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On input-to-state stabilization of switching retarded control systems

Ihab Haidar* and Pierdomenico Pepe†

Abstract

In this paper we address input-to-state stabilization-type results for nonlinear switching retarded control systems. A methodology based on Fréchet differentiable Lyapunov-Krasovskii functionals is developed in order to compute input-to-state stabilizing controllers, with respect to actuator errors, for globally asymptotically stabilizable systems. The problem of input-to-state practical stabilization is also investigated and an efficient method is proposed in this context. Two examples are reported in order to show the effectiveness of the proposed methodologies.

Keywords: input-to-state stability, Input-to-state practical stability, Lyapunov–Krasovskii functionals, Retarded functional differential equations; Switching systems.

1 Introduction

The stabilization problem of switching retarded systems has attracted considerable interest in the literature of control theory (see, e.g., [1, 8]). The trajectory based approach is used in [1] to design state feedback stabilizing control for linear switching time-varying delay systems. In [8], an approach based on polytopic uncertain systems is used to synthesize feedback control for time-varying linear switching systems. The input-state and input-output feedback linearization approach, that is initially introduced for ordinary differential control systems, can be found in the literature (see, e.g., [3, 4] for time-delay systems, and [12] for switching discrete-time systems). Another technique, based on control Lyapunov functions [2] and universal formulas such as Sontag’s formula [18], initially introduced for ordinary differential control systems, is also developed for retarded control systems (see, e.g., [9, 15, 16]). The importance of this approach lies in its ability to ensure the robustness of a stabilizing control, as disturbances in actuation can compromise the performance and even lead to instabilities [11]. However, a principal problem with this approach concerns the existence and construction of suitable Lyapunov-Krasovskii functionals.

Here, we focus on the class of control-affine nonlinear switching retarded systems. We present two main contributions addressing the input-to-state stabilization problem. First, we prove that a globally stabilizable switching retarded system admits an input-to-state stabilizing (with respect to actuator disturbances) control law, provided that the control system obtained by closing the loop with the stabilizing (in the case of disturbance equal to zero) state feedback control law admits a suitable Fréchet differentiable Lyapunov–Krasovskii functional. The second contribution concerns the problem of input-to-state practical stabilization of switching retarded systems. We give a Sontag’s type formula that achieves practical attenuation of the actuator disturbance, in the sense that for bounded disturbances the closed-loop solution’s system can reach any arbitrarily fixed neighbourhood of the origin by increasing a tuning parameter of the control law. These two contributions provide a non-trivial extension of the ones developed in [14, 15] for non-switching and constant-delay control systems, to the case of switching systems with Lebesgue measurable input, switching and time-delay signals and are obtained by the help of converse Lyapunov-Krasovskii theorems recently developed in [5, 6].

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2 Switching retarded systems

2.1 Notation

By \mathbb{R} we denote the set of real numbers, \mathbb{R}_+ the set of non-negative real numbers, and $\overline{\mathbb{R}}$ the extended real line. $(\mathbb{R}^n, |\cdot|)$ denotes the n -dimensional Euclidean space, where n is a positive integer and $|\cdot|$ is the Euclidean norm. By 1_I we denote the indicator function of a non-empty subset I of \mathbb{R} . Given $\Delta > 0$, $\mathcal{C} = (\mathcal{C}([-\Delta, 0], \mathbb{R}^n), \|\cdot\|_\infty)$ denotes the Banach space of continuous functions from $[-\Delta, 0]$ into \mathbb{R}^n , where $\|\cdot\|_\infty$ is the norm of uniform convergence. For a function $x : [-\Delta, b) \rightarrow \mathbb{R}^n$, with $0 < b \leq +\infty$, for $t \in [0, b)$, $x_t : [-\Delta, 0] \rightarrow \mathbb{R}^n$ denotes the history function defined by $x_t(\theta) = x(t + \theta)$, $-\Delta \leq \theta \leq 0$. For $H > 0$ and $\phi \in \mathcal{C}$, $\mathcal{C}_H(\phi)$ denotes the subset $\{\psi \in \mathcal{C} : \|\phi - \psi\|_\infty \leq H\}$. We simply denote $\mathcal{C}_H(0)$ by \mathcal{C}_H . A measurable function $u : \mathbb{R}_+ \rightarrow \mathbb{R}^m$, m positive integer, is said to be essentially bounded if $\text{ess sup}_{t \geq 0} |u(t)| < +\infty$. We keep using the symbol $\|\cdot\|_\infty$ to indicate the essential supremum norm of an essentially bounded function. For given times $0 \leq t_1 < t_2$, $u_{[t_1, t_2]} : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ indicates the function given by $u_{[t_1, t_2]} = u(t)1_{[t_1, t_2)}(t)$ for $t \geq 0$. A function $u : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ is said to be locally essentially bounded if, for any $t > 0$, $u_{[0, t]}$ is essentially bounded. A function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be of class \mathcal{K} if it is continuous, strictly increasing and $\alpha(0) = 0$; it is said to be of class \mathcal{K}_∞ if it is of class \mathcal{K} and unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be of class \mathcal{KL} if $\beta(\cdot, t)$ is of class \mathcal{K} for each $t \geq 0$ and, for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and converges to zero as t tends to $+\infty$. With $\|\cdot\|_a$ we indicate any functional in \mathcal{C} such that, for \mathcal{K}_∞ functions $\underline{\gamma}_a$ and $\overline{\gamma}_a$:

$$\underline{\gamma}_a(|\phi(0)|) \leq \|\phi\|_a \leq \overline{\gamma}_a(\|\phi\|_\infty), \forall \phi \in \mathcal{C}.$$

2.2 Definitions and assumptions

Consider the nonlinear switching retarded system

$$\begin{aligned} \dot{x}(t) &= f_{\sigma(t)}(x_t) + g_{\sigma(t)}(x_t)v(t), & \text{a.e. } t \geq 0, \\ x(\theta) &= x_0(\theta), & \theta \in [-\Delta, 0], \end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^n$; n is a positive integer; $\Delta > 0$ is the maximum involved delay; $x_0 \in \mathcal{C}$ is the initial state; $\sigma : \mathbb{R}_+ \rightarrow \mathcal{S}$ is the switching signal; \mathcal{S} is a non-empty index set; $f_s : \mathcal{C} \rightarrow \mathbb{R}^n$ and $g_s : \mathcal{C} \rightarrow \mathbb{R}^{n \times m}$, m positive integer, are uniformly (with respect to s) Lipschitz on bounded subsets of \mathcal{C} , and $f_s(0) = 0$, for $s \in \mathcal{S}$; $t \mapsto v(t) \in \mathbb{R}^m$ is Lebesgue measurable and locally essentially bounded input.

We denote by \mathcal{U} the set of Lebesgue measurable locally essentially bounded inputs from \mathbb{R}_+ to \mathbb{R}^m and by \mathcal{S} the set of measurable signals from \mathbb{R}_+ to \mathcal{S} .

Assumption 1. For each $\phi \in \mathcal{C}$, $\sigma \in \mathcal{S}$ and $v \in \mathcal{U}$, the function $f_{\sigma(\cdot)}(\phi) + g_{\sigma(\cdot)}(\phi)v(\cdot)$ is Lebesgue measurable.

Definition 1 ([6]). We say that system (1) is globally asymptotically stable at 0 (0-GAS for short) if there exists a class \mathcal{KL} function β such that for any $x_0 \in \mathcal{C}$ and any $\sigma \in \mathcal{S}$, the corresponding solution with $v \equiv 0$ exists in \mathbb{R}_+ and satisfies the inequality

$$|x(t)| \leq \beta(\|x_0\|_\infty, t), \quad \forall t \geq 0.$$

Definition 2 ([6]). We say that system (1) is input-to-state stable (ISS for short) if there exist a class \mathcal{KL} function β and a class \mathcal{K} function γ such that, for any $x_0 \in \mathcal{C}$, any locally essentially bounded input v , and any $\sigma \in \mathcal{S}$, the corresponding solution exists in \mathbb{R}_+ and satisfies the inequality

$$|x(t)| \leq \beta(\|x_0\|_\infty, t) + \gamma(\|v_{[0, t]}\|_\infty), \quad \forall t \geq 0. \quad (2)$$

Definition 3 ([6]). For a continuous functional $V : \mathcal{C} \rightarrow \mathbb{R}_+$, its Driver's form derivative, $D^+V : \mathcal{C} \times \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$, is defined, for system (1), for $\phi \in \mathcal{C}$ and $v \in \mathbb{R}^m$, as $D^+V(\phi, v) = \sup_{s \in \mathcal{S}} D_s^+V(\phi, v)$, where

$$D_s^+V(\phi, v) := \limsup_{h \rightarrow 0^+} \frac{V(\phi_h^s) - V(\phi)}{h},$$

and $\phi_h^s : [-\Delta, 0] \rightarrow \mathbb{R}^n$ is defined as follows

$$\phi_h^s(\theta) = \begin{cases} \phi(\theta + h) & \theta \in [-\Delta, -h) \\ \phi(0) + (\theta + h)(f_s(\phi) + g_s(\phi)v) & \theta \in [-h, 0]. \end{cases}$$

We give the following two definitions which are a variation of the ones given in [14] and [15].

Definition 4. A Lipschitz on bounded sets functional $V : \mathcal{C} \rightarrow \mathbb{R}^+$ is called uniformly regular for system (1) if it is Fréchet differentiable and, in addition, for each $s \in S$, the map $p_s : \mathcal{C} \rightarrow \mathbb{R}^m$ defined by

$$p_s(\phi) = \limsup_{h \rightarrow 0^+} \left(D_F V(\phi) \frac{1}{h} \phi_h^{g_s} e_1, \dots, D_F V(\phi) \frac{1}{h} \phi_h^{g_s} e_m \right)^T, \quad \phi \in \mathcal{C}, \quad (3)$$

is uniformly (with respect to s) Lipschitz on bounded subsets of \mathcal{C} , where $(e_i)_{1 \leq i \leq m}$ is the canonical bases of \mathbb{R}^m , $D_F V(\phi)$ is the Fréchet derivative of V at ϕ and $\phi_h^{g_s}$, for $h > 0$, is given by the continuous function

$$\phi_h^{g_s}(\theta) = \begin{cases} 0_{n \times m}, & \theta \in [-\Delta, -h) \\ (\theta + h)g_s(\phi), & \theta \in [-h, 0]. \end{cases}$$

Definition 5. A Lipschitz on bounded sets functional $V : \mathcal{C} \rightarrow \mathbb{R}^+$ is called robust universal Lyapunov-Krasovskii functional for system (1) if there exist class \mathcal{K}_∞ functions $\alpha_1, \alpha_2, \alpha_3$, positive reals r, p , and, for every $s \in S$, there exist Lipschitz on bounded sets functionals $a_s : \mathcal{C} \rightarrow \mathbb{R}^+$ and $b_s : \mathcal{C} \rightarrow \mathbb{R}^m$ such that

$$(A1) \quad \alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(\|\phi\|_a), \quad \forall \phi \in \mathcal{C};$$

(A2) the Driver derivative of V satisfies the following inequality

$$D_s^+ V(\phi, v) \leq a_s(\phi) + b_s^T(\phi)v, \quad \forall v \in \mathbb{R}^m, \forall \phi \in \mathcal{C};$$

$$(A3) \quad b_s(0) = 0; \text{ if } b_s(\phi) = 0 \text{ then } a_s(\phi) \leq 0;$$

$$(A4) \quad a_s(\phi)^2 + |b_s(\phi)|^4 \geq \alpha_3^2(\|\phi\|_a), \quad \forall \phi \in \mathcal{C};$$

$$(A5) \quad \sup_{\phi \in \mathcal{C}, 0 < |b_s(\phi)| \leq r} \frac{a_s(\phi)}{|b_s(\phi)|} \leq p.$$

The points (A1), (A2) and (A3) are standard in the theory of control Lyapunov functionals (see, e.g., [9, 15, 18]). The point (A4) allows the controller obtained in Section 3.2 to ensure that the Driver derivative of V satisfies the standard ISS inequality (see, e.g., [6]). The point (A5) guarantees that the obtained controller is locally Lipschitz.

3 Main results

Consider the system Σ defined by

$$\Sigma : \begin{cases} \dot{x}(t) = f_{\sigma(t)}(x_t) + g_{\sigma(t)}(x_t)(u(t) + d(t)), & \text{a.e. } t \geq 0, \\ x(\theta) = x_0(\theta), & \theta \in [-\Delta, 0], \end{cases}$$

where u and d are Lebesgue measurable and locally essentially bounded functions representing the control and the disturbance signals, respectively.

3.1 Input-to-state stabilization of 0-GAS systems

Theorem 1. *Suppose that there exist uniformly Lipschitz on bounded sets functionals $k_s : \mathcal{C} \rightarrow \mathbb{R}^m$, $s \in \mathbb{S}$, such that the disturbance free system Σ in closed-loop with $u(t) = k_{\sigma(t)}(x_t)$ is 0-GAS, and let V be an associated Lyapunov–Krasovskii functional. If V is uniformly regular (see Definition 4) then system Σ in closed-loop with*

$$u(t) = k_{\sigma(t)}(x_t) - p_{\sigma(t)}(x_t), \quad (4)$$

where $p_{\sigma(t)}$ is given by (3), is ISS.

Remark 1. *By converse results from [5, 6], the functional V in Theorem 1 satisfies (14)-(15). The result given in Theorem 1 holds if in (14)-(15) instead of $\|\cdot\|_\infty$ a functional $\|\cdot\|_a$ is used. See the proof of Theorem 1.*

3.2 Practical input-to-state stabilization

We introduce the functionals $k_{r,s} : \mathcal{C} \rightarrow \mathbb{R}^n$ given, for each $s \in \mathbb{S}$ and $r > 0$, by

$$k_{r,s}(\phi) = \begin{cases} -\frac{a_s(\phi) + \sqrt{a_s^2(\phi) + |b_s(\phi)|^4}}{|b_s(\phi)|^2} b_s(\phi), & |b_s(\phi)| > r, \\ -\frac{a_s(\phi) + \sqrt{a_s^2(\phi) + |b_s(\phi)|^4}}{r^2} b_s(\phi), & |b_s(\phi)| \leq r, \end{cases}$$

where a_s and b_s are given in Definition 5.

Theorem 2. *Suppose that system Σ admits a robust universal Lyapunov-Krasovskii functional (see Definition 5). Then, there exist functions β of class \mathcal{KL} and γ of class \mathcal{K} , both independent of r and p , such that, chosen any $q > 0$ for the closed-loop system Σ with*

$$u(t) = k_{r,\sigma(t)}(x_t) - qb_{\sigma(t)}(x_t), \quad (5)$$

the solution exists for all $t \geq 0$ and satisfies

$$|x(t)| \leq \beta(\|x_0\|_\infty, t) + \gamma\left(\sqrt{\frac{2}{q}}\|d_{[0,t]}\|_\infty\right) + \gamma\left(\sqrt{\frac{2}{q}}(2p+r)\right). \quad (6)$$

4 Examples

Example 1. *Consider the switching control system*

$$\dot{x}(t) = b(t)x(t - \tau(t)) + c(t)(1 + x^2(t - \tau(t)))(u(t) + d(t)) \quad (7)$$

where $t \geq 0$, $x(t) \in \mathbb{R}$; $\tau : \mathbb{R}_+ \rightarrow [0, \Delta]$, $b : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $c : \mathbb{R}_+ \rightarrow \mathbb{R}$ are Lebesgue measurable functions such that $c(t) \neq 0$, for all $t \geq 0$.

Let $\delta, \nu > 0$ and let $a : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a bounded Lebesgue measurable function such that $a(t) \geq \delta > 0$ and $(2a(t) - \delta)\delta > e^{\nu\Delta}b^2(t)$, for all $t \geq 0$. Let $\sigma(t) = (a(t), b(t), c(t), \tau(t))$ be the switching signal which take values in a compact set $\mathbb{S} \subset \mathbb{R}_+ \times \mathbb{R}^2 \times [0, \Delta]$. For each $s = (s_1, s_2, s_3, s_4) \in \mathbb{S}$, let the functionals $f_s, g_s : \mathcal{C} \rightarrow \mathbb{R}$ be defined, for $\phi \in \mathcal{C}$, as

$$f_s(\phi) = s_2\phi(-s_4) \quad \text{and} \quad g_s(\phi) = s_3(1 + \phi^2(-s_4)).$$

System (7) can be equivalently written as system Σ . Consider the state feedback control law defined by

$$k_s(\phi) = -\frac{1}{s_3} \frac{s_2(\phi(-s_4) + \phi(-\Delta)) + s_1\phi(0)}{1 + \phi(-s_4)^2}. \quad (8)$$

In the case where $d \equiv 0$, system (7) in closed-loop with $u(t) = k_{\sigma(t)}(x_t)$ is given by

$$\dot{x}(t) = -a(t)x(t) - b(t)x(t - \Delta). \quad (9)$$

By a slight variation of the proof given in [7, Chapter 5], system (9) can be proved 0-GAS using the Lyapunov-Krasovskii functional

$$V(\phi) = \frac{1}{2}\phi(0)^2 + \frac{\delta}{2} \int_{-\Delta}^0 e^{\nu\theta} \phi^2(\theta) d\theta,$$

which satisfies (14)-(15) with $\|\cdot\|_a$ (instead of $\|\cdot\|_\infty$) where $\|\cdot\|_a$ is given by

$$\|\phi\|_a = \frac{1}{2}\phi^2(0) + \frac{\delta}{2} \int_{-\Delta}^0 \phi^2(\theta) d\theta, \quad \forall \phi \in \mathcal{C}.$$

The Fréchet derivative of V is given, for $\phi, \psi \in \mathcal{C}$, by

$$D_F V(\phi)\psi = \phi(0)\psi(0) + \delta \int_{-\Delta}^0 e^{\nu\theta} \phi(\theta)\psi(\theta) d\theta. \quad (10)$$

In this case one can easily check that, for each $s \in \mathbb{S}$ and $\phi \in \mathcal{C}$, the function p_s defined in (3) is given by $p_s(\phi) = \phi(0)g_s(\phi)$. By consequence, the feedback control law defined in (4) and given, in this case, by

$$u(t) = k_{\sigma(t)}(x_t) - c(t)x(t)(1 + x^2(t - \tau(t))) \quad (11)$$

guarantees that (7) in closed-loop with (11) is ISS.

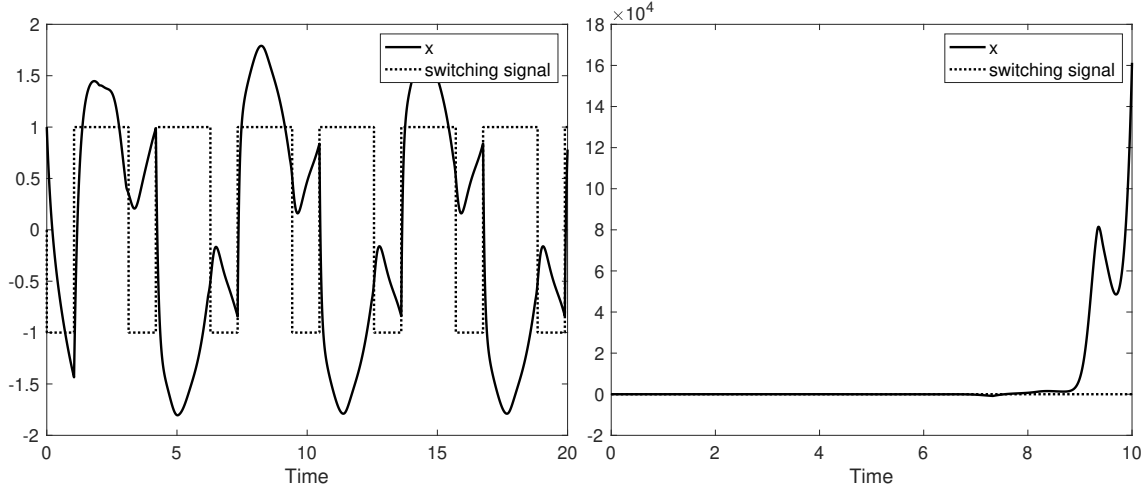


Figure 1: Example 1 with $\tau \equiv 2$, $a \equiv 2$, $b \equiv 1$, $c(t)$ switches in $\{-1, 1\}$ and $d(t) = 5 \sin(t)$. Top: $u(t)$ is given by (11) and bottom: $u(t) = k_{\sigma(t)}(x_t)$.

Remark 2. In the case where $b \equiv c \equiv 1$ and $\tau \equiv 0$, system (7) turns to be the motivating example provided in [17] to show destabilizing effect of added disturbances.

Example 2. Consider the time-delay control system

$$\dot{x}(t) = x^3(t - \tau(t)) + (x(t) + \frac{1}{2}x(t - \tau(t)))^3(u(t) + d(t)), \quad (12)$$

where $t \geq 0$, $x(t) \in \mathbb{R}$; $\tau : \mathbb{R}_+ \rightarrow [0, \Delta]$ is a Lebesgue measurable function. Let $\sigma \equiv \tau$ and $\mathbb{S} = [0, \Delta]$. Let the functionals $f_s, g_s : \mathcal{C} \rightarrow \mathbb{R}$, $s \in \mathbb{S}$, be defined as

$$f_s(\phi) = \phi^3(-s), \quad g_s(\phi) = (\phi(0) + \frac{1}{2}\phi(-s))^3.$$

System (12) can be equivalently written as system Σ . We define the functional $V : \mathcal{C} \rightarrow \mathbb{R}_+$ given by

$$V(\phi) = \frac{1}{2}\phi^2(0) + \int_{-\Delta}^0 \left(-\frac{\tau}{\Delta} + 2\frac{\tau+\Delta}{\Delta} \right) \phi^4(\tau) d\tau.$$

The functionals a_s and b_s in Definition 5 are given by

$$\begin{aligned} a_s(\phi) &= \phi(0)\phi^3(-s) + 2\phi^4(0) - \phi^4(-\Delta) - \frac{1}{\Delta} \int_{-\Delta}^0 \phi^4(\theta) d\theta \\ b_s(\phi) &= \phi(0)(\phi(0) + \frac{1}{2}\phi(-s))^3. \end{aligned}$$

We define the functional $\|\cdot\|_a$ given by

$$\|\phi\|_a = \frac{1}{2}\phi^2(0) + \int_{-\Delta}^0 \phi^4(\theta) d\theta, \quad \forall \phi \in \mathcal{C}.$$

One can easily verify that the functional V satisfies the point A1) of Definition 5. The point A2) and A3) of Definition 5 clearly hold true. Concerning the point A4), let us define the function $h : \mathbb{R}^4 \rightarrow \mathbb{R}$ given by

$$h(y) = (y_1 y_2^3 + 2y_1^4 - y_3^4 - \frac{1}{\Delta} y_4^2)^2 + y_1^4 (y_1 + \frac{1}{2} y_2)^{12}.$$

Since h is positive definite and radially unbounded, then by [10, Lemma 4.3] there exists a class \mathcal{K}_∞ function α_3 such that the point A4) holds true. As far as the point A5) in Definition 5 is concerned, let us define the set

$$\Lambda = \{y = (y_1, y_2, y_3, y_4)^T \in \mathbb{R}^4 : y_1(y_1 + \frac{1}{2}y_2) \neq 0\}$$

and let $h : \Lambda \rightarrow \mathbb{R}$ be the function defined by

$$h(y) = \frac{y_1 y_2^3 + 2y_1^4 - y_3^4 - \frac{1}{\Delta} y_4^2}{|y_1(y_1 + \frac{1}{2}y_2)^3|}, \quad \forall y \in \Lambda.$$

The point A5) in Definition 5 holds true provided that, for suitable positive reals r, p , the following inequality

$$\sup_{0 < |y_1(y_1 + \frac{1}{2}y_2)^3| < r} h(y_1, y_2, y_3, y_4) \leq p, \quad (13)$$

holds. Set $y_2 = cy_1$, for $c \in \mathbb{R} \setminus \{-2\}$, and observe that (13) holds provided that $\sup_{c \neq -2} \frac{c^3 + 2}{|1 + \frac{1}{2}c|^3} \leq p$. As choice for p is $p = 8$. Thus, for any chosen $r, q > 0$, inequality (6) holds for the system (12)-(5), with $p = 8$.

5 Conclusion

We develop two methodologies for input-to-state stabilization of nonlinear switching retarded systems. The first methodology, based on Fréchet differentiable Lyapunov–Krasovskii functionals, is developed in order to compute input-to-state stabilizing controllers. The second methodology, based on Sontag’s type formula, is given for input-to-state practical stabilization. Two academic examples are provided to demonstrate the practical application of these methodologies.

6 Proofs of the main results

6.1 Proof of Theorem 1

Let us denote by Σ_0 the system Σ in closed-loop with $u(t) = k_{\sigma(t)}(x_t)$. Knowing that Σ_0 is 0-GAS then (see the proof of [5, Theorem 1]) there exists a Lipschitz on bounded sets functional $V : \mathcal{C} \rightarrow \mathbb{R}^+$ and

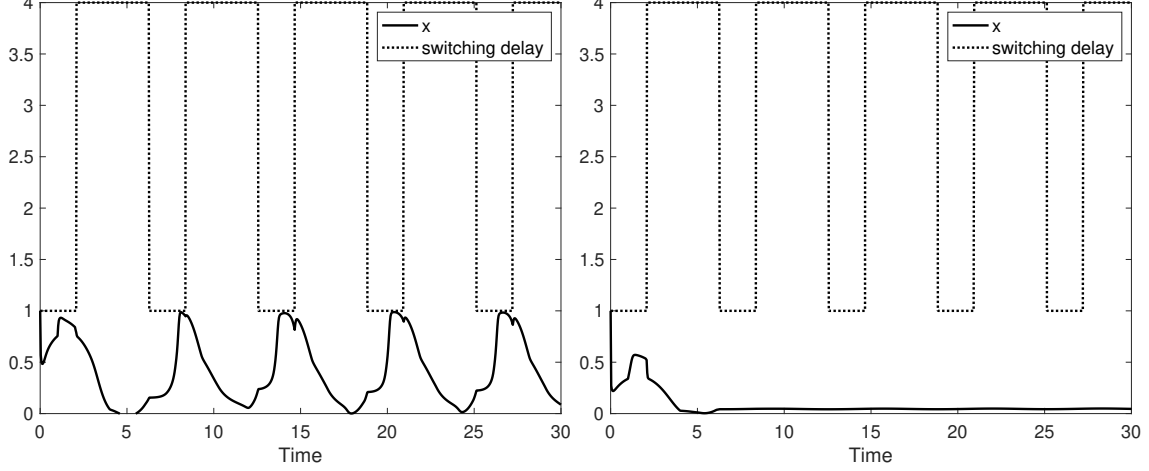


Figure 2: Example 2 in the case where τ switches in $\{1, 4\}$, $d(t) = 10 \sin(t)$ and $r = 0.1$. Top ($q = 5$) and bottom ($q = 50$).

$\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that for every $\phi \in \mathcal{C}$ we have

$$\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(\|\phi\|_\infty), \quad (14)$$

$$\limsup_{h \rightarrow 0^+} \frac{V(\phi_{h,0}^{\Sigma_0,s}) - V(\phi)}{h, 0} \leq -\alpha_3(\|\phi\|_\infty), \quad (15)$$

where the function $\phi_{h,0}^{\Sigma_0,s}$ is given by

$$\phi_{h,0}^{\Sigma_0,s}(\theta) = \begin{cases} \phi(\theta + h), & \theta \in [-\Delta, -h) \\ \phi(0) + (\theta + h)(f_s(\phi) + g_s(\phi)k_s(\phi)), & \theta \in [-h, 0]. \end{cases}$$

Suppose that V is uniformly regular. Let $\phi \in \mathcal{C}$ be a fixed globally Lipschitz function and let switching and input signals be piecewise-constant. We introduce, for $d \in \mathbb{R}^m$ and $s \in \mathbb{S}$, the function $\phi_{h,d}^{\Sigma,s} : [-\Delta, 0] \rightarrow \mathbb{R}^n$ given by

$$\phi_{h,d}^{\Sigma,s}(\theta) = \begin{cases} \phi(\theta + h), & \theta \in [-\Delta, -h) \\ \phi(0) + (\theta + h)(f_s(\phi) + g_s(\phi)(\kappa_s(\phi) + d)), & \theta \in [-h, 0] \end{cases} \quad (16)$$

with $\kappa_s(\phi) := k_s(\phi) - p_s(\phi)$. We have

$$\begin{aligned} \limsup_{h \rightarrow 0^+} \frac{V(\phi_{h,d}^{\Sigma,s}) - V(\phi)}{h} &= \limsup_{h \rightarrow 0^+} \left(\frac{V(\phi_{h,0}^{\Sigma_0,s}) - V(\phi)}{h} + \frac{V(\phi_{h,d}^{\Sigma,s}) - V(\phi_{h,0}^{\Sigma_0,s})}{h} \right) \\ &\leq \limsup_{h \rightarrow 0^+} \frac{V(\phi_{h,0}^{\Sigma_0,s}) - V(\phi)}{h} + \limsup_{h \rightarrow 0^+} \frac{V(\phi_{h,d}^{\Sigma,s}) - V(\phi_{h,0}^{\Sigma_0,s})}{h}. \end{aligned} \quad (17)$$

Let us introduce the following notations: $\psi_h^s := \phi_{h,d}^{\Sigma,s} - \phi$ and $\omega_h^s := \phi_{h,0}^{\Sigma_0,s} - \phi$. Taking into account the fact that V is Fréchet differentiable, after introducing

$$\begin{aligned} A(h) &:= V(\psi_h^s + \phi) - V(\phi) - D_F V(\phi) \psi_h^s \\ B(h) &:= V(\omega_h^s + \phi) - V(\phi) - D_F V(\phi) \omega_h^s \end{aligned}$$

it follows that

$$\begin{aligned} \limsup_{h \rightarrow 0^+} \frac{V(\phi_{h,d}^{\Sigma,s}) - V(\phi_{h,0}^{\Sigma_0,s})}{h} &= \limsup_{h \rightarrow 0^+} \left(\frac{A(h)}{h} - \frac{B(h)}{h} + D_F V(\phi) \frac{1}{h} (\psi_h^s - \omega_h^s) \right) \\ &\leq \limsup_{h \rightarrow 0^+} \frac{A(h)}{h} + \limsup_{h \rightarrow 0^+} \frac{-B(h)}{h} + \limsup_{h \rightarrow 0^+} D_F V(\phi) \frac{1}{h} (\psi_h^s - \omega_h^s). \end{aligned} \quad (18)$$

Observe that

$$\limsup_{h \rightarrow 0^+} \frac{A(h)}{h} = 0 \quad \text{and} \quad \limsup_{h \rightarrow 0^+} \frac{-B(h)}{h} = 0. \quad (19)$$

Indeed, concerning the first equality in (19), we have

$$\begin{aligned} \left| \limsup_{h \rightarrow 0^+} \frac{A(h)}{h} \right| &\leq \limsup_{h \rightarrow 0^+} \left| \frac{A(h)}{h} \right| = \limsup_{h \rightarrow 0^+, \|\psi_h^s\|_\infty \neq 0} \frac{|A(h)|}{\|\psi_h^s\|} \left\| \frac{\psi_h^s}{h} \right\|_\infty \\ &\leq \limsup_{h \rightarrow 0^+, \|\psi_h^s\|_\infty \neq 0} \frac{|A(h)|}{\|\psi_h^s\|} \limsup_{h \rightarrow 0^+, \|\psi_h^s\|_\infty \neq 0} \frac{\|\psi_h^s\|_\infty}{h} \\ &\leq \limsup_{h \rightarrow 0^+, \|\psi_h^s\|_\infty \neq 0} \frac{|A(h)|}{\|\psi_h^s\|} \limsup_{h \rightarrow 0^+, \|\psi_h^s\|_\infty \neq 0} \left\| \frac{\psi_h^s}{h} \right\|_\infty. \end{aligned}$$

Knowing that $\frac{\psi_h^s}{h}$ is uniformly bounded with respect to $h \in (0, \Delta)$ and taking into account that V is Fréchet differentiable we get the first equality in (19). The same reasoning leads to the second equality in (19). Therefore, from (18)-(19), it follows that

$$\begin{aligned} \limsup_{h \rightarrow 0^+} \frac{V(\phi_{h,d}^{\Sigma,s}) - V(\phi_{h,0}^{\Sigma_0,s})}{h} &\leq \limsup_{h \rightarrow 0^+} \frac{1}{h} D_F V(\phi) (\psi_h^s - \omega_h^s) \\ &= \limsup_{h \rightarrow 0^+} \frac{1}{h} D_F V(\phi) \phi_h^{g_s} (-p_s(\phi) + d) \\ &= \limsup_{h \rightarrow 0^+} (p_{s,1}, \dots, p_{s,m}) (-p_s(\phi) + d) \\ &= p_s(\phi)^T (-p_s(\phi) + d). \end{aligned} \quad (20)$$

By consequence, (17) together with (20) lead to the following inequality

$$\begin{aligned} \limsup_{h \rightarrow 0^+} \frac{V(\phi_{h,d}^{\Sigma,s}) - V(\phi)}{h} &\leq -\alpha_3(\|\phi\|_\infty) + p_s(\phi)^T (-p_s(\phi) + d) \\ &\leq -\alpha_3(\|\phi\|_\infty) - |p_s(\phi)|^2 + |p_s(\phi)|^2 + \frac{1}{4}|d|^2, \end{aligned}$$

from which we conclude that

$$D^+ V(\phi, d) \leq -\alpha_3(\|\phi\|_\infty) + \frac{1}{4}|d|^2. \quad (21)$$

So, by [6, Theorems 3.1 and 3.2], system Σ in closed-loop with (4) is ISS with respect to globally Lipschitz initial states. The equivalence of ISS with respect to continuous initial state and continuously differentiable initial state can be invoked to conclude the proof (see [13]).

6.2 Proof of Theorem 2

Lemma 1. *For any $r > 0$ and $s \in \mathcal{S}$ the functional $k_{r,s}$ is Lipschitz on bounded subsets of \mathcal{C} .*

Proof. Let $r > 0$ be fixed. We have to prove that for each $H > 0$ there exists $L_{r,H} > 0$ such that for every $\phi, \psi \in \mathcal{C}_H$ the following inequality holds

$$|k_{r,s}(\phi) - k_{r,s}(\psi)| \leq L_{r,H} \|\phi - \psi\|_\infty, \quad \forall s \in \mathbb{S}. \quad (22)$$

In order to simplify the presentation of the proof let us introduce, for $s \in \mathbb{S}$ and $\phi \in \mathcal{C}$, the functional

$$l_s(\phi) = - \left(a_s(\phi) + \sqrt{a_s^2(\phi) + |b_s(\phi)|^4} \right) b_s(\phi).$$

We distinguish three different cases:

1) $|b_s(\phi)| > r$ and $|b_s(\psi)| > r$: in this case we have

$$\begin{aligned} |k_{r,s}(\phi) - k_{r,s}(\psi)| &\leq \frac{|l_s(\phi)|b_s(\psi)|^2 - l_s(\psi)|b_s(\phi)|^2|}{r^4} \\ &\leq \frac{|l_s(\phi)| \left| |b_s(\psi)|^2 - |b_s(\phi)|^2 \right|}{r^4} + \frac{|b_s(\phi)|^2 |l_s(\phi) - l_s(\psi)|}{r^4}. \end{aligned}$$

Using the uniform (with respect to s) Lipschitzianity of $a_s(\cdot)$ and $b_s(\cdot)$, one can straightforwardly deduce the existence of $L_{r,H} > 0$ such (22) holds.

2) $|b_s(\phi)| \leq r$ and $|b_s(\psi)| \leq r$: in this case we have

$$|k_{r,s}(\phi) - k_{r,s}(\psi)| = \frac{1}{r^2} |l_s(\phi) - l_s(\psi)| \leq L_{r,H} \|\phi - \psi\|_\infty,$$

for some $L_{H,r} > 0$, and then (22) holds.

3) $|b_s(\phi)| > r$ and $|b_s(\psi)| \leq r$: let us introduce the function Ω_λ given by $\Omega_\lambda = \lambda\phi + (1-\lambda)\psi$, $\lambda \in [0, 1]$. Knowing that $\phi \mapsto b_s(\phi)$ is continuous, there exists $\lambda \in [0, 1]$ such that $|b_s(\Omega_\lambda)| = r$. Using the results of the previous two cases, we have:

$$\begin{aligned} |k_{r,s}(\phi) - k_{r,s}(\psi)| &= |k_{r,s}(\phi) - k_{r,s}(\Omega_\lambda) + k_{r,s}(\Omega_\lambda) - k_{r,s}(\psi)| \\ &\leq |k_{r,s}(\phi) - k_{r,s}(\Omega_\lambda)| + |k_{r,s}(\psi) - k_{r,s}(\Omega_\lambda)| \\ &\leq L_{r,H} \|\phi - \Omega_\lambda\|_\infty + L_{r,H} \|\psi - \Omega_\lambda\|_\infty \\ &= L_{r,H} (1-\lambda) \|\phi - \psi\|_\infty + L_{r,H} \lambda \|\phi - \psi\|_\infty \\ &= L_{r,H} \|\phi - \psi\|_\infty, \end{aligned}$$

and inequality (22) holds also in this case.

Hence the proof of the lemma. \square

Lemma 2. Let $r > 0$ be fixed. For each $s \in \mathbb{S}$, let $k_s : \mathcal{C} \rightarrow \mathbb{R}^n$ be defined as

$$k_s(\phi) = \begin{cases} -\frac{a_s(\phi) + \sqrt{a_s^2(\phi) + |b_s(\phi)|^4}}{|b_s(\phi)|^2} b_s(\phi), & b_s(\phi) \neq 0, \\ 0, & b_s(\phi) = 0. \end{cases}$$

We have the following inequality

$$|k_s(\phi) - k_{r,s}(\phi)| \leq 2p + r, \quad \forall \phi \in \mathcal{C}, \forall s \in \mathbb{S}, \quad (23)$$

where p is given by point (A5) in Definition 5.

Proof. If $|b_s(\phi)| > r$ or $|b_s(\phi)| = 0$, then $k_s(\phi) = k_{r,s}(\phi)$, and thus the inequality (23) holds. If $0 < |b_s(\phi)| \leq r$ then the following equality holds

$$\begin{aligned} |k_s(\phi) - k_{r,s}(\phi)| &= |k_s(\phi)| \left| 1 - \frac{|b_s(\phi)|^2}{r^2} \right| \leq |k_s(\phi)| \\ &\leq |b_s(\phi)| \left| \frac{a_s(\phi)}{|b_s(\phi)|^2} + \frac{|a_s(\phi)|}{|b_s(\phi)|^2} + 1 \right| \leq 2p + r, \end{aligned}$$

and the lemma is proved. \square

Proof of Theorem 2

Consider the functional V given in Definition 5. Let $\phi \in \mathcal{C}$ and $d \in \mathbb{R}^m$. From (5) together with (A2)-(A4) in Definition 5, it follows that

$$\begin{aligned} D^+V(\phi, d) &\leq \sup_{s \in \mathcal{S}} (a_s(\phi) + b_s^T(\phi)u + b_s^T(\phi)d) \\ &= \sup_{s \in \mathcal{S}} (a_s(\phi) - b_s^T(\phi)(k_s(\phi) - k_{r,s}(\phi)) + b_s^T(\phi)k_s(\phi) - qb_s^T(\phi)b_s(\phi) + b_s^T(\phi)d) \\ &\leq \sup_{s \in \mathcal{S}} (-\alpha_3(\|\phi\|_a) - b_s^T(\phi)(k_s(\phi) - k_{r,s}(\phi)) - q|b_s(\phi)|^2 + b_s^T(\phi)d). \end{aligned}$$

Then, thanks to Lemma 2, it follows that

$$D^+V(\phi, d) \leq \sup_{s \in \mathcal{S}} (-\alpha_3(\|\phi\|_a) + |b_s(\phi)||k_s(\phi) - k_{r,s}(\phi)| - q|b_s(\phi)|^2 + |b_s(\phi)||d|),$$

from which, using the Young's inequality, we obtain

$$\begin{aligned} D^+V(\phi, d) &\leq \sup_{s \in \mathcal{S}} (-\alpha_3(\|\phi\|_a) + \frac{(2p+r+|d|)^2}{4q} + q|b_s(\phi)|^2 - q|b_s(\phi)|^2) \\ &= -\alpha_3(\|\phi\|_a) + \left(\frac{2p+r+|d|}{2\sqrt{q}} \right)^2. \end{aligned} \quad (24)$$

Now, let $\sigma \in \mathcal{S}$ and $d \in \mathcal{U}$ be two piecewise-constant functions. Let x be the associated solution of Σ in closed-loop with (5) in a maximal interval $[0, b)$, $0 < b \leq +\infty$. Let $w : [0, b) \rightarrow \mathbb{R}_+$ be defined as

$$w(t) = V(x_t(\phi, d, \sigma)), \quad \forall t \in [0, b).$$

Knowing that d and σ are piecewise-constant, then, following the same reasoning as in the proof of [6, Theorem 3.2], we have

$$D^+w(t) \leq D^+V(x_t(\phi, d, \sigma), d(t)), \quad \forall t \in [0, b), \quad (25)$$

where the expression of $\phi_{h,d}^{\Sigma,s}$ is given by equation (16). From (24) and (25) it follows that

$$D^+w(t) \leq -\alpha_3(\|x_t\|_a) + \left(\frac{2p+r+|d(t)|}{2\sqrt{q}} \right)^2, \quad \forall t \in [0, b).$$

By analogous reasoning as in [6], it follows that the solution of the closed-loop system exists for all $t \geq 0$ and there exist functions β of class \mathcal{KL} and γ of class \mathcal{K} such that the following inequality holds for every $t \geq 0$

$$|x(t)| \leq \beta(\|\phi\|_\infty, t) + \gamma \left(\frac{\|d_{[0,t]}\|_\infty + 2p + r}{\sqrt{2q}} \right). \quad (26)$$

Knowing that $\gamma(v_1 + v_2) \leq \gamma(2v_1) + \gamma(2v_2)$, $v_1, v_2 \in \mathbb{R}_+$, inequality (6) follows from (26) for all piecewise-constant signals $\sigma \in \mathcal{S}$ and $d \in \mathcal{U}$. The ISS equivalence property given in [6, Theorem 1] allows to conclude.

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