

defines a norm on  $\mathcal{S}(G) \cap \text{dom}(\mathcal{R}^{s/\nu})$ . Moreover (see [37, Section 4.4]), if  $\mathcal{R}_1$  and  $\mathcal{R}_2$  are two positive Rockland operators, we obtain as in Proposition 2.27 (6),

$$\exists C = C_{s,G,\mathcal{R}_1,\mathcal{R}_2} > 0, \quad \forall f \in \mathcal{S}(G), \quad \|f\|_{L_{s,\mathcal{R}_1}^2(G)} \leq C \|f\|_{L_{s,\mathcal{R}_2}^2(G)}.$$

with a similar remark as in Remark 2.28 regarding the constant  $C$  above. Therefore, the subspace of tempered distributions obtained as the completion of  $\mathcal{S}(G) \cap \text{dom}(\mathcal{R}^{s/\nu})$  for the norm  $\|\cdot\|_{\dot{L}_s^2(G), \mathcal{R}}$  is independent of a choice of a positive Rockland operator  $\mathcal{R}$ . It is called the homogeneous Sobolev subspace  $\dot{L}_s^2(G)$ . Moreover [37, Section 4.4], the inclusion  $\dot{L}_s^2(G) \subset \mathcal{S}'(G)$  is continuous. In the case when  $s \in \mathbb{N}$  is a common multiple of the weights of the gradation, fixing a basis  $\mathbb{V}$  of  $\mathfrak{g}$  adapted to the gradation,  $\dot{L}_s^2(G)$  contains  $\mathcal{S}(G)$  and is the closure of  $\mathcal{S}(G)$  in  $\mathcal{S}'(G)$  for  $f \mapsto \sum_{|\alpha|=s} \|\mathbb{V}^\alpha f\|_{L^2(G)}$ .

## 2.4. Invariant symbol classes.

2.4.1. *Definitions.* We start with the definition of difference operators:

**Definition 2.29.** Let  $q \in C^\infty(G)$  and let  $\sigma$  be an invariant symbol on  $\widehat{G}$ . We say that  $\sigma$  is  $\Delta_q$ -differentiable when  $\sigma \in L_{a,b}^\infty(\widehat{G})$  and  $q\mathcal{F}_G^{-1}\sigma \in \mathcal{K}_{c,d}(G)$  for some  $a, b, c, d \in \mathbb{R}$ . In this case, we set

$$\Delta_q \sigma = \mathcal{F}_G(q\mathcal{F}_G^{-1}\sigma).$$

We can now define our classes of symbols.

**Definition 2.30.** An invariant symbol  $\sigma$  on  $\widehat{G}$  is of order  $m$  if  $\sigma \in L_{0,-m}^\infty(\widehat{G})$  and moreover,  $\Delta_{x^\alpha} \sigma \in L_{0,-m+|\alpha]}^\infty(\widehat{G})$  for any indices  $\alpha \in \mathbb{N}_0^n$ .

In the above definition, the  $x^\alpha$ 's are the monomials defined with respect to a basis  $\mathbb{V}$  adapted to the gradation (see Example 2.16). It is independent of the choice of adapted basis  $\mathbb{V}$ .

*Notation 2.31.* We denote by  $S^m(\widehat{G})$  the space of invariant symbols on  $\widehat{G}$  of order  $m$ .

The space  $S^m(\widehat{G})$  is naturally equipped with a Fréchet structure given by the semi-norms

$$(2.7) \quad \begin{aligned} \|\sigma\|_{S^m(\widehat{G}), N, \mathbb{V}, \mathcal{R}} &:= \|\sigma\|_{S^m(\widehat{G}), N, \mathcal{R}} := \max_{|\alpha| \leq N} \|\Delta_{x^\alpha} \sigma\|_{L_{0,-m+|\alpha]}^\infty(\widehat{G}), \mathcal{R}} \\ &= \max_{|\alpha| \leq N} \|(I + \mathcal{R})^{\frac{|\alpha|-m}{\nu}} \Delta_{x^\alpha} \sigma\|_{L^\infty(\widehat{G})} \end{aligned}$$

with  $N \in \mathbb{N}_0$ , having fixed a positive Rockland operator  $\mathcal{R}$  on  $G$  (of homogeneous order  $\nu$ ) and the Haar measure induced by  $\mathbb{V}$ . With a Haar measure fixed, the distribution  $\kappa := \mathcal{F}_G^{-1}\sigma \in \mathcal{S}'(G)$  is called the convolution kernel of  $\sigma$ .

*Example 2.32.* If  $\mathbb{V}$  is an adapted basis of  $\mathfrak{g}$ , then for any  $\alpha \in \mathbb{N}_0^n$ , the symbol

$$\widehat{\mathbb{V}}^\alpha = \{\pi(\mathbb{V})^\alpha, \pi \in \widehat{G}\}$$

is in  $S^{|\alpha]}(\widehat{G})$ .

We have the continuous inclusions:

$$(2.8) \quad m_1 \leq m_2 \implies S^{m_1}(\widehat{G}) \subset S^{m_2}(\widehat{G}).$$

Moreover, the properties of interpolation of Sobolev spaces imply natural properties of interpolation for the space  $S^m(\widehat{G})$ .

2.4.2. *Symbolic properties.* At the symbolic level, the symbol classes form a  $*$ -algebra:

**Theorem 2.33.** *Assume that an adapted basis  $\mathbb{V}$  of  $\mathfrak{g}$  has been fixed.*

- (1) *If  $\sigma \in S^m(\widehat{G})$ , then  $\Delta_{x^\alpha}\sigma \in L_{\gamma, -m+[\alpha]+\gamma}^\infty$  for any  $\alpha \in \mathbb{N}_0^n$  and  $\gamma \in \mathbb{R}$ . Moreover, the following map is a continuous semi-norm on  $S^m(\widehat{G})$ :*

$$\sigma \longmapsto \max_{[\alpha] \leq N} \|\Delta_{x^\alpha}\sigma\|_{L_{\gamma, -m+[\alpha]+\gamma}^\infty(\widehat{G}, \mathcal{R})}, \quad N \in \mathbb{N}_0, \gamma \in \mathbb{R}.$$

- (2) *The composition and adjoint maps*

$$\left\{ \begin{array}{l} S^{m_1}(\widehat{G}) \times S^{m_2}(\widehat{G}) \longrightarrow S^{m_1+m_2}(\widehat{G}) \\ (\sigma_1, \sigma_2) \longmapsto \sigma_1\sigma_2 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} S^m(\widehat{G}) \longrightarrow S^m(\widehat{G}) \\ \sigma \longmapsto \sigma^* \end{array} \right. ,$$

*are continuous. Moreover, denoting by  $\kappa_1, \kappa_2$  the convolution kernels of  $\sigma_1, \sigma_2$ , their convolution product  $\kappa_2 \star_G \kappa_1$  makes sense and is the convolution kernel of  $\sigma_1\sigma_2$ . Denoting by  $\kappa$  the convolution kernel of  $\sigma$ , then the convolution kernel of  $\sigma^*$  is given by  $v \mapsto \bar{\kappa}(v^{-1})$ .*

The proof of Theorem 2.33 may be found in [37], see Theorems 5.2.20 and 5.5.22 therein (with the difference that the symbols in [37] also depend on  $x \in G$ ).

*Remark 2.34.* The proof of Theorem 2.33 implies in particular that the constants describing the continuity of the linear map for the adjoint and the bilinear map for the composition depend on the Lie structure only via  $\|[\cdot, \cdot]\|_{\mathbb{V}}$  after a choice of a basis  $\mathbb{V}$  adapted to the gradation. Let us explain what this means for the continuity of the adjoint. We fix a basis  $\mathbb{V}$  adapted to the gradation. Then for any  $N \in \mathbb{N}_0$  and  $m \in \mathbb{R}$ , there exist  $C > 0$  and  $N' \in \mathbb{N}$  such that

$$\forall \sigma \in S^m(\widehat{G}), \quad \|\sigma^*\|_{S^m(\widehat{G}), N, \mathcal{R}_{\mathbb{V}}} \leq C \|\sigma\|_{S^m(\widehat{G}), N', \mathcal{R}_{\mathbb{V}}}.$$

We could have taken any positive Rockland operator but we choose  $\mathcal{R}_{\mathbb{V}}$  the one associated with  $\mathbb{V}$  (see Example 2.26) to make the constant structural. The constants  $C$  and  $N'$  can be written as increasing functions of  $\|[\cdot, \cdot]\|_{\mathbb{V}}$ . Naturally, these increasing functions depend on what has been fixed, that is, beside  $m$  and  $N$ , on the gradation of the vector space underlying  $\mathfrak{g}$  and the adapted basis  $\mathbb{V}$ .

2.4.3. *Asymptotic sums of symbol.*

**Definition 2.35.** Let  $\sigma \in S^m(\widehat{G})$  and  $\sigma_j, j \in \mathbb{N}_0$ , be a sequence of symbols such that  $\sigma_j \in S^{m_j}(\widehat{G})$  with  $m_0 = m$  and  $m_j$  strictly decreasing to  $-\infty$ . We say that  $\sum_j \sigma_j$  is an asymptotic expansion for  $\sigma$  and we write

$$\sigma \sim \sum_j \sigma_j,$$

when we have for any  $N \in \mathbb{N}_0$ ,

$$\sigma - \sum_{j=0}^N \sigma_j \in S^{m_{N+1}}(\widehat{G}).$$

Given a sequence  $(\sigma_j)_{j \in \mathbb{N}_0}$  as in Definition 2.35, we can construct a symbol  $\sigma \in S^{m_0}(\widehat{G})$  following the classical ideas due to Borel (see e.g. [37, Section 5.5.1]), that will satisfy  $\sigma \sim \sum_j \sigma_j$ . Moreover,  $\sigma$  is unique up to  $S^{-\infty}(\widehat{G})$ .

2.4.4. *Kernel estimates.* The convolution kernel associated with a symbol in some  $S^m(\widehat{G})$  will be Schwartz away from the origin 0, but may have a singularity at 0 [37, Theorem 5.4.1]

**Theorem 2.36.** *Let  $\sigma \in S^m(\widehat{G})$  and denote its convolution kernel by  $\kappa = \mathcal{F}_G^{-1}\sigma$ . Then  $\kappa$  is smooth away from the origin. Moreover, fixing a quasinorm  $|\cdot|$  on  $G$ , we have the following kernel estimates:*

(1) *The convolution kernel  $\kappa$  decays faster than any polynomial away from the origin:*

$$\forall N \in \mathbb{N}_0, \quad \exists C = C_{\sigma, N} > 0 : \quad \forall y \in G, \\ |y| \geq 1 \implies |\kappa(y)| \leq C|y|^{-N}.$$

(2) *If  $Q + m < 0$  then  $\kappa$  is continuous and bounded on  $G$ :*

$$\exists C = C_\sigma > 0, \quad \sup_{y \in G} |\kappa(y)| \leq C.$$

(3) *If  $Q + m > 0$ , then*

$$\exists C = C_\sigma > 0 : \quad \forall y \in G, \quad 0 < |y| \leq 1 \implies |\kappa(y)| \leq C|y|^{-(Q+m)}.$$

(4) *If  $Q + m = 0$ , then*

$$\exists C = C_\sigma > 0 : \quad \forall y \in G, \quad 0 < |y| \leq 1/2 \implies |\kappa(y)| \leq -C \ln |y|.$$

*In all the estimates above, the constant  $C = C_\sigma$  may be chosen of the form*

$$C = C_1 \|\sigma\|_{S^m(\widehat{G}), N', \mathcal{R}},$$

*where  $\mathcal{R}$  is a positive Rockland operator; the constant  $C_1$  and the semi-norm  $\|\cdot\|_{S^m(\widehat{G}), N', \mathcal{R}}$  are independent of  $\sigma$ .*

*Remark 2.37.* (1) We can be even more precise. Having fixed a basis  $\mathbb{V}$ , consider the associated quasinorm  $|\cdot|_{\mathbb{V}}$  of Example 2.13 and the positive Rockland operator  $\mathcal{R}_{\mathbb{V}}$  of Example 2.26. The constant  $C_1$  in each item of Theorem 2.36 is an increasing function of  $\|[\cdot, \cdot]\|_{\mathbb{V}}$ . Furthermore, these increasing functions depend on  $m$  and, in Part (1), also on  $N$ , as well as the gradation of the underlying subspace of the Lie algebra  $\mathfrak{g}$  and on the adapted basis  $\mathbb{V}$ , but not on the Lie structure.

(2) Note that if  $\kappa \in \mathcal{S}'(G)$  is the convolution kernel of  $\sigma \in S^m(\widehat{G})$ , then  $x^\alpha \mathbb{V}^{\beta_1} \widetilde{\mathbb{V}}^{\beta_2} \kappa$  is the convolution kernel of the symbol  $\Delta_{x^\alpha} \widehat{\mathbb{V}}^{\beta_1} \sigma \widehat{\mathbb{V}}^{\beta_2}$  which is in  $S^{m-[\alpha]+[\beta_1]+[\beta_2]}(\widehat{G})$  by Theorem 2.33. Recall that the left-invariant differential operators  $\mathbb{V}^\alpha$  (resp. right-invariant  $\widetilde{\mathbb{V}}^\alpha$ ) have been introduced in Example 2.10.

**Corollary 2.38.** *Assume that we have fixed a basis  $\mathbb{V}$  of  $\mathfrak{g}$  adapted to the gradation and the corresponding Haar measure. Let  $\sigma \in S^m(\widehat{G})$  with  $m \in \mathbb{R}$ .*

- (1) If  $\alpha \in \mathbb{N}_0^n$ ,  $\beta \in \mathbb{N}_0^n$ ,  $p \in [1, \infty]$  with  $m - [\alpha] + [\beta] < Q(\frac{1}{p} - 1)$  (with the convention  $\frac{1}{\infty} = 0$ ), then the quantity

$$\|\sigma\|_{\alpha, \beta, p} := \|x^\alpha \nabla^\beta \kappa\|_{L^p(G)}, \quad \sigma = \mathcal{F}_G \kappa$$

is finite. Moreover, the map  $\sigma \mapsto \|\sigma\|_{\alpha, \beta, p}$  is a continuous semi-norm on  $S^m(\widehat{G})$ .

- (2) If  $N, M_1 \in \mathbb{N}_0$  such that  $-m + N \leq M_1$  with  $M_1 \in \mathbb{N}_0$  a common multiple of the dilations' weights, then there exists a constant  $C > 0$  such that

$$\|\sigma\|_{S^m(\widehat{G}), N, \mathcal{R}} \leq C \max_{[\alpha'] + [\beta'] \leq N + M_1} \|\sigma\|_{\alpha', \beta', 1},$$

for any  $\sigma \in S^m(\widehat{G})$  such that the right-hand side is finite.

- (3) Let  $\sigma \in S^m(\widehat{G})$ . Its convolution kernel  $\kappa = \mathcal{F}_G^{-1} \sigma$  is a distribution smooth away from the origin. At the origin, its order is less than or equal to the integer

$$p_\sigma := \max\{|\alpha| : \alpha \in \mathbb{N}_0^n, [\alpha] \leq s'\}$$

where  $s'$  is the smallest non-negative integer multiple of the dilation weights such that  $s' > Q/2 + m$ . Moreover, we have for any compact neighbourhood  $\mathcal{C}_0$  of 0 in  $\mathbb{R}^n$ ,

$$\exists C > 0, \quad \forall \phi \in C_c^\infty(\mathcal{C}_0), \quad \left| \int_{\mathbb{R}^n} \kappa(v) \phi(v) dv \right| \leq C \max_{|\alpha| \leq p_\sigma} \|\partial^\alpha \phi\|_{L^\infty(\mathcal{C}_0)}.$$

Above,  $C$  may be chosen of the form

$$C = C_1 \|\sigma\|_{S^m(\widehat{G}), N, \mathcal{R}},$$

with  $C_1 > 0$ ,  $N \in \mathbb{N}_0$ .

*Proof.* (1) The first part follows readily from the kernel estimates in Theorem 2.36 and the second point in Remark 2.37. Indeed, only the singularity in 0 has to be considered when  $m - [\alpha] + [\beta] \geq -Q$ . Observing that  $S^m(\widehat{G}) \subset S^{m+\epsilon}(\widehat{G})$  for all  $\epsilon > 0$ , we deduce from Theorem 2.36 (3):

$$0 < |y| \leq 1, \quad |y|^{[\alpha]} \nabla^\beta \kappa(y) \leq C |y|^{-(Q+m+\epsilon-[\alpha]+[\beta])} \in L^p(G) \quad \text{for } p(Q+m+\epsilon-[\alpha]+[\beta]) > Q.$$

Note that it is possible to find such an  $\epsilon$  as soon as  $p(Q+m-[\alpha]+[\beta]) > Q$ , which concludes the proof.

- (2) For the second part, the properties of Sobolev spaces (see [38] or Proposition 2.27)

$$\|\sigma\|_{S^m(\widehat{G}), N, \mathcal{R}} = \max_{[\alpha] \leq N} \|\Delta_{x^\alpha} \sigma\|_{L_{0, -m+[\alpha]}^\infty(\widehat{G}), \mathcal{R}} = \max_{[\alpha] \leq N} \|(I + \widehat{\mathcal{R}})^{-\frac{m+[\alpha]}{\nu}} \sigma\|_{L^\infty(\widehat{G})}$$

Then since  $-m + [\alpha] \leq M_1 < 0$  and by boundedness of  $(I + \widehat{\mathcal{R}})^{-a/\nu}$  for  $a > 0$ , we have

$$\|(I + \widehat{\mathcal{R}})^{-\frac{m+[\alpha]}{\nu}} \sigma\|_{L^\infty(\widehat{G})} \lesssim \|(I + \widehat{\mathcal{R}})^{\frac{M_1}{\nu}} \sigma\|_{L^\infty(\widehat{G})} \lesssim \sum_{[\beta] \leq M_1} \|\widehat{\mathcal{V}}^\beta \Delta_{x^\alpha} \sigma\|_{L^\infty(\widehat{G})}.$$

Finally it is easy to see that

$$\|\widehat{\mathcal{V}}^\beta \Delta_{x^\alpha} \sigma\|_{L^\infty(\widehat{G})} \lesssim \max_{[\alpha] \leq N} \|\sigma\|_{\alpha, \beta, 1}$$

whence the result.

(3) If  $m < -Q$ , then by Theorem 2.36,  $\kappa$  is a continuous bounded function on  $G$ , so it is a distribution of order 0. Assume  $m \geq -Q$ . For any  $\phi \in C_c^\infty(G)$ , we have by the Sobolev embeddings (see Proposition 2.27 (7)) with  $s > Q/2$

$$\|\phi * \kappa\|_{L^\infty(G)} \lesssim \|(I + \tilde{\mathcal{R}})^{s/\nu}(\phi * \kappa)\|_{L^2(G)};$$

above  $\tilde{\mathcal{R}}$  denotes the right-invariant operator corresponding to a fixed Rockland operator  $\mathcal{R}$  of homogeneous degree  $\nu$ . The properties of the convolution yield:

$$(I + \tilde{\mathcal{R}})^{s/\nu}(\phi * \kappa) = ((I + \tilde{\mathcal{R}})^{\frac{s}{\nu}}\phi) * \kappa = ((I + \tilde{\mathcal{R}})^{\frac{s+m}{\nu}}\phi) * ((I + \mathcal{R})^{-\frac{m}{\nu}}\kappa),$$

so by the Plancherel formula,

$$\|(I + \tilde{\mathcal{R}})^{s/\nu}(\phi * \kappa)\|_{L^2(G)} \leq \|(I + \hat{\mathcal{R}})^{-\frac{m}{\nu}}\sigma\|_{L^\infty(\hat{G})} \|(I + \tilde{\mathcal{R}})^{\frac{s+m}{\nu}}\phi\|_{L^2(G)}.$$

The properties of the Sobolev spaces (see Proposition 2.27) imply

$$\|(I + \tilde{\mathcal{R}})^{\frac{s+m}{\nu}}\phi\|_{L^2(G)} \lesssim \sum_{|\alpha| \leq s'} \|\tilde{X}^\alpha \phi\|_{L^2(G)} \lesssim \sum_{|\alpha| \leq p_\sigma} \|\partial^\alpha \phi\|_{L^\infty(G)},$$

having chosen  $s$  very close to  $Q/2$  and  $s'$  and  $p_\sigma$  as in the statement, with the last implicit constant depending on the support of  $\phi$ . This shows Part (3) for  $m \geq -Q$ .  $\square$

*Remark 2.39.* The proof of Corollary 2.38 implies in particular that the constant  $C$  and the constants describing the continuity of  $\sigma \mapsto \|\sigma\|_{\alpha, \beta, p}$  depend on the Lie structure only via  $\|[\cdot, \cdot]\|_{\mathbb{V}}$  after a choice of a basis  $\mathbb{V}$  adapted to the gradation and the choice of the Rockland operator  $\mathcal{R} = \mathcal{R}_{\mathbb{V}}$ , beside  $m, \alpha, \beta, p$  for the first part and  $m, N, M_1$  for the second.

The semi-norms  $\|\cdot\|_{\alpha, \beta, p}$  defined in Corollary 2.38 may not generate the topology on  $S^m(\hat{G})$ , but they have the advantage of being homogeneous in the following sense. If  $\sigma = \mathcal{F}_G \kappa$  then  $\sigma(r \cdot \pi) = \mathcal{F}_G \kappa^{(r)}$  with  $\kappa^{(r)}(x) = r^{-Q} \kappa(r^{-1}x)$  and we have

$$\|\sigma(r \cdot)\|_{\alpha, \beta, p} = r^{Q(\frac{1}{p}-1) - [\beta] + [\alpha]} \|\sigma\|_{\alpha, \beta, p}.$$

Moreover, the inequality in the second part of Corollary 2.38 together with judicious expansions are often used to recover the membership to  $S^m(\hat{G})$  as well as other properties.

**2.4.5. Spectral multipliers.** Another important example of classes of symbols are given by the multipliers in the symbols of a positive Rockland operator.

*Notation 2.40.* Let  $\mathcal{G}^m(\mathbb{R})$  be the space of smooth functions  $\phi : \mathbb{R} \rightarrow \mathbb{C}$  growing at rate  $m \in \mathbb{R}$  in the sense that

$$\forall k \in \mathbb{N}_0, \quad \exists C = C_{k, \phi}, \quad \forall \lambda \in \mathbb{R}, \quad |\partial_\lambda^k \phi(\lambda)| \leq C(1 + |\lambda|)^{m-k}.$$

The set  $\mathcal{G}^m(\mathbb{R})$  is a Fréchet space when equipped with the semi-norms given by

$$\|\phi\|_{\mathcal{G}^m, N} := \max_{k=0, \dots, N} \sup_{\lambda \in \mathbb{R}} (1 + |\lambda|)^{-m+k} |\partial_\lambda^k \phi(\lambda)|, \quad N \in \mathbb{N}_0.$$

Many spectral multipliers in positive Rockland operators are in our symbol classes [37, Proposition 5.3.4]:

**Theorem 2.41.** *Let  $\phi \in \mathcal{G}^m(\mathbb{R})$  and let  $\mathcal{R}$  be a positive Rockland operator on  $G$ . Then  $\phi(\widehat{\mathcal{R}}) \in S^{m\nu}(\widehat{G})$  where  $\nu$  is the homogeneous degree of  $\mathcal{R}$ . Moreover, the map*

$$\mathcal{G}^m(\mathbb{R}) \ni \phi \mapsto \phi(\widehat{\mathcal{R}}) \in S^{m\nu}(\widehat{G})$$

*is continuous.*

*Remark 2.42.* Again, the proof of the theorem shows that the constants describing the continuity of the linear map  $\mathcal{G}^m(\mathbb{R}) \ni \phi \mapsto \phi(\widehat{\mathcal{R}}) \in S^{m\nu}(\widehat{G})$  depend on the Lie structure only via  $\|[\cdot, \cdot]\|_{\mathbb{V}}$  after a choice of a basis  $\mathbb{V}$  adapted to the gradation.

## 2.5. Smoothing symbols.

**Definition 2.43.** The class of *smoothing symbols*

$$S^{-\infty}(\widehat{G}) := \bigcap_{m \in \mathbb{R}} S^m(\widehat{G}).$$

is equipped with the induced topology of projective limit.

The topology is well defined thanks to the continuous inclusions (2.8).

Then, the results recalled above, or equivalently [37, Lemme 5.4.13 and Theorem 5.4.9], imply the following:

**Proposition 2.44.** *The group Fourier transform is an isomorphism between the topological vector spaces  $\mathcal{S}(G)$  and  $S^{-\infty}(\widehat{G})$ .*

*Remark 2.45.* Again, the proof of Proposition 2.44 shows that the constants describing the continuity of the linear maps  $\mathcal{F}_G : \mathcal{S}(G) \rightarrow S^{-\infty}(\widehat{G})$  depend on the Lie structure only via  $\|[\cdot, \cdot]\|_{\mathbb{V}}$  after a choice of a basis  $\mathbb{V}$  adapted to the gradation.

In many circumstances, it will be useful to handle the convolution kernels of symbols as if they were smooth functions and not distributions. The following property will allow us to do so.

We fix a positive Rockland operator  $\mathcal{R}$  and a function  $\chi \in \mathcal{D}(\mathbb{R})$  valued in  $[0, 1]$  and with  $\chi = 1$  on  $[0, 1]$ . We define the symbols

$$(2.9) \quad \sigma_\ell := \sigma \chi(\ell^{-1}\widehat{\mathcal{R}}), \quad \ell \in \mathbb{N}.$$

By Theorem 2.41, the symbols  $\chi(\ell^{-1}\widehat{\mathcal{R}})$  are smoothing, and so is  $\sigma_\ell$  by the algebraic symbolic properties of Theorem 2.33 (2).

**Proposition 2.46.** *Let  $\sigma \in S^m(\widehat{G})$  with  $m \in \mathbb{R}$ . The sequence of smoothing symbols  $(\sigma_\ell)_{\ell \in \mathbb{N}} \subset S^{-\infty}(\widehat{G})$  defined in (2.9) satisfies the following properties:*

(1) *For any  $m_1 > m$ , we have the convergence*

$$\lim_{\ell \rightarrow \infty} \sigma_\ell = \sigma \quad \text{in } S^{m_1}(\widehat{G}).$$

*In other words, the space  $S^{-\infty}(\widehat{G})$  equipped with the  $S^{m_1}(\widehat{G})$ -topology is dense in  $S^m(\widehat{G})$ .*

(2) For any semi-norm  $\|\cdot\|_{S^m(\widehat{G}),N}$  of  $S^m(\widehat{G})$ , there exist a constant  $C > 0$  and  $N' \geq N$  such that

$$\forall \ell \in \mathbb{N}, \quad \|\sigma_\ell - \sigma\|_{S^m(\widehat{G}),N} \leq C \|\sigma\|_{S^m(\widehat{G}),N'}.$$

Consequently,

$$\forall \ell \in \mathbb{N}, \quad \|\sigma_\ell\|_{S^m(\widehat{G}),N} \leq (1 + C) \|\sigma\|_{S^m(\widehat{G}),N'}.$$

(3) Denoting by  $\kappa$  and  $\kappa_\ell$  the convolution kernels of  $\sigma$  and  $\sigma_\ell$  respectively, we have the convergences  $\lim_{\ell \rightarrow \infty} \kappa_\ell = \kappa$  in  $\mathcal{S}'(G)$  and in  $C^\infty(U)$  for any open subset  $U \subset G \setminus \{0\}$ .

(4) We have the following relation between  $S^m(\widehat{G})$ -semi-norms:

$$\liminf_{\ell \rightarrow \infty} \|\sigma_\ell\|_{S^m(\widehat{G}),N} \geq \|\sigma\|_{S^m(\widehat{G}),N}.$$

*Proof of Proposition 2.46.* Parts 1 and 2. By the symbolic properties of symbols in Theorem 2.33 (2), given a semi-norm  $\|\cdot\|_{S^{m_1}(\widehat{G}),N}$ , there exist  $C > 0$  and  $N'', N' \geq N$  such that

$$\|\sigma - \sigma_\ell\|_{S^{m_1}(\widehat{G}),N} = \left\| \sigma (1 - \chi)(\ell^{-1}\widehat{\mathcal{R}}) \right\|_{S^{m_1}(\widehat{G}),N} \leq C \|\sigma\|_{S^m(\widehat{G}),N'} \|(1 - \chi)(\ell^{-1}\widehat{\mathcal{R}})\|_{S^{m_1-m}(\widehat{G}),N''}.$$

If  $m_1 \geq m$ , by Theorem 2.41, we have for some  $N_1 \in \mathbb{N}$

$$\|(1 - \chi)(\ell^{-1}\widehat{\mathcal{R}})\|_{S^{m_1-m}(\widehat{G}),N''} \lesssim \|(1 - \chi)(\ell^{-1}\cdot)\|_{\mathcal{G}^{\frac{m_1-m}{\nu}},N_1},$$

and we check

$$\forall \ell \in \mathbb{N}, \quad \|(1 - \chi)(\ell^{-1}\cdot)\|_{\mathcal{G}^{\frac{m_1-m}{\nu}},N_1} \lesssim \ell^{\frac{m-m_1}{\nu}}.$$

This shows Part (1) and the first estimate in Part (2), the second being obtained by the triangle inequality.

Part 3. The kernel estimates in Theorem 2.36 and the estimate of the  $S^{m_1}(\widehat{G})$ -semi-norm of  $\sigma - \sigma_\ell$  for  $m_1 > m$  gives the convergence in  $C^\infty(U)$  for any open subset  $U$  of  $G \setminus \{0\}$ . Recall that  $\mathcal{F}^{-1}S^{m_1}(\widehat{G}) \subset \mathcal{K}_{m_1,0}$  and that the inclusion  $\mathcal{K}_{a,b} \subset \mathcal{S}'(G)$  is continuous [37, Proposition 5.1.17(3)]. This and Part (1) imply the convergence  $\kappa_\ell \rightarrow \kappa$  in  $\mathcal{S}'(G)$ .

Part 4. To prove Part (4), we observe that by functional analysis, we have

$$\begin{aligned} \|(I + \widehat{\mathcal{R}})^{-\frac{m}{\nu}} \sigma\|_{L^\infty(\widehat{G})} &\geq \|(I + \widehat{\mathcal{R}})^{-\frac{m}{\nu}} \sigma \chi(\ell^{-1}\widehat{\mathcal{R}})\|_{L^\infty(\widehat{G})} \\ &\geq \sup_{\substack{\pi \in \widehat{G} \\ |\mathcal{H}_\pi| = 1}} \sup_{v \in 1_{[0,\ell]}(\pi(\mathcal{R}))} |(I + \pi(\mathcal{R}))^{-\frac{m}{\nu}} \sigma(\pi)v|_{\mathcal{H}_\pi}, \end{aligned}$$

so,

$$(2.10) \quad \lim_{\ell \rightarrow \infty} \|(I + \widehat{\mathcal{R}})^{-\frac{m}{\nu}} \sigma \chi(\ell^{-1}\widehat{\mathcal{R}})\|_{L^\infty(\widehat{G})} = \|(I + \widehat{\mathcal{R}})^{-\frac{m}{\nu}} \sigma\|_{L^\infty(\widehat{G})}.$$

For  $|\alpha| = 1$ , the Leibniz property implies

$$\Delta_\alpha \sigma_\ell = (\Delta_\alpha \sigma) \chi(\ell^{-1}\widehat{\mathcal{R}}) + \sigma(\Delta_\alpha \chi(\ell^{-1}\widehat{\mathcal{R}})),$$

so

$$\|(I + \widehat{\mathcal{R}})^{-\frac{m-|\alpha|}{\nu}} \Delta_\alpha \sigma_\ell\|_{L^\infty(\widehat{G})} \geq E_1 - E_2,$$

where

$$\begin{aligned} E_1 &:= \|(I + \widehat{\mathcal{R}})^{-\frac{m-[\alpha]}{\nu}} (\Delta_\alpha \sigma) \chi(\ell^{-1} \widehat{\mathcal{R}})\|_{L^\infty(\widehat{G})} \\ E_2 &:= \|(I + \widehat{\mathcal{R}})^{-\frac{m-[\alpha]}{\nu}} \sigma (\Delta_\alpha \chi(\ell^{-1} \widehat{\mathcal{R}}))\|_{L^\infty(\widehat{G})} \\ &\lesssim \|\sigma\|_{S^{m, N_1}} \|\chi(\ell^{-1} \widehat{\mathcal{R}})\|_{S^{[\alpha]}(\widehat{G}), N_2} \lesssim_{\sigma, \chi} \ell^{-\frac{[\alpha]}{\nu}}, \end{aligned}$$

for some integers  $N_i$ 's, having used the symbolic properties for composition and for spectral multipliers (see Theorems 2.33 and 2.41). Applying (2.10) to  $E_1$ , we obtain,

$$\liminf_{\ell \rightarrow \infty} \|(I + \widehat{\mathcal{R}})^{-\frac{m-[\alpha]}{\nu}} \Delta_\alpha \sigma_\ell\|_{L^\infty(\widehat{G})} \geq \|(I + \widehat{\mathcal{R}})^{-\frac{m-[\alpha]}{\nu}} (\Delta_\alpha \sigma)\|_{L^\infty(\widehat{G})},$$

for  $|\alpha| = 1$ . The same ideas and method yield inductively the case of any index  $\alpha$ , proving Part (4).  $\square$

*Remark 2.47.* Again, the proof of Proposition 2.46 shows that the constants involved in the statement depend on the Lie structure only via  $\|[\cdot, \cdot]\|_{\mathbb{V}}$  after a choice of a basis  $\mathbb{V}$  adapted to the gradation.

## 2.6. Homogeneous and polyhomogeneous symbol classes.

2.6.1. *Homogeneous classes of symbols.* Recall that the dilations act on  $\widehat{G}$  via

$$(2.11) \quad r \cdot \pi(x) = \pi(rx), \quad \pi \in \widehat{G}, \quad x \in G, \quad r > 0.$$

**Definition 2.48.** An invariant symbol  $\sigma$  on  $\widehat{G}$  is  $m$ -homogeneous when  $\sigma(r \cdot \pi) = r^m \sigma(\pi)$  for almost all  $\pi \in \widehat{G}$  and  $r > 0$ .

*Example 2.49.* Let  $\mathcal{R}$  be a positive Rockland operator. We denote by  $\nu$  its homogeneous degree. For any  $s \in \mathbb{R}$ , the properties of homogeneous Sobolev spaces (Section 2.3.3) imply that the field of spectrally defined operators  $\widehat{\mathcal{R}}^{m/\nu} = \{\pi(\mathcal{R})^{m/\nu} : \pi \in \widehat{G}\}$  is a well-defined invariant symbol, and we check readily that it is  $m$ -homogeneous.

We define the following subspace of invariant symbols

$$\sigma \in \dot{L}_{a,b}^\infty(\widehat{G}) \quad \text{if and only if} \quad \sigma_{a,b} := \widehat{\mathcal{R}}^{b/\nu} \sigma \widehat{\mathcal{R}}^{-a/\nu} \in L^\infty(\widehat{G});$$

by this, we mean that there exists a unique element  $\sigma_{a,b}$  of  $L^\infty(\widehat{G})$  that is composable on the left and the right by  $\widehat{\mathcal{R}}^{-b/\nu}$  and  $\widehat{\mathcal{R}}^{a/\nu}$  respectively such that  $\sigma = \widehat{\mathcal{R}}^{-b/\nu} \sigma_{a,b} \widehat{\mathcal{R}}^{a/\nu}$ . Equipped with the norm

$$\|\sigma\|_{L_{a,b}^\infty(\widehat{G}), \mathcal{R}} := \|\widehat{\mathcal{R}}^{b/\nu} \sigma \widehat{\mathcal{R}}^{-a/\nu}\|_{L^\infty(\widehat{G})},$$

the space  $\dot{L}_{a,b}^\infty(\widehat{G})$  becomes a Banach space.

Since the inclusion  $\dot{L}_s^2(G) \subset \mathcal{S}'(G)$  is continuous for any  $s \in \mathbb{R}$ , by the Schwartz kernel theorem, any symbol in  $L_{0,s}^\infty(\widehat{G})$  has a convolution kernel; more generally, if  $M_0 \in \mathbb{N}_0$  is a multiple of the dilation's weights, any symbol in  $L_{M_0,s}^\infty(\widehat{G})$  has a convolution kernel. We can therefore apply at least formally a difference operator to these symbols. This allows us to consider the following definition:

**Definition 2.50.** An  $m$ -homogeneous symbol  $\sigma$  is said to be *regular* when  $\sigma \in \dot{L}_{0,-m}^\infty(\widehat{G})$  and furthermore  $\Delta_{x^\alpha} \sigma \in \dot{L}_{0,-m+[\alpha]}^\infty(\widehat{G})$  for all  $\alpha \in \mathbb{N}_0^n$ . We denote by  $\dot{S}^m(\widehat{G})$  the space of  $m$ -homogeneous regular symbols.

The space  $\dot{S}^m(\widehat{G})$  is Fréchet when equipped with the semi-norms given by

$$(2.12) \quad \|\sigma\|_{\dot{S}^m(\widehat{G}),N,\mathcal{R}} := \max_{[\alpha] \leq N} \|\Delta_{x^\alpha} \sigma\|_{\dot{L}_{0,-m+[\alpha]}^\infty(\widehat{G}),\mathcal{R}},$$

having fixed a positive Rockland operator  $\mathcal{R}$ .

In [29], we proved the following property. We shall say that a function  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $\psi \equiv 1$  on a neighbourhood of  $+\infty$  when there exists  $\Lambda > 0$  such that  $\psi \equiv 1$  on  $(\Lambda, +\infty)$ .

**Proposition 2.51.** *Let  $\sigma$  be a  $m$ -homogeneous invariant symbol on  $\widehat{G}$ . The following properties are equivalent:*

- (1) For all  $\alpha \in \mathbb{N}_0^n$ ,  $\Delta_{x^\alpha} \sigma \in \dot{L}_{0,-m+[\alpha]}^\infty(\widehat{G})$ .
- (2) For all  $\gamma \in \mathbb{R}$  and  $\alpha \in \mathbb{N}_0^n$ ,  $\Delta_{x^\alpha} \sigma \in \dot{L}_{\gamma,-m+[\alpha]+\gamma}^\infty(\widehat{G})$ .
- (3) For one (and then any)  $\psi \in C^\infty(\mathbb{R})$  with  $\psi \equiv 0$  on a neighbourhood of 0 and  $\psi \equiv 1$  on a neighbourhood of  $+\infty$  and for one (and then any) positive Rockland operator  $\mathcal{R}$ ,  $\psi(\widehat{\mathcal{R}}) \sigma \in S^m(\widehat{G})$  and  $\sigma \psi(\widehat{\mathcal{R}}) \in S^m(\widehat{G})$ .

In this case, the following map

$$\sigma \longmapsto \Delta_{x^\alpha} \sigma, \quad \dot{S}^0(\widehat{G}) \longrightarrow \dot{L}_{\gamma,-m+[\alpha]+\gamma}^\infty(\widehat{G}),$$

is continuous for each  $\gamma \in \mathbb{R}$  and  $\alpha \in \mathbb{N}_0^n$ . And in the case  $m = 0$ , the maps

$$\sigma \longmapsto \psi(\widehat{\mathcal{R}})\sigma, \quad \text{and} \quad \sigma \longmapsto \sigma\psi(\widehat{\mathcal{R}}),$$

are also continuous  $\dot{S}^0(\widehat{G}) \rightarrow S^0(\widehat{G})$  and injective.

*Remark 2.52.* (1) An improved presentation of the proof showing that the property in Proposition 2.51 (3) is independent of a choice of a positive Rockland operator  $\mathcal{R}$  and a function  $\psi$  is obtained by considering the proof in Section 5.5.2.

- (2) The constants describing the continuity properties in Proposition 2.51 depend on the Lie structure only via  $\|\cdot, \cdot\|_{\mathbb{V}}$  after a choice of a basis  $\mathbb{V}$  adapted to the gradation.

In [29], we also showed that the classes of regular homogeneous symbols enjoy the algebraic properties for composition and adjoint as described for the inhomogeneous case in Theorem 2.33. Furthermore, a remark regarding constants similar to Remark 2.34 is true.

*Example 2.53.* If  $\mathbb{V}$  is an adapted basis of  $\mathfrak{g}$ , then for any  $\alpha \in \mathbb{N}_0^n$ , the symbol

$$\widehat{\mathbb{V}}^\alpha = \{\pi(\mathbb{V})^\alpha, \pi \in \widehat{G}\}$$

is in  $\dot{S}^{[\alpha]}(\widehat{G})$ .

### 2.6.2. Polyhomogeneous expansion and symbol classes.

**Definition 2.54.** A symbol  $\sigma \in S^m(\widehat{G})$  admits the polyhomogeneous expansion

$$\sigma \sim_h \sum_{j=0}^{\infty} \sigma_{m-j}$$

when each  $\sigma_{m-j}$  is homogeneous and in  $\dot{S}^m(\widehat{G})$ , and  $\sigma$  admits the inhomogeneous expansion (in the sense of Definition 2.35)

$$\sigma \sim \sum_{j=0}^{\infty} \sigma_{m-j} \psi(\widehat{\mathcal{R}}),$$

for some positive Rockland operator  $\mathcal{R}$  and  $\psi \in C^\infty(\mathbb{R})$  with  $\psi \equiv 0$  on a neighbourhood of 0 and  $\psi \equiv 1$  on a neighbourhood of  $+\infty$ .

By the symbolic calculus on groups, this definition is independent of the choice of the Rockland operator  $\mathcal{R}$  (see [29, Proposition 4.6 ] or Proposition 2.51 above) and we have an equivalent definition considering  $\sum_{j=0}^{\infty} \psi(\widehat{\mathcal{R}}) \sigma_{m-j}$ .

Unlike the expansion of a general symbol of order  $m$  (in the sense of Definition 2.35) that may be perturbed by any term of lower order, if  $\sigma \in S^m(\widehat{G})$  admits a polyhomogeneous expansion  $\sigma \sim \sum_{j=0}^{\infty} \sigma_{m-j} \psi(\widehat{\mathcal{R}})$ , then the homogenous symbols  $\sigma_{m-j}$ ,  $j = 0, 1, 2, \dots$  are unique. This follows readily from the injectivity of the maps mentioned in Proposition 2.51.

**Definition 2.55.** The polyhomogeneous symbol classes of order  $m \in \mathbb{R}$  is the space  $S_{ph}^m(\widehat{G})$  of symbols  $\sigma \in S^m(\widehat{G})$  admitting a polyhomogeneous expansion.

The properties of the homogeneous and inhomogeneous symbol classes imply that the product of a symbol in  $S_{ph}^{m_1}(\widehat{G})$  with a symbol in  $S_{ph}^{m_2}(\widehat{G})$  is in  $S_{ph}^{m_1+m_2}(\widehat{G})$ . Similarly, the adjoint of a symbol in  $S_{ph}^m(\widehat{G})$  is in  $S_{ph}^m(\widehat{G})$ .

*Example 2.56.* For any positive Rockland operator  $\mathcal{R}$ , we have  $(I + \widehat{\mathcal{R}})^{m/\nu} \in S_{ph}^m(\widehat{G})$  where  $\nu$  is the homogeneous degree of  $\mathcal{R}$ . Indeed, if  $\psi \in C^\infty(\mathbb{R})$  satisfies  $\psi \equiv 0$  on a neighbourhood of 0 and  $\psi \equiv 1$  on a neighbourhood of  $+\infty$ , by spectral calculus

$$\begin{aligned} (I + \widehat{\mathcal{R}})^{m/\nu} \psi(\widehat{\mathcal{R}}) &= (I + \widehat{\mathcal{R}}^{-1})^{m/\nu} \psi(\widehat{\mathcal{R}}) \widehat{\mathcal{R}}^{m/\nu} \\ &\sim \widehat{\mathcal{R}}^{m/\nu} \psi(\widehat{\mathcal{R}}) + \frac{m}{\nu} \widehat{\mathcal{R}}^{-1+m/\nu} \psi(\widehat{\mathcal{R}}) + \dots, \end{aligned}$$

having used the Taylor series of  $(1+x)^{m/\nu}$ .

**2.7. Difference operators and Leibniz property.** An important property of the difference operator  $\Delta_{x^\alpha}$  introduced in Definition 2.29 is that they satisfy a form of Leibniz property which is a direct consequence of properties regarding monomials:

**Lemma 2.57.** *Let  $G$  be a graded Lie group and  $\mathbb{V}$  an adapted basis.*

(1) For any  $\alpha \in \mathbb{N}_0^n$ , we have the following relations between  $\mathbb{V}$ -monomials:

$$(xy)^\alpha = \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1, \alpha_2}^{(\alpha)} x^{\alpha_1} y^{\alpha_2}$$

for unique coefficients  $c_{\alpha_1, \alpha_2}^{(\alpha)} \in \mathbb{R}$  satisfying

$$c_{\alpha_1, 0}^{(\alpha)} = \begin{cases} 1 & \text{if } \alpha_1 = \alpha, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad c_{0, \alpha_2}^{(\alpha)} = \begin{cases} 1 & \text{if } \alpha_2 = \alpha, \\ 0 & \text{otherwise} \end{cases}$$

Moreover, the coefficients  $c_{\alpha_1, \alpha_2}^{(\alpha)}$  are universal polynomial expressions in the structural constants of  $G$  for  $\mathbb{V}$ , i.e. in the constant  $c_{i,j,k}$  defined via  $[V_i, V_k] = \sum_k c_{i,j,k} V_k$ .

(2) For any  $\sigma_1, \sigma_2 \in S^{-\infty}(\widehat{G})$ ,

$$\Delta_{x^\alpha}(\sigma_1 \sigma_2) = \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1, \alpha_2}^{(\alpha)} \Delta_{x^{\alpha_1}} \sigma_1 \Delta_{x^{\alpha_2}} \sigma_2.$$

*Proof.* Part (1) follows from [37, Proposition 5.2.3 (4)]. For Part (2), the convolution kernel of  $\sigma_1 \sigma_2$  is  $\kappa_2 * \kappa_1$  where  $\kappa_i$  is the convolution kernel of  $\sigma_i$ ,  $i = 1, 2$ . We have

$$\begin{aligned} x^\alpha \kappa_2 * \kappa_1(x) &= \int_G (y(y^{-1}x))^\alpha \kappa_2(y) \kappa_1(y^{-1}x) dy \\ &= \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1, \alpha_2}^{(\alpha)} \int_G y^{\alpha_1} \kappa_2(y) (y^{-1}x)^{\alpha_2} \kappa_1(y^{-1}x) dy \end{aligned}$$

The conclusion follows by taking the group Fourier transform.  $\square$

From the definition of the symbol classes, it follows that  $\Delta_{x^\alpha}$  is a continuous operator  $S^m(\widehat{G}) \rightarrow S^{m-[\alpha]}(\widehat{G})$  for any  $m \in \mathbb{R} \cup \{-\infty\}$  and any  $\alpha \in \mathbb{N}_0^n$ . But we can also show the continuity of any difference operator as defined in Definition 2.29:

**Proposition 2.58.** *Let  $q \in C_c^\infty(G)$ . Then  $\Delta_q$  is a continuous operator  $S^m(\widehat{G}) \rightarrow S^m(\widehat{G})$  for any  $m \in \mathbb{R} \cup \{-\infty\}$ . Moreover, writing the Taylor series of  $q$  at 0 as  $\mathbb{T}_0 q(x) \sim \sum_\alpha c_\alpha x^\alpha$ , the map*

$$\sigma \longmapsto \Delta_q \sigma - \sum_{[\alpha] \leq N} c_\alpha \Delta_{x^\alpha} \sigma$$

is continuous  $S^m(\widehat{G}) \rightarrow S^{m-(N+1)}(\widehat{G})$  for any  $N \in \mathbb{N}_0$ .

*Proof of Proposition 2.58.* By Corollary 2.38, it suffices to prove that for any  $m \in \mathbb{R}$ , for any multi-indices  $\alpha_0, \beta_0 \in \mathbb{N}_0^n$ , there exists  $N_0 \in \mathbb{N}_0$  such that for any  $N \in \mathbb{N}_0$  with  $N \geq N_0$ , we have

$$\forall \sigma \in S^{-\infty}(\widehat{G}), \quad \|\Delta_q \sigma - \sum_{[\alpha] \leq N} c_\alpha \Delta_{x^\alpha} \sigma\|_{\alpha_0, \beta_0, 1} \leq C_{q, N, \alpha_0, \beta_0} \|\sigma\|_{S^m(\widehat{G}), N', \mathcal{R}_V},$$

for some (large) integer  $N' \in \mathbb{N}_0$  depending on  $N$  and other constants but not  $\sigma$ . Moreover, by modifying  $\sigma$  and  $m$ , it suffices to show the case  $\alpha_0 = \beta_0 = 0$ . Hence, we are led to analyse

$$\|\Delta_q \sigma - \sum_{[\alpha] \leq N} c_\alpha \Delta_{x^\alpha} \sigma\|_{0, 0, 1} = \|q\kappa - \sum_{[\alpha] \leq N} c_\alpha x^\alpha \kappa\|_{L^1(G)}.$$

We fix a function  $\chi \in C_c^\infty(G)$  such that  $\chi = 1$  on a neighbourhood of  $\text{supp } q$  and of 0. Let  $\kappa \in \mathcal{S}(G)$ . We may write

$$q\kappa - \sum_{[\alpha] \leq N} c_\alpha x^\alpha \kappa = \rho_1 + \rho_2, \quad \text{where } \rho_1 := \kappa\chi(q - \mathbb{P}_{G,q,0,N}), \quad \rho_2 := (\chi - 1)\kappa\mathbb{P}_{G,q,0,N}.$$

As  $\kappa$  is Schwartz away from 0 (see Theorem 2.36), the  $L^1$ -norm of  $\rho_2$  is finite. When choosing  $N_0$  large enough, this is also the case for  $\rho_1$  from the kernel estimates near 0 (see Theorem 2.36) together with Taylor's estimate (see Theorem 2.17). Moreover, quantitatively, each  $\|\rho_i\|_{L^1(G)}$ ,  $i = 1, 2$ , is bounded up to a constant by an  $S^m(\widehat{G})$ -semi-norm in  $\sigma$ . The conclusion follows.  $\square$

*Remark 2.59.* Again, the proof of Proposition 2.58 shows that the constants describing the continuity of the linear map above depend on the Lie structure only via  $\|[\cdot, \cdot]\|_{\mathbb{V}}$  after a choice of a basis  $\mathbb{V}$  adapted to the gradation.

A direct consequence of Proposition 2.58 is the following asymptotic expansion:

$$\Delta_q \sigma \sim \sum_{[\alpha] \leq N} c_\alpha \Delta_{x^\alpha} \sigma.$$

**2.8. Transformation of symbols by morphisms.** By a *morphism*  $\theta$  of the graded nilpotent Lie group, we mean a morphism of the nilpotent Lie group whose associated Lie algebra morphism (for which we keep the same notation) respects the gradation of the Lie algebra in the sense that it maps each  $\mathfrak{g}_j$  on itself. Using the notation of Lemma 2.21, we have:

**Proposition 2.60.** *Let  $\theta$  be an automorphism of the graded nilpotent Lie group  $G$ . If  $\sigma \in S^m(\widehat{G})$ , then  $\theta_*\sigma \in S^m(\widehat{G})$  while if  $\sigma \in \dot{S}^m(\widehat{G})$ , then  $\theta_*\sigma \in \dot{S}^m(\widehat{G})$ . Moreover, the map  $\sigma \mapsto \theta_*\sigma$  is an automorphism of the topological vector space  $S^m(\widehat{G})$  and also of the topological vector space  $\dot{S}^m(\widehat{G})$ .*

As a consequence of the continuity of  $\sigma \mapsto \theta_*\sigma$ , once we have chosen a Rockland operator  $\mathcal{R}$  and  $N \in \mathbb{N}$ , there exists a constant  $C > 0$  such that for all  $\sigma \in S^m(\widehat{G})$ ,

$$\|\theta_*\sigma\|_{S^m(\widehat{G}),N,\mathcal{R}} \leq C \|\sigma\|_{S^m(\widehat{G}),N,\mathcal{R}}.$$

The proof below will show that the constant  $C$  depends on various quantities of the group.

*Remark 2.61.* Assume as in Remark 2.28 that the gradation of the vector space underlying  $\mathfrak{g}$  as well as an adapted basis  $\mathbb{V}$  are fixed. Then, the constant  $C$  may be chosen as

$$C = \widetilde{C}(\max(s, \|[\cdot, \cdot]_{\mathfrak{g}}\|_{\mathbb{V}}, C_{\mathbb{V}}(\mathcal{R}), \|\theta\|_{\mathcal{L}(\mathfrak{g})})$$

where  $\widetilde{C} : [0, \infty) \rightarrow (0, \infty)$  is an increasing function of the structural constant  $\|[\cdot, \cdot]\|_{\mathbb{V}}$ , of the constant  $C_{\mathbb{V}}(\mathcal{R})$  introduced in Remark 2.28, and of the norm of  $\theta$  as a linear map of  $\mathfrak{g}$ .

*Proof.* We keep the same notation  $\theta$  for the automorphism of the graded nilpotent Lie group  $G$ , and the corresponding automorphisms of the Lie algebra  $\mathfrak{g}$  and of its enveloping algebra  $\mathcal{U}(\mathfrak{g})$ . We observe that if  $\mathcal{R}$  is a Rockland operator viewed as an element of

$\mathcal{U}(\mathfrak{g})$ , then  $\mathcal{R}_\theta := \theta(\mathcal{R})$  is also a Rockland operator since  $\pi(\mathcal{R}_\theta) = \pi_1(\mathcal{R})$  with  $\pi_1 = \pi \circ \theta$  for any  $\pi \in \widehat{G}$ . For instance, if  $\mathcal{R}$  is as in Example 2.26,

$$\mathcal{R} = \sum_j (-1)^{\frac{M_0}{v_j}} V_j^{2\frac{M_0}{v_j}}, \quad \text{then} \quad \mathcal{R}_\theta = \sum_j (-1)^{\frac{M_0}{v_j}} (\theta(V_j))^{2\frac{M_0}{v_j}}.$$

If in addition of being Rockland,  $\mathcal{R}$  is also positive, then  $\pi(\mathcal{R}_\theta) = \pi_1(\mathcal{R})$  is positive for any  $\pi \in \widehat{G}$ , so  $\mathcal{R}_\theta$  is also positive. We check readily

$$\|\theta_*\sigma\|_{L_{a,b}^\infty(\widehat{G}),\mathcal{R}} = \|\sigma\|_{L_{a,b}^\infty(\widehat{G}),\mathcal{R}_\theta} \quad \text{and} \quad \Delta_q(\theta_*\sigma) = \theta_*(\Delta_{q\circ\theta^{-1}}\sigma).$$

As  $x_j \circ \theta^{-1}$  is a linear combination of  $x_k$ 's of the same degree as  $x_j$ , we have

$$\|\Delta^\alpha \theta_*\sigma\|_{L_{a,b}^\infty(\widehat{G}),\mathcal{R}} \asymp \sum_{[\beta]=[\alpha]} \|\Delta^\alpha \sigma\|_{L_{a,b}^\infty(\widehat{G}),\mathcal{R}_\theta}.$$

We deduce the relation about the semi-norms in the inhomogeneous case. The homogeneous case is similar.  $\square$

### 3. FILTERED MANIFOLDS

In this section, we recall properties of filtered manifolds, and discuss their frames, geometric exponential maps, differential operators, and group exponentiation.

#### 3.1. Filtered manifolds and their osculating bundles.

**Definition 3.1.** A *filtered manifold* is a smooth manifold  $M$  whose tangent bundles admits a nested sequence of sub-bundles  $\{0\} = H^0 \subseteq H^1 \subseteq \dots \subseteq H^r = TM$  that respects the Lie bracket of vector fields:

$$[\Gamma(H^i), \Gamma(H^j)] \subseteq \Gamma(H^{i+j}), \quad 0 \leq i, j \leq r,$$

with the convention  $H^\ell = H^r$  whenever  $\ell \geq r$ .

By convention, in this paper, a smooth manifold is also assumed to be second countable, paracompact Hausdorff.

*Example 3.2.* A fundamental example consists in a manifold  $M$  equipped with a distribution  $\mathcal{D}$  generating  $TM$  in the sense that any vector field may be obtained by iterated commutator brackets of elements of  $\Gamma(H_1)$  and that the filtration is obtained as follows. We set  $H_1 := \mathcal{D}$ , and for each  $j > 1$ ,  $H_j$  the subspace of  $TM$  such that  $\Gamma(H_j)$  is the  $C^\infty(M)$ -module generated by  $[V_1, [V_2, \dots, [V_\ell, \dots]] \dots]$ , with  $V_1, \dots, V_\ell \in \Gamma(H_1)$ ,  $\ell = 2, \dots, j$ . When the rank  $\dim H_j$  of the vector bundle  $H_j$  is constant for each  $j$ ,  $M$  is a filtered manifold. When  $H_1$  generates  $TM$ ,  $r = 2$  and  $\dim H_2 = 1$ , the manifold is *contact*. When  $H_1$  generates  $TM$  and is equipped with a metric  $g$ ,  $M$  is an *equivariable subRiemannian manifold*  $(M, H_1, g)$  (references about subRiemannian manifold include [1] for example).

From now on, we fix a filtered manifold  $M$  as in Definition 3.1. We set

$$n = \dim M.$$

For each  $x \in M$ , the vector space

$$\mathrm{gr}(T_x M) := \bigoplus_{i \geq 1} (H_x^i / H_x^{i-1})$$

is naturally graded and the gradation of the underlying vector space is independent of  $x \in M$ . Its *weights* (see Definition 2.2) are denoted by

$$0 < v_1 \leq \dots \leq v_n,$$

counted with multiplicity, and, when counted without multiplicity,

$$0 < v_1 = w_1 < \dots < w_s = v_n,$$

where we denote by  $s$  the *step of the gradation* ( $s \leq r$ ). These weights, counted with multiplicity or not, are constant functions on  $M$ . We denote the dimension of each of these subspaces in the decomposition  $\mathrm{gr}(T_x M) = \bigoplus_{i=1}^s H^{w_i} / H^{w_i-1}$ , by

$$d_i := \dim H^{w_i} / H^{w_i-1}, \quad i = 1, \dots, s;$$

these are independent of  $x \in M$ .

For each  $x \in M$ , the direct sum of vector spaces  $\mathrm{gr}(T_x M) = \bigoplus_i (H_x^i / H_x^{i-1})$  is naturally equipped with a *Lie bracket* that we denote  $[\cdot, \cdot]_{\mathfrak{g}_x M}$ . Indeed, take  $U, V \in \mathrm{gr}(T_x M)$  and consider  $X \in \Gamma(H^{w_i})$  such that  $U = X(x) \bmod H_x^{w_i}$  and  $Y \in \Gamma(H^{w_j})$  such that  $V = Y(x) \bmod H_x^{w_j}$ . Then

$$[U, V]_{\mathfrak{g}_x M} := [X, Y]_x \bmod H_x^{w_i+w_j}$$

is independent of the choice of representatives  $X$  and  $Y$  of  $U, V \in \mathrm{gr}(T_x M)$ .

When  $\mathrm{gr}(T_x M)$  is equipped with this Lie bracket, we denote the resulting *Lie algebra*

$$\mathfrak{g}_x M = (\mathrm{gr}(T_x M), [\cdot, \cdot]_{\mathfrak{g}_x M}).$$

It is naturally graded, and the gradation of the underlying vector space is independent of  $x \in M$ .

For each  $x \in M$ , we denote by  $G_x M$  the corresponding *nilpotent Lie group*, by  $\mathcal{U}(\mathfrak{g}_x M)$  the corresponding *universal enveloping Lie algebra* to  $\mathfrak{g}_x M$ . The unions

$$GM := \bigcup_{x \in M} G_x M \quad \text{and} \quad \mathfrak{g}M := \bigcup_{x \in M} \mathfrak{g}_x M$$

are naturally equipped with a structure of smooth vector bundles; they are called the *bundles of osculating groups and Lie algebras* over  $M$ . Moreover, the *group bundle exponential map*

$$\mathrm{Exp} : \mathfrak{g}M \rightarrow GM, \quad \mathrm{Exp}_x := \mathrm{Exp}_{G_x M} : \mathfrak{g}_x M \rightarrow G_x M,$$

is smooth and bijective. We also denote by

$$\mathcal{U}(\mathfrak{g}M) := \bigcup_{x \in M} \mathcal{U}(\mathfrak{g}_x M)$$

the bundle of universal enveloping Lie algebras over  $M$ , and by

$$\mathcal{U}_N(\mathfrak{g}M) := \bigcup_{x \in M} \mathcal{U}_N(\mathfrak{g}_x M), \quad \text{and} \quad \mathcal{U}_{\leq N}(\mathfrak{g}M) := \bigcup_{x \in M} \mathcal{U}_{\leq N}(\mathfrak{g}_x M),$$

the linear sub-bundles of  $\mathcal{U}(\mathfrak{g})$  of elements of homogeneous degree equal to  $N$  and of linear combination of elements of homogeneous degree at most equal to  $N$  respectively. They can be described in terms of adapted local frames, as we will see in the next section. This yields a natural smooth structure for the vector bundles  $\mathcal{U}_N(\mathfrak{g}M)$  and  $\mathcal{U}_{\leq N}(\mathfrak{g}M)$  of finite rank. Moreover, the natural inclusions

$$\mathcal{U}_{\leq N_1}(\mathfrak{g}M) \hookrightarrow \mathcal{U}_{\leq N_2}(\mathfrak{g}M), \quad N_1 \leq N_2,$$

are smooth bundle morphisms. Hence, their direct limit

$$\mathcal{U}(\mathfrak{g}M) = \varinjlim_N \mathcal{U}_{\leq N}(\mathfrak{g}M)$$

inherits a structure of smooth vector bundle (of infinite rank).

### 3.2. Local frames adapted to the filtration.

3.2.1. *Generalities.* Recall that a frame  $\mathbb{X} := (X_1, \dots, X_n)$  on an open subset  $U$  of a manifold  $M$  yields a local trivialization of the tangent space on  $U$  set via the linear map

$$\mathbb{R}^n \longrightarrow TM, \quad v = (v_1, \dots, v_n) \longmapsto \sum_{i=1}^n v_i X_i.$$

Local frames exist on open neighbourhoods of any points of the manifold  $M$ . Although  $M$  can always be covered by such open sets equipped with frames,  $M$  may not admit global frames.

If  $\mathbb{X}$  and  $\mathbb{Y}$  are two frames on the same open subset  $U \subset M$ , then we may write

$$X_i(x) = \sum_{1 \leq k \leq n} T_{k,i}(x) Y_k(x), \quad 1 \leq i \leq n,$$

for uniquely defined smooth maps  $x \mapsto T_{k,i}(x)$  on  $U$ ,  $1 \leq k, i \leq n$ . These define the change of frame matrix  $T = T^{\mathbb{Y}, \mathbb{X}} = (T_{i,j})_{1 \leq i, j \leq n}$  as a smooth map  $U \rightarrow \text{GL}(n, \mathbb{R})$ . Viewing  $\mathbb{X}$  and  $\mathbb{Y}$  as column matrices, we have  $\mathbb{X} = T^t \mathbb{Y}$ . If  $\mathbb{Z}$  is another adapted frame on  $U$ , then  $T_x^{\mathbb{Z}, \mathbb{X}} = T_x^{\mathbb{Z}, \mathbb{Y}} T_x^{\mathbb{Y}, \mathbb{X}}$ .

3.2.2. *Frames adapted to the filtration.* When the manifold  $M$  is filtered, a frame  $\mathbb{X} := (X_1, \dots, X_n)$  of an open subset  $U \subset M$  is said to be *adapted to the filtration*  $H$  when for every  $x \in U$ ,  $X_{1,x}, \dots, X_{d_1,x}$  is a basis of  $H_x^{w_1}$ ,  $X_{1,x}, \dots, X_{d_1+d_2,x}$  is a basis of  $H_x^{w_2}$ , and so on. In this case, we say that  $\mathbb{X} = (X_1, \dots, X_n)$  is a *local adapted frame* or an *adapted frame* on  $U$ . Such local frames exist in neighbourhoods of any points of the manifold  $M$ . Thus, every smooth manifold  $M$  has a covering by open sets equipped with adapted frames.

We also associate to an adapted frame a basis of the osculating Lie algebras. More precisely, if  $\mathbb{X}$  is an adapted frame on  $U$ , we set for each  $x \in U$ ,

$$\begin{aligned} \langle X_i \rangle_x &:= X_i(x) \bmod H_x^{w_1-1}, \quad i = 1, \dots, d_1, \\ \langle X_i \rangle_x &:= X_i(x) \bmod H_x^{w_2-1}, \quad i = d_1 + 1, \dots, d_1 + d_2, \quad \text{etc.} \end{aligned}$$

Note that each  $\langle X_i \rangle_x$  has weight  $v_i$  in the graded Lie algebra  $\mathfrak{g}_x M$ , i.e.  $\delta_r \langle X_i \rangle_x = r^{v_i} \langle X_i \rangle_x$ . In other words, the basis  $\langle X_1 \rangle_x, \dots, \langle X_n \rangle_x$  is adapted to the gradation of  $\mathfrak{g}_x M$ , as defined in

Definition 2.2. Moreover, each  $\langle X_i \rangle \in \Gamma(\mathcal{W}_{v_i}(\mathfrak{g}M|U))$  is a smooth section over the bundle of osculating universal Lie algebras. More generally,

$$\langle \mathbb{X} \rangle^\alpha := \langle X_1 \rangle^{\alpha_1} \dots \langle X_n \rangle^{\alpha_n}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n,$$

defines an element of  $\Gamma(\mathcal{W}_{[\alpha]}(\mathfrak{g}M|U))$  where

$$[\alpha] := v_1 \alpha_1 + \dots + v_n \alpha_n.$$

The element of  $\Gamma(\mathcal{W}_0(\mathfrak{g}M|U))$  corresponding to  $\alpha = 0$  is  $\langle \mathbb{X} \rangle^\alpha = 1$ . Any element in  $\Gamma(\mathcal{W}(\mathfrak{g}M|U))$  may be written in a unique way as a finite linear combination over  $C^\infty(U)$  of  $\langle \mathbb{X} \rangle^\alpha$ . In other words,  $\langle \mathbb{X} \rangle^\alpha$ ,  $\alpha \in \mathbb{N}_0^n$ , is a free basis of the  $C^\infty(U)$ -module  $\Gamma(\mathcal{W}(\mathfrak{g}M|U))$ . Furthermore,  $\Gamma(\mathcal{W}_N(\mathfrak{g}M|U))$  is a submodule with basis  $\langle \mathbb{X} \rangle^\alpha$ ,  $[\alpha] = N$ .

3.2.3. *Transition between two adapted frames.* Let  $BU(d_1, \dots, d_s)$  be the group of invertible real block-upper-triangular matrices for which the  $j$ th diagonal block has size  $d_j \times d_j$ .

**Lemma 3.3.** *Let  $\mathbb{X}$  and  $\mathbb{Y}$  be two adapted frames on an open subset  $U \subset M$ .*

- (1) *The transition matrix  $T = T^{\mathbb{Y}, \mathbb{X}}$  (see Section 3.2.1) is valued in  $BU(d_1, \dots, d_s)$ .*
- (2) *We have*

$$\langle \mathbb{X} \rangle_x = \text{diag}(T_x)^t \langle \mathbb{Y} \rangle_x,$$

where  $\text{diag}(T_x)$  is the block-diagonal part of  $T_x \in BU(d_1, \dots, d_s)$ .

- (3) *We may write  $\langle \mathbb{X} \rangle^\alpha$  as:*

$$\langle \mathbb{X} \rangle^\alpha = \sum_{[\beta]=[\alpha]} \tilde{T}_{\alpha, \beta} \langle \mathbb{Y} \rangle^\beta, \quad \tilde{T}_{\alpha, \beta} \in C^\infty(U).$$

Then the coefficients  $\tilde{T}_{\alpha, \beta}$  are determined via

$$(\text{diag}(T_x)v)^\alpha = \sum_{[\beta]=[\alpha]} \tilde{T}_{\alpha, \beta}(x)v^\beta \quad x \in U, \quad v \in \mathbb{R}^n,$$

where  $v^\alpha = v_1^{\alpha_1} \dots v_n^{\alpha_n}$ .

*Proof of Lemma 3.3.* For each  $i = 1, \dots, n$ , let  $j \in \{1, \dots, s\}$  be the level of the gradation for which  $d_{j-1} < i \leq d_j$ , i.e. for which  $w_j = v_i$ , then

$$(3.1) \quad X_i(x) = \sum_{1 \leq k \leq d_j} T_{k,i}(x)Y_k(x), \quad \text{or in other words } X_i(x) = \sum_{v_k \leq v_i} T_{k,i}(x)Y_k(x).$$

We deduce the structure of the matrix  $T$ , and Part (1) follows.

Quotienting (3.1) by  $H^{w_j-1}$  gives

$$\langle X_i \rangle_x = \sum_{d_{j-1} < k \leq d_j} T_{k,i}(x) \langle Y_k \rangle_x = \sum_{v_k = v_i} T_{k,i}(x) \langle Y_k \rangle_x \quad \text{in } \mathfrak{g}M|U,$$

so  $\langle \mathbb{X} \rangle_x = \text{diag}(T_x)^t \langle \mathbb{Y} \rangle_x$ . This is Part (2). Part (3) is an easy consequence of Part (2).  $\square$

**3.3. Geometric exponentiation.** In this section, we recall the definition of geometric exponentiation for any local frame on any smooth manifold  $M$  as well as some of its properties. This notion does not require the manifold  $M$  to be filtered or the local frame to be adapted to the filtration, and the results hold in this general context.

3.3.1. *Definitions of  $\exp$  and  $\exp^{\mathbb{X}}$ .* The exponential map is defined as follows. If  $X$  is a vector field on a smooth manifold  $M$  and  $x \in M$ , then  $\exp_x X$  is the time-one flow, when defined, of the vector field  $X$ , that is, the solution at time 1 of the Ordinary Differential Equation (ODE) defined by  $X$  and starting from  $x$ :

$$\dot{\gamma}(t) = X(\gamma(t)), \quad \gamma(0) = x, \quad \exp_x X := \gamma(1).$$

Given a local frame  $(\mathbb{X}, U)$  on a manifold, we set

$$\exp^{\mathbb{X}}(x, v) := \exp_x^{\mathbb{X}}(v) := \exp_x \sum_j v_j X_j, \quad x \in U, \quad v \in \mathbb{R}^n,$$

when defined. This yields a *geometric exponential map*  $\exp^{\mathbb{X}}$  associated with the frame  $(\mathbb{X}, U)$ . It is defined on a subset of  $U \times \mathbb{R}^n$ . In fact, there exists an open neighbourhood  $U_{\mathbb{X}} \subset U \times \mathbb{R}^n$  of the zero section  $U \times \{0\}$  on which the map

$$(3.2) \quad \exp^{\mathbb{X}} : \begin{cases} U_{\mathbb{X}} & \longrightarrow & M \times M \\ (x, v) & \longmapsto & (x, \exp_x^{\mathbb{X}}(v)) \end{cases}$$

is a  $C^\infty$ -diffeomorphism onto its image in  $U \times U$ .

Without restriction, we may assume that  $U_{\mathbb{X}}$  is such that for any  $(x, v) \in U_{\mathbb{X}}$ , there exists a neighbourhood  $I$  of  $[0, 1]$  such that  $(x, tv) \in U_{\mathbb{X}}$  for  $t \in I$ . We will always assume so for technical reasons that will become apparent below.

3.3.2. *Some fundamental properties.* The theory of ODEs has the following two implications. Firstly, for any  $x, y \in M$  such that  $\ln_x^{\mathbb{X}} y$  and  $\ln_y^{\mathbb{X}} x$  exist, we have

$$(3.3) \quad \ln_x^{\mathbb{X}} y = -\ln_y^{\mathbb{X}} x,$$

and also that for any  $x \in U$ , we have:

$$\partial_{v_j} \exp_x^{\mathbb{X}} v \Big|_{v=0} = X_j, \quad j = 1, \dots, n.$$

In particular  $\text{jac}_0 \exp_x^{\mathbb{X}} = 1 = \text{jac}_x \ln_x^{\mathbb{X}}$  when we take the volume form obtained by the pullback of  $dv$  from  $\mathbb{R}^n$  to  $U \subset M$ .

Secondly, the definition of the geometric exponential implies

$$(3.4) \quad \frac{d}{dt} f(\exp_x^{\mathbb{X}} tv) = \left( \sum_i v_i X_i f \right) (\exp_x^{\mathbb{X}} tv), \quad f \in C^\infty(U), \quad (x, v) \in U_{\mathbb{X}},$$

for any  $t$  in a neighbourhood of  $[0, 1]$ .

3.3.3. *Some Taylor estimates.* Using inductively the derivation in (3.4) will allow us to obtain the following (Euclidean) Taylor estimates:

**Lemma 3.4.** *Let  $\mathbb{X}$  be a frame on an open set  $U \subset M$ . Let  $f \in C^\infty(U)$  and  $N \in \mathbb{N}$ . For any  $(x, v) \in U_{\mathbb{X}}$ , we have:*

$$\left| f(\exp_x^{\mathbb{X}} v) - \sum_{k \leq N} \frac{1}{k!} \left( \sum_i v_i X_i \right)^k f(x) \right| \leq \frac{1}{(N+1)!} \sup_{t \in [0,1]} \left| \left( \sum_i v_i X_i \right)^{N+1} f \right| (\exp_x^{\mathbb{X}} tv).$$

*Proof of Lemma 3.4.* Set  $F(t) := f(\exp_x^{\mathbb{X}} tv)$  for  $t$  in a neighbourhood of  $[0, 1]$ . We have

$$F^{(k)}(t) = \left( \left( \sum_i v_i X_i \right)^k f \right) (\exp_x^{\mathbb{X}} tv),$$

for  $k = 1$  by (3.4), and inductively for any  $k \in \mathbb{N}$ . We conclude with Taylor's estimate for  $F$  at order  $N$  from  $t = 0$  to  $t = 1$ .  $\square$

Using again (3.4), we can also obtain Taylor expansions related to changes of frames:

**Lemma 3.5.** *Let  $\mathbb{X}$  and  $\mathbb{Y}$  be two frames on an open subset  $U \subset M$ . Let  $U_{\mathbb{X}, \mathbb{Y}}$  be an open neighbourhood of  $U \times \{0\}$  in  $U_{\mathbb{X}} \cap U_{\mathbb{Y}} \subset U \times \mathbb{R}^n$  such that for any  $(x, v) \in U_{\mathbb{X}, \mathbb{Y}}$ , the points  $(x, tv)$  lie in  $U_{\mathbb{X}, \mathbb{Y}}$  for  $t$  in some neighbourhood of  $[0, 1]$ . Consider the function  $u := u^{\mathbb{Y}, \mathbb{X}}$  defined via*

$$(x, u(x, v)) = \ln^{\mathbb{Y}}(\exp^{\mathbb{X}}(x, v)), \quad (x, v) \in U_{\mathbb{X}, \mathbb{Y}}.$$

*Then the map  $u = u^{\mathbb{Y}, \mathbb{X}} : U_{\mathbb{X}, \mathbb{Y}} \rightarrow \mathbb{R}^n$  is smooth and given via*

$$(3.5) \quad u(x, v) := u_x(v) = \ln_x^{\mathbb{Y}} \circ \exp_x^{\mathbb{X}}(v), \quad (x, v) \in U_{\mathbb{X}, \mathbb{Y}}.$$

*It satisfies*

$$u_x(0) = 0 \quad \text{and} \quad D_0 u_x(v) = \left. \frac{d}{dt} \right|_{t=0} u_x(tv) = T_x v,$$

*where  $T$  is their change of frame matrix as in Section 3.2.1. Moreover, the Taylor series at  $v \sim 0$  of  $u_x(v)$  is given by*

$$u_x(v) \sim T_x v + \sum_{k \geq 2} \frac{1}{k!} \left( \sum_j v_j X_j \right)^{k-1} T_x v.$$

*Proof of Lemma 3.5.* We check readily that the map  $u$  is smooth, given by (3.5) and that it satisfies  $u_x(0) = 0$ . Below, we will show that for any  $(x, v) \in U_{\mathbb{X}, \mathbb{Y}}$  and  $t$  in a neighbourhood of  $[0, 1]$ , we have:

$$(3.6) \quad \frac{d^k}{dt^k} u_x(tv) = \left( \sum_{j=1}^n v_j X_{j,z} \right)^{k-1} \left. T_z v \right|_{z=\exp_x^{\mathbb{X}}(tv)} \quad k = 1, 2, 3 \dots$$

From this, we can obtain the Taylor series of  $u_x(tv)$  at  $t \sim 0$ , and deduce the Taylor series of  $u_x(v)$  at  $v \sim 0$ ; hence, showing (3.6) will conclude the proof.

On the one hand, we have the equality in (3.4) and on the other, since  $\exp_x^{\mathbb{X}} tv = \exp_x^{\mathbb{Y}} u(x, tv)$ , the left-hand side in (3.4) is also equal to

$$\frac{d}{dt} f(\exp_x^{\mathbb{Y}} u(x, tv)) = \left( \sum_k \left( \frac{d}{dt} u(x, tv) \right)_k Y_k \right) f(\exp_x^{\mathbb{Y}} u(x, tv)).$$

We have  $\sum_j v_j X_j = \sum_k (Tv)_k Y_k$  in (3.4), and these equalities are valid for any  $f \in C^\infty(U)$ . It allows us to identify the first  $t$ -derivative of  $u_x(tv)$  and we obtain (3.6) for  $k = 1$ . We then obtain (3.6) inductively on  $k = 1, 2, \dots$   $\square$

**3.4. Homogeneous order and principal part of a differential operator.** A differential operator on a manifold  $M$  is a uniformly finite linear combination over  $C^\infty(M)$  of products of vector fields and the identity. If  $\mathbb{X} = (X_1, \dots, X_n)$  is a frame over an open subset  $U$  of a manifold  $M$ , then any differential operator  $P$  on  $U$  may be written as a non-commutative polynomial in the  $X_j$ 's, that is, as a linear combination over  $C^\infty(U)$  of  $X_{i_1} X_{i_2} \dots X_{i_\ell}$ ,  $1 \leq i_1, \dots, i_\ell \leq n$ . This writing is not unique. We say that  $P$  is of degree at most  $d$ , if  $P$  coincides on  $U$  with a linear combination of  $X_{i_1} X_{i_2} \dots X_{i_\ell}$  with  $1 \leq i_1, \dots, i_\ell \leq n$  and  $\ell \leq d$ . The smallest of these  $d$  over all the local frames  $(\mathbb{X}, U)$  of  $M$  defines the degree of  $P$ .

We now go back to the case of a filtered manifold  $M$  and an adapted frame  $(\mathbb{X}, U)$ . We can express the differential operators in terms of the ordered products

$$\mathbb{X}^\alpha := X_1^{\alpha_1} \dots X_n^{\alpha_n}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n,$$

with the convention  $\mathbb{X}^0 = I$  for  $\alpha = 0$  as in the preceding section. Moreover, this writing will have the advantage of being unique:

**Lemma 3.6.** *Let  $\mathbb{X}$  be an adapted frame on the open subset  $U \subset M$ . Then any differential operator on  $M$  may be uniquely written on  $U$  as a finite linear combination over  $C^\infty(U)$  of  $\mathbb{X}^\alpha$ ,  $\alpha \in \mathbb{N}_0^n$ , on  $U$ .*

*Proof of Lemma 3.6.* We observe that given  $i_1, \dots, i_\ell \in \{1, \dots, n\}$ , we can define the associated multi-index  $\alpha = (\alpha_1, \dots, \alpha_n)$  with  $\alpha_k = |\{j : i_j = k\}|$  and we have

$$X_{i_1} X_{i_2} \dots X_{i_\ell} = \mathbb{X}^\alpha + \sum_{|\beta| < |\alpha|} c_\beta \mathbb{X}^\beta, \quad c_\beta \in C^\infty(U),$$

This is trivially true for the length  $\ell = 1$ . For the length  $\ell = 2$ , this is a consequence of the frame  $\mathbb{X}$  being adapted to the filtration since  $X_i X_j = X_j X_i + [X_i, X_j]$  and  $[X_i, X_j] \in \Gamma(H^{v_i+v_j}|_U)$  may be written as a linear combination of  $X_k$ ,  $k \leq \dim H^{v_i+v_j}$ . We then obtain the property inductively over the length  $\ell$ . This implies that any differential operator may be written as a finite linear combination over  $C^\infty(U)$  of  $\mathbb{X}^\alpha$ ,  $\alpha \in \mathbb{N}_0^n$ .

The uniqueness of the writing is equivalent to the property that if such a finite linear combination coincides with the zero operator, then all the coefficients are equal to zero. For this, it suffices to construct for each  $x_0 \in U$  and each  $\alpha \in \mathbb{N}_0^n$  a smooth function  $f_{\alpha, x_0}$  in a neighbourhood of  $x_0$  satisfying

$$\mathbb{X}^\beta f_{\alpha, x_0}(x_0) = \begin{cases} 0 & \text{if } [\beta] \leq [\alpha] \text{ with } \beta \neq \alpha, \\ 1 & \text{if } \beta = \alpha. \end{cases}$$

The natural choices are  $f_{0,x_0} = 1$  for  $\alpha = 0$  and

$$f_{j_0,x_0} := [\ln_{x_0}^{\mathbb{X}} x]_{j_0} \quad \text{for } |\alpha| = 1.$$

We check inductively on  $[\alpha]$  that the functions

$$f_{\alpha,x_0} := f_{1,x_0}^{\alpha_1} \cdots f_{n,x_0}^{\alpha_n}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0,$$

satisfy the properties described above.  $\square$

Lemma 3.6 allows us to define the notion of *homogeneous order at most  $N$*  for differential operator ( $N \in \mathbb{N}$ ).

**Definition 3.7.** A differential operator on  $M$  is of *homogeneous order at most  $N$*  when it can be written as a linear combination over  $C^\infty(U)$  of  $\mathbb{X}^\alpha$ ,  $[\alpha] \leq N$ , with respect to any adapted frame  $(\mathbb{X}, U)$ .

The following statement shows that if a differential operator is written as a  $C^\infty(U)$ -linear combination of  $\mathbb{X}^\alpha$ ,  $[\alpha] \leq N$ , with respect to one frame  $(\mathbb{X}, U)$ , then it will be so for any frame  $\mathbb{Y}$  on  $U$ .

**Lemma 3.8.** *Let  $\mathbb{X}$  and  $\mathbb{Y}$  be two adapted frames on the same open subset  $U \subset M$ .*

- (1) *For any non-zero multi-index  $\alpha \in \mathbb{N}_0^n$ , we have for any  $x \in U$*

$$\mathbb{X}_x^\alpha = \sum_{[\beta]=[\alpha]} \tilde{T}_{\alpha,\beta} \mathbb{Y}^\beta + \sum_{[\beta]<[\alpha]} r_\beta(x) \mathbb{Y}^\beta,$$

*where the coefficients  $\tilde{T}_{\alpha,\beta}$ ,  $[\beta] = N$ , were defined in Lemma 3.3, and  $r_\beta \in C^\infty(U)$  are some functions also depending on  $\alpha$ .*

- (2) *Consider a differential operator on  $P$  written as  $P = \sum_{[\alpha] \leq N} c_\alpha^\mathbb{X}(x) \mathbb{X}^\alpha$  with respect to the frame  $\mathbb{X}$ . Then  $P$  is written as  $P = \sum_{[\beta] \leq N} c_\beta^\mathbb{Y}(x) \mathbb{Y}^\beta$  with respect to the frame  $\mathbb{Y}$  for some unique coefficients  $c_\beta^\mathbb{Y}$ , and we have above any  $x \in U$*

$$\sum_{[\alpha]=N} c_\alpha^\mathbb{X}(x) \langle \mathbb{X} \rangle_x^\alpha = \sum_{[\beta]=N} c_\beta^\mathbb{Y}(x) \langle \mathbb{Y} \rangle_x^\alpha \quad \text{in } \mathcal{U}_N(\mathfrak{g}_x M).$$

*Proof.* Let  $T$  be the transition matrix as in Lemma 3.3. We have  $\mathbb{X} = T^t \mathbb{Y}$ , so

$$\mathbb{X}^\alpha = [T^t \mathbb{Y}]_1^{\alpha_1} \cdots [T^t \mathbb{Y}]_n^{\alpha_n}.$$

Part (1) follows by the Leibniz property of vector fields and Lemma 3.3 (3). Part (2) follows by linearity.  $\square$

We denote by  $\text{DO}(M)$  the space of differential operators on  $M$ , and by  $\text{DO}^{\leq N}(M)$  the subspace of differential operators of homogeneous order at most  $N$ . Lemma 3.6 and 3.8 show that if  $(\mathbb{X}, U)$  is an adapted frame, then the set of monomials  $\mathbb{X}^\alpha$ ,  $\alpha \in \mathbb{N}_0^n$ , is a basis of the  $C^\infty(U)$ -module  $\text{DO}(U)$  while the set of monomials  $\mathbb{X}^\alpha$ ,  $[\alpha] \leq N$ , is a basis of the submodule  $\text{DO}^{\leq N}(U)$ .

Let us open a brief parenthesis for the case of  $H_1$  generating  $TM$  as in Example 3.2:

**Corollary 3.9.** *Assume that  $H_1$  generates  $TM$ . Let  $\mathbb{X}$  be an adapted frame on the open subset  $U \subset M$ . Then  $\text{DO}(U)$  is the  $C^\infty(U)$ -module generated by*

$$X_{i_1} \dots X_{i_\ell}, \quad 1 \leq i_1, \dots, i_\ell \leq d_1 = \dim H_1, \quad \ell \in \mathbb{N}_0.$$

*Moreover,  $\text{DO}^{\leq N}(U)$  is the  $C^\infty(U)$ -module generated by  $X_{i_1} \dots X_{i_\ell}$ ,  $1 \leq i_1, \dots, i_\ell \leq d_1$ ,  $\ell = 0, \dots, N$ .*

In other words, any differential operator of degree  $\leq N$  may be written on  $U$  as a combination over  $C^\infty(U)$  of  $X_{i_1} \dots X_{i_\ell}$ ,  $1 \leq i_1, \dots, i_\ell \leq d_1$ ,  $\ell = 0, \dots, N$ ; however, this writing may not be unique.

*Proof of Corollary 3.9.* The first claim of the statement follows readily from Lemma 3.6 and any  $X_j$ ,  $j > d_1$  being obtained as a  $C^\infty(U)$ -combination of nested bracket of  $X_i$ ,  $i = 1, \dots, d_1$ .

As  $H_1$  generates  $TM$  (see Example 3.2), each  $X_j$ ,  $j = 1, \dots, n$  may be written as a  $C^\infty$ -combination of  $X_{i_1} \dots X_{i_\ell}$  with  $\ell \leq v_j$ . This allows us to prove the rest of the statement inductively on  $N$  by writing for every  $\alpha \in \mathbb{N}_0 \setminus \{0\}$ ,  $\mathbb{X}^\alpha = X_j \mathbb{X}^{\alpha'}$  for some  $j = 1, \dots, n$  and  $\alpha' \in \mathbb{N}_0^n$  uniquely determined by  $\alpha$  and satisfying  $[\alpha] = v_j + [\alpha']$ .  $\square$

If  $P \in \text{DO}^{\leq N}(M)$  and  $(\mathbb{X}, U)$  is an adapted frame, then for each  $x \in U$ , we define the element of  $\mathcal{U}_N(\mathfrak{g}_x M)$

$$(3.7) \quad \text{Princ}_{N,x}(P) := \sum_{[\alpha]=N} c_\alpha(x) \langle \mathbb{X} \rangle_x^\alpha,$$

where the coefficients  $c_\alpha$  come from the unique expression  $P = \sum_{[\alpha] \leq N} c_\alpha \mathbb{X}^\alpha$  on  $U$ . By Lemma 3.8 (2), this defines an element of  $\mathcal{U}_N(\mathfrak{g}_x M)$  that does not depend on the choice of the adapted frame  $\mathbb{X}$  and that is smooth in  $x$ . This allows us to define the principal part of a differential operator of homogeneous order at most  $N$  as the morphism of  $C^\infty(M)$ -module:

$$\text{Princ}_N : \text{DO}^{\leq N}(M) \rightarrow \Gamma(\mathcal{U}_N(\mathfrak{g}M)).$$

We may say that  $P \in \text{DO}^{\leq N}(M)$  is of *homogeneous order  $N$*  when its principal part  $\text{Princ}_N(P)$  is non-zero in  $\Gamma(\mathcal{U}_N(\mathfrak{g}M))$ . Note that the degree of  $P$  as a differential operator is less or equal to its homogeneous order  $N$ ; it is usually much smaller than  $N$ .

**3.5. Group exponentiation and BCH formula.** If an adapted frame  $(\mathbb{X}, U)$  is fixed, we use the adapted basis  $\langle \mathbb{X} \rangle_x$  to write the *group exponentiation*:

$$(3.8) \quad \text{Exp}_x^{\mathbb{X}}(v) := \text{Exp}_x \left( \sum_i v_i \langle X_i \rangle_x \right) \in G_x M.$$

This yields the smooth mapping

$$\text{Exp}^{\mathbb{X}} : \begin{cases} U \times \mathbb{R}^n & \longrightarrow GM|_U \\ (x, v) & \longmapsto \text{Exp}_x \left( \sum_{i=1}^n v_i \langle X_i \rangle_x \right) \end{cases}$$

with inverse

$$\text{Ln}^{\mathbb{X}} := (\text{Exp}^{\mathbb{X}})^{-1} : \text{Exp}^{\mathbb{X}}(U \times \mathbb{R}^n) \longrightarrow U \times \mathbb{R}^n.$$

This allows us to extend the definition of the dilations  $\delta_r$  to  $U \times \mathbb{R}^n$ . Moreover, the Lebesgue density  $|dv|$  on  $\mathbb{R}^n$  gives us a frame-dependent choice of volume density on  $TM|_U$  and a Haar system for  $GM|_U$  by pushing forward through the trivialization.

The next lemma explains what happens when the adapted frame is changed:

**Lemma 3.10.** *Let  $\mathbb{X}$  and  $\mathbb{Y}$  be two adapted frames  $\mathbb{X}$  and  $\mathbb{Y}$  on an open subset  $U \subset M$ , and  $T$  the matrix of change of frame as in Lemma 3.3. Then, the filtration preserving map  $\text{diag}(T_x)$  satisfies*

$$\forall (x, v) \in U \times \mathbb{R}^n \quad \text{Ln}_x^{\mathbb{Y}} \circ \text{Exp}_x^{\mathbb{X}}(v) = \text{diag}(T_x).$$

*Proof of Lemma 3.10.* Since  $\langle \mathbb{X} \rangle_x = \text{diag}(T_x)^t \langle \mathbb{Y} \rangle_x$ , we have

$$\sum_i v_i \langle X_i \rangle_x = \sum_i v_i \sum_{v_i=v_k} T_{ki}(x) \langle Y_k(x) \rangle_x = \sum_k (\text{diag}(T_x)v)_k \langle Y_k \rangle_x,$$

so  $\text{Exp}_x^{\mathbb{X}}v = \text{Exp}_x^{\mathbb{Y}}(\text{diag}(T_x)v)$ . □

The basis  $\langle X_1 \rangle_x, \dots, \langle X_n \rangle_x$  together with the group exponential map  $\text{Exp}_x^{\mathbb{X}}$  allows for a precise group isomorphism between the group fiber  $G_xM$  with  $\mathbb{R}^n$ .

*Remark 3.11.* We have already given natural examples of quasinorms on graded groups associated with an adapted basis, see Example 2.13. With the above isomorphism in mind, it is natural to consider the quasinorm on  $\mathbb{R}^n$  given by

$$|v|_{\mathbb{X}} = \left( \sum_{i=1}^n |v_i|^{\alpha/v_i} \right)^{1/\alpha}, \quad v = (v_j)_{j=1}^n \in \mathbb{R}^n,$$

$\alpha = 2M_0$  where  $M_0$  is a common multiple of the weights  $v_1, \dots, v_n$ , the least one to fix the idea. We will use this quasinorm throughout the rest of the work.

To describe more precisely the group law inherited by  $\mathbb{R}^n$ , we introduce the map given at every point  $x$  by the adjoint map  $\text{ad}_{\mathfrak{g}_xM}$  in the  $\langle \mathbb{X} \rangle$ -coordinates:

**Definition 3.12.** Let  $\mathbb{X}$  be an adapted frame on an open set  $U$  of the filtered manifold  $M$ . For each  $x \in U$ , and for any  $v, w \in \mathbb{R}^n$ , we denote by  $\text{ad}_x^{\mathbb{X}}(v)(w)$  the vector in  $\mathbb{R}^n$  whose coordinates  $(u_1, \dots, u_n)$  satisfy

$$\sum_{k=1}^n u_k \langle X_k \rangle_x := \left[ \sum_{i=1}^n v_i \langle X_i \rangle_x, \sum_{j=1}^n w_j \langle X_j \rangle_x \right]_{\mathfrak{g}_xM}.$$

This defines a skew-symmetric map  $\text{ad}_x^{\mathbb{X}} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  we call the *adjoint map in the  $\langle \mathbb{X} \rangle$ -coordinates*.

This equips  $\mathbb{R}^n$  with a graded Lie algebra structure isomorphic to the one of  $\mathfrak{g}_xM$ , the isomorphism being given by the choice of basis  $\langle \mathbb{X} \rangle_x$ . Moreover, the map

$$\text{ad}_x^{\mathbb{X}} : x \longmapsto \text{ad}_x^{\mathbb{X}}, \quad U \longrightarrow C^\infty(U, \mathcal{L}(\mathbb{R}^n)),$$

is smooth. It describes the group law of  $G_xM$  in exponential coordinates (3.8):

**Lemma 3.13.** *Let  $\mathbb{X}$  be an adapted frame on an open set  $U$  of the filtered manifold  $M$ . For any  $x \in U$  and  $v, w \in \mathbb{R}^n$ , we have:*

$$\mathrm{Exp}_x^{\mathbb{X}}(v) \mathrm{Exp}_x^{\mathbb{X}}(w) = \mathrm{Exp}_x^{\mathbb{X}}(z(x; v, w)) = \mathrm{Exp}_x \left( \sum_i z_i(x; v, w) \langle X_i \rangle_x \right)$$

where  $z(x; v, w) = (z_k(x; v, w))_{k=1}^n$  is given by

$$z(x; v, w) = \mathrm{BCH}_{s_0}(v, w; \mathrm{ad}_x^{\mathbb{X}}), \quad x \in U, \quad v, w \in \mathbb{R}^n,$$

where  $s_0 \geq s$ ,  $s$  being the step of every  $\mathfrak{g}_x M$ . For each  $k = 1, \dots, n$ , the function  $z_k(x; v, w)$  is polynomial in  $v, w$  with coefficients depending smoothly on  $x \in U$ . Moreover,  $z_k(x; v, w)$  is  $v_k$ -homogeneous, in the sense that  $z_k(\delta_r v, \delta_r w; x) = r^{v_k} z_k(x; v, w)$ .

Recall that that the step of the graded spaces  $\mathfrak{g}_x M$  was defined in Section 3.1 and that  $\mathrm{BCH}_{s_0}$  has been introduced in Notation 2.12.

We denote by

$$(3.9) \quad v *_x^{\mathbb{X}} w := \mathrm{Ln}_x^{\mathbb{X}}(\mathrm{Exp}_x^{\mathbb{X}}(v) \mathrm{Exp}_x^{\mathbb{X}}(w)) \quad v, w \in \mathbb{R}^n$$

the law of the group  $\mathbb{R}^n$  isomorphic to  $G_x M$  via the group exponential map  $\mathrm{Exp}_x$  and the choice of the basis  $\langle \mathbb{X} \rangle_x$ . By Lemma 3.13 (and with its notation), we have

$$v *_x^{\mathbb{X}} w = z(x; v, w) = \mathrm{BCH}_{s_0}(v, w; \mathrm{ad}_x^{\mathbb{X}}),$$

so that

$$\mathrm{Exp}_x^{\mathbb{X}}(v *_x^{\mathbb{X}} w) = \mathrm{Exp}_x^{\mathbb{X}}(v) \mathrm{Exp}_x^{\mathbb{X}}(w).$$

If the adapted frame  $\mathbb{X}$  is fixed, we will allow ourselves to use the shorthand  $*_x^{\mathbb{X}} = *_x$ . The reader should keep in mind that star-notation is used in various ways in this paper:  $*_x$  gives the product law of the group  $G_x M$  after suitable identifications while the convolution product on a group  $G$  will be denoted by  $\star_G$ , in accordance with (2.3).

#### 4. ACTION OF GEOMETRIC MAPS ON THE OSCULATING GROUP

In this section we analyse the composition of geometric exponential maps. After defining the notion of higher order maps between graded vector spaces in Section 4.1, we use this notion on filtered manifolds to describe various algebraic properties in Section 4.2, and eventually the composition of geometric exponential mappings in Section 4.3.

The notion of terms of higher order is crucial in this analysis. It is also present in various forms in many papers studying the composition of geometric exponential on filtered or subRiemannian manifolds, for instance in [13, 14].

**4.1. Higher order terms on graded spaces.** Here, we define what higher order means for maps between graded vector spaces. This notion will be at the core of the comparison of the commutator brackets of vector fields at  $x$  with the structure of the Lie algebra  $\mathfrak{g}_x$ , and in various other places.

In this section, we consider two graded vector spaces  $\mathcal{V}_1$  and  $\mathcal{V}_2$  and denote by  $\delta_{1,r}$  and  $\delta_{2,r}$  for  $r > 0$  the associated dilations (see Definition 2.2).

**Definition 4.1.** A function  $f : \mathcal{V}_1 \rightarrow \mathcal{V}_2$  is *homogeneous* of degree  $m$  (for the dilations  $\delta_{1,r}, \delta_{2,r}, r > 0$ ) when  $f(\delta_{1,r}v) = \delta_{2,r^m}f(v)$  for any  $r > 0$  and  $v \in \mathcal{V}_1$ . In the case  $m = 1$ , we say that  $f$  is homogeneous.

*Example 4.2.* If  $G$  is a graded Lie group with a fixed adapted basis  $\mathbb{V}$  for its Lie algebra, then the group law expressed as a map  $(v, w) \mapsto v *_G w, \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a homogeneous map.

We fix bases  $(V_1, \dots, V_{\dim \mathcal{V}_1})$  and  $(W_1, \dots, W_{\dim \mathcal{V}_2})$  for  $\mathcal{V}_1$  and  $\mathcal{V}_2$  adapted to the gradation, and we denote by  $v_j$  the weight of  $W_j, j = 1, \dots, \dim \mathcal{V}_2$ . It is easy to check that a function  $f : \mathcal{V}_1 \rightarrow \mathcal{V}_2$  is homogeneous if and only if, for each  $j = 1, \dots, \dim \mathcal{V}_2$ , the  $j$ -components  $f_j : \mathcal{V}_1 \rightarrow \mathbb{R}$  of  $f$  is an  $v_j$ -homogeneous map. This motivates the following definition:

- Definition 4.3.**
- (1) We say that a polynomial map  $r : \mathcal{V}_1 \rightarrow \mathcal{V}_2$  is *higher order* (for the gradations) when the degrees of homogeneity (for the dilations on  $\mathcal{V}_1$ ) of the monomials occurring in its  $j$ -component  $r_j$  are  $> v_j$ . The notion of higher order extends to formal power series.
  - (2) We say that a smooth function  $r$  defined on a neighbourhood of 0 in  $\mathcal{V}_1$  and valued in  $\mathcal{V}_2$  is higher order when its Taylor series at 0 is a higher order power series.

It is not difficult to construct higher order polynomial maps from homogeneous ones:

*Example 4.4.* Let  $q : \mathcal{V}_1 \rightarrow \mathcal{V}_2$  and  $r : \mathcal{V}_1 \rightarrow \mathcal{V}_2$  be polynomial maps that are respectively homogeneous and higher order.

- (1) If  $p : \mathcal{V}_1 \rightarrow \mathbb{C}$  is a polynomial satisfying  $p(0) = 0$ , then  $pq$  and  $pr$  are higher order polynomial maps.
- (2) This first example can be generalised as follow. We fix a basis  $(W_1, \dots, W_{\dim \mathcal{V}_2})$  adapted to the gradation of  $\mathcal{V}_1$ , so that we may write  $q = (q_k)_{k=1}^{\dim \mathcal{V}_2}$  and  $r = (r_k)_{k=1}^{\dim \mathcal{V}_2}$  in the corresponding coordinates. Given any family of polynomials  $p_k : \mathcal{V}_1 \rightarrow \mathbb{C}, k = 1, \dots, \dim \mathcal{V}_2$ , then the polynomial maps  $(p_k r_k)_{k=1}^{\dim \mathcal{V}_2}$  are higher order. If in addition  $p_k(0) = 0$  for every  $k = 1, \dots, \dim \mathcal{V}_2$ , then the polynomial maps  $(p_k q_k)_{k=1}^{\dim \mathcal{V}_2}$  are higher order.

The following properties characterise the higher order maps. In particular, the characterisations (2) or (3) imply that the notion of higher order does not depend on the chosen graded bases.

**Proposition 4.5.** *Let  $r : \mathcal{V}_1 \rightarrow \mathcal{V}_2$  be a smooth map. The following properties are equivalent:*

- (1) *The function  $r$  is higher order.*
- (2) *For one (and then any) neighbourhood  $V_0$  of 0, we have for any  $v \in V_0$*

$$\lim_{\varepsilon \rightarrow 0} \delta_{\varepsilon^{-1}} r(\delta_{\varepsilon} v) = 0.$$

- (3) *There exists a unique smooth map  $R : \mathbb{R} \times \mathcal{V}_1 \rightarrow \mathcal{V}_2$  satisfying  $\delta_{\varepsilon^{-1}} r(\delta_{\varepsilon} v) = \varepsilon R(\varepsilon, v)$ .*

*If an adapted basis of  $\mathcal{V}_2$  has been chosen, then it is also equivalent to:*

(4) For  $j = 1, \dots, \dim \mathcal{V}_2$ , we have  $\partial_v^\alpha r_j(0) = 0$ ,  $[\alpha] \leq v_j$ ; equivalently, this means that the Taylor series of  $r_j$  about 0 is

$$r_j(v) \sim_{v \sim 0} \sum_{[\alpha] > v_j} \partial^\alpha r_j(0) v^\alpha.$$

*Proof.* As above, we fix bases  $(V_1, \dots, V_{\dim \mathcal{V}_1})$  and  $(W_1, \dots, W_{\dim \mathcal{V}_2})$  for  $\mathcal{V}_1$  and  $\mathcal{V}_2$  adapted to the gradation, with  $v_j$  the weight of  $W_j$ . We denote by  $r_j$  the  $j$ th component of  $r$ . By definition,  $r$  is higher order if and only if Property (4) holds.

Whether  $r$  is higher order or not, we may write,

$$r_j(v) = \sum_{[\alpha] \leq v_j} \partial^\alpha r_j(0) v^\alpha + R_j(v),$$

with  $R_j \in C^\infty(\mathcal{V}_1)$ . Fixing a quasinorm  $|\cdot|_1$  on  $\mathcal{V}_1$ , by the Taylor estimate due to Folland and Stein at order  $v_j$  (see [41] or Theorem 2.17), for any compact neighbourhood  $W$  of 0 in  $\mathcal{V}_1$ , there exists a constant  $C_j > 0$  such that

$$\forall v \in W, \quad |R_j(v)| \leq C_j |v|_1^{v_j+1}.$$

If Property (1) holds, that is, if the Taylor series of  $r$  at 0 is higher order, then for any  $j = 1, \dots, \dim \mathcal{V}_2$

$$|\varepsilon^{-v_j} r_j(\delta_\varepsilon v)| = |\varepsilon^{-v_j} R_j(\delta_\varepsilon v)| \leq \varepsilon^{-v_j} C_j |\delta_\varepsilon v|_1^{v_j+1} = C_j \varepsilon |v|_1^{v_j+1} \xrightarrow{\varepsilon \rightarrow 0} 0,$$

so Property (2) hold.

If Property (2) holds, or equivalently if for any  $j = 1, \dots, \dim \mathcal{V}_2$  and  $v \in \mathcal{V}_1$

$$0 = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-v_j} r_j(\varepsilon v) = \lim_{\varepsilon \rightarrow 0} \sum_{[\alpha] \leq v_j} \varepsilon^{-v_j+[\alpha]} \partial^\alpha r_j(0) v^\alpha + \varepsilon^{-v_j} R_j(\delta_\varepsilon v),$$

then  $\partial^\alpha r_j(0) = 0$ ,  $[\alpha] \leq v_j$  and Property (1) holds. Moreover, the smoothness of the  $r_j$ 's implies

$$r_j(v) = \sum_{|\alpha| \leq N_0} \partial^\alpha r_j(0) v^\alpha + \sum_{|\alpha| = N_0+1} v^\alpha \rho_{j,\alpha}(v), \quad \text{with } \rho_{j,\alpha} \in C^\infty(\mathcal{V}_1).$$

Choosing  $N_0 = \max\{|\alpha| : \alpha \in \mathbb{N}_0, [\alpha] \leq v_j\}$ , since  $\partial^\alpha r_j(0) = 0$ ,  $[\alpha] \leq v_j$ , we obtain

$$\varepsilon^{-v_j} r_j(\delta_\varepsilon v) = \sum_{|\alpha| \leq N_0, [\alpha] \leq v_j} \varepsilon^{-v_j+[\alpha]} \partial^\alpha r_j(0) v^\alpha + \sum_{|\alpha| = N_0+1} \varepsilon^{-v_j+[\alpha]} v^\alpha \rho_{j,\alpha}(\delta_\varepsilon v),$$

implying Property (3). We have obtained (1)  $\iff$  (2)  $\implies$  (3). Clearly, (3)  $\implies$  (2). The proof is complete.  $\square$

Let us present two corollaries of Proposition 4.5 and its proof. The first one gives estimates of a higher order function:

**Corollary 4.6.** *Let  $r : \mathcal{V}_1 \rightarrow \mathcal{V}_2$  be a smooth map of higher order. Having fixed quasinorms  $|\cdot|_1$  and  $|\cdot|_2$  on  $\mathcal{V}_1$  and  $\mathcal{V}_2$ , for any bounded neighbourhood  $W$  of 0 in  $\mathcal{V}_1$ , there exists  $C > 0$  such that*

$$\forall v \in W, \quad |r(v)|_2 \leq C |v|_1^{1+\frac{1}{v_n}}.$$

Note that in the isotropic case, i.e. when  $v_1 = \dots = v_n = 1$ , we recover the Euclidean or Riemannian exponent  $1 + \frac{1}{v_n} = 1 + 1 = 2$ . However, this is not generally the case; for instance, it won't be the case for a regular subRiemannian manifold as in Example 3.2.

*Proof of Corollary 4.6.* Keeping the notation in the proof of Proposition 4.5, the estimates therein show that when  $r$  is higher order, we have for any  $v \in W$

$$|r_j(v)| = |R_j(v)| \leq C_j |v|_1^{v_j+1},$$

so

$$\max_{j=1, \dots, \dim \mathcal{V}_2} |r_j(v)|^{\frac{1}{v_j}} \leq \max_{j=1, \dots, \dim \mathcal{V}_2} C_j^{\frac{1}{v_j}} |v|_1^{1+\frac{1}{v_j}} \lesssim |v|_1^{1+\frac{1}{v_n}}.$$

As  $u \mapsto \max_{j=1, \dots, \dim \mathcal{V}_2} |u|^{1/v_j}$  is a quasinorm on  $\mathcal{V}_2$  and all the quasinorms are equivalent, the statement is proved.  $\square$

In the second corollary, we consider higher order functions depending continuously and smoothly on a parameter:

**Corollary 4.7.** (1) *Let  $A$  be a topological vector space and  $(a, v) \mapsto r(a, v)$  be a continuous map  $A \times \mathcal{V}_1 \rightarrow \mathcal{V}_2$  such that for every  $a \in A$ ,  $r(a, \cdot)$  is a smooth function on  $\mathcal{V}_1$  that is higher order. Then there exists a unique continuous function  $R : A \times \mathbb{R} \times \mathcal{V}_1 \rightarrow \mathcal{V}_2$  such that for every  $a \in A$ ,  $R(a, \cdot, \cdot)$  satisfies Property (3) of Proposition 4.5, that is,  $R(a, \cdot, \cdot)$  is a smooth function  $\mathbb{R} \times \mathcal{V}_1 \rightarrow \mathcal{V}_2$  satisfying*

$$\forall \varepsilon, v \in \mathbb{R} \setminus \{0\} \times \mathcal{V} \quad \delta_{\varepsilon^{-1}} r(a, \delta_\varepsilon v) = \varepsilon R(a, \varepsilon, v).$$

(2) *Let  $(t, v) \mapsto r(t, v)$  be a smooth map  $I \times \mathcal{V}_1 \rightarrow \mathcal{V}_2$ ,  $I$  an interval of  $\mathbb{R}$ . Assume that for every  $t \in I$ ,  $r$  is higher order in  $v$ . Then  $\partial_{t=0} r(t, v)$  is higher order.*

*Proof.* We use the notation in the proof of Proposition 4.5 and consider bases for  $\mathcal{V}_1$  and  $\mathcal{V}_2$  that are adapted to the gradation where  $r$  has coordinates  $r_j$ . Part (1) follows from Property (4). For Part (2), by Proposition 4.5,  $\partial_v^\alpha r_j(t, 0) = 0$  for any  $t \in I$  and for any  $\alpha$  with  $[\alpha] \leq v_j$ , so it is also the case for  $\partial_{t=0} r(t, v)$  by smoothness and  $\partial_{t=0} r(t, v)$  is higher order.  $\square$

**4.2. Higher order remainders on filtered manifolds.** The notion of higher order term defined above allows us to now investigate the algebraic structure of a filtered manifold: up to remainder terms of higher order, it will be given by homogeneous polynomials or series. Following Corollary 4.7 (1), we extend the notion of higher order to maps depending smoothly on a point  $x$  of the manifold and we will say that such a  $x$ -dependent polynomial or a  $x$ -dependent remainder are homogeneous or higher order above some open subset  $U \subset M$  of interest, when they are thus for all  $x \in U$ .

**4.2.1. Nested commutator brackets on filtered manifolds.** The notion of higher order map defined above together with the map  $\text{ad}_x^\times$  (see Definition 3.12) allow for the description of the commutator bracket on  $M$  modulo higher order terms:

**Proposition 4.8.** *Let  $\mathbb{X}$  be an adapted frame on an open set  $U$  of the filtered manifold  $M$ . The commutator brackets of vector fields may be written as:*

$$\text{ad} \left( \sum_{i=1}^n v_i X_i \right) \left( \sum_{j=1}^n w_j X_j \right) (x) = \sum_{j=1}^n (q_j(x; v, w) + r_j(x; v, w)) X_j,$$

where the  $q_j(x; v, w)$ 's and  $r_j(x; v, w)$ 's are polynomials in  $v, w$  with smooth coefficients in  $x \in U$ ; the  $q_j$ 's are  $v_j$ -homogeneous and described by

$$(q_j(x; v, w))_{j=1}^n = \text{ad}_x^{\mathbb{X}}(v)(w),$$

and  $r(x; v, w) = (r_j(x; v, w))_{j=1}^n$  is higher order in  $v, w$ .

Before entering into the proof of Proposition 4.8, let us state some structural properties of the commutator brackets for  $\mathbb{X}$ . We denote by  $c_{i,j,k}(x)$  the structural constants of the commutator brackets for  $\mathbb{X}$ , that is,

$$\text{ad}(X_i)(X_j)(x) = [X_i, X_j](x) = \sum_{k=1}^n c_{i,j,k}(x) X_k(x);$$

the  $c_{i,j,k}(x)$  are smooth functions of  $x \in U$ . As  $\mathbb{X}$  is adapted to the filtration, we see:

- (a)  $c_{i,j,k} = 0$  when  $v_k > v_i + v_j$ ,
- (b) when  $v_k = v_i + v_j$ , the  $c_{i,j,k}(x)$ 's are the structural constants of  $[\cdot, \cdot]_{\mathfrak{g}_x}$  for  $\langle \mathbb{X} \rangle_x$ :

$$[\langle X_i \rangle_x, \langle X_j \rangle_x]_{\mathfrak{g}_x} = \sum_{v_k=v_i+v_j} c_{i,j,k}(x) \langle X_k \rangle_x,$$

or in other words, using the map  $\text{ad}_x^{\mathbb{X}} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  (see Definition 3.12) and denoting the canonical basis of  $\mathbb{R}^n$  by  $(e_1, \dots, e_n)$ ,

$$\text{ad}_x^{\mathbb{X}}(e_i, e_j) = \sum_{v_k=v_i+v_j} c_{i,j,k} e_k$$

- (c) the polynomial  $(c_{i,j,k}(x) v_i w_j)_{k=1}^n$  from the components of

$$\sum_{v_k < v_i + v_j} c_{i,j,k}(x) v_i w_j X_k(x) = [v_i X_i, w_j X_j](x) - \sum_{v_k = v_i + v_j} c_{i,j,k}(x) v_i w_j X_k(x),$$

are higher order.

*Proof.* Using the structural constants  $c_{i,j,k}$ , we can write

$$\left[ \sum_{i=1}^n v_i X_i, \sum_{j=1}^n w_j X_j \right] (x) = \sum_{v_i+v_j \leq v_k} v_i w_j c_{i,j,k}(x) X_k(x) = \sum_{k=1}^n (q_k(x; v, w) + r_k(x; v, w)) X_k(x),$$

where

$$q_k(x; v, w) := \sum_{v_k=v_i+v_j} c_{i,j,k}(x) v_i w_j.$$

Hence the maps  $q(x; v, w) := (q_k(x; v, w))_{k=1}^n$  and  $r(x; v, w) := (r_k(x; v, w))_{k=1}^n$  are polynomials in  $v, w$  with smooth coefficients in  $x \in U$ . From the observation in (b) above,

$q(x; v, w) = \text{ad}_x^{\mathbb{X}}(v)(w)$  is homogeneous in  $v, w$ , while from the observation in (c),  $r(x; v, w)$  is higher order in  $v, w$ .  $\square$

Proposition 4.8 implies inductively the following property of nested commutator brackets:

**Corollary 4.9.** *Let  $\mathbb{X}$  be an adapted frame on an open set  $U$  of the filtered manifold  $M$ . Let  $\ell \in \mathbb{N}_0$ ,  $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{N}_0^\ell$  and  $\beta = (\beta_1, \dots, \beta_\ell) \in \mathbb{N}_0^\ell$ . We can write*

$$\begin{aligned} & \left( \text{ad} \sum_{i=1}^n v_i X_i \right)^{\alpha_1} \left( \text{ad} \sum_{i=1}^n w_i X_i \right)^{\beta_1} \dots \left( \text{ad} \sum_{i=1}^n v_i X_i \right)^{\alpha_\ell} \left( \text{ad} \sum_{i=1}^n w_i X_i \right)^{\beta_\ell} (x) \\ &= \sum_{k=1}^n (q_k^{\alpha, \beta}(x; v, w) + r_k^{\alpha, \beta}(x; v, w)) X_k(x), \end{aligned}$$

with  $q_k^{\alpha, \beta}(x; v, w)$  and  $r_k^{\alpha, \beta}(x; v, w)$  polynomials in  $v, w$  with smooth coefficients in  $x \in U$ ; the polynomial map  $q(x; v, w) = (q_k^{\alpha, \beta}(x; v, w))_{k=1}^n$  is  $([\alpha] + [\beta])$ -homogeneous and is defined via

$$q^{\alpha, \beta}(x; v, w) := (\text{ad}_x^{\mathbb{X}} v)^{\alpha_1} (\text{ad}_x^{\mathbb{X}} w)^{\beta_1} \dots (\text{ad}_x^{\mathbb{X}} v)^{\alpha_\ell} (\text{ad}_x^{\mathbb{X}} w)^{\beta_\ell} (x)$$

while the polynomial map  $r^{\alpha, \beta}(x; v, w) = (r_k^{\alpha, \beta}(x; v, w))_{k=1}^n$  is higher order.

*Proof.* The proof follows by induction from Proposition 4.8, together with the observations from Examples 4.4 and the fact that if  $q(x; v, w)$  and  $r(x; v, w)$  are polynomial maps that are respectively homogeneous and higher order, then so are their  $x$ -derivatives  $X_\ell q(x; v, w)$  and  $X_\ell r(x; v, w)$ ,  $\ell = 1, \dots, n$  (for the latter, see Corollary 4.7 (2)).  $\square$

4.2.2. *Examples and further properties of higher order remainders.* In this section, we give two further examples and some general properties of higher order remainders.

The first example concerns the laws of the osculating groups, or rather the law  $*_x^{\mathbb{X}}$  of the isomorphic group  $\mathbb{R}^n$  (see (3.9)). Later on, we will need the following property:

**Lemma 4.10.** *Let  $(\mathbb{X}, U)$  be an adapted frame on a filtered manifold  $M$ .*

- (1) *The map  $(x; v, w) \mapsto v *_x^{\mathbb{X}} w$  is a homogeneous polynomial map in  $v, w$  with coefficients depending smoothly in  $x \in U$ , and so are the maps  $(x; v, w) \mapsto \mathbb{X}_x^\beta(v *_x^{\mathbb{X}} w)$ .*
- (2) *The maps defined by*

$$\begin{aligned} r_1(x; v, w) &:= w *_x^{\mathbb{X}} \exp_x^{\mathbb{X}} v - w *_x^{\mathbb{X}} v \\ r_2(x; v, w) &:= -(w *_x^{\mathbb{X}} v) *_x^{\mathbb{X}} \left( w *_x^{\mathbb{X}} \exp_x^{\mathbb{X}} v \right) \end{aligned}$$

*are smooth on an open neighbourhood of the zero section of  $U \times \mathbb{R}^n \times \mathbb{R}^n$ . They are higher order in  $v, w$ .*

*Proof.* Part (1) follows from Lemma 3.13 and the homogeneity of the group law (see Example 4.2). For Part (2) and  $r_1$ , we have by the homogeneity of the group law

$$\begin{aligned} \delta_\varepsilon^{-1} r_1(x; \delta_\varepsilon v, \delta_\varepsilon w) &= w *_x^{\mathbb{X}} \exp_x^{\mathbb{X}} \delta_\varepsilon v - w *_x^{\mathbb{X}} \delta_\varepsilon v \\ &= \text{BCH}_{s_0}(v, w; \text{ad}_{\exp_x^{\mathbb{X}} \delta_\varepsilon v}^{\mathbb{X}}) - \text{BCH}_{s_0}(v, w; \text{ad}_x^{\mathbb{X}}); \end{aligned}$$

this last expression tends to 0 as  $\varepsilon \rightarrow 0$  since  $\text{BCH}_{s_0}(v, w; \text{ad}_x^{\mathbb{X}})$  is a polynomial expression in  $\text{ad}_x^{\mathbb{X}}v$  and  $\text{ad}_x^{\mathbb{X}}w$  and  $\lim_{\varepsilon \rightarrow 0} \text{ad}_{\exp_x^{\mathbb{X}} \delta_\varepsilon v}^{\mathbb{X}} = \text{ad}_x^{\mathbb{X}}$ . For  $r_2$ , we see

$$\delta_\varepsilon^{-1} r_2(x; \delta_\varepsilon v, \delta_\varepsilon w) = \left( -(w *_x^{\mathbb{X}} v) \right) *_x^{\mathbb{X}} \left( w *_x^{\mathbb{X}} \exp_x^{\mathbb{X}} \delta_\varepsilon v \right) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

We conclude with Proposition 4.5. □

Before giving the second example, let us show some properties of higher order remainders:

**Lemma 4.11.** *Let  $(\mathbb{X}, U)$  be an adapted frame on a filtered manifold  $M$ . Let  $r : U \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a smooth higher order map. Then, the maps*

$$(v, w) \mapsto r_x(v *_x^{\mathbb{X}} w, v), \quad (v, w) \mapsto r_{\exp_x^{\mathbb{X}} v}(v, w), \quad (v, w) \mapsto (-v) *_x^{\mathbb{X}} (v + r_x(v, w)),$$

are also smooth and higher order.

Note that this lemma extends to functions  $r_x$  depending only on the variable  $v$ .

The proof relies on the characterizations of higher order functions in Proposition 4.5, and its extensions to functions depending on a parameter in Corollary 4.7.

*Proof.* For analyzing the function,  $(v, w) \mapsto r_x(v *_x^{\mathbb{X}} w, v)$ , we observe that by the properties of dilation with respect to the group law, for  $v, w \in \mathbb{R}^n$ ,

$$\delta_\varepsilon^{-1} r_x(\delta_\varepsilon v *_x^{\mathbb{X}} \delta_\varepsilon w, \delta_\varepsilon v) = \delta_\varepsilon^{-1} r_x(\delta_\varepsilon(v *_x^{\mathbb{X}} w), \delta_\varepsilon v).$$

Therefore

$$\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon^{-1} r_x(\delta_\varepsilon v *_x^{\mathbb{X}} \delta_\varepsilon w, \delta_\varepsilon v) = 0,$$

and  $(v, w) \mapsto r_x(v *_x^{\mathbb{X}} w, v)$  is higher order by Proposition 4.5.

Regarding the function  $(v, w) \mapsto r_{\exp_x^{\mathbb{X}} v}(v, w)$ , consider the function  $R = R(x, v, w, \varepsilon)$  such that  $\delta_\varepsilon^{-1} r(x, \delta_\varepsilon v, \delta_\varepsilon w) = \varepsilon R(x, v, w, \varepsilon)$ . By Corollary 4.7 (1),  $R$  is continuous in  $x, v, w, \varepsilon$  and we have

$$\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon^{-1} r(\exp_x^{\mathbb{X}} \delta_\varepsilon v, \delta_\varepsilon v, \delta_\varepsilon w) = \lim_{\varepsilon \rightarrow 0} \varepsilon R(\exp_x^{\mathbb{X}} \delta_\varepsilon v, v, w) = 0,$$

showing that  $(v, w) \mapsto r_{\exp_x^{\mathbb{X}} v}(v, w)$  is higher order by Proposition 4.5.

Consider now  $(v, w) \mapsto (-v) *_x^{\mathbb{X}} (v + r_x(v, w)) =: \tilde{r}_x(v, w)$ . By homogeneity of the group laws, we have:

$$\delta_\varepsilon^{-1} \tilde{r}_x(\delta_\varepsilon v, \delta_\varepsilon w) = (-v) *_x (v + \delta_\varepsilon^{-1} r_x(\delta_\varepsilon v, \delta_\varepsilon w)) = \text{BCH}_{s_0}(-v, v + \delta_\varepsilon^{-1} r_x(\delta_\varepsilon v, \delta_\varepsilon w), \text{ad}_x^{\mathbb{X}})$$

As  $r$  is higher order, by Proposition 4.5,

$$\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon^{-1} r_x(\delta_\varepsilon v, \delta_\varepsilon w) = 0 \quad \text{so} \quad \lim_{\varepsilon \rightarrow 0} \delta_\varepsilon^{-1} \tilde{r}_x(\delta_\varepsilon v, \delta_\varepsilon w) = (-v) *_x v = 0,$$

and  $\tilde{r}_x$  is higher order. □

The second example of higher order remainders we will use later on occurs when considering changes of adapted frames:

**Proposition 4.12.** *We consider two local frames  $\mathbb{X}$  and  $\mathbb{Y}$  as in Lemma 3.5 (whose notation we keep), with the additional assumption that they are adapted to the filtration of  $M$ . The map  $r = r(x, v) = r_x(v)$  defined via*

$$(4.1) \quad -\text{diag}(T_x)^{-1}u_x(-v) = v *_x^{\mathbb{X}} r(x, v), \quad (x, v) \in U_{\mathbb{X}, \mathbb{Y}}$$

*is smooth on  $U_{\mathbb{X}, \mathbb{Y}}$  and higher order.*

*Proof.* By Lemma 3.5, the Taylor series of  $u_x(v) - \text{diag}(T_x)v$  about  $v \sim 0$  is given by

$$u_x(v) - \text{diag}(T_x)v \sim (T_x - \text{diag}(T_x))v + \sum_{k \geq 2} \frac{1}{k!} \left( \sum_j v_j X_j \right)^{k-1} T_x v.$$

Since  $(T_x - \text{diag}(T_x))v$  and each term of the series  $\sum_{k > 2}$  are higher order in  $v$ ,  $u_x(v) - \text{diag}(T_x)v$  is higher order. Hence, the function

$$\tilde{r}_x(v) := -\text{diag}(T_x)^{-1}u_x(-v) - v$$

is smooth on  $U_{\mathbb{X}, \mathbb{Y}}$  and higher order. By Lemma 4.11,  $r_x(v) = (-v) *_x (v + \tilde{r}_x(v))$  is higher order.  $\square$

**4.3. Composition of geometric exponential mappings.** In this section, we study the composition of geometric exponential mappings, comparing it with the law of the osculating group. Then, we derive technical properties that we shall use later.

**4.3.1. Relation with the law group.** The core of this section is the next result which is expressed using the notion of higher order developed in Section 4.1 and the structure of the osculating groups via the group law  $*_x^{\mathbb{X}}$  defined in (3.9):

**Theorem 4.13.** *Let  $\mathbb{X}$  be an adapted frame on an open set  $U \subset M$ . Then  $\ln_x^{\mathbb{X}}(\exp_{\exp_x^{\mathbb{X}}} w(v))$  may be written in the form:*

$$(4.2) \quad \ln_x^{\mathbb{X}}(\exp_{\exp_x^{\mathbb{X}}} w(v)) = w *_x^{\mathbb{X}} v + r(x; v, w);$$

*the map  $r(x; v, w)$  is smooth on an open neighbourhood of the 0-section of  $U \times \mathbb{R}^n \times \mathbb{R}^n$ , valued in  $\mathbb{R}^n$  and higher order in  $v, w$ .*

*Proof of Theorem 4.13.* By [55, Appendix], for any  $f \in C_c^\infty(U)$ , we have the following equality between Taylor series for  $v, w \sim 0$ :

$$(4.3) \quad f(\exp_{\exp_x^{\mathbb{X}}} w(v)) \sim f(\exp_x Z(v, w)),$$

where  $\exp_x Z$  denotes the 1-time flow of a vector field  $Z$  on  $U$  with initial data  $x$  (see Section 3.3) and  $Z(v, w)$  formally given by the BCH formula in (2.1) at  $x$ ; in the BCH, the adjoint operation  $\text{ad}$  refers to the commutator bracket between vector fields,  $a = \sum_i w_i X_i$  and  $b = \sum_i v_i X_i$ . The meaning of (4.3) is that for any  $N \in \mathbb{N}_0$ , we have:

$$(4.4) \quad \begin{aligned} f(\exp_{\exp_x^{\mathbb{X}}} w(v)) &= f(\exp_x(Z^{(N)}(v, w))) + O(|v|^{N+1} + |w|^{N+1}), \\ &= \sum_{k=0}^N \frac{1}{k!} ((Z^{(N)}(v, w))^k f)(x) + O(|v|^{N+1} + |w|^{N+1}), \end{aligned}$$

where  $Z^{(N)}(v, w)$  denotes the truncation in  $(v, w)$  at order  $N$  of the formal series defining  $Z(v, w)$ , i.e. with Notation 2.12:

$$Z^{(N)}(v, w) = \text{BCH}_N\left(\sum_i w_i X_i, \sum_i v_i X_i, \text{ad}\right).$$

Note that  $Z^{(N)}(v, w)$  is indeed a vector field on  $U$ , and its  $\mathbb{X}$ -coordinates, i.e.

$$z^{(N)}(x; v, w) = (z_j^{(N)}(x; v, w))_{j=1}^n, \quad Z^{(N)}(v, w)(x) = \sum_{j=1}^n z_j^{(N)}(x; v, w) X_j(x),$$

are polynomial in  $v, w$  and smooth in  $x \in U$ . Because of this dependence in  $x$ , in general, we expect that  $\exp_x^{\mathbb{X}}(z^{(N)}(x; v, w))$  be different from  $\exp_x(Z^{(N)}(v, w))$ .

We will need the following two observations.

(i) Firstly, Corollary 4.9 implies that

$$z^{(N)}(x; v, w) = \tilde{z}^{(N)}(x; v, w) + r^{(N)}(x; v, w),$$

where  $r^{(N)}$  and  $\tilde{z}^{(N)}$  are  $N$ -truncated power series in  $v, w$  with smooth coefficients in  $x \in U$  with  $r^{(N)}$  of higher order and  $\tilde{z}^{(N)}$  given by BCH for  $\text{ad}_x^{\mathbb{X}}$ . By Proposition 4.8, if  $N > s$  where  $s$  is the step of the gradation, we have

$$\tilde{z}^{(N)} := \text{BCH}_N(w, v; \text{ad}_x^{\mathbb{X}}) = w *_x^{\mathbb{X}} v.$$

(ii) Secondly, omitting the writing of the dependence in  $v, w$ , we have

$$\begin{aligned} (Z^{(N)})^2 &= \sum_{j_1, j_2} \left( z_{j_1}^{(N)} X_{j_1} z_{j_2}^{(N)} X_{j_2} \right) \\ &= \sum_{j_1, j_2} z_{j_1}^{(N)} z_{j_2}^{(N)} X_{j_1} X_{j_2} + \sum_{j_1} z_{j_1}^{(N)} \sum_j (X_{j_1} z_j^{(N)}) X_j. \end{aligned}$$

We write  $\mathbb{X}_I := X_{i_1} \dots X_{i_k}$  for any multi-index  $I = (i_1, \dots, i_k) \in \{1, \dots, n\}^k$  of length  $|I| = k$ . Then, inductively, we can write for  $k = 2, 3, \dots$

$$(4.5) \quad (Z^{(N)})^k = \sum_{(j_1, \dots, j_k)=[I, I_1, \dots, I_k]} (\mathbb{X}_{I_1} z_{j_1}^{(N)}) \dots (\mathbb{X}_{I_k} z_{j_k}^{(N)}) \mathbb{X}_I,$$

where the sum is over the multi-indices  $I, I_1, \dots, I_k$  with at least one of the  $I_1, \dots, I_k$  empty but  $I \neq \emptyset$ , and such that their concatenation  $[I, I_1, \dots, I_k] = (j_1, \dots, j_k)$  is in  $\{1, \dots, n\}^k$  with  $j_i \notin I_i$  and  $(j_1, \dots, j_k)$  runs over  $\{1, \dots, n\}^k$ . This relation is proved in the Appendix A.

We can apply the formula in (4.4) to functions  $f$  that are valued in a finite dimensional vector space. We choose  $f$  as  $\ln_x^{\mathbb{X}}$  locally. Proceeding for instance as in the proof of Lemma 3.4, we see that for  $f = \ln_x^{\mathbb{X}}$ , we have  $f(x) = 0$  and for any  $u \in \mathbb{R}^n$

$$(4.6) \quad \sum_j u_j X_j f(x) = u = \sum_j u_j e_j,$$

with  $(e_1, \dots, e_n)$  denoting the canonical basis of  $\mathbb{R}^n$ . Moreover, for  $k > 1$ ,

$$(4.7) \quad \left( \sum_j u_j X_j \right)^k f(x) = \sum_{j_1, \dots, j_k} u_{j_1} \dots u_{j_k} (X_{j_1} \dots X_{j_k} f)(x) = 0.$$

In other words, the (non-commutative) polynomial expression

$$(4.8) \quad \sum_{I=(j_1, \dots, j_\ell)} u_{j_1} \dots u_{j_\ell} \mathbb{X}_{(j_1, \dots, j_\ell)}$$

vanishes when applied to  $f = \ln_x^{\mathbb{X}}$  at  $x$  if  $\ell > 1$ . Hence, we focus in identifying such expressions in (4.5). For this we will use the third following observation:

(iii) Reorganising the terms of  $(Z^{(N)})^k$  in terms of the cardinal of the indices  $I$  in equation (4.5), and writing

$$(4.9) \quad (Z^{(N)})^k = \sum_{\ell=1}^k \sum_{I=(j_1, \dots, j_\ell)} a_I \mathbb{X}_I,$$

the terms  $\sum_{I=(j_1, \dots, j_\ell)} a_I \mathbb{X}_I$  is the sum of terms that have the property (4.8) for  $\ell \geq 1$ . This is proved in the Appendix A.

In view of (iii) and of (4.7), when applying  $(Z^{(N)})^k$  written as in (4.9) to the function  $f = \ln_x^{\mathbb{X}}$  at  $x$ , the  $\ell$ -th term vanishes when  $\ell > 1$ . Hence, we are left with the sum over the  $I$ 's with only 1 element:

$$\begin{aligned} (Z^{(N)})^k f(x) &= \sum_{j=1}^n \sum_{(j_1, \dots, j_k) \in \{\{j\}, I_1, \dots, I_k\}} (\mathbb{X}_{I_1} z_{j_1}) \dots (\mathbb{X}_{I_k} z_{j_k}) X_j f(x) \\ &= \sum_{j=1}^n \sum_{(j_1, \dots, j_k) \in \{\{j\}, I_1, \dots, I_k\}} (\mathbb{X}_{I_1} z_{j_1}) \dots (\mathbb{X}_{I_k} z_{j_k}) e_j. \end{aligned}$$

If  $k > 1$ , each term in the sum above is a product of  $k$   $x$ -derivatives of the higher order polynomials  $z_{j_i}(x; v, w)$ 's in  $(v, w)$ ; so  $(Z^{(N)})^k f(x)$  is of higher order.

Therefore, using  $f(x) = 0$  and equation (4.6), we obtain for any  $N > s$

$$\ln_x^{\mathbb{X}}(\exp_{\exp_x^{\mathbb{X}}}^{\mathbb{X}} w(v)) = z^{(N)}(x; v, w) + \sum_{k=2}^N \frac{1}{k!} (Z^{(N)}(v, w)^k f)(x) + O(|v|^{N+1} + |w|^{N+1}),$$

with  $z^{(N)}(x; v, w) = \tilde{z}(x; v, w) + r^{(N)}(x; v, w)$ , the polynomial  $\tilde{z}(x; v, w) = w *_x^{\mathbb{X}} v$  being homogeneous while  $r^{(N)}(x; v, w)$  and every  $((Z^{(N)}(v, w))^k f)(x)$ ,  $k > 1$ , are of higher order. The conclusion follows.  $\square$

**4.3.2. Consequences of Theorem 4.13.** We will use Theorem 4.13 via the following statement which will bring key arguments while proving the main properties of the pseudodifferential calculus. This section can be skipped at first reading. We recall that when a frame  $(\mathbb{X}, U)$  has been fixed, for  $x \in U$ ,  $*_x^{\mathbb{X}}$  describes the law of the group  $G_x M$ . We use the shorthand  $*_x^{\mathbb{X}} = *_x$  in this section.

**Corollary 4.14.** *We continue with the setting of Theorem 4.13.*

(1) *We may write*

$$-\ln_{\exp_x^{\mathbb{X}} v}^{\mathbb{X}} \exp_x^{\mathbb{X}}(v *_x (-w)) = w *_x \tilde{r}(x; v, w);$$

above,  $\tilde{r}$  is a smooth function on a 0-section of  $M \times \mathbb{R}^n \times \mathbb{R}^n$  where  $(x, v, w) \mapsto -\ln_{\exp_x^{\mathbb{X}} v}^{\mathbb{X}} \exp_x^{\mathbb{X}}(v *_x (-w))$  is defined and smooth. Moreover,  $\tilde{r}(x; v, w)$  is higher order in  $(v, w)$  and vanishes at  $v = 0$  and at  $w = 0$ , so we have locally

$$|\tilde{r}(x; v, w)|_{\mathbb{X}} = O(|v|_{\mathbb{X}}), \quad O(|w|_{\mathbb{X}}), \quad O(|v|_{\mathbb{X}} + |w|_{\mathbb{X}})^{1+\frac{1}{v_n}}.$$

(2) *We may write*

$$-\ln_x^{\mathbb{X}} \exp_{\exp_x^{\mathbb{X}} -v}^{\mathbb{X}}(v *_x (-w)) = w *_x \check{r}(x; v, w);$$

above  $\check{r}$  is a smooth function on a 0-section of  $M \times \mathbb{R}^n \times \mathbb{R}^n$  where  $(x, v, w) \mapsto -\ln_x^{\mathbb{X}} \exp_{\exp_x^{\mathbb{X}} -v}^{\mathbb{X}}((-w) *_x v)$  is defined and smooth. Moreover,  $\check{r}(x; v, w)$  is higher order in  $(v, w)$  and vanishes at  $v = 0$  and at  $w = 0$ , so we have locally

$$|\check{r}(x; v, w)|_{\mathbb{X}} = O(|v|_{\mathbb{X}}), \quad O(|w|_{\mathbb{X}}), \quad O(|v|_{\mathbb{X}} + |w|_{\mathbb{X}})^{1+\frac{1}{v_n}}.$$

Above, we have used the quasinorm  $|\cdot|_{\mathbb{X}}$  defined in Remark 3.11.

Before entering into the proof of Corollary 4.14, let us explain what is meant by ‘locally’ in the statement. Actually, the function  $\tilde{r}$  above is simply defined as

$$(4.10) \quad \tilde{r}(x; v, w) = (-w) *_x \left( -\ln_{\exp_x^{\mathbb{X}} v}^{\mathbb{X}} \exp_x^{\mathbb{X}}(v *_x (-w)) \right),$$

and the fact that the inequality above holds locally means that for any compact subset  $\mathcal{K}$  of  $U_{\mathbb{X}}$  satisfying

$$\forall x \in U, \quad w, v \in \mathbb{R}^n, \quad (x, w), (x, v) \in \mathcal{K} \implies (\exp_x^{\mathbb{X}} v, \exp_x^{\mathbb{X}}(v *_x (-w))) \in \exp^{\mathbb{X}}(U_{\mathbb{X}}),$$

there exists  $C > 0$  such that for any  $x \in U$ ,  $v, w \in \mathbb{R}^n$  with  $(x, w)$  and  $(x, v)$  lying in  $\mathcal{K}$  we have

$$\begin{aligned} |\tilde{r}(x; v, w)|_{\mathbb{X}} &\leq C|v|_{\mathbb{X}}, \quad |r_1(x; v, w)|_{\mathbb{X}} \leq C|w|_{\mathbb{X}}, \\ |\tilde{r}(x; v, w)|_{\mathbb{X}} &\leq C(|v|_{\mathbb{X}} + |w|_{\mathbb{X}})^{1+\frac{1}{v_n}}, \end{aligned}$$

the constant  $C$  depends on  $N, \mathcal{K}, \mathbb{X}, M$ , while the hypotheses on  $\mathcal{K}$ , besides its compactness, ensures that the left-hand side of the inequality above makes sense. The meaning of the local estimates in Part 2 of Corollary 4.14 is similar to the one of Part 1.

*Proof of Corollary 4.14.* By Theorem 4.13 (and using its notation), we may write:

$$\ln_{\exp_x^{\mathbb{X}} v}^{\mathbb{X}} \exp_x^{\mathbb{X}}(v *_x (-w)) = r(\exp_x v; v *_x (-w), -v) - w,$$

with  $r$  higher order. The function  $\tilde{r}$  defined via (4.10) is smooth and also higher order by Lemmata 4.10 and 4.11. We also have

$$\begin{aligned} \tilde{r}(x; v, 0) &= \ln_{\exp_x^{\mathbb{X}} v}^{\mathbb{X}} \exp_x^{\mathbb{X}} v = 0, \\ \tilde{r}(x; 0, w) &= (-w) *_x (\ln_x^{\mathbb{X}} \exp_x^{\mathbb{X}} w) = (-w) *_x w = 0. \end{aligned}$$

The first two local estimates follow from applying the Mean Value Theorem of [41] (see Theorem 2.17 (1) and Remark 2.18). For the third local estimate, we apply Corollary 4.6 to  $\mathcal{V}_1 = \mathbb{R}^n \times \mathbb{R}^n$  and  $\mathcal{V}_2 = \mathbb{R}^n$ , with  $|\cdot|_{\mathbb{X}}$  the quasinorm on  $\mathcal{V}_2$ , and  $(v, w) \mapsto |v| + |w|$  the quasinorm on  $\mathcal{V}_1$ ; the estimate is indeed locally uniform by Corollary 4.7 (1) and the proof of Corollary 4.6. This shows Part 1.

For Part 2, we proceed as for Part 1 having observed the two following formulae:

$$\begin{aligned} \check{r}(x; v, w) &= (-w) *_x \left( -\ln_x^{\mathbb{X}} \exp_{\exp_x^{\mathbb{X}} -v}^{\mathbb{X}} v *_x (-w) \right), \\ \ln_x^{\mathbb{X}} \exp_{\exp_x^{\mathbb{X}} -v}^{\mathbb{X}} v *_x (-w) &= -w + r(x; v *_x (-w), -v). \end{aligned}$$

□

## 5. SYMBOLS CLASSES ON FILTERED MANIFOLDS

In this section, we introduce the notion of symbols on filtered manifolds in Section 5.1 and define symbol classes in Section 5.2. We study their algebraic properties in Section 5.3 and focus on two important types of symbols: on the one hand, spectral multipliers in Section 5.4 and, on the other hand, polyhomogeneous symbols in Section 5.5. The quantization process of these symbols will be described in Section 7, after studying their convolution kernels in Section 6.

### 5.1. The symbols and their convolution kernels.

#### 5.1.1. Definitions.

**Definition 5.1.** A *symbol*  $\sigma$  on  $M$  is a collection of invariant symbols

$$\sigma(x) = \{\sigma(x, \pi) : \mathcal{H}_{\pi}^{+\infty} \rightarrow \mathcal{H}_{\pi}^{-\infty} : \pi \in \widehat{G}_x M\}$$

on  $\widehat{G}_x M$  parametrised by  $x \in M$ .

Alternatively, we will see a symbol as a collection of fields of operators

$$\sigma = \{\sigma(x, \pi) : \mathcal{H}_{\pi}^{+\infty} \rightarrow \mathcal{H}_{\pi}^{-\infty} : x \in M, \pi \in \widehat{G}_x M\}.$$

The operators are defined up to the unitary equivalence of representation (see Section 2.2) and are parametrised by

$$\widehat{G}M := \{(x, \pi) : x \in M, \pi \in \widehat{G}_x M\}.$$

Our definition of convolution kernels relies on smooth Haar systems whose definition we now recall:

**Definition 5.2.** A smooth Haar system  $\mu = \{\mu_x\}_{x \in M}$  for  $GM$  is a choice of Haar measure  $\mu_x$  for each of the group fibers  $G_x M$  such that the map  $x \mapsto \int_{G_x M} f(x, v) d\mu_x(v)$  is smooth on  $M$  for any  $f \in C_c^\infty(GM)$ .

*Remark 5.3.* Alternatively, a smooth Haar system  $\mu$  is the image under the group bundle exponential mapping  $\text{Exp} : \mathfrak{g}M \rightarrow GM$  of a positive 1-density of the vector bundle  $\mathfrak{g}M$ , i.e.  $\mu = \text{Exp}_* w$ ,  $w \in |\Omega|^+(\mathfrak{g}M)$  (see Appendix B.1 for these notions).

**Definition 5.4.** Let  $\{\mu_x\}_{x \in M}$  be a smooth Haar system. Let  $\sigma$  be a symbol on  $M$ . If for every  $x \in M$ , the following makes sense

$$(5.1) \quad \kappa_{\sigma,x}^\mu = \mathcal{F}_{G_x M, \mu_x}^{-1} \sigma(x),$$

where  $\mathcal{F}_{G_x M, \mu_x}$  denotes the Fourier transform on  $G_x M$  with respect to the Haar measure  $\mu_x$ , then we say that  $\sigma$  admits a convolution kernel. The distribution  $\kappa_{\sigma,x}^\mu \in \mathcal{S}'(G_x M)$  is called the convolution kernel of  $\sigma$  at  $x \in M$  with respect to  $\mu$ .

*Example 5.5.* If  $\sigma$  is a symbol such that for any  $x \in M$ , we have  $\sigma(x) \in L_{a,b}^\infty(\widehat{G}_x M)$  for some  $a, b \in \mathbb{R}$  (possibly depending on  $x$ ), then  $\sigma$  admits a convolution kernel. For instance, this is the case for a symbol such that  $\sigma(x)$  is in  $S^m(\widehat{G}_x M)$  for every  $x \in M$ .

Let  $\mu' = \{\mu'_x\}_{x \in M}$  be another smooth Haar system. Then  $\mu'_x$  is another Haar measure for  $G_x M$ , which is therefore equal to  $\mu_x$  up to a constant  $c(x) > 0$ . This defines a positive smooth function  $c \in C^\infty(M)$  satisfying

$$(5.2) \quad \mu'_x = c(x)\mu_x.$$

Hence, if  $\sigma$  admits a convolution kernel with respect to one smooth Haar system  $\mu$ , then it admits a convolution kernel with respect to any smooth Haar system  $\mu'$ . Moreover, even though  $\kappa_{\sigma,x}^\mu$  depends on the choice of such a Haar system, its existence is independent of which Haar system is chosen and we have:

$$(5.3) \quad \kappa_{\sigma,x}^\mu = c(x)\kappa_{\sigma,x}^{\mu'}, \quad x \in M.$$

5.1.2. *Convolution kernels in an adapted frame.* If  $\mathbb{X}$  is an adapted frame on an open set  $U \subset M$ , this induces a smooth Haar system  $\mu^\mathbb{X} = \{\mu_x^\mathbb{X}\}_{x \in U}$  on  $U$  given by

$$\mu^\mathbb{X} = (\text{Exp}^\mathbb{X})_* dv, \quad \text{i.e.} \quad \int_{G_x M} f d\mu_x^\mathbb{X} = \int_{\mathbb{R}^n} f(\text{Exp}_x^\mathbb{X} v) dv,$$

where  $dv$  is the Lebesgue measure on  $\mathbb{R}^n$ .

Let  $\sigma$  be a symbol on  $M$  admitting a convolution kernel. We consider the convolution kernel  $\kappa_{\sigma,x}^{\mu^\mathbb{X}} = \kappa_x^{\mu^\mathbb{X}} \in \mathcal{S}'(G_x M)$  at  $x \in U$  corresponding to the induced Haar measure  $\mu^\mathbb{X}$ , or, equivalently, its composition with the group exponential mapping:

$$(5.4) \quad \kappa_x^\mathbb{X}(v) := \kappa_x^{\mu^\mathbb{X}}(\text{Exp}_x^\mathbb{X} v).$$

Note that

$$\kappa_x^\mathbb{X}(v) dv = (\text{Exp}^\mathbb{X})^*(\kappa_x^{\mu^\mathbb{X}} d\mu_x^\mathbb{X}),$$

and at least formally:

$$\sigma(x, \pi) = \int_{\mathbb{R}^n} \kappa_x^\mathbb{X}(v) \pi(\text{Exp}_x^\mathbb{X} - v) dv,$$

in the sense that  $\sigma(x, \cdot)$  is the group Fourier transform of  $\kappa_x^{\mu^\mathbb{X}} \in \mathcal{S}'(G_x M)$  for the Haar measure  $\mu^\mathbb{X}$ .

**Definition 5.6.** We call the map  $\kappa_x^{\mu^\mathbb{X}} \in \mathcal{S}'(G_x M)$  or  $\kappa_x^\mathbb{X} \in \mathcal{S}'(\mathbb{R}^n)$  (as in (5.4)) the *convolution kernel in the  $\mathbb{X}$ -coordinates* at  $x$ , depending on whether we see this object as a distribution on  $G_x M$  or on  $\mathbb{R}^n$ .

The relation between the convolution kernels in two different frames is explained in the following statement.

**Lemma 5.7.** *Let  $\mathbb{X}$  and  $\mathbb{Y}$  be two adapted frames on an open subset  $U$  of  $M$ . Let  $\sigma$  be a symbol admitting a convolution kernel. Then, using the notation of Lemma 3.3, we have:*

$$\kappa_{\sigma,x}^{\mathbb{X}}(v) = \det(\theta_x) \kappa_{\sigma,x}^{\mathbb{Y}}(\theta_x v), \quad \text{where } \theta_x := \text{diag } T_x.$$

In other words, considering the notation of Lemma 2.21 and the isomorphism  $\theta_x = \text{diag } T_x$  of the Lie group  $G$ , we have

$$(\theta_x)_* \mathcal{F}_{G_x M}(\kappa_{\sigma,x}^{\mathbb{X}} \circ \text{Ln}_x^{\mathbb{X}}) = \mathcal{F}_{G_x M}(\kappa_{\sigma,x}^{\mathbb{Y}} \circ \text{Ln}_x^{\mathbb{Y}}).$$

*Proof.* The formula follows from

$$\sigma(x, \pi) = \int_{\mathbb{R}^n} \kappa_{\sigma,x}^{\mathbb{X}}(v) \pi(\text{Exp}_x^{\mathbb{X}} v) dv = \int_{\mathbb{R}^n} \kappa_{\sigma,x}^{\mathbb{Y}}(v_1) \pi(\text{Exp}_x^{\mathbb{Y}} v_1) dv_1,$$

and Lemma 3.10. □

5.1.3. *First examples.* Let us describe some examples that we will be develop along this section.

*Example 5.8.* A function  $f \in C^\infty(M)$  defines the symbol

$$f\mathbf{I} = \{f(x)\mathbf{I}_{\mathcal{H}_\pi}, (x, \pi) \in \widehat{G}M\},$$

with convolution kernel  $\kappa_f^{\mathbb{X}} := f(x)\delta_0$  in any  $\mathbb{X}$ -coordinates.

The following example follows readily from Example 2.32:

*Example 5.9.* The symbol

$$\widehat{\langle X_j \rangle} := \{\pi(\langle X_j \rangle_x) : x \in U, \pi \in \widehat{G}_x M\}$$

admits  $\kappa_{\widehat{\langle X_j \rangle}}^{\mathbb{X}} := \partial_{v_j} \delta_0$  as convolution kernel in the  $\mathbb{X}$ -coordinates.

In order to generalise this example to higher order, we introduce the following notation:

*Notation 5.10.* We denote by  $L_{\langle X_j \rangle_x}$  and  $R_{\langle X_j \rangle_x}$  the vector fields of  $\mathbb{R}^n$  corresponding to the left and right-invariant vector fields associated with  $\langle X_j \rangle_x \in \mathfrak{g}_x M$  on the Lie group  $G_x M$ : for  $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$L_{\langle X_j \rangle_x} \varphi(v) = (\langle X_j \rangle(\varphi \circ \text{Ln}_x^{\mathbb{X}}))(\text{Exp}_x^{\mathbb{X}} v) \quad \text{and} \quad R_{\langle X_j \rangle_x} \varphi(v) = (\widetilde{\langle X_j \rangle}(\varphi \circ \text{Ln}_x^{\mathbb{X}}))(\text{Exp}_x^{\mathbb{X}} v).$$

We shall use the short-hand:

$$L_{\langle \mathbb{X} \rangle_x}^\beta := L_{\langle X_1 \rangle_x}^{\beta_1} \cdots L_{\langle X_n \rangle_x}^{\beta_n}, \quad R_{\langle \mathbb{X} \rangle_x}^\gamma := L_{\langle X_1 \rangle_x}^{\gamma_1} \cdots L_{\langle X_n \rangle_x}^{\gamma_n}.$$

*Example 5.11.* Revisiting Example 5.9, we observe that the symbol

$$\widehat{\langle \mathbb{X} \rangle}^\alpha := \{\pi(\langle X \rangle_x^\alpha) : x \in U, \pi \in \widehat{G}_x M\}$$

admits  $\kappa_{\widehat{\langle \mathbb{X} \rangle}^\alpha}^{\mathbb{X}} := L_{\langle \mathbb{X} \rangle_x}^\alpha \delta_0$  as convolution kernel in the  $\mathbb{X}$ -coordinates.

5.1.4. *Smooth symbols and their derivatives.*

**Definition 5.12.** Let  $\sigma$  be a symbol on  $M$  admitting a convolution kernel. The symbol  $\sigma$  or its convolution kernel is *smooth in the variable  $x$*  (or just smooth) when for any adapted frame  $(\mathbb{X}, U)$ , the convolution kernel in the  $\mathbb{X}$ -coordinates is smooth in  $x$  in the sense that

$$\kappa^{\mathbb{X}} \in C^\infty(U, \mathcal{S}'(\mathbb{R}^n)).$$

A symbol admitting a convolution kernel is smooth in the variable  $x$  on a open subset  $U \subset M$  small enough (to admit adapted frames) if there is one adapted frame  $\mathbb{X}$  on  $U$  where  $\kappa^{\mathbb{X}} \in C^\infty(U, \mathcal{S}'(\mathbb{R}^n))$ . This notion is invariantly defined since Lemma 5.7 implies that  $\kappa^{\mathbb{Y}} \in C^\infty(U, \mathcal{S}'(\mathbb{R}^n))$  for any other adapted frame  $\mathbb{Y}$  on  $U$ . We will give it an invariant meaning in Section 5.1.5 below.

We observe that the space of smooth symbols is a module over  $C^\infty(M)$ .

*Notation 5.13.* For any smooth symbol  $\sigma$  and any multi-index  $\beta \in \mathbb{N}_0^n$ , we adopt the notation

$$D_{\mathbb{X}}^\beta \sigma$$

for the symbol (when it exists) whose convolution kernel in the  $\mathbb{X}$ -coordinates is  $\mathbb{X}_x^\beta \kappa_x^{\mathbb{X}}$  at any  $x \in U$ . If  $|\beta| = 1$ , that is, if  $\beta = e_j$  is the index which is 0 everywhere except at the  $j$ th entry where it is 1, we may write  $D_{X_j}$  instead of  $D_{\mathbb{X}}^\beta$ . As we shall see later (see Remark 5.26), the derivation  $D_{\mathbb{X}}$  satisfies Leibniz rules.

5.1.5. *Invariant meaning of convolution kernels.* Here, we discuss the invariant definition of convolution kernels as distributional densities. This viewpoint is not required in practice in our subsequent analysis. However, it sheds a geometric light on the above considerations. We will use the language of vector bundles and of their functional spaces recalled in Appendix B. We will apply it to the vector bundle  $\mathfrak{g}M$  comprised of the osculating Lie algebras.

We observe that an adapted frame  $(\mathbb{X}, U)$  yields a local trivialisation of the vector bundle  $\mathfrak{g}M$ , that is,

$$(5.5) \quad \phi_{\mathbb{X}} : U \times \mathbb{R}^n \rightarrow \mathfrak{g}M, \quad \phi_{\mathbb{X}}(x, v) = (x, \sum_j v_j \langle X_j \rangle_x).$$

Conversely, given a local trivialisation of the vector bundle  $\mathfrak{g}M$ , we may operate a linear change of coordinates and assume that it comes from an adapted frame. The properties of the change of frames in Lemma 5.7 together with (5.3) may be summarised in the following statement:

**Lemma 5.14.** *If  $\sigma$  is a smooth symbol on  $M$ , then there exists a unique tempered vertical density  $\kappa$  in  $\mathcal{S}'(\mathfrak{g}M, |\Omega|(\mathfrak{g}M))$  such that  $\phi_{\mathbb{X}}^* \kappa(x, v) = \kappa_x^{\mathbb{X}}(v) |dv|$  for any trivialisation  $\phi_{\mathbb{X}}$  induced by an adapted frame  $(\mathbb{X}, U)$  as in (5.5).*

It is more natural for our future analysis to consider objects on the group bundle  $GM$  rather than on the algebra bundle  $\mathfrak{g}M$ . This was already the case for smooth Haar systems.

Indeed, we preferred them over 1-densities on  $\mathfrak{g}M$ , although the two notions are in one-to-one correspondence via the group bundle exponential mapping  $\text{Exp} : \mathfrak{g}M \rightarrow GM$  (see Remark 5.3):

$$\{\mu \text{ smooth Haar system}\} = \text{Exp}^*|\Omega|^+(\mathfrak{g}M).$$

For this reason, we define the convolution kernel of a smooth symbol in the following way:

**Definition 5.15.** The *convolution kernel density* (or convolution kernel for short) of a smooth symbol  $\sigma$  on  $M$  is  $\tilde{\kappa} := \text{Exp}^*\kappa$  with  $\kappa$  as in Lemma 5.14.

We have

$$(\text{Exp}_x^{\mathbb{X}})_*\tilde{\kappa}_x(v) = (\text{Ln}_x^{\mathbb{X}})^*\tilde{\kappa}_x(v) = \kappa_x^{\mathbb{X}}(v)|dv|,$$

and at least formally,

$$\sigma(x, \pi) = \int_{G_x M} \tilde{\kappa}_x(w) \pi(w)^*.$$

We define the space

$$\mathcal{S}'(GM; |\Omega|(GM)) := \text{Exp}^*\mathcal{S}'(\mathfrak{g}M; |\Omega|(\mathfrak{g}M)) = \{\text{Exp}^*\kappa, \kappa \in \mathcal{S}'(\mathfrak{g}M; |\Omega|(\mathfrak{g}M))\},$$

and call it the space of convolution kernel densities.

**Definition 5.16.** Let  $m \in \mathbb{R}$ . An element  $\tilde{\kappa} \in \mathcal{S}'(GM; |\Omega|(GM))$  is  $m$ -homogeneous when  $\kappa = \text{Ln}^*\tilde{\kappa} \in \mathcal{S}'(\mathfrak{g}M; |\Omega|(\mathfrak{g}M))$  is  $m$ -homogeneous (as defined via Lemma B.6).

Given a smooth Haar system  $\mu$ , we denote by  $\kappa_x^\mu \in \mathcal{S}'(G_x M)$  the distribution corresponding to  $\kappa = \text{Ln}^*\tilde{\kappa}$  and  $\text{Ln}^*\mu \in |\Omega|(\mathfrak{g}M)$  via Lemma B.5. In the case of a convolution kernel densities associated with a smooth kernel, this coincides with the notation defined in Section 5.1.1.

We check that  $\tilde{\kappa} \in \mathcal{S}'(GM; |\Omega|(GM))$  is  $m$ -homogeneous when the distribution  $\kappa_x^\mu \in \mathcal{S}'(G_x M)$  is  $m$ -homogeneous.

It will be handy to consider Schwartz convolution kernels densities:

$$(5.6) \quad \mathcal{S}(GM; |\Omega|(GM)) := \text{Exp}^*\mathcal{S}(\mathfrak{g}M; |\Omega|(\mathfrak{g}M)) = \{\kappa \circ \text{Exp}, \kappa \in \mathcal{S}(\mathfrak{g}M; |\Omega|(\mathfrak{g}M))\}.$$

Both  $\mathcal{S}'(GM; |\Omega|(GM))$  and  $\mathcal{S}(GM; |\Omega|(GM))$  are naturally equipped with structures of topological vector spaces, with  $\mathcal{S}(GM; |\Omega|(GM))$  a dense subspace of  $\mathcal{S}'(GM; |\Omega|(GM))$ .

5.1.6. *Semi-norms and difference operators.* In Example 5.5, we have considered symbols belonging in  $S^m(\widehat{G}_x M)$  associated with each  $x \in M$ . By (5.3), the membership of  $\sigma(x, \cdot)$  in  $S^m(\widehat{G}_x M)$  is a property that holds independently of the choice of a smooth Haar system  $\mu$ . However, its  $S^m(\widehat{G}_x M)$ -semi-norms require fixing a graded basis of the Lie algebra and a Rockland operator on  $G_x M$ . We can now consider semi-norms of  $S^m(\widehat{G}_x M)$  defined in (2.7) using adapted frames:

**Lemma 5.17.** *Let  $\mathbb{X}$  and  $\mathbb{Y}$  be two adapted frames on an open subset  $U$  of  $M$ . For any  $x \in U$ , any  $N \in \mathbb{N}$  and any Rockland operator  $\mathcal{R}$  on  $G_x M$ , there exists a constant  $C$  such that for any symbol  $\sigma$  admitting a convolution kernel, we have*

$$\|\sigma(x, \cdot)\|_{S^m(\widehat{G}_x M), N, \langle \mathbb{Y} \rangle_x, \mathcal{R}} \leq C \|\sigma(x, \cdot)\|_{S^m(\widehat{G}_x M), N, \langle \mathbb{X} \rangle_x, \mathcal{R}}.$$

In the inequalities above, the semi-norms may not exist in the sense that they are infinite. The meaning of the estimates above is that if the right-hand side is finite, then so is the left-hand side and the inequality holds. This will be so for all the lemmata in this section.

*Proof.* The result follows from Lemmata 3.10 and 5.7 and Proposition 2.60.  $\square$

In Lemma 5.17, we have used implicitly the difference operators defined in each group  $G_x M$ . We can extend the definition of the difference operators in the following way.

**Definition 5.18.** Let  $\mathbb{X}$  be an adapted frame on an open subset  $U \subset M$  and let  $\sigma$  be a symbol on  $U$  admitting a convolution kernel. If  $\sigma(x, \cdot) \in S^m(\widehat{G}_x M)$  for some  $m \in \mathbb{R} \cup \{-\infty\}$  and any  $x \in U$ , then for any  $\alpha \in \mathbb{N}_0^n$ ,  $\Delta_{\mathbb{X}}^\alpha \sigma$  is the symbol on  $U$  whose convolution kernel in the  $\mathbb{X}$ -coordinates is  $u^\alpha \kappa_{\sigma, x}^{\mathbb{X}}(u)$ .

5.1.7. *Rockland operators over GM.* The following definition will allow us to choose a positive Rockland operator above each group fiber  $G_x M$  depending smoothly in  $x \in M$ :

**Definition 5.19.** An element  $\mathcal{R} \in \Gamma(\mathcal{U}(\mathfrak{g}M))$  is *Rockland* on  $GM$ , resp. *positive Rockland* on  $GM$ , when each  $\mathcal{R}_x$  is so on the fiber  $\mathcal{U}(\mathfrak{g}_x M)$ . Denoting by  $\nu$  its degree of homogeneity, we then call *Rockland symbol* and denote by  $\pi(\mathcal{R})$  the collection

$$\pi(\mathcal{R}_x) \in S^\nu(\widehat{G}_x M), \quad x \in M, \quad \pi \in \widehat{G}_x M.$$

It follows from Section 3.2 that any Rockland operator on  $GM$  may be written with respect to any adapted frame  $(\mathbb{X}, U)$  as  $\mathcal{R} = \sum_{[\alpha]=\nu} c_\alpha \langle \mathbb{X} \rangle^\alpha$ . The homogeneous degree  $\nu$  is independent of the frame  $\mathbb{X}$ .

We will often use in this text the positive Rockland element on  $GM|_U$  given by

$$(5.7) \quad \mathcal{R}_{\mathbb{X}} = \sum_j (-1)^{\frac{M_0}{v_j}} \langle X_j \rangle^{2\frac{M_0}{v_j}},$$

where  $M_0$  is a (fixed) multiple of  $v_1, \dots, v_n$  that we will generally choose as the least common multiple of  $v_1, \dots, v_n$ . The resulting Rockland symbol is

$$\widehat{\mathcal{R}}_{\mathbb{X}} = \{\pi(\mathcal{R}_{\mathbb{X}, x}), (x, \pi) \in \widehat{GM}|_U\}.$$

From this, we can readily construct a positive Rockland element on  $GM$ , see Remark 9.36.

*Remark 5.20.* Although we will not consider the particular case of an equiregular subRiemannian manifold in detail in this paper, let us mention that the more natural and global positive Rockland operator is given by the subLaplacian symbol in that setting. Indeed, recall that a regular subRiemannian manifold  $(M, \mathcal{D}, g)$  is naturally a filtered manifold with, furthermore,  $H_1 = \mathcal{D}$  equipped with a metric  $g$  (see Example 3.2). The subLaplacian symbol  $\widehat{\mathcal{L}}$  is then globally defined as follows: on any open set  $U \subset M$  equipped with an adapted frame  $\mathbb{X} = (X_1, \dots, X_n)$ , it is given by

$$\widehat{\mathcal{L}}|_U = - \sum_{j=1}^{d_1} \widehat{X}_j^2,$$

with  $X_{1,x}, \dots, X_{d_1,x}$  a  $g_x$ -orthonormal basis of  $H_1(x)$  (if this is not so, a Graham-Schmidt procedure allows us to assume so). We check readily that it defines a global object that is the symbol of a positive Rockland operator on  $GM$ .

The next lemma shows that the semi-norms involved in the definition of the symbol classes  $S^m(G_x M)$  are uniform in  $x$ , once chosen a Rockland operator on  $M$ , in the sense of Definition 5.18. This point will be important below (see Section 5.2) for defining global symbol classes on  $M$ .

**Lemma 5.21.** *Let  $\mathbb{X}$  and  $\mathbb{Y}$  be two adapted frames on an open subset  $U$  of  $M$ . Let  $\mathcal{R}$  and  $\mathcal{S}$  be two positive Rockland elements on  $GM|_U$ . For any  $x \in U$  and any  $N \in \mathbb{N}$ , there exists a constant  $C$  such that for any symbol  $\sigma$  admitting a convolution kernel, we have*

$$\|\sigma(x, \cdot)\|_{S^m(\widehat{G}_x M), N, \langle \mathbb{Y} \rangle_x, \mathcal{R}_x} \leq C \|\sigma(x, \cdot)\|_{S^m(\widehat{G}_x M), N, \langle \mathbb{X} \rangle_x, \mathcal{S}_x}.$$

Moreover, the constant  $C$  depends locally uniformly on  $x \in U$ , that is, it is uniform when  $x$  remains in a compact subset  $\mathcal{C}$  of  $U$ .

*Proof.* This follows from the properties of Rockland operators, see Proposition 2.27 (6). The fact that the constant depends only locally uniformly in  $x$  is a consequence of Remarks 2.28 and 2.61 following Propositions 2.27 and 2.60 respectively.  $\square$

5.1.8. *Derivatives and  $S^m(G_x M)$ -membership.* The existence of the  $x$ -derivatives of a symbol in  $S^m(G_x M)$  is independent of a choice of frame in the following sense:

**Lemma 5.22.** *Let  $\mathbb{X}$  and  $\mathbb{Y}$  be two adapted frames on an open subset  $U$  of  $M$ . If  $\sigma$  is a smooth symbol such that  $D_{X_j} \sigma$  makes sense for any  $j = 1, \dots, n$  with  $D_{X_j} \sigma(x, \cdot) \in S^m(\widehat{G}_x M)$  for any  $x \in U$ , then  $D_{Y_\ell} \sigma$  also makes sense for any  $\ell = 1, \dots, n$  with  $D_{Y_\ell} \sigma(x, \cdot) \in S^m(\widehat{G}_x M)$  for any  $x \in U$ . Moreover, fixing a Rockland element  $\mathcal{R}$  on  $GM|_U$ , we have*

$$\begin{aligned} & \|D_{Y_\ell} \sigma(x, \cdot)\|_{S^m(\widehat{G}_x M), N, \langle \mathbb{X} \rangle_x, \mathcal{R}_x} \\ & \leq C \left( \|\sigma(x, \cdot)\|_{S^m(\widehat{G}_x M), N', \langle \mathbb{X} \rangle_x, \mathcal{R}_x} + \sum_j \|D_{X_j} \sigma(x, \cdot)\|_{S^m(\widehat{G}_x M), N, \langle \mathbb{X} \rangle_x, \mathcal{R}_x} \right), \end{aligned}$$

where  $N' \in \mathbb{N}_0$  and the constant  $C > 0$  are independent of  $\sigma$ ,  $N'$  is independent of  $x \in U$  while  $C$  depends on  $x \in U$  locally uniformly.

*Proof.* As in the proof of Lemma 3.3 (see (3.1)), we may write

$$Y_\ell = \sum_{j: v_j \leq v_\ell} c_j X_j, \quad c_j \in C^\infty(U).$$

Therefore, by Lemma 5.7,

$$Y_{\ell,x} \kappa_{\sigma,x}^{\mathbb{Y}} = \sum_j c_j(x) X_{j,x} \kappa_{\sigma,x}^{\mathbb{Y}} = \sum_j c_j(x) X_{j,x} (\det \theta_x^{-1} \kappa_{\sigma,x}^{\mathbb{X}} \circ \theta_x^{-1}),$$

and we compute

$$\begin{aligned} X_{j,x}(\det \theta_x^{-1} \kappa_{\sigma,x}^{\mathbb{X}} \circ \theta_x^{-1}) \\ = (X_{j,x} \det \theta_x^{-1}) \kappa_{\sigma,x}^{\mathbb{X}} \circ \theta_x^{-1} + \det \theta_x^{-1} (X_{j,x} \kappa_{\sigma,x}^{\mathbb{X}}) \circ \theta_x^{-1} + \det \theta_x^{-1} X_{j,x_1=x} \kappa_{\sigma,x}^{\mathbb{X}} \circ \theta_{x_1}^{-1}. \end{aligned}$$

For the last expression, vector calculus in  $\mathbb{R}^n$  yields

$$X_{j,x_1=x} \kappa_{\sigma,x}^{\mathbb{X}}(\theta_{x_1}^{-1}(v)) = \sum_{i=1}^n (X_{j,x} \theta_x^{-1}(v))_i (\partial_i \kappa_{\sigma,x}^{\mathbb{X}})(\theta_{x_1}^{-1}(v)).$$

Now, recall that  $\theta_x : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is linear and block diagonal for every  $x \in U$ , so  $(X_{j,x} \theta_x^{-1}(v))_i$  is a polynomial in  $v$  that is homogeneous of degree  $v_i$  and smooth in  $x \in U$ . Furthermore, using [41, Proposition 1.26] or [37, Section 3.1.5], we have

$$\partial_i = L_{\langle X_i \rangle_x} + \sum_{i': v_{i'} < v_i} p_{i,i'}(x, v) L_{\langle X_{i'} \rangle_x},$$

where  $p_{i,i'}(x, v)$  is a polynomial in  $v$  that is  $(v_i - v_{i'})$  homogeneous for the graded structure, and smooth in  $x$ . We recall that the vector fields  $L_{\langle X_i \rangle_x}$  have been introduced in Notation 5.10. Hence, we obtain

$$\begin{aligned} X_{j,x_1=x} \kappa_{\sigma,x}^{\mathbb{X}}(\theta_{x_1}^{-1}(v)) &= \sum_i (X_{j,x} \theta_x^{-1}(v))_i \left( L_{\langle X_i \rangle_x} \kappa_{\sigma,x}^{\mathbb{X}} + \sum_{i': v_{i'} < v_i} p_{i,i'}(x, v) L_{\langle X_{i'} \rangle_x} \kappa_{\sigma,x}^{\mathbb{X}} \right) (\theta_{x_1}^{-1}(v)) \\ &= \sum_{k=1}^n q_{j,k}(x, v) (L_{\langle X_k \rangle_x} \kappa_{\sigma,x}^{\mathbb{X}})(\theta_{x_1}^{-1}(v)), \end{aligned}$$

with  $q_{j,k}(x, v)$  being a polynomial in  $v$  that is homogeneous for the graded structure of degree at least  $v_k$ , smooth in  $x$ . We set  $\tilde{q}_{j,k}(x, v) = q_{j,k}(x, \theta_x^{-1}v)$ ; it is a polynomial in  $v$  smooth in  $x$  with the same properties as  $q_{j,k}(x, v)$ . Denoting by  $\rho_{j,k}$  the symbol with convolution kernel in the  $\mathbb{X}$ -coordinates given by  $\tilde{q}_{j,k}(x, \cdot) (L_{\langle X_k \rangle_x} \kappa_{\sigma,x}^{\mathbb{X}})$ , the properties of  $S^m(\widehat{G}_x M)$  imply that for any  $N \in \mathbb{N}_0$ ,

$$\|\rho_{j,k}(x, \cdot)\|_{S^m(\widehat{G}_x M), N, \langle \mathbb{X} \rangle_x, \mathcal{R}} \leq C \|\sigma(x, \cdot)\|_{S^m(\widehat{G}_x M), N', \langle \mathbb{X} \rangle_x, \mathcal{R}},$$

where  $N' \in \mathbb{N}_0$  and the constant  $C$  are independent of  $\sigma$  while  $N'$  is independent of  $x \in U$  and  $C$  depends on it locally uniformly. This remains so even if we replace a fixed Rockland operator  $\mathcal{R}_x$  on  $G_x M$  with a Rockland element on  $GM|_U$  as in Lemma 5.21. Since we have

$$\begin{aligned} D_{Y_\ell} \sigma(x, \cdot) &= \sum_j c_j(x) \frac{X_{j,x} \det \theta_x^{-1}}{\det \theta_x^{-1}} (\theta_x)_* \sigma(x, \cdot) + c_j(x) (\theta_x)_* (D_{X_j} \sigma(x, \cdot)) \\ &\quad + c_j(x) \sum_k (\theta_x)_* \rho_{j,k}(x, \cdot), \end{aligned}$$

we conclude as in the proofs of Lemmata 5.17 and 5.21.  $\square$

**5.2. The classes of symbols  $S^m(\widehat{G}M)$ .** We now have set the notation required for defining symbol classes on the filtered manifold  $M$ .

5.2.1. *Definition of the class  $S^m(\widehat{GM})$ .*

**Definition 5.23.** A symbol  $\sigma$  is of order  $m$  on  $M$  when

- (1)  $\sigma(x, \cdot) \in S^m(\widehat{G}_x M)$  for each  $x \in M$  and  $\sigma$  is smooth in the variable  $x \in M$ ,
- (2) for any adapted frame  $U$  on an open subset  $U \subset M$  and for any multi-index  $\beta \in \mathbb{N}_0^n$ , the symbol  $D_{\mathbb{X}}^\beta \sigma$  exists with  $D_{\mathbb{X}}^\beta \sigma(x, \cdot) \in S^m(\widehat{G}_x M)$  for any  $x \in U$ ,
- (3) for any local adapted frame  $\mathbb{X}$  on  $U \subset M$ , for any compact subset  $\mathcal{C} \subset U$ , for any positive Rockland element  $\mathcal{R}$  on  $GM|_U$ , and for any  $N \in \mathbb{N}_0$  and  $\beta \in \mathbb{N}_0^n$ , the following quantity is finite:

$$\sup_{x \in \mathcal{C}} \|D_{\mathbb{X}}^\beta \sigma(x, \cdot)\|_{S^m(\widehat{G}_x M), N, \langle \mathbb{X} \rangle_x, \mathcal{R}_x} < \infty.$$

We denote by  $S^m(\widehat{GM})$  the space of symbols of order  $m$  on  $M$ .

In Property (3), we use the semi-norms defined in (2.7) on the group  $G_x M$  for the choice of graded basis  $\langle \mathbb{X} \rangle_x$ , and we do so for each  $x \in U$ . Property (1) implies that  $\sigma$  admits a convolution kernel and is smooth. By Lemmata 5.17 and 5.21, and, inductively, by Lemma 5.22, if Property (2) holds for one adapted frame  $\mathbb{X}$  on an open subset  $U \subset M$ , it will hold for any other adapted frame  $\mathbb{Y}$  on  $U$ . This will be also true for Property (3) for any other choice of positive Rockland operator  $\mathcal{S}$  on  $GM|_U$  and we have

$$\sup_{x \in \mathcal{C}} \|D_{\mathbb{Y}}^{\beta_1} \sigma(x, \cdot)\|_{S^m(\widehat{G}_x M), N, \langle \mathbb{Y} \rangle_x, \mathcal{S}_x} \leq C \sup_{x \in \mathcal{C}} \sum_{|\beta| \leq |\beta_1|} \|D_{\mathbb{X}}^\beta \sigma(x, \cdot)\|_{S^m(\widehat{G}_x M), N, \langle \mathbb{X} \rangle_x, \mathcal{R}_x},$$

with a constant  $C$  dependent on  $M, H, N, \mathbb{X}, \mathbb{Y}, \mathcal{R}, \mathcal{S}, \mathcal{C}, \beta_1$  but not on  $\sigma$ .

We equip  $S^m(\widehat{GM})$  with the topology generated by the semi-norms given in Property (3) of Definition 5.23. This topology is in fact Fréchet as we assume that the manifold  $M$  is second countable. Indeed, if  $\mathbb{X}$  is an adapted frame on an open set  $U \subset M$  and if  $\mathcal{C}$  is a compact of  $U$ , we set

$$(5.8) \quad \|\sigma\|_{S^m(\widehat{GM}), \langle \mathbb{X}, U \rangle, \mathcal{C}, N} := \max_{|\beta| \leq N} \sup_{x \in \mathcal{C}} \|D_{\mathbb{X}}^\beta \sigma(x, \cdot)\|_{S^m(\widehat{G}_x M), N, \langle \mathbb{X} \rangle_x, \mathcal{R}_x}, \quad N \in \mathbb{N}_0,$$

where  $\mathcal{R}_x = \mathcal{R}_{\mathbb{X}, x}$  is the positive Rockland operator given in  $x \in U$  by (5.7). We now fix a countable sequence of compact subsets  $\mathcal{C}_k$ ,  $k = 0, 1, 2, \dots$  covering  $M = \cup_k \mathcal{C}_k$  and such that each compact is included in a bounded open subset  $U_k$  on which an adapted frame  $\mathbb{X}_k$  exists. The continuous semi-norms given by

$$(5.9) \quad \|\sigma\|_{S^m(\widehat{GM}), N} := \max_{k \leq N} \|\sigma\|_{S^m(\widehat{GM}), \langle \mathbb{X}_k, U_k \rangle, \mathcal{C}_k, N}, \quad N \in \mathbb{N}_0,$$

form a countable fundamental basis for the topology of  $S^m(\widehat{GM})$ , yielding the Fréchet topology.

The properties of symbol classes on groups imply readily the continuous inclusions:

$$(5.10) \quad m_1 \leq m_2 \implies S^{m_1}(\widehat{GM}) \subset S^{m_2}(\widehat{GM}),$$

and the natural properties of interpolation for the spaces  $S^m(\widehat{G})$ ,  $m \in \mathbb{R}$ .

5.2.2. *Examples from Section 5.1.3.* If  $f \in C^\infty(M)$  as in Example 5.8, then the symbol  $f\mathbf{I} = \{f(x) \text{id}_{\mathcal{H}_\pi}, (x, \pi) \in \widehat{GM}\}$  satisfies  $D_{\mathbb{X}}^\beta(f\mathbf{I}) = (\mathbb{X}^\beta f)\mathbf{I}$  for all  $\beta$  and  $f\mathbf{I} \in S^0(\widehat{GM})$ .

Similarly, the symbol  $\widehat{\langle X_j \rangle} := \{\pi(\langle X_j \rangle_x) : x \in U, \pi \in \widehat{G}_x M\}$  introduced in Example 5.9 satisfies  $D_{\mathbb{X}}^\beta \widehat{\langle X_j \rangle} = 0$  for all  $\beta$ , and  $\{\pi(\langle X_j \rangle_x), \pi \in \widehat{G}\} \in S^{v_j}(\widehat{G}_x M)$  for any  $x \in U$ . Consequently,  $\widehat{\langle X_j \rangle} \in S^{v_j}(\widehat{GM}|_U)$ .

5.2.3. *Smoothing symbols.*

**Definition 5.24.** The class of smoothing symbols

$$S^{-\infty}(\widehat{GM}) := \bigcap_{m \in \mathbb{R}} S^m(\widehat{GM})$$

is equipped with the induced topology of projective limit.

This topology makes sense thanks to the continuous inclusions in (5.10).

The advantage of smoothing symbols is that their convolution kernels are given by Schwartz densities. Indeed, the convolution kernel  $\kappa_{\sigma,x}^{\mathbb{X}}(v)$  of a smoothing symbol  $\sigma \in S^{-\infty}(\widehat{GM})$  in the coordinates of an adapted frame  $(\mathbb{X}, U)$  is Schwartz in  $v$  and depends smoothly in  $x \in U$ . Therefore, using the notation in (5.6),

$$\kappa_\sigma \in \mathcal{S}(GM, |\Omega|(GM)).$$

**5.3. Operations on symbols.** We analyse the different operations that we can perform on symbols on filtered manifolds and that preserve their membership to the symbol classes defined above. This section relies on the analogue results for symbols and kernels on groups stated in Section 2. Indeed, by definition of our symbol classes on  $M$ , for any  $\sigma \in S^m(\widehat{GM})$ , any smooth system of Haar measure  $\mu = \{\mu_x, x \in G\}$  and any  $x \in G$ , the symbol  $\sigma(x)$  is in  $S^m(G_x M)$  with convolution kernel  $\kappa_{\sigma,x}^\mu$ .

5.3.1. *Symbolic algebraic properties.* From the Leibniz property of vector fields and Theorem 2.33 together with Remark 2.34, it follows readily:

**Proposition 5.25.** *For any  $m_1, m_2, m \in \mathbb{R} \cup \{-\infty\}$ , the composition and adjoint operations*

$$\left\{ \begin{array}{l} S^{m_1}(\widehat{GM}) \times S^{m_2}(\widehat{GM}) \longrightarrow S^{m_1+m_2}(\widehat{GM}) \\ (\sigma_1, \sigma_2) \longmapsto \sigma_1 \sigma_2 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} S^m(\widehat{GM}) \longrightarrow S^m(\widehat{GM}) \\ \sigma \longmapsto \sigma^* \end{array} \right.$$

are continuous maps. Moreover, for any smooth Haar system  $\mu = \{\mu_x\}_{x \in M}$ , the convolution kernels of  $\sigma_1 \sigma_2$  and  $\sigma^*$  are formally given by

$$(5.11) \quad \kappa_{\sigma_1 \sigma_2, x}^\mu(v) = \kappa_{\sigma_2, x}^\mu \star_{G_x M, \mu_x} \kappa_{\sigma_1, x}^\mu(v) = \int_{G_x M} \kappa_{\sigma_2, x}^\mu(vw^{-1}) \kappa_{\sigma_1, x}^\mu(w) d\mu_x(w),$$

$$(5.12) \quad \kappa_{\sigma^*, x}^\mu(v) = \bar{\kappa}_{\sigma, x}^\mu(v^{-1}),$$

where  $\star_{G_x M, \mu_x}$  denotes the convolution product on  $G_x M$  with respect to the Haar measure  $\mu_x$ .

Note that the convolution kernel  $\kappa_{\sigma_1\sigma_2,x}^\mu$  in (5.11) is the convolution product of two tempered distributions on  $G_xM$  for the Haar measure  $\mu_x$ . Moreover, it makes sense on the group  $G_xM$  as each distribution can be written as the sum of a Schwartz function with a distribution with compact support, see Theorem 2.36.

*Proof of Proposition 5.25.* Consider a smooth Haar system  $\mu = \{\mu_x\}_{x \in M}$  and  $x \in M$ . The properties of composition and taking the adjoint of symbols on groups imply that the symbol  $\sigma_1(x)\sigma_2(x)$  on the group  $G_xM$  is in  $S^{m_1+m_2}(G_xM)$  with convolution kernel given by the convolution product  $\kappa_{\sigma_2,x}^\mu \star_{G_xM,\mu_x} \kappa_{\sigma_1,x}^\mu$ , see Theorem 2.33.

If  $\mu' = \{\mu'_x\}_{x \in M}$  is another smooth Haar system, it differs from  $\mu = \{\mu_x\}_{x \in M}$  by a positive constant  $c(x)$ , see (5.2), and so do the convolution kernels of symbols, see (5.3). As we check readily

$$\begin{aligned}\kappa_{\sigma_2,x}^\mu \star_{G_xM,\mu_x} \kappa_{\sigma_1,x}^\mu &= c(x) \kappa_{2,x}^{\mu'} \star_{G_xM,\mu'_x} \kappa_{1,x}^{\mu'}, \\ \kappa_{\sigma^*,x}^\mu &= c(x) \kappa_{\sigma^*,x}^{\mu'},\end{aligned}$$

the convolution kernels of the symbols  $\sigma_1\sigma_2$  and  $\sigma^*$  are well defined as densities independently of the choice of a smooth Haar system. This shows that these symbols satisfy Part (1) of Definition 5.23.

Once a frame  $\mathbb{X}$  is fixed on an open subset  $U \subset M$ , then equation (5.11) reads for  $v \in \mathbb{R}^n$  at  $x \in U$ ,

$$(5.13) \quad \kappa_{\sigma_1\sigma_2,x}^{\mathbb{X}}(v) = \int_{\mathbb{R}^n} \kappa_{\sigma_2,x}^{\mathbb{X}}(vw^{-1}) \kappa_{\sigma_1,x}^{\mathbb{X}}(w) dw.$$

Routine checks using the Leibniz properties of vector fields and Theorem 2.33 imply that  $\sigma^*$  and  $\sigma_1\sigma_2$  satisfy Parts (2) and (3) of Definition 5.23.  $\square$

*Remark 5.26.* Note that the proof of Proposition 5.25 (namely equation (5.13)) shows that the derivation  $D_{\mathbb{X}}$  satisfies Leibniz rules in the sense that for  $(\sigma_1, \sigma_2) \in S^{m_1}(\widehat{GM}) \times S^{m_2}(\widehat{GM})$  and  $\beta \in \mathbb{N}^d$  with  $|\beta| = 1$ ,

$$D_{\mathbb{X}}^\beta(\sigma_1\sigma_2) = (D_{\mathbb{X}}^\beta\sigma_1)\sigma_2 + \sigma_1(D_{\mathbb{X}}^\beta\sigma_2).$$

Recall that a smooth function  $f \in C^\infty(M)$  gives rise to a symbol  $fI \in S^0(M)$ , see Example 5.8. For any  $\sigma \in S^m(\widehat{GM})$ , the symbol  $f\sigma$  may be viewed as the composition of  $fI \in S^0(M)$  with  $\sigma \in S^m(\widehat{GM})$ . Hence it is also in  $S^m(\widehat{GM})$  by Proposition 5.25. In other words,  $S^m(\widehat{GM})$  is stable under multiplication by  $C^\infty(M)$ . It is clearly stable under addition, hence  $S^m(\widehat{GM})$  is a  $C^\infty(M)$ -module. By Proposition 5.25,  $S^m(\widehat{GM})$  is also stable under taking the adjoint and the space of symbols  $\cup_{m \in \mathbb{R}} S^m(\widehat{GM})$  is an  $*$ -algebra over the ring  $C^\infty(M)$ .

In Example 5.9, we showed that given an adapted frame  $\mathbb{X}$  on an open subset  $U \subset M$ , the symbol  $\langle X_j \rangle$  is in  $S^{v_j}(\widehat{GM}|_U)$ . We can describe more precisely the composition with  $\langle X_j \rangle$ .

**Lemma 5.27.** *Let  $\mathbb{X}$  be an adapted frame on an open subset  $U \subset M$ . Let  $m \in \mathbb{R} \cup \{-\infty\}$ . If  $\sigma \in S^m(\widehat{GM}|_U)$  and  $j = d_{i-1} + 1, \dots, d_i$ , then  $\langle X_j \rangle \sigma$  and  $\sigma \langle X_j \rangle$  are in  $S^{m+w_j}(\widehat{GM}|_U)$  with*

respective convolution kernels in the  $\mathbb{X}$ -coordinates:

$$L_{\langle X_j \rangle_x} \kappa_{\sigma, x}^{\mathbb{X}} \quad \text{and} \quad R_{\langle X_j \rangle_x} \kappa_{\sigma, x}^{\mathbb{X}}.$$

*Proof.* This follows readily from the group case, see Remark 2.37 (2).  $\square$

The composition property allows us to generalise Examples 5.8 and 5.11 with the following two further sets of examples:

*Example 5.28.* Any element in  $\Gamma(H^{w_i}/H^{w_{i-1}})$  is in  $S^{w_i}(\widehat{GM})$ ,  $i = 1, 2, \dots$ . Indeed, it can be written in some  $\mathbb{X}$ -coordinates as a  $C^\infty(M)$ -combination of  $\widehat{\langle X_j \rangle}$ ,  $j = d_{i-1} + 1, \dots, d_i$ , or equivalently, as a sum of products of symbols  $fI \in S^0(\widehat{GM})$  (as in Example 5.8) with symbols  $\widehat{\langle X_j \rangle} \in S^{v_j}(\widehat{GM})$  (as in Examples 5.9 and 5.11).

*Example 5.29.* If  $\mathbb{X}$  is an adapted frame on an open subset  $U \subset M$ , for any  $\alpha \in \mathbb{N}_0$ , the symbol  $\widehat{\langle \mathbb{X} \rangle}^\alpha$  is in  $S^{[\alpha]}(\widehat{GM}|_U)$  and admits

$$\kappa_{\widehat{\langle \mathbb{X} \rangle}^\alpha, x}^{\mathbb{X}} = L_{\langle \mathbb{X}_1 \rangle_x}^{\alpha_1} \dots L_{\langle \mathbb{X}_n \rangle_x}^{\alpha_n} \delta_0 = L_{\langle \mathbb{X} \rangle_x}^\alpha \delta_0,$$

as convolution kernel in the  $\mathbb{X}$ -coordinates. In particular,  $I = \widehat{\langle \mathbb{X} \rangle}^\alpha$  with  $\alpha = 0$  is an element of  $\Gamma(\mathcal{U}(\mathfrak{g}M))$  of zero-th order, with convolution kernel in the  $\mathbb{X}$ -coordinates given by  $\delta_0$ .

This may be generalised in the following observation:

**Lemma 5.30.** *Let  $T \in \Gamma_{\leq m}(\mathcal{U}(\mathfrak{g}M))$ . If  $\mathbb{X}$  is an adapted frame on an open subset  $U \subset M$ , then any element  $T$  of  $\Gamma_{\leq m}(\mathcal{U}(\mathfrak{g}M))$  can be written uniquely as*

$$T = \sum_{[\alpha] \leq m} c_\alpha \widehat{\langle \mathbb{X} \rangle}^\alpha, \quad c_\alpha \in C^\infty(U).$$

The symbol associated to  $T$  then is

$$\widehat{T} = \sum_{[\alpha] \leq m} c_\alpha \widehat{\langle \mathbb{X} \rangle}^\alpha \in S^m(\widehat{GM}|_U),$$

and the convolution kernel of  $\widehat{T}$  is given in the  $\mathbb{X}$ -coordinates by  $\kappa_{\widehat{T}, x}^{\mathbb{X}} = L_{T_x} \delta_0$ , where  $L_{T_x} = \sum_{[\alpha] \leq m} c_\alpha(x) L_{\langle \mathbb{X} \rangle}^\alpha$  is the differential operator on  $\mathbb{R}^n$  identified with the left-invariant operator corresponding to the element  $T_x \in \mathcal{U}(\mathfrak{g}_x M)$ , once  $G_x M$  is identified with  $\mathbb{R}^n$  via  $\text{Exp}_x^{\mathbb{X}}$ .

*Proof.* The writing of  $T$  is explained in Section 3.2.2. The rest of the statement follows readily from the property of composition of symbols (Proposition 5.25) and Examples 5.8 and 5.29.  $\square$

**5.3.2. Difference operators.** The definition of the difference operators performed in Section 5.1.6 extends naturally in the following way. For any adapted frame  $\mathbb{X}$  on an open subset  $U \subset M$  and any  $\sigma \in S^m(\widehat{GM}|_U)$  where  $m \in \mathbb{R} \cup \{-\infty\}$ , for any  $\alpha \in \mathbb{N}_0^n$ , we denote by  $\Delta_{\mathbb{X}}^\alpha \sigma$  the symbol on  $U$  whose convolution kernel in the  $\mathbb{X}$ -coordinates is  $u^\alpha \kappa_{\sigma, x}^{\mathbb{X}}(u)$ . We check readily:

**Lemma 5.31.** *Let  $\mathbb{X}$  be an adapted frame on an open subset  $U \subset M$ . For any  $\alpha \in \mathbb{N}_0^n$  and  $m \in \mathbb{R} \cup \{-\infty\}$ , the following map is continuous*

$$\begin{cases} S^m(\widehat{GM}) & \longrightarrow S^{m-[\alpha]}(\widehat{GM}|_U) \\ \sigma & \longmapsto \Delta_{\mathbb{X}}^\alpha \sigma \end{cases}$$

It follows from the group case (see Lemma 2.57) that the difference operators  $\Delta_{\mathbb{X}}^\alpha$  satisfy a Leibniz property:

**Corollary 5.32.** *Let  $\mathbb{X}$  be an adapted frame on an open subset  $U$  of  $M$ . There exist smooth functions  $c_{\alpha_1, \alpha_2}^{(\alpha)} \in C^\infty(U)$ ,  $\alpha, \alpha_1, \alpha_2 \in \mathbb{N}_0^n$  satisfying*

$$\forall \sigma_1, \sigma_2 \in S^{-\infty}(\widehat{GM}), \quad \Delta_{\mathbb{X}}^\alpha(\sigma_1 \sigma_2) = \sum_{[\alpha_1] + [\alpha_2] = [\alpha]} c_{\alpha_1, \alpha_2}^{(\alpha)} \Delta_{\mathbb{X}}^{\alpha_1} \sigma_1 \Delta_{\mathbb{X}}^{\alpha_2} \sigma_2.$$

It also follows from the group case (see Proposition 2.58 and Remark 2.59) that locally compactly supported functions on  $GM$  (in the following sense) yields continuous generalised difference operators:

**Definition 5.33.** A function  $\chi_0 \in C^\infty(M \times \mathbb{R}^{n'})$  is  *$M$ -locally compactly supported* in  $\mathbb{R}^{n'}$  when for any compact subset  $\mathcal{C} \subset M$ , there exists a compact neighbourhood  $V \subset \mathbb{R}^{n'}$  of  $v = 0$  such that  $\chi_0(x, v) = 0$  for  $(x, v) \notin \mathcal{C} \times V$ .

**Corollary 5.34.** *Let  $\mathbb{X}$  be an adapted frame on an open subset  $U$  of  $M$ . Let  $q \in C^\infty(U \times \mathbb{R}^n)$  be  $U$ -locally compactly supported. Then the operation  $\Delta_q$  defined via*

$$\Delta_q \sigma(x, \pi) = \mathcal{F}_{G_x M, \mu^{\mathbb{X}}} (q(x, \text{Ln}_x^{\mathbb{X}} \cdot) \kappa_x^{\mu^{\mathbb{X}}}), \quad \sigma = \mathcal{F}_{G_x M, \mu^{\mathbb{X}}} \kappa_x^{\mu^{\mathbb{X}}}$$

is a continuous operator  $S^m(\widehat{GM}) \rightarrow S^m(\widehat{GM})$  for any  $m \in \mathbb{R} \cup \{-\infty\}$ . Moreover, fixing an adapted frame  $\mathbb{X}$  on an open subset  $U$  of  $M$  and writing the Taylor series of  $q(x, \cdot)$  at  $0$  as  $\mathbb{T}_0 q(x, v) \sim \sum_{\alpha} c_{\alpha}(x) v^{\alpha}$ , produces smooth functions  $c_{\alpha} \in C^\infty(U)$ ,  $\alpha \in \mathbb{N}_0^n$ , and the map

$$\sigma \longmapsto \Delta_q \sigma - \sum_{[\alpha] \leq N} c_{\alpha} \Delta_{\mathbb{X}}^{\alpha} \sigma$$

is continuous  $S^m(\widehat{GM}|_U) \rightarrow S^{m-(N+1)}(\widehat{GM}|_U)$  for any  $N \in \mathbb{N}_0$ .

We may summarise parts of Corollary 5.34 as

$$\Delta_q \sigma \sim \sum_{\alpha \in \mathbb{N}_0^n} c_{\alpha} \Delta_{\mathbb{X}}^{\alpha} \sigma \quad \text{in } S^m(\widehat{GM}|_U),$$

having used the following natural notion of expansion of a symbol:

**Definition 5.35.** A symbol  $\sigma \in S^m(\widehat{GM})$  admits the (inhomogeneous) expansion

$$\sigma \sim \sum_{j=0}^{\infty} \sigma_{m-j} \quad \text{in } S^m(\widehat{GM})$$

when for all  $N \in \mathbb{N}_0$ ,  $\sigma - \sum_{j=0}^N \sigma_{m-j} \in S^{m-N-1}(\widehat{GM})$ .

Given a sequence  $(\sigma_j)$  as in Definition 5.35, we can construct a symbol  $\sigma \in S^m(\widehat{GM})$  following the classical ideas due to Borel (see e.g. [37, Section 5.5.1]), that will satisfy  $\sigma \sim \sum_j \sigma_j$ . Moreover,  $\sigma$  is unique up to  $S^{-\infty}(\widehat{GM})$ .

We will often apply Corollary 5.34 with functions  $q$  whose first terms in their Taylor series is zero. To describe this precisely, we adapt the notion of functions vanishing at a certain order at 0 to the anisotropic setting:

**Definition 5.36.** Let  $f$  be a (scalar-valued) function defined on an open subset of  $\mathbb{R}^n$  containing 0. Here,  $\mathbb{R}^n$  is equipped with the dilations  $\delta_r$ . We say that  $f$  *vanishes at homogeneous order*  $N \in \mathbb{N}_0$  when  $f(v) = O(|v|^{N+1})$  for one (and then any) homogeneous quasinorm  $|\cdot|$  on  $\mathbb{R}^n$ .

We abuse the vocabulary for the case  $N = -1$  where  $f(v) = O(1)$ .

We readily adapt the usual properties of functions vanishing at a certain order by using the Taylor expansion due to Folland and Stein [41] (see also Theorem 2.17).

**Lemma 5.37.** *Let  $f$  be a (scalar-valued) function defined and smooth on an open subset  $V$  of  $\mathbb{R}^n$  containing 0. Let  $N \in \mathbb{N}_0$ . The following properties are equivalent:*

- (1)  $f$  vanishes at homogeneous order  $N$ .
- (2)  $\partial^\alpha f(0) = 0$  for any  $\alpha \in \mathbb{N}_0$ ,  $[\alpha] \leq N$ .
- (3) for any fixed  $v \in V$ , we have  $f(\delta_\varepsilon v) = O(\varepsilon^{N+1})$  for  $\delta \in \mathbb{R}$ .

If it is the case, then for any  $\alpha_0 \in \mathbb{N}_0^n$ ,  $\partial^{\alpha_0} f$  vanishes at homogeneous order  $\max(N - [\alpha_0], -1)$ .

We observe that in our context, Property (2) is equivalent to

- (2)'  $L_{(\mathbb{X})x}^\alpha f(0) = 0$  for any  $\alpha \in \mathbb{N}_0$ ,  $[\alpha] \leq N$ .

In any case, Property (2) allows to detect when the first coefficients  $c_\alpha$  in the asymptotics given by Corollary 5.34 vanish.

5.3.3. *Differentiation of symbols.* The differentiation of symbols introduced in Notation 5.13 preserves the symbol classes.

**Lemma 5.38.** *Let  $\mathbb{X}$  be an adapted frame on an open subset  $U \subset M$ . For any  $\beta \in \mathbb{N}_0^n$  and  $m \in \mathbb{R} \cup \{-\infty\}$ , the following map is continuous*

$$\begin{cases} S^m(\widehat{GM}) & \longrightarrow & S^m(\widehat{GM}|_U) \\ \sigma & \longmapsto & D_{\mathbb{X}}^\beta \sigma \end{cases}$$

*Proof.* The convolution kernel of  $D_{\mathbb{X}}^\beta \sigma$  in  $\mathbb{X}$ -coordinates is  $\mathbb{X}^\beta \kappa_{\sigma,x}^{\mathbb{X}}(u)$ . For every  $\beta' \in \mathbb{N}^n$  there are functions  $c_\gamma^{\beta\beta'} \in C^\infty(U)$  such that  $\mathbb{X}^{\beta'} \mathbb{X}^\beta = \sum_{[\gamma] \leq [\beta] + [\beta']} c_\gamma^{\beta\beta'} \mathbb{X}^\gamma$ . Then we have

$$\|D_{\mathbb{X}}^\beta \sigma\|_{S^m(\widehat{GM}), C, N, \mathcal{R}} \leq C \|\sigma\|_{S^m(\widehat{GM}), C, N + [\beta], \mathcal{R}}$$

where  $C \leq \sum_{[\gamma] \leq [\beta] + [\beta']} \sup_{x \in C} |c_\gamma^{\beta\beta'}(x)|$ . □

To sum-up the elements of this section and the previous one, we can generalise easily this property.

**Corollary 5.39.** *Let  $\mathbb{X}$  be an adapted frame on an open subset  $U \subset M$ . For any  $\alpha, \beta, \beta', \gamma \in \mathbb{N}_0^n$  and  $m \in \mathbb{R} \cup \{-\infty\}$ , if  $\sigma \in S^m(\widehat{GM})$ , then the symbol  $\Delta_{\mathbb{X}}^{\alpha} D_{\mathbb{X}}^{\beta'} \widehat{\langle \mathbb{X} \rangle}^{\beta} \sigma \widehat{\langle \mathbb{X} \rangle}^{\gamma}$  belongs to the symbol class  $S^{m-[\alpha]+[\beta]+[\gamma]}(\widehat{GM}|_U)$  with convolution kernel in the  $\mathbb{X}$ -coordinates*

$$(-u)^{\alpha} \mathbb{X}^{\beta'} L_{\langle \mathbb{X} \rangle}^{\beta} R_{\langle \mathbb{X} \rangle}^{\gamma} \kappa_{\sigma, x}^{\mathbb{X}}(u).$$

Moreover, the following map is continuous:

$$\begin{cases} S^m(\widehat{GM}) & \longrightarrow & S^{m-[\alpha]+[\beta]+[\gamma]}(\widehat{GM}|_U) \\ \sigma & \longmapsto & \Delta_{\mathbb{X}}^{\alpha} D_{\mathbb{X}}^{\beta'} \widehat{\langle \mathbb{X} \rangle}^{\beta} \sigma \widehat{\langle \mathbb{X} \rangle}^{\gamma} \end{cases}$$

**5.4. Spectral multipliers of Rockland symbols.** A fundamental example of symbols consists in the *spectral multipliers* of positive Rockland symbols. Here, we will generalise the properties of spectral multipliers obtained in Theorem 2.41 on groups and show that it holds on filtered manifolds:

**Proposition 5.40.** *Let  $\phi \in \mathcal{G}^m(\mathbb{R})$  with  $m \in \mathbb{R}$  (see Notation 2.40), and let  $\mathcal{R}$  be a positive Rockland element on  $GM$ . Then the symbol*

$$\phi(\widehat{\mathcal{R}}) = \{\phi(\pi(\mathcal{R}_x)) : (x, \pi) \in \widehat{GM}\}$$

is in  $S^{m\nu}(\widehat{GM})$  where  $\nu$  is the homogeneous degree of  $\mathcal{R}$ . Moreover, the map

$$\phi \longmapsto \phi(\widehat{\mathcal{R}}), \quad \mathcal{G}^m(\mathbb{R}) \longrightarrow S^{m\nu}(\widehat{GM}),$$

is continuous.

This section will be mainly devoted to the proof of Proposition 5.40, which follows the arguments in [36, Sections 4.7 & 4.8] and [37, Section 5.3]. The strategy is to decompose the multiplier function  $\phi$  in dyadic pieces  $\psi_j(2^{-j}\cdot)$ , having shown beforehand enough properties for spectral multipliers of the form  $\psi(t\widehat{\mathcal{R}})$  with  $t \in (0, 1]$  and  $\psi \in C_c^\infty(\mathbb{R})$ . The analysis of  $\psi(t\widehat{\mathcal{R}})$  may be obtained with two different methods: in [37, Section 5.3] via heat kernel properties, and in [36, Sections 4.7 & 4.8] via the Helffer-Sjöstrand formula. Only the latter analysis lends itself readily to a generalisation on manifolds. Hence, we will start with recalling the Helffer-Sjöstrand formula [23, 43].

After presenting the proof we will observe that essentially the same arguments will allow for the multiplier function to depend on  $x$  and also that Proposition 5.40 implies the density of smoothing symbols.

**5.4.1. The Helffer-Sjöstrand formula.** If  $T$  is a self-adjoint operator densely defined on a separable Hilbert space  $\mathcal{H}$ , and if  $\psi \in C_c^\infty(\mathbb{R})$ , then the spectrally defined operator  $\psi(T) \in \mathcal{L}(\mathcal{H})$  coincides with

$$(5.14) \quad \psi(T) = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\psi}(z) (T - z)^{-1} L(dz).$$

Here,  $\tilde{\psi}$  is any almost analytic extension of  $\psi$  such that  $\int_{\mathbb{C}} |\bar{\partial} \tilde{\psi}(z)| |\operatorname{Im} z|^{-1} L(dz)$  is finite, and  $L(dz) = dx dy$ ,  $z = x + iy$ , is the Lebesgue measure on  $\mathbb{C}$ . Recall that an almost analytic

extension of a function  $\psi \in C^\infty(\mathbb{R})$  is any function  $\tilde{\psi} \in C^\infty(\mathbb{C})$  satisfying

$$\tilde{\psi}|_{\mathbb{R}} = \psi \quad \text{and} \quad \bar{\partial}\tilde{\psi}|_{\mathbb{R}} = 0, \quad \text{where} \quad \bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y).$$

This can be extended to smooth functions  $\psi$  not necessarily with compact support but with enough decay at infinity [36, Appendix]:

**Lemma 5.41.** *Let  $\psi \in \mathcal{G}^m(\mathbb{R})$  with  $m < -1$ . Then, there exists an almost analytic extension  $\tilde{\psi} \in C^\infty(\mathbb{C})$  to  $\psi$  for which the Helffer-Sjöstrand formula in (5.14) holds for any self-adjoint operator  $T$ . Moreover, we have for all  $N \in \mathbb{N}_0$ ,*

$$\int_{\mathbb{C}} |\bar{\partial}\tilde{\psi}(z)| \left( \frac{1+|z|}{|\text{Im } z|} \right)^N L(dz) \leq C_N \|\psi\|_{\mathcal{G}^{m, N+3}},$$

with the constant  $C_N > 0$  depending on  $N$  and  $m$ , but not on  $\psi$ .

5.4.2. *Properties of the resolvent.* In order to use the Helffer-Sjöstrand formula, we show some properties for the resolvent of  $\widehat{\mathcal{R}}$ :

**Lemma 5.42.** *Let  $\mathcal{R}$  be a positive Rockland element on  $GM$ .*

- (1) *For any  $N \in \mathbb{Z}$ , the symbol  $(\mathbf{I} + \widehat{\mathcal{R}})^N = \{(\mathbf{I} + \pi(\mathcal{R}_x))^N : (x, \pi) \in \widehat{GM}\}$  is in  $S^{N\nu}(\widehat{GM})$  where  $\nu$  is the homogeneous degree of  $\mathcal{R}$ .*
- (2) *For any  $z \in \mathbb{C} \setminus \mathbb{R}$ , the symbol*

$$(\widehat{\mathcal{R}} - z)^{-1} = \{(\pi(\mathcal{R}_x) - z)^{-1} : (x, \pi) \in \widehat{GM}\}$$

*is in  $S^{-\nu}(\widehat{GM})$ . Moreover, for any seminorm  $\|\cdot\|_{S^{-\nu}(\widehat{GM}), (\mathbb{X}, U), \mathcal{C}, N}$ , there exist  $C > 0$  and  $p \in \mathbb{N}_0$  such that*

$$\forall z \in \mathbb{C} \setminus \mathbb{R}, \quad \|(\widehat{\mathcal{R}} - z)^{-1}\|_{S^{-\nu}(\widehat{GM}), (\mathbb{X}, U), \mathcal{C}, N} \leq C \left( 1 + \frac{1+|z|}{|\text{Im } z|} \right)^p.$$

*Proof of Lemma 5.42.* For Part (1), the case of  $N = 1$  by Lemma 5.30 and the properties of composition of symbols (Proposition 5.25) imply that it suffices to show the case  $N = -1$ . By Theorem 2.41,  $(\mathbf{I} + \widehat{\mathcal{R}}_x)^{-1} \in S^{-\nu}(\widehat{G}_x M)$  for every  $x \in M$ , and from Remark 2.42, we obtain that the 0-seminorm defined in (5.8), i.e.

$$\max_{x \in \mathcal{C}} \|(\mathbf{I} + \widehat{\mathcal{R}}_x)^{-1}\|_{S^{-\nu}(\widehat{G}_x M), (\mathbb{X}, U), \mathcal{C}, 0} < \infty,$$

is finite; we could also obtain this fact from the analysis of the Bessel kernels in [37, Section 4.3] without resorting to Theorem 2.41 and Remark 2.42.

Differentiating with respect to  $x$ , we obtain

$$(5.15) \quad D_{X_j}(\mathbf{I} + \widehat{\mathcal{R}}_x)^{-1} = (\mathbf{I} + \widehat{\mathcal{R}}_x)^{-1} (D_{X_j} \widehat{\mathcal{R}})_x (\mathbf{I} + \widehat{\mathcal{R}}_x)^{-1}$$

in a local adapted frame  $(\mathbb{X}, U)$  (at first formally in terms of convolution kernels). Inductively, we obtain an expression for  $D_{\mathbb{X}}^\beta (\mathbf{I} + \widehat{\mathcal{R}})^{-1}$ ,  $\beta \in \mathbb{N}_0^n$ , as a linear combination of products of  $(\mathbf{I} + \widehat{\mathcal{R}}_x)^{-1}$  and  $D_{\mathbb{X}}^{\beta'} \widehat{\mathcal{R}}$ ,  $\beta' \in \mathbb{N}_0^n$ . By Lemma 5.30,  $\widehat{\mathcal{R}} \in S^\nu(\widehat{GM})$ , so  $D_{\mathbb{X}}^{\beta'} \widehat{\mathcal{R}} \in S^\nu(\widehat{GM})$ . This yields suitable seminorm bounds for  $D_{\mathbb{X}}^\beta (\mathbf{I} + \widehat{\mathcal{R}})^{-1}$ , showing that  $(\mathbf{I} + \mathcal{R})^{-1} \in S^{-\nu}(\widehat{GM})$ . This concludes the proof of Part (1).

Let us show Part (2). By [36, Proposition 4.33] and its proof,  $(\widehat{\mathcal{R}}_x - z)^{-1} \in S^{-\nu}(\widehat{G}_x M)$  for any  $x \in M$  with for any  $\pi \in \widehat{G}_x M$

$$\begin{aligned} \|(I + \pi(\mathcal{R}_x))^{\frac{\nu+\gamma}{\nu}} (\pi(\mathcal{R}_x) - z)^{-1} (I + \pi(\mathcal{R}_x))^{-\frac{\gamma}{\nu}}\|_{\mathcal{L}(\mathcal{H}_\pi)} &= \|(\pi(\mathcal{R}_x) - z)^{-1} (I + \pi(\mathcal{R}_x))\|_{\mathcal{L}(\mathcal{H}_\pi)} \\ &\leq \sup_{\lambda \geq 0} \left| \frac{1 + \lambda}{\lambda - z} \right| \leq 1 + \frac{1 + |z|}{|\operatorname{Im} z|}, \end{aligned}$$

by functional analysis. Using (5.15) in a local adapted frame  $(\mathbb{X}, U)$  and arguing inductively, we obtain an expression for  $D_{\mathbb{X}}^\beta \Delta_{\mathbb{X}}^\alpha (\widehat{\mathcal{R}} - z)^{-1}$ ,  $\beta, \alpha \in \mathbb{N}_0^n$ , with suitable seminorm bounds by proceeding as above (see the proof of [36, Proposition 4.33]). This concludes the proof of Part (2).  $\square$

5.4.3. *Proof of Proposition 5.40.* The main step in the proof of Proposition 5.40 is the following statement:

**Lemma 5.43.** *Let  $\mathcal{R}$  be a positive Rockland operator on  $GM$ . We denote by  $\nu$  its homogeneous degree.*

- (1) *For any  $\psi \in \mathcal{G}^m(\mathbb{R})$  with  $m < -1$ , the symbol  $\psi(\widehat{\mathcal{R}})$  is in  $S^{-\nu}(\widehat{GM})$ . Moreover, for any seminorm  $\|\cdot\|_{S^{-\nu}(\widehat{GM}),(\mathbb{X},U),C,N}$ , there exist  $C > 0$  and a seminorm  $\|\cdot\|_{\mathcal{G}^m, N'}$  such that*

$$\forall \psi \in \mathcal{G}^m(\mathbb{R}), \forall t \in (0, 1], \|\psi(t\widehat{\mathcal{R}})\|_{S^{-\nu}(\widehat{GM}),(\mathbb{X},U),C,N} \leq Ct^{-1} \|\psi\|_{\mathcal{G}^m, N'}.$$

- (2) *If  $\psi \in \mathcal{S}(\mathbb{R})$ , then  $\psi(\widehat{\mathcal{R}}) \in S^{-\infty}(\widehat{GM})$ . Moreover, the map  $\psi \mapsto \psi(\widehat{\mathcal{R}})$  is continuous  $\mathcal{S}(\mathbb{R}) \rightarrow S^{-\infty}(\widehat{GM})$ .*

- (3) *For any  $m \in \mathbb{R}$  and any seminorm  $\|\cdot\|_{S^{m\nu}(\widehat{GM}),(\mathbb{X},U),C,N}$ , there exist  $C > 0$  and  $k \in \mathbb{N}$  such that for any  $\psi \in C_c^\infty(\frac{1}{2}, 2)$ ,*

$$\forall t \in (0, 1], \|\psi(t\widehat{\mathcal{R}})\|_{S^{m\nu}(\widehat{GM}),(\mathbb{X},U),C,N} \leq Ct^m \max_{k'=0,\dots,k} \sup_{\lambda \in \mathbb{R}} |\psi^{(k')}(\lambda)|.$$

*Proof of Lemma 5.43.* Let us prove (1). Let  $\psi \in \mathcal{G}^m(\mathbb{R})$ ,  $m < -1$ . We consider the almost analytical extension  $\tilde{\psi}$  constructed as in Lemma 5.41. By Lemma 5.42 (2) and the formula in (5.14), we have

$$\begin{aligned} \|\psi(\widehat{\mathcal{R}})\|_{S^{-\nu}(\widehat{GM}),(\mathbb{X},U),C,N} &\leq \frac{1}{\pi} \int_{\mathbb{C}} |\bar{\partial} \tilde{\psi}(z)| \|(\widehat{\mathcal{R}} - z)^{-1}\|_{S^{-\nu}(\widehat{GM}),(\mathbb{X},U),C,N} L(dz) \\ &\lesssim \int_{\mathbb{C}} |\bar{\partial} \tilde{\psi}(z)| \left(1 + \frac{1 + |z|}{|\operatorname{Im} z|}\right)^p L(dz) \lesssim \|\psi\|_{\mathcal{G}^m, p+3}. \end{aligned}$$

This provides the estimate for  $t = 1$ . To obtain the estimate for any  $t \in (0, 1]$ , it suffices to observe

$$\psi(t\widehat{\mathcal{R}}) = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\psi}(z) (t\widehat{\mathcal{R}} - z)^{-1} L(dz) = \frac{t^{-1}}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\psi}(z) (\widehat{\mathcal{R}} - t^{-1}z)^{-1} L(dz),$$

and to proceed as above (see the proof of [36, Corollary 4.37]). This shows Part (1).

Let us sketch the proof of Parts (2) and (3) which follows the argument of the proof of [36, Lemma 4.38]. For (2), we consider  $\psi \in \mathcal{S}(\mathbb{R})$ . As  $\widehat{\mathcal{R}} \geq 0$ , we may assume that

$\psi(\lambda) = 0$  for  $\lambda \leq -1/2$ . Then for any  $N \in \mathbb{N}_0$ ,  $\psi_N(\lambda) := (1 + \lambda)^N \psi(\lambda)$  defines a function  $\psi_N \in \mathcal{S}(\mathbb{R})$ . Applying Part (1) to  $\psi_N$  and using the properties of composition of symbols (Proposition 5.25) to

$$\psi(\widehat{\mathcal{R}}) = (\mathbf{I} + \widehat{\mathcal{R}})^{-N} \psi_N(\widehat{\mathcal{R}}),$$

thanks to Lemma 5.42 (1), imply Part (2).

Finally, let  $\psi \in C_c^\infty(\frac{1}{2}, 2)$ . The case  $m = -1$  follows from Part (1). Then, for  $m = N - 1$  with  $N \in \mathbb{N}$ , one applies Part (1) to  $\phi_N(\lambda) := \lambda^{-N} \psi(\lambda)$  and uses the properties of composition of symbols (Proposition 5.25) on

$$\psi(\widehat{\mathcal{R}}) = \widehat{\mathcal{R}}^N \phi_N(\widehat{\mathcal{R}}),$$

together with  $\widehat{\mathcal{R}}^N \in S^{N\nu}(\widehat{GM})$ . Writing  $\psi = \psi\chi$  with  $\chi \in C_c^\infty(\frac{1}{4}, 4)$  such that  $\chi = 1$  on  $(1/2, 2)$  and using the properties of composition of symbols (Proposition 5.25) on

$$\psi(\widehat{\mathcal{R}}) = \psi(\widehat{\mathcal{R}})\chi(\widehat{\mathcal{R}}),$$

yields the case of  $m = -2$ . Inductively, we obtain the case  $m = -2, -3, \dots$  Part (3) follows by interpolation.  $\square$

We can now show Proposition 5.40. Our proof will follow closely [36, Section 4.8].

*Proof of Proposition 5.40.* We only require to prove the result for  $m < -1$ . Indeed, for  $\phi \in \mathcal{G}^m(\mathbb{R})$ , with  $m \geq -1$ , choosing  $N \in \mathbb{N}_0$  such that  $N > m + 1$ , one can find  $\psi \in \mathcal{G}^{m-N}(\mathbb{R})$  such that for all  $\lambda \in [0, +\infty)$ ,  $\phi(\lambda) = (1 + \lambda)^N \psi(\lambda)$ . The result follows by the properties of symbols with respect to product (see Proposition 5.25) and Lemma 5.42 (1). Thus we focus on  $m < -1$ . By Lemma 5.43 (2), we may assume that  $\phi$  is supported on  $[1, \infty)$ . Let  $(\eta_j)$  be a dyadic decomposition of  $[1, +\infty)$ , that is,  $\eta_0 \in C_c^\infty(\frac{1}{2}, 2)$  with

$$(5.16) \quad \sum_{j=0}^{\infty} \eta_j(\lambda) = 1 \text{ for all } \lambda \geq 1, \quad \text{where } \eta_j(\lambda) := \eta_0(2^{-j}\lambda).$$

We may write for any  $\lambda \geq 0$

$$\phi(\lambda) = \sum_{j=0}^{\infty} \eta_j(\lambda) \phi(\lambda) = \sum_{j=0}^{\infty} 2^{jm} \psi_j(2^{-j}\lambda), \quad \text{where } \psi_j(\mu) := 2^{-jm} \phi(2^j\mu) \eta_0(\mu).$$

We observe that

$$\psi_j \in C_c^\infty\left(\frac{1}{2}, 2\right) \quad \text{and} \quad \sup_{\lambda \geq 0} |\psi_j^{(k)}(\lambda)| \lesssim_k \|\phi\|_{\mathcal{G}^{m,k}}$$

for any  $k \in \mathbb{N}_0$  with an implicit constant independent of  $j$ . Let  $(\mathbb{X}, U)$  be an adapted frame, let  $\alpha, \beta \in \mathbb{N}_0^n$ ,  $x \in M$  and  $\pi \in \widehat{G}_x$ . For each  $j \in \mathbb{N}_0$ , let us consider the operator

$$T_j(\alpha, \beta, \gamma; x, \pi) := T_j := 2^{jm} (\mathbf{I} + \pi(\mathcal{R}_x))^{-\frac{m\nu + |\alpha| + \gamma}{\nu}} \Delta_{\mathbb{X}}^\alpha D_{\mathbb{X}}^\beta \psi_j(2^{-j}\pi(\mathcal{R}_x)) (\mathbf{I} + \pi(\mathcal{R}_x))^{-\frac{\gamma}{\nu}}.$$

The properties of composition and adjoint for symbols in Proposition 5.25 imply

$$\begin{aligned} \|T_i^* T_j\|_{\mathcal{L}(\mathcal{H}_\pi)} &= 2^{(i+j)m} \|(I + \pi(\mathcal{R}_x))^{-\frac{\gamma}{\nu}} \Delta_{\mathbb{X}}^\alpha D_{\mathbb{X}}^\beta \psi_i(2^{-i}\pi(\mathcal{R}_x))(I + \pi(\mathcal{R}_x))^{\frac{[\alpha]+\gamma}{\nu}} \times \\ &\quad \times (I + \pi(\mathcal{R}_x))^{-\frac{2m\nu+[\alpha]+\gamma}{\nu}} \Delta_{\mathbb{X}}^\alpha D_{\mathbb{X}}^\beta \psi_j(2^{-j}\pi(\mathcal{R}_x))(I + \pi(\mathcal{R}_x))^{-\frac{\gamma}{\nu}}\|_{\mathcal{L}(\mathcal{H}_\pi)} \\ &\lesssim 2^{(i+j)m} \|\psi_i(2^{-i}\pi(\mathcal{R}_x))\|_{S^0(\widehat{G}_x M), [\alpha], [\beta], c} \|\psi_j(2^{-j}\pi(\mathcal{R}_x))\|_{S^{2m\nu}(\widehat{G}_x M), [\alpha], [\beta], c}, \end{aligned}$$

for some  $c > 0$ . Lemma 5.43 (3) yields that for some  $k \in \mathbb{N}_0$ , we have

$$\begin{aligned} \|T_i^* T_j\|_{\mathcal{L}(\mathcal{H}_\pi)} &\lesssim 2^{(i+j)m} 2^{-j(2m)} \max_{k'=0, \dots, k} \sup_{\lambda \geq 0} |\psi_i^{(k')}(\lambda)| \max_{k'=0, \dots, k} \sup_{\lambda \geq 0} |\psi_j^{(k')}(\lambda)| \\ &\lesssim 2^{(i-j)m} \|\phi\|_{\mathcal{G}^m, k}^2, \end{aligned}$$

and similarly for  $\|T_i T_j^*\|_{\mathcal{L}(\mathcal{H}_\pi)}$ . By the Cotlar-Stein Lemma [61, §VII.2],  $\sum_j T_j$  converges in the strong operator topology to a bounded operator on  $\mathcal{H}_\pi$  with norm  $\lesssim \|\phi\|_{\mathcal{G}^m, k}$ , the implicit constant being independent of  $(x, \pi) \in G \times \widehat{G}$ . This implies that

$$\sup_{(x, \pi) \in \widehat{G}M} \|(I + \pi(\mathcal{R}_x))^{-\frac{m\nu+[\alpha]+\gamma}{\nu}} \Delta_{\mathbb{X}}^\alpha D_{\mathbb{X}}^\beta \phi(\pi(\mathcal{R}_x))(I + \pi(\mathcal{R}_x))^{-\frac{\gamma}{\nu}}\|_{\mathcal{L}(\mathcal{H}_\pi)} \lesssim \|\phi\|_{\mathcal{G}^m, k}$$

for some  $k \in \mathbb{N}$ . As this holds for any  $\alpha, \beta, (\mathbb{X}, U)$ , the conclusion follows.  $\square$

5.4.4. *Multipliers depending on  $x$ .* We observe that the proof above is robust enough to allow a dependence of the multiplier function  $\phi$  in  $x$ . More precisely, we can consider multiplier functions in the space

$$\mathcal{G}^m(M \times \mathbb{R}) = C^\infty(M) \otimes \mathcal{G}^m(\mathbb{R}),$$

equipped with the tensor topology; equivalently, this is the Fréchet space of smooth functions  $\phi : M \times \mathbb{R} \rightarrow \mathbb{C}$  generated by the semi-norms given by

$$\|\phi\|_{\mathcal{G}^m, N} := \max_{k, j, [\alpha] \leq N} \sup_{(x, \lambda) \in \mathcal{C}_k \times \mathbb{R}} (1 + |\lambda|)^{-m+j} |\mathbb{X}_k^\alpha \partial_\lambda^j \phi(x, \lambda)|,$$

where a countable sequence of compact subsets  $\mathcal{C}_k$ ,  $k = 0, 1, 2, \dots$  covering  $M = \cup_k \mathcal{C}_k$  has been fixed, with each compact  $\mathcal{C}_k$  included in a bounded open subset  $U_k$  on which an adapted frame  $\mathbb{X}_k$  exists.

Adapting the proof above, we readily obtain:

**Corollary 5.44.** *Let  $\phi \in \mathcal{G}^m(M \times \mathbb{R})$  with  $m \in \mathbb{R}$ , and let  $\mathcal{R}$  be a positive Rockland element on  $GM$ . Then the symbol*

$$\phi(\widehat{\mathcal{R}}) := \{\phi(x, \pi(\mathcal{R}_x)) : (x, \pi) \in \widehat{G}M\}$$

*is in  $S^{m\nu}(\widehat{G}M)$  where  $\nu$  is the homogeneous degree of  $\mathcal{R}$ . Moreover, the map*

$$\phi \longmapsto \phi(\widehat{\mathcal{R}}), \quad \mathcal{G}^m(M \times \mathbb{R}) \longrightarrow S^{m\nu}(\widehat{G}M),$$

*is continuous.*

5.4.5. *Density of smoothing symbols.* The proofs of Propositions 2.44 and 2.46, with Remarks 2.45 and 2.47, can be readily adapted to the manifold case by using Proposition 5.40. We obtain the following result:

**Proposition 5.45.** (1) *The fiberwise group Fourier transform is an isomorphism from the topological vector space of the vertical Schwartz densities  $\mathcal{S}(GM; |\Omega|(GM))$  onto  $S^{-\infty}(\widehat{GM})$ .*

(2) *Let  $\sigma \in S^m(\widehat{GM})$  with  $m \in \mathbb{R}$ . We can construct a sequence of smoothing symbols  $(\sigma_\ell)_{\ell \in \mathbb{N}} \subset S^{-\infty}(\widehat{GM})$  satisfying the following properties:*

(a) *For any  $m_1 > m$ , we have the convergence*

$$\lim_{\ell \rightarrow \infty} \sigma_\ell = \sigma \quad \text{in } S^{m_1}(\widehat{GM}).$$

*In other words, the space  $S^{-\infty}(\widehat{GM})$  equipped with the  $S^{m_1}(\widehat{GM})$ -topology is dense in  $S^m(\widehat{GM})$ .*

(b) *For any semi-norm  $\|\cdot\|_{S^m(\widehat{G}),(\mathbb{X},U),\mathcal{C},N}$  of  $S^m(\widehat{G})$ , there exist a constant  $C > 0$  and  $N' \geq N$  such that*

$$\forall \ell \in \mathbb{N}, \quad \|\sigma_\ell - \sigma\|_{S^m(\widehat{G}),(\mathbb{X},U),\mathcal{C},N} \leq C \|\sigma\|_{S^m(\widehat{G}),(\mathbb{X},U),\mathcal{C},N'}.$$

*Consequently,*

$$\forall \ell \in \mathbb{N}, \quad \|\sigma_\ell\|_{S^m(\widehat{G}),(\mathbb{X},U),\mathcal{C},N} \leq (1 + C) \|\sigma\|_{S^m(\widehat{G}),(\mathbb{X},U),\mathcal{C},N'}.$$

(c) *Denoting respectively by  $\kappa^{\mathbb{X}}$  and  $\kappa_\ell^{\mathbb{X}}$  the convolution kernels of  $\sigma$  and  $\sigma_\ell$  in the  $\mathbb{X}$ -coordinates, we have the convergences*

$$\lim_{\ell \rightarrow \infty} \kappa_\ell^{\mathbb{X}} = \kappa^{\mathbb{X}} \quad \text{in } \Gamma(\mathcal{S}'(\mathbb{R}^n)) \quad \text{and in } \Gamma(C^\infty(U_1)),$$

*for any open subset of  $U_1 \subset G \setminus \{0\}$ .*

(d) *Moreover,  $\liminf_{\ell \rightarrow \infty} \|\sigma_\ell\|_{S^m(\widehat{GM}),(\mathbb{X},U),\mathcal{C},N} \geq \|\sigma\|_{S^m(\widehat{GM}),(\mathbb{X},U),\mathcal{C},N}$ .*

*Remark 5.46.* The above result will be useful in many proofs. Indeed, it will allow us to assume that symbols are smoothing, and hence have Schwartz convolution kernels.

For instance, instead of manipulating distributions in the composition of symbols in Proposition 5.25, we can instead consider Schwartz functions by considering smoothing symbols and applying the result of density in Proposition 5.45. Indeed, if  $\sigma_1, \sigma_2$  and  $\sigma$  are smoothing, given a smooth Haar measure  $\mu$  and fixing  $x \in M$ , the formulae for  $\kappa_{\sigma_1 \sigma_2, x}^\mu$  and  $\kappa_{\sigma^*, x}^\mu$  in Proposition 5.25 make sense as Schwartz functions on  $G_x M$ . We check as above that these define Schwartz densities independently of a choice of a smooth Haar measure. The estimates with respect to suitable seminorms of symbol classes are checked using the results on  $G_x M$ ,  $x \in M$ . And we can conclude by density (Proposition 5.45).

## 5.5. Homogeneous and polyhomogeneous symbols.

5.5.1. *Homogeneous symbols.* We define the class  $\dot{S}^m(\widehat{GM})$  of homogeneous symbol of order  $m$  by replacing  $S^m(\widehat{G}_xM)$  with  $\dot{S}^m(\widehat{G}_xM)$  in Definition 5.23 (see Section 2.6.1 for the definition of homogeneous symbols on a group). As Lemmata 5.17, 5.21 and 5.22 can easily be adapted to hold for  $\dot{S}^m(\widehat{G}_xM)$ -semi-norms, this notion is intrinsically defined. We denote by  $\dot{S}^m(\widehat{GM})$  the space of homogeneous symbols of order  $m$  on  $M$ . If  $\mathbb{X}$  is an adapted frame on an open set  $U \subset M$  and  $\mathcal{C}$  a compact subset of  $U$ , we define the associated seminorm  $\|\cdot\|_{\dot{S}^m(\widehat{GM}),(\mathbb{X},U),\mathcal{C},N}$  as in (5.8) by replacing the  $S^m(\widehat{G}_xM)$ -semi-norm with the corresponding one on  $\dot{S}^m(\widehat{G}_xM)$  (see (2.12)). It is a Fréchet space for the countable family of semi-norms analogous to (5.9).

*Example 5.47.* Adapting Example 5.11, the symbol  $\widehat{\langle \mathbb{X} \rangle}^\alpha := \{\pi(\langle X \rangle_x^\alpha) : x \in U, \pi \in \widehat{G}_xM\}$  is in  $\dot{S}^{[\alpha]}(\widehat{GM})$ .

*Example 5.48.* Extending Example 2.56 to the manifold setting, for any positive Rockland operator  $\mathcal{R}$ , we have  $(I + \widehat{\mathcal{R}})^{m/\nu} \in S_{ph}^m(\widehat{GM})$  where  $\nu$  is the homogeneous degree of  $\mathcal{R}$ , and

$$(I + \widehat{\mathcal{R}})^{m/\nu} \sim_h \widehat{\mathcal{R}}^{m/\nu} + \frac{m}{\nu} \widehat{\mathcal{R}}^{-1+m/\nu} + \dots$$

By construction, every symbol  $\sigma \in \dot{S}^m(\widehat{GM})$  admits a convolution kernel density that is  $m$ -homogeneous in the sense of Definition 5.16.

Adapting the proof for the inhomogeneous symbol classes, we obtain that the homogeneous classes  $\dot{S}^m(\widehat{GM})$  enjoy the algebraic properties for composition and adjoint as described in Proposition 5.25 for the inhomogeneous case.

5.5.2. *Links between homogeneous and inhomogeneous symbol classes.*

**Proposition 5.49.** *Let  $\mathcal{R}$  be a positive Rockland element of  $\mathcal{W}(\mathfrak{g}M)$ . Let  $\psi_0 \in C^\infty(M \times \mathbb{R})$  be such that  $\psi_0(x, \lambda) = 0$  for  $\lambda$  in a neighbourhood of  $(-\infty, 0]$  and  $\psi_0(x, \lambda) = 1$  on a neighbourhood of  $+\infty$ , these two  $\lambda$ -neighbourhoods depending locally uniformly on  $x \in M$ . Then for any  $\sigma \in \dot{S}^{m_1}(\widehat{GM})$  and  $\psi \in \mathcal{G}^{m_2}(\mathbb{R})$ , the symbols*

$$\sigma(\psi\psi_0)(\widehat{\mathcal{R}}) = \{\sigma(x, \pi)(\psi\psi_0)(x, \pi(\mathcal{R}_x)) : x \in M, \pi \in \widehat{G}_xM\}$$

and

$$(\psi\psi_0)(\widehat{\mathcal{R}})\sigma = \{(\psi\psi_0)(x, \pi(\mathcal{R}_x))\sigma(x, \pi) : x \in M, \pi \in \widehat{G}_xM\}$$

are in  $S^{m_1+m_2\nu}(\widehat{GM})$  where  $\nu$  is the homogeneous degree of  $\mathcal{R}$ . Moreover, the maps

$$(\sigma, \psi) \mapsto (\psi\psi_0)(\widehat{\mathcal{R}})\sigma \quad \text{and} \quad (\sigma, \psi) \mapsto \sigma(\psi\psi_0)(\widehat{\mathcal{R}}),$$

are continuous  $\dot{S}^{m_1}(\widehat{GM}) \times \mathcal{G}^{m_2}(\mathbb{R}) \rightarrow S^{m_1+m_2\nu}(\widehat{GM})$ .

Below, we will use Proposition 5.49 with  $\psi \equiv 1$ , that is,  $\sigma\psi_0(\widehat{\mathcal{R}}) \in S^{m_1}(\widehat{GM})$  if  $\sigma \in \dot{S}^{m_1}(\widehat{GM})$ . In the proof of Proposition 5.49 and in the remainder of this section, we will use the notation

$$\widehat{GM}^* = \bigcup_{x \in M} (\widehat{G}_xM \setminus \{1\})$$

and write  $\widehat{G}_xM^*$  for the fibers of  $\widehat{GM}^*$ .

*Proof of Proposition 5.49.* From the group case (see Section 2.6.1 and Proposition 4.6 in [29]), we see that for any  $x \in M$ , the symbols  $\sigma(x, \cdot)(\psi\psi_0)(x, \widehat{\mathcal{R}}_x)$  and  $(\psi\psi_0)(x, \widehat{\mathcal{R}}_x)\sigma(x, \cdot)$  are in  $S^m(\widehat{G}_x M)$ . Moreover, setting  $m := m_1 + m_2\nu$ , the  $S^m(\widehat{G}_x M)$ -semi-norms of these symbols are bounded by some  $S^{m_1}(\widehat{G}_x M)$ -semi-norms of  $\sigma$  and  $\mathcal{G}^{m_2}(\mathbb{R})$ -semi-norms in  $\psi$  up to a constant locally uniformly in  $x \in M$ . The main issue is the  $x$ -derivatives. Let  $\mathbb{X}$  be an adapted frame on an open subset  $U$  of  $M$ . By the Leibniz rule in  $x$ , we have at least formally with  $|\beta| = 1$

$$D_{\mathbb{X}}^\beta(\sigma(\psi\psi_0)(\widehat{\mathcal{R}})) = D_{\mathbb{X}}^\beta\sigma(\psi\psi_0)(\widehat{\mathcal{R}}) + \sigma D_{\mathbb{X}}^\beta(\psi\psi_0)(\widehat{\mathcal{R}}).$$

By construction,  $D_{\mathbb{X}}^\beta\sigma \in \dot{S}^m(\widehat{G}M|_U)$  and therefore the first term may be analysed fiberwise as above. For the second term, we can not in general claim that  $D_{\mathbb{X}}^\beta(\psi\psi_0)(\widehat{\mathcal{R}})$  vanishes identically for low frequencies. To deal with this issue, we modify the dyadic decomposition (5.16) into a dyadic decomposition on  $[0, +\infty)$ : we fix  $\eta_0 \in C^\infty([-1, 1], [0, 1])$  and  $\eta_1 \in C_c^\infty([\frac{1}{2}, \frac{3}{2}], [0, 1])$  such that

$$1 = \sum_{j=0}^{\infty} \eta_j(\lambda) \quad \text{for any } \lambda \geq 0, \quad \text{with } \eta_j(\lambda) := \eta_1(2^{j-1}\lambda).$$

We then write

$$\sup_{(x, \pi) \in \widehat{G}M^*} \|\sigma(x, \pi) D_{\mathbb{X}}^\beta(\psi\psi_0)(\widehat{\mathcal{R}}_x) (\mathbf{I} + \pi(\mathcal{R}_x))^{-\frac{m}{\nu}}\|_{\mathcal{L}(\mathcal{H}_\pi)} \leq C_0 + C_{\psi, \beta} \sum_{j=1}^{\infty} E_j,$$

where

$$\begin{aligned} C_0 &:= \sup_{(x, \pi) \in \widehat{G}M^*} \|\sigma(x, \pi) \eta_0(\pi(\mathcal{R}_x)) D_{\mathbb{X}}^\beta(\psi\psi_0)(\widehat{\mathcal{R}}_x) (\mathbf{I} + \pi(\mathcal{R}_x))^{-\frac{m}{\nu}}\|_{\mathcal{L}(\mathcal{H}_\pi)}, \\ E_j &:= \sup_{(x, \pi) \in \widehat{G}M^*} \|\sigma(x, \pi) \eta_j(\pi(\mathcal{R}_x))\|_{\mathcal{L}(\mathcal{H}_\pi)}, \\ C_{\psi, \beta} &:= \sup_{(x, \pi) \in \widehat{G}M^*} \|D_{\mathbb{X}}^\beta(\psi_0\psi)(\widehat{\mathcal{R}}_x) (\mathbf{I} + \pi(\mathcal{R}_x))^{-\frac{m}{\nu}}\|_{\mathcal{L}(\mathcal{H}_\pi)}. \end{aligned}$$

The constant  $C_0$  is finite since we have  $C_0 \leq C'_0 \|\sigma\|_{\dot{S}^{m_1}(\widehat{G}M), 0}$  with the constant

$$\begin{aligned} C'_0 &:= \sup_{(x, \pi) \in \widehat{G}M^*} \|\widehat{\mathcal{R}}_x^{\frac{m_1}{\nu}} \eta_0(\pi(\mathcal{R}_x)) (\mathbf{I} + \pi(\mathcal{R}_x))^{-\frac{m_1}{\nu}}\|_{\mathcal{L}(\mathcal{H}_\pi)} \\ &\quad \times \|(\mathbf{I} + \pi(\mathcal{R}_x))^{\frac{m_1}{\nu}} D_{\mathbb{X}}^\beta(\psi\psi_0)(\widehat{\mathcal{R}}_x) (\mathbf{I} + \pi(\mathcal{R}_x))^{-\frac{m}{\nu}}\|_{\mathcal{L}(\mathcal{H}_\pi)}, \end{aligned}$$

being finite and in fact bounded by the product of a  $S^0(\widehat{G}M)$ -semi-norm in  $\eta_0(\widehat{\mathcal{R}})$  with a  $S^{m_2\nu}(\widehat{G}M)$ -semi-norm in  $(\psi\psi_0)(\widehat{\mathcal{R}})$  up to a constant.

By Corollary 5.44,  $C_{\psi,\beta}$  is a finite constant since  $D_{\mathbb{X}}^\beta(\psi_0\psi) = -D_{\mathbb{X}}^\beta(1 - \psi_0\psi)$  and  $1 - \psi_0\psi \in C^\infty(M) \otimes C_c^\infty(\mathbb{R})$ . Regarding the  $E_j$ -s, we write

$$\begin{aligned} E_j &= \sup_{(x,\pi) \in \widehat{GM}^*} \|\sigma(x, \pi) \eta_1(2^{j-1} \pi(\mathcal{R}_x))\|_{\mathcal{L}(\mathcal{H}_\pi)} \\ &= \sup_{(x,\pi) \in \widehat{GM}^*} \|\sigma(x, \pi) \eta_1(2^{\frac{j-1}{\nu}} \cdot \pi(\mathcal{R}_x))\|_{\mathcal{L}(\mathcal{H}_\pi)} \\ &= \sup_{(x,\pi') \in \widehat{GM}^*} \|\sigma(x, 2^{-\frac{j-1}{\nu}} \cdot \pi') \eta_1(\pi'(\mathcal{R}_x))\|_{\mathcal{L}(\mathcal{H}_{\pi'})} \\ &= 2^{-m_1 \frac{j-1}{\nu}} \sup_{(x,\pi') \in \widehat{GM}^*} \|\sigma(x, \pi') \eta_1(\pi'(\mathcal{R}_x))\|_{\mathcal{L}(\mathcal{H}_{\pi'})} \end{aligned}$$

where we have used the homogeneity of  $\sigma$ . We obtain

$$E_j \leq 2^{-m_1 \frac{j-1}{\nu}} \|\sigma\|_{\dot{S}^{m_1}(\widehat{GM})_0} \sup_{\lambda > 0} (1 + \lambda)^{\frac{m_1}{\nu}} |\eta_1(\lambda)|$$

by functional analysis; alternatively, we can estimate  $S^{-m_1}(\widehat{GM})$ -seminorms of  $\eta_1(\pi'(\mathcal{R}_x))$  by  $\sup_{\lambda > 0} \lambda^{\frac{m_1}{\nu}} |\eta_1(\lambda)|$  using Lemma 5.43. Hence for  $m_1 > 0$ ,  $\sum_j E_j$  is finite.

If  $m_1 \leq 0$ , we modify the analysis of the second term by considering  $\sigma_1 := \sigma \widehat{\mathcal{R}}^{N_1}$  and  $\psi_1(\lambda) := \lambda^{-N_1} \psi$  for  $N_1$  large enough. Indeed, as  $\sigma_1$  will be in  $\dot{S}^{m_1 + N_1 \nu}$ , we will choose  $N_1 > -m_1/\nu$  and we will still have  $\sigma\psi(\widehat{\mathcal{R}}) = \sigma_1\psi_1(\widehat{\mathcal{R}})$ .

Adapting the reasoning above to any derivative  $\mathbb{X}^\beta$ ,  $\beta \in \mathbb{N}_0^n$ , and  $\Delta_{\mathbb{X}}^\alpha$ ,  $\alpha \in \mathbb{N}_0^n$ , we obtain that the quantity  $\|\sigma(\psi_0\psi)(\widehat{\mathcal{R}})\|_{S^m(\widehat{GM})_N}$  is finite for any  $N \in \mathbb{N}_0$ , and therefore  $\sigma(\psi_0\psi)(\widehat{\mathcal{R}}) \in S^m(\widehat{GM})$ . The continuity follows from tracking the constant in the above estimates. The result for the symbol  $(\psi_0\psi)(\widehat{\mathcal{R}})\sigma$  is proved in a similar manner.  $\square$

The following statement shows that modulo smoothing symbols, the precise choices of the positive Rockland element  $\mathcal{R}$  or the cut-off  $\psi_0$  for the low frequencies in Proposition 5.49 are irrelevant:

**Lemma 5.50.** *Let  $\psi_1, \psi_2$  be two smooth functions on  $M \times \mathbb{R}$  satisfying for  $i = 1, 2$ ,  $\psi_i(x, \lambda) = 0$  for  $\lambda$  in a neighbourhood of  $(-\infty, 0]$  and  $\psi_i(x, \lambda) = 1$  on a neighbourhood of  $+\infty$ , these two  $\lambda$ -neighbourhoods depending locally uniformly on  $x \in M$ . Let  $\mathcal{R}$  and  $\mathcal{S}$  be two positive Rockland elements on  $GM$ .*

(1) *The symbol  $\psi_1(\mathcal{R}) - \psi_2(\mathcal{S})$  is smoothing:*

$$\psi_1(\mathcal{R}) - \psi_2(\mathcal{S}) \in S^{-\infty}(\widehat{GM}).$$

(2) *More generally, for any  $\sigma \in \dot{S}^m(\widehat{GM})$ , the symbols*

$$\sigma\psi_1(\mathcal{R}) - \sigma\psi_2(\mathcal{S}) \quad \text{and} \quad \psi_1(\mathcal{R})\sigma - \psi_2(\mathcal{S})\sigma$$

*are smoothing, that is, in  $S^{-\infty}(\widehat{GM})$ .*

*Proof of Lemma 5.50.* By Corollary 5.44 and Proposition 5.49, we may assume  $\psi_1 = \psi_2 = \psi_0$ . Part (1) is a consequence of Part (2) with  $\sigma = \text{I}$ . Let us prove Part (2). We consider

a dyadic decomposition on  $[0, +\infty)$  as introduced in the proof of Proposition 5.49: we fix  $\eta_0 \in C^\infty([-1, 1], [0, 1])$  and  $\eta_1 \in C_c^\infty([\frac{1}{2}, \frac{3}{2}], [0, 1])$  such that

$$1 = \sum_{j=0}^{\infty} \eta_j(\lambda) \quad \text{for any } \lambda \geq 0, \quad \text{with } \eta_j(\lambda) := \eta_1(2^{j-1}\lambda).$$

For each  $j \in \mathbb{N}_0$ , the symbol  $\sigma \eta_j(\widehat{\mathcal{R}}) \left( \psi_0(\widehat{\mathcal{R}}) - \psi_0(\widehat{\mathcal{S}}) \right)$  is smoothing by Proposition 5.49 and the properties of composition of symbols stated in Proposition 5.25. The properties of the supports of  $\psi_0$  and  $\eta_0, \eta_1$  imply the existence of  $j_0 \in \mathbb{N}$  such that for all  $j \geq j_0$ ,  $\eta_j \psi_0 = \eta_j$ . Hence, for  $j \geq j_0$ ,

$$\eta_j(\widehat{\mathcal{R}}) \left( \psi_0(\widehat{\mathcal{R}}) - \psi_0(\widehat{\mathcal{S}}) \right) = \eta_j(\widehat{\mathcal{R}}) - \eta_j(\widehat{\mathcal{R}}) \psi_0(\widehat{\mathcal{S}}) = \eta_j(\widehat{\mathcal{R}}) (1 - \psi_0)(\widehat{\mathcal{S}}).$$

For any semi-norm  $\|\cdot\|_{S^{-N_1}(\widehat{GM}), N_2}$ , we have

$$\|\sigma \eta_j(\widehat{\mathcal{R}}) (1 - \psi_0)(\widehat{\mathcal{S}})\|_{S^{-N_1}(\widehat{GM}), N_2} \lesssim \|\sigma \eta_j(\widehat{\mathcal{R}})\|_{S^{m-\nu}(\widehat{GM}), N'_2} \|(1 - \psi_0)(\widehat{\mathcal{S}})\|_{S^{-N_1-m+\nu}(\widehat{GM}), N''_2},$$

for some  $N'_2, N''_2 \in \mathbb{N}_0$ , where  $\nu$  is the homogeneous degree of  $\mathcal{R}$ . By Proposition 5.40,  $\|(1 - \psi_0)(\widehat{\mathcal{S}})\|_{S^{-N_1-m+\nu}(\widehat{GM}), N''_2}$  is finite since  $1 - \psi_0 \in C_c^\infty(\mathbb{R})$ , while we have by Lemma 5.43 (3):

$$\|\sigma \eta_j(\widehat{\mathcal{R}})\|_{S^{m-\nu}(\widehat{GM}), N'_2} \lesssim \|\sigma\|_{S^m(\widehat{GM}), \tilde{N}_2} \|\eta_j(\widehat{\mathcal{R}})\|_{S^{-\nu}(\widehat{GM}), N''_2} \lesssim \|\eta_j\|_{\mathcal{G}^{-1}, N_3} \lesssim 2^{-j},$$

for some  $\tilde{N}_2, N''_2, N_3 \in \mathbb{N}_0$ . Hence, the sum

$$\sum_{j \geq j_0} \|\eta_j(\widehat{\mathcal{R}}) \left( \psi_0(\widehat{\mathcal{R}}) - \psi_0(\widehat{\mathcal{S}}) \right)\|_{S^{-N_1}(\widehat{GM}), N_2}$$

is finite for any semi-norm  $\|\cdot\|_{S^{-N_1}(\widehat{GM}), N_2}$ . This concludes the proof for  $\sigma \psi_1(\mathcal{R}) - \sigma \psi_2(\mathcal{S})$ . The case of  $\psi_1(\mathcal{R})\sigma - \psi_2(\mathcal{S})\sigma$  may be dealt with in a similar manner.  $\square$

The following statement shows that modulo smoothing symbols, cutting the low frequencies on the left or on the right makes no difference:

**Lemma 5.51.** *Let  $\psi_0$  be a smooth function on  $M \times \mathbb{R}$  satisfying  $\psi_0(x, \lambda) = 0$  for  $\lambda$  in a neighbourhood of  $(-\infty, 0]$  and  $\psi_0(x, \lambda) = 1$  on a neighbourhood of  $+\infty$ , these two  $\lambda$ -neighbourhoods depending locally uniformly on  $x \in M$ . Let  $\mathcal{R}$  be a positive Rockland element on  $GM$ . For any  $\sigma \in \dot{S}^m(\widehat{GM})$ , the symbol  $\sigma \psi_0(\widehat{\mathcal{R}}) - \psi_0(\widehat{\mathcal{R}})\sigma$  is smoothing, that is, in  $S^{-\infty}(\widehat{GM})$ .*

*Proof.* Let  $\psi_1$  be a smooth function on  $M \times \mathbb{R}$  satisfying  $\psi_1(x, \lambda) = 0$  for  $\lambda$  in a neighbourhood of  $(-\infty, 0]$  (locally uniformly in  $x \in M$ ) and  $\psi_1 = 1$  on the support of  $\psi_0$ . We may compute modulo smoothing symbols by Corollary 5.44 and Proposition 5.49

$$\begin{aligned} \sigma \psi_0(\widehat{\mathcal{R}}) - \psi_0(\widehat{\mathcal{R}})\sigma &= \psi_1(\widehat{\mathcal{R}}) \left( \sigma \psi_0(\widehat{\mathcal{R}}) - \psi_0(\widehat{\mathcal{R}})\sigma \right) + S^{-\infty}(\widehat{GM}) \\ &= \psi_1(\widehat{\mathcal{R}})\sigma (\psi_0 - 1)(\widehat{\mathcal{R}}) - \psi_1(\psi_0 - 1)(\widehat{\mathcal{R}})\sigma + S^{-\infty}(\widehat{GM}) \\ &\in S^{-\infty}(\widehat{GM}), \end{aligned}$$

concluding the proof.  $\square$

5.5.3. *Polyhomogeneous expansion and symbols.* We extend to filtered manifolds the notion of polyhomogeneous expansion of symbols (see Section 2.6.2 for the groups case).

**Definition 5.52.** A symbol  $\sigma \in S^m(\widehat{GM})$  admits the *polyhomogeneous expansion*

$$\sigma \sim_h \sum_{j=0}^{\infty} \sigma_{m-j}, \quad \sigma_{m-j} \in \dot{S}^{m-j}(\widehat{GM}),$$

if for a function  $\psi \in C^\infty(M \times \mathbb{R})$  satisfying  $\psi(x, \lambda) = 0$  for  $\lambda$  in a neighbourhood of  $(-\infty, 0]$  and  $\psi_0(\lambda) = 1$  on a neighbourhood of  $+\infty$ , these two  $\lambda$ -neighbourhoods depending locally uniformly on  $x \in M$ , and for a positive Rockland element  $\mathcal{R}$ , the symbol  $\sigma$  admits the following expansion in  $S^m(\widehat{GM})$  (in the sense of Definition 5.35):

$$\sigma \sim \sum_{j=0}^N \psi(\widehat{\mathcal{R}}) \sigma_{m-j}.$$

By Lemma 5.50, if the property in Definition 5.52 holds for one such function  $\psi$  and one positive Rockland element  $\mathcal{R}$  then it holds for any of them. By Lemma 5.51, the property above is also equivalent to

$$\sigma - \sum_{j=0}^N \psi(\widehat{\mathcal{R}}) \sigma_{m-j} \in S^{m-N-1}(\widehat{GM}).$$

**Definition 5.53.** A symbol is *polyhomogeneous* of order  $m \in \mathbb{R}$  when it is a symbol in  $S^m(\widehat{GM})$  admitting a polyhomogeneous expansion. We denote by  $S_{ph}^m(\widehat{GM})$  the classes of polyhomogeneous symbols of order  $m \in \mathbb{R}$ .

By construction, if  $\sigma \in S_{ph}^m(\widehat{GM})$ , then  $\sigma(x, \cdot) \in S_{ph}^m(\widehat{G}_x M)$  for every  $x \in M$ . Consequently, the polyhomogeneous expansion on  $\widehat{GM}$ , when it exists, is unique as it is unique in the group case (Section 2.6.2).

The properties of the homogeneous and inhomogeneous symbol classes imply that the polyhomogeneous symbol classes enjoy the following algebraic properties of composition and adjoint:

- (a) the product of a symbol in  $S_{ph}^{m_1}(\widehat{GM})$  with a symbol in  $S_{ph}^{m_2}(\widehat{GM})$  is in  $S_{ph}^{m_1+m_2}(\widehat{GM})$ ,
- (b) the adjoint of a symbol in  $S_{ph}^m(\widehat{GM})$  is in  $S_{ph}^m(\widehat{GM})$ .

## 6. KERNEL ESTIMATES AND RELATED RESULTS

As for operators of Calderon-Zygmund's type, a deep analysis of the kernel will be at the roots of the analysis of the operators obtained by quantization of the symbols in  $\cup_{m \in \mathbb{R}} S^m(\widehat{GM})$ . In this section, we first present in Section 6.1 kernel estimates that are really similar to those of the group case. Then, in the next two sections, we develop consequences of kernels estimates regarding kernel functions that will appear when developing a pseudodifferential calculus in the rest of the article. Such terms do not appear in the group case. They are mainly due to the fact that the coordinates provided by the exponential map on the manifold

do not coincide exactly with the variables of the osculating group. This difficulty will be overcome by identifying terms of higher order, that we will be able to treat. In Section 6.2, we derive technical lemma that we will use in Sections 7 to analyse the invariance for the change of frame. In Section 6.3, we focus on identifying convolution structure up to higher order terms. This section can be skipped at first reading.

*Notation 6.1.* If an adapted frame  $\mathbb{X}$  has been fixed on an open subset  $U \subset M$ , then we consider the family  $(q_\alpha(x, v))_{\alpha \in \mathbb{N}_0^n}$  consisting in the homogeneous polynomials  $q_\alpha(x, v) = q_{x, \alpha}(v)$  on  $\mathbb{R}^n$  that form a basis dual to  $(L_{(\mathbb{X})_x}^\beta)_{\beta \in \mathbb{N}_0^n}$ . We also consider the quasinorm  $|\cdot|_{\mathbb{X}}$  defined in Remark 3.11.

**6.1. Kernel estimates.** As a consequence of Theorem 2.36 and Remark 2.37, we have estimates on the convolution kernel of a symbol.

**Theorem 6.2.** *Let  $\mathbb{X}$  be an adapted frame on an open set  $U \subset M$ . Let  $\sigma \in S^m(\widehat{GM})$  and  $m \in \mathbb{R}$ . Then for each  $x \in U$ , the distribution  $\kappa_x^{\mathbb{X}}$  is Schwartz away from the origin. Moreover, if  $\mathcal{C}$  is a fixed compact subset of  $U$ , we have the following kernel estimates:*

(1) *For any  $N \in \mathbb{N}_0$ , there exists  $C > 0$  such that for all  $x \in \mathcal{C}$  and  $v \in \mathbb{R}^n$ , we have:*

$$|v|_{\mathbb{X}} \geq 1 \implies |\kappa_x^{\mathbb{X}}(v)| \leq C|v|_{\mathbb{X}}^{-N}.$$

(2) *If  $Q + m < 0$  then  $\kappa_x^{\mathbb{X}}$  is continuous on  $G$  and bounded: there exists  $C > 0$  such that*

$$\sup_{(x, v) \in \mathcal{C} \times \mathbb{R}^n} |\kappa_x^{\mathbb{X}}(v)| < C.$$

(3) *If  $Q + m > 0$ , then there exists  $C > 0$  such that for all  $x \in \mathcal{C}$  and  $v \in \mathbb{R}^n$ , we have*

$$0 < |v|_{\mathbb{X}} \leq 1 \implies |\kappa_x^{\mathbb{X}}(v)| \leq C|v|_{\mathbb{X}}^{-(Q+m)}.$$

(4) *If  $m = -Q$ , then there exists  $C > 0$  such that for all  $x \in \mathcal{C}$  and  $v \in \mathbb{R}^n$ , we have*

$$0 < |v|_{\mathbb{X}} \leq 1/2 \implies |\kappa_x^{\mathbb{X}}(v)| \leq -C \ln |v|_{\mathbb{X}}.$$

*In all the estimates above, the constant  $C$  may be chosen of the form*

$$C = C_1 \|\sigma\|_{S^m(\widehat{GM}), (\mathbb{X}, U), \mathcal{C}, N'}$$

*with  $C_1 > 0$ ,  $N' \in \mathbb{N}_0$ .*

Corollaries 5.39 and Theorem 6.2 yield also estimates for  $(-u)^\alpha L_{(\mathbb{X})}^\beta R_{(\mathbb{X})}^\gamma \mathbb{X}_x^{\beta'} \kappa_{\sigma, x}^{\mathbb{X}}(u)$ . They, together with Corollary 2.38, and its subsequent remarks, also yield the following notion of homogeneous semi-norms. Recall that the definition of the semi-norms is in (5.8).

**Corollary 6.3.** *Let  $\sigma \in S^m(\widehat{GM})$  with  $m \in \mathbb{R}$ ,  $\alpha, \beta, \beta' \in \mathbb{N}_0^n$ ,  $p \in [1, \infty]$ . If  $m - [\alpha] + [\beta] < Q(\frac{1}{p} - 1)$  then the quantity*

$$(6.1) \quad \|\sigma\|_{(\mathbb{X}, U), \mathcal{C}, \alpha, \beta, \beta', p} := \left\| \sup_{x \in \mathcal{C}} |v^\alpha L_{(\mathbb{X})}^\beta \mathbb{X}^{\beta'} \kappa_{\sigma, x}^{\mathbb{X}}| \right\|_{L^p(\mathbb{R}^n)}$$

is finite. Moreover, the map  $\sigma \mapsto \|\sigma\|_{(\mathbb{X},U),\mathcal{C},\alpha,\beta,\beta',p}$  is a continuous semi-norm on  $S^m(\widehat{GM})$  while if  $m \leq -M_0$  with  $M_0 \in \mathbb{N}_0$  a common multiple of the dilations' weight, we have for all  $N \in \mathbb{N}$

$$\|\sigma\|_{S^m(\widehat{GM}),(\mathbb{X},U),\mathcal{C},N} \leq C \max_{\substack{[\alpha]+[\beta] \leq N+M_0 \\ |\beta'| \leq N}} \|\sigma\|_{(\mathbb{X},U),\mathcal{C},\alpha,\beta,\beta',1},$$

From the kernel estimate above via Corollary 2.38 (3), we also obtain an estimate for the order of the distribution  $\kappa_x^{\mathbb{X}}$ . We will need the following quantification of this property:

**Corollary 6.4.** *Let  $\sigma \in S^m(\widehat{GM})$ . Let  $\mathbb{X}$  be an adapted frame on an open set  $U \subset M$ . For any  $x \in U$ , the distribution  $\kappa_x^{\mathbb{X}}$  is smooth away from the origin. At the origin, its order is the integer*

$$p_\sigma := \max\{|\alpha| : \alpha \in \mathbb{N}_0, [\alpha] \leq s'\}$$

where  $s'$  is the smallest non-negative integer such that  $s' > Q/2 + m$ . Moreover, for any compact  $\mathcal{C} \subset U$ , we have for any compact neighbourhood  $\mathcal{C}_0$  of 0 in  $\mathbb{R}^n$ ,

$$\exists C > 0, \quad \forall \phi \in C_c^\infty(\mathcal{C}_0), \quad \forall x \in \mathcal{C}, \quad \left| \int_{\mathbb{R}^n} \kappa_x^{\mathbb{X}}(v) \phi(v) dv \right| \leq C \max_{|\alpha| \leq p_\sigma} \|\partial^\alpha \phi\|_{L^\infty(\mathcal{C}_0)}.$$

Above,  $C$  may be chosen of the form

$$C = C_1 \|\sigma\|_{S^m(\widehat{GM}),(\mathbb{X},U),\mathcal{C}_0,N'},$$

with  $C_1 > 0$ ,  $N' \in \mathbb{N}_0$ .

Finally, the next corollary identifies the neighbourhood of 0 as the singular part of the kernel of a symbol, in terms of smoothing symbols. We obtain it here as a consequence of the kernel estimates, although we can also deduce it from Proposition 5.45 (1).

**Corollary 6.5.** *Let  $\sigma \in S^m(\widehat{GM})$  for some  $m \in \mathbb{R}$ .*

- (1) *If its convolution kernel  $\kappa_\sigma^{\mathbb{X}}$  in an adapted frame  $(\mathbb{X}, U)$  is a smooth function on  $U \times \mathbb{R}^n$  then the restriction of  $\sigma$  to  $U$  is smoothing, i.e.  $\sigma|_U \in S^{-\infty}(\widehat{GM}|_U)$ .*
- (2) *If  $\sigma$  is smoothing, i.e.  $\sigma \in S^{-\infty}(\widehat{GM})$ , then its convolution kernel in any adapted frame  $(\mathbb{X}, U)$  is a smooth function on  $U \times \mathbb{R}^n$ .*
- (3) *If  $q \in C^\infty(GM)$  is such that  $q \equiv 0$  in a neighbourhood of  $M \times \{0\}$ , then  $\Delta_q \sigma \in S^{-\infty}(\widehat{GM})$ .*

*Proof.* Part (1) follows from Corollary 6.3. Part (3) follows from Part (1). Part (2) follows from the kernel estimates in Theorem 6.2 together with Corollary 5.39.  $\square$

**6.2. Kernel estimates and higher order terms.** We first analyse convolution kernels of the form  $(x, v) \mapsto \kappa_x^{\mathbb{X}}(v *_{x, \mathbb{X}}^{\mathbb{X}} r(x, v))$  with  $r$  of higher order. We introduce the notation:

**Definition 6.6.** For  $m \in \mathbb{N}_0$ , we define

$$|m|_{min} := \min\{|\alpha| : \alpha \in \mathbb{N}^n, [\alpha] = m\}.$$

For  $\alpha \in \mathbb{N}_0^n$  with  $[\alpha] = m$ , we write  $|\alpha|_{min} := |m|_{min}$ .

*Example 6.7.* (1) In the case of a stratification, which implies  $\{v_1, \dots, v_n\} = \{1, \dots, s\}$  with  $s$  the gradation step, we compute readily:  $|1|_{min} = 1$ , and more generally

$$|j|_{min} = 1 \quad \text{for } j \leq s, \quad |j|_{min} = 1 + \text{Int}[j/(s+1)] \quad \text{for } j \in \mathbb{N},$$

above,  $\text{Int}[x]$  denotes the largest integer less or equal to  $x > 0$ .

(2) Let us consider  $\mathbb{R}^3$  with the gradation with weights  $(2, 3, 5)$ . We compute:

$$|1|_{min} = 0, \quad |2|_{min} = |3|_{min} = |5|_{min} = 1 \quad \text{but} \quad |4|_{min} = 2.$$

Hence, in the graded but not stratified case, the map  $M \mapsto |M|_{min}$  may not be increasing.

Definition 6.6 is motivated by the following observations:

**Lemma 6.8.** *Let  $q$  be a homogeneous polynomial on  $\mathbb{R}^n$  that is  $m$ -homogeneous for  $\delta_r$ .*

(1) *Then for any fixed  $v \in \mathbb{R}^n$ , we have for the isotropic dilations:*

$$q(\varepsilon v) = O(\varepsilon^{|m|_{min}}).$$

(2) *If  $r$  is a smooth higher order map on  $\mathbb{R}^n$ , then  $p = q \circ r$  vanishes at homogeneous order  $m + |m|_{min} - 1$  (in the sense of Definition 5.36).*

*Proof.* Part (1). Writing  $q(v) = \sum_{[\alpha]=m} c_\alpha v^\alpha$ , we have

$$q(\varepsilon v) = \sum_{[\alpha]=m} c_\alpha (\varepsilon v)^\alpha = \sum_{[\alpha]=m} c_\alpha \varepsilon^{|\alpha|} v^\alpha.$$

This proves Part (1).

Part (2). By Proposition 4.5, we may write  $\delta_\varepsilon^{-1} r(\delta_\varepsilon v) = \varepsilon R(v, \varepsilon)$ , with  $R$  smooth. Hence, we have

$$p(\delta_\varepsilon v) = q(\delta_\varepsilon \delta_\varepsilon^{-1} r(\delta_\varepsilon v)) = \varepsilon^m q(\delta_\varepsilon^{-1} r(\delta_\varepsilon v)) = \varepsilon^m q(\varepsilon R(v, \varepsilon)) = O(\varepsilon^{m+|m|_{min}}),$$

by Part (1). We conclude the proof of Part (2) with Lemma 5.37.  $\square$

In our analysis of the local calculus on  $M$ , we will often consider a symbol  $\sigma$  modified into a symbol  $\tau$  formally given by its convolution kernel  $\kappa_{\tau,x}^{\mathbb{X}} : v \mapsto \kappa_{\sigma,x}^{\mathbb{X}}(v *_{x,x}^{\mathbb{X}} r(x, v))$  with  $r$  higher order. The following statement gives an approximation of  $\tau$  by an asymptotic sum of symbols related to  $\sigma$ :

**Proposition 6.9.** *Let  $\sigma \in S^m(\widehat{GM})$ . Let  $\mathbb{X}$  be an adapted frame on an open set  $U \subset M$ . Let  $r \in C^\infty(U \times \mathbb{R}^n)$  be higher order. Let  $\chi \in C^\infty(U \times \mathbb{R}^n)$  be a function  $U$ -locally compactly supported in  $\mathbb{R}^n$ . We also assume that  $\chi = 1$  on a neighbourhood of  $\{(x, 0), x \in U\}$ . For every  $\alpha \in \mathbb{N}_0$ , with Notation 6.1 for  $q_\alpha$ , we set*

$$p_\alpha(x, v) := q_\alpha(x, r(x, v)), \quad x \in U, v \in \mathbb{R}^n,$$

*The following holds:*

(1) For each  $\alpha \in \mathbb{N}_0$ , the symbol  $\Delta_{\chi p_\alpha} \widehat{\langle \mathbb{X} \rangle}^\alpha \sigma$  is in  $S^{m-|\alpha|_{\min}}(\widehat{GM}|_U)$ , and the map

$$\sigma \longmapsto \Delta_{\chi p_\alpha} \widehat{\langle \mathbb{X} \rangle}^\alpha \sigma, \quad S^m(\widehat{GM}) \longrightarrow S^{m-|\alpha|_{\min}}(\widehat{GM}|_U),$$

is continuous.

(2) The symbol

$$\tau = \sigma^{(r, \chi)}$$

given by its convolution kernel in the  $\mathbb{X}$ -coordinates

$$(6.2) \quad \kappa_{\tau, x}^{\mathbb{X}}(v) := \chi(x, v) \kappa_{\sigma, x}^{\mathbb{X}}(v *_x^{\mathbb{X}} r(x, v)), \quad v \in \mathbb{R}^n,$$

is in  $S^m(\widehat{GM}|_U)$ . Moreover, it admits the asymptotic expansion in  $S^m(\widehat{GM})$

$$(6.3) \quad \tau \sim \sum_j \tau_j, \quad \tau_j := \sum_{j=|\alpha|_{\min}} \Delta_{\chi p_\alpha} \widehat{\langle \mathbb{X} \rangle}^\alpha \sigma \in S^{m-j}(\widehat{GM}|_U),$$

and, for each  $N \in \mathbb{N}_0$ , the map

$$\sigma \longmapsto \tau - \sum_{j=0}^N \tau_j, \quad S^m(\widehat{GM}) \rightarrow S^{m-N-1}(\widehat{GM}|_U),$$

is continuous.

When the filtered manifold  $M$  is equiregular subRiemannian as in Example 3.2, the gradation is in fact a stratification so the indices  $|\alpha|_{\min}$  are computed in Example 6.7 (1). However, if the gradation is not a stratification (for instance as in Example 6.7 (2)), there may be terms with very large  $|\alpha|$  contributing to the  $S^{m-1}$  term in the asymptotic expansion (6.3) in that case.

Note however that the sum defining  $\tau_j$  in (6.3) is finite. Indeed, for  $j \in \mathbb{N}$  fixed, the set  $\mathcal{A}_j = \{\alpha \in \mathbb{N}^n; j = |\alpha|_{\min}\}$  cannot contain a sequence of multi-indices  $(\alpha^{(\ell)})_{\ell \in \mathbb{N}}$  such that  $|\alpha^{(\ell)}| \rightarrow_{\ell \rightarrow +\infty} +\infty$ . Indeed, if it would, then we could find  $N$  such that for  $\ell \geq N$ , we have  $|\alpha^{(\ell)}| \geq |\alpha^{(\ell)}| > (j+1)v_n$ . Choosing then  $\beta$  such that  $[\beta] = [\alpha^{(\ell)}]$  and  $|\beta| = |\alpha|_{\min} = j$ , we have simultaneously  $[\beta] \leq v_n |\beta| = v_n j$ , and  $[\beta] = [\alpha^{(\ell)}] > (j+1)v_n$ , which is a contradiction.

The proof of Proposition 6.9 relies on a structural property that we now state. For various choices of symbols  $\sigma$  and cut-off  $\chi$ , we use the notation  $\tau(\sigma, \chi)$  to denote the symbol determined by its kernel in the  $\mathbb{X}$ -coordinates given formally by (6.2).

**Lemma 6.10.** *Let  $\sigma \in S^m(\widehat{GM})$ . Let  $\mathbb{X}$  be an adapted frame on an open set  $U \subset M$ . Let  $r \in C^\infty(U \times \mathbb{R}^n)$  be higher order. Let  $\chi \in C^\infty(U \times \mathbb{R}^n)$  be a function  $U$ -locally compactly supported in  $\mathbb{R}^n$ . We assume that  $\chi(x, v)$  vanishes at order  $N_0$  in  $v$  for any  $x \in U$ , and we consider the symbol  $\tau = \tau(\sigma, \chi)$  determined by its kernel in the  $\mathbb{X}$ -coordinates given formally by (6.2). We can construct functions*

$$\chi_{\beta, \beta_1}, \chi'_{\beta', \beta'_1, \beta'_2} \in C^\infty(U \times \mathbb{R}^n), \quad \beta, \beta', \beta_1, \beta'_1, \beta'_2 \in \mathbb{N}_0^n, \quad |\beta_1| \leq |\beta|, \quad |\beta'_1| + |\beta'_2| \leq |\beta'|,$$

vanishing at homogeneous order  $\max(N_0 + [\beta_1] - [\beta], -1)$  and  $\max(N_0 + [\beta'_2], -1)$  respectively, and such that

$$L_{\langle \mathbb{X} \rangle_x}^\beta \kappa_{\tau(\sigma, \chi), x} = \sum_{|\beta_1| \leq |\beta|} \kappa_{\tau_{\beta, \beta_1}, x} \quad \text{where} \quad \tau_{\beta, \beta_1} := \tau(\langle \widehat{\mathbb{X}} \rangle_x^{\beta_1} \sigma, \chi_{\beta, \beta_1}),$$

$$\mathbb{X}_x^{\beta'} \kappa_{\tau(\sigma, \chi), x} = \sum_{|\beta'_1| + |\beta'_2| \leq |\beta'|} \kappa_{\tau'_{\beta'_1, \beta'_1, \beta'_2}, x} \quad \text{where} \quad \tau'_{\beta'_1, \beta'_1, \beta'_2} := \tau(D_{\mathbb{X}}^{\beta'_1} \langle \widehat{\mathbb{X}} \rangle_x^{\beta'_2} \sigma, \chi'_{\beta'_1, \beta'_1, \beta'_2}).$$

*Remark 6.11.* The numerology above is explained by the following observation: if there was no terms in  $r$ , that is if  $r = 0$ , it is then straightforward that  $\tau \in S^m(\widehat{GM}|U)$ . Moreover, the function  $L_{\langle \mathbb{X} \rangle_x}^\beta \kappa_{\tau(\sigma, \chi), x}$  is the kernel associated with a symbol of order  $m - N_0 + [\beta]$  (by Corollary 5.34) and so is any  $\tau_{\beta, \beta_1}$ . Similarly,  $\mathbb{X}^{\beta'} \kappa_{\tau(\sigma, \chi), x}$  is a kernel associated with a symbol of order  $m - N_0$  and so are  $\tau'_{\beta'_1, \beta'_1, \beta'_2}$ . The result of Lemma 6.10 consists in proving that these properties still holds with a non zero term in  $r$ , when  $r$  is higher order. Indeed, the order of the vanishing of the functions  $\chi_{\beta, \beta_1}$  and  $\chi_{\beta'_1, \beta'_1, \beta'_2}$  is exactly compensating the loss of regularity of the kernel when it is differentiated in  $v$ .

The proof of Lemma 6.10 relies on Lemma 2.19 reformulated in the context of the groups  $G_x M$  for  $x \in M$  as follows: if  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are smooth, then

$$(6.4) \quad L_{\langle X_j \rangle_x} f \circ g(v) = \sum_{\ell=1}^n h_\ell(v) (L_{\langle X_\ell \rangle_x} f)(g(v)),$$

where  $h_\ell(v) := \partial_{t=0} [(-g(v)) *_{x}^{\mathbb{X}} g(v *_{x}^{\mathbb{X}} t e_j)]_\ell$ .

*Proof of Lemma 6.10.* Thanks to the Leibniz property of vector fields, inductively, it suffices to show the result for  $|\beta| = |\beta'| = 1$ , that is for  $L_{\langle X_j \rangle_x}$  and for  $X_{j,x}$ .

Let us analyse the  $L_{\langle X_j \rangle_x}$ -derivative, which means that we choose  $\beta$  such that  $[\beta] = v_j$ . We have

$$L_{\langle X_j \rangle_x} \kappa_{\tau, x}^{\mathbb{X}}(v) = L_{\langle X_j \rangle_x} \chi(x, v) \kappa_{\sigma, x}^{\mathbb{X}}(v *_{x}^{\mathbb{X}} r(x, v)) + \chi(x, v) L_{\langle X_j \rangle_x} (\kappa_{\sigma, x}^{\mathbb{X}}(v *_{x}^{\mathbb{X}} r(x, v)))$$

As  $L_{\langle X_j \rangle_x} \chi(x, v)$  vanishes at homogeneous order  $\max(N_0 - v_j, -1)$ , the first term is of the correct form. By (6.4), the second writes

$$L_{\langle X_j \rangle_x} (\kappa_{\sigma, x}^{\mathbb{X}}(v *_{x}^{\mathbb{X}} r(x, v))) = \sum_{\ell=1}^n f_\ell(x, v) (L_{\langle X_\ell \rangle_x} \kappa_{\sigma, x}^{\mathbb{X}}) (v *_{x}^{\mathbb{X}} r(x, v)),$$

where

$$f_\ell(x, v) := \partial_{t=0} [(-(v *_{x}^{\mathbb{X}} r(x, v))) *_{x}^{\mathbb{X}} v *_{x}^{\mathbb{X}} t e_j *_{x}^{\mathbb{X}} r(x, v *_{x}^{\mathbb{X}} t e_j)]_\ell.$$

In the sum above, the terms for which  $v_\ell < v_j$  are already of the correct form. Indeed, the derivatives  $L_{\langle X_\ell \rangle_x}$  induces loss of regularities smaller or equal to  $v_j = [\beta]$ . It is not the case for the other terms. However, we can conclude for them once we prove that for  $v_\ell \geq v_j$ , the function  $f_\ell$  vanishes at homogeneous order  $v_\ell - v_j$ . Let us do so. By homogeneity, we have:

$$\varepsilon^{-v_\ell + v_j} f_\ell(x, v) = \partial_{t=0} [\delta_\varepsilon^{-1} ((-(v *_{x}^{\mathbb{X}} r(x, v))) *_{x}^{\mathbb{X}} v *_{x}^{\mathbb{X}} \varepsilon^{v_j} t e_j *_{x}^{\mathbb{X}} r(x, v *_{x}^{\mathbb{X}} \varepsilon^{v_j} t e_j))]_\ell,$$

so

$$(6.5) \quad \varepsilon^{-v_\ell+v_j} f_\ell(x, \delta_\varepsilon v) = \partial_{t=0} [(-v *_x^\mathbb{X} s_0(\varepsilon)) *_x^\mathbb{X} v *_x^\mathbb{X} t e_j *_x^\mathbb{X} s_t(\varepsilon)]_\ell,$$

where  $s_t(\varepsilon) := \delta_\varepsilon^{-1} r(x, \delta_\varepsilon(v *_x^\mathbb{X} t e_j))$ ; here, we have fixed  $(x, v)$ . As  $r$  is smooth and higher order, by Proposition 4.5,  $(t, \varepsilon) \mapsto s_t(\varepsilon)$  is smooth on  $\mathbb{R} \times \mathbb{R}$  and vanishes at  $\varepsilon = 0$  for any  $t$ . The RHS in (6.5) is smooth in  $\varepsilon$  and taken as  $\varepsilon = 0$ , it is equal to

$$\partial_{t=0}[(-v) *_x^\mathbb{X} v *_x^\mathbb{X} t e_j]_\ell = \partial_{t=0}[t e_j]_\ell = [e_j]_\ell = \delta_{j=\ell}.$$

We have obtained

$$\forall j \neq \ell, \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{-v_\ell+v_j} f_\ell(x, \delta_\varepsilon v) = 0.$$

Consequently,  $f_\ell$  vanishes at homogeneous order  $v_j - v_\ell$  when  $v_j \leq v_\ell$ . We can now conclude that  $L_{\langle X_j \rangle_x} \kappa_{\tau, x}^\mathbb{X}$  is of the correct form.

We now analyse the  $X_{j,x}$ -derivatives. We have

$$\begin{aligned} X_{j,x} \kappa_{\tau, x}^\mathbb{X}(v) &= X_{j,x} \chi(x, v) \kappa_{\sigma, x}^\mathbb{X}(v *_x^\mathbb{X} r(x, v)) + \chi(x, v) (X_j \kappa_{\sigma, x}^\mathbb{X})(v *_x^\mathbb{X} r(x, v)) \\ &\quad + \chi(x, v) X_{j, x_1=x} (\kappa_{\sigma, x}^\mathbb{X}(v *_x^\mathbb{X} r(x, v))) + \chi(x, v) X_{j, x_1=x} (\kappa_{\sigma, x}^\mathbb{X}(v *_x^\mathbb{X} r(x_1, v))). \end{aligned}$$

The first two terms are already in the correct form. Let us analyse the third term. We first transform it in order to highlight higher order terms:

$$\begin{aligned} X_{j, x_1=x} (\kappa_{\sigma, x}^\mathbb{X}(v *_x^\mathbb{X} r(x, v))) &= X_{j, x_1=x} \kappa_{\sigma, x}^\mathbb{X}(v *_x^\mathbb{X} r(x, v) *_x^\mathbb{X} s(x, x_1, v)) \\ &= \sum_{\ell=1}^n [X_{j, x_1=x} s(x, x_1, v)]_\ell (L_{\langle X_\ell \rangle_x} \kappa_{\sigma, x}^\mathbb{X})(v *_x^\mathbb{X} r(x, v)), \end{aligned}$$

where

$$s(x, x_1, v) := (-v *_x^\mathbb{X} r(x, v)) *_x (v *_x^\mathbb{X} r(x, v)).$$

We observe that

$$\delta_\varepsilon^{-1} s(x, x_1, \delta_\varepsilon v) = (-v *_x^\mathbb{X} \delta_\varepsilon^{-1} r(x, \delta_\varepsilon v)) *_x (v *_x^\mathbb{X} \delta_\varepsilon^{-1} r(x, \delta_\varepsilon v)) \longrightarrow_{\varepsilon \rightarrow 0} 0,$$

since  $\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon^{-1} r(x, \delta_\varepsilon v) = 0$  by Proposition 4.5,  $r$  being higher order. Using again Proposition 4.5 implies that  $s$  is higher order in  $v$ . By Corollary 4.7, so is  $X_{j, x_1=x} s(x, x_1, v)$ . Hence,  $[X_{j, x_1=x} s(x, x_1, v)]_\ell$  vanishes at homogeneous order  $v_\ell$ , implying that the third term is of the correct form.

Let us analyse the fourth term. As before, we start by exhibiting higher order terms:

$$\begin{aligned} X_{j, x_1=x} (\kappa_{\sigma, x}^\mathbb{X}(v *_x^\mathbb{X} r(x_1, v))) &= X_{j, x_1=x} \kappa_{\sigma, x}^\mathbb{X}(v *_x^\mathbb{X} r(x, v) *_x^\mathbb{X} (-r(x, v)) *_x^\mathbb{X} r(x_1, v)) \\ &= \sum_{\ell=1}^n [X_{j, x_1=x} \tilde{s}(x, x_1, v)]_\ell L_{\langle X_\ell \rangle_x} \kappa_{\sigma, x}^\mathbb{X}(v *_x^\mathbb{X} r(x, v)), \end{aligned}$$

where

$$\tilde{s}(x, x_1, v) := (-r(x, v)) *_x^\mathbb{X} r(x_1, v).$$

We observe that

$$\delta_\varepsilon^{-1} \tilde{s}(x, x_1, \delta_\varepsilon v) = (-\delta_\varepsilon^{-1} r(x, \delta_\varepsilon v)) *_x^\mathbb{X} \delta_\varepsilon^{-1} r(x_1, \delta_\varepsilon v) \longrightarrow_{\varepsilon \rightarrow 0} 0,$$

since  $\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon^{-1} r(x, \delta_\varepsilon v) = 0$  by Proposition 4.5,  $r$  being higher order. Using again Proposition 4.5 implies that  $\tilde{s}$  is higher order in  $v$ . By Corollary 4.7, so is  $X_{j,x_1=x} \tilde{s}(x, x_1, v)$ . Hence,  $[X_{j,x_1=x} \tilde{s}(x, x_1, v)]_\ell$  vanishes at homogeneous order  $v_\ell$ , implying that the fourth term is of the correct form. We conclude that  $X_{j,x} \kappa_{\tau,x}^{\mathbb{X}}$  is of the correct form.  $\square$

We can now present the proof of Proposition 6.9.

*Proof of Proposition 6.9.* By Lemma 6.8 (2),  $v \mapsto p_\alpha(x, v)$  vanishes at order  $[\alpha] + |\alpha|_{\min} - 1$ . Part (1) then follows from Corollaries 5.34 and 5.39. Let us now prove Point (2). We may assume that on the support of  $\chi$ , there exists a constant  $C_0 > 0$  such that

$$\forall (x, v) \in U \times B_0, \quad \forall w \in \mathbb{R}^n, \quad |w|_{\mathbb{X}} \leq |r(x, v)|_{\mathbb{X}} \implies |v *_x^{\mathbb{X}} w|_{\mathbb{X}} \geq C_0 |v|_{\mathbb{X}}.$$

We now apply the Taylor estimates due to Folland and Stein [41] on the group  $G_x M$  and recalled in Theorem 2.17 (2) (see also Remark 2.18 for the local uniformity in  $x$ ). Using the notation  $[N]$  and  $\eta$  of Theorem 2.17, we have for any  $r \in \mathbb{R}^n$

$$(6.6) \quad \begin{aligned} & |\kappa_{\sigma,x}^{\mathbb{X}}(v *_x^{\mathbb{X}} r) - \sum_{[\alpha] \leq N} q_\alpha(r) L_{\langle \mathbb{X} \rangle_x}^\alpha \kappa_x^{\mathbb{X}}(v)| \\ & \leq C \sum_{\substack{|\alpha| \leq [N]+1 \\ [\alpha] > N}} |r|_{\mathbb{X}}^{[\alpha]} \sup_{|w|_{\mathbb{X}} \leq \eta^{[N]+1} |r|_{\mathbb{X}}} \left| (L_{\langle \mathbb{X} \rangle_x}^\alpha \kappa_x^{\mathbb{X}})(v *_x^{\mathbb{X}} w) \right|, \end{aligned}$$

with a constant  $C = C_{N,x}$  depending locally uniformly in  $x \in U$ ; we may assume  $\eta \geq 1$ . We set

$$(6.7) \quad \kappa_{N,\sigma,\chi,x}(v) := \kappa_{N,x}(v) := \sum_{[\alpha] \leq N} \chi(x, v) q_\alpha(x, r(x, v)) L_{\langle \mathbb{X} \rangle_x}^\alpha \kappa_{\sigma,x}^{\mathbb{X}}(v),$$

and, by Part 1, the kernels  $(x, v) \mapsto \chi(x, v) q_\alpha(x, r(x, v)) L_{\langle \mathbb{X} \rangle_x}^\alpha \kappa_{\sigma,x}^{\mathbb{X}}(v)$  are convolution kernels in the  $\mathbb{X}$ -coordinates of symbols in  $S^{m-|\alpha|_{\min}}(\widehat{GM}|_U)$ .

Let us now analyse  $\kappa_{N,x}$ . We choose  $N$  large enough so that

$$m + \varepsilon_0 N > -Q \quad \text{with} \quad \varepsilon_0 := 1/v_n.$$

For  $|v|_{\mathbb{X}} \leq 1$ , we have by Corollary 4.6 and the kernel estimates from Theorem 6.2:

$$\begin{aligned} |\kappa_{\tau,x}(v) - \kappa_{N,x}(v)| & \leq C \sum_{\substack{|\alpha| \leq [N]+1 \\ [\alpha] > N}} |r(x, v)|_{\mathbb{X}}^{[\alpha]} \sup_{|w|_{\mathbb{X}} \leq \eta^{[N]+1} |r(x, v)|_{\mathbb{X}}} |(L_{\langle \mathbb{X} \rangle_x}^\alpha \kappa_x^{\mathbb{X}})(v *_x^{\mathbb{X}} w)| \\ & \lesssim \sum_{\substack{|\alpha| \leq [N]+1 \\ [\alpha] > N}} |v|_{\mathbb{X}}^{[\alpha](1+\varepsilon_0)} \sup_{|w|_{\mathbb{X}} \leq \eta^{[N]+1} |r(x, v)|_{\mathbb{X}}} |v *_x^{\mathbb{X}} w|_{\mathbb{X}}^{-Q-m-[\alpha]} \|\sigma\|_{S^m(\widehat{GM}, (\mathbb{X}, U), \mathcal{C}, N')} \\ & \lesssim |v|_{\mathbb{X}}^{N\varepsilon_0 - m - Q} \|\sigma\|_{S^m(\widehat{GM}, (\mathbb{X}, U), \mathcal{C}, N')}. \end{aligned}$$

More precisely, we have obtained the existence of a constant  $C > 0$  depending locally uniformly in  $x \in U$  such that for  $|v|_{\mathbb{X}} \leq 1$ ,

$$(6.8) \quad |\kappa_{\tau,x}(v) - \kappa_{N,x}(v)| \leq C |v|_{\mathbb{X}}^{\varepsilon_0 N - Q - m} \max_{\substack{|\alpha| \leq [N]+1 \\ [\alpha] > N}} \sup_{v' \in \mathbb{R}^n \setminus \{0\}} \|v'\|_{\mathbb{X}}^{-Q-m-[\alpha]} |L_{\langle \mathbb{X} \rangle_x}^\alpha \kappa_x^{\mathbb{X}}(v')|.$$

As the supports of both  $\kappa_{\tau,x}(v)$  and  $\kappa_{N,x}(v)$  are included in the support of the function  $\chi(x, v)$ , the kernel estimates from Theorem 6.2 imply that there exist  $C_1 > 0$  and  $N' \in \mathbb{N}_0$  such that

$$\|\tau - \sum_{[\alpha] \leq N} \Delta_{\chi p_\alpha} \widehat{\langle \mathbb{X} \rangle}^\alpha \sigma\|_{(\mathbb{X}, U), \mathcal{C}, 0, 0, 0, \infty} \leq C_1 \|\sigma\|_{S^m(\widehat{GM}), (\mathbb{X}, U), \mathcal{C}, N'}$$

for the semi-norm defined in (6.1), as soon as  $N > (Q + m)/\epsilon_0$ .

We now introduce derivatives in  $x$  and  $v$ . By Lemma 6.10,

$$L_{\langle \mathbb{X} \rangle_x}^\beta \mathbb{X}_x^{\beta'} \kappa_{\tau(\sigma, \chi), x}(v) = \sum_{|\beta_1| \leq |\beta|} \sum_{|\beta'_1| \leq |\beta'|} \kappa_{\tau_{\beta_1, \beta'_1}, x}(v), \quad \text{where } \tau_{\beta_1, \beta'_1} := \tau(\widehat{\langle \mathbb{X} \rangle}_x^{\beta_1} D_{\mathbb{X}}^{\beta'_1} \sigma, \chi_{\beta_1, \beta'_1}),$$

and we can perform the same analysis as above on each  $\kappa_{\tau_{\beta_1, \beta'_1}, x}$ . Extending the notation in (6.7), we obtain:

$$\begin{aligned} & |\kappa_{\tau_{\beta_1, \beta'_1}, x}(v) - \kappa_{N, \widehat{\langle \mathbb{X} \rangle}_x^{\beta_1} D_{\mathbb{X}}^{\beta'_1} \sigma, \chi_{\beta_1, \beta'_1}, x}(v)| \\ & \leq C |v|_{\mathbb{X}}^{\epsilon_0 N - Q - m} \max_{\substack{|\alpha| \leq [N] + 1 \\ [\alpha] > N}} \sup_{v' \in \mathbb{R}^n \setminus \{0\}} \|v'\|_{\mathbb{X}}^{-Q - m - [\alpha]} |L_{\langle \mathbb{X} \rangle_x}^\alpha L_{\langle \mathbb{X} \rangle_x}^{\beta_1} \mathbb{X}_x^{\beta'_1} \kappa_x^{\mathbb{X}}(v')|. \end{aligned}$$

In particular, we have

$$(6.9) \quad |L_{\langle \mathbb{X} \rangle_x}^\beta \mathbb{X}_x^{\beta'} (\kappa_\tau(v) - \kappa_{N, \sigma, \chi, x}(v))| \lesssim |v|_{\mathbb{X}}^{\epsilon_0 N - Q - m - [\beta]},$$

with an implicit constant depending on a seminorm of  $\sigma \in S^m(\widehat{GM})$ , thereby finishing the proof, once we show

$$(6.10) \quad |L_{\langle \mathbb{X} \rangle_x}^\beta \mathbb{X}_x^{\beta'} \kappa_{N, \sigma, \chi, x}(v) - \sum_{|\beta_1| \leq |\beta|} \sum_{|\beta'_1| \leq |\beta'|} \kappa_{N, \widehat{\langle \mathbb{X} \rangle}_x^{\beta_1} D_{\mathbb{X}}^{\beta'_1} \sigma, \chi_{\beta_1, \beta'_1}, x}(v)| \lesssim |v|_{\mathbb{X}}^{\epsilon_0 N - Q - m - [\beta]},$$

again with an implicit constant depending on seminorms of  $\sigma \in S^m(\widehat{GM})$ . This will follow from the fact that the Taylor polynomials at order  $N_1 \leq \epsilon_0 N - Q - m - [\beta]$  of  $L_{\langle \mathbb{X} \rangle_x}^\beta \mathbb{X}_x^{\beta'} \kappa_{N, \sigma, \chi, x}(v)$  and  $\sum_{|\beta_1| \leq |\beta|} \sum_{|\beta'_1| \leq |\beta'|} \kappa_{N, \widehat{\langle \mathbb{X} \rangle}_x^{\beta_1} D_{\mathbb{X}}^{\beta'_1} \sigma, \chi_{\beta_1, \beta'_1}, x}(v)$  coincide:

$$(6.11) \quad \mathbb{P}_{N_1}^{L_{\langle \mathbb{X} \rangle_x}^\beta \mathbb{X}_x^{\beta'} \kappa_{N, x}}(v) = \sum_{|\beta_1| \leq |\beta|} \sum_{|\beta'_1| \leq |\beta'|} \mathbb{P}_{N_1}^{\kappa_{N, \widehat{\langle \mathbb{X} \rangle}_x^{\beta_1} D_{\mathbb{X}}^{\beta'_1} \sigma, \chi_{\beta_1, \beta'_1}, x}}(v).$$

Indeed, in this case, (6.10) follows readily from the Taylor estimates due to Folland-Stein at order  $N_1$  for  $L_{\langle \mathbb{X} \rangle_x}^\beta \mathbb{X}_x^{\beta'} \kappa_{N, \sigma, \chi, x}(v)$  and  $\sum_{|\beta_1| \leq |\beta|} \sum_{|\beta'_1| \leq |\beta'|} \kappa_{N, \widehat{\langle \mathbb{X} \rangle}_x^{\beta_1} D_{\mathbb{X}}^{\beta'_1} \sigma, \chi_{\beta_1, \beta'_1}, x}(v)$ .

It remains to show (6.11). We start by observing that the analysis above implies that the (Folland-Stein) Taylor polynomial for  $\kappa_{\tau, x}(v)$  and for  $\kappa_{N, x}(x)$  coincide up to order  $N_1 < \epsilon_0 N - Q - m$ :  $\mathbb{P}_{N_1}^{\kappa_{\tau, x}}(v) = \mathbb{P}_{N_1}^{\kappa_{N, x}}(v)$ . This implies that the  $(x, v)$ -derivatives of these polynomials also coincide:

$$L_{\langle \mathbb{X} \rangle_x}^\beta \mathbb{X}_x^{\beta'} \mathbb{P}_{N_1}^{\kappa_{\tau, x}}(v) = L_{\langle \mathbb{X} \rangle_x}^\beta \mathbb{X}_x^{\beta'} \mathbb{P}_{N_1}^{\kappa_{N, x}}(v),$$

for any  $\beta, \beta'$ . From the general properties of Taylor polynomials, we have for  $N_2 \geq [\beta]$

$$L_{\langle \mathbb{X} \rangle_x}^\beta \mathbb{X}_x^{\beta'} \mathbb{P}_{N_2}^{\kappa_{\tau, x}}(v) = \mathbb{P}_{N_2 - [\beta]}^{L_{\langle \mathbb{X} \rangle_x}^\beta \mathbb{X}_x^{\beta'} \kappa_{\tau, x}}(v), \quad L_{\langle \mathbb{X} \rangle_x}^\beta \mathbb{X}_x^{\beta'} \mathbb{P}_{N_2}^{\kappa_{N, x}}(v) = \mathbb{P}_{N_2 - [\beta]}^{L_{\langle \mathbb{X} \rangle_x}^\beta \mathbb{X}_x^{\beta'} \kappa_{N, x}}(v).$$

Hence, we have for  $N_1 < \epsilon_0 N - Q - m - [\beta]$

$$\mathbb{P}_{N_1}^{L_{(\mathbb{X})x}^\beta \mathbb{X}_x^{\beta'} \kappa_{\tau,x}}(v) = \mathbb{P}_{N_1}^{L_{(\mathbb{X})x}^\beta \mathbb{X}_x^{\beta'} \kappa_{N,x}}(v).$$

By Lemma 6.10, the LHS is equal to

$$\sum_{|\beta_1| \leq |\beta|} \sum_{|\beta'_1| \leq |\beta'|} \mathbb{P}_{N_1}^{\kappa_{\tau, \beta_1, \beta'_1, x}}(v) = \sum_{|\beta_1| \leq |\beta|} \sum_{|\beta'_1| \leq |\beta'|} \mathbb{P}_{N_1}^{\kappa_{\sigma, \beta_1, \beta'_1, x, \beta_1, \beta'_1, x}}(v),$$

having applied the above result to  $\sigma_{\beta_1, \beta'_1} = \widehat{\langle \mathbb{X} \rangle}_x^{\beta_1} D_{\mathbb{X}}^{\beta'_1} \sigma$  with  $\chi_{\beta_1, \beta'_1}$ . This shows (6.11) and concludes the proof.  $\square$

**6.3. Kernel estimates and perturbed convolution.** The following technical result analyses an integral expression given by the perturbation of the convolution of a kernel with a function.

**Lemma 6.12.** *Let  $\mathbb{X}$  be an adapted frame on an open set  $U \subset M$ . Let  $J \in C^\infty(U \times \mathbb{R}^n \times \mathbb{R}^n)$  be  $U$ -locally compactly supported in  $\mathbb{R}^n \times \mathbb{R}^n$ . Let  $\sigma \in S^m(\widehat{GM})$  and denote its integral kernel in the  $\mathbb{X}$  coordinates by  $\kappa_\sigma^\mathbb{X}$ . For any  $f \in \mathcal{S}(\mathbb{R}^n)$ , we define in the distributional sense:*

$$(6.12) \quad \kappa_{\sigma, f, J, x}^\mathbb{X}(v) := \int_{\mathbb{R}^n} \kappa_{\sigma, x}^\mathbb{X}(w) f(v *_x^\mathbb{X}(-w)) J(x, v, w) dw, \quad x \in U, v \in \mathbb{R}^n.$$

We define the coefficients  $c_\alpha(x, v)$  described via the Taylor expansion about  $w = 0$

$$J(x, v, w) \sim \sum_{\alpha} c_\alpha(x, v) w^\alpha.$$

The following convolution makes sense in the sense of distribution as the convolution of a distribution with a Schwartz function  $f$ :

$$f \star_{G_x M}^\mathbb{X} (\cdot^\alpha \kappa_{\sigma, x}^\mathbb{X})(v) = \int_{\mathbb{R}^n} w^\alpha \kappa_{\sigma, x}^\mathbb{X}(w) f(v *_x^\mathbb{X}(-w)) dw, \quad x \in U, v \in \mathbb{R}^n.$$

For any  $p \in [1, \infty]$  and any  $N \in \mathbb{N}_0$  with  $N \geq m + Q/p$ , for any compact subset  $\mathcal{C} \subset U$ , there exist  $C > 0$  and  $N_1 \in \mathbb{N}_0$  (independent of  $f, \sigma$ ) such that the following inequality holds uniformly in  $x \in \mathcal{C}$ :

$$(6.13) \quad \|\kappa_{\sigma, f, J, x}^\mathbb{X} - \sum_{|\alpha| \leq N} c_\alpha(x, \cdot) f \star_{G_x M}^\mathbb{X} (\cdot^\alpha \kappa_{\sigma, x}^\mathbb{X})\|_{L^p(\mathbb{R}^n)} \leq C \|\sigma\|_{S^m(\widehat{GM}, (\mathbb{X}, U), \mathcal{C}, N_1)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Recall that the notation  $\star_{G_x M}$  has been introduced in (2.3). The mention of the frame  $\mathbb{X}$  is added in order to settle the choice of coordinates.

*Proof.* We fix  $\chi_1, \chi_2 \in C^\infty(U \times \mathbb{R}^n)$  valued in  $[0, 1]$  and  $U$ -locally compactly supported such that  $\chi_1(x, v) = 1 = \chi_2(x, w)$  on the support of  $J(x; v, w)$ . We can insert  $\chi_1(x; v)\chi_2(x, w)$  in the integral expression (6.12) of  $\kappa_{\sigma, f, x}^\mathbb{X}(v)$ . Moreover, the kernel estimates and the properties

of the convolution on groups allow us to replace  $\kappa_{\sigma,x}^{\mathbb{X}}(w)$  with  $\chi_1(x,v)\chi_2(x,w)\kappa_{\sigma,x}^{\mathbb{X}}(w)$  in the desired inequality (6.13). Hence, we are now analysing:

$$\begin{aligned}\kappa_{N,x}^{\mathbb{X}}(v) &:= \kappa_{\sigma,f,J,x}^{\mathbb{X}}(v) - \sum_{[\alpha] \leq N_1} (\chi_1 c_\alpha)(x,v) f \star_{G_x M}^{\mathbb{X}} (\cdot^\alpha \chi_2(x,\cdot) \kappa_{\sigma,x}^{\mathbb{X}})(v) \\ &= \int_{\mathbb{R}^n} \kappa_{\sigma,x}^{\mathbb{X}}(w) f(v \star_x^{\mathbb{X}}(-w)) R_{N_1}(x,v,w) dw,\end{aligned}$$

where

$$R_{N_1}(x,v,w) := J(x,v,w) - \sum_{[\alpha] \leq N_1} (\chi_1 c_\alpha)(x,v) w^\alpha \chi_2(x,w).$$

By the Taylor estimates due to Folland and Stein (see Theorem 2.17 and Remark 2.18), we have

$$|R_{N_1}(x,v,w)| \lesssim \chi_1(x,v) |w|_{\mathbb{X}}^{N_1+1},$$

with an implicit constant depending locally uniformly on  $x \in U$ . Hence, by Hölder's inequality, with  $1 = \frac{1}{p} + \frac{1}{p'}$ ,

$$\begin{aligned}|\kappa_{N,x}^{\mathbb{X}}(v)| &\leq \|f(v \star_x^{\mathbb{X}}(-\cdot))\|_{L^p(\mathbb{R}^n)} \|\kappa_{\sigma,x}^{\mathbb{X}} R_{N_1}(x,v,\cdot)\|_{L^{p'}(\mathbb{R}^n)} \\ &\lesssim \|f\|_{L^p(\mathbb{R}^n)} \chi_1(x,v) \|\kappa_{\sigma,x}^{\mathbb{X}} |w|_{\mathbb{X}}^{N_1+1}\|_{L^{p'}(\mathbb{R}^n)}.\end{aligned}$$

We can now conclude with the kernel estimates of Theorem 6.2.  $\square$

## 7. THE LOCAL QUANTIZATION

This section is devoted to the local quantization of the classes of symbols  $S^m(\widehat{GM})$  defined in Section 5. After its definition in Section 7.1, we discuss its  $L^2$ -boundedness for symbols of order 0 and its dependence on local items in Sections 7.2 and 7.3 respectively.

### 7.1. Local quantization.

7.1.1. *Definition.* We consider a frame  $\mathbb{X}$  on an open set  $U \subset M$  and a  $\mathbb{X}$ -cut-off  $\chi$  in the following sense (with the notation of Section 3.3):

**Definition 7.1.** Let  $(\mathbb{X}, U)$  be a local adapted frame. A *cut-off for  $\exp^{\mathbb{X}}$* , or a *cut-off for the frame  $(\mathbb{X}, U)$* , is a smooth function  $\chi \in C^\infty(U \times U)$  valued in  $[0, 1]$ , properly supported in  $\exp^{\mathbb{X}}(U_{\mathbb{X}}) \subset U \times U$  for some  $U_{\mathbb{X}}$ , and identically equal to 1 on a neighbourhood of the diagonal of  $U \times U$ .

Equivalently, a function  $\chi \in C^\infty(U \times U)$  is a cut-off for  $(\mathbb{X}, U)$  when

$$\chi_0(x,v) = \chi(x, \exp_x v),$$

defines a function  $\chi_0 \in C^\infty(U \times \mathbb{R}^n)$  that satisfies  $\chi_0 = 1$  on a neighbourhood of  $U \times \{0\}$  and that is  $U$ -locally compactly supported in the sense of Definition 5.33. Given a cut-off  $\chi$  for  $\exp^{\mathbb{X}}$ , we will often use the shorthand:

$$\chi_x(y) := \chi(x, y).$$

For a symbol  $\sigma \in S^m(\widehat{GM})$  with associated convolution kernel density  $\kappa \in \mathcal{S}'(GM; |\Omega|(GM))$ , we define the operator  $\text{Op}^{\mathbb{X}, \chi}(\sigma)$  at any function  $f \in C_c^\infty(U)$  via

$$(7.1) \quad \text{Op}^{\mathbb{X}, \chi}(\sigma)f(x) := \int_{w \in G_x M} \kappa_x(w) (\chi_x f)(\exp_x^{\mathbb{X}}(-\text{Ln}_x^{\mathbb{X}} w)), \quad x \in U,$$

with an integration in the density sense. When we consider a smooth Haar system  $\mu = \{\mu_x\}_{x \in U}$  on  $U$ , this formula becomes

$$\text{Op}^{\mathbb{X}, \chi}(\sigma)f(x) = \int_{w \in G_x M} \kappa_x^\mu(w) (\chi_x f)(\exp_x^{\mathbb{X}}(-\text{Ln}_x^{\mathbb{X}} w)) d\mu_x(w).$$

In particular, for the Haar system induced by  $\mathbb{X}$ , we obtain

$$\text{Op}^{\mathbb{X}, \chi}(\sigma)f(x) = \int_{v \in \mathbb{R}^n} \kappa_x^{\mathbb{X}}(v) (\chi_x f)(\exp_x^{\mathbb{X}} -v) dv,$$

in terms of the convolution kernel  $\kappa^{\mathbb{X}}$  of  $\sigma$  in the  $\mathbb{X}$ -coordinates.

From now on, we assume that we have fixed a positive 1-density, that is, a smooth positive measure on  $M$  denoted by  $dx$  whenever  $x$  is the integrated variable on  $M$ ; this means that  $dx$  is a positive Borel measure such that the image of  $dx$  in any local coordinates has a smooth Radon-Nykodym derivative with respect to the Lebesgue measure.

Performing the change of variable  $y = \exp_x^{\mathbb{X}}(-v)$ , or more accurately, considering the pullback of  $dv$  via  $\exp_x^{\mathbb{X}} -v$ , the quantization formula is also given by

$$(7.2) \quad \text{Op}^{\mathbb{X}, \chi}(\sigma)f(x) = \int_M \kappa_x^{\mathbb{X}}(-\text{ln}_x^{\mathbb{X}} y) (\chi_x f)(y) \text{jac}_y(\text{ln}_x^{\mathbb{X}}) dy;$$

here  $\text{jac}_y(\text{ln}_x^{\mathbb{X}})$  is the (smooth) Radon-Nykodym derivative  $\frac{d \text{ln}_x^{\mathbb{X}}}{dy}$  of  $(\exp_x^{\mathbb{X}} -v)^* dv$  against  $dy$ . The latter formula shows that the integral kernel of  $\text{Op}^{\mathbb{X}, \chi}(\sigma)$  is formally given by the map

$$(x, y) \longmapsto \kappa_x^{\mathbb{X}}(-\text{ln}_x^{\mathbb{X}} y) \chi_x(y) \text{jac}_y(\text{ln}_x^{\mathbb{X}}).$$

### 7.1.2. Action on smooth compactly supported functions.

**Lemma 7.2.** *Let  $(\mathbb{X}, U)$  be an adapted local frame and  $\chi$  a cut-off function for  $(\mathbb{X}, U)$ . Let  $m \in \mathbb{R}$  and  $\sigma \in S^m(\widehat{GM}|_U)$ , then  $\text{Op}^{\mathbb{X}, \chi}(\sigma)$  acts continuously on  $C_c^\infty(U)$ . Moreover,  $\sigma \mapsto \text{Op}^{\mathbb{X}, \chi}(\sigma)$  is a continuous morphism of topological vector spaces  $S^m(\widehat{GM}|_U) \rightarrow \mathcal{L}(C_c^\infty(U))$ .*

*Proof.* The quantization formula may be rephrased for  $f \in C_c^\infty(U)$ ,  $x \in U$ , as

$$(7.3) \quad \text{Op}^{\mathbb{X}, \chi}(\sigma)f(x) = (\kappa_x^{\mathbb{X}}, g)_{\mathcal{D}'(\mathbb{R}^n), C_c^\infty(\mathbb{R}^n)}, \quad \text{with } g(v) := (\chi_x f)(\exp_x^{\mathbb{X}} -v).$$

Note that  $g = g_{x, f} \in C_c^\infty(\mathbb{R}^n)$  and that the maps  $x \mapsto \kappa_x^{\mathbb{X}}$  and  $x \mapsto g_{x, f}$  are smooth  $U \rightarrow \mathcal{D}'(\mathbb{R}^n)$  and  $U \rightarrow C_c^\infty(\mathbb{R}^n)$  respectively. Hence, the function  $x \mapsto \text{Op}^{\mathbb{X}, \chi}(\sigma)f(x)$  is smooth in  $x \in U$ .

Let  $\mathcal{C}$  be a compact subset of  $U$  with non-empty interior. Let  $\theta \in C_c^\infty(U)$  with  $\theta = 1$  on a neighbourhood of  $\mathcal{C}$  and  $\text{supp } \chi(x, \cdot)$ ,  $x \in \mathcal{C}$ . Then for any  $f \in C_c^\infty(U)$  supported in  $\mathcal{C}$ , we have

$$\text{Op}^{\mathbb{X}, \chi}(\sigma)f = \theta \text{Op}^{\mathbb{X}, \chi}(\sigma)\theta f = \text{Op}^{\mathbb{X}, \chi}(\theta\sigma)\theta f.$$

Hence,  $\text{Op}^{\mathbb{X},\chi}(\sigma)f \in C_c^\infty(U)$ .

It follows readily from (7.3) and Corollary 6.4 that the map  $f \mapsto \text{Op}^{\mathbb{X},\chi}(\sigma)f$  is continuous  $C_c^\infty(U) \rightarrow C_c^\infty(U)$ . Moreover, the estimates obtained from the application of Corollary 6.4 also imply that the map  $\sigma \mapsto \text{Op}^{\mathbb{X},\chi}(\sigma)$  is continuous  $S^m(\widehat{GM}|_U) \rightarrow \mathcal{L}(C_c^\infty(U))$ .  $\square$

We will extend the action of the operators  $\text{Op}^{\mathbb{X},\chi}(\sigma)$ ,  $\sigma \in S^m(\widehat{GM})$ , to distributions in Corollary 8.10.

**7.1.3. Differential operators as pseudodifferential operators.** With the preceding definitions, we obtain the differential calculus as the quantization of symbols given by elements of  $\Gamma(\mathcal{U}(\mathfrak{g}M))$ :

**Proposition 7.3.** *Let  $(\mathbb{X}, U)$  be an adapted frame. Let  $T \in \Gamma(\mathcal{U}(\mathfrak{g}M))$ . With the notation of Lemma 5.30, writing  $T = \sum_{[\alpha] \leq m} c_\alpha \langle \mathbb{X} \rangle^\alpha$  in  $\Gamma(\mathcal{U}(\mathfrak{g}M|_U))$ , we have for any  $f \in C^\infty(U)$*

$$\text{Op}^{\mathbb{X},\chi}(\widehat{T})f(x) = L_{T_x}(f \circ \exp_x)(0) = \sum_{[\alpha] \leq m} c_\alpha(x) L_{\langle \mathbb{X} \rangle_x}^\alpha(f \circ \exp_x)(0).$$

In particular, if  $T = \text{I}$ , then  $\text{Op}^{\mathbb{X},\chi}(\text{I}) = \text{I}$ .

*Proof of Proposition 7.3.* The convolution kernel of  $\widehat{T}$  in the  $\mathbb{X}$ -coordinates is  $L_{T_x} \delta_0$ , so the quantization formula yields:

$$\begin{aligned} \text{Op}^{\mathbb{X},\chi}(\widehat{T})f(x) &= \int_{v \in \mathbb{R}^n} L_{T_x} \delta_0(\chi_x f)(\exp_x^\mathbb{X} -v) dv \\ &= L_{T_x}^t(\chi_x f)(\exp_x^\mathbb{X} -v)|_{v=0} = L_{T_x}(f \circ \exp_x)(0) \end{aligned}$$

$\square$

It follows from Proposition 7.3 that

$$\forall T \in \Gamma(\mathcal{U}(\mathfrak{g})), \quad \forall \alpha \in \mathbb{N}_0, \quad \text{Op}^{\mathbb{X},\chi}(\widehat{T}) \text{Op}^{\mathbb{X},\chi}(\widehat{\langle \mathbb{X} \rangle}^\alpha) = \text{Op}^{\mathbb{X},\chi}(\widehat{T} \widehat{\langle \mathbb{X} \rangle}^\alpha).$$

However, as in the Euclidean setting, this will not be the case for the composition of two symbols in general.

**7.2.  $L^2$  boundedness for the local quantization.** In this section, we show the  $L^2$ -boundedness of the operators with symbols in  $S^0(\widehat{GM})$ , the action of the operators with symbols of orders  $m \neq 0$  being discussed later in Section 8. Actually, we will consider the local  $L^2$  space defined on  $M$  as

$$(7.4) \quad L_{loc}^2(M) = \{f \in \mathcal{D}'(M) : \theta f \in L^2(M) \text{ for any } \theta \in C_c^\infty(M)\}.$$

We recall that  $L_{loc}^2(M)$  is a topological vector space naturally equipped with a Fréchet structure given by the seminorms

$$f \mapsto \max_{j \leq N} \|\theta_j f\|_{L^2(M)}, \quad N \in \mathbb{N},$$

having fixed a sequence  $(\theta_j)_{j \in \mathbb{N}}$  in  $C_c^\infty(M)$  valued in  $[0, 1]$  such that  $M = \cup_{j \in \mathbb{N}} \{\theta_j = 1\}$ .

**Theorem 7.4.** *Let  $\mathbb{X}$  be an adapted frame on  $U$  and  $\chi$  a cut-off for  $\mathbb{X}$ . Let  $\sigma \in S^0(\widehat{GM})$ . Then  $\text{Op}^{\mathbb{X},\chi}(\sigma)$  is continuous on  $L^2_{loc}(U)$  and the linear map  $\sigma \mapsto \text{Op}^{\mathbb{X},\chi}(\sigma)$  is continuous  $S^0(\widehat{GM}) \rightarrow \mathcal{L}(L^2_{loc}(U))$ .*

Slightly more precisely, we will show that for any  $\chi_1 \in C_c^\infty(U)$  and  $\sigma \in S^0(\widehat{GM})$ , the operator  $\chi_1 \text{Op}^{\mathbb{X},\chi}(\sigma)$  is bounded on  $L^2(U)$ , and furthermore that there exist  $C > 0$  and  $N \in \mathbb{N}$  (both independent of  $\sigma$ ) such that

$$\forall \sigma \in S^0(\widehat{GM}), \quad \|\chi_1 \text{Op}^{\mathbb{X},\chi}(\sigma)\|_{\mathcal{L}(L^2(U))} \leq C \|\sigma\|_{S^0(\widehat{GM}),(\mathbb{X},U),\text{supp}\chi_1,N}.$$

The rest of this section is devoted to the proof of Theorem 7.4. The proof relies on the kernel estimates stated in Theorem 6.2 which show that the operators  $\text{Op}^{\mathbb{X},\chi}(\sigma)$  have singularities à la Calderón-Zygmund. It is inspired by Stein's argument in [61, ch.VI §2] (see also [37, Section 5.4.4]),

*Proof of Theorem 7.4.* If  $\sigma \in S^m(\widehat{GM})$  with  $m < -Q$ , then using the Cauchy-Schwartz inequality, we have:

$$\begin{aligned} \|\chi_1 \text{Op}^{\mathbb{X},\chi}(\sigma)f\|_{L^2(U)}^2 &\leq \int_{x \in U} |\chi_1(x)|^2 \int_{u_1 \in \mathbb{R}^n} |\kappa_{\sigma,x}^{\mathbb{X}}(u_1)|^2 du_1 \int_{u_2 \in \mathbb{R}^n} |(\chi_x f)(\exp_x^{\mathbb{X}} - u_2)|^2 du_2 dx \\ &\lesssim \|\sigma\|_{S^m(\widehat{GM}),(\mathbb{X},U),\mathcal{C}_1,N'}^2 \|f\|_{L^2(U)}^2, \end{aligned}$$

by the kernel estimates in Theorem 6.2 (1). Hence, the linear map  $\sigma \mapsto \text{Op}^{\mathbb{X},\chi}(\sigma)$  is continuous  $S^m(\widehat{GM}) \rightarrow \mathcal{L}(L^2_{loc}(U))$ .

The proof will use several  $U$ -locally compactly supported functions, and we will assume that their supports are small enough so that the expression involving the geometric exponential makes sense. For instance, we fix a  $U$ -locally compactly supported function  $\chi_2 \in C^\infty(U \times \mathbb{R}^n)$  valued in  $[0, 1]$  such that  $\exp_x(-v)$  exists where  $\chi_2(x, v) \neq 0$ , and  $\chi_2(x, v) = 1$  when  $\chi(x, \exp_x(-v)) \neq 0$ .

By Corollary 5.34,  $\Delta_{1-\chi_2}\sigma \in S^{-\infty}(\widehat{GM})$ , so  $\text{Op}^{\mathbb{X},\chi}(\Delta_{1-\chi_2}\sigma)$  is bounded on  $L^2_{loc}(U)$  by the result for  $m < -Q$  above. Hence, we now focus on  $\sigma_0 := \Delta_{\chi_2}\sigma$ . We denote by  $\kappa_{0,x}^{\mathbb{X}}$  its convolution kernel in the  $\mathbb{X}$ -coordinates. Let  $\chi_1 \in C_c^\infty(U)$ . Up to decomposing  $\chi_1$  into a finite sum of functions with smaller supports and modifying  $\chi_2$ , we may assume that

$$\chi_1(x)\chi_2(x, u)\chi_x(\exp_x^{\mathbb{X}} - u) = \chi_1(x)\chi_2(x, u).$$

We then have with  $\mathcal{C}_1 := \text{supp } \chi_1$ ,

$$\begin{aligned} I_0 &:= \|\chi_1 \text{Op}^{\mathbb{X},\chi}(\sigma_0)f\|_{L^2(U)}^2 \\ &\leq \int_{x \in U} |\chi_1(x)|^2 \sup_{x_1 \in \mathcal{C}_1} \left| \int_{u \in \mathbb{R}^n} \kappa_{0,x_1}^{\mathbb{X}}(u) f(\exp_x^{\mathbb{X}} - u) du \right|^2 dx \\ &\lesssim \sum_{|\alpha| \leq 1+n/2} \int_{x \in U} |\chi_1(x)|^2 \int_{x_1 \in \mathcal{C}_1} \left| \int_{u \in \mathbb{R}^n} \mathbb{X}_{x_1}^\alpha \kappa_{0,x_1}^{\mathbb{X}}(u) f(\exp_x^{\mathbb{X}} - u) du \right|^2 dx_1 dx, \end{aligned}$$

having used the Sobolev inequality on a compact manifold of dimension  $n$ :

$$\|g\|_{L^\infty(M)}^2 \leq C \sum_{|\alpha| \leq 1+n/2} \|\mathbb{X}^\alpha g\|_{L^2}^2.$$

Swapping the integral over  $x$  and  $x_1$ , we obtain

$$I_0 \lesssim \sum_{|\alpha| \leq 1+n/2} \sup_{x_1 \in \mathcal{C}_1} I_\alpha(x_1),$$

where

$$I_\alpha(x_1) := \int_{x \in U} |\chi_1(x)|^2 \left| \int_{u \in \mathbb{R}^n} \mathbb{X}_{x_1}^\alpha \kappa_{0,x_1}^\mathbb{X}(u) f(\exp_x^\mathbb{X} -u) du \right|^2 dx.$$

The integrals  $I_\alpha(x_1)$  are compactly supported in  $x$  and  $u$  and we now transform them with changes of variables. The first one aims at identifying all the points of the integral in terms of the base point  $x_1$  and the map  $\exp_{x_1}^\mathbb{X}$ . In the variable  $x$ , we set  $x = \exp_{x_1}^\mathbb{X} v$ , which gives

$$I_\alpha(x_1) = \int_{v \in \mathbb{R}^n} |\chi_1(\exp_{x_1}^\mathbb{X} v)|^2 \left| \int_{u \in \mathbb{R}^n} \mathbb{X}_{x_1}^\alpha \kappa_{0,x_1}^\mathbb{X}(u) f(\exp_{\exp_{x_1}^\mathbb{X} v}^\mathbb{X} -u) du \right|^2 \text{jac}_v \exp_{x_1}^\mathbb{X} dv,$$

and in the variable  $u$ , we make the substitution

$$u' := \varphi_v(u) := \ln_{x_1}^\mathbb{X} \exp_{\exp_{x_1}^\mathbb{X} v}^\mathbb{X} -u,$$

and obtain:

$$I_\alpha(x_1) = \int_{\mathbb{R}^n} |\chi_1(\exp_{x_1}^\mathbb{X} v)|^2 \left| \int_{\mathbb{R}^n} \mathbb{X}_{x_1}^\alpha \kappa_{0,x_1}^\mathbb{X}(\varphi_v^{-1}(u')) f(\exp_{x_1}^\mathbb{X} u') \text{jac}_{u'} \varphi_v^{-1} du' \right|^2 \text{jac}_v \exp_{x_1}^\mathbb{X} dv.$$

We now introduce a convolution structure via the change of variables  $u' \mapsto w := (-u') *_{x_1}^\mathbb{X} v$ . We write

$$\varphi_v^{-1}(u') = -\ln_{\exp_{x_1}^\mathbb{X} v}^\mathbb{X} \exp_{x_1}^\mathbb{X} u' = -\ln_{\exp_{x_1}^\mathbb{X} v}^\mathbb{X} \exp_{x_1}^\mathbb{X} v *_{x_1}^\mathbb{X} (-w) = w *_{x_1}^\mathbb{X} r_{x_1}(v, w),$$

with  $r_{x_1}(v, w) = r_1(x_1, v, w)$  as in Corollary 4.14 (1). At the end, we are left with

$$I_\alpha(x_1) = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \mathbb{X}_{x_1}^\alpha \kappa_{0,x_1}^\mathbb{X}(w *_{x_1}^\mathbb{X} r_{x_1}(v, w)) \tilde{f}(v *_{x_1}^\mathbb{X} (-w)) J(x_1; v, w) dw \right|^2 \text{jac}_v \exp_{x_1}^\mathbb{X} dv,$$

where

$$\begin{aligned} \tilde{f}(v) &:= \tilde{f}_{x_1}(v) := f(\exp_{x_1}^\mathbb{X} v) \chi_3(x_1, v), \\ J(x_1; v, w) &:= \chi_4(x; v, w) \text{jac}_{v *_{x_1}^\mathbb{X} (-w)} \varphi_v^{-1} \text{jac}_v \exp_{x_1}^\mathbb{X}, \end{aligned}$$

and  $\chi_3 \in C^\infty(U \times \mathbb{R}^n)$ ,  $\chi_4 \in C^\infty(U \times \mathbb{R}^n \times \mathbb{R}^n)$  are  $U$ -locally compactly supported functions satisfying  $\chi_3 = 1$  on a neighbourhood of  $\text{supp } \chi_2$  and  $\chi_4 = 1$  on a neighbourhood of the support of  $(x_1, v, w) \mapsto \chi_2(x_1, w *_{x_1}^\mathbb{X} r_{x_1}(v, w)) \chi_3(x_1, v *_{x_1}^\mathbb{X} (-w))$ .

Using the notation of Proposition 6.9, we observe

$$\mathbb{X}_{x_1}^\alpha \kappa_{0,x_1}^\mathbb{X}(w *_{x_1}^\mathbb{X} r_{x_1}(v, w)) = \kappa_{(D_x^\alpha \sigma)^{\tilde{r}_v, \chi_2}, x_1}^\mathbb{X}$$

with  $\tilde{r}_v(x, w) = r_x(v, w)$  at  $v$  fixed. Therefore, using the analysis in the proof of Proposition 6.9, more precisely the inequality (6.8), we obtain  $v$ -uniform estimates leading to:

$$I_\alpha(x_1) \leq \sum_{[\alpha'] \leq N} I_{\alpha, \alpha'}(x_1) + C_\sigma R$$

where

$$I_{\alpha, \alpha'} := \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \mathbb{X}_{x_1}^\alpha L_{\langle \mathbb{X} \rangle_x}^{\alpha'} \kappa_{0, x_1}^{\mathbb{X}}(w) \tilde{f}(v *_{x_1}^{\mathbb{X}}(-w)) J_{\alpha'}(x; v, w) dw \right|^2 \text{jac}_v \exp_{x_1} dv,$$

$$R := \left| \int_{\mathbb{R}^n} |\tilde{f}(v *_{x_1}^{\mathbb{X}}(-w)) J(x; v, w)| dw \right|^2 \text{jac}_v \exp_{x_1} dv$$

for  $N$  with  $v_n N - Q \geq 0$ , with a constant  $C_\sigma$  given by kernel estimates uniformly in  $v$  and locally uniformly in  $x_1 \in U$ , and with

$$J_{\alpha'}(x; v, w) := q_{\alpha'}(x, r(x; v, w)) J(x; v, w),$$

having used Notation 6.1. We readily check

$$R \lesssim \|\tilde{f}\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(U)},$$

and we have

$$I_{\alpha, \alpha'} \lesssim \|\kappa_{D_{\mathbb{X}}^\alpha(\widehat{\mathbb{X}})_x}^{\alpha'} \sigma, \tilde{f}, J_{\alpha'}\|_{L^2(\mathbb{R}^n)},$$

with the notation of Lemma 6.12. Applying this lemma, we obtain:

$$\|\kappa_{D_{\mathbb{X}}^\alpha(\widehat{\mathbb{X}})_x}^{\alpha'} \sigma, \tilde{f}, J_{\alpha'}\|_{L^2(\mathbb{R}^n)} \lesssim \sum_{\alpha''} \|\tilde{f} \star_{G_{x_1}^{\mathbb{X}}}^{\mathbb{X}} (\cdot^{\alpha''} \mathbb{X}_{x_1}^\alpha L_{\langle \mathbb{X} \rangle_x}^{\alpha'} \kappa_{\sigma, x_1}^{\mathbb{X}})\|_{L^2(\mathbb{R}^n)} + C'_\sigma \|\tilde{f}\|_{L^2(\mathbb{R}^n)},$$

with  $C'_\sigma$  a constant depending on some seminorm in  $\sigma \in S^0(\widehat{GM})$ ; above the sum is over  $\alpha''$  such that  $[\alpha'] < [\alpha''] \leq Q/2$  when  $\alpha' \neq 0$  and  $[\alpha''] \leq Q/2$  when  $\alpha' = 0$ . For the terms in this sum over  $\alpha''$ , we notice (see Section 2.2.3 for the Fourier transform of distributions)

$$\|\tilde{f} \star_{G_{x_1}^{\mathbb{X}}}^{\mathbb{X}} (\cdot^{\alpha''} \mathbb{X}_{x_1}^\alpha L_{\langle \mathbb{X} \rangle_x}^{\alpha'} \kappa_{\sigma, x_1}^{\mathbb{X}})\|_{L^2(\mathbb{R}^n)} \leq \|\Delta_{\mathbb{X}}^{\alpha''} D_{\mathbb{X}}^\alpha(\widehat{\mathbb{X}})_x^{\alpha'} \sigma(x_1, \cdot)\|_{L^\infty(\widehat{G}_{x_1} M)} \|\tilde{f}\|_{L^2(\mathbb{R}^n)}.$$

Gathering the estimates above yields with  $N_1 \in \mathbb{N}_0$  large enough

$$I_\alpha(x_1) \lesssim \|\sigma\|_{S^0(\widehat{GM}, \mathbb{X}, c_1, N_1)}^2 \|f\|_{L^2(M)}^2.$$

This concludes the proof.  $\square$

**7.3. Independence of the local quantization in the cut-off and the frame.** In this section, we use the kernel estimates to analyse the consequences of a change of cut-off or of frames in the local quantization process.

7.3.1. *Independence in the cut-off.* The local quantization is independent in the cut-off modulo smoothing symbols in the following sense:

**Proposition 7.5.** *Let  $\mathbb{X}$  be an adapted frame on an open subset  $U \subset M$ . Let  $\chi_1$  and  $\chi_2$  be two cut-offs for  $\mathbb{X}$ . Then for any  $\sigma \in S^m(\widehat{GM})$ ,  $m \in \mathbb{R} \cup \{-\infty\}$ , there exists  $\rho \in S^{-\infty}(\widehat{GM})$  such that*

$$\text{Op}^{\mathbb{X},\chi_1}(\sigma) = \text{Op}^{\mathbb{X},\chi_2}(\sigma) + \text{Op}^{\mathbb{X},\chi_2}(\rho).$$

Moreover, we can construct  $\rho$  such that the map  $\sigma \mapsto \rho$  is continuous  $S^m(\widehat{GM}) \rightarrow S^{-\infty}(\widehat{GM})$ .

This proposition is a consequence of the following more technical lemma:

**Lemma 7.6.** *Let  $\mathbb{X}$  be an adapted frame on an open subset  $U \subset M$ . Let  $m \in \mathbb{R}$ .*

- (1) *Let  $\chi_1$  and  $\chi_2$  be  $\mathbb{X}$ -cut-offs such that  $\chi_1$  is supported in the interior of  $\{\chi_2 = 1\}$ . For any  $\sigma \in S^m(\widehat{GM})$ , defining the symbol  $\tau \in S^m(\widehat{GM}|_U)$  via its convolution kernel in the  $\mathbb{X}$ -coordinates:*

$$\kappa_{\tau,x}^{\mathbb{X}}(u) := \kappa_{\sigma,x}^{\mathbb{X}}(u)\chi_{1,x}(\exp_x^{\mathbb{X}} - u),$$

we have

$$\text{Op}^{\mathbb{X},\chi_1}\sigma = \text{Op}^{\mathbb{X},\chi_2}\tau.$$

Moreover, the map  $\sigma \mapsto \tau$  is continuous  $S^m(\widehat{GM}) \rightarrow S^m(\widehat{GM}|_U)$ .

- (2) *Let  $\underline{\chi}, \chi, \bar{\chi}$  be three  $\mathbb{X}$ -cut-offs such that  $\underline{\chi}$  is supported in the interior of  $\{\chi = 1\}$  and  $\chi$  is supported in the interior of  $\{\bar{\chi} = 1\}$ . For any  $\sigma \in S^m(\widehat{GM})$ , defining the symbol  $\rho \in S^{-\infty}(\widehat{GM}|_U)$  via its convolution kernel in the  $\mathbb{X}$ -coordinates:*

$$\kappa_{\rho,x}^{\mathbb{X}}(u) := \kappa_{\sigma,x}^{\mathbb{X}}(u)(\chi_x - \underline{\chi}_x)(\exp_x^{\mathbb{X}} - u),$$

we have

$$\text{Op}^{\mathbb{X},\chi}\sigma = \text{Op}^{\mathbb{X},\underline{\chi}}\sigma + \text{Op}^{\mathbb{X},\bar{\chi}}\rho.$$

Moreover, the map  $\sigma \mapsto \rho$  is continuous  $S^m(\widehat{GM}) \rightarrow S^{-\infty}(\widehat{GM}|_U)$ .

The proof of Lemma 7.6 follows from the kernel estimates in Corollary 6.3 and Proposition 5.45 (1), keeping in mind that the map  $x \mapsto \exp_x^{\mathbb{X}} u$  defined in (3.2) is smooth.

Then, the proof of Proposition 7.5 follows from Lemma 7.6 together with constructing two  $\mathbb{X}$ -cut-offs  $\underline{\chi}, \bar{\chi}$  that, having been given the two  $\mathbb{X}$ -cut-offs  $\chi_1$  and  $\chi_2$  in the statement, are such that  $\underline{\chi}$  is supported in the interior of  $\{\chi_1 = 1\} \cup \{\chi_2 = 1\}$  and  $\text{supp}\chi_1 \cup \text{supp}\chi_2$  is included in the interior of  $\{\bar{\chi} = 1\}$ ,

7.3.2. *Independence in the frame.* The independence of the local quantization in the frame only holds up to lower order terms. It is also the case when working on Riemannian manifolds with the standard pseudodifferential calculus via change of charts (see e.g. [70]).

**Proposition 7.7** (Independence in the adapted frames). *Let  $\mathbb{X}$  and  $\mathbb{Y}$  be two adapted frames on the same open subset  $U \subset M$ . Let  $\chi_1$  and  $\chi_2, \chi$  be cut-off functions for  $\exp^{\mathbb{X}}$*

and  $\exp^{\mathbb{Y}}$  respectively. Then for any  $\sigma \in S^m(\widehat{GM})$ ,  $m \in \mathbb{R} \cup \{-\infty\}$ , there exist a symbol  $\rho \in S^{m-1}(\widehat{GM}|_U)$ , such that

$$\text{Op}^{\mathbb{Y}, \chi_2}(\sigma) = \text{Op}^{\mathbb{X}, \chi_1}(\sigma) + \text{Op}^{\mathbb{X}, \chi}(\rho).$$

Moreover, we can construct  $\rho$  such that the map

$$\sigma \longmapsto \rho, \quad S^m(\widehat{GM}|_U) \rightarrow S^{m-1}(\widehat{GM}|_U),$$

is continuous. Furthermore, we can also construct a sequence of symbols  $\rho_j \in S^{m-1-j}(\widehat{GM}|_U)$ ,  $j \in \mathbb{N}_0$ , such that  $\rho$  admits the asymptotic expansion  $\rho \sim \sum_{j \in \mathbb{N}_0} \rho_j$  and the following maps are continuous

$$\sigma \longmapsto \rho - \sum_{j=0}^N \rho_j, \quad S^m(\widehat{GM}) \longrightarrow S^{m-1-(N+1)}(\widehat{GM}|_U), \quad N \in \mathbb{N}_0.$$

*Proof.* It suffices to consider the case of a finite order  $m \in \mathbb{R}$ . By Proposition 7.5, we may assume  $\chi = \chi_1 = \chi_2$ . We adopt the notation of Lemma 3.5: with the two frames  $\mathbb{X}$  and  $\mathbb{Y}$  on the open subset  $U \subset M$  is associated an open neighbourhood  $U_{\mathbb{X}, \mathbb{Y}} \subset U_{\mathbb{X}} \cap U_{\mathbb{Y}} \subset U \times \mathbb{R}^n$  such that for any  $(x, v) \in U_{\mathbb{X}, \mathbb{Y}}$ , the point  $(x, tv)$  lie in  $U_{\mathbb{X}, \mathbb{Y}}$  for  $t$  in some neighbourhood of  $[0, 1]$ , and we consider the function  $u := u^{\mathbb{Y}, \mathbb{X}}$  defined via

$$(x, u(x, v)) = \ln^{\mathbb{Y}}(\exp^{\mathbb{X}}(x, v)), \quad (x, v) \in U_{\mathbb{X}, \mathbb{Y}}.$$

We fix a function  $\bar{\chi} \in C^\infty(U \times \mathbb{R}^n)$   $U$ -locally compactly supported such that  $\bar{\chi}(x, -v) = 1$  on a zero section of  $U \times \mathbb{R}^n$  containing the support of  $(x, v) \mapsto \chi(x, \text{Exp}_x^{\mathbb{X}}v)$ .

We set

$$\kappa_{\rho, x}^{\mathbb{X}}(v) := \bar{\chi}^2(x, v) (\kappa_x^{\mathbb{Y}}(-u_x(-v)) \text{jac}_{-v}(u_x^{\mathbb{Y}, \mathbb{X}}) - \kappa_x^{\mathbb{X}}(v))$$

We check readily that  $\kappa_{\rho, x}^{\mathbb{X}}(v)$  is well defined. Moreover, the formula between operators given in the statement holds formally after the change of variable given by  $u_x$  since  $\kappa^{\mathbb{X}}$  and  $\kappa^{\mathbb{Y}}$  correspond to the same convolution kernel  $\kappa$  viewed in the coordinates for  $\mathbb{X}$  and  $\mathbb{Y}$  respectively. It remains to check that the symbol  $\rho$  associated with  $\kappa_{\rho, x}^{\mathbb{X}}(v)$  is well defined in  $S^{m-1}(\widehat{GM})$  and depends continuously on  $\sigma$ .

By Lemma 3.10, we have

$$\kappa_x^{\mathbb{Y}}(v) = \kappa_x^{\mathbb{X}}(\text{diag}(T_x)^{-1}v) |\det T_x|^{-1},$$

and therefore,

$$\kappa_{\rho, x}^{\mathbb{X}}(v) = \bar{\chi}^2(x, v) (\kappa_x^{\mathbb{X}}(-\text{diag}(T_x)^{-1}u_x(-v)) \text{jac}_{-v}(u_x) |\det T_x|^{-1} - \kappa_x^{\mathbb{X}}(v)).$$

We set

$$\begin{aligned} \kappa_{\tau, x}^{\mathbb{X}}(v) &:= \bar{\chi}(x, v) \kappa_x^{\mathbb{X}}(-\text{diag}(T_x)^{-1}u_x(-v)) \\ &= \bar{\chi}(x, v) \kappa_x^{\mathbb{X}}(v *_{\mathbb{X}}^{\mathbb{X}} r(x, v)), \end{aligned}$$

by Proposition 4.12, and using its notation. By Proposition 6.9, this is the convolution kernel in the  $\mathbb{X}$ -coordinates of a symbol  $\tau \in S^m(\widehat{GM}|_U)$ . We can now write

$$\rho = \Delta_q \tau - \Delta_{\bar{\chi}^2} \sigma,$$

where  $q \in C^\infty(U \times \mathbb{R}^n)$  is the function  $U$ -locally compactly supported given by:

$$q(x, v) := \bar{\chi}(x, v) \text{jac}_{-v}(u_x) |\det T_x|^{-1}.$$

By Proposition 6.9 and Corollary 5.34, since  $q(x, 0) = 1$  and  $\bar{\chi}^2(x, 0) = 1$ , both  $\Delta_q \tau$  and  $\Delta_{\bar{\chi}^2} \sigma$  admit asymptotic expansions whose first terms are  $\sigma$ . The properties of continuity follow from the ones in Proposition 6.9 and Corollary 5.34.  $\square$

## 8. SYMBOLIC CALCULUS FOR THE LOCAL QUANTIZATION

In this section, we prove that the operators obtained by the local quantization on  $M$  enjoy a symbolic calculus. The proof relies on the notion of amplitudes that notably simplify the arguments and is presented in Section 8.1 while Section 8.2 is devoted to their quantization and main asymptotic properties. Then, we analyse the adjoints of pseudodifferential operators in Section 8.3, and their compositions in Section 8.4.

**8.1. Amplitudes.** In this section, we define the notion of amplitudes and give important examples.

8.1.1. *Definition.*

**Definition 8.1.** An *amplitude*  $\tau$  of order  $m$  is a smooth map

$$y \longmapsto \tau_y, \quad M \longrightarrow S^m(\widehat{GM}).$$

We denote by  $\mathcal{T}^m(\widehat{GM})$  the Fréchet space of amplitudes of order  $m$ .

The topology on  $\mathcal{T}^m(\widehat{GM})$  is given by the family of seminorms

$$(8.1) \quad \|\tau\|_{\mathcal{T}^m(\widehat{GM}), (\mathbb{X}, U), \mathcal{C}, N} := \max_{|\beta| \leq N} \sup_{y \in \mathcal{C}} \|\mathbb{X}_y^\beta \tau_y\|_{S^m(\widehat{GM}), (\mathbb{X}, U), \mathcal{C}, N},$$

where  $N \in \mathbb{N}_0$ ,  $\mathbb{X}$  is an adapted frame on an open set  $U \subset M$  and  $\mathcal{C}$  is a compact subset of  $U$ ; the seminorms on  $S^m(\widehat{GM})$  are defined in (5.8). We will often view an amplitude  $\tau \in \mathcal{T}^m(\widehat{GM})$  as being given by a family of symbols

$$\tau(x, y) = \{\tau(x, y, \pi), \pi \in \widehat{G}_x M\} \in S^m(\widehat{G}_x M)$$

smoothly parametrised by  $x, y \in M$ .

If  $\mu$  is a smooth Haar system over  $M$ , the corresponding convolution kernels are the distributions

$$\mathcal{S}'(G_x M) \ni \kappa_{\tau, x, y}^\mu := \mathcal{F}_{G_x M, \mu_x}^{-1} \tau(x, y, \cdot), \quad x, y \in M.$$

As in Section 5.1, if  $\mu'$  is another smooth Haar system, then  $\mu' = c\mu$  for a positive function  $c \in C^\infty(M)$  and we have

$$\kappa_{\tau, x, y}^\mu = c(x) \kappa_{\tau, x, y}^{\mu'}, \quad x, y \in M.$$

This allows us to define the convolution kernel of  $\tau$  independently of any Haar system as distributional vertical densities smoothly parametrised by  $y \in M$ :

$$\kappa_\tau : y \longmapsto \kappa_{\tau, y} \quad \text{in} \quad C^\infty(M, \mathcal{S}'(GM; |\Omega|(GM))).$$

If  $\mathbb{X}$  is an adapted frame on an open set  $U \subset M$ , the distributions

$$\kappa_{\tau,x,y}^{\mathbb{X}} \in \mathcal{S}'(\mathbb{R}^n), \quad \kappa_{\tau,x,y}^{\mathbb{X}}(v) := \kappa_{\tau,x,y}^{\mu^{\mathbb{X}}}(\text{Exp}_x^{\mathbb{X}}v), \quad x, y \in M,$$

define a map

$$\kappa_{\tau}^{\mathbb{X}} \in C^{\infty}(U \times M, \mathcal{S}'(\mathbb{R}^n)),$$

called the *convolution kernel* of  $\tau$  in the  $\mathbb{X}$ -coordinates. At least formally, we have

$$\tau(x, y, \pi) = \mathcal{F}_{G_x M, \mu^{\mathbb{X}}} \kappa_{\tau,x,y}^{\mathbb{X}}(\pi), \quad x, y \in U, \pi \in \widehat{G}_x M.$$

The following properties are easily checked using the relations of symbols and amplitudes with their convolution kernels:

**Lemma 8.2.** (1) *If  $\tau \in \mathcal{T}^m(\widehat{G}M)$  then the symbol  $\sigma = \sigma_{\tau}$  given by*

$$\sigma(x, \pi) = \tau(x, x, \pi), \quad x \in M, \pi \in \widehat{G}_x M,$$

*is in  $S^m(\widehat{G}M)$  with convolution kernel in the  $\mathbb{X}$ -coordinates given by  $\kappa_{\sigma,x} = \kappa_{\tau,x,x}$ . Moreover, the map  $\tau \mapsto \sigma_{\tau}$  is continuous  $\mathcal{T}^m(\widehat{G}M) \rightarrow S^m(\widehat{G}M)$ .*

(2) *Let  $\mathbb{X}$  be an adapted frame on an open set  $U \subset M$ . If  $\tau \in \mathcal{T}^m(\widehat{G}M)$  and  $\alpha, \beta \in \mathbb{N}_0^n$ , then the amplitude  $\tau^{(\alpha,\beta)}$  defined via*

$$\tau^{(\alpha,\beta)}(x, y, \pi) = \Delta_{\alpha}^{\mathbb{X}} D_{\mathbb{X},y}^{\beta} \tau(x, y, \pi) \quad x, y \in U, \pi \in G_x M,$$

*is in  $\mathcal{T}^{m-[\alpha]}(\widehat{G}M|_U)$  with convolution kernel in the  $\mathbb{X}$ -coordinates formally given by*

$$\kappa_{\tau^{(\alpha,\beta)},x,y}(v) = v^{\alpha} \mathbb{X}_y^{\beta} \kappa_{\tau,x,y}(v), \quad x, y \in U, v \in \mathbb{R}^n.$$

*Moreover, the map  $\tau \mapsto \tau^{(\alpha,\beta)}$  is continuous  $\mathcal{T}^m(\widehat{G}M) \rightarrow \mathcal{T}^{m-[\alpha]}(\widehat{G}M|_U)$ .*

### 8.1.2. Some examples of amplitudes.

**Lemma 8.3.** *Let  $\sigma_1 \in S^{m_1}(\widehat{G}M)$  and  $\sigma_2 \in S^{m_2}(\widehat{G}M)$ . Then the amplitude  $\tau_{\sigma_1,\sigma_2}$  defined via*

$$\tau_{\sigma_1,\sigma_2}(x, y) = \sigma_1(x)\sigma_2(y)$$

*makes sense and is in  $\mathcal{T}^{m_1+m_2}(\widehat{G}M)$ . Moreover, the map*

$$(\sigma_1, \sigma_2) \mapsto \tau_{\sigma_1,\sigma_2}, \quad S^{m_1}(\widehat{G}M) \times S^{m_2}(\widehat{G}M) \longrightarrow \mathcal{T}^{m_1+m_2}(\widehat{G}M)$$

*is continuous.*

Continuing with the setting of Lemma 8.3, let  $\kappa_1, \kappa_2$  and  $\kappa_{\tau}$  denote the convolution kernels densities of  $\sigma_1, \sigma_2, \tau$  respectively. For any smooth Haar system  $\mu = \{\mu_x\}_{x \in M}$ , we have for any  $x, y \in M$ ,

$$\kappa_{\tau,x,y}^{\mu} = \kappa_{2,y}^{\mu} \star_{G_x M, \mu_x} \kappa_{1,x}^{\mu},$$

that is, with  $v \in G_x M$

$$\kappa_{\tau,x,y}^{\mu}(v) = \int_{G_x M} \kappa_{2,x}^{\mu}(w) \kappa_{1,x}^{\mu}(w^{-1}v) d\mu_x(w) = \int_{G_x M} \kappa_{2,x}^{\mu}(vw^{-1}) \kappa_{1,x}^{\mu}(w) d\mu_x(w).$$

The meaning of this convolution is similar to the one for the product of symbols, see the proof of Proposition 5.25: this is a convolution between distributions which are sums of a

Schwartz function and a compactly supported distribution. Alternatively, this can be defined first for Schwartz kernels  $\kappa_{1,x}^\mu, \kappa_{2,x}^\mu$ , i.e. for smoothing symbols  $\sigma_1, \sigma_2$ , and then extended to any symbols in some symbol classes  $S^{m_j}(\widehat{GM})$ ,  $j = 1, 2$ , by density (see Remark 5.46).

We now generalise Lemma 8.3 with an amplitude described by the perturbation of the convolution of convolution kernels. Without details, we straightforwardly extend the notion of asymptotic expansion stated for symbols in Definition 5.35 to the case of amplitudes.

**Proposition 8.4.** *Let  $\sigma_1 \in S^{m_1}(\widehat{GM})$  and  $\sigma_2 \in S^{m_2}(\widehat{GM})$  with convolution kernels  $\kappa_1$  and  $\kappa_2$  respectively. Let  $(\mathbb{X}, U)$  a local frame and  $(x, y; v, w) \mapsto J(x, y; v, w)$  a  $U \times U$ -locally compactly supported function in  $C^\infty(U \times U \times \mathbb{R}^n \times \mathbb{R}^n)$ . Then, the convolution kernel*

$$\kappa_{\tau, x, y}^{\mathbb{X}}(v) = \int_{\mathbb{R}^n} \kappa_{1,x}^{\mathbb{X}}(w) \kappa_{2,y}^{\mathbb{X}}(v *_{\mathbb{X}}(-w)) J(x, y; v, w) dw, \quad x, y \in U, \quad v \in \mathbb{R}^n,$$

defines an amplitude

$$(8.2) \quad \tau = \tau(\sigma_1, \sigma_2, J) \in \mathcal{T}^{m_1+m_2}(\widehat{GM})$$

admitting the asymptotics

$$\tau \sim \sum_{j \in \mathbb{N}_0} \tau_j, \quad \text{with } \tau_j = \tau_j(\sigma_1, \sigma_2, J) \in \mathcal{T}^{m_1+m_2-j}(\widehat{GM}), \quad j \in \mathbb{N}_0.$$

Moreover, we may choose  $\tau_0 := J(x, y; 0, 0)\sigma_1(x)\sigma_2(y)$ , and more generally

$$\tau_j(x, y) := \sum_{[\alpha_1]+[\alpha_2]=j} c_{\alpha_1, \alpha_2}(x, y) \Delta_{\mathbb{X}}^{\alpha_1} \sigma_1(x) \Delta_{\mathbb{X}}^{\alpha_2} \sigma_2(y).$$

Above, the coefficients  $c_{\alpha_1, \alpha_2}(x, y)$  come from the Taylor expansion at  $(v, w) \sim 0$  of

$$\tilde{J}(x, y; v, w) := J(x, y; v *_{\mathbb{X}} w, w), \quad \tilde{J}(x, y; v, w) \sim \sum_{\alpha_1, \alpha_2} c_{\alpha_1, \alpha_2}(x, y) v^{\alpha_2} w^{\alpha_1}.$$

With these choices, the following maps are continuous:

$$\begin{aligned} (\sigma_1, \sigma_2) &\longmapsto \tau(\sigma_1, \sigma_2, J), & S^{m_1}(\widehat{GM}) \times S^{m_2}(\widehat{GM}) &\longrightarrow \mathcal{T}^{m_1+m_2}(\widehat{GM}), \\ (\sigma_1, \sigma_2) &\longmapsto \tau_j(\sigma_1, \sigma_2, J), & S^{m_1}(\widehat{GM}) \times S^{m_2}(\widehat{GM}) &\longrightarrow \mathcal{T}^{m_1+m_2-j}(\widehat{GM}), \\ (\sigma_1, \sigma_2) &\longmapsto \tau - \sum_{j=0}^N \tau_j, & S^{m_1}(\widehat{GM}) \times S^{m_2}(\widehat{GM}) &\longrightarrow \mathcal{T}^{m_1+m_2-(N+1)}(\widehat{GM}), \end{aligned}$$

*Strategy of the proof of Proposition 8.4.* By Lemma 8.3 and the properties of  $\Delta_{\mathbb{X}}^{\alpha}$  as stated in Lemma 5.31, the amplitude  $\tau_j = \tau_j(\sigma_1, \sigma_2)$  defined in the statement is in  $S^{m_1+m_2-j}(\widehat{GM})$ . Moreover, the map  $(\sigma_1, \sigma_2) \mapsto \tau_j(\sigma_1, \sigma_2)$  is continuous from  $S^{m_1}(\widehat{GM}) \times S^{m_2}(\widehat{GM})$  to the set  $\mathcal{T}^{m_1+m_2-j}(\widehat{GM})$ .

We may assume that  $\sigma_1$  and  $\sigma_2$  are smoothing, so that  $\kappa_{\tau, x, y}^{\mathbb{X}}(v)$  is a Schwartz function in  $v$  depending smoothly in  $x, y$ . Then, we will extend the result to any symbols with the density property of smoothing symbols (see Proposition 5.45) as well as by proving the continuities of the maps  $(\sigma_1, \sigma_2) \mapsto \tau$  and  $(\sigma_1, \sigma_2) \mapsto \tau - \sum_{j=0}^N \tau_j$ .

The strategy is to prove kernel estimates using Lemma 6.12 together with the properties of  $\tau(\sigma_1, \sigma_2, J)$  obtained in Lemma 8.5 below. We postpone the development of this proof after stating and proving Lemma 8.5.  $\square$

We first prove some algebraic properties for the amplitude  $\tau$  defined in Proposition 8.4:

**Lemma 8.5.** *We continue with the setting and notation of Proposition 8.4 and consider an amplitude  $\tau$  given by (8.2) with  $\sigma_1, \sigma_2 \in S^{-\infty}(\widehat{GM})$ .*

(1) For any  $\alpha \in \mathbb{N}_0^n$ ,

$$\Delta_{\mathbb{X}}^\alpha \tau = \sum_{[\alpha_1] + [\alpha_2] = [\alpha]} c_{\alpha_1, \alpha_2}^{(\alpha)} \tau(\Delta_{\mathbb{X}}^{\alpha_1} \sigma_1, \Delta_{\mathbb{X}}^{\alpha_2} \sigma_2, J)$$

where the functions  $c_{\alpha_1, \alpha_2}^{(\alpha)}$  are the same as in Corollary 5.32.

(2) For any  $j = 1, \dots, n$ ,

$$X_{j,y} \tau = \tau(\sigma_1, D_{X_j} \sigma_2, J) + \tau(\sigma_1, \sigma_2, X_{j,y} J).$$

(3) For any  $j = 1, \dots, n$ ,

$$\widehat{\langle X_j \rangle}^\beta \tau = \tau(\widehat{\langle X_j \rangle}^\beta \sigma_1, \sigma_2, J).$$

(4) For any  $j = 1, \dots, n$ ,

$$D_{X_{j,x}} \tau = \tau(D_{X_j} \sigma_1, \sigma_2, J) + \tau(\sigma_1, \sigma_2, X_{j,x} J) + \sum_{\ell=1}^n \tau(\sigma_1, \widehat{\langle X_j \rangle} \sigma_2, g_\ell J),$$

where  $g_\ell(x; v, w)$  is a polynomial in  $v, w$  with smooth coefficients in  $x \in U$  vanishing in  $(v, w)$  at order  $\max(v_\ell - v_j, -1)$ .

(5) Let  $\mathcal{R}$  be a symmetric positive Rockland operator on  $GM$  of homogenous degree  $\nu$  (see Definition 5.19).

$$\tau = \sum_{[\alpha_1] + [\alpha_2] = \nu N} \tau(\sigma_1 \widehat{\langle \mathbb{X} \rangle}^{\alpha_2}, (\mathbf{I} + \pi(\mathcal{R}_x))^{-N} \sigma_2, c_{\mathcal{R}, \alpha_1, \alpha_2} \tilde{\mathbb{X}}_x^{\alpha_2} J),$$

where the functions  $c_{\mathcal{R}, \alpha_1, \alpha_2} \in C^\infty(U)$  come from

$$(\mathbf{I} + \tilde{\mathcal{R}}_x)^N (g_1 \times g_2) = \sum_{[\alpha_1] + [\alpha_2] = \nu N} c_{\mathcal{R}, \alpha_1, \alpha_2}(x) \tilde{\mathbb{X}}_x^{\alpha_1} g_1 \times \tilde{\mathbb{X}}_x^{\alpha_2} g_2, \quad g_1, g_2 \in C^\infty(\mathbb{R}^n).$$

Recall that the notation  $\tilde{\mathcal{R}}_x$  denotes the right-invariant operator corresponding to  $\mathcal{R}_x$  (see Notation 2.11).

*Proof of Lemma 8.5.* Part (1) follows from a modification of the proof of Corollary 5.32 which relies on Lemma 2.57. Part (2) is a consequence of the Leibniz property of vector fields. For Part (3), we write

$$\kappa_{\tau, x, y}^{\mathbb{X}}(v) := \int_{\mathbb{R}^n} \kappa_{1, x}^{\mathbb{X}}(w) F(x, y; v *_{\mathbb{X}}(-w), w) dw, \quad x \in U, v \in \mathbb{R}^n,$$

where

$$F(x, y; v', w) := \kappa_{2, y}(v') J(x, y; v' *_{\mathbb{X}} w, w).$$

Let us recall the following general property valid for any smooth function  $\varphi : G \rightarrow \mathbb{C}$ :

$$\begin{aligned} L_{\langle X_j \rangle_x, v} (\varphi(v *_x^{\mathbb{X}}(-w))) &= \partial_{t=0} \varphi(v *_x^{\mathbb{X}} te_j *_x^{\mathbb{X}}(-w)) \\ &= \partial_{t=0} \varphi(v *_x^{\mathbb{X}}(-(w *_x^{\mathbb{X}} - te_j))) \\ &= -L_{\langle X_j \rangle_x, w} (\varphi(v *_x^{\mathbb{X}}(-w))). \end{aligned}$$

Hence, with an integration by parts, this leads to

$$\begin{aligned} L_{\langle X_j \rangle_x, v} \kappa_{\tau, x, y}(v) &= \int_{\mathbb{R}^n} \kappa_{1, x}^{\mathbb{X}}(w) L_{\langle X_j \rangle_x, v} (F(x, y; v *_x^{\mathbb{X}}(-w), w)) dw \\ &= \int_{\mathbb{R}^n} L_{\langle X_j \rangle_x, w} \kappa_{\sigma, x}^{\mathbb{X}}(w) F(x, y; v *_x^{\mathbb{X}}(-w), w) dw. \end{aligned}$$

This shows Part (3).

Let us show Part (4). We have

$$\begin{aligned} X_{j, x} \kappa_{\tau, x, y}(v) &= \int_{\mathbb{R}^n} X_{j, x} \kappa_{1, x}^{\mathbb{X}}(w) \kappa_{2, y}^{\mathbb{X}}(v *_x^{\mathbb{X}}(-w)) J(x, y; v, w) dw \\ &\quad + \int_{\mathbb{R}^n} \kappa_{2, x}^{\mathbb{X}}(w) \kappa_{2, y}^{\mathbb{X}}(v *_x^{\mathbb{X}}(-w)) X_{j, x} J(x, y; v, w) dw \\ &\quad + \int_{\mathbb{R}^n} \kappa_{1, x}^{\mathbb{X}}(w) X_{j, x} (\kappa_{2, y}^{\mathbb{X}}(v *_x^{\mathbb{X}}(-w))) J(x, y; v, w) dw. \end{aligned}$$

We recognise the first and second term as the convolution kernels of  $\tau(D_{X_j} \sigma_1, \sigma_2, J)$  and  $\tau(\sigma_1, \sigma_2, X_{j, x} J)$  respectively. It remains to analyse the third. We write for any function  $\varphi$ :

$$\begin{aligned} X_{j, x} (\varphi((-w) *_x v)) &= \partial_{t=0} \varphi(v *_x^{\mathbb{X}} \exp_x^{\mathbb{X}} te_j (-w)) \\ &= \sum_{\ell=1}^n g_{\ell}(x; v, w) (R_{\langle X_{\ell} \rangle_x} \varphi)(v *_x^{\mathbb{X}}(-w)) \end{aligned}$$

with

$$(g_{\ell}(x; v, w))_{\ell=1}^n := g(x; v, w) := \partial_{t=0} (w *_x^{\mathbb{X}}(-v)) *_x^{\mathbb{X}} (v *_x^{\mathbb{X}} \exp_x^{\mathbb{X}} te_j (-w)).$$

The BCH formula implies that  $g_{\ell}(x; v, w)$  is a polynomial in  $v, w$  with smooth coefficients in  $x \in U$ . We also observe

$$\varepsilon^{v_j} \delta_{\varepsilon}^{-1} g(x; \delta_{\varepsilon}(v, w)) = g(x; v, w), \quad \varepsilon > 0,$$

so that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{v_j - v_{\ell}} g_{\ell}(x; \delta_{\varepsilon}(v, w)) = 0 \quad \text{for } v_{\ell} > v_j.$$

This shows that the third term is as stated, and concludes the proof of Part (4).

Let us show Part (5). We use the notation from Definition 5.19 and identify  $\mathcal{R}_x$  with a differential operator on  $\mathbb{R}^n \sim G_x M$  satisfying  $\mathcal{R}_x^t = \mathcal{R}_x$ . Then, for any  $\varphi \in \mathcal{S}(\mathbb{R}^n)$ , we have

$$(\mathbf{I} + \mathcal{R}_x)^N (\mathbf{I} + \mathcal{R}_x)^{-N} \varphi = \varphi,$$

so

$$\varphi(v *_x^{\mathbb{X}}(-w)) = (\mathbf{I} + \tilde{\mathcal{R}}_x)_w^N ((\mathbf{I} + \mathcal{R}_x)^{-N} \varphi)(v *_x^{\mathbb{X}}(-w)).$$

Hence,

$$\begin{aligned}\kappa_{\tau,x,y}^{\mathbb{X}}(v) &= \int_{\mathbb{R}^n} \kappa_{1,x}^{\mathbb{X}}(w) (I + \tilde{\mathcal{R}}_x)|_w^N ((I + \mathcal{R}_x)^{-N} \kappa_{2,y}^{\mathbb{X}}) (v *_x^{\mathbb{X}}(-w)) J(x,y;v,w) dw \\ &= \int_{\mathbb{R}^n} (I + \tilde{\mathcal{R}}_x)^N (\kappa_{1,x}^{\mathbb{X}}(w) J(x,y;v,w)) ((I + \mathcal{R}_x)^{-N} \kappa_{2,y}^{\mathbb{X}}) (v *_x^{\mathbb{X}}(-w)) dw,\end{aligned}$$

after integrating by parts. Developing the differential operator  $(I + \tilde{\mathcal{R}}_x)^N$  and using the Leibniz properties of vector fields, yield the result.  $\square$

Let us now show Proposition 8.4.

*Proof of Proposition 8.4.* Let  $m_1, m_2 \in \mathbb{R}$ . It suffices to show that for any compact subset  $\mathcal{C} \subset U$ , given  $\alpha_0, \beta_0, \beta'_0, \beta''_0 \in \mathbb{N}_0^n$ , there exists  $N_0 \in \mathbb{N}_0$  such that for any  $N \geq N_0$ , there exist  $C > 0$ ,  $N'_1, N'_2 \in \mathbb{N}$  for which the following holds

$$\begin{aligned}\left\| \mathbb{X}_y^{\beta''_0} \left( \tau(\cdot, y) - \sum_{j=0}^N \tau_j(\cdot, y) \right) \right\|_{(\mathbb{X}, U), \mathcal{C}, \alpha_0, \beta_0, \beta'_0, \infty} \\ \leq C \|\sigma_1\|_{S^{m_1}(\widehat{GM}), (\mathbb{X}, U), \mathcal{C}, N'_1} \|\sigma_2\|_{S^{m_2}(\widehat{GM}), (\mathbb{X}, U), \mathcal{C}, N'_2},\end{aligned}$$

for any  $\sigma_1, \sigma_2 \in S^{-\infty}(\widehat{GM})$ .

We observe that the properties described in Lemma 8.5 hold for  $\tau$  but also for each  $\tau_j$ ; indeed, the latter can be viewed as a finite sum of particular cases for  $J(x,y,v,w) = c_{\alpha_1, \alpha_2} w^{\alpha_1} (v *_x^{\mathbb{X}}(-w))^{\alpha_2}$  for which the results in Lemma 8.5 hold with a proof identical to the one given for this lemma. Furthermore, Parts (1), (2), (3) and (4) of Lemma 8.5 imply that we may assume  $\alpha_0 = \beta_0 = \beta'_0 = \beta''_0 = 0$  while Part (5) and the symbolic properties of the calculus (see Proposition 5.45 as well as Theorem 2.33) imply that it suffices to consider the case of  $m_2$  as negative as we want. The case of  $m_2 < -Q$  and  $\alpha_0 = \beta_0 = \beta'_0 = \beta''_0 = 0$  follows from Lemma 6.12 and the kernel estimates (see Theorem 6.2), concluding the proof.  $\square$

8.1.3. *Some deformations of amplitudes.* In Lemma 6.12, we analysed some deformation of the symbol obtained by convolution of two convolution kernels. In our proof of composition (Proposition 8.13), we will also need to consider further deformations of amplitudes. They will be covered by the following lemma:

**Lemma 8.6.** *Let  $m \in \mathbb{R}$ . Let  $v_1 \mapsto \tau(v_1)$  be a smooth map  $\mathbb{R}^n \rightarrow \mathcal{T}^m(\widehat{GM})$ . We assume that  $\tau_{x,y}(v_1)$  is compactly supported in  $v_1 \in \mathbb{R}^n$  locally in  $(x,y) \in M \times M$ . Let  $(\mathbb{X}, U)$  be a local frame. Consider the amplitude  $\tau_0$  given via*

$$\kappa_{\tau_0,x,y}^{\mathbb{X}}(v) = \kappa_{\tau(v_1),x,y}^{\mathbb{X}}(v)|_{v_1=v}, \quad x, y \in U, \quad v \in \mathbb{R}^n.$$

(1) *The amplitude  $\tau_0$  is in  $\mathcal{T}^m(\widehat{GM}|_U)$  and admits the asymptotics*

$$\tau_0 \sim \sum_j \sum_{[\alpha]=j} \Delta_{q_\alpha} L_{(\mathbb{X}, v_1)}^\alpha \tau(v_1)|_{v_1=0} \quad \text{in } \mathcal{T}^m(\widehat{GM}|_U),$$

*with  $q_\alpha$  as in Notation 6.1.*

(2) For each  $j' \in \mathbb{N}_0$ , we consider a sequence of smooth maps

$$v_1 \mapsto \tau_{j'}(v_1), \quad \mathbb{R}^n \longrightarrow \mathcal{T}^{m-j'}(\widehat{GM}),$$

with  $\tau_{j',x,y}(v_1)$  compactly supported in  $v_1 \in \mathbb{R}^n$  locally in  $(x,y) \in M \times M$ . We assume that the asymptotics  $\tau(v_1) \sim \sum_{j'} \tau_{j'}(v_1)$  holds in  $\mathcal{T}^m(\widehat{GM})$  for each  $v_1 \in \mathbb{R}^n$ , and furthermore that for each  $N' \in \mathbb{N}_0$ , the maps  $v_1 \mapsto \tau(v_1) - \sum_{j'=0}^{N'} \tau_{j'}(v_1)$  are smooth  $\mathbb{R}^n \rightarrow \mathcal{T}^{m-(N+1)}(\widehat{GM})$ . Then  $\tau_0$  admits the asymptotics

$$\tau_0 \sim \sum_{j',j} \sum_{[\alpha]=j} \Delta_{q_\alpha} L_{\langle \mathbb{X} \rangle, v_1}^\alpha \tau_{j'}(v_1)|_{v_1=0} \quad \text{in } \mathcal{T}^m(\widehat{GM}|_U).$$

*Proof.* Before starting the proof, let us observe that the asymptotics make sense since

$$\Delta_{q_\alpha} L_{\langle \mathbb{X} \rangle, v_1}^\alpha \tau(v_1)|_{v_1=0} \in \mathcal{T}^{m-[\alpha]}(\widehat{GM}|_U).$$

We apply the Taylor estimates due to Folland and Stein [41] on the group  $G_x M$ , and recalled in Theorem 2.17 (2) (see also Remark 2.18 for the local uniformity in  $x$ ). We will use the notation  $[N]$  and  $\eta$  of Theorem 2.17. We have

$$\begin{aligned} & \left| \kappa_{\tau(v),x,y}^{\mathbb{X}}(v) - \sum_{[\alpha] \leq N} q_\alpha(v) L_{\langle \mathbb{X} \rangle, v_1}^\alpha \kappa_{\tau(v_1),x,y}^{\mathbb{X}}(v)|_{v_1=0} \right| \\ & \leq C \sum_{\substack{|\alpha| \leq [N]+1 \\ [\alpha] > N}} |v|_{\mathbb{X}}^{[\alpha]} \sup_{|v_1|_{\mathbb{X}} \leq \eta^{[N]+1}|v|_{\mathbb{X}}} \left| (L_{\langle \mathbb{X} \rangle, v_1}^\alpha \kappa_{\tau(v_1),x,y}^{\mathbb{X}}(v)) \right|, \end{aligned}$$

with a constant  $C = C_{N,x}$  depending locally uniformly in  $x \in U$ . Using the kernel estimates from Theorem 6.2 and the uniformity with respect to  $v_1 \in \mathbb{R}^n$ , we obtain for any compact subset  $\mathcal{C} \subset U$  and any  $N$  sufficiently large,

$$\left\| \tau_0 - \sum_{[\alpha] \leq N} \Delta_{q_\alpha} L_{\langle \mathbb{X} \rangle, v_1}^\alpha \tau(v_1)|_{v_1=0} \right\|_{(\mathbb{X}, U), \mathcal{C}, \alpha_0, \beta_0, \beta'_0, \infty} \leq C' \sup_{v_1 \in \mathbb{R}^n, |\alpha'| \leq N'} \|\partial_{v_1}^{\alpha'} \tau(v_1)\|_{S^m(\widehat{GM}), (\mathbb{X}, U), \mathcal{C}, N'_1}$$

for  $\alpha_0 = \beta_0 = \beta'_0 = \beta''_0 = 0$ ; above,  $N'$  is an integer depending on  $N$  and the structural constant of  $M$  while  $C' > 0$  is constant depending on  $N$ , on the structural constant of  $M$  on  $\mathcal{C}$ . This also holds for any given multi-indices  $\alpha_0, \beta_0, \beta'_0, \beta''_0 \in \mathbb{N}_0^n$  by applying the above estimate to  $\Delta_{\mathbb{X}}^{\alpha_0} \langle \mathbb{X} \rangle^{\beta_0} D_{\mathbb{X},x}^{\beta'_0} D_{\mathbb{X},y}^{\beta''_0} \tau_{x,y}(v_1)$  which satisfy similar hypotheses to  $\tau(v_1)$ . This proves Part (1).

Under the hypotheses of Part (2), we observe

$$\begin{aligned} & \left| \kappa_{\tau(v),x,y}^{\mathbb{X}}(v) - \sum_{j' \leq N'} \sum_{[\alpha] \leq N} q_\alpha(v) L_{\langle \mathbb{X} \rangle, v_1}^\alpha \kappa_{\tau_{j'}(v_1),x,y}^{\mathbb{X}}(v)|_{v_1=0} \right| \\ & \leq \left| \kappa_{\tau(v),x,y}^{\mathbb{X}}(v) - \sum_{[\alpha] \leq N} q_\alpha(v) L_{\langle \mathbb{X} \rangle, v_1}^\alpha \kappa_{\tau(v_1),x,y}^{\mathbb{X}}(v)|_{v_1=0} \right| \\ & \quad + \sum_{[\alpha] \leq N} \left| q_\alpha(v) L_{\langle \mathbb{X} \rangle, v_1}^\alpha \left( \kappa_{\tau(v_1),x,y}^{\mathbb{X}}(v) - \sum_{j'=0}^{N'} \kappa_{\tau_{j'}(v_1),x,y}^{\mathbb{X}}(v) \right) \Big|_{v_1=0} \right|. \end{aligned}$$

Using the estimate from the proof of Part (1) for the first term, and the kernel estimates together with the smoothness of  $v_1 \mapsto \tau(v_1) - \sum_{j'=0}^{N'} \tau_{j'}(v_1)$ , we can conclude as for Part 1.  $\square$

**8.2. Local quantization of amplitudes.** The local quantization of amplitudes is defined as follows. Let  $\mathbb{X}$  be a frame on an open set  $U \subset M$  and let  $\chi$  be a  $\mathbb{X}$ -cut-off. For an amplitude  $\tau \in \mathcal{T}^m(\widehat{GM})$ , we define the operator  $\text{AOp}^{\mathbb{X},\chi}(\tau)$  at any function  $f \in C_c^\infty(M)$  and any point  $x \in U$  via

$$\begin{aligned} \text{AOp}^{\mathbb{X},\chi}(\tau)f(x) &:= \int_M \kappa_{\tau,x,y}^{\mathbb{X}}(-\ln_x^{\mathbb{X}} y) (\chi_x f)(y) \text{jac}_y(\ln_x^{\mathbb{X}}) dy \\ &= \int_{\mathbb{R}^n} \kappa_{\tau,x,\exp_x^{\mathbb{X}}(-w)}^{\mathbb{X}}(w) (\chi_x f)(\exp_x^{\mathbb{X}}(-w)) dw; \end{aligned}$$

as before,  $\text{jac}_y(\ln_x^{\mathbb{X}})$  is the (smooth) Radon-Nykodym derivative  $\frac{d\ln_x^{\mathbb{X}}}{dy}$  of  $(\exp_x^{\mathbb{X}} - u)^* du$  against the measure  $dy$  that we have fixed on  $M$ . In other words, the integral kernel of the operator  $\text{AOp}^{\mathbb{X},\chi}(\tau)$  is given by

$$(8.3) \quad (x, y) \mapsto \kappa_{\tau,x,y}^{\mathbb{X}}(-\ln_x^{\mathbb{X}} y) \chi_x(y) \text{jac}_y(\ln_x^{\mathbb{X}}).$$

The local quantizations for amplitudes and symbols are related by the following asymptotic property:

**Proposition 8.7.** *Let  $\mathbb{X}$  be an adapted frame on an open set  $U \subset M$  and  $\chi$  an adapted cut-off. Let  $\tau \in \mathcal{T}^m(\widehat{GM}|_U)$  be an amplitude. Then there exists a symbol  $\sigma \in S^m(\widehat{GM}|_U)$  such that the quantizations for amplitudes and symbols are related by*

$$\text{AOp}^{\mathbb{X},\chi}(\tau) = \text{Op}^{\mathbb{X},\chi}(\sigma),$$

and the asymptotic expansion  $\sigma \sim \sum_{j \in \mathbb{N}_0} \sigma_j$  for some  $\sigma_j \in S^{m-j}(\widehat{GM}|_U)$  holds. Moreover, the first symbol  $\sigma_0(x, \cdot)$  coincides with  $\tau(x, x, \cdot)$ , and for each  $N \in \mathbb{N}$ , the map

$$\tau \longmapsto \sigma - \sum_{j=0}^{N-1} \sigma_j, \quad \mathcal{T}^m(\widehat{GM}) \longrightarrow S^{m-N}(\widehat{GM}|_U)$$

is continuous.

Proposition 8.7 is a consequence of the following more technical property and its proof:

**Lemma 8.8.** *We continue with the setting of Proposition 8.7. We fix an  $\mathbb{X}$ -cut-off function  $\tilde{\chi}$  such that  $\tilde{\chi} \equiv 1$  on the support of  $\chi$  and we assume that the support of  $\chi$  and  $\tilde{\chi}$  are sufficiently close to the diagonal. The symbol  $\sigma$  in Proposition 8.7 may be constructed via its convolution kernel in the  $\mathbb{X}$ -coordinates which is given by:*

$$\kappa_{\sigma,x}^{\mathbb{X}}(u) = \kappa_{\tau,x,\exp_x^{\mathbb{X}}(-u)}^{\mathbb{X}}(u) \tilde{\chi}(x, \exp_x^{\mathbb{X}}(-u)), \quad x \in U, u \in \mathbb{R}^n.$$

In this case, the  $\sigma_j$ 's are given by the following finite sum over the indices  $\alpha, \beta$

$$\sigma_j(x, \pi) = \sum_{[\alpha]=j, |\beta| \leq |\alpha|} c_{\alpha,\beta}(x) \Delta_{\mathbb{X}}^\alpha D_{\mathbb{X},y=x}^\beta \tau(x, x, \pi),$$

where the coefficients  $c_{\alpha,\beta} \in C^\infty(U)$ ,  $\alpha, \beta \in \mathbb{N}_0^n$ , are determined by

$$\left(\sum_{i=1}^n -u_i X_{i,y}\right)^k = \sum_{|\alpha|=k, |\beta| \leq k} c_{\alpha,\beta}(y) u^\alpha \mathbb{X}_y^\beta;$$

in particular,  $c_{0,0} = 1$ , and  $c_{0,\beta} = 0$  for  $\beta \neq 0$ .

*Proof of Lemma 8.8.* At least formally, the quantization for the amplitudes, especially (8.3), shows that  $\text{AOp}^{\mathbb{X},\mathbb{X}}(\tau) = \text{Op}^{\mathbb{X},\mathbb{X}}(\sigma)$ , with  $\sigma$  as in Lemma 8.8. We will prove that  $\sigma$  is a well-defined symbol in  $S^m(\widehat{GM}|_U)$  that satisfies the properties stated in Lemma 8.8 and Proposition 8.7. The hypothesis in Lemma 8.8 on the cut-offs imposes no restriction on the proof of Proposition 8.7 because of the independence of the quantization in the cut-off, see Proposition 7.5. In particular, we may assume that the function

$$\chi_0(x, u) := \widetilde{\chi}(x, \exp_x -u), \quad x \in U, \quad u \in \mathbb{R}^n,$$

has  $U$ -local compact support where  $\exp^{\mathbb{X}}$  is defined and such that that if  $(x, v)$  is in  $\text{supp } \chi_0$  then so is  $(x, tv)$  for any  $t \in [-1, 1]$ . Moreover, by Lemma 8.2, each  $\sigma_j$  is a well defined symbol in  $S^{m-j}(\widehat{GM})$ . As the distribution  $\kappa_{\sigma,x}^{\mathbb{X}}(u)$  is  $U$ -locally compactly supported on  $\text{supp } \chi_0$  and  $\sigma_j \in S^{m-j}(\widehat{GM})$ , Lemma 8.8 and Proposition 8.7 will be proved once we have shown the following estimates: for any  $\beta, \beta' \in \mathbb{N}_0$ , there exist  $N_0, N'_0 \in \mathbb{N}_0$  such that for any compact subsets  $\mathcal{C}, \mathcal{C}_0 \subset U$  with  $\mathcal{C}$  being a neighbourhood of  $\mathcal{C}_0$ , for any  $N \geq N_0$ , there exist  $C > 0$  and  $N_1 \in \mathbb{N}_0$  for which we have for  $x \in \mathcal{C}_0$  and  $u \in V_{\mathcal{C}_0}$ :

$$(8.4) \quad \left| L_{(\mathbb{X})x,u}^\beta \mathbb{X}_x^{\beta'} \left( \kappa_{\sigma,x}^{\mathbb{X}}(u) - \sum_{j=0}^N \kappa_{\sigma_j,x}^{\mathbb{X}}(u) \right) \right| \leq C |u|_{\mathbb{X}}^{N-N'_0} \|\tau\|_{\mathcal{T}^m(\widehat{GM}, (\mathbb{X}, U), \mathcal{C}, N_1)};$$

above  $V_{\mathcal{C}_0}$  is the neighbourhood  $V_{\mathcal{C}_0} := \cup_{x \in \mathcal{C}_0} \text{supp } \chi_0(x, \cdot)$  of 0 in  $\mathbb{R}^n$  and  $|\cdot|_{\mathbb{X}}$  is the fixed quasinorm associated with the gradation of  $\mathbb{R}^n$ , see Notation 6.1.

Let us start by showing (8.4) for  $\beta = \beta' = 0$ . Lemma 3.4 yields the following Taylor expansion in  $v \sim 0$  (for  $x_0, u_0$  fixed)

$$\kappa_{\tau,x_0,\exp_{x_0}^{\mathbb{X}}(v)}^{\mathbb{X}}(u_0) \sim_{v \sim 0} \sum_k \frac{1}{k!} \left( \sum_{i=1}^n v_i X_{i,y} \right)^k \kappa_{\tau,x_0,y}^{\mathbb{X}}(u_0) \Big|_{y=x_0},$$

and we define the corresponding (Euclidean) Taylor polynomial at order  $N$ :

$$\begin{aligned} \mathbb{P}_N(\tau, x_0; u_0, v) &:= \sum_{k \leq N} \frac{1}{k!} \left( \sum_{i=1}^n v_i X_{i,y} \right)^k \kappa_{\tau,x_0,y}^{\mathbb{X}}(u_0) \Big|_{y=x_0} \\ &= \sum_{k \leq N} \frac{1}{k!} \sum_{|\alpha|=k, |\beta| \leq k} c_{\alpha,\beta}(x) (-v)^\alpha \mathbb{X}_{y=x}^\beta \kappa_{\tau,x,y}^{\mathbb{X}}(u_0), \end{aligned}$$

with the notation of the statement. With  $\mathcal{C}, \mathcal{C}_0, V_{\mathcal{C}_0}$  as above, the same lemma gives the following (Euclidean) estimate for the remainder for any  $x \in \mathcal{C}_0$ ,  $v \in V_{\mathcal{C}_0}$

$$(8.5) \quad \left| \kappa_{\tau,x_0,\exp_{x_0}^{\mathbb{X}}(v)}^{\mathbb{X}}(u_0) - \mathbb{P}_N(\tau, x_0; u_0, v) \right| \lesssim_{N,\mathcal{C},\mathcal{C}_0} |v|^{N+1} \max_{|\alpha|=N+1} \sup_{y \in \mathcal{C}} \left| \mathbb{X}_y^\alpha \kappa_{\tau,x_0,y}^{\mathbb{X}}(u_0) \right|,$$

where  $|v|$  is the Euclidean norm of  $v$ . This implies for any  $N \geq Q + m$ ,  $x \in \mathcal{C}_0$  and  $u \in V_{\mathcal{C}_0}$ ,

$$\begin{aligned} |\kappa_{\sigma,x}^{\mathbb{X}}(u) - \sum_{j=0}^N \kappa_{\sigma_j,x}^{\mathbb{X}}(u)| &\leq |\chi_0(x, u)| |\kappa_{\tau,x_0,\exp_{x_0}^{\mathbb{X}}}^{\mathbb{X}}(u) - \mathbb{P}_N(\tau, x_0; u, -u)| \\ &+ |(1 - \chi_0)(x, u)| \sum_{|\alpha| < N, |\beta| \leq |\alpha|} \frac{1}{|\alpha|!} |c_{\alpha,\beta}(x) u^{\alpha} \mathbb{X}_{y=x}^{\beta} \kappa_{\tau,x,y}^{\mathbb{X}}(u)| \\ &+ |\chi_0(x, u)| \sum_{[\alpha] > N, |\alpha| < N, |\beta| \leq |\alpha|} \frac{1}{|\alpha|!} |c_{\alpha,\beta}(x) u^{\alpha} \mathbb{X}_{y=x}^{\beta} \kappa_{\tau,x,y}^{\mathbb{X}}(u)| \\ &\lesssim_{N,N_2,\mathcal{C}} (|u|^{N+1} + |u|_{\mathbb{X}}^{N+1}) |u|_{\mathbb{X}}^{-\max(Q+m,0)} \|\tau\|_{\mathcal{T}^m(\widehat{GM},(\mathbb{X},U),\mathcal{C},N_1)}, \end{aligned}$$

for some  $N_1 \in \mathbb{N}_0$  obtained from using the kernel estimates (see Theorem 6.2). Since  $|u| \lesssim |u|_{\mathbb{X}}$  for  $u$  in a bounded neighbourhood of 0, this yields (8.4) in the case  $\beta = \beta' = 0$ .

To consider the general case, we first prove that an estimate similar to (8.5) holds after derivation in  $x_0, v, u_0$ . By the properties (especially uniqueness) of Taylor polynomials, such a Taylor estimate holds, however with a bound for which we know nothing of its dependence in  $\tau$ . This is not sufficient for our proof: it is essential to know that the constant is given by a kernel estimate for  $\tau$ . To obtain this dependence, we observe the following properties:

$$\begin{aligned} L_{\langle \mathbb{X} \rangle_x, u}^{\beta} \kappa_{\tau,x,\exp_x^{\mathbb{X}}}^{\mathbb{X}}(u) &= \widehat{\kappa}_{\langle \mathbb{X} \rangle_x}^{\beta} \tau, x, \exp_x^{\mathbb{X}}(v)(u), \\ \mathbb{X}_x^{\beta'} \kappa_{\tau,x,\exp_x^{\mathbb{X}}}^{\mathbb{X}}(u) &= \sum_{|\beta'_1| + |\beta'_2| \leq |\beta'|} c'_{\beta'_1, \beta'_1, \beta'_2}(x, v) \kappa_{D_{\mathbb{X}_x}^{\beta'_1} \mathbb{X}_y^{\beta'_2} \tau, x, \exp_x^{\mathbb{X}}(v)}^{\mathbb{X}}(u), \\ R_{\langle \mathbb{X} \rangle_x, v}^{\gamma} \kappa_{\tau,x,\exp_x^{\mathbb{X}}}^{\mathbb{X}}(u) &= \sum_{|\gamma_1| \leq |\gamma|} c_{\gamma, \gamma_1}(x, v) \kappa_{\mathbb{X}_y^{\gamma_1} \tau, x, \exp_x^{\mathbb{X}}(v)}^{\mathbb{X}}(u), \end{aligned}$$

for some families of functions  $c'_{\beta'_1, \beta'_1, \beta'_2}, c_{\gamma, \gamma_1} \in C^\infty(U \times \mathbb{R}^n)$  that we can construct by induction. Consequently, denoting by  $\mathbb{P}_N^f(v)$  the (Euclidean) Taylor polynomial at order  $N$  of a smooth function  $f$  in  $v \sim 0$ , the (Euclidean) Taylor polynomial at order  $N$  in  $v \sim 0$  of

$$R_{\langle \mathbb{X} \rangle_x, v}^{\gamma} \mathbb{X}_x^{\beta'} L_{\langle \mathbb{X} \rangle_x, u}^{\beta} \kappa_{\tau,x,\exp_x^{\mathbb{X}}}^{\mathbb{X}}(u) = \sum_{\substack{|\beta'_1| + |\beta'_2| \leq |\beta'| \\ |\gamma_1| \leq |\gamma|}} (c'_{\beta'_1, \beta'_1, \beta'_2} c_{\gamma, \gamma_1})(x, v) \kappa_{\mathbb{X}_y^{\gamma_1} D_{\mathbb{X}_x}^{\beta'_1} \mathbb{X}_y^{\beta'_2} \widehat{\langle \mathbb{X} \rangle_x}^{\beta} \tau, x, \exp_x^{\mathbb{X}}(v)}^{\mathbb{X}}(u),$$

is

$$\begin{aligned} &\sum_{\substack{|\beta'_1| + |\beta'_2| \leq |\beta'| \\ |\gamma_1| \leq |\gamma|}} \sum_{N_1 + N_2 = N} \mathbb{P}_{N_1}^{(c'_{\beta'_1, \beta'_1, \beta'_2} c_{\gamma, \gamma_1})(x, \cdot)}(v) \mathbb{P}_{N_2}(\mathbb{X}_y^{\gamma_1} D_{\mathbb{X}_x}^{\beta'_1} \mathbb{X}_y^{\beta'_2} \widehat{\langle \mathbb{X} \rangle_x}^{\beta}, x; u, v), \\ &= R_{\langle \mathbb{X} \rangle_x, v}^{\gamma} \mathbb{X}_x^{\beta'} L_{\langle \mathbb{X} \rangle_x, u}^{\beta} \mathbb{P}_{N+|\gamma|}(\tau, x; u, v), \end{aligned}$$

by uniqueness of Taylor polynomials; furthermore, by the preceding step of the proof, its remainder for  $x \in \mathcal{C}_0$  and  $v \in V$  is estimated by

$$\lesssim_{N,\mathcal{C},V} |v|^{N+1} \max_{\beta_1 \leq |\beta|} \max_{|\alpha| \leq N+1+|\gamma|} \sup_{y \in \mathcal{C}} \left| \mathbb{X}_y^{\alpha} \mathbb{X}_x^{\beta_1} L_{\langle \mathbb{X} \rangle_x}^{\beta} \kappa_{\tau,x,y}^{\mathbb{X}}(u) \right|.$$

This gives the generalisation of (8.5) for derivatives. This allows us to estimate

$$\begin{aligned}
& \left| L_{\langle \mathbb{X} \rangle_x, u}^\beta \mathbb{X}_x^{\beta'} \left( \kappa_{\sigma, x}^{\mathbb{X}}(u) - \sum_{j=0}^N \kappa_{\sigma_j, x}^{\mathbb{X}}(u) \right) \right| \\
& \leq |\chi_0(x, u)| \left| L_{\langle \mathbb{X} \rangle_x, u}^\beta \mathbb{X}_x^{\beta'} \left( \kappa_{\tau, x, \exp_x^{\mathbb{X}} - u}^{\mathbb{X}}(u) - \sum_{j=0}^N \kappa_{\sigma_j, x}^{\mathbb{X}}(u) \right) \right| \\
& \quad + \sum_{\substack{\beta_1 + \beta_2 = \beta \\ \beta'_1 + \beta'_2 = \beta' \\ \beta_1 + \beta'_1 > 0}} \binom{\beta_1}{\beta} \binom{\beta'_1}{\beta'} \left| L_{\langle \mathbb{X} \rangle_x}^{\beta_1} \mathbb{X}_x^{\beta'_1} \chi_0(x, u) \right| \left| L_{\langle \mathbb{X} \rangle_x, u}^{\beta_2} \mathbb{X}_x^{\beta'_2} \left( \kappa_{\tau, x, \exp_x^{\mathbb{X}} - u}^{\mathbb{X}}(u) \right) \right|.
\end{aligned}$$

For the first term on the RHS, we proceed as in the case  $\beta = \beta' = 0$  above using the generalisation of (8.5) for derivatives we just obtained. Due to the derivatives on  $\chi_0$ , the second term is supported on a  $U$ -locally compact neighbourhood of 0 and away from a smaller  $U$ -locally compact neighbourhood of 0. The kernel estimates (see Theorem 6.2) allow us to obtain (8.4) for any  $\beta, \beta'$ . This concludes the proof.  $\square$

**8.3. The adjoints of locally quantized operators.** This section is devoted to showing the following property for the local quantization:

**Proposition 8.9.** *Let  $\mathbb{X}$  be an adapted frame on an open subset  $U \subset M$ . If  $\sigma \in S^m(\widehat{GM})$ , then there exists a unique symbol  $\sigma^{(*)} \in S^m(\widehat{GM}|_U)$  such that*

$$\text{Op}^{\mathbb{X}, \chi}(\sigma)^* = \text{Op}^{\mathbb{X}, \chi}(\sigma^{(*)}).$$

Moreover, the following map is continuous

$$\sigma \longmapsto \sigma^{(*)}, \quad S^m(\widehat{GM}) \longrightarrow S^m(\widehat{GM}|_U),$$

and  $\sigma^{(*)}$  admits an asymptotic expansion

$$\sigma^{(*)} \sim \sum_{j \geq 0} \rho_j$$

such that  $\rho_0 = \sigma^*$  and for each  $N \in \mathbb{N}_0$ , the following map is continuous

$$\sigma \longmapsto \sigma^{(*)} - \sum_{j=0}^N \rho_j, \quad S^m(\widehat{GM}) \longrightarrow S^{m-N-1}(\widehat{GM}).$$

Before discussing the proof of Proposition 8.9, we observe that it implies together with the properties of the adjoint and the continuous action of  $\text{Op}^{\mathbb{X}, \chi}(\sigma)$ ,  $\sigma \in \cup_m S^m(\widehat{G})$ , on  $C_c^\infty(U)$  (see Lemma 7.2) that the action of these operators can be extended to distributions:

**Corollary 8.10.** *Let  $(\mathbb{X}, U)$  be an adapted local frame and  $\chi$  a cut-off function for  $(\mathbb{X}, U)$ . Let  $m \in \mathbb{R}$ . If  $\sigma \in S^m(\widehat{GM}|_U)$ , the operator  $\text{Op}^{\mathbb{X}, \chi}(\sigma)$  acts continuously on  $C_c^\infty(U)$  and  $\mathcal{D}'(U)$ . Furthermore, the map  $\sigma \mapsto \text{Op}^{\mathbb{X}, \chi}(\sigma)$  is an injective morphism of topological spaces*

$$S^m(\widehat{GM}|_U) \rightarrow \mathcal{L}(C_c^\infty(U)) \quad \text{and} \quad S^m(\widehat{GM}|_U) \rightarrow \mathcal{L}(\mathcal{D}'(U)).$$

*Proof.* If  $f \in \mathcal{D}'(U)$ , the distribution  $\text{Op}^{\mathbb{X},\chi}(\sigma)f$  is defined by

$$\langle \text{Op}^{\mathbb{X},\chi}(\sigma)f, g \rangle = \overline{\langle f, \text{Op}^{\mathbb{X},\chi}(\sigma^*)\bar{g} \rangle}, \quad g \in \mathcal{D}(M) := C_c^\infty(M).$$

Here  $\langle \cdot, \cdot \rangle$  denotes the duality bracket in distributions. Hence the statement follows by duality and Lemma 7.2.  $\square$

The proof of Proposition 8.9 follows from the properties of invariance by change of cut-off functions (see Proposition 7.5) and the following lemma:

**Lemma 8.11.** *Let  $\mathbb{X}$  be an adapted frame on an open subset  $U \subset M$ , and  $\chi$  be a  $\mathbb{X}$ -cut-off and we have:*

(1) *The function  $\chi^*$  given by*

$$\chi^*(x, y) = \chi(y, x), \quad x, y \in U,$$

*is also a cut-off for  $\mathbb{X}$  and if  $\sigma \in S^m(\widehat{GM})$ , then*

$$\text{Op}^{\mathbb{X},\chi}(\sigma)^* = \text{Op}^{\mathbb{X},\chi^*}(\sigma^{(*)}),$$

*where  $\sigma^{(*)} \in S^m(\widehat{GM}|_U)$  is the symbol whose convolution kernel in the  $\mathbb{X}$ -coordinates is given by:*

$$\kappa_{\sigma^{(*)},x}^{\mathbb{X}}(u) = \bar{\kappa}_{\sigma, \exp_x^{\mathbb{X}} - u}(-u) \frac{\text{jac}_x(\ln_{\exp_x^{\mathbb{X}} - u}^{\mathbb{X}})}{\text{jac}_{\exp_x^{\mathbb{X}} - u}(\ln_x^{\mathbb{X}})}.$$

(2) *If  $\sigma \in S^m(\widehat{GM})$ , then*

$$\sigma^{(*)} \sim \sum_{j \in \mathbb{N}} \sum_{[\alpha]=j, \beta} d_{\alpha, \beta}(x) \Delta_\alpha^{\mathbb{X}} D_{\mathbb{X}}^\beta \sigma^*,$$

*where the coefficients  $d_{\alpha, \beta}$  are obtained from the coefficients  $c_{\alpha, \beta}$  of Lemma 8.8:*

$$c_{\alpha, \beta}(x) \mathbb{X}_{y=x}^\beta \left( \frac{\text{jac}_x(\ln_y^{\mathbb{X}})}{\text{jac}_y(\ln_x^{\mathbb{X}})} f(y) \right) = \sum_{|\beta'| \leq |\beta|} d_{\alpha, \beta'}(x) \mathbb{X}^{\beta'} f(x), \quad x \in U, \quad f \in C^\infty(U).$$

*Moreover, for each  $N \in \mathbb{N}$ , the following map is continuous*

$$\sigma \longmapsto \sigma^{(*)} - \sum_{j \leq N} \sum_{[\alpha]=j, \beta} d_{\alpha, \beta}(x) \Delta_\alpha^{\mathbb{X}} D_{\mathbb{X}}^\beta \sigma^*, \quad S^m(\widehat{GM}) \longleftrightarrow S^{m-N-1}(\widehat{GM}|_U).$$

*Proof of Lemma 8.11. Part (1).* For any  $f \in C_c^\infty(U)$  and  $x \in M$

$$\text{Op}^{\mathbb{X},\chi}(\sigma)^* f(x) = \int_M \bar{\kappa}_{\sigma, y}^{\mathbb{X}}(-\ln_y^{\mathbb{X}} x) (\chi_x^* f)(y) \text{jac}_x(\ln_y^{\mathbb{X}}) dy.$$

Therefore, the convolution kernel  $\kappa_{\sigma^{(*)},x}^{\mathbb{X}}$  of the operator  $\text{Op}^{\mathbb{X},\chi}(\sigma)^*$  satisfies for  $x, y \in M$

$$\kappa_{\sigma^{(*)},x}^{\mathbb{X}}(-\ln_x^{\mathbb{X}} y) = \bar{\kappa}_{\sigma, y}^{\mathbb{X}}(-\ln_y^{\mathbb{X}} x) \frac{\text{jac}_x(\ln_y^{\mathbb{X}})}{\text{jac}_y(\ln_x^{\mathbb{X}})}.$$

The change of variable  $u = -\ln_x^{\mathbb{X}} y$  gives

$$\kappa_{\sigma^{(*)},x}^{\mathbb{X}}(u) = \bar{\kappa}_{\sigma, \exp_x^{\mathbb{X}} -u}^{\mathbb{X}}(-\ln_{\exp_x^{\mathbb{X}} -u}^{\mathbb{X}} x) \frac{\text{jac}_x(\ln_{\exp_x^{\mathbb{X}} -u}^{\mathbb{X}})}{\text{jac}_{\exp_x^{\mathbb{X}} -u}(\ln_x^{\mathbb{X}})}.$$

We then conclude by observing that

$$-\ln_{\exp_x^{\mathbb{X}} -u}^{\mathbb{X}} x = \ln_x^{\mathbb{X}} \exp_x^{\mathbb{X}} -u = -u.$$

*Part (2).* By Part (1), we can write at least formally  $\text{Op}^{\mathbb{X},\chi}(\sigma)^* = \text{AOp}^{\mathbb{X},\chi}(\tau)$  where  $\tau$  is the amplitude given by the convolution kernel in the  $\mathbb{X}$ -coordinates:

$$\kappa_{\tau,x,y}^{\mathbb{X}}(v) = \bar{\kappa}_{\sigma,y}^{\mathbb{X}}(-v) \frac{\text{jac}_x(\ln_y^{\mathbb{X}})}{\text{jac}_y(\ln_x^{\mathbb{X}})}, \quad x, y \in U.$$

We check readily that  $\tau \in \mathcal{T}^m(\widehat{GM}|_U)$  with

$$c_{\alpha,\beta}(x) \Delta_{\alpha}^{\mathbb{X}} D_{\mathbb{X},y=x}^{\beta} \tau(x, y, \pi) = \sum_{|\beta'| \leq |\beta|} d_{\alpha,\beta'}(x) \Delta_{\alpha}^{\mathbb{X}} D_{\mathbb{X}}^{\beta} \sigma^*(x, \pi), \quad x \in U, \pi \in G_x M,$$

by Lemma 8.2. We then conclude with Proposition 8.7.  $\square$

Usually,  $\sigma^{(*)}$  is different from  $\sigma^*$ . However, the symbols  $\widehat{\langle X_j \rangle}$  have special properties:

**Lemma 8.12.** *Let  $\mathbb{X}$  be an adapted frame on an open subset  $U \subset M$ , and  $\chi$  be a cut-off for  $\mathbb{X}$ . Then*

$$\text{Op}^{\mathbb{X},\chi}(\widehat{\langle X_j \rangle})^* = -\text{Op}^{\mathbb{X},\chi}(\widehat{\langle X_j \rangle}).$$

In other words,  $\widehat{\langle X_j \rangle}^{(*)} = \widehat{\langle X_j \rangle}^* = -\widehat{\langle X_j \rangle}$ .

*Proof of Lemma 8.12.* Let  $f, g \in C_c^{\infty}(U)$ . Proposition 7.3 implies

$$\begin{aligned} (\text{Op}^{\mathbb{X},\chi}(\widehat{\langle X_j \rangle})f, g)_{L^2(U)} &= \partial_j|_{v=0} \int_U f(\exp_x^{\mathbb{X}} v) \bar{g}(x) dx \\ &= \partial_j|_{v=0} \int_U f(y) \bar{g}(\exp_y^{\mathbb{X}} -v) \text{jac}_y \exp_y^{\mathbb{X}}(-v) dy \\ &= - \int_U f(y) \partial_j|_{v=0} \bar{g}(\exp_y^{\mathbb{X}} -v) dy, \end{aligned}$$

since  $\text{jac}_y \exp_y^{\mathbb{X}} 0 = 1$  and  $\partial_j|_{v=0} \text{jac}_y \exp_y^{\mathbb{X}}(-v) = 0$  because we have

$$D_{v=0} D_y \exp_y^{\mathbb{X}}(-v) = D_y D_{v=0} \exp_y^{\mathbb{X}}(-v) = -D_y(\text{I}) = 0.$$

$\square$

**8.4. The composition of locally quantized operators.** This section is devoted to showing properties regarding composition for the local quantization:

**Proposition 8.13.** *Let  $\mathbb{X}$  be an adapted frame on an open subset  $U \subset M$  and let  $\chi$  be a cut-off function for  $\mathbb{X}$ . If  $\sigma_1 \in S^{m_1}(\widehat{GM})$  and  $\sigma_2 \in S^{m_2}(\widehat{GM})$ , then there exists a unique symbol*

$$\sigma_1 \diamond \sigma_2 = \sigma_1 \diamond_{\mathbb{X}, \chi} \sigma_2 \in S^{m_1+m_2}(\widehat{GM}|_U)$$

such that

$$\text{Op}^{\mathbb{X}, \chi}(\sigma_1 \diamond \sigma_2) = \text{Op}^{\mathbb{X}, \chi}(\sigma_1) \text{Op}^{\mathbb{X}, \chi}(\sigma_2).$$

Moreover, the following properties hold:

(1) *The following map is continuous*

$$(\sigma_1, \sigma_2) \longmapsto \sigma_1 \diamond \sigma_2, \quad S^{m_1}(\widehat{GM}) \times S^{m_2}(\widehat{GM}) \longrightarrow S^{m_1+m_2}(\widehat{GM}|_U).$$

(2) *The symbol  $\sigma_1 \diamond \sigma_2$  admits an asymptotic expansion*

$$\sigma_1 \diamond \sigma_2 \sim \sum_{j \geq 0} \rho_j \quad \text{in } S^{m_1+m_2}(\widehat{GM}|_U),$$

satisfying  $\rho_0 = \sigma_1 \sigma_2$  and such that for each  $N \in \mathbb{N}$ , the following map is continuous

$$(\sigma_1, \sigma_2) \longmapsto \sigma_1 \diamond \sigma_2 - \sum_{j=0}^N \rho_j, \quad S^{m_1}(\widehat{GM}) \times S^{m_2}(\widehat{GM}) \longrightarrow S^{m_1+m_2-N-1}(\widehat{GM}).$$

Let us first explain the strategy of the proof of Proposition 8.13 (which follows from standard ideas developed in the Euclidean case). By Proposition 8.9, there exists a symbol  $\sigma_2^{(*)}$  such that

$$\text{Op}^{\mathbb{X}, \chi}(\sigma_2) = (\text{Op}^{\mathbb{X}, \chi}(\sigma_2)^*)^* = \text{Op}^{\mathbb{X}, \chi}(\sigma_2^{(*)})^*.$$

Therefore, it is enough to be able to prove that for all symbol  $\sigma_3 \in S^m(\widehat{GM})$ , the operator  $\text{Op}^{\mathbb{X}, \chi}(\sigma_1) \text{Op}^{\mathbb{X}, \chi}(\sigma_3)^*$  is pseudodifferential in the sense that it has a symbol in a suitable class, and that this symbol can be written as an asymptotic sum of symbols with suitable continuity properties. The advantage of considering this new problem is that a simple amplitude is associated with the operator  $\text{Op}^{\mathbb{X}, \chi}(\sigma_1) \text{Op}^{\mathbb{X}, \chi}(\sigma_3)^*$ . Then Proposition 8.7 will allow us to conclude and to identify the first term of the asymptotic expansion.

With this strategy in mind, Proposition 8.13 will derive from the following lemma:

**Lemma 8.14.** *We continue with the setting and notation of Proposition 8.13. The symbol  $\sigma_1 \diamond (\sigma_2^{(*)})$  admits an asymptotic expansion*

$$\sigma_1 \diamond (\sigma_2^{(*)}) \sim \sum_{j \geq 0} \rho_j \quad \text{in } S^{m_1+m_2}(\widehat{GM}|_U),$$

satisfying  $\rho_0 = \sigma_1 \sigma_2^*$  and such that for each  $J \in \mathbb{N}$ , the following map is continuous

$$(\sigma_1, \sigma_2) \longmapsto \sigma_1 \diamond (\sigma_2^{(*)}) - \sum_{j=0}^J \rho_j, \quad S^{m_1}(\widehat{GM}) \times S^{m_2}(\widehat{GM}) \longrightarrow S^{m_1+m_2-J-1}(\widehat{GM}).$$

Moreover, there exist two families of smooth functions  $(c_{\alpha_1, \alpha_2, \gamma_1, \gamma_2})_{\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in \mathbb{N}^n}$  in  $C^\infty(U)$  and  $(p_{\alpha_3})_{\alpha_3 \in \mathbb{N}^n}$  in  $C^\infty(U \times \mathbb{R}^n)$  such that

$$\rho_j = \sum_{[\alpha_1] + [\alpha_2] + [\alpha_3]_{\min} = j} \sum_{\gamma_1 + \gamma_2 = \alpha_3, |\beta| \leq |\gamma_2|} c_{\alpha_1, \alpha_2, \gamma_1, \gamma_2}(x) \Delta_{p_{\alpha_3}} \widehat{\langle X \rangle}^{\alpha_3} \Delta_{\mathbb{X}}^{\alpha_1 + \gamma_1} \sigma_1 \Delta_{\mathbb{X}}^{\alpha_2 + \gamma_2} D_{\mathbb{X}}^\beta \sigma_2^*.$$

*Proof of Lemma 8.14.* By density of smoothing symbols (see Proposition 5.45), and, since we will prove estimates for continuous seminorms, we may assume that the symbols  $\sigma_1, \sigma_2$  are smoothing; this justifies the manipulations below.

The integral kernel of the operator  $\text{Op}^{\mathbb{X}, \chi}(\sigma_1) \text{Op}^{\mathbb{X}, \chi}(\sigma_2)^*$  is given by

$$\begin{aligned} (x, y) &\mapsto \int_M \kappa_{1,x}^{\mathbb{X}}(-\ln_x^{\mathbb{X}} z) \bar{\kappa}_{2,y}^{\mathbb{X}}(-\ln_y^{\mathbb{X}} z) \text{jac}_z(\ln_x^{\mathbb{X}}) \text{jac}_z(\ln_y^{\mathbb{X}}) \chi_x(z) \bar{\chi}_y(z) dz \\ &= \tilde{\chi}(x, y) \int_{\mathbb{R}^n} \kappa_{1,x}^{\mathbb{X}}(-v) \bar{\kappa}_{2,y}^{\mathbb{X}}(-\ln_y^{\mathbb{X}}(\exp_x^{\mathbb{X}} v)) \text{jac}_{\exp_x^{\mathbb{X}} v}(\ln_y^{\mathbb{X}}) (\chi_x \bar{\chi}_y)(\exp_x^{\mathbb{X}} v) dv, \end{aligned}$$

where  $\tilde{\chi}$  is an  $\mathbb{X}$ -cut-off function with  $\tilde{\chi} = 1$  on a neighbourhood of  $\cup_{z \in U} \text{supp}(\chi(\cdot, z) \bar{\chi}(\cdot, z))$ . Consequently,

$$\text{Op}^{\mathbb{X}, \chi}(\sigma_1) \text{Op}^{\mathbb{X}, \chi}(\sigma_2)^* = \text{Op}^{\mathbb{X}, \tilde{\chi}}(\sigma) = \text{AOp}^{\mathbb{X}, \tilde{\chi}}(\tau),$$

where  $\tau$  is the amplitude whose convolution kernel in the  $\mathbb{X}$ -coordinates is given by

$$\kappa_{\tau, x, y}^{\mathbb{X}}(u) = \int_{\mathbb{R}^n} \kappa_{1,x}^{\mathbb{X}}(-v) \bar{\kappa}_{2,y}^{\mathbb{X}}(-\ln_{\exp_x^{\mathbb{X}} - u}^{\mathbb{X}}(\exp_x^{\mathbb{X}} v)) \frac{\text{jac}_{\exp_x^{\mathbb{X}} v}(\ln_y^{\mathbb{X}})}{\text{jac}_y(\ln_x^{\mathbb{X}})} (\chi_x \bar{\chi}_y)(\exp_x^{\mathbb{X}} v) dv.$$

After the change of variable

$$w = \varphi_{x,u}(v) = \ln_{\exp_x^{\mathbb{X}} - u}^{\mathbb{X}}(\exp_x^{\mathbb{X}} v), \quad v = \varphi_{x,u}^{-1}(w) = \ln_x^{\mathbb{X}} \exp_{\exp_x^{\mathbb{X}} - u}^{\mathbb{X}} w,$$

we obtain:

$$\kappa_{\tau, x, y}^{\mathbb{X}}(u) = \int_{\mathbb{R}^n} \kappa_{1,x}^{\mathbb{X}}(-\varphi_{x,u}^{-1}(w)) \kappa_{2*,y}^{\mathbb{X}}(w) (\chi_x \bar{\chi}_y)(\exp_{\exp_x^{\mathbb{X}} - u}^{\mathbb{X}} w) \frac{\text{jac}_{\exp_{\exp_x^{\mathbb{X}} - u}^{\mathbb{X}} w}(\ln_y^{\mathbb{X}})}{\text{jac}_y(\ln_x^{\mathbb{X}})} \text{jac}_w \varphi_{x,u}^{-1} dw,$$

where  $\kappa_{2*,y}^{\mathbb{X}}(u) := \bar{\kappa}_{2,y}^{\mathbb{X}}(-u)$  is the convolution kernel of the symbol  $\sigma_2^*$  in the  $\mathbb{X}$ -coordinates. We then make the substitution  $w = u *_x (-w_1)$ , and define as in Corollary 4.14 (2)

$$\check{r}(x; u, w_1) = (-w_1) *_x (-\ln_x^{\mathbb{X}} \exp_{\exp_x^{\mathbb{X}} - u}^{\mathbb{X}} u *_x (-w_1)),$$

to obtain

$$\kappa_{\tau, x, y}^{\mathbb{X}}(u) = \kappa_{\tau(u_1), x, y}(u) \Big|_{u_1 = u},$$

where

$$\kappa_{\tau(u_1), x, y}^{\mathbb{X}}(u) := \int_{\mathbb{R}^n} \chi(x; u_1, w_1) \kappa_{1,x}^{\mathbb{X}}(w_1 *_x \check{r}(x; u_1, w_1)) \kappa_{2*,y}^{\mathbb{X}}(u *_x (-w_1)) J(x, y; u, w_1) dw_1.$$

Above, we adopted the short-hand

$$J(x, y; u, w_1) := (\chi_x \bar{\chi}_y)(\exp_{\exp_x^{\mathbb{X}} - u}^{\mathbb{X}} u *_x (-w_1)) \frac{\text{jac}_{\exp_{\exp_x^{\mathbb{X}} - u}^{\mathbb{X}} u *_x (-w_1)}(\ln_y^{\mathbb{X}})}{\text{jac}_y(\ln_x^{\mathbb{X}})} \text{jac}_{u *_x (-w_1)} \varphi_{x,u}^{-1}$$

for the expression involving the cut-off functions and the Jacobians; we also inserted a suitable  $U$ -locally compactly supported function  $\chi_1 \in C^\infty(U \times \mathbb{R}^n \times \mathbb{R}^n)$ . Hence, with the notation of Lemma 8.6, we have

$$\tau = \tau_0 \quad \text{where} \quad \tau(u_1) := \tau(\sigma_1^{(\check{r}_{u_1}, \chi_1, u_1)}, \sigma_2^*, J),$$

having used the notation of Propositions 8.4 and 6.9, and set

$$\check{r}_{u_1}(x; w_1) := \check{r}(x; u_1, w_1), \quad \chi_{1, u_1}(x; w_1) := \chi_1(x; u_1, w_1).$$

These statements, together with the link between amplitudes and symbols (see Proposition 8.7) and the continuity of taking the adjoint of symbols (see Proposition 5.25), allow us to conclude the proof.  $\square$

## 9. PSEUDODIFFERENTIAL CALCULUS ON $M$

In this section we define a pseudodifferential calculus on filtered manifolds. In Section 9.1 we introduce a passage from the local quantization to the definition of operators defined on the whole manifold  $M$ . It follows classical lines and is inspired by the construction of quantization on manifolds presented in [70, 46]. Then, we verify that the operators we have introduced satisfy the expected properties: composition rules when they are properly supported (Section 9.2), a notion of principal symbol (Section 9.3), and the existence of parametrices when the symbols satisfy adequate conditions (Section 9.4). As usual, associated with this set of pseudodifferential calculus comes a notion of (local) Sobolev spaces that we describe in Section 9.5, and on which pseudodifferential operators act continuously. Finally, in Section 9.6, we discuss what is the classical pseudodifferential calculus deriving from our construction and prove in Section 9.7 that it coincides with the abstract class of pseudodifferential operators constructed by van Erp and Yuncken calculus in [67, 68] via groupoids.

**9.1. Pseudodifferential operators on  $M$ .** We recall that the smooth manifold  $M$  is assumed second countable, paracompact Hausdorff, and that we have fixed a measure  $dx$  on  $M$ .

9.1.1. *Definition.*

**Definition 9.1.** Let  $M$  be a filtered manifold. Let  $m \in \mathbb{R}$ . A linear operator

$$P : C_c^\infty(M) \longrightarrow C^\infty(M)$$

is a pseudodifferential operator of order  $m$  on the filtered manifold  $M$  when

- (1) for any adapted frame  $\mathbb{X}$  on an open set  $U \subset M$ , for any  $\mathbb{X}$ -cut-off  $\chi$ , there exists a symbol  $\sigma \in S^m(\widehat{GM}|_U)$  such that

$$\varphi P(\psi f) = \varphi \text{Op}^{\mathbb{X}, \chi}(\sigma)(\psi f),$$

for any  $f \in C_c^\infty(M)$  and  $\varphi, \psi \in C_c^\infty(U)$ ,

(2) there exists a smooth function  $K \in C^\infty(M \times M \setminus D_M)$  defined away from the diagonal  $D_M := \{(x, x) : x \in M\}$  such that

$$\varphi_1 P(\varphi_2)(x) = \varphi_1(x) \int_M K(x, y) \varphi_2(y) dy, \quad x \in M,$$

for any  $\varphi_1, \varphi_2 \in C_c^\infty(M)$  with disjoint supports (i.e.  $\text{supp } \varphi_1 \cap \text{supp } \varphi_2 = \emptyset$ ).

The properties in the definition above and those of the local quantization (especially Lemma 7.2) imply that  $P$  is a continuous operator  $C_c^\infty(M) \rightarrow \mathcal{D}'(M)$ . By the Schwartz kernel theorem, it admits an integral kernel  $K_0 \in \mathcal{D}'(M \times M)$ . Part (2) above says that  $K_0$  coincides with a smooth function on  $M \times M \setminus D_M$ , i.e. away from the diagonal. When this integral kernel is smooth on  $M \times M$ , we will follow the traditional convention (as well as in coherence with what we have done in the preceding sections) of calling the operator smoothing:

**Definition 9.2.** Let  $M$  be a smooth manifold. An operator  $P$  on  $M$  is *smoothing* when it admits a smooth integral kernel, that is, there exists  $K \in C^\infty(M \times M)$  such that

$$Pf(x) = \int_M f(y) K(x, y) dy, \quad x \in M.$$

We denote by  $\Psi^m(M)$  the space of pseudodifferential operators of order  $m \in \mathbb{R}$  on  $M$  and by  $\Psi^{-\infty}(M)$  the space of smoothing operators on  $M$ . We check readily for  $m, m_1, m_2 \in \mathbb{R}$  with  $m_1 \leq m_2$

$$\Psi^{-\infty}(M) \subset \Psi^m(M) \quad \text{and} \quad \Psi^{m_1}(M) \subset \Psi^{m_2}(M).$$

We also denote by

$$\Psi(M) := \cup_{m \in \mathbb{R}} \Psi^m(M)$$

the space of pseudodifferential operators (of any order) on  $M$ .

From Definitions 9.1 and 9.2 together with the properties of the local quantization, we check readily that  $\Psi^m(M)$  for each  $m \in \mathbb{R} \cup \{-\infty\}$  is a vector space and furthermore a left and right  $C^\infty(M)$ -module:

$$\forall \varphi, \psi \in C^\infty(M), \quad \forall P \in \Psi^m(M) \quad \varphi P \psi \in \Psi^m(M).$$

Moreover, it is stable under the formal adjoint:

**Lemma 9.3.** *If  $P \in \Psi^m(M)$  with  $m \in \mathbb{R} \cup \{-\infty\}$ , then its formal adjoint  $P^*$ , that is, the operator defined via*

$$\forall f, g \in C_c^\infty(M) \quad (P^* f, g)_{L^2(M)} = (f, P g)_{L^2(M)},$$

*is a well defined operator in  $\Psi^m(M)$ .*

*Proof.* We observe that if  $P$  is an operator  $C_c^\infty(M) \rightarrow C^\infty(M)$  with integral kernel  $K \in \mathcal{D}'(M \times M)$ , then  $P^*$  is also an operator  $C_c^\infty(M) \rightarrow \mathcal{D}'(M)$  with integral kernel  $K^* \in \mathcal{D}'(M \times M)$  given by  $K^*(x, y) = \bar{K}(y, x)$ . This observation implies the statement for smoothing operators (i.e. the case  $m = -\infty$ ) as well as the second part in Definition 9.1 for the formal

adjoint  $P^*$  of an operator  $P \in \Psi^m(M)$  with  $m \in \mathbb{R}$ . The first part in Definition 9.1 for  $P^*$  is a consequence of Proposition 8.9. This concludes the proof.  $\square$

From Definition 9.1 and the properties of the local quantization (especially Lemma 7.2 and Corollary 8.10), we check readily that a pseudodifferential operator on  $M$  acts linearly continuously  $C_c^\infty(M) \rightarrow C^\infty(M)$  and also  $\mathcal{E}'(M) \rightarrow \mathcal{D}'(M)$  by Lemma 9.3; here  $\mathcal{E}'(M)$  stands for the space of distributions with compact support. Note that the space  $\Psi^{-\infty}(M)$  may be described equivalently as the space of continuous linear operators  $\mathcal{E}'(M) \rightarrow C^\infty(M)$ , as well as

$$\Psi^{-\infty}(M) = \bigcap_{m \in \mathbb{R}} \Psi^m(M).$$

9.1.2. *Examples.* Here, we give some examples of pseudodifferential operators. Our first class of examples results from the local quantizations defined in Section 7.1.1:

**Lemma 9.4.** *Let  $m \in \mathbb{R} \cup \{-\infty\}$ . Let  $(\mathbb{X}, U)$  be an adapted frame with  $\mathbb{X}$ -cut-off  $\chi$ .*

- (1) *If  $\sigma \in S^m(\widehat{GM}|_U)$ , then  $\text{Op}^{\mathbb{X}, \chi}(\sigma) \in \Psi^m(U)$ .*
- (2) *Moreover, if  $\sigma \in \bigcup_{m' \in \mathbb{R} \cup \{-\infty\}} S^{m'}(\widehat{GM}|_U)$  satisfies  $\text{Op}^{\mathbb{X}, \chi}(\sigma) \in \Psi^m(U)$ , then  $\sigma \in S^m(\widehat{GM}|_U)$ .*

*Proof.* The characterisation of smoothing symbols (see Proposition 5.45 (1), see also Corollary 6.5) implies the case  $m = -\infty$ . We now prove the case  $m \in \mathbb{R}$ .

Let  $\sigma \in S^m(\widehat{GM})$ . The kernel estimates (see Theorem 6.2) imply that the Schwartz kernel  $K(x, y)$  of  $\text{Op}^{\mathbb{X}, \chi}(\sigma)$  is smooth away from the diagonal when  $m \in \mathbb{R}$ . Hence, Part (2) of Definition 9.1 holds.

Let  $\mathbb{Y}$  be another adapted frame on an open set  $V \subset U$ , with  $\mathbb{Y}$ -cut-off  $\tilde{\chi}$ . Then by Propositions 7.5 and 7.7, there exists  $\tilde{\sigma} \in S^m(\widehat{GM}|_U)$  such that  $\text{Op}^{\mathbb{X}, \chi}(\sigma) = \text{Op}^{\mathbb{Y}, \tilde{\chi}}(\tilde{\sigma})$ . This shows that Part (1) of Definition 9.1 holds. Therefore  $\text{Op}^{\mathbb{X}, \chi}(\sigma) \in \Psi^m(U)$ .

Let  $m' \in \mathbb{R}$  and  $\sigma \in S^{m'}(\widehat{GM}|_U)$ . We assume  $\text{Op}^{\mathbb{X}, \chi}(\sigma) \in \Psi^m(U)$ . We may assume  $m' < m$ . By Part (1) of Definition 9.1, there exists  $\tilde{\sigma} \in S^{m'}(\widehat{GM}|_U)$  such that

$$\forall \varphi, \psi \in C_c^\infty(U) \quad \varphi \text{Op}^{\mathbb{X}, \chi}(\sigma)\psi = \varphi \text{Op}^{\mathbb{X}, \chi}(\tilde{\sigma})\psi.$$

This readily implies that  $\kappa_\sigma^{\mathbb{X}} - \kappa_{\tilde{\sigma}}^{\mathbb{X}}$  is smooth on  $U \times \mathbb{R}^n$ , so  $\sigma - \tilde{\sigma} \in S^{-\infty}(\widehat{GM})$  by Corollary 6.5. This shows Part (2).  $\square$

We will often identify a function with compact support in  $U \subset M$  with a function trivially extended to  $M$ . And conversely, if a function on  $M$  is supported in  $U$ , we may view it as a function on  $U$ . With these conventions in mind, Lemma 9.4 and its proof imply readily:

**Corollary 9.5.** *Let  $m \in \mathbb{R} \cup \{-\infty\}$ . Let  $(\mathbb{X}, U)$  be an adapted frame with  $\mathbb{X}$ -cut-off  $\chi$ . For any  $\sigma \in S^m(\widehat{GM}|_U)$  and  $\psi \in C_c^\infty(U)$ , the operator given by*

$$f \longmapsto \text{Op}^{\mathbb{X}, \chi}(\sigma)(\psi f), \quad C_c^\infty(M) \longrightarrow C^\infty(M).$$

*is in  $\Psi^m(M)$ . That is,*

$$\forall \psi \in C_c^\infty(U), \quad \text{Op}^{\mathbb{X}, \chi}(\sigma)\psi \in \Psi^m(M).$$

Moreover, if  $\sigma \in \cup_{m' \in \mathbb{R} \cup \{-\infty\}} S^{m'}(\widehat{GM}|_U)$  satisfies  $\text{Op}^{\mathbb{X}, \chi}(\sigma)\psi \in \Psi^m(M)$  for any  $\psi \in C_c^\infty(U)$ , then  $\sigma \in S^m(\widehat{GM}|_U)$ .

*Remark 9.6.* Lemma 9.4 and its proof imply that if the property described in Part (1) of Definition 9.1 holds for a family of open subsets  $(U_\alpha)_\alpha$  of  $M$ , each  $U_\alpha$  being equipped with an adapted frame  $\mathbb{X}_\alpha$  and a  $\mathbb{X}_\alpha$ -cut-off, with the family of open subsets covering  $M$ , then Part (1) of Definition 9.1 holds for any  $(\mathbb{X}, U)$ .

Our second class of examples comprises differential operators:

**Lemma 9.7.** *Any differential operator is pseudodifferential, and its integral kernel is supported on the diagonal. Moreover, if  $P$  is a differential operator, then  $P \in \Psi^N(M)$  with  $N$  being its homogeneous order.*

The homogeneous order was defined in Definition 3.7. This statement implies:

$$\text{DO}(M) \subset \Psi(M), \quad \text{and more precisely,} \quad \text{DO}^{\leq N}(M) \subset \Psi^N(M).$$

*Proof.* A differential operator  $P$  is a linear operator  $P : C_c^\infty(M) \rightarrow C^\infty(M)$  with integral kernel supported on the diagonal. It remains to check Part (1) of Definition 9.1. We denote by  $N$  the homogeneous order of  $P$  and by  $\sigma_N \in \Gamma(\mathcal{Z}_N(\mathfrak{g}M))$  its principal part in the sense of Section 3.4.

Let  $\mathbb{X}$  be an adapted frame on an open set  $U \subset M$  and let  $\chi$  be a  $\mathbb{X}$ -cut-off. The operator  $\text{Op}^{\mathbb{X}, \chi}(\sigma_N)$  is differential by Proposition 7.3 and in  $\Psi^N(M)$  by Lemma 9.4. Moreover, Proposition 7.3 implies that  $P|_U - \text{Op}^{\mathbb{X}, \chi}(\sigma_N)$  is a differential operator on  $U$  of homogeneous order  $\leq N - 1$ . Inductively, we construct a finite sequence of symbols

$$\sigma_N, \sigma_{N-1}, \dots, \sigma_0 \quad \text{respectively in } \Gamma(\mathcal{Z}_N(\mathfrak{g}M)), \Gamma(\mathcal{Z}_{N-1}(\mathfrak{g}M)), \dots, \Gamma(\mathcal{Z}_0(\mathfrak{g}M)),$$

such that the differential operator  $P$  coincides on  $U$  with  $\text{Op}^{\mathbb{X}, \chi}(\sigma_N + \sigma_{N-1} + \dots + \sigma_0)$ . This implies Part (2) of Definition 9.1 and concludes the proof.  $\square$

9.1.3. *Characterisation via atlases.* Remark 9.6 above leads us to adapt the notion of atlas to the setting of filtered manifold in the following way:

**Definition 9.8.** The collection  $\mathcal{A} = (U_\alpha, \mathbb{X}_\alpha, \chi_\alpha, \psi_\alpha)_{\alpha \in A}$  is an *atlas* of the filtered manifold  $M$  when it consists in

- (a) a countable family of open subsets  $U_\alpha$  providing a locally finite covering of  $M = \cup_\alpha U_\alpha$ , and
- (b) for each  $\alpha \in A$ , an adapted frame  $\mathbb{X}_\alpha$  on  $U_\alpha$ , a  $\mathbb{X}_\alpha$ -cut-off  $\chi_\alpha$  and a function  $\psi_\alpha \in C_c^\infty(U_\alpha : [0, 1])$  such that the functions  $(x, y) \mapsto \psi_\alpha(x)\psi_\alpha(y)$  are supported in the interior of the set  $\{\chi_\alpha(x, y) = 1\}$ , and satisfying  $\sum_\alpha \psi_\alpha^2 = 1$ .

It is possible to construct such atlases via routine arguments omitted here.

This notion of atlas on  $M$  allows us to characterise the pseudodifferential operators on  $M$  in the following way:

**Proposition 9.9.** *Let  $P$  be a continuous linear operator  $C_c^\infty(M) \rightarrow \mathcal{D}'(M)$ . Then  $P \in \Psi^m(M)$  if and only if there exists one (and then any) atlas  $\mathcal{A} = (U_\alpha, \mathbb{X}_\alpha, \chi_\alpha, \psi_\alpha)_{\alpha \in A}$  for the filtered manifold  $M$ , a family of symbols  $(\sigma_\alpha)_{\alpha \in A}$  with  $\sigma_\alpha \in S^m(\widehat{GM}|_{U_\alpha})$  and a smoothing operator  $R \in \Psi^{-\infty}(M)$  such that*

$$(9.1) \quad P = R + \sum_{\alpha} \psi_{\alpha} \operatorname{Op}^{\mathbb{X}_{\alpha}, \chi_{\alpha}}(\sigma_{\alpha}) \psi_{\alpha}.$$

The equality in (9.1) above means that for any  $f \in C_c^\infty(M)$  and  $x \in M$ , we have:

$$Pf(x) = Rf(x) + \sum_{\alpha} \psi_{\alpha}(x) \operatorname{Op}^{\mathbb{X}_{\alpha}, \chi_{\alpha}}(\sigma_{\alpha})(\psi_{\alpha}f)(x).$$

We will see in Corollary 9.25 that we can in fact consider one symbol  $\sigma$  instead of a family  $(\sigma_{\alpha})_{\alpha \in A}$ . In any case, the family of symbols  $(\sigma_{\alpha})_{\alpha \in A}$  and the smoothing operator  $R$  depend on the atlas  $\mathcal{A}$ . However, even if we fix the atlas  $\mathcal{A}$ , they are not unique. Indeed, we may modify a symbol  $\sigma_{\alpha}$  in (9.1) with a smoothing symbol on  $U_{\alpha}$ . Moreover, we observe that the integral kernel of  $\psi_{\alpha} \operatorname{Op}^{\mathbb{X}_{\alpha}, \chi_{\alpha}}(\sigma_{\alpha}) \psi_{\alpha}$  is supported near the diagonal, more precisely in  $\operatorname{supp} \chi_{\alpha} \cap \operatorname{supp}((x, y) \mapsto \psi_{\alpha}(x) \psi_{\alpha}(y))$ . As the integral kernel of  $P$  may not be supported near the diagonal, the presence of the smoothing operator  $R$  is essential for the characterisation of  $\Psi(M)$  in terms of the local quantization.

*Proof of Proposition 9.9.* If  $P$  is of the form (9.1), then  $P \in \Psi^m(M)$  by Lemma 9.4. Let us prove the converse. Let  $P \in \Psi^m(M)$ . We denote by  $\sigma_{\alpha}$  the symbol in  $S^m(\widehat{GM}|_{U_{\alpha}})$  obtained from Part (1) of Definition 9.1 for the open set  $U_{\alpha}$ . We start the analysis with writing  $P$  as

$$P = \sum_{\alpha, \alpha'} \psi_{\alpha'}^2 P \psi_{\alpha}^2 = \sum_{\alpha'} \psi_{\alpha'}^2 P \varphi_{\alpha'} + R_1$$

with

$$R_1 := \sum_{\substack{\alpha, \alpha' \\ U_{\alpha} \cap U_{\alpha'} = \emptyset}} \psi_{\alpha'}^2 P \psi_{\alpha}^2, \quad \text{and} \quad \varphi_{\alpha'} := \sum_{\alpha: U_{\alpha} \cap U_{\alpha'} \neq \emptyset} \psi_{\alpha}^2 \in C_c^\infty(M).$$

Part (2) of Definition 9.1 implies that  $R_1$  is smoothing. We now write the sum over  $\alpha'$  as

$$\sum_{\alpha'} \psi_{\alpha'}^2 P \varphi_{\alpha'} = \sum_{\alpha'} \psi_{\alpha'}^2 P \psi_{\alpha'}^2 \varphi_{\alpha'} + R_2$$

where  $R_2$  is the smoothing operator given by:

$$R_2 := \sum_{\alpha'} \psi_{\alpha'}^2 P (1 - \psi_{\alpha'}^2) \varphi_{\alpha'}.$$

We check readily that the symbol

$$\sigma'_{\alpha'} := \sum_{\alpha, \alpha'} \psi_{\alpha'} (\sigma_{\alpha'} \diamond_{\mathbb{X}_{\alpha'}, \chi_{\alpha'}} \psi_{\alpha} \varphi_{\alpha'})$$

is in  $S^m(\widehat{GM}|_{U_{\alpha}})$  and that we have

$$\sum_{\alpha'} \psi_{\alpha'} \operatorname{Op}^{\mathbb{X}_{\alpha'}, \chi_{\alpha'}}(\sigma'_{\alpha'}) \psi_{\alpha'} = \sum_{\alpha'} \psi_{\alpha'}^2 \operatorname{Op}^{\mathbb{X}_{\alpha'}, \chi_{\alpha'}}(\sigma_{\alpha'}) \psi_{\alpha'}^2 \varphi_{\alpha'} = \sum_{\alpha'} \psi_{\alpha'}^2 P \psi_{\alpha'}^2 \varphi_{\alpha'},$$

concluding the proof.  $\square$

This characterisation together with the properties of the local quantization imply that pseudodifferential operators of non-negative order act continuously on  $L^2_{loc}(M)$ :

**Corollary 9.10.** *If  $P \in \Psi^m(M)$  with  $m \in (-\infty, 0] \cup \{-\infty\}$ , then  $P$  acts continuously on the space  $L^2_{loc}(M)$  (see (7.4) for a precise definition).*

*Proof.* Consider the case of  $P \in \Psi^{-\infty}(M)$  smoothing. This means that its integral kernel  $K(x, y)$  is a smooth function in  $x, y \in M$ . This implies that  $P$  acts continuously on  $L^2_{loc}(M)$  for instance by the Cauchy-Schwartz inequality. The case of  $m \in (-\infty, 0]$  follows from the characterisation in Proposition 9.9 together with the  $L^2$ -boundedness (see Theorem 7.4).  $\square$

**9.2. Properly supported pseudodifferential operators and composition.** Already in the traditional case of a trivial filtration on a manifold  $M$ , two pseudodifferential operators may not be composable as they are maps between functional spaces which are not compatible with composition; for instance, one of them is an operator  $C_c^\infty(M) \rightarrow C^\infty(M)$  while the other is  $\mathcal{E}'(M) \rightarrow \mathcal{D}'(M)$ . The same technical obstruction appears here, and we adapt the usual line of approach to solve this issue by considering properly supported pseudodifferential operators.

9.2.1. *Properly supported pseudodifferential operators.*

**Definition 9.11.** A pseudodifferential operator  $P$  is said to be *properly supported* when its integral kernel  $K \in \mathcal{D}'(M \times M)$  is properly supported or, in other words, when the two projection maps  $\text{supp } K \rightarrow M$  are proper (i.e. preimages of compact sets are compact).

With Definition 9.11 in mind together with Lemma 9.3, it follows readily that a properly supported pseudodifferential operator on  $M$  acts on  $C_c^\infty(M)$ :

**Lemma 9.12.** *If  $P$  is a properly supported pseudodifferential operator on  $M$ , then  $P$  is a linear continuous operator  $C_c^\infty(M) \rightarrow C_c^\infty(M)$ ,  $\mathcal{D}'(M) \rightarrow \mathcal{D}'(M)$  and  $\mathcal{E}'(M) \rightarrow \mathcal{E}'(M)$ . If  $P$  is moreover smoothing, then  $P$  is also a linear continuous operator  $\mathcal{E}'(M) \rightarrow C_c^\infty(M)$ .*

Consequently, properly supported pseudodifferential operators on  $M$  are composable with other pseudifferential operators, and we will study their composition soon. However, one drawback of considering properly supported pseudodifferential operators is that, by definition, their integral kernels are supported close to the diagonal. Hence, unless the manifold is compact, they comprise a strictly smaller subspace of the space of pseudodifferential operators.

Here are some fundamental examples of properly supported pseudodifferential operators:

- Example 9.13.*
- (1) If the manifold  $M$  is compact, then any pseudodifferential operator is properly supported.
  - (2) If  $P \in \Psi^m(M)$  and if  $\varphi, \psi \in C_c^\infty(M)$ , then  $\psi P \varphi \in \Psi^m(M)$  is properly supported.
  - (3) Whether  $M$  is compact or not, any differential operator is a properly supported pseudodifferential operator by Lemma 9.7.

- (4) Considering an atlas  $\mathcal{A} = (U_\alpha, \mathbb{X}_\alpha, \chi_\alpha, \psi_\alpha)_{\alpha \in A}$  for the filtered manifold  $M$  and a family of symbols  $(\sigma_\alpha)_{\alpha \in A}$  with  $\sigma_\alpha \in S^m(\widehat{GM}|_{U_\alpha})$ , then the pseudodifferential operator

$$\sum_{\alpha} \psi_\alpha \text{Op}^{\mathbb{X}_\alpha, \chi_\alpha}(\sigma_\alpha) \psi_\alpha \in \Psi^m(M)$$

is properly supported.

As a consequence of the last example and the characterisation of pseudodifferential operators on  $M$  in Proposition 9.9, any pseudodifferential operator on  $M$  is properly supported modulo a smoothing operator:

**Lemma 9.14.** *For any  $P \in \Psi^m(M)$ , there exists a properly supported pseudodifferential operator  $P' \in \Psi^m(M)$  and a smoothing operator  $R \in \Psi^{-\infty}(M)$  such that  $P = P' + R$ .*

9.2.2. *Composition of pseudodifferential operators.* We can now consider the composition of pseudodifferential operators on the filtered manifold  $M$ :

**Theorem 9.15.** *Let  $m_1, m_2 \in \mathbb{R} \cup \{-\infty\}$ . Let  $P_1 \in \Psi^{m_1}(M)$  and  $P_2 \in \Psi^{m_2}(M)$ . If one of them is properly supported then their composition  $P_1 P_2$  makes sense and is a pseudodifferential operator  $P_1 P_2 \in \Psi^{m_1+m_2}(M)$ . If they are both properly supported then so is  $P_1 P_2$ .*

In particular, on a compact filtered manifold  $M$ ,  $\Psi^{m_1}(M) \circ \Psi^{m_2}(M) \subset \Psi^{m_1+m_2}(M)$ .

The proof will use the following observations.

- Lemma 9.16.** (1) *If  $P \in \Psi^m(M)$  with  $m \in \mathbb{R} \cup \{-\infty\}$  is properly supported, then  $P^* \in \Psi^m(M)$  is also properly supported.*  
(2) *Let  $P \in \Psi^m(M)$  with*

$$P = R + \sum_{\alpha} \psi_\alpha \text{Op}^{\mathbb{X}_\alpha, \chi_\alpha}(\sigma_\alpha) \psi_\alpha.$$

*for  $R \in \Psi^{-\infty}(M)$ , an atlas  $\mathcal{A} = (U_\alpha, \mathbb{X}_\alpha, \chi_\alpha, \psi_\alpha)_{\alpha \in A}$  and a family of symbols  $(\sigma_\alpha)_{\alpha \in A}$  with  $\sigma_\alpha \in S^m(\widehat{GM}|_{U_\alpha})$ . If  $P$  is properly supported, so it is for  $R$ .*

*Proof of Lemma 9.16.* Part (1) follows readily from Lemma 9.3 and its proof, while Part (2) comes from Example 9.13 (2).  $\square$

*Proof of Theorem 9.15.* Lemmata 9.3 and 9.16 (1) allow us to assume  $P_2$  properly supported. Then  $P_1 P_2$  makes sense and is a continuous linear operator  $C_c^\infty(M) \rightarrow C^\infty(M)$ . Denoting by  $K, K_1, K_2$  the integral kernels of  $P_2 P_1, P_1$  and  $P_2$ , we have in the sense of distribution:

$$(9.2) \quad K(x, y) = \int_M K_1(x, z) K_2(z, y) dz.$$

Lemma 9.16 (2) and the vector space structure of  $\Psi(M)$  imply that it suffices to consider the three following cases:

- (a)  $P_2 \in \Psi^{-\infty}(M)$  (i.e.  $K_2$  smooth),
- (b)  $P_1 \in \Psi^{-\infty}(M)$  (i.e.  $K_1$  smooth) and  $P_2$  given by a local quantization, i.e. of the form  $P_2 = \psi \text{Op}^{\mathbb{X}, \chi}(\sigma) \psi$ , and

- (c) when both  $P_1$  and  $P_2$  are properly supported pseudodifferential operators and given by a local quantization.

Let us analyse these three cases.

(a) If  $P_2$  is also smoothing (in addition of being properly supported), then  $K_2$  is smooth and properly supported, and for fixed  $y \in M$ ,  $K$  is the image of the compactly supported function  $z \mapsto K(z, y)$  by  $P_1$ , and thus smooth since  $P_1$  is a pseudodifferential operator. In this case, we obtain  $P_1 P_2 \in \Psi^{-\infty}(M)$ .

(b) If  $P_1 \in \Psi^{-\infty}(M)$  and

$$P_2 = \psi \operatorname{Op}^{\mathbb{X}, \chi}(\sigma) \psi,$$

where  $\mathbb{X}$  is an adapted frame on an open subset  $U \subset M$ ,  $\chi$  is a  $\mathbb{X}$ -cut-off,  $\psi \in C_c^\infty(U)$  and  $\sigma \in S^m(\widehat{GM}|_U)$ . Denoting by  $\tilde{K}_2$  the integral kernel of  $\operatorname{Op}^{\mathbb{X}, \chi}(\sigma)$ , we have in the sense of distribution

$$K_2(z, y) = \psi(z) \tilde{K}_2(z, y) \psi(y).$$

Hence, the expression in (9.2) shows that  $K$  is smooth. In this case, again,  $P_1 P_2 \in \Psi^{-\infty}(M)$ .

(c) We now consider the case of  $m_1, m_2$  finite and  $P_1, P_2$ , given for  $j = 1, 2$  by

$$(9.3) \quad P_j = \psi_j \operatorname{Op}^{\mathbb{X}_j, \chi_j}(\sigma_j) \psi_j,$$

where  $\mathbb{X}_j$  is an adapted frame on an open subset  $U_j \subset M$ ,  $\chi_j$  is a  $\mathbb{X}_j$ -cut-off,  $\psi_j \in C_c^\infty(U_j)$  and  $\sigma_j \in S^m(\widehat{GM}|_{U_j})$ . If  $\psi_1 \psi_2 = 0$ , then  $P_1 P_2 = 0$ . Hence, we may assume  $\psi_1 \psi_2 \neq 0$  and we fix a function  $\tilde{\psi} \in C_c^\infty(U_1 \cap U_2)$  such that  $\tilde{\psi} = 1$  on a neighbourhood of  $\operatorname{supp}(\psi_1 \psi_2)$ .

Let us show that we can write

$$(9.4) \quad \psi_1 \operatorname{Op}^{\mathbb{X}_1, \chi_1}(\sigma_1) \tilde{\psi}^2 = \psi_1 \operatorname{Op}^{\mathbb{X}_2, \chi_2}(\tilde{\psi} \sigma_1) \tilde{\psi} + \psi_1 \operatorname{Op}^{\mathbb{X}_2, \chi_2}(\rho) \tilde{\psi},$$

for some  $\rho \in S^{m_1-1}(\widehat{GM})$ . By the symbolic properties of the local quantization (see Proposition 8.13), there exists  $\rho_1 \in S^{m_1-1}(\widehat{GM})$

$$\psi_1 \operatorname{Op}^{\mathbb{X}_1, \chi_1}(\sigma_1) \tilde{\psi}^2 = \psi_1 \operatorname{Op}^{\mathbb{X}_1, \chi_1}(\tilde{\psi} \sigma_1 + \rho_1) \tilde{\psi}$$

By the properties of invariance of the local quantization for the frame and the cut-off (see Propositions 7.7 and 7.5), there exists  $\rho'_1 \in S^{m_1-1}(\widehat{GM})$  such that

$$\psi_1 \operatorname{Op}^{\mathbb{X}_1, \chi_1}(\tilde{\psi} \sigma_1) \tilde{\psi} = \psi_1 \operatorname{Op}^{\mathbb{X}_2, \chi_2}(\tilde{\psi} \sigma_1 + \rho'_1) \tilde{\psi}.$$

This implies (9.4).

Using (9.4) in the expression for  $P_1 P_2$ , we obtain

$$\begin{aligned} P_1 P_2 &= \psi_1 \operatorname{Op}^{\mathbb{X}_1, \chi_1}(\sigma_1) \tilde{\psi}^2 \psi_1 \psi_2 \operatorname{Op}^{\mathbb{X}_2, \chi_2}(\sigma_2) \psi_2 \\ &= \left( \psi_1 \operatorname{Op}^{\mathbb{X}_2, \chi_2}(\tilde{\psi} \sigma_1) \tilde{\psi} + \psi_1 \operatorname{Op}^{\mathbb{X}_2, \chi_2}(\rho) \tilde{\psi} \right) \psi_1 \psi_2 \operatorname{Op}^{\mathbb{X}_2, \chi_2}(\tilde{\psi} \sigma_2) \psi_2 \\ &= \psi_1 \operatorname{Op}^{\mathbb{X}_2, \chi_2}((\tilde{\psi} \sigma_1) \diamond_{\mathbb{X}_2, \chi_2} (\psi_1 \psi_2 \sigma_2)) \psi_2 + \psi_1 \operatorname{Op}^{\mathbb{X}_2, \chi_2}(\sigma_1 \diamond_{\mathbb{X}_2, \chi_2} \psi_1 \psi_2 \rho) \psi_2. \end{aligned}$$

By the properties of composition in Proposition 8.13 and Lemma 9.4, the first term in the sum of the right-hand side is in  $\Psi^{m_1+m_2}(M)$  while the second is in  $\Psi^{m_1+m_2-1}(M)$ . This implies the case of two operators  $P_1, P_2$  of the form (9.3).

The case where they are both properly supported follows readily from (9.2).  $\square$

### 9.2.3. The global quantization $\text{Op}^{\mathcal{A}}$ .

**Lemma 9.17.** *We fix an atlas  $\mathcal{A} = (U_\alpha, \mathbb{X}_\alpha, \chi_\alpha, \psi_\alpha)_{\alpha \in A}$  for the filtered manifold  $M$ . The operators defined via*

$$(9.5) \quad \text{Op}^{\mathcal{A}}(\sigma) := \sum_{\alpha} \psi_{\alpha} \text{Op}^{\mathbb{X}_{\alpha}, \chi_{\alpha}}(\sigma) \psi_{\alpha}, \quad \sigma \in \cup_{m \in \mathbb{R} \cup \{-\infty\}} S^m(\widehat{GM}),$$

are properly supported pseudodifferential. More precisely, we have for any  $m \in \mathbb{R} \cup \{-\infty\}$ ,

$$\forall \sigma \in S^m(\widehat{GM}), \quad \text{Op}^{\mathcal{A}}(\sigma) \in \Psi^m(M).$$

Moreover, it satisfies the following two properties:

- (1) If  $\sigma \in \cup_{m' \in \mathbb{R} \cup \{-\infty\}} S^{m'}(\widehat{GM})$  satisfies  $\text{Op}^{\mathcal{A}}(\sigma) \in \Psi^m(M)$ , then  $\sigma \in S^m(\widehat{GM})$ .
- (2) If  $\sigma$  and  $\sigma'$  are two symbols in  $S^m(\widehat{GM})$  such that  $\text{Op}^{\mathcal{A}}(\sigma) = \text{Op}^{\mathcal{A}}(\sigma')$ , then  $\sigma - \sigma' \in S^{-\infty}(\widehat{GM})$  is smoothing.

*Proof.* From Example 9.13 (4), it follows readily that  $\text{Op}^{\mathcal{A}}(\sigma) \in \Psi^m(M)$  for any  $\sigma \in S^m(\widehat{GM})$ . Part (1) is proved by proceeding as in the proof of Lemma 9.4 (2). Part (2) follows.  $\square$

The class of operators  $\text{Op}^{\mathcal{A}}(S^m(\widehat{GM}))$ ,  $m \in \mathbb{R}$ , enjoy a symbolic calculus modulo lower order terms as a consequence of the proof of Theorem 9.15 and the consideration on the adjoint.

**Corollary 9.18.** *Let  $\mathcal{A} = (U_\alpha, \mathbb{X}_\alpha, \chi_\alpha, \psi_\alpha)_{\alpha \in A}$  be an atlas for  $M$  and let  $\text{Op}^{\mathcal{A}}$  be its associated global quantization given via (9.5). Let  $m_1, m_2, m \in \mathbb{R} \cup \{-\infty\}$ .*

- (1) *If  $\sigma \in S^m(\widehat{GM})$  then*

$$(\text{Op}^{\mathcal{A}}(\sigma))^* = \text{Op}^{\mathcal{A}}(\sigma^{(*)}) = \text{Op}^{\mathcal{A}}(\sigma^*) + R,$$

where  $R \in \Psi^{m-1}(M)$ .

- (2) *If  $\sigma_1 \in S^{m_1}(\widehat{GM})$  and  $\sigma_2 \in S^{m_2}(\widehat{GM})$ , then*

$$\text{Op}^{\mathcal{A}}(\sigma_1)\text{Op}^{\mathcal{A}}(\sigma_2) = \text{Op}^{\mathcal{A}}(\sigma_1\sigma_2) + S,$$

where  $S \in \Psi^{m_1+m_2-1}(M)$ .

*Proof of Corollary 9.18.* We check readily the properties of the adjoint. For the composition, we first observe that continuing with the proof of Theorem 9.15 Case (c), we have in  $S^{m_1+m_2}(\widehat{GM}|_{U_2})$

$$(\tilde{\psi}\sigma_1) \diamond_{\mathbb{X}_2, \chi_2} (\psi_1\psi_2\sigma_2) = \psi_1^2\psi_2\sigma_1\sigma_2 + \rho',$$

for some  $\rho' \in S^{m_1+m_2-1}(\widehat{GM}|_{U_2})$ . Therefore, we have obtained:

$$\psi_1 \text{Op}^{\mathbb{X}_1, \chi_1}(\sigma_1) \psi_1 \psi_2 \text{Op}^{\mathbb{X}_2, \chi_2}(\sigma_2) \psi_2 = \psi_2 \text{Op}^{\mathbb{X}_2, \chi_2}(\psi_1^2 \sigma_1 \sigma_2) \psi_2 + R_{\psi_1, \psi_2}$$

where  $R_{\psi_1, \psi_2} \in \Psi^{m_1+m_2-1}(M)$  is given by

$$R_{\psi_1, \psi_2} = \psi_1 \text{Op}^{\mathbb{X}_2, \chi_2}(\rho') \psi_2 + \psi_1 \text{Op}^{\mathbb{X}_2, \chi_2}(\sigma_1 \diamond_{\mathbb{X}_2, \chi_2} \psi_1 \psi_2 \rho) \psi_2.$$

Performing this for every  $\psi_{\alpha_1}, \psi_{\alpha_2}$ , we have

$$\begin{aligned} \text{Op}^A(\sigma_1)\text{Op}^A(\sigma_2) &= \sum_{\alpha_1, \alpha_2} \psi_{\alpha_1} \text{Op}^{\mathbb{X}_{\alpha_1}, \chi_{\alpha_1}}(\sigma) \psi_{\alpha_1} \psi_{\alpha_2} \text{Op}^{\mathbb{X}_{\alpha_2}, \chi_{\alpha_2}}(\sigma) \psi_{\alpha_2} \\ &= \sum_{\alpha_2, \alpha_1} \psi_{\alpha_2} \text{Op}^{\mathbb{X}_2, \chi_2}(\psi_{\alpha_1}^2 \sigma_1 \sigma_2) \psi_{\alpha_2} + S, \end{aligned}$$

with  $\text{Op}^A(\sigma_1 \sigma_2)$  for the first term since  $\sum_{\alpha_1} \psi_{\alpha_1}^2 = 1$ , and for the second term,

$$S := \sum_{\alpha_1, \alpha_2} R_{\psi_{\alpha_1}, \psi_{\alpha_2}}.$$

We check readily  $S \in \Psi^{m_1+m_2-1}(M)$  and this concludes the proof.  $\square$

**9.3. Principal Symbol.** We start by defining a local notion of principal symbol, attached to open sets small enough to be included into open sets endowed with a local frame. Then, in a second step, we will define a global notion of principal symbol.

*9.3.1. Local principal symbol.* We introduce the following notation:

**Definition 9.19.** Let  $m \in \mathbb{R}$ , let  $P \in \Psi^m(M)$  and let  $V_0$  be an open subset of  $M$  small enough to be included in an open set equipped with adapted frames. Then we denote by

$$\text{Symb}_{m, V_0}(P) \in S^m(\widehat{GM}|_{V_0})/S^{m-1}(\widehat{GM}|_{V_0})$$

the class of symbols modulo  $S^{m-1}(\widehat{GM}|_{V_0})$  of  $\sigma|_{V_0}$ :

$$\text{Symb}_{m, V_0}(P) := \sigma|_{V_0} \text{ mod } S^{m-1}(\widehat{GM}|_{V_0}),$$

where  $\sigma|_{V_0} \in S^m(\widehat{GM}|_{V_0})$  is the restriction to  $V_0$  of a symbol  $\sigma \in S^m(\widehat{GM})$  satisfying

$$(9.6) \quad \phi P \phi = \text{Op}^{\mathbb{X}, \chi}(\phi \sigma) \phi,$$

for some adapted frame  $\mathbb{X}$  on an open set  $U$  with  $V_0 \subset U \subset M$ ,  $\mathbb{X}$ -cut-off  $\chi$  and function  $\phi \in C_c^\infty(U)$  with  $\phi = 1$  on a neighbourhood of  $V_0$ .

For  $P \in \Psi^m(M)$ , the existence of symbols  $\sigma$  as in Definition 9.19 is guaranteed by (1) of Definition 9.1. The fact that it leads to a well-defined and unique class of symbol follows from the following property:

**Lemma 9.20.** *Let  $m \in \mathbb{R}$  and  $P \in \Psi^m(M)$ . Let  $V$  be an open subset of an open set  $U \subset M$  admitting adapted frames. Let  $\phi, \phi' \in C_c^\infty(U)$  be two functions such that  $\phi = \phi' = 1$  on  $\overline{V}$ , let  $\mathbb{X}, \mathbb{X}'$  be two adapted frames on  $U$ , let  $\chi, \chi'$  be two cut-offs for  $\mathbb{X}$  and  $\mathbb{X}'$ . If  $\sigma$  and  $\sigma'$  are symbols in  $S^m(\widehat{GM})$  satisfying (9.6) for  $\phi, \mathbb{X}, \chi$  and  $\phi', \mathbb{X}', \chi'$  respectively, then  $(\phi' \phi)^2 (\sigma - \sigma') \in S^{m-1}(\widehat{GM})$ .*

The proof of Lemma 9.20 relies mainly on the invariances of the local quantization stated in Section 7.3.

*Proof of Lemma 9.20.* We keep the notation of the statement and we have

$$0 = \phi' \phi P \phi \phi' - \phi \phi' P \phi' \phi = \phi' \phi \text{Op}^{\mathbb{X}, \chi}(\sigma) \phi \phi' - \phi \phi' \text{Op}^{\mathbb{X}', \chi'}(\sigma') \phi' \phi.$$

By invariance of the local quantization in frames and cut-offs (see Section 7.3), we have

$$\text{Op}^{\mathbb{X}', \chi'}(\sigma') = \text{Op}^{\mathbb{X}, \chi}(\sigma' + \rho),$$

for some  $\rho \in S^{m-1}(\widehat{GM}|_U)$ . The equality above implies

$$\text{Op}^{\mathbb{X}, \chi}((\phi' \phi (\sigma - \sigma' - \rho)) \diamond_{\mathbb{X}, \chi} (\phi \phi')) = 0,$$

so by Lemma 9.4 (2) applied to the smoothing operator 0,

$$(\phi' \phi (\sigma - \sigma' - \rho)) \diamond_{\mathbb{X}, \chi} (\phi \phi') \in S^{-\infty}(\widehat{GM}|_U).$$

Therefore,

$$\begin{aligned} 0 &= (\phi' \phi (\sigma - \sigma' - \rho)) \diamond_{\mathbb{X}, \chi} (\phi \phi') \text{ mod } S^{m-1}(\widehat{GM}|_U) \\ &= (\phi' \phi)^2 (\sigma - \sigma' - \rho) \text{ mod } S^{m-1}(\widehat{GM}|_U) \\ &= (\phi' \phi)^2 (\sigma - \sigma') \text{ mod } S^{m-1}(\widehat{GM}|_U), \end{aligned}$$

by the symbolic properties of the local quantization. This concludes the proof.  $\square$

The local principal symbol enjoys the following properties:

**Lemma 9.21.** *Let  $V_0$  be an open subset of  $M$  small enough to be included in an open set equipped with adapted frames.*

(1) *Let  $m \in \mathbb{R}$ . The map*

$$\text{Symb}_{m, V_0}: P \longmapsto \text{Symb}_{m, V_0}(P), \quad \Psi^m(M) \longrightarrow S^m(\widehat{GM}|_{V_0})/S^{m-1}(\widehat{GM}|_{V_0}),$$

*is linear and vanishes on  $\Psi^{m-1}(M)$ .*

(2) *Let  $\sigma \in S^m(\widehat{GM})$  with  $m \in \mathbb{R}$ . Let  $\mathbb{X}$  be an adapted frame on an open set  $U \subset M$ , let  $\chi$  be a  $\mathbb{X}$ -cut-off and let  $\psi \in C_c^\infty(U)$ . Then*

$$\text{Symb}_{m, V_0}(\text{Op}^{\mathbb{X}, \chi}(\sigma) \psi) = (\psi \sigma)|_{V_0} \text{ mod } S^{m-1}(\widehat{GM}|_{V_0}).$$

(3) *For any  $P \in \Psi(M)$ , we have*

$$\text{Symb}_{m, V_0}(P^*) = (\text{Symb}_{m, V_0}(P))^*.$$

*Proof of Lemma 9.21.* By construction, Parts (1) and (3) hold and we have for Part (2),

$$\text{Symb}_{m, V_0}(\text{Op}^{\mathbb{X}, \chi}(\sigma) \psi) = \sigma \diamond \psi|_{V_0} \text{ mod } S^{m-1}(\widehat{GM}|_{V_0}).$$

By the symbolic calculus, we have

$$\sigma \diamond \psi = \psi \sigma \text{ mod } S^{m-1}(\widehat{GM}|_U).$$

Part (2) follows.  $\square$

9.3.2. *Principal symbol.* We can now define the notion of principal symbol.

**Theorem 9.22.** *Let  $m \in \mathbb{R}$ . Let  $P \in \Psi^m(M)$ . There exists a unique element*

$$\text{Princ}_m(P) \in S^m(\widehat{GM})/S^{m-1}(\widehat{GM})$$

*such that for any open subset  $V_0$  of  $M$  small enough to be included in an open set equipped with adapted frames,  $\text{Symb}_{m,V_0}(P)$  is the class obtained by restriction to  $V_0$  of any representative of  $\text{Princ}_m(P)$ , i.e.*

$$(9.7) \quad \text{Symb}_{m,V_0}(P) = \sigma|_{V_0} \text{ mod } S^{m-1}(\widehat{GM}|_{V_0})$$

*for any  $\sigma \in S^m(\widehat{GM})$  satisfying  $\sigma \text{ mod } S^{m-1}(\widehat{GM}) = \text{Princ}_m(P)$ .*

*This defines a linear map*

$$\text{Princ}_m : \Psi^m(M) \rightarrow S^m(\widehat{GM})/S^{m-1}(\widehat{GM})$$

*with the following properties:*

- (1) *The map  $\text{Princ}_m$  is surjective.*
- (2)  *$\ker \text{Princ}_m = \Psi^{m-1}(M)$ .*
- (3) *The adjoint is naturally defined on  $S^m(\widehat{GM})/S^{m-1}(\widehat{GM})$  and we have*

$$\forall P \in \Psi(M), \quad \text{Princ}_m(P^*) = (\text{Princ}_m(P))^*.$$

- (4) *When  $m = N \in \mathbb{N}_0$ , the restriction of  $\text{Princ}_m$  to the space  $\text{DO}^{\leq N}$  of differential operators of homogeneous order at most  $N$  coincides with the map also denoted by  $\text{Princ}_N$  and defined in Section 3.4:*

$$\text{Princ}_N|_{\text{DO}^{\leq N}} = \text{Princ}_N : \text{DO}^{\leq N} \longrightarrow \Gamma(\mathcal{U}_N(\mathfrak{g}M))$$

**Definition 9.23.** Let  $P \in \Psi^m(M)$ . The class of symbols  $\text{Princ}_m(P)$  is called the *principal symbol* of  $P$  at order  $m$ .

The proof relies on the characterisation of pseudodifferential operators stated in Proposition 9.9 and the properties of  $\text{Symb}_{m,V_0}(P)$  given in Lemma 9.21.

*Proof of Theorem 9.22.* We fix an atlas  $\mathcal{A} = (U_\alpha, \mathbb{X}_\alpha, \chi_\alpha, \psi_\alpha)_{\alpha \in A}$  on  $M$ . Let  $P \in \Psi^m(M)$ . By Proposition 9.9, there exist a family of symbols  $(\sigma_\alpha)_{\alpha \in A}$  with  $\sigma \in S^m(\widehat{GM}|_{U_\alpha})$ , and a smoothing operator  $R \in \Psi^{-\infty}(M)$  such that (9.1) holds. We check readily that the symbol given via

$$\sigma_P := \sum_{\alpha} \psi_\alpha^2 \sigma_\alpha$$

is well defined and in  $S^m(\widehat{GM})$ . Moreover, by Lemma 9.21 (1) and (2), it satisfies for any open subset  $V_0 \subset M$  small enough,

$$\text{Symb}_{m,V_0}(P) = \sigma_P|_{V_0} \text{ mod } S^{m-1}(\widehat{GM}|_{V_0}).$$

Consequently,  $\sigma_P$  represents a class satisfying (9.7) for  $P$ .

By definition, any two symbols in  $S^m(\widehat{GM})$  satisfying (9.7) for  $P$  must coincide modulo  $S^{m-1}(\widehat{GM}|_{V_0})$  on any open subset  $V_0 \subset M$  small enough; this shows the uniqueness of the

class satisfying (9.7). Hence  $\text{Princ}_m$  is a well defined map. Lemma 9.21 also implies that it is linear, with kernel containing  $\Psi^{m-1}(M)$  and respecting the adjoint, i.e. Part (3) holds.

Recall that for the above family  $(\sigma_\alpha)_{\alpha \in A}$ ,  $\sigma_\alpha$  may be obtained in the proof of Proposition 9.9 from Part (1) of Definition 9.1 for the open set  $U_\alpha$ . Hence, if  $P \in \ker \text{Princ}_m$ , then by (9.7),  $\sigma_\alpha \in S^{m-1}(\widehat{GM}|_{U_\alpha})$ , and (9.1) implies  $P \in \Psi^{m-1}(M)$ . This shows  $\ker \text{Princ}_m = \Psi^{m-1}(M)$ , i.e. Part (2) holds.

If  $\sigma \in S^m(\widehat{GM})$ , we check readily that

$$\text{Princ}_m(\text{Op}^A(\sigma)) = \sigma \bmod S^{m-1}(\widehat{GM})$$

where  $\text{Op}^A(\sigma) \in \Psi^m(M)$  is defined via (9.5). This shows the surjectivity of  $\text{Princ}_m$ , i.e. Part (1) holds.

Let us prove Part (4). Let  $P$  be a differential operator on  $M$ . Denote by  $N$  its homogeneous degree. Choose a symbol  $\sigma$  representing the class in  $S^N(\widehat{GM})$  for the principal part of  $P$  in the sense of Section 3.4. The considerations in that section (in fact, the very definition of principal part in Section 3.4, see also the proof of Lemma 9.7) imply for any open subset  $V_0 \subset M$  small enough

$$\text{Symb}_{N, V_0}(P) = \sigma|_{V_0} \bmod S^{N-1}(\widehat{GM}|_{V_0}).$$

Consequently,

$$\text{Princ}_N P = \sigma \bmod S^{N-1}(\widehat{GM}).$$

This shows Part (4), and concludes the proof of Theorem 9.22.  $\square$

Theorem 9.22 and its proof imply readily the following relation between principal symbols and global quantization (see Section 9.2.3 for the latter):

**Corollary 9.24.** *We fix an atlas  $\mathcal{A} = (U_\alpha, \mathbb{X}_\alpha, \chi_\alpha, \psi_\alpha)_{\alpha \in A}$  for  $M$ . For any  $P \in \Psi^m(M)$ , if  $\sigma \in S^m(\widehat{GM})$  is a representative of  $\text{Princ}_m(P)$ , then*

$$P = \text{Op}^A(\sigma) \bmod \Psi^{m-1}(M).$$

Hence, when considering principal symbols, that is, when working mod  $\Psi^{m-1}(M)$ , we may assume that the pseudodifferential operators are given by the global quantization  $\text{Op}^A(\sigma)$  of a symbol  $\sigma \in S^m(\widehat{GM})$  on  $M$ .

When considering pseudodifferential operators modulo smoothing operators, we may assume that the pseudodifferential operators are given by the global quantization:

**Corollary 9.25.** *We fix an atlas  $\mathcal{A} = (U_\alpha, \mathbb{X}_\alpha, \chi_\alpha, \psi_\alpha)_{\alpha \in A}$  for  $M$ . Let  $m \in \mathbb{R}$ . For any  $P \in \Psi^m(M)$ , there exist a symbol  $\sigma \in S^m(\widehat{GM})$  and a smoothing operator  $R \in \Psi^{-\infty}(M)$  such that*

$$P = \text{Op}^A(\sigma) + R.$$

Moreover, the symbol  $\sigma \in S^m(\widehat{GM})$  is unique modulo  $S^{-\infty}(\widehat{GM})$ .

*Proof.* Using Corollary 9.24, we construct inductively  $\sigma_0, \dots, \sigma_N$  in  $S^m(\widehat{GM}), \dots, S^{m-N}(\widehat{GM})$  such that

$$P - \text{Op}^A(\sigma_0 + \dots + \sigma_N) \in \Psi^{m-N-1}(M),$$

with  $\sigma_{N+1} \in S^{m-(N+1)}(\widehat{GM})$  being the representative of the principal symbol of the above operator. Let  $\sigma \in S^m(\widehat{GM})$  be a symbol with asymptotic  $\sigma_0 + \dots + \sigma_N + \dots$ . By construction,  $P - \text{Op}^A(\sigma) \in \Psi^{-\infty}(M)$ . The uniqueness of the symbol modulo smoothing symbols follows from Lemma 9.17 (2).  $\square$

9.3.3. *Examples of principal symbols.* Our main examples of pseudodifferential operators in Section 9.1.2 were differential operators and operators obtained via local quantization (see Corollary 9.5). The principal parts of the former are described in Part (4) above. For the latter, we have:

**Corollary 9.26.** *For any  $\mathbb{X}$  on an open subset  $U \subset M$  and  $\mathbb{X}$ -cut-off  $\chi$ , we have for any  $m \in \mathbb{R}$  and  $\sigma \in S^m(\widehat{GM})$ , and for any  $\psi \in C_c^\infty(U)$ ,*

$$\text{Princ}_m(\text{Op}^{\mathbb{X},\chi}(\sigma)\psi) = \psi\sigma \text{ mod } S^{m-1}(\widehat{GM}).$$

*Proof.* This follows from Lemma 9.21 (2) and Theorem 9.22.  $\square$

9.3.4. *Principal symbols and composition.* The next proposition states that principal symbols respect composition of operators and symbols modulo lower order terms. The sense of the composition of symbols modulo lower order terms comes from the following observation: the properties of composition between symbols (Proposition 5.25) implies readily that the following map

$$\left( \sigma_1 \text{ mod } S^{m_1-1}(\widehat{GM}), \sigma_2 \text{ mod } S^{m_2-1}(\widehat{GM}) \right) \longmapsto \sigma_1\sigma_2 \text{ mod } S^{m_1+m_2-1}(\widehat{GM}),$$

is well defined and continuous  $\frac{S^{m_1}(\widehat{GM})}{S^{m_1-1}(\widehat{GM})} \times \frac{S^{m_2}(\widehat{GM})}{S^{m_2-1}(\widehat{GM})} \rightarrow \frac{S^{m_1+m_2}(\widehat{GM})}{S^{m_1+m_2-1}(\widehat{GM})}$ . We have:

**Proposition 9.27.** *Let  $m_1, m_2 \in \mathbb{R}$ . For any two pseudodifferential operators  $P_1 \in \Psi^{m_1}(M)$  and  $P_2 \in \Psi^{m_2}(M)$ , at least one of them being properly supported, we have*

$$\text{Princ}_{m_1+m_2}(P_1P_2) = \text{Princ}_{m_1}(P_1)\text{Princ}_{m_2}(P_2).$$

*Proof.* By Corollary 9.24, the property of composition of pseudodifferential operators (Theorem 9.15), and the properties of  $\text{Princ}_m$  in Theorem 9.22, it suffices to consider Case (c) in the proof of Theorem 9.15. With its setting, we have

$$\begin{aligned} \text{Princ}_{m_1+m_2}(P_1P_2) &= \text{Princ}_{m_1+m_2}(\psi_1 \text{Op}^{\mathbb{X}_1,\chi_1}(\sigma_1 \diamond (\psi_1\psi_2\sigma_2))\psi_2) \\ &= \psi_1\psi_2(\sigma_1 \diamond (\psi_1\psi_2\sigma_2)) \text{ mod } S^{m_1+m_2-1}(\widehat{GM}) \end{aligned}$$

by Corollary 9.26. By the properties of the symbolic calculus (Proposition 8.13), we have

$$\psi_1\psi_2(\sigma_1 \diamond (\psi_1\psi_2\sigma_2)) = \psi_1^2\psi_2^2\sigma_1\sigma_2 \text{ mod } S^{m_1+m_2-1}(\widehat{GM}),$$

concluding the proof.  $\square$

9.4. **Parametrics.** In this section, we discuss left parametrices, following mainly the presentation in [36]. First let us recall the notion of left parametrix:

**Definition 9.28.** An operator  $Q$  admits a *left parametrix* in  $\Psi^m(M)$  when  $Q \in \Psi^m(M)$  and there exists  $P \in \Psi^{-m}(M)$  properly supported such that  $PQ = I \bmod \Psi^{-\infty}(M)$ .

The existence of a left parametrix for an operator  $A$  is an important quality as it readily implies for instance hypoellipticity and subelliptic estimates in terms of the Sobolev spaces defined below. We will show below that we are able to construct left parametrices when principal symbols are invertible in a sense we now define. First, let us recall the notion of invertibility for high frequencies on a fixed graded nilpotent Lie group (see [36, Section 4.6]):

**Definition 9.29.** Let  $G$  be a graded nilpotent Lie group. A symbol  $\sigma \in S^m(\widehat{G})$  is *invertible for the frequencies of a positive Rockland symbol  $\widehat{\mathcal{R}}$  higher than  $\Lambda \in \mathbb{R}$*  (or just *invertible for high frequencies*) when for any  $\gamma \in \mathbb{R}$  and almost-every  $\pi \in \widehat{G}$ , we have

$$(9.8) \quad \forall v \in \mathcal{H}_{\pi, \mathcal{R}, \Lambda} \quad \|(I + \pi(\mathcal{R}))^{\frac{\gamma}{\nu}} \sigma(\pi) v\|_{\mathcal{H}_{\pi}} \geq C_{\gamma} \|(I + \pi(\mathcal{R}))^{-\frac{\gamma+m}{\nu}} v\|_{\mathcal{H}_{\pi}}$$

with  $C_{\gamma} = C_{\gamma, \sigma, G} > 0$  a constant independent of  $\pi$ . Above,  $\nu$  is the homogeneous degree of  $\mathcal{R}$  and  $\mathcal{H}_{\pi, \mathcal{R}, \Lambda}$  is the subspace  $\mathcal{H}_{\pi, \mathcal{R}, \Lambda} := E_{\pi}[\Lambda, \infty) \mathcal{H}_{\pi}$  of  $\mathcal{H}_{\pi}$  where  $E_{\pi}$  is the spectral decomposition of  $\pi(\mathcal{R}) = \int_{\mathbb{R}} \lambda dE_{\pi}(\lambda)$  as in (2.5).

*Remark 9.30.* (1) The definition seems to require a Plancherel measure to have been fixed on  $\widehat{G}$ . However, the choice of a Plancherel measure is in one-to-one correspondence with the choice of a Haar measure, and two choices would differ by a constant. Hence, if a symbol is invertible for the high frequency of a Rockland operator for one Plancherel measure, it is so for any.

(2) Another way of formulating the condition in (9.8) is the fact that the operator

$$(I + \pi(\mathcal{R}))^{\frac{\gamma}{\nu}} \sigma(\pi) E_{\pi}[\Lambda, \infty) (I + \pi(\mathcal{R}))^{-\frac{\gamma+m}{\nu}}$$

is invertible with a bounded inverse whose  $\mathcal{H}_{\pi}$ -operator norm is  $\leq C_{\gamma}^{-1}$ . In other words, with the notation in (2.6)

$$\|E_{\pi}[\Lambda, \infty) \sigma(\pi)^{-1}\|_{L^{\infty}_{\gamma, \gamma+m}(\widehat{G})} \leq C_{\gamma}^{-1}.$$

(3) As explained in [36, Section 4.6], Definition 9.29 is the same notion but for invariant symbols as [37, Definition 5.8.1], that was called ellipticity there. However, we prefer to use the vocabulary regarding ellipticity for polyhomogeneous symbols, as is customary in the traditional case of a trivially filtered manifold. In any case, the properties explained in [37, Section 8.1] hold. For instance, in Definition 9.29, it suffices to prove the estimate for a sequence of  $\gamma = \gamma_{\ell}$ ,  $\ell \in \mathbb{Z}$ , with  $\lim_{\ell \rightarrow \pm\infty} \gamma_{\ell} = \pm\infty$ . From the examples given in [37, Section 5.8], if  $\widehat{\mathcal{R}}$  is a positive Rockland symbol of homogeneous degree  $\nu$ , then  $\widehat{\mathcal{R}} \in S^{\nu}(\widehat{G})$  and  $(I + \widehat{\mathcal{R}})^{m/\nu} \in S^m(\widehat{G})$ , are operators with invertible symbols for the high frequencies of  $\widehat{\mathcal{R}}$ .

(4) Finally, let us point out that the invertibility we discuss here could be improved by localisation along ‘directions’ of the set  $\widehat{G}M$ . Following [29], these directions are

defined via the equivalence relation given by the dilations: for  $x \in M$ ,  $\pi_1, \pi_2 \in G_x M$ ,  $\pi_1 \sim \pi_2$  if and only if there exists  $r > 0$  such that for all  $w \in G_x M$  we have  $\pi_1(\delta_r w) = \pi_2(w)$ . Refined microlocalisations in conical neighbourhoods of the phase space  $\widehat{G}M$  could be pursued with these ideas in mind, at least on concrete examples.

We can now define the invertibility of a symbol on a filtered manifold.

**Definition 9.31.** A symbol  $\sigma \in S^m(\widehat{G}M)$  is *invertible for the high frequencies of a positive Rockland symbol*  $\widehat{\mathcal{R}}$  (see Definition 5.19) when at every  $x \in M$ , the symbol  $\sigma(x) \in S^m(\widehat{G}_x M)$  is invertible for frequencies higher than  $\Lambda_x \in \mathbb{R}$  of the Rockland symbol  $\widehat{\mathcal{R}}_x$ , with both  $\Lambda_x$  and the constant  $C_\gamma$  in (9.8) depending locally uniformly in  $x \in M$  (for one and then any smooth Haar system  $\mu = \{\mu_x\}_{x \in M}$ ).

*Example 9.32.* Let  $\widehat{\mathcal{R}}$  be a positive Rockland symbol. We denote its homogeneous degree  $\nu$ . For any  $m \in \mathbb{R}$ , the symbol  $(I + \widehat{\mathcal{R}})^{m/\nu} \in S^m(\widehat{G}M)$  is invertible for all the frequencies of  $\widehat{\mathcal{R}}$ .

We can now adapt the traditional construction of parametrices to our context under the following hypothesis:

**Theorem 9.33.** *Let  $Q \in \Psi^m(M)$  and  $\sigma \in S^m(\widehat{G}M)$  such that  $\text{Princ}_m Q = \sigma \text{ mod } S^{m-1}(\widehat{G}M)$ . If  $\sigma$  is invertible for the high frequencies of a positive Rockland symbol, then  $Q$  admits a left parametrix.*

The proof of Theorem 9.33 is classical and relies on the following statement:

**Lemma 9.34.** *Let  $\sigma \in S^m(\widehat{G}M)$  be invertible for the high frequencies of a positive Rockland symbol  $\widehat{\mathcal{R}}$ . Let us fix  $\psi \in C^\infty(M \times \mathbb{R}^n)$  be such that for every  $x \in \mathbb{R}$ , we have with  $\psi_x(\lambda) := \psi(x, \lambda)$ ,*

- $\psi_x(\lambda) = 1$  on a neighbourhood of  $+\infty$ ,
- $\psi_x(\lambda) = 0$  on a neighbourhood of  $[0, \Lambda_x]$  for some  $\Lambda_x$  as in Definition 9.31.

The symbol

$$\psi(\widehat{\mathcal{R}})\sigma^{-1} = \{\psi_x(\pi(\mathcal{R}_x))\sigma(x, \pi)^{-1} : x \in M, \pi \in \widehat{G}_x M\}$$

makes sense and is in  $S^{-m}(\widehat{G}M)$ . Moreover, it satisfies

$$\psi(\widehat{\mathcal{R}})\sigma^{-1} \sigma = \psi(\widehat{\mathcal{R}}) = I + \rho, \quad \text{with } \rho := (1 - \psi(\widehat{\mathcal{R}})) \in S^{-\infty}(\widehat{G}M).$$

*Proof of Lemma 9.34.* By Corollary 5.44,  $\psi(\widehat{\mathcal{R}}) \in S^0(\widehat{G}M)$  and  $(1 - \psi)(\widehat{\mathcal{R}}) \in S^{-\infty}(\widehat{G}M)$ . Moreover, for any adapted frame  $(\mathbb{X}, U)$  and any  $\beta \in \mathbb{N}_0^n$  with  $\beta \neq 0$ ,  $D_{\mathbb{X}}^\beta \psi(\widehat{\mathcal{R}}) = -D_{\mathbb{X}}^\beta (1 - \psi)(\widehat{\mathcal{R}}) \in S^{-\infty}(\widehat{G}M|_U)$ . By [36, Theorem 4.28] (see also [37, Theorem 5.8.7]), for each  $x \in M$ ,  $\psi_x(\widehat{\mathcal{R}}_x)\sigma(x)^{-1}$  makes sense as a symbol in  $S^{-m}(\widehat{G}_x M)$ . It remains to analyse its  $x$ -derivatives. For this, we sketch the adaptation of the argument explained in the proof of [37, Theorem 5.8.7] to take into account our manifold setting and the presence of the smoothing terms  $D_{\mathbb{X}}^\beta \psi(\widehat{\mathcal{R}})$ .

We consider an open set  $U \subset M$  small enough so that the symbol  $\sigma(x) \in S^m(\widehat{G}_x M)$  is invertible for frequencies higher than  $\Lambda$  of the Rockland symbol  $\widehat{\mathcal{R}}_x$ , for some  $\Lambda \in \mathbb{R}$  and  $C_\gamma$  independent of  $x \in U$  (see equation (9.8)). Moreover, we also assume that  $\psi_x(\lambda) = 1$  on

$(\Lambda - 1, +\infty)$  and that  $U$  is equipped with an adapted frame  $\mathbb{X}$ . By the Leibniz properties (see Remark 5.26), we have (at first in a distributional sense for the  $x$ -derivatives on the  $\mathbb{X}$ -convolution kernels) with  $|\beta| = 1$

$$D_{\mathbb{X}}^{\beta}(\psi(\widehat{\mathcal{R}})\sigma^{-1})\sigma = D_{\mathbb{X}}^{\beta}(\psi(\widehat{\mathcal{R}})) - \psi(\widehat{\mathcal{R}})\sigma^{-1}D_{\mathbb{X}}^{\beta}\sigma;$$

multiplying both side by  $E[\Lambda, +\infty)$  and using the invertibility of  $\sigma E[\Lambda, +\infty)$ , we obtain

$$D_{\mathbb{X}}^{\beta}(\psi(\widehat{\mathcal{R}})\sigma^{-1}) = \left( D_{\mathbb{X}}^{\beta}(\psi(\widehat{\mathcal{R}})) - \psi(\widehat{\mathcal{R}})\sigma^{-1}D_{\mathbb{X}}^{\beta}\sigma \right) E[\Lambda, +\infty)\sigma^{-1}.$$

By Remark 9.30 (2), we obtain

$$\begin{aligned} & \|D_{\mathbb{X}}^{\beta}(\psi(\widehat{\mathcal{R}})\sigma^{-1})(x)\|_{L_{\gamma, m+\gamma}^{\infty}(\widehat{G}_x M)} \\ & \leq C_{\gamma}^{-1} \left( \|D_{\mathbb{X}}^{\beta}(\psi(\widehat{\mathcal{R}}))\|_{L_{m+\gamma, m+\gamma}^{\infty}(\widehat{G}_x M)} + \|\psi\|_{L^{\infty}} C_{\gamma}^{-1} \|D_{\mathbb{X}}^{\beta}\sigma\|_{L_{m+\gamma, \gamma}^{\infty}(\widehat{G}_x M)} \right). \end{aligned}$$

More generally, by induction on  $|\alpha|$  and  $|\beta|$ , we write  $\Delta_{\mathbb{X}}^{\alpha} D_{\mathbb{X}}^{\beta}(\psi(\widehat{\mathcal{R}})\sigma^{-1})$  as a sum of terms involving  $\Delta_{\mathbb{X}}^{\alpha_1} D_{\mathbb{X}}^{\beta_1} \psi(\widehat{\mathcal{R}})$  and  $\Delta_{\mathbb{X}}^{\alpha_2} D_{\mathbb{X}}^{\beta_2}(\psi(\widehat{\mathcal{R}})\sigma^{-1})$  with  $|\alpha_2| < |\alpha|$ ,  $|\beta_2| < |\beta|$  and  $|\alpha_2| + |\beta_2| \neq 0$ . A similar analysis as above implies then that  $\Delta_{\mathbb{X}}^{\alpha} D_{\mathbb{X}}^{\beta}(\psi(\widehat{\mathcal{R}})\sigma^{-1}) \in L_{\gamma, m-|\alpha|+\gamma}^{\infty}(\widehat{G}_x M)$  for every  $x \in U$ .

We have obtained  $\psi(\widehat{\mathcal{R}})\sigma^{-1} \in S^{-m}(\widehat{G}M)$  and the rest of the statement follows by the symbolic calculus.  $\square$

*Proof of Theorem 9.33.* We keep the notation of the statement and fix an atlas  $\mathcal{A}$  on  $M$ . By Corollary 9.24, we can write

$$Q = \text{Op}^{\mathcal{A}}(\sigma) + S,$$

for some  $S \in \Psi^{m-1}(M)$ . The construction of a function  $\psi$  as in Lemma 9.34 follows routine arguments. By Lemma 9.34,  $\psi(\widehat{\mathcal{R}})\sigma^{-1} \in S^{-m}(\widehat{G}M)$  and we have by the symbolic property of the global quantization (see Corollary 9.18)

$$\begin{aligned} \text{Op}^{\mathcal{A}}(\psi(\widehat{\mathcal{R}})\sigma^{-1})Q &= \text{Op}^{\mathcal{A}}(\psi(\widehat{\mathcal{R}})\sigma^{-1})\text{Op}^{\mathcal{A}}(\sigma) + \text{Op}^{\mathcal{A}}(\psi(\widehat{\mathcal{R}})\sigma^{-1})S \\ &= \text{Op}^{\mathcal{A}}(\text{I}) + R_{-1}, \end{aligned}$$

for some  $R_{-1} \in \Psi^{-1}(M)$ . Note that  $\text{Op}^{\mathcal{A}}(\text{I}) = \text{I}$ . By Corollary 9.24, we may write  $R_{-1}$  in the form

$$R_{-1} = \text{Op}^{\mathcal{A}}(\rho_{-1}) + S_{-1},$$

where  $\rho_{-1} \in S^{-1}(\widehat{G}M)$  is a symbol representing  $\text{Princ}_{-1}R_{-1}$  and  $S_{-1} \in \Psi^{-2}(M)$ . We will then have

$$\text{Op}^{\mathcal{A}}\left((1 - \rho_{-1})\psi(\widehat{\mathcal{R}})\sigma^{-1}\right)Q = \text{I} + R_{-2},$$

for some  $R_{-2} \in \Psi^{-2}(M)$ . Inductively, we construct  $\rho_{-1}, \dots, \rho_{-N}$  in  $S^{-1}(\widehat{G}M), \dots, S^{-N}(\widehat{G}M)$  and  $R_{-1}, \dots, R_{-N-1}$  in  $\Psi^{-1}(\widehat{G}M), \dots, \Psi^{-N-1}(\widehat{G}M)$  respectively, such that

$$\text{Op}^{\mathcal{A}}\left((1 - \rho_{-N}) \dots (1 - \rho_{-1}) \psi(\widehat{\mathcal{R}})\sigma^{-1}\right)Q = \text{I} + R_{-N-1}.$$

We observe that the formal expansion of the product  $(1 - \rho_{-N}) \dots (1 - \rho_{-1}) \psi(\widehat{\mathcal{R}})\sigma^{-1}$  as  $N \rightarrow \infty$  provides an asymptotic in  $S^{-m}(\widehat{GM})$ , and we consider a symbol  $\tau \in S^{-m}(\widehat{GM})$  with this asymptotic. By construction, we have for any  $N \in \mathbb{N}$

$$\tau = (1 - \rho_{-N}) \dots (1 - \rho_{-1}) \psi\sigma^{-1} \bmod S^{-m-N-1}(\widehat{GM}).$$

This implies with  $P = \text{Op}^A(\tau)$ ,

$$PQ = \text{I} \bmod \Psi^{-m-N-1}(M),$$

for any  $N \in \mathbb{N}$ , concluding the proof.  $\square$

The proof above shows how to construct parametrices locally or globally for the invertible symbol in Example 9.32:

**Corollary 9.35.** *Let  $m \in \mathbb{R}$ ,*

- (1) *For any atlas  $\mathcal{A}$  on  $M$  and for any positive Rockland symbol  $\widehat{\mathcal{R}}$  on  $GM$ , there exists a symbol  $\tau \in S^{-m}(\widehat{GM})$  such that we have:*

$$\text{Op}^A(\tau) \text{Op}^A\left(\text{I} + \widehat{\mathcal{R}}^{m/\nu}\right) - \text{I} \in \Psi^{-\infty}(M),$$

*where  $\nu$  denotes the homogeneous degree of  $\mathcal{R}$ .*

- (2) *For any adapted frame  $\mathbb{X}$  on an open subset  $U \subset M$ , for any  $\mathbb{X}$ -cut-off, for any positive Rockland symbol  $\widehat{\mathcal{R}}$  on  $GM|_U$ , there exists a symbol  $\tau \in S^{-m}(\widehat{GM}|_U)$  such that we have:*

$$\text{Op}^{\mathbb{X},\chi}(\tau) \text{Op}^{\mathbb{X},\chi}\left(\text{I} + \widehat{\mathcal{R}}^{m/\nu}\right) - \text{I} \in \Psi^{-\infty}(U),$$

*where  $\nu$  denotes its homogeneous degree.*

*Proof.* We consider  $\tau$  obtained from the proof of Theorem 9.33 for  $\sigma = (\text{I} + \mathcal{R})^{m/\nu}$  and  $S = 0$ .  $\square$

*Remark 9.36.* For Part (2) above, the existence of such positive Rockland symbol  $\widehat{\mathcal{R}}$  on  $GM|_U$  is given by  $\mathcal{R}_{\mathbb{X}}$  constructed in Example 5.7. For Part (1) (with its notation), we can construct such a positive Rockland symbol by writing

$$\widehat{\mathcal{R}} = \sum_{\alpha} \psi_{\alpha}(x)^2 \widehat{\mathcal{R}}_{\mathbb{X}_{\alpha}}.$$

On a regular subRiemannian manifold, the natural global positive Rockland symbol is the subLaplacian symbol, see Remark 5.20.

**9.5. Local Sobolev spaces.** Following the now classical ideas developed on nilpotent Lie groups [38, 39], we introduce Sobolev-type spaces that take into account the filtration of the manifold  $M$ . Naturally, the spaces we define here will be local in nature, unless the manifold  $M$  is compact. An alternative construction was introduced in [22].

9.5.1. *Definition and first properties.*

**Definition 9.37** (Local Sobolev spaces). Let  $r \in \mathbb{R}$ . We denote by  $L_{r,loc}^2(M)$  the space of distributions  $f \in \mathcal{D}'(M)$  satisfying

$$\forall P \in \Psi^r(M), \quad \forall \theta \in C_c^\infty(M), \quad P(\theta f) \in L_{loc}^2(M).$$

We call it the *local Sobolev space of order  $r$*  on the filtered manifold  $M$ .

This definition makes sense since pseudodifferential operators map  $\mathcal{E}'(M)$  to  $\mathcal{D}'(M)$ , and  $L_{loc}^2(M)$  is a subspace of  $\mathcal{D}'(M)$  (see its definition in (7.4)). The properties of  $\Psi(M)$  then imply readily:

**Proposition 9.38.** (1) *For any  $r \in \mathbb{R}$ , the vector space  $L_{loc,r}^2(M)$  is a  $C^\infty(M)$ -module: if  $f \in L_{loc,r}^2(M)$  and  $g \in C^\infty(M)$  then  $fg \in L_{loc,r}^2(M)$ .*

(2) *We have  $L_{loc,0}^2(M) = L_{loc}^2(M)$  and the inclusion*

$$r_1 \geq r_2 \implies L_{loc,r_1}^2(M) \subset L_{loc,r_2}^2(M).$$

(3) *Let  $r, m \in \mathbb{R}$ . We have*

$$\forall P \in \Psi^m(M), \quad \forall f \in L_{r,loc}^2(M), \quad \forall \theta \in C_c^\infty(M), \quad P(\theta f) \in L_{r-m,loc}^2(M).$$

(4) *We have for any  $r \in \mathbb{R}$*

$$C^\infty(M) \subset L_{loc,r}^2(M),$$

*and for  $P \in \Psi^{-\infty}(M)$ ,*

$$\forall f \in L_{r,loc}^2(M), \quad \forall \theta \in C_c^\infty(M), \quad P(\theta f) \in C^\infty(M).$$

*Proof.* Part (1) follows from  $\Psi^m(M)$  being a vector space and a  $C^\infty(M)$ -module. If  $r = 0$ , taking  $P = I$  in Definition 9.37 implies  $L_{loc,0}^2(M) \subset L_{loc}^2(M)$ , while the reverse inclusion follows from Corollary 9.10. The inclusion in Part (2) follows from  $\Psi^{r_1}(M) \subset \Psi^{r_2}(M)$ . Part (3) follows from the properties of properly supported and of pseudodifferential operators and of their composition (see Lemma 9.14 and Theorem 9.15). Part (4) follows from the basic properties of pseudodifferential operators, see Section 9.1.1.  $\square$

9.5.2. *Characterisation of local Sobolev spaces.* Local Sobolev spaces may be characterised by the action of local or global pseudodifferential operators, or by the action of the local or global quantization of powers of positive Rockland operators:

**Theorem 9.39.** *Let  $r \in \mathbb{R}$  and let  $f \in \mathcal{D}'(M)$ . The following are equivalent:*

- (i)  $f \in L_{r,loc}^2(M)$ .
- (ii) *For any adapted frame  $\mathbb{X}$  on an open subset  $U \subset M$ , for any  $\mathbb{X}$ -cut-off  $\chi$ , we have*

$$\forall \sigma \in S^r(\widehat{GM}), \quad \forall \theta \in C_c^\infty(U), \quad \text{Op}^{\mathbb{X},\chi}(\sigma)(\theta f) \in L^2(U).$$

- (iii) *For one (and then any) atlas  $\mathcal{A}$  for  $M$ , we have*

$$\forall \sigma \in S^r(\widehat{GM}), \quad \forall \theta \in C_c^\infty(M), \quad \text{Op}^{\mathcal{A}}(\sigma)(\theta f) \in L^2(M).$$

(iv) For one (and then any) positive Rockland symbol  $\widehat{\mathcal{R}}$  on  $GM$ ,  $\nu$  being its homogeneous degree, for any adapted frame  $\mathbb{X}$  on an open subset  $U \subset M$  and any  $\mathbb{X}$ -cut-off  $\chi$ , we have

$$\forall \theta \in C_c^\infty(U), \quad \text{Op}^{\mathbb{X}, \chi} \left( (I + \widehat{\mathcal{R}})^{\frac{r}{\nu}} \right) (\theta f) \in L^2(U),$$

(v) For one (and then any) atlas  $\mathcal{A}$  for  $M$ , for one (and then any) positive Rockland symbol  $\widehat{\mathcal{R}}$  on  $GM$ ,  $\nu$  being its homogeneous degree,

$$\forall \theta \in C_c^\infty(M), \quad \text{Op}^{\mathcal{A}} \left( (I + \widehat{\mathcal{R}})^{\frac{r}{\nu}} \right) (\theta f) \in L^2(M).$$

(vi) For any adapted frame  $\mathbb{X}$  on an open subset  $U \subset M$ , for any  $\mathbb{X}$ -cut-off  $\chi$ , we have

$$\forall \theta \in C_c^\infty(U), \quad \text{Op}^{\mathbb{X}, \chi} \left( (I + \widehat{\mathcal{R}}_U)^{\frac{r}{\nu}} \right) (\theta f) \in L^2(U),$$

for one (and then any) positive Rockland operator  $\mathcal{R}_U$  on  $GM|_U$ ,  $\nu$  being its homogeneous degree.

The local and global existence of positive Rockland symbols has been discussed in Remark 9.36.

*Proof.* The implication (i)  $\Rightarrow$  (ii) follows from Corollary 9.5, and the converse by Proposition 9.9. Similarly, (i)  $\Rightarrow$  (iii) follows from Corollary 9.5, and the converse from Corollary 9.24 and the inclusion in Proposition 9.38 (2).

The implication (iii)  $\Rightarrow$  (v) is easy. For the converse, we observe that, using the setting and notation of Corollary 9.35 and writing:

$$Q_m = \text{Op}^{\mathcal{A}} \left( (I + \mathcal{R})^{m/\nu} \right) \quad \text{and} \quad P_{-m} = \text{Op}^{\mathcal{A}} (\tau),$$

the operator  $R := P_{-m}Q_m - I \in \Psi^{-\infty}(M)$  is smoothing and properly supported. Hence, we may write any  $P \in \Psi^m(M)$  as

$$P = PP_{-m}Q_m - PR, \quad \text{with} \quad PP_{-m} \in \Psi^0(M) \quad \text{and} \quad PR \in \Psi^{-\infty}(M).$$

This readily implies that if  $f \in \mathcal{D}'(M)$  is as in (v), then  $P(\theta f) \in L_{loc}^2(M)$  for any  $\theta \in C_c^\infty(M)$  by Corollary 9.10. We conclude that the equivalence (iii)  $\Leftrightarrow$  (v) holds. Finally, the equivalence (ii)  $\Leftrightarrow$  (iv) is proved in a similar way.

Let us show that the property in (vi) is indeed independent of the positive Rockland operator  $\mathcal{R}_U$  on  $GM|_U$ . This will readily imply the equivalence (iv)  $\Leftrightarrow$  (vi) because of the equivalence (ii)  $\Leftrightarrow$  (iv) that is already proved. If  $\mathcal{R}_1$  and  $\mathcal{R}_2$  are two positive Rockland operators on  $GM|_U$  of homogeneous degree  $\nu_1$  and  $\nu_2$ , then by Corollary 9.35, there exists  $\tau_1 \in S^{-r}(\widehat{GM}|_U)$  and  $R_1 \in \Psi^{-\infty}(U)$  such that

$$\text{Op}^{\mathbb{X}, \chi}(\tau_1) \text{Op}^{\mathbb{X}, \chi} \left( (I + \mathcal{R}_1)^{r/\nu_1} \right) = I + R_1,$$

so

$$\text{Op}^{\mathbb{X}, \chi} \left( (I + \mathcal{R}_2)^{r/\nu_2} \right) = P_1 \text{Op}^{\mathbb{X}, \chi} \left( (I + \mathcal{R}_1)^{r/\nu_1} \right) + R',$$

with

$$\begin{aligned} P_1 &:= \text{Op}^{\mathbb{X},\chi} \left( (I + \mathcal{R}_2)^{r/\nu_2} \right) \text{Op}^{\mathbb{X},\chi}(\tau_1) \in \Psi^0(U) \\ R' &= -R_1 \text{Op}^{\mathbb{X},\chi} \left( (I + \mathcal{R}_2)^{r/\nu_2} \right) \in \Psi^{-\infty}(U). \end{aligned}$$

This together with Proposition 9.38 (3) and (4) imply readily that for any  $f \in \mathcal{D}'(M)$  and  $\theta \in C_c^\infty(U)$ , if  $\theta f$  satisfies the property in (vi) for  $\mathcal{R}_1$ , then it will also do so for  $\mathcal{R}_2$ . This complete the proof of Theorem 9.39.  $\square$

In the case of an integer that is also a multiple of the dilation's weights, we obtain a further characterisation:

**Theorem 9.40.** *Let  $r \in \mathbb{N}$  be a common multiple of the dilation's weights. Let  $f \in \mathcal{D}'(M)$ . The following are equivalent:*

- (i)  $f \in L_{r,loc}^2(M)$ .
- (vii)  $Pf \in L_{loc}^2(M)$  for any  $P \in \text{DO}^{\leq r}(M)$ .
- (viii) For any adapted frame  $\mathbb{X}$  on an open subset  $U \subset M$ , we have  $\mathbb{X}^\alpha f \in L_{loc}^2(U)$  for any  $\alpha \in \mathbb{N}_0^n$  with  $[\alpha] \leq r$ .

*Proof.* Let  $r \in \mathbb{N}$  be a common multiple of the dilation's weights. Proposition 9.38 (2) & (3) and Definition 9.37 imply readily (i)  $\Rightarrow$  (vii) and (viii). We now prove the converses.

Let  $f \in \mathcal{D}'(M)$  be such that (vii) holds. Choosing  $P = I$  in (vii), we have  $f \in L_{loc}^2(M)$ . We consider an adapted frame  $\mathbb{X}$  on a subset  $U \subset M$  and  $\theta \in C_c^\infty(U)$ , and the Rockland symbol  $\widehat{\mathcal{R}}_{\mathbb{X}} \in S^{2r}(\widehat{G}M)$  given in Example 5.7. We have by the properties of the local quantization (Propositions 8.9 and 8.13),

$$\left( \text{Op}^{\mathbb{X},\chi} \left( (I + \widehat{\mathcal{R}}_{\mathbb{X}})^{1/2} \right) \right)^* \text{Op}^{\mathbb{X},\chi} \left( (I + \widehat{\mathcal{R}}_{\mathbb{X}})^{1/2} \right) = \text{Op}^{\mathbb{X},\chi} \left( I + \widehat{\mathcal{R}}_{\mathbb{X}} \right) + \text{Op}^{\mathbb{X},\chi}(\rho),$$

with  $\rho \in S^{2r-1}(\widehat{G}M|_U)$ , and by Corollary 9.35 (2),

$$\text{Op}^{\mathbb{X},\chi}(\tau) \text{Op}^{\mathbb{X},\chi} \left( I + \widehat{\mathcal{R}}_{\mathbb{X}} \right) = I + R,$$

with  $\tau \in S^{-2r}(\widehat{G}M|_U)$  and  $R \in \Psi^{-\infty}(U)$ , so

$$\left( \text{Op}^{\mathbb{X},\chi} \left( (I + \widehat{\mathcal{R}}_{\mathbb{X}})^{1/2} \right) \right)^* \text{Op}^{\mathbb{X},\chi} \left( (I + \widehat{\mathcal{R}}_{\mathbb{X}})^{1/2} \right) = P' \text{Op}^{\mathbb{X},\chi} \left( I + \widehat{\mathcal{R}}_{\mathbb{X}} \right) + R'',$$

with

$$\begin{aligned} P' &:= I + \text{Op}^{\mathbb{X},\chi}(\rho) \text{Op}^{\mathbb{X},\chi}(\tau) \in \Psi^0(U), \\ R'' &:= -\text{Op}^{\mathbb{X},\chi}(\rho) R \in \Psi^{-\infty}(U), \end{aligned}$$

by the properties of the calculus. Therefore, for any  $\theta \in C_c^\infty(U)$ ,

$$\begin{aligned} &\| \text{Op}^{\mathbb{X},\chi} \left( (I + \widehat{\mathcal{R}}_{\mathbb{X}})^{1/2} \right) (\theta f) \|_{L^2(U)}^2 \\ &= \left( \left( \text{Op}^{\mathbb{X},\chi} \left( (I + \widehat{\mathcal{R}}_{\mathbb{X}})^{1/2} \right) \right)^* \text{Op}^{\mathbb{X},\chi} \left( (I + \widehat{\mathcal{R}}_{\mathbb{X}})^{1/2} \right) (\theta f), (\theta f) \right)_{L^2(U)} \\ &= \left( P' \text{Op}^{\mathbb{X},\chi} \left( I + \widehat{\mathcal{R}}_{\mathbb{X}} \right) (\theta f), (\theta f) \right)_{L^2(U)} + (R''(\theta f), (\theta f))_{L^2(U)}. \end{aligned}$$

As  $\text{Op}^{\mathbb{X},\mathbb{X}}\left(\mathbb{I} + \widehat{\mathcal{R}}_{\mathbb{X}}\right)$  is a differential operator (see Lemma 9.7), the hypothesis on  $f$  and Proposition 9.38 (3) imply that the first term on the right hand-side is finite. Moreover, Proposition 9.38 (4) and  $f \in L_{loc}^2(M)$  imply that the second term in the right hand side is finite. We have thus obtained  $\text{Op}^{\mathbb{X},\mathbb{X}}\left((\mathbb{I} + \widehat{\mathcal{R}}_{\mathbb{X}})^{1/2}\right)(\theta f) \in L^2(U)$ . By Theorem 9.39 (vi),  $f \in L_{loc,r}^2(M)$ . We have proved (vii)  $\Rightarrow$  (i).

With Lemma 3.6, the same arguments as above show (viii)  $\Rightarrow$  (i). This concludes the proof.  $\square$

The same proof as above readily implies the following characterisation in the case of  $H_1$  generating  $TM$  (for instance on a regular subRiemannian manifolds, see Example 3.2 and Corollary 3.9).

**Corollary 9.41.** *Assume that  $H_1$  generates  $TM$ . Then for any  $r \in \mathbb{N}$ ,  $L_{loc,r}^2(M)$  is the  $C^\infty(M)$ -module of distributions  $f \in \mathcal{D}'(M)$  such that for any adapted frame  $\mathbb{X}$  on an open subset  $U \subset M$ , we have  $X_{i_1} \dots X_{i_{r'}} f \in L_{loc}^2(U)$  for any  $1 \leq i_1, \dots, i_{r'} \leq d_1 = \dim H_1$  and  $r' \leq r$ .*

For a general filtered manifold, Theorem 9.40 implies readily

$$\forall r \in M_0\mathbb{N}, \quad C^\infty(M) \subset L_{loc,r}^2(M),$$

where  $M_0$  is a common multiple of the dilation's weights, and furthermore,

$$C^\infty(M) = \bigcap_{r \in M_0\mathbb{N}} L_{loc,r}^2(M).$$

The inclusion in Proposition 9.38 (2) then yields

$$\forall r \in \mathbb{R}, \quad C^\infty(M) \subset L_{loc,r}^2(M),$$

and

$$C^\infty(M) = \bigcap_{r \in \mathbb{R}} L_{loc,r}^2(M).$$

*Remark 9.42.* Note that, in this special case where  $H_1$  generates  $TM$ , one can use Corollary 9.41 for defining Sobolev spaces in terms of adapted frames.

9.5.3. *Topology.* We equip  $L_{loc,r}^2(M)$  with the smallest topology ensuring the continuity of the map

$$f \longmapsto P(\theta f), \quad L_{loc,r}^2(M) \longrightarrow L_{loc}^2(M),$$

for every  $P \in \Psi^r(M)$  and  $\theta \in C_c^\infty(M)$ . The arguments in the proofs of Proposition 9.38 and Theorems 9.39 and 9.40 routinely imply the following properties:

**Lemma 9.43.** (1) *The topological vector space  $L_{loc,r}^2(M)$  is Fréchet. Fixing an atlas  $\mathcal{A}$  for  $M$ , a positive Rockland symbol  $\widehat{\mathcal{R}}$  on  $GM$ ,  $\nu$  denoting its homogeneous degree, and a sequence  $(\theta_j)_{j \in \mathbb{N}}$  in  $C_c^\infty(M : [0, 1])$  such that  $M = \cup_{j \in \mathbb{N}} \{\theta_j = 1\}$ , a generating family of seminorms is given by*

$$f \longmapsto \max_{j \leq N} \left\| \text{Op}^{\mathcal{A}} \left( (\mathbb{I} + \widehat{\mathcal{R}})^{\frac{\nu}{2}} \right) (\theta_j f) \right\|_{L^2(M)}, \quad N \in \mathbb{N}.$$

- (2) The inclusion  $L^2_{loc,r_1}(M) \subset L^2_{loc,r_2}(M)$  with  $r_1 \geq r_2$  is continuous.
- (3) Let  $r, m \in \mathbb{R}$ . Any  $P \in \Psi^m(M)$  maps  $L^2_{r,loc}(M)$  to  $L^2_{r-m,loc}(M)$  continuously.
- (4) Assume that  $r \in \mathbb{N}$  is a common multiple of the dilation's weights. Let  $(\theta_j)_{j \in \mathbb{N}}$  be a sequence of functions in  $C_c^\infty(M)$  valued in  $[0, 1]$  such that  $\text{supp } \theta_j$  is included in an open set  $U_j$  equipped with an adapted frame  $\mathbb{X}_j$  and such that  $M = \cup_{j \in \mathbb{N}} \{\theta_j = 1\}$ . Then the maps

$$f \longmapsto \|\mathbb{X}^\alpha(\theta_j f)\|_{L^2(M)}, \quad j \in \mathbb{N}, \quad [\alpha] \leq r,$$

are continuous on the Fréchet  $L^2_{loc,r}(M)$ , and provide a countable family of seminorms generating its topology.

When  $H_1$  generates  $TM$ , we may consider any integer  $r \in \mathbb{N}$  with the family of seminorms

$$f \longmapsto \|X_{i_1} \dots X_{i_{r'}}(\theta_j f)\|_{L^2(M)}, \quad j \in \mathbb{N}, \quad 1 \leq i_1, \dots, i_{r'} \leq d_1, \quad r' \leq r.$$

**Corollary 9.44.** (1) The natural topology of  $C^\infty(M)$  coincides with the one obtained as the inductive limit of

$$C^\infty(M) = \bigcap_{r \in \mathbb{R}} L^2_{loc,r}(M).$$

- (2) For any  $r \in \mathbb{R}$ , the inclusions  $C^\infty(M) \subset L^2_{loc,r}(M)$  and  $C_c^\infty(M) \subset L^2_{loc,r}(M)$  are continuous. Moreover,  $C^\infty(M)$  and  $C_c^\infty(M)$  are dense subspaces of  $L^2_{loc,r}(M)$ .

*Proof.* Let  $M_0$  be the least common multiple of the dilation's weights. Part (2) when  $r \in M_0\mathbb{N}_0$  follows readily from Lemma 9.43 (4). This also shows that the topology of  $C^\infty(M)$  coincides with the inductive limit of  $\bigcap_{r \in M_0\mathbb{N}} L^2_{loc,r}(M)$ . This together with the continuous inclusions in Lemma 9.43 (2) imply Part (1) with the intersection over  $r \in \mathbb{R}$  and Part (2) for any  $r \in \mathbb{R}$ .  $\square$

9.5.4. *Duality.* If  $f \in L^2_{loc,r'}(M)$  for some  $r' \in \mathbb{R}$ , then it is a distribution, meaning that the map  $\varphi \mapsto (f, \varphi)_{\mathcal{D}' \times C_c^\infty}$  formally given by the distributional duality

$$(f, \varphi)_{\mathcal{D}' \times C_c^\infty} = \int_M f(x) \varphi(x) dx, \quad \varphi \in C_c^\infty(M),$$

is a well defined linear functional  $C_c^\infty(M) \rightarrow \mathbb{C}$  that is continuous. The following statement allows us to identify the topological dual of  $L^2_{loc,r}(M)$  with  $L^2_{loc,-r}(M)$ :

$$(L^2_{loc,r}(M))' \cong L^2_{loc,-r}(M),$$

via the natural extension of the distributional duality.

**Lemma 9.45.** Let  $r \in \mathbb{R}$ .

- (1) If  $f \in L^2_{loc,-r}(M)$ , then the linear functional  $\varphi \mapsto (f, \varphi)_{\mathcal{D}' \times C_c^\infty}$  extends uniquely to a continuous linear functional on  $L^2_{loc,r}(M)$ .
- (2) Conversely, given a continuous linear functional  $\ell$  on  $L^2_{loc,r}(M)$ , there exists a unique distribution  $f \in L^2_{loc,-r}(M)$  such that

$$\forall \varphi \in C_c^\infty(M) \quad \ell(\varphi) = (f, \varphi)_{\mathcal{D}' \times C_c^\infty}.$$

*Proof.* Part (1) follows from Corollary 9.44 (2). Conversely, let  $\ell$  be a continuous linear functional on  $L^2_{loc,r}(M)$ . If  $r' \in \mathbb{N}$  is a multiple of the dilation's weights such that  $r' \geq r$ , then  $\ell$  restrict to a continuous linear functional on  $L^2_{loc,r'}(M)$  by Lemma 9.43 (2), and the seminorms in Lemma 9.43 (3) shows that  $\ell$  also restrict to a continuous linear functional on  $C_c^\infty(M)$ . Therefore, there exists  $f \in \mathcal{D}'(M)$  such that  $\ell(\varphi) = (f, \varphi)_{\mathcal{D}' \times C_c^\infty}$  for any  $\varphi \in C_c^\infty(M)$ . Let  $\theta \in C_c^\infty(M)$ . We fix an atlas  $\mathcal{A}$ . We have  $\theta f \in \mathcal{E}'(M)$ , so  $\text{Op}^{\mathcal{A}}(\sigma)\theta f \in \mathcal{E}'(M)$  for any  $\sigma \in S^{-r}(\widehat{GM})$ , with

$$(\text{Op}^{\mathcal{A}}(\sigma)\theta f, \varphi)_{\mathcal{D}' \times C_c^\infty} = (f, \overline{\theta \text{Op}^{\mathcal{A}}(\sigma)^* \varphi})_{\mathcal{D}' \times C_c^\infty} = \ell \left( \overline{\theta \text{Op}^{\mathcal{A}}(\sigma)^* \varphi} \right).$$

We now use the seminorms for  $L^2_{loc,r}(M)$  described in Lemma 9.43 (1) with its notation. The continuity of  $\ell : L^2_{loc,r}(M) \rightarrow \mathbb{C}$  imply the existence of  $C > 0$  and  $N \in \mathbb{N}$  such that

$$\left| \ell \left( \overline{\theta \text{Op}^{\mathcal{A}}(\sigma)^* \theta' \varphi} \right) \right| \leq C \max_{j \leq N} \|P_j(\varphi)\|_{L^2(M)},$$

where

$$P_j := \theta_j \text{Op}^{\mathcal{A}} \left( (I + \widehat{\mathcal{R}})^{\frac{j}{\nu}} \right) \overline{\theta \text{Op}^{\mathcal{A}}(\sigma)^*}.$$

By the properties of the global quantization, especially for composition and adjoint (see Corollary 9.18),  $P_j \in \Psi^0(M)$  is a properly supported pseudodifferential operator of order 0. Therefore,

$$\max_{j \leq N} \|P_j(\varphi)\|_{L^2(M)} \leq C' \|\varphi\|_{L^2(M)},$$

for some  $C' > 0$  independent of  $\varphi$ . This implies that  $\text{Op}^{\mathcal{A}}(\sigma)\theta f \in L^2(M)$ . By Theorem 9.39, this shows that  $f \in L^2_{loc,-r}(M)$ . This concludes the proof of Part (2).  $\square$

**9.6. The classical  $H$ -calculus.** In this section, we adapt to the setting of the filtered manifold  $M$  the traditional lines of approach regarding polyhomogeneous pseudodifferential operators (Section 9.6.1) and classical calculus (Section 9.6.2). We discuss the link with van Erp and Yuncken calculus [67, 68] in Section 9.7.

**9.6.1. Polyhomogeneous operators.**

**Definition 9.46.** A pseudodifferential  $P \in \Psi^m(M)$  of order  $m \in \mathbb{R}$  on  $M$  is *polyhomogeneous* when the symbol in Part (1) of Definition 9.1 is polyhomogeneous in the sense of Definition 5.53, that is, for any adapted frame  $\mathbb{X}$  on an open set  $U \subset M$ , for any  $\mathbb{X}$ -cut-off  $\chi$ , there exists  $\sigma \in S^m_{ph}(\widehat{GM}|_U)$  such that  $\varphi P \psi = \varphi \text{Op}^{\mathbb{X},\chi}(\sigma) \psi$  for any  $\varphi, \psi \in C_c^\infty(U)$ .

We denote by  $\Psi^m_{ph}(M)$  the space of polyhomogeneous pseudodifferential operators of order  $m \in \mathbb{R}$ .

*Example 9.47.* (1) Any differential operator is polyhomogeneous with a finite asymptotic, see the proof of Lemma 9.7:  $\text{DO}^{\leq N} \subset \Psi^N_{ph}(M)$ .

(2) Let  $m \in \mathbb{R}$ . Let  $(\mathbb{X}, U)$  be an adapted frame with  $\mathbb{X}$ -cut-off  $\chi$ . If  $\sigma \in S^m_{ph}(\widehat{GM}|_U)$ , then  $\text{Op}^{\mathbb{X},\chi}(\sigma) \in \Psi^m_{ph}(U)$  by the proof of Lemma 9.4 while  $\psi \in C_c^\infty(U)$ ,  $\text{Op}^{\mathbb{X},\chi}(\sigma) \psi \in \Psi^m_{ph}(M)$  for any  $\psi \in C_c^\infty(U)$  by the proof of Corollary 9.5.

The characterisations via atlases of polyhomogeneous pseudodifferential operators together with the notion of homogeneous principal symbol are obtained by modifying the ones in Proposition 9.9 and Corollary 9.25 by considering polyhomogeneous symbols:

**Proposition 9.48.** *Let  $P$  be a continuous linear operator  $C_c^\infty(M) \rightarrow \mathcal{D}'(M)$ . The following are equivalent.*

- (i)  $P \in \Psi_{ph}^m(M)$ .
- (ii) *For one (and then any) atlas  $\mathcal{A} = (U_\alpha, \mathbb{X}_\alpha, \chi_\alpha, \psi_\alpha)_{\alpha \in A}$  for the filtered manifold  $M$ , there exists a family of polyhomogeneous symbols  $(\sigma_\alpha)_{\alpha \in A}$  with  $\sigma_\alpha \in S_{ph}^m(\widehat{GM}|_{U_\alpha})$  and a smoothing operator  $R \in \Psi^{-\infty}(M)$  such that*

$$P = R + \sum_{\alpha} \psi_\alpha \text{Op}^{\mathbb{X}_\alpha, \chi_\alpha}(\sigma_\alpha) \psi_\alpha$$

- (iii) *For one (and then any) atlas  $\mathcal{A}$  on  $M$ , there exists a polyhomogeneous symbol  $\sigma \in S_{ph}^m(\widehat{GM})$  and a smoothing operator  $R \in \Psi^{-\infty}(M)$  such that  $P = R + \text{Op}^{\mathcal{A}}(\sigma)$ .*

Moreover, in this case, there exists a unique symbol  $\sigma_m \in \dot{S}^m(\widehat{GM})$  such that  $\text{Princ}_m(P) = \sigma_m \phi(\widehat{\mathcal{R}})$  for some positive Rockland symbol  $\widehat{\mathcal{R}}$  on  $\widehat{GM}$  and for some function  $\phi \in C^\infty(M \times \mathbb{R})$  satisfying  $\phi(x, \lambda) = 0$  for  $\lambda$  in a neighbourhood of  $(-\infty, 0)$  and  $\phi(x, \lambda) = 1$  on a neighbourhood of  $+\infty$ , these two  $\lambda$ -neighbourhoods depending locally uniformly on  $x \in M$ .

Recall that by Corollary 5.44, the symbol  $\phi(\widehat{\mathcal{R}}) = \{\phi(x, \pi(\mathcal{R}_x)) : (x, \pi) \in \widehat{GM}\}$  is in  $S^0(\widehat{GM})$  while  $(1 - \phi)(\widehat{\mathcal{R}}) \in S^{-\infty}(\widehat{GM})$ .

*Beginning of the proof of Proposition 9.48.* For the equivalence (i)  $\iff$  (ii), we readily modify the proof of Proposition 9.9.  $\square$

For the equivalence (i)  $\iff$  (iii), the modification in the proof of Corollary 9.25 requires to refine the notion of principal symbols which we start here. By Definitions 9.46 and 9.19, for open subset  $V_0 \subset M$  small enough, we have for some  $\sigma \in S_{ph}^m(\widehat{GM})$ :

$$\begin{aligned} \text{Symb}_{m, V_0} &= \sigma|_{V_0} \bmod S^{m-1}(\widehat{GM}|_{V_0}) \\ &= \sigma_{m, V_0} \phi(\widehat{\mathcal{R}})|_{V_0} \bmod S^{m-1}(\widehat{GM}|_{V_0}) \end{aligned}$$

where  $\sigma_{m, V_0}$  is the first term in the expansion of  $\sigma$  with  $\phi, \widehat{\mathcal{R}}$  as in the statement. Moreover, by the uniqueness of a polyhomogeneous asymptotic expansion (see Section 5.5.3),  $\sigma_{m, V_0}$  is uniquely determined on  $V_0$ . Proceeding as in the first part of the proof of Theorem 9.22, this allows us to construct  $\sigma_m \in \dot{S}^m(\widehat{GM})$  such that  $\sigma_m|_{V_0} = \sigma_{m, V_0}|_{V_0}$ .

**Definition 9.49.** For  $P \in \Psi_{ph}^m(M)$ , the unique symbol  $\sigma_m \in \dot{S}^m(\widehat{GM})$  as in (the above proof of) Proposition 9.48 is denoted by

$$\sigma_m =: \text{Princ}_{m, h}(P).$$

It is called the homogeneous principal symbol of  $P$ .

*Example 9.50.* If  $\mathcal{A}$  is an atlas and if  $\sigma \in S_{ph}^m(\widehat{GM})$  with  $\sigma \sim_h \sum_{j \geq 0} \sigma_{m-j}$ , then

$$\text{Princ}_{m, h}(\text{Op}^{\mathcal{A}}(\sigma)) = \sigma_m.$$

From Theorem 9.22 and its proof together with Example 9.50, we obtain readily that the map

$$\text{Princ}_{m,h} : \Psi_{ph}^m(M) \rightarrow \dot{S}^m(\widehat{GM})$$

is surjective, with kernel  $\Psi_{ph}^{m-1}(M)$ . Furthermore,

$$\forall P \in \Psi_{ph}(M), \quad \text{Princ}_{m,h}(P^*) = (\text{Princ}_{m,h}(P))^*,$$

and for any  $P_1 \in \Psi_{ph}^{m_1}(M)$ ,  $P_2 \in \Psi_{ph}^{m_2}(M)$  one of them being properly supported,

$$P_1 P_2 \in \Psi_{cl}^{m_1+m_2}(M) \quad \text{with} \quad \text{Princ}_{m,h}(P_1 P_2) = \text{Princ}_{m,h}(P_1) \text{Princ}_{m,h}(P_2).$$

When  $m = N \in \mathbb{N}_0$ , the restriction of  $\text{Princ}_{m,h}$  to the space  $\text{DO}^{\leq N}$  of differential operators of homogeneous order at most  $N$  coincides with the map also denoted by  $\text{Princ}_N$  and defined in Section 3.4 and in Theorem 9.22.

*End of the proof of Proposition 9.48.* From Example 9.50 and the properties of  $\text{Princ}_{m,h}$ , we obtain the equivalence (i)  $\iff$  (iii).  $\square$

We also obtain the uniqueness of the polyhomogeneous expansion:

**Corollary 9.51.** *Let  $P \in \Psi_{ph}^m(M)$ . We fix an atlas  $\mathcal{A}$ , a positive Rockland symbol  $\widehat{\mathcal{R}}$  on  $\widehat{GM}$  and a function  $\phi \in C^\infty(M \times \mathbb{R})$  as in Proposition 9.48. Then the polyhomogeneous expansion of  $\sigma \in S_{ph}^m(\widehat{GM})$  in Proposition 9.48 (3) is unique.*

*Proof.* By Example 9.50,  $\text{Princ}_{m,h}(P - \text{Op}^{\mathcal{A}}(\sigma_m \phi(\widehat{\mathcal{R}}))) = 0$ , so  $P_1 = P - \text{Op}^{\mathcal{A}}(\sigma_m \phi(\widehat{\mathcal{R}})) \in \Psi_{ph}^{m-1}(M)$  and we set  $\sigma_{m-1} = \text{Princ}_{m-1,h}(P_1)$ . Inductively, we construct

$$\sigma_m, \dots, \sigma_{m-N} \text{ in } \dot{S}^m(\widehat{GM}), \dots, \dot{S}^{m-N}(\widehat{GM}),$$

such that

$$P - \text{Op}^{\mathcal{A}}\left((\sigma_m + \dots + \sigma_{m-N}) \phi(\widehat{\mathcal{R}})\right) \in \Psi_{ph}^{m-(N+1)}(M).$$

This concludes the proof of Proposition 9.48. If  $P$  is smoothing, then every  $\sigma_{m-j}$  is zero, and this shows the uniqueness of the polyhomogeneous expansion in Proposition 9.48 (3).  $\square$

We conclude this paragraph with the notion of homogeneous elliptic symbol.

**Definition 9.52.** (1) A  $m$ -homogeneous symbol  $\sigma$  is *elliptic* when for one (and then any) positive Rockland symbol  $\widehat{\mathcal{R}}$  and for any  $\gamma \in \mathbb{R}$ ,  $x \in M$  and almost-every  $\pi \in \widehat{G}$ , the inverse of  $\pi(\mathcal{R}_x)^{\frac{\gamma}{\nu}} \sigma(x, \pi) \pi(\mathcal{R}_x)^{-\frac{\gamma+m}{\nu}}$  exists as a bounded operator on  $\pi$  with

$$\sup_{(x,\pi) \in \widehat{GM}} \left\| \left( \pi(\mathcal{R}_x)^{\frac{\gamma}{\nu}} \sigma(x, \pi) \pi(\mathcal{R}_x)^{-\frac{\gamma+m}{\nu}} \right)^{-1} \right\|_{\mathcal{L}(\mathcal{H}_\pi)} < \infty.$$

(2) An operator  $P \in \Psi_{ph}^m(M)$  is *elliptic* when its homogeneous principal symbol is elliptic.

**Proposition 9.53.** *Any elliptic operator  $Q \in \Psi_{ph}^m(M)$  admits a left parametrix  $P \in \Psi_{ph}^{-m}(M)$ .*

*Proof.* Let  $Q \in \Psi_{ph}^m(M)$  be elliptic and set  $\sigma_m = \text{Princ}_{m,h}(Q) \in \dot{S}^m(\widehat{GM})$ . We fix  $\phi$  and  $\mathcal{R}$  as in Proposition 9.48. We check readily that  $\sigma_m \phi(\widehat{\mathcal{R}})$  is invertible for the high frequencies of  $\widehat{\mathcal{R}}$ . We construct a parametrix as in the proof of Theorem 9.33, and we check that it is polyhomogeneous.  $\square$

9.6.2. *Definition and properties of the classical  $H$ -calculus.* We denote by  $\Psi_{cl}^m(M)$  the space of properly supported polyhomogeneous operator of order  $m$ , and by  $\Psi_{cl}(M) = \cup_{m \in \mathbb{R}} \Psi_{cl}^m(M)$  the space of properly supported polyhomogeneous operators of any order. Clearly,  $\Psi_{cl}^m(M)$  is a left and right  $C^\infty(M)$ -module. Moreover, the space  $\Psi_{cl}(M)$  is a classical pseudodifferential calculus in the following sense:

(1) It is stable under composition and taking the formal adjoint. More precisely, we have:

$$\forall P_1 \in \Psi_{cl}^{m_1}(M), P_2 \in \Psi_{cl}^{m_2}(M), \quad P_1 P_2 \in \Psi_{cl}^{m_1+m_2}(M) \quad \text{and} \quad P_1^* \in \Psi_{cl}^{m_1}(M).$$

(2) It contains the differential calculus:  $\text{DO}(M) \subset \Psi_{cl}(M)$ . More precisely,  $\text{DO}^{\leq N} \subset \Psi_{cl}^N(M)$  for any  $N \in \mathbb{N}_0$ .

(3) It acts on the adapted local Sobolev spaces, and the loss of Sobolev derivatives is bounded by the order. More precisely, any  $P \in \Psi_{cl}^m(M)$  acts continuously  $L_s^2(M) \rightarrow L_{s-m}^2(M)$  for any  $m, s \in \mathbb{R}$ .

(4) The smoothing operators in the classical  $H$ -calculus are the operators with smooth integral kernels that are properly supported. Denoting this space by

$$\Psi_{cl}^{-\infty}(M) := \Psi^{-\infty}(M) \cap \Psi_{cl}(M),$$

it may be equivalently described as

$$\Psi_{cl}^{-\infty}(M) = \cap_{m \in \mathbb{R}} \Psi_{cl}^m(M).$$

(5) Any elliptic operator admits a left parametrix. More precisely, if  $Q \in \Psi_{cl}^m(M)$  is elliptic, then we can construct  $P \in \Psi_{cl}^{-m}(M)$  such that  $PQ - I \in \Psi_{cl}^{-\infty}(M)$ .

*Example 9.54.* Considering Example 5.48, an example of elliptic operator is  $\text{Op}^{\mathcal{A}} \left( (I + \widehat{\mathcal{R}})^{m/\nu} \right)$ , where  $\widehat{\mathcal{R}}$  is a positive Rockland symbol on  $\widehat{GM}$ ,  $\nu$  its homogeneous degree and  $\mathcal{A}$  an atlas for  $M$ . By Proposition 9.53, it admits a left parametrix in the classical  $H$ -calculus.

## 9.7. $H$ -calculus and tangent groupoid.

9.7.1. *The tangent groupoid and  $H$ -pseudodifferential kernels.* Before briefly presenting the tangent groupoid adapted to the filtered manifold  $M$  as constructed by Van Erp and Yuncken in [67, 68], let us explain what they mean by a pseudodifferential calculus. We will allow ourselves to identify densities and functions as we have fixed a measure  $dx$  on  $M$ .

Van Erp and Yuncken blur the distinction between a linear continuous operator  $P : C_c^\infty(M) \rightarrow \mathcal{D}'(M)$  and its integral kernel given by the distribution  $K_P$ . Moreover, they consider pseudodifferential operators modulo smoothing operators, that is, modulo operators with smooth integral kernels:

$$P \bmod \Psi^{-\infty}(M) \longleftrightarrow K_P \bmod C^\infty(M \times M).$$

Hence, instead of describing classes of operators, they describe classes of properly supported integral kernels.

Let us now recall some elements of their construction, the references to their works being [67, 68]. The  $H$ -tangent groupoid (the letter  $H$  refers to the filtration  $M$  is equipped with) is

the union of the osculating group bundle  $GM$  with  $M \times M$  together with a dilation parameter  $t \in \mathbb{R}_+ = [0, +\infty)$ :

$$\mathbb{T}_H M = GM \times \{0\} \cup (M \times M) \times \mathbb{R}_+^*$$

where  $\mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\}$ . Van Erp and Yuncken prove in [67] that  $\mathbb{T}_H M$  may be equipped with a natural smooth structure that makes it a Lie groupoid. The description of this smooth structure on  $\mathbb{T}_H M$  is in terms of exponential maps associated with connections. It is equivalent (and perhaps simpler) to express it in terms of geometric exponential associated with an adapted frame.

On  $\mathbb{T}_H M$  is defined the set of  $r$ -fibred distribution,  $\mathcal{E}'_r(\mathbb{T}_H M)$ , which is the set of continuous  $C^\infty(M \times \mathbb{R}_+)$ -linear maps  $C^\infty(\mathbb{T}_H M) \rightarrow C^\infty(M \times \mathbb{R}_+)$ . This space is naturally equipped with a convolution product law (with the standard precautions about the fact that the involved distributions are properly supported), and it is called the *convolution algebra* of  $\mathbb{T}_H M$ .

The zoom action  $\alpha$  of  $\mathbb{R}_+^*$  on  $\mathbb{T}_H M$  is defined according to

$$\begin{cases} \alpha_\lambda(x, y, t) &= (x, y, \lambda^{-1}t) \text{ if } (x, y) \in M \times M \text{ and } t \neq 0, \\ \alpha_\lambda(x, v, 0) &= (x, \delta_\lambda v, 0) \text{ if } x \in M \text{ and } v \in \mathfrak{g}_x M. \end{cases}$$

The zoom action induces a one-parameter family of automorphisms  $\alpha_{\lambda*}$ ,  $\lambda \in \mathbb{R}_+^*$  on the convolution algebra  $\mathcal{E}'_r(\mathbb{T}_H M)$ , and preserves the ideal  $C_p^\infty(GM; \Omega_r)$  of properly supported smooth densities.

Van Erp and Yuncken then define the following two notions [68, Definitions 18 & 19]:

**Definition 9.55.** (1) A properly supported  $r$ -fibred distribution  $\mathbb{P} \in \mathcal{E}'_r(\mathbb{T}_H M)$  is called *essentially homogeneous of weight  $m \in \mathbb{R}$*  if

$$\forall \lambda \in \mathbb{R}_+^*, \quad \alpha_{\lambda*} \mathbb{P} - \lambda^m \mathbb{P} \in C_p^\infty(\mathbb{T}_H M; \Omega_r).$$

- (2) A  *$H$ -pseudodifferential kernel of order  $\leq m$*  is a distribution  $K \in \mathcal{E}'_r(M \times M)$  satisfying  $K = \mathbb{P}|_{t=1}$  for some properly supported  $r$ -fibred distribution  $\mathbb{P} \in \mathcal{E}'_r(\mathbb{T}_H M)$  which is essentially homogeneous of weight  $m \in \mathbb{R}$ .
- (3) We denote by  $\Psi_{vEY}^m(M)$  the space of properly supported operators  $P : C_c^\infty(M) \rightarrow C^\infty(M)$  whose integral kernels are  $H$ -pseudodifferential of order  $\leq m$ .

Let us show that van Erp and Yuncken's construction coincides with  $\Psi_{cl}^m(M)$  in the following sense:

**Theorem 9.56.** *For any  $m \in \mathbb{R}$ ,  $\Psi_{cl}^m(M) = \Psi_{vEY}^m(M)$ .*

In other words, our classical calculus  $\Psi_{cl}(M)$  and the pseudodifferential calculus constructed via groupoid coincide in exactly the same way that the Hörmander classical calculus on a manifold corresponds to the abstract operator classes defined via the groupoid approach (see Section 11 in [68], see also Debord and Skandalis [24]).

The rest of the section is devoted to the proof of Theorem 9.56, that is, for the two inclusions of the two pseudodifferential calculi. We start with the observation that both calculi may be viewed modulo smoothing operators as:

*Remark 9.57.* As in the case of  $\Psi_{cl}(M)$ , the set  $\bigcap_{m \in \mathbb{R}} \Psi_{vEY}^m(M)$  coincides with the space of smoothing properly supported operators [68, Corollary 53]:

$$\bigcap_{m \in \mathbb{R}} \Psi_{vEY}^m(M) = \Psi_{cl}^{-\infty}(M).$$

9.7.2. *The proof of the inclusion  $\Psi_{cl}^m(M) \subset \Psi_{vEY}^m(M)$ .* The main idea behind the proof of the inclusion  $\Psi_{cl}^m(M) \subset \Psi_{vEY}^m(M)$  is the construction of an  $r$ -fibred distribution associated with a classical symbol, and this will use the following lemma:

**Lemma 9.58.** *Let  $\sigma \in S_{ph}^m(\widehat{GM})$ .*

- (1) *For any  $t > 0$ , the density  $\kappa_\sigma \circ \delta_t - t^m \kappa_\sigma$  is smooth.*
- (2) *Write the polyhomogeneous expansion of  $\sigma$  as  $\sigma \sim_h \sigma_m + \sigma_{m-1} + \dots$ . Then, for any adapted frame  $(\mathbb{X}, U)$ , for every  $x \in U$ , we have the convergence in the sense of distribution*

$$t^{m+Q} \kappa_{\sigma, x}^{\mathbb{X}}(\delta_t v) \xrightarrow{t \rightarrow 0} \kappa_{\sigma_m, x}^{\mathbb{X}}(v).$$

*Furthermore, this convergence holds for any  $x$ -derivatives locally uniformly in  $x$ .*

*Proof of Lemma 9.58.* We start the proof with some preliminary analysis. For a symbol  $\tau$  in  $\cup_{m' \in \mathbb{R}} S^{m'}(\widehat{GM})$ , we denote by  $\tau \circ \delta_{t-1}$  the symbol obtained using the dilations in (2.11):

$$\forall (x, \pi) \in \widehat{GM}, \quad \forall t > 0, \quad \tau \circ \delta_{t-1}(x, \pi) = \tau(x, \pi \circ \delta_{t-1}).$$

We check readily that the convolution kernels satisfy

$$\kappa_{\tau \circ \delta_{t-1}} = t^{m'} \kappa_\tau \circ \delta_t, \quad \lambda > 0,$$

as densities. If a smooth Haar system  $\mu = \{\mu_x\}_{x \in M}$  has been fixed, then this becomes:

$$\kappa_{\tau \circ \delta_{t-1}, x}^\mu = t^{m'+Q} \kappa_{\tau, x}^\mu \circ \delta_t.$$

We fix  $\widehat{\mathcal{R}}$  and  $\phi$  as in Proposition 9.48, that is,  $\widehat{\mathcal{R}}$  is a positive Rockland symbol on  $\widehat{GM}$  and  $\phi \in C^\infty(M \times \mathbb{R})$  satisfies  $\phi(x, \lambda) = 0$  for  $\lambda$  in a neighbourhood of  $(-\infty, 0]$  and  $\phi(x, \lambda) = 1$  on a neighbourhood of  $+\infty$ , these two  $\lambda$ -neighbourhoods depending locally uniformly on  $x \in M$ . We denote by  $\nu$  the homogeneous degree of  $\widehat{\mathcal{R}}$ . Then

$$\phi(\widehat{\mathcal{R}}) \circ \delta_{t-1} = \phi(t^{-\nu} \widehat{\mathcal{R}}).$$

If now  $\tau \in S^{m'}(\widehat{G})$  is  $m'$ -homogeneous for some  $m' \in \mathbb{R}$ , then

$$t^{-m'} (\tau \phi(\widehat{\mathcal{R}})) \circ \delta_{t-1} = \tau \phi(t^{-\nu} \widehat{\mathcal{R}}) = \tau \phi(\widehat{\mathcal{R}}) + \rho_{\tau, t}$$

where  $\rho_{\tau, t}$  is the symbol given by

$$(9.9) \quad \rho_{\tau, t} := \tau \left( \phi(t^{-\nu} \widehat{\mathcal{R}}) - \phi(\widehat{\mathcal{R}}) \right).$$

Since  $\lambda \mapsto \phi(t^{-\nu} \lambda) - \phi(\lambda)$  is compactly supported, Corollary 5.44 implies that  $\rho_{\tau, t}$  is smoothing for all  $t > 0$ .

Let us now prove (1). We consider  $\sigma \in S_{ph}^m(\widehat{GM})$ . Writing its homogeneous expansion  $\sigma \sim_h \sigma_m + \sigma_{m-1} + \dots$ , it admits the inhomogeneous expansion  $\sigma \sim \sigma_m \phi(\widehat{\mathcal{R}}) + \sigma_{m-1} \phi(\widehat{\mathcal{R}}) + \dots$ . By the kernels estimates of Theorem 6.2, the function

$$R_x^N(v) := \kappa_{\sigma,x}(v) - \sum_{j=0}^N \kappa_{\sigma_{m-j}\phi(\widehat{\mathcal{R}}),x}(v)$$

is continuous in  $v$  when  $m - (N + 1) < -Q$  for each  $x \in M$ . In fact, Theorem 6.2 also shows that it is in  $L_s^2(G_x M)$  for any  $s \in \mathbb{R}$  such that  $2(s + m - (N + 1) + Q) < Q$ , i.e. for  $s < -Q/2 - m + (N + 1)$ . With these properties, we have

$$\begin{aligned} t^Q \kappa_{\sigma,x} \circ \delta_t - t^m \kappa_{\sigma,x} &= \sum_{j=0}^N \left( t^Q \kappa_{\sigma_{m-j}\phi(\widehat{\mathcal{R}}),x} \circ \delta_t - t^m \kappa_{\sigma_{m-j}\phi(\widehat{\mathcal{R}}),x} \right) + t^Q R_x^N \circ \delta_t - t^m R_x^N \\ &= \sum_{j=0}^N \kappa_{(\sigma_{m-j}\phi(\widehat{\mathcal{R}})) \circ \delta_{t^{-1},x} - t^m \sigma_{m-j}\phi(\widehat{\mathcal{R}})} + t^Q R_x^N \circ \delta_t - t^m R_x^N \\ &= \sum_{j=0}^N t^j \kappa_{\rho_{\sigma_{m-j},t,x}} + t^Q R_x^N \circ \delta_t - t^m R_x^N, \end{aligned}$$

using the notation (9.9). By the preliminary analysis, the kernel  $\sum_{j=0}^N t^j \kappa_{\rho_{\sigma_{m-j},t,x}}$  is smoothing. Moreover, the kernel  $t^Q R_x^N \circ \delta_t - t^m R_x^N$  is in  $L_s^2(G_x M)$  for all  $s$  such that  $s + m < -Q/2 + (N + 1)$ . The properties of Sobolev spaces on groups imply that  $t^Q \kappa_{\sigma,x} \circ \delta_t - t^m \kappa_{\sigma,x}$  is in  $C^r$  on  $G_x M$  with  $r = -m - Q + N$ , and this for all  $N \in \mathbb{N}$ , thus smooth. One concludes the proof by applying this to  $D_{\mathbb{X}}^\beta \sigma$  for any  $\beta \in \mathbb{N}_0^n$ , which shows (1).

Taking  $t \rightarrow 0$  shows the convergence at every  $x \in U$ . Again, applying this to  $D_{\mathbb{X}}^\beta \sigma$  for any  $\beta \in \mathbb{N}_0^n$  shows (2).  $\square$

*Proof of the inclusion  $\Psi_{cl}^m(M) \subset \Psi_{vEY}^m(M)$  in Theorem 9.56.* We fix an atlas  $\mathcal{A}$  on  $M$ . Let  $P \in \Psi_{cl}^m(M)$ . By Proposition 9.48, we may assume  $P = \text{Op}^A(\sigma)$  for some  $\sigma \in S_{ph}^m(\widehat{GM})$ . It follows from our construction that its integral kernel  $K_P$  is given by

$$K_P(x, y) = \sum_{\alpha \in A} \kappa_{\sigma,x}^{\mathbb{X}_\alpha}(-\ln_x^{\mathbb{X}_\alpha} y) \psi_\alpha(x) \psi_\alpha(y) \text{jac}_y(\ln_x^{\mathbb{X}_\alpha}), \quad x, y \in M.$$

It is natural to deform this convolution kernel into the following family of distributions on  $M \times M$  parametrised by  $t > 0$ :

$$\mathbb{P}(x, y, t) := \sum_{\alpha \in A} t^m \kappa_{\sigma,x}^{\mathbb{X}_\alpha}(-\delta_t \ln_x^{\mathbb{X}_\alpha} y) \psi_\alpha(x) \psi_\alpha(\exp_x^{\mathbb{X}_\alpha}(\delta_t \ln_x^{\mathbb{X}_\alpha} y)) \text{jac}_{\exp_x^{\mathbb{X}_\alpha}(\delta_t \ln_x^{\mathbb{X}_\alpha} y)}(\ln_x^{\mathbb{X}_\alpha}).$$

We observe for any  $v \in \mathbb{R}^n$ ,

$$\begin{aligned} D_{\exp_x^{\mathbb{X}_\alpha}(\delta_t \ln_x^{\mathbb{X}_\alpha} y)}(\ln_x^{\mathbb{X}_\alpha})(v) &= \partial_{s=0} \ln_x^{\mathbb{X}_\alpha} \exp_x^{\mathbb{X}_\alpha}((\delta_t \ln_x^{\mathbb{X}_\alpha} \exp_y^{\mathbb{X}_\alpha}(sv))) \\ &= \delta_t \partial_{s=0} \ln_x^{\mathbb{X}_\alpha} \exp_y^{\mathbb{X}_\alpha}(sv) \\ &= \delta_t D_y \ln_x^{\mathbb{X}_\alpha}(v), \end{aligned}$$

so

$$\text{jac}_{\exp_x^{\mathbb{X}_\alpha}(\delta_t \ln_x^{\mathbb{X}_\alpha} y)}(\ln_x^{\mathbb{X}_\alpha}) = t^Q \text{jac}_y \ln_x^{\mathbb{X}_\alpha}.$$

Therefore, we have

$$(9.10) \quad \mathbb{P}(x, y, t) = \sum_{\alpha \in A} t^{m+Q} \kappa_{\sigma, x}^{\mathbb{X}_\alpha}(-\delta_t \ln_x^{\mathbb{X}_\alpha} y) J_\alpha(x, y, t),$$

where

$$(9.11) \quad J_\alpha(x, y, t) := \psi_\alpha(x) \psi_\alpha(\exp_x^{\mathbb{X}_\alpha}(\delta_t \ln_x^{\mathbb{X}_\alpha} y)) \text{jac}_y(\ln_x^{\mathbb{X}_\alpha}).$$

Hence, we are lead to extend  $\mathbb{P}$  to  $\mathbb{T}_H M$  via

$$\mathbb{P}(x, v, 0) = \sum_{\alpha \in A} \psi_\alpha(x)^2 \kappa_{\sigma_m, x}^{\mathbb{X}_\alpha}(v).$$

By Lemma 9.58 (2), the equations 9.10 and 9.11 define a distribution  $\mathbb{P} \in \mathcal{E}'_r(\mathbb{T}_H M)$ . It remains to show that  $\mathbb{P}$  is essentially homogeneous of weight  $m \in \mathbb{R}$ .

We already have for  $t = 0$

$$\alpha_{\lambda^*} \mathbb{P}(x, v, 0) = \lambda^{m+Q} \mathbb{P}(x, v, 0).$$

We now consider for  $t, \lambda > 0$

$$\begin{aligned} \alpha_{\lambda^*} \mathbb{P}(x, y, t) &= \lambda^Q \mathbb{P}(x, y, \lambda^{-1}t) \\ &= \lambda^m \sum_{\alpha \in A} t^{Q+m} \kappa_{\sigma, x}^{\mathbb{X}_\alpha}(-\delta_{\lambda^{-1}t} \ln_x^{\mathbb{X}_\alpha} y) J_\alpha(x, y, \lambda^{-1}t). \end{aligned}$$

We have

$$(9.12) \quad \begin{aligned} &(\alpha_{\lambda^*} \mathbb{P} - \lambda^m \mathbb{P})(x, y, t) \\ &= R(x, y, t, \lambda) + \lambda^m \sum_{\alpha \in A} t^{Q+m} (\kappa_{\sigma, x}^{\mathbb{X}_\alpha}(-\delta_{\lambda^{-1}t} \ln_x^{\mathbb{X}_\alpha} y) - \kappa_{\sigma, x}^{\mathbb{X}_\alpha}(-\delta_t \ln_x^{\mathbb{X}_\alpha} y)) J_\alpha(x, y, \lambda^{-1}t), \end{aligned}$$

with

$$R(x, y, t, \lambda) := \lambda^m \sum_{\alpha \in A} t^{-(Q+m)} \kappa_{\sigma, x}^{\mathbb{X}_\alpha}(-\delta_t \ln_x^{\mathbb{X}_\alpha} y) (J_\alpha(x, y, \lambda^{-1}t) - J_\alpha(x, y, t)).$$

Moreover, the sum over  $\alpha \in A$  in (9.12) is smooth by Lemma 9.58 (1). We are left with showing the smoothness of  $R(x, y, t)$ . Recall that  $\kappa_{\sigma, x}$  is smooth away from 0. Since  $J_\alpha(x, y, \lambda^{-1}t) - J_\alpha(x, y, t) = 0$  indentially when  $x, y$  are near the diagonal,  $R$  is smooth on  $M \times M \times \mathbb{R}_+^* \times \mathbb{R}_+^*$ . We deduce that  $\alpha_{\lambda^*} \mathbb{P} - \lambda^m \mathbb{P}$  is smooth on  $M \times M$ .  $\square$

9.7.3. *The reverse inclusion.* For the proof of the reverse inclusion, we need to discuss the notion of *co-principal symbol* in the sense of van Erp and Yuncken [68, Section 6]: for any  $m \in \mathbb{R}$ , the co-principal symbol of an  $H$ -pseudodifferential kernel  $K \in \mathcal{E}'_r(M \times M)$  of order  $\leq m$  may be represented uniquely by a vertical distributional density  $\kappa \in \mathcal{S}'(GM; |\Omega|(GM))$  that is  $m$ -homogeneous in the sense of Definition 5.16; in this paper,  $\kappa$  is called the *homogeneous co-principal symbol* of  $K$ . This follows from choosing an essentially homogeneous distribution  $\mathbb{P} \in \mathcal{E}'_r(\mathbb{T}_H M)$  of weight 0 satisfying  $K = \mathbb{P}|_{t=1}$  and that is homogeneous on the nose outside of  $(-1, 1)$  [68, Proposition 42], and then applying [68, Proposition 43]. Moreover,  $\kappa$  is smooth away from the 0-section of  $GM$ .

The main point of the proof of the reverse inclusion of Theorem 9.56 is the natural correspondence between the 0-homogeneous co-principal symbols in  $\Psi_{vEY}(M)$  and the 0-homogeneous symbols in  $\Psi_{cl}(M)$ :

**Lemma 9.59.** *If  $\kappa \in \mathcal{S}'(GM; |\Omega|(GM))$  is 0-homogeneous and is smooth away from the 0-section of  $GM$ , then it is the convolution kernel of a symbol  $\sigma \in \dot{S}^0(\widehat{GM})$ .*

We will not need the converse, which is true, and follows from the properties of symbols in  $\dot{S}^0(\widehat{GM})$ : if the symbol  $\sigma$  is in  $\dot{S}^0(\widehat{GM})$ , then its convolution kernel  $\kappa$  is 0-homogeneous and is smooth away from the 0-section of  $GM$ . The proof of both implications follow from the study of the group case in [29].

*Proof of Lemma 9.59.* Let  $\kappa \in \mathcal{S}'(GM; |\Omega|(GM))$  be a 0-homogeneous distribution density that is smooth away from the 0-section of  $GM$ . It follows from Lemma 2.25 and Corollary 5.4 in [29] that we may define a symbol  $\sigma_0$  on  $M$  with convolution kernel  $\kappa$  such that  $\sigma_0(x, \cdot) \in \dot{S}^0(\widehat{G}_x M)$  with seminorms depending locally uniformly in  $x \in M$ . We then check that, for any adapted frame  $(\mathbb{X}, U)$  and  $\beta \in \mathbb{N}_0^n$ , the distribution  $\mathbb{X}^\beta \kappa^\mathbb{X} \in \mathcal{S}'(GM|_U; |\Omega|(GM|_U))$  is still 0-homogeneous and smooth away from the origin, so by uniqueness of the Fourier transform, above each  $x \in M$ , it is the convolution kernel of  $D_{\mathbb{X}}^\beta \sigma_0(x, \cdot) \in \dot{S}^0(\widehat{G}_x M)$ . This implies  $\sigma \in \dot{S}^0(\widehat{GM})$ .  $\square$

The properties of  $\Psi_{cl}(M)$  and the inclusion  $\Psi_{cl}(M) \subseteq \Psi_{vEY}(M)$  then imply:

**Corollary 9.60.** *We fix an atlas  $\mathcal{A}$  on  $M$ , and  $\widehat{\mathcal{R}}$  and  $\phi$  as in Proposition 9.48. Let  $m \in \mathbb{R}$  and let  $P \in \Psi_{vEY}^m(M)$ . Then there exists a symbol  $\sigma_m \in \dot{S}^m(\widehat{GM})$  such that*

$$P - \text{Op}^{\mathcal{A}}(\sigma_m \phi(\widehat{\mathcal{R}})) \in \Psi_{vEY}^{m-1}(M).$$

The proof of Corollary 9.60 follows from Lemma 9.59 and standard manipulations of pseudodifferential theory. Before giving it, let us conclude the proof of Theorem 9.56.

*End of the proof of Theorem 9.56.* It remains to show the inclusion  $\Psi_{vEY}^m(M) \subset \Psi_{cl}^m(M)$  for any  $m \in \mathbb{R}$ . Let  $P = P_m \in \Psi_{vEY}^m(M)$ . Let  $\sigma_m$  as in Corollary 9.60. Inductively, we construct a sequence of symbols  $\sigma_{m-j} \in \dot{S}^{m-j}$ ,  $j \in \mathbb{N}_0$ , such that

$$P_{m-(j+1)} = P_{m-j} - \text{Op}^{\mathcal{A}}(\sigma_{m-j} \phi(\widehat{\mathcal{R}})) \in \Psi_{vEY}^{m-(j+1)}(M),$$

By construction, we have for any  $N \in \mathbb{N}$ ,

$$P = \sum_{j=0}^N \text{Op}^{\mathcal{A}}(\sigma_{m-j}\phi(\widehat{\mathcal{R}})) + P_{m-(N+1)}$$

Consider a symbol  $\sigma \in S^m(\widehat{GM})$  with asymptotic expansion  $\sigma \sim \sum_j \sigma_{m-j}\phi(\widehat{\mathcal{R}})$ . Then  $\sigma \in S_{ph}^m(\widehat{GM})$  and for any  $N \in \mathbb{N}$ , the operator

$$P - \text{Op}^{\mathcal{A}}(\sigma) = \text{Op}^{\mathcal{A}}\left(\sum_{j=0}^N \sigma_{m-j}\phi(\widehat{\mathcal{R}}) - \sigma\right) + P_{m-(N+1)},$$

is in  $\Psi_{vEY}^{m-(N+1)}(M)$  since  $\sum_{j=0}^N \sigma_{m-j}\phi(\widehat{\mathcal{R}}) - \sigma \in S^{m-(N+1)}(\widehat{GM})$ . By Remark 9.57, this implies that  $P - \text{Op}^{\mathcal{A}}(\sigma) \in \Psi_{cl}^{-\infty}(M)$ .  $\square$

It remains to show Corollary 9.60. The case  $m = 0$  is essentially a consequence of Lemma 9.59:

*Proof of Corollary 9.60 when  $m = 0$ .* Let  $\kappa \in \mathcal{S}'(GM; |\Omega|(GM))$  be the homogeneous principal co-symbol of the integral kernel  $K_P$  of  $P$ . As we are in the case  $m = 0$ , by Lemma 9.59,  $\kappa$  is the convolution of a symbol  $\sigma_0 \in \dot{S}^0(\widehat{GM})$ . We now observe that the proof of the implication (2)  $\Rightarrow$  (1) in Theorem 9.56 above shows  $\text{Op}^{\mathcal{A}}(\sigma_0\phi(\widehat{\mathcal{R}})) \in \Psi_{vEY}^m(M)$ , and moreover that the homogeneous co-principal symbol of its integral kernel is  $\kappa_{\sigma_0} = \kappa \in \mathcal{S}'(GM; |\Omega|(GM))$ . This implies that the homogeneous co-principal symbol of the integral kernel of  $P - \text{Op}^{\mathcal{A}}(\sigma_0\phi(\widehat{\mathcal{R}}))$  is zero. Therefore, by [68, Corollary 38],  $P - \text{Op}^{\mathcal{A}}(\sigma_0\phi(\widehat{\mathcal{R}})) \in \Psi_{vEY}^{-1}(M)$ .  $\square$

Traditional manipulations in pseudodifferential theory allow us to pass from the case  $m = 0$  to the case of general  $m$ .

*Proof of Corollary 9.60 for any  $m \in \mathbb{R}$ .* As we have proved  $\Psi_{cl}^{m'}(M) \subseteq \Psi_{vEY}^{m'}(M)$  for any  $m' \in \mathbb{R}$ ,  $\text{Op}^{\mathcal{A}}((I + \widehat{\mathcal{R}})^{-m/\nu}) \in \Psi_{vEY}^{-m}(M)$ . By the composition properties of the calculus built by van Erp and Yuncken [68, Corollary 49],  $\text{Op}^{\mathcal{A}}((I + \widehat{\mathcal{R}})^{-m/\nu})P \in \Psi_{vEY}^0(M)$ . Denoting  $\tilde{\sigma}_0$  the corresponding symbol as in the case  $m = 0$ , we have

$$\text{Op}^{\mathcal{A}}((I + \widehat{\mathcal{R}})^{-m/\nu})P - \text{Op}^{\mathcal{A}}(\tilde{\sigma}_0\phi(\widehat{\mathcal{R}})) \in \Psi_{vEY}^{-1}(M).$$

Applying Corollary 9.35 to  $-m$ , we obtain a symbol  $\tau \in S_{ph}^m(\widehat{GM})$  satisfying

$$\text{Op}^{\mathcal{A}}(\tau)\text{Op}^{\mathcal{A}}((I + \widehat{\mathcal{R}})^{-m/\nu}) - I \in \Psi_{cl}^{-\infty}(M).$$

Moreover, an inspection of the construction shows that  $\text{Princ}_{m,h}(\tau) = \widehat{\mathcal{R}}^{m/\nu}$ . Hence, we have

$$\text{Princ}_{h,m}\left(\text{Op}^{\mathcal{A}}(\tau)\text{Op}^{\mathcal{A}}(\tilde{\sigma}_0\phi(\widehat{\mathcal{R}}))\right) = \text{Princ}_{h,m}\left(\text{Op}^{\mathcal{A}}(\tau)\right)\text{Princ}_{h,m}\left(\text{Op}^{\mathcal{A}}(\tilde{\sigma}_0\phi(\widehat{\mathcal{R}}))\right) = \widehat{\mathcal{R}}^{m/\nu}\tilde{\sigma}_0.$$

Consequently,

$$\text{Op}^{\mathcal{A}}(\tau)\text{Op}^{\mathcal{A}}(\tilde{\sigma}_0\phi(\widehat{\mathcal{R}})) = \text{Op}^{\mathcal{A}}(\widehat{\mathcal{R}}^{m/\nu}\tilde{\sigma}_0\phi(\widehat{\mathcal{R}})) \bmod \Psi_{cl}^{m-1}(M).$$

The inclusion  $\Psi_{cl}^{m'}(M) \subseteq \Psi_{vEY}^{m'}(M)$  for any  $m' \in \mathbb{R}$  together with the properties of the calculus built by van Erp and Yuncken, especially its composition [68, Corollary 49] and for the smoothing operators (see Remark 9.57), imply

$$\begin{aligned} P &= \text{Op}^A(\tau)\text{Op}^A((I + \widehat{\mathcal{R}})^{-m/\nu})P \bmod \Psi_{cl}^{-\infty}(M) \\ &= \text{Op}^A(\tau)\text{Op}^A(\widetilde{\sigma}_0\phi(\widehat{\mathcal{R}})) \bmod \Psi_{vEY}^{m-1}(M) \\ &= \text{Op}^A(\widehat{\mathcal{R}}^{m/\nu}\widetilde{\sigma}_0\phi(\widehat{\mathcal{R}})) \bmod \Psi_{vEY}^{m-1}(M), \end{aligned}$$

which terminates the proof.  $\square$

## APPENDIX A. COMBINATORIC

In this section we prove equation (4.5) and analyse the form of the terms after reorganisation by the number of elements in the set  $I$ , in order to obtain (4.9) with the afferent properties.

The proof of (4.5) is done by induction. The formula (4.5) holds for  $k = 1$  since, in that case, there is only  $n$  configuration of indices sets that satisfy the assumptions:  $I = \{j\}$ ,  $I_1 = \emptyset$ , which gives the term  $z_j X_j$ , for  $j = 1, \dots, n$ . It also holds for  $k = 2$  with the sets of indices:

$$\begin{aligned} I &= \{j_1, j_2\}, \quad I_1 = I_2 = \emptyset, \quad \text{whence the term } z_{j_1} z_{j_2} X_{j_1} X_{j_2}, \\ I &= \{j_1\}, \quad I_1 = \{j_2\}, \quad I_2 = \emptyset, \quad \text{whence } (X_{j_2} z_{j_1}) z_{j_2} X_{j_1}, \end{aligned}$$

with  $j_1, j_2$  describing  $1, \dots, n$ . The configuration  $I = \{j_1\}$ ,  $I_1 = \emptyset$ ,  $I_2 = \{j_2\}$  is forbidden.

Assume now that the formula (4.5) holds for some  $k \geq 1$ , let us prove that the result also holds for  $k + 1$ . We derive

$$\begin{aligned} (Z^{(N)})^{k+1} &= \sum_{j=1}^n \sum_{(j_1, \dots, j_k)=[I, I_1, \dots, I_k]} \left( z_j (\mathbb{X}_{I_1} z_{j_1}) \dots (\mathbb{X}_{I_k} z_{j_k}) \mathbb{X}_{\{j\}, I} \right. \\ &\quad \left. + \sum_{1 \leq p \leq k} z_j (\mathbb{X}_{I_1} z_{j_1}) \dots (\mathbb{X}_{\{j\}, I_p} z_{j_p}) \dots (\mathbb{X}_{I_k} z_{j_k}) \right). \end{aligned}$$

Denoting by  $j_{k+1}$  the new index, we obtain

$$\begin{aligned} (Z^{(N)})^{k+1} &= \sum_{(j_1, \dots, j_k)=[I, I_1, \dots, I_k]} \sum_{j_{k+1}=1}^n \left( (\mathbb{X}_{I_1} z_{j_1}) \dots (\mathbb{X}_{I_k} z_{j_k}) z_{j_{k+1}} \mathbb{X}_{\{j_{k+1}\}, I} \right. \\ &\quad \left. + \sum_{1 \leq p \leq k} (\mathbb{X}_{I_1} z_{j_1}) \dots (\mathbb{X}_{\{j_{k+1}\}, I_p} z_{j_p}) \dots (\mathbb{X}_{I_k} z_{j_k}) z_{j_{k+1}} \mathbb{X}_I \right). \end{aligned}$$

We then observe the equality of the set of indices  $(j_1, \dots, j_{k+1}) = [I, I_1, \dots, I_{k+1}]$  (equipped with the associated constraints), with the disjoint union

$$\begin{aligned} & \bigcup_{j=1}^n \left\{ (j_1, \dots, j_{k+1}) = [[\{j_{k+1}\}, I], I_1, \dots, I_k, \emptyset] \right\} \\ & \cup \bigcup_{p=1}^k \bigcup_{j=1}^n \left\{ (j_1, \dots, j_{k+1}) = [I, I_1, \dots, I_{p-1}, [\{j_{k+1}\}, I_p], I_{p+1}, \dots, I_k, \emptyset] \right\}. \end{aligned}$$

This concludes the proof.

Let us now reorganise the terms of  $(Z^{(N)})^k$  in terms of the cardinal of the indices set  $I$  and write

$$(A.1) \quad (Z^{(N)})^k = \sum_{\ell=1}^k \sum_{I=(j_1, \dots, j_\ell)} a_I^{(\ell)} \mathbb{X}_I.$$

Our aim is to prove that for all  $\ell \geq 1$ , the term  $\sum_{I=(j_1, \dots, j_\ell)} a_I^{(\ell)} \mathbb{X}_I$  writes as the sum of terms having the property (4.8), that is such that there exists  $v_1^{\{\ell\}}, \dots, v_\ell^{\{\ell\}}$  such that  $a_I^{(\ell)} = v_{j_1}^{(\ell)} \dots v_{j_\ell}^{(\ell)}$  for  $I = (j_1, \dots, j_\ell)$ ,  $1 \leq \ell \leq k$ . We argue by a recursion and starts by observing that the result holds when  $k = 1$ . Let us now assume that the property holds for  $k \in \mathbb{N}$ , with

$$a_{j_1, \dots, j_\ell} = v_{j_1}^{(\ell)} \dots v_{j_\ell}^{(\ell)}, \quad 1 \leq \ell \leq k.$$

We want to prove that the same property holds for  $(Z^{(N)})^{k+1}$ . It is enough to apply  $Z^{(N)}$  to one of the terms of the sum and to prove that it has the desired structure. We write

$$\begin{aligned} Z^{(N)} \left( \sum_{I=(j_1, \dots, j_\ell)} a_I^{(\ell)} \mathbb{X}_I \right) &= I_1 + I_2 \\ I_1 &= \sum_{j=1}^n \sum_{I=(j_1, \dots, j_\ell)} z_j^{(N)} a_I^{(\ell)} X_{\{\{j\}, I\}} \quad \text{and} \quad I_2 = \sum_{I=(j_1, \dots, j_\ell)} (Z^{(N)} a_I^{(\ell)}) X_I \end{aligned}$$

where  $I_1$  contains derivatives of length  $\ell + 1$  and  $I_2$  derivatives of length  $\ell$ .

The term  $I_1$  writes

$$I_1 = \sum_{I=(j_1, \dots, j_\ell)} (z_1^{(N)} a_I^{(\ell)} X_1 X_I + \dots + z_n^{(N)} a_I^{(\ell)} X_n X_I) = \sum_{I'=(j_1, \dots, j_{\ell+1})} a_{I'}^{(\ell+1)} \mathbb{X}_{I'}$$

where the coefficients  $a_{I'}^{(\ell+1)}$  are constructed as follows: let  $I' = (j_1, \dots, j_{\ell+1})$ ,  $I' = (\{j_1\}, I)$  then  $a_{I'}^{(\ell+1)} = v_{j_1}^{(\ell+1)} \dots v_{j_{\ell+1}}^{(\ell+1)}$  with  $v_{j_1}^{(\ell+1)} = z_{j_1}$  and  $v_{j_p}^{(\ell+1)} = v_{j_{p-1}}^{(\ell)}$  for  $p = 2, \dots, \ell + 1$ . Therefore,  $I_1$  has the property (4.8).

Let us now consider the term  $I_2$ . It is the sum of  $\ell$  terms of the form

$$\sum_{I=(j_1, \dots, j_\ell)} v_{j_1}^{(\ell)} \dots (Z^{(N)} v_{j_p}^{(\ell)}) \dots v_{j_\ell}^{(\ell)} X_I, \quad 1 \leq p \leq \ell.$$

We observe

$$\sum_{I=(j_1, \dots, j_\ell)} v_{j_1}^{(\ell)} \dots (Z^{(N)} v_{j_p}^{(\ell)}) \dots v_{j_\ell}^{(\ell)} X_I = \sum_{I=(j_1, \dots, j_\ell)} v_{j_1}^{(\ell+1,p)} \dots v_{j_\ell}^{(\ell+1,p)} X_I$$

with  $v_{j_{p'}}^{(\ell+1,p)} = v_{j_{p'}}^{(\ell)}$  if  $p' \neq p$  and  $v_{j_p}^{(\ell+1,p)} = Z^{(N)} v_{j_p}^{(\ell)}$ . Therefore,  $I_2$  is the sum of  $k$  terms with the property (4.8), which concludes the proof.

## APPENDIX B. FUNCTIONAL SPACES ON A VECTOR BUNDLE

Here, we recall the definitions of spaces of Schwartz functions and tempered distributions associated with a (smooth, real, finite dimensional) vector bundle  $E$  over a (smooth) manifold  $M$ . We denote by  $d = \dim E$  the dimension of (the fiber of) the vector bundle  $E$ . The functional spaces we want to introduce will use the concept of vertical densities we first recall.

**B.1. Vertical densities on  $E$ .** We denote by  $|\Omega|(E)$  the space of 1-densities on  $E$  whose definition we recall:

**Definition B.1.** • A 1-density on  $\mathbb{R}^d$  is a map  $\omega : (\mathbb{R}^d)^d \rightarrow \mathbb{C}$  satisfying

$$\forall v_1, \dots, v_d \in \mathbb{R}^d, \forall A \in \text{GL}_d(\mathbb{R}), \quad \omega(Av_1, \dots, Av_d) = |\det A| \omega(v_1, \dots, v_d).$$

- A 1-density on  $E$  is map  $\omega : E^{\times d} \rightarrow \mathbb{C}$  such that  $\phi^* \omega(x, \cdot)$  is a 1-density on  $\mathbb{R}^d$  for every trivialising chart  $\phi : U \times \mathbb{R}^d \rightarrow E$  and every  $x \in U$ .

A 1-density on  $E$  is sometimes called a *vertical density* as it yields above each  $x \in M$  a density on the vector space  $E_x$ .

*Example B.2.* (1) If  $\tau \in \Gamma(\wedge^n \mathbb{T}M)$  is an  $n$ -form on  $M$  with  $n = \dim M$ , then  $|\tau| \in |\Omega|(\mathbb{T}M)$  is a density on the tangent bundle  $E = \mathbb{T}M$ .

- (2) More generally, for any vector bundle  $E$ , any  $\tau \in \Gamma(\wedge^d E)$  yields the 1-density  $|\tau| \in |\Omega|(E)$  on  $E$ .

The space  $|\Omega|(E)$  is a line bundle over  $M$ . Moreover, if  $\omega \in |\Omega|(E)$  never vanishes, then any other 1-density  $\omega' \in |\Omega|(E)$  differs by a smooth function  $c \in C^\infty(M)$ :

$$(B.1) \quad \forall (x, v) \in E \quad \omega'(x, v) = c(x) \omega(x, v).$$

We denote by  $|\Omega|^+(E)$  the set of positive 1-densities on  $E$ , i.e. the set of  $\omega \in |\Omega|(E)$  with  $\omega(x, v) > 0$  for any  $(x, v) \in E$ . We can readily construct positive 1-densities using local trivialising charts and subordinated covering of the manifold. As Example B.2 suggests, a positive 1-density on  $E$  may be viewed as an element of volume  $|dv|$  of the fiber  $E_x$  depending smoothly on  $x \in M$ . In the case of two positive 1-densities, the smooth function  $c \in C^\infty(M)$  they differ from in (B.1) is positive:  $c(x) > 0$  for all  $x \in M$ .

## B.2. Schwartz functions and vertical tempered densities on $E$ .

**Definition B.3.** A *Schwartz function*  $f$  on  $E$  is a smooth function  $f : E \rightarrow \mathbb{C}$  such that for every trivialising chart  $\phi : U \times \mathbb{R}^d \rightarrow E$ ,  $\phi^*f$  may be written as  $\phi^*f(x, v) = f_\phi(x, v)$  with

$$(B.2) \quad f_\phi \in C^\infty(U \times \mathbb{R}^d) \quad \text{and} \quad x \mapsto \phi^*f(x, \cdot) \in C^\infty(U; \mathcal{S}(\mathbb{R}^d)).$$

We denote by  $\mathcal{S}(E)$  the set of Schwartz functions on  $E$ . It is naturally equipped with a structure of topological vector space.

**Definition B.4.** A *vertical tempered density* on  $E$  is a continuous map  $\mathcal{S}(E) \rightarrow C^\infty(M)$  that is  $C^\infty(M)$ -linear.

A vertical tempered density  $\kappa$  is described in any trivialising chart  $\phi : U \times \mathbb{R}^d \rightarrow E$  by  $\phi^*\kappa(x, v) = \kappa_\phi(x, v)|dv|$  with

$$\kappa_\phi \in \mathcal{D}'(U \times \mathbb{R}^d) \quad \text{and} \quad x \mapsto \kappa_\phi(x, \cdot) \in C^\infty(U, \mathcal{S}'(\mathbb{R}^d)).$$

We denote by  $\mathcal{S}'(E; |\Omega|(E))$  the set of vertical tempered densities on  $E$ . It is naturally equipped with a structure of topological vector space. By definition, the map

$$\mathcal{S}(E) \times \mathcal{S}'(E; |\Omega|(E)) \longrightarrow C^\infty(M), \quad (\varphi, \kappa) \longmapsto (x \mapsto \int_E \varphi(x, v)\kappa(x, v)),$$

is continuous and  $C^\infty(M)$ -bilinear.

**Lemma B.5.** *We assume that a positive 1-density  $\omega \in |\Omega|^+(E)$  has been fixed on  $E$ .*

- (1) *A vertical tempered density  $\kappa \in \mathcal{S}'(E; |\Omega|(E))$  is described above every  $x \in M$  by a unique distribution  $\kappa_x^\omega \in \mathcal{S}'(E_x)$  satisfying*

$$\forall \varphi \in \mathcal{S}(E) \quad \int_E \varphi(x, v)\kappa(x, v) = \int_{E_x} \varphi(x, v)\kappa_x^\omega(v)d\omega_x(v).$$

- (2) *Moreover, if  $\omega' \in |\Omega|^+(E)$  is another positive 1-density on  $E$ , then*

$$\kappa_x^\omega = c(x)\kappa_x^{\omega'} \in \mathcal{S}'(E_x),$$

where  $c$  is the function defined via (B.1).

*Proof.* The distribution  $\kappa_x^\omega \in \mathcal{S}'(E_x)$  is defined invariantly via trivialisation charts, showing Part (1). Part (2) follows by uniqueness.  $\square$

The following statement is readily checked and defines the notion of *m-homogeneous vertical distributional density*.

**Lemma B.6.** *Here, we assume that the vector bundle  $E$  is graded, i.e. each fiber  $E_x$  admits a gradation in the sense of Definition 2.1, and the corresponding dilations  $\delta_{E_x}$  yield a map  $\delta : E \rightarrow E$  that is smooth (and hence is a morphism of the vector bundle  $E$ ).*

*Let  $m \in \mathbb{R}$ . The following properties of a vertical tempered density  $\kappa \in \mathcal{S}'(E; |\Omega|(E))$  are equivalent:*

- (1) *For one (and then any) positive 1-density  $\omega \in |\Omega|^+(E)$  on  $E$ , for every  $x \in M$ , the distribution  $\kappa_x^\omega \in \mathcal{S}'(E_x)$  is m-homogeneous for the dilations  $\delta_{E_x}$ .*

- (2) For every trivialising chart  $\phi : U \times \mathbb{R}^d \rightarrow E$  and ever  $x \in M$ , the distribution  $\kappa_\phi(x, \cdot)$  is  $m$ -homogeneous for the dilations for  $\phi^* \delta_{E_x}$

### B.3. Vertical Schwartz densities.

**Definition B.7.** A *Schwartz vertical density* on  $E$  is a smooth section  $f \in C^\infty(E; |\Omega|(E))$  such that for every trivialising chart  $\phi : U \times \mathbb{R}^d \rightarrow E$ , we may write  $\phi^* f(x, v) := f_\phi(x, v) |dv|$  with  $f_\phi$  as in (B.2).

The set  $\mathcal{S}(E; |\Omega|(E))$  of Schwartz vertical densities on  $E$  is naturally equipped with a structure of topological vector space and may be viewed naturally as a subspace of  $\mathcal{S}'(E; |\Omega|(E))$ . In fact, it is dense in  $\mathcal{S}'(E; |\Omega|(E))$ .

### REFERENCES

- [1] Andrei Agrachev, Davide Barilari and Ugo Boscain. *A Comprehensive Introduction to Sub-Riemannian Geometry*, Cambridge Studies in Advanced Mathematics, 181, Cambridge University Press, 2019.
- [2] Serge Alinhac and Patrick Gérard. *Pseudodifferential operators and the Nash-Moser theorem*. Graduate Studies in Mathematics, 82. American Mathematical Society, Providence, RI, 2007.
- [3] Iakovos Androulidakis and Omar Mohsen and Robert Yuncken. A pseudodifferential calculus for maximally hypoelliptic operators and the Helffer-Nourrigat conjecture, 2022, Arxiv:2201.12060.
- [4] Victor Arnaiz and Gabriel Rivière. Quantum limits of perturbed sub-Riemannian contact Laplacians in dimension 3, preprint arXiv:2306.10757.
- [5] Hajer Bahouri, Clotilde Fermanian-Kammerer and Isabelle Gallagher, *Phase-space analysis and pseudodifferential calculus on the Heisenberg group*, Astérisque, 342, 2012.
- [6] Claude Bardos, Gilles Lebeau, and Jeff Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.*, 30:1024–1065, 1992.
- [7] Richard Beals and Peter C. Greiner. Calculus on Heisenberg manifolds, *Annals of Mathematics Studies*, 119. Princeton University Press, Princeton, NJ, 1988.
- [8] Karine Beauchard, Jérémie Dardé and Sulvain Ervedoza. Minimal time issues for the observability of Grushin-type equations, *Ann. Inst. Fourier (Grenoble)*, 70(1):247–312, 2020.
- [9] André Bellaïche. The tangent space in sub-Riemannian geometry, in *Sub-Riemannian Geometry*, André Bellaïche, Jean-Jacques Risler Edts., Birkhäuser Verlag, 1991.
- [10] Lino Benedetto. Quantum ergodicity for contact metric structures. arXiv:2507.18216.
- [11] Nicolas Burq and Chenmin Sun. Time optimal observability for Grushin Schrödinger equation, *Anal. PDE*, 15(6):1487–1530, 2022.
- [12] Woocheol Choi and Raphaël Ponge. Tangent maps and tangent groupoid for Carnot manifolds, *Differential Geom. Appl.*, 62: 136–183, 2019.
- [13] Woocheol Choi and Raphaël Ponge. Privileged coordinates and nilpotent approximation of Carnot manifolds, I. General results *Journal of Dynamical and Control Systems* 25:109-157, 2019.
- [14] Woocheol Choi and Raphaël Ponge. Privileged coordinates and nilpotent approximation for Carnot manifolds, II. Carnot coordinates *Journal of Dynamical and Control Systems*, 25:631-670, 2019.
- [15] Michael Christ, Daryl Geller, Pawel Glowacki, and Larry Polin. Pseudodifferential operators on groups with dilations. *Duke Math. J.*, 68(1):3165, 1992.
- [16] Yves Colin de Verdière. Ergodicité et fonctions propres du laplacien, *Comm. Math. Phys.* 102(3):497-502, 1985.
- [17] Yves Colin de Verdière, Luc Hillairet and Emmanuel Trélat. Spectral asymptotics for sub-Riemannian Laplacians, I: Quantum ergodicity and quantum limits in the 3-dimensional contact case. *Duke Math. J.*, 167(1):109 174, 2018.

- [18] Yves Colin de Verdière, Luc Hillairet and Emmanuel Trélat. Small-time asymptotics of hypoelliptic heat kernels near the diagonal, nilpotentization and related results, *Ann. H. Lebesgue*, 4:897–971, 2021.
- [19] Yves Colin de Verdière and Cyril Letrouit. Propagation of well-prepared states along Martinet singular geodesics, *J. Spectr. Theory* 12(3):1235–1253, 2022.
- [20] Alain Connes. *Noncommutative geometry*. Academic Press, Inc., San Diego, CA, 1994.
- [21] Lawrence J. Corwin and Frederik P. Greenleaf. *Representations of nilpotent Lie groups and their applications*. Part I, Cambridge Studies in Advanced Mathematics, 18, Cambridge University Press, Cambridge, 1990.
- [22] Shantanu Dave and Stefan Haller, The heat asymptotics on filtered manifolds, *J. Geom. Anal.*, 30(1):337–389, 2020.
- [23] Brian E. Davies. *Spectral theory and differential operators*. Cambridge Studies in Advanced Mathematics, Cambridge University Press, 42, 1995.
- [24] Claire Debord and Georges Skandalis. Adiabatic groupoid, crossed product by  $\mathbb{R}$  and pseudodifferential calculus *Adv. Math.*, 257:66-91, 2014.
- [25] Claire Debord and George Skandalis, Pseudodifferential extensions and adiabatic deformation of smooth groupoid actions, *J Bull. Sci. Math.*, 139(7):750-776, 2015.
- [26] Jacques Dixmier,  *$C^*$ -algebras*, Translated from the French by Francis Jellet, North-Holland Mathematical Library, 15, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.
- [27] Jacques Dixmier. *von Neumann algebras*, Translated from the second French edition by Francis Jellet, North-Holland Mathematical Library, 27, North-Holland Publishing Co., Amsterdam, 1981.
- [28] Charles Epstein and Richard Melrose. The Heisenberg algebra, index theory and homology. Unpublished manuscript, available at weblink
- [29] Clotilde Fermanian Kammerer and Véronique Fischer. Defect measures on graded Lie groups, *Ann. Sc. Norm. Super. Pisa.*, 21(5):207-291, 2020.
- [30] Clotilde Fermanian Kammerer and Véronique Fischer. Semi-classical analysis on H-type groups, *Science China, Mathematics*, 62(6)1057-1086, 2019.
- [31] Clotilde Fermanian Kammerer and Véronique Fischer. Quantum evolution and sub-laplacian operators on groups of Heisenberg type, *Journal of Spectral Theory*, 11(3):1313-1367, 2021.
- [32] Clotilde Fermanian Kammerer, Véronique Fischer and Steven Flynn. Geometric invariance of the semi-classical calculus on nilpotent graded Lie groups, *The Journal of Geometric Analysis*, 33(4), 2023.
- [33] Clotilde Fermanian Kammerer, Véronique Fischer and Steven Flynn. Some remarks on semi-classical analysis on two-step Nilmanifolds, in *Quantum Mathematics I and II*, INdAM Series, vol. 57 and 58.
- [34] Clotilde Fermanian Kammerer and Cyril Letrouit. Observability and controllability for the Schrodinger equation on quotients of groups of Heisenberg type, *Journal de l'École Polytechnique, Math.*, 8:1459-1513, 2021.
- [35] Clotilde Fermanian Kammerer and Jérôme Le Rousseau. Semi-classical analysis, arXiv:2407.01998.
- [36] Véronique Fischer and Søren Mikkelsen, Semiclassical functional calculus on nilpotent Lie groups and their compact nilmanifolds, arXiv:2409.05520.
- [37] Véronique Fischer and Mickael Ruzhansky, *Quantization on nilpotent Lie groups*, Progress in Mathematics, 314, Birkhäuser Basel, 2016.
- [38] Véronique Fischer and Michael Ruzansky. Sobolev spaces on graded Lie groups, *Ann. Inst. Fourier (Grenoble)*, 67(4):1671–1723, 2017.
- [39] Gerald Folland. Subelliptic estimates and function spaces on nilpotent Lie groups, *Ark. Mat.*, 13(2):161-207, 1975.
- [40] Gerald Folland. Applications of analysis on nilpotent groups to partial differential equations. *Bull. Amer. Math. Soc.*, 83(5):912930, 1977.
- [41] Gerald Folland and Elias Stein. Hardy spaces on homogeneous groups, *Mathematical Notes*, 28, Princeton University Press, 1982.

- [42] Patrick Gérard. Microlocal defect measures. *Commun. in PDE*, 16(11):1761-1794, 1991.
- [43] Bernard Helffer and Johannes Sjöstrand. Équation de Schrödinger avec champ magnétique et équation de Harper, in *Schrödinger operators (Sønderborg, 1988)*, Lecture Notes in Phys. 345:118–197, Springer, Berlin, 1989.
- [44] Bernard Helffer and Jean Nourrigat, Caractérisation des opérateurs hypoelliptiques homogènes invariants à gauche sur un groupe de Lie nilpotent gradué. *Comm. Partial Differential Equations*, 4(8):899-958, 1979.
- [45] Bernard Helffer and Jean Nourrigat, *Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs*, Progress in Mathematics, vol. 58, Birkhuser, Boston, 1985.
- [46] Peter Hintz. An introduction to microlocal analysis. Lecture notes (weblink).
- [47] Lars Hörmander. Hypoelliptic second order differential equations, *Acta Math.*, 119:147–171, 1967.
- [48] Lars Hörmander. *The Analysis of Linear Partial Differential Operators I-III*, Springer Verlag (1983-85).
- [49] Andrzej Hulanicki, Joe Jenkins, Jean Ludwig. Minimum eigenvalues for positive Rockland operators. *Proceedings of the American Mathematical Society*, 94(4):718-720, 1985.
- [50] Alexandre A. Kirillov. *Lectures on the orbit method*, Graduate Studies in Mathematics, 64, American Mathematical Society, Providence, RI, 2004.
- [51] Joseph J. Kohn and Louis Nirenberg, An algebra of pseudo-differential operators, *Comm. Pure Appl. Math.*, 18:269–305, 1965.
- [52] Cyril Letrouit. Subelliptic wave equations are never observable, *Anal. PDE* 16(3):643–678, 2023.
- [53] Anders Melin. Lie filtrations and pseudodifferential operators. Preprint, 1982.
- [54] Detlef Müller, Fulvio Ricci and Elias Stein, Marcinkiewicz multipliers and multi-parameter structure on Heisenberg (-type) groups. I, *Invent. Math.*, 119, 2:199–233, 1995.
- [55] Alexander Nagel, Elias M. Stein, and Stephen Wainger, Balls and metrics defined by vector fields. I. Basic properties, *Acta Math.*, 155(1-2):103-147, 1985.
- [56] Louis Nirenberg, *Pseudo-differential operators*, C.I.M.E. 47, Lectures from CIME 1968 Summer School held in Stresa (reprint of the 1969 original).
- [57] Lawrence W. Polin, *Pseudodifferential operators on nilpotent Lie groups with dilations*, ProQuest LLC, Ann Arbor, MI, 1989; MR2638145
- [58] Raphaël S. Ponge. Heisenberg calculus and spectral theory of hypoelliptic operators on Heisenberg manifolds. *Mem. Amer. Math. Soc.*, 194(906):viii+ 134, 2008.
- [59] Gabriel Rivière. Asymptotic regularity of sub-Riemannian eigenfunctions in dimension 3: the periodic case, preprint arXiv:2311.02990.
- [60] Linda P. Rothschild and Elias M. Stein. Hypoelliptic differential operators and nilpotent groups. *Acta Math.*, 137(34):247320, 1976.
- [61] Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.
- [62] Brian Street, *Multi-parameter singular integrals*, Annals of Mathematics Studies, 189, Princeton University Press, Princeton, NJ, 2014.
- [63] Brian Street, *Maximal subellipticity*, De Gruyter Studies in Mathematics, 93, 2023.
- [64] Luc Tartar, H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations, *Proc. Roy. Soc. Edinburgh, Sect. A*, 115:193-230, 1990.
- [65] Michael Taylor. *Noncommutative harmonic analysis*, Mathematical Surveys and Monographs, 22, American Mathematical Society, Providence, RI, 1986.
- [66] Erik Van Erp. The Atiyah-Singer index formula for subelliptic operators on contact manifolds. Part I, *Annals of Mathematics*, 171: 16471681, 2010.
- [67] Erik Van Erp and Robert Yuncken. On the tangent groupoid of a filtered manifold. *Bull. of the L.M.S.* 49 (6):1000-1012, 2017.

- [68] Erik Van Erp and Robert Yuncken. A groupoid approach to pseudodifferential operators. *Crelle's Journal*, 759:151-182, 2019.
- [69] Steve Zelditch. Uniform distribution of eigenfunctions on compact hyperbolic surfaces, *Duke Math. J.*, 55(4):919-941, 1987.
- [70] Maciej Zworski. *Semiclassical analysis*, Graduate Studies in Mathematics, 138, American Mathematical Society, Providence, RI, 2012.