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A dynamic analysis of criminal networks

Luca Colombo^a, Paola Labrecciosa^{b,*}, Agnieszka Rusinowska^c 

^a Rennes School of Business, 2 rue Robert d'Arbrissel, CS 76522 35065, Rennes, France

^b ESSCA School of Management (Paris Campus), 55 Quai Alphonse le Gallo, 92100 Boulogne-Billancourt, France

^c Centre d'Economie de la Sorbonne, CNRS, University Paris 1 Panthéon-Sorbonne, Paris School of Economics, 106-112 Boulevard de l'Hôpital, 75647 Paris, France

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ABSTRACT

We take a novel approach based on differential games to the study of criminal networks. We extend the static crime network game (Ballester et al., 2006, 2010) to a dynamic setting where criminal activities negatively impact the accumulation of total wealth in the economy. We derive a Markov Feedback Equilibrium and show that, unlike in the static crime network game, the vector of equilibrium crime rates is not necessarily proportional to the vector of Bonacich centralities. Next, we conduct a comparative dynamic analysis with respect to the network size, the network density, and the marginal expected punishment, finding results in contrast with those arising in the static crime network game. We also shed light on a novel issue in the network theory literature, i.e., the existence of a *voracity effect*. Finally, we study the problem of identifying the optimal target in the population of criminals when the planner's objective is to minimize aggregate crime at each point in time. Our analysis shows that the key player in the dynamic and the static setting may differ, and that the key player in the dynamic setting may change over time.

1. Introduction

It is natural to think about criminal and delinquent activities in terms of networks, and more specifically, social networks. Indeed, as argued in Lindquist and Zenou (2019), social network analysis can be quite useful for understanding more about the root causes of crime and delinquency, and for designing crime prevention policies. Not surprisingly, there exists a vast literature devoted to crime and networks (see, e.g., Lindquist and Zenou, 2019, for an overview). However, dynamic considerations in the context of criminal networks have not received enough attention so far. The present paper contributes to the network and crime literature, by proposing a novel (dynamic) approach to the study of criminal networks, i.e., a differential game approach.

The natural fit between crime and social networks comes particularly from the fact that crime is primarily considered as a group activity and that social interactions heavily affect criminal behavior.¹ Indeed, the importance of social networks and peer influences in criminal activities has been acknowledged for a long time in the criminology and sociology literature (e.g., Sutherland, 1947; Haynie, 2001; Sarnecki, 2001; Warr, 2002). Also, the economic literature is very active in the study of peer and network effects in crime. Sah (1991) and Glaeser et al. (1996) were the first to develop economic models of social interactions and crime, and were followed by others that proposed various theoretical foundations on peer and network effects in criminal activities (e.g., Calvó-Armengol and Zenou, 2004; Ballester et al., 2006, 2010; Cortés et al., 2019). In parallel to theoretical investigations, there is also strong empirical

* Corresponding author.

E-mail addresses: luca.colombo@rennes-sb.com (L. Colombo), paola.labrecciosa@essca.fr (P. Labrecciosa), agnieszka.rusinowska@univ-paris1.fr (A. Rusinowska).

¹ A classical example of crime as a group/family activity is that of Italian Mafias (e.g., Calderoni, 2012; Allum et al., 2019).

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evidence of peer effects in crime (e.g., Ludwig et al., 2001; Kling et al., 2005; Patacchini and Zenou, 2012; Bayer et al., 2009; Damm and Dustmann, 2014).

There exists a sizeable literature on applications of differential games in the field of crime and crime control (e.g., Feichtinger, 1983; Dawid and Feichtinger, 1996; Dubovik and Parakhonyak, 2014; Faria et al., 2019), government corruption (e.g., Kemp and Long, 2009; Ngendakuriyo and Zaccour, 2013, 2017), counterfeiting (e.g., Crettez et al., 2020) and terrorism (e.g., Nova et al., 2010; Wrzaczek et al., 2017). This literature has been able to shed light on a number of important issues related to the dynamics of illegal activities carried out by individuals, firms, and governments. However, it has abstracted from the widely recognized fact that criminals are embedded in social networks (see Ballester et al., 2010). In this paper, we aim to fill this gap in the literature by merging two so far disjoint strands of research, namely, the research on the dynamics of crime without social networks, and the research on social networks without dynamics. Indeed, to the best of our knowledge, ours is the first analysis of criminal networks in a full-fledged dynamic game.

Identifying the optimal targets and key players in social networks is a fundamental problem in various social, economic and political situations. In parallel to information diffusion, technology adoption, marketing or political campaigning, this problem is of particular importance in crime, delinquency and terrorism; for overviews on targeting and pricing, and key players in different contexts, see Bloch (2016) and Zenou (2016), respectively. Numerous studies in the economics literature characterize optimal targets by well established or new centrality measures (e.g., Ballester et al., 2006; Galeotti and Goyal, 2009; Candogan et al., 2012; Bloch and Querou, 2013; Banerjee et al., 2013, 2019; Bimpikis et al., 2016; Demange, 2017; Grabisch et al., 2018; Galeotti et al., 2020). Some works consider network formation in the setting of Ballester et al. (2006). Liu et al. (2012) develop a network formation model to determine key criminals, i.e., those who, once removed, generate the highest possible reduction in aggregate crime level in a network. At each period of time, a criminal is chosen at random and decides with whom she/he wants to form a link, anticipating the criminal effort game played by all criminals after a new link has been added. König et al. (2014) develop a two-stage game where agents play the game of Ballester et al. (2006) in the first stage, which is followed by a linking-formation process in the second stage. Network formation is also considered by Lee et al. (2021), who empirically identify the key player defined in Ballester et al. (2006).

Our analysis takes a different (dynamic) road and is conducted in terms of a differential game (see Başar and Olsder, 1995; Dockner et al., 2000; Haurie et al., 2012, and Long, 2010 for concepts and applications). As is well known, differential games are particularly useful for modeling economic problems which involve both dynamics and strategic behavior. We propose and analyze an infinite-horizon linear quadratic differential game based on the seminal papers by Ballester et al. (2006, 2010).² In our differential game, the state variable is the stock of total wealth legally produced in the economy. At each point in time, criminals embedded in a social network decide on the level of criminal activities, taking as given the criminal activities of the other criminals. Aggregate crime negatively affects the evolution of the state variable. As such, part of total wealth in the economy is transferred from the legal to the illegal sector.³ We assume that players use Markov strategies, i.e., they condition their level of criminal activities on the current state variable, and derive a Markov Feedback Equilibrium. Next, we perturb the equilibrium by changing the network size, the network density, the marginal expected punishment, and the implicit growth rate. Finally, we study the problem of identifying the key player, i.e., the player who, if removed, leads to the largest drop in aggregate crime.

Our main results can be summarized as follows. First, in the static game by Ballester et al. (2006), each player's action in a Nash equilibrium is proportional to the Bonacich centrality (Bonacich, 1987). In our dynamic setting, instead, this proportionality does not hold in general. However, we do recover the result by Ballester et al. (2006) as a particular case, when the shadow price of total wealth in the economy is the same for all players (e.g., in a regular network). Second, we show that a social multiplier effect, which occurs when an increase in the number of criminals, or links, or both, leads to an increase in aggregate crime, does not necessarily arise. In general, the results of our comparative dynamic analysis with respect to the network size and density⁴ and with respect to the expected marginal punishment suggest that some of the conclusions reached in the static literature on criminal networks do not necessarily carry over to a Markovian environment. Conditions exist such that more criminals or more connected criminals induce lower crime in the economy, and conditions exist such that the impact on aggregate crime of an increase in the marginal expected punishment in the static and the dynamic setting differ. This holds true both in the short run and at the steady-state equilibrium. In order to sharpen our intuition and highlight the network effect, we consider the two extreme network structures, namely, the empty and the complete network, and find results in contrast with the static analysis. Third, we show that a faster growing economy (in the absence of crime) may cause an increase in aggregate crime, which, in the end, may dampen economic growth. This is related to the so called voracity effect (see Tornell and Lane, 1999), which, to the best of our knowledge, has not been studied in the network literature so far. Fourth, we extend the analysis of the key player in Ballester et al. (2006, 2010), and show that the identification of the key player is more nuanced than in the static setting. In our dynamic game, the key player does not necessarily correspond to the player with the highest intercentrality measure. This is so because, compared with the static game, for each criminal, there exists an additional intertemporal marginal cost captured by the shadow price of total wealth in the economy. Moreover, in our dynamic game, it is possible that the key player changes over time. Beyond theoretical interest, this finding has clear policy implications:

² On the class of linear-quadratic differential games see Dockner et al. (2000, Chapter 7). Some examples of applications of linear-quadratic differential games in economics include Fershtman and Kamien (1987), Tsutsui and Mino (1990), Dockner and Long (1993), Dockner and Sorger (1996), Benchekroun (2003, 2008), Jun and Vives (2004), and Colombo and Labrecciosa (2015).

³ In a similar vein, there exists a (static) literature studying situations where power and coercion govern the exchange of resources, and stronger agents are able to take resources from weaker agents (see, e.g., Piccione and Rubinstein, 2004; Jordan, 2006; Piccione and Rubinstein, 2007).

⁴ The density of a network is a relative fraction of possible links that are present in the network. In other words, it is the average degree of all n nodes in the network divided by $n - 1$.

under certain circumstances, it is optimal (from an aggregate crime minimization perspective) to imprison a specific criminal only for a finite time, after which the same criminal should be reintegrated into society, and “substituted in jail” with a different criminal, either temporarily or ad infinitum.

The rest of the paper is organized as follows. In Section 2, we present the model, first by recalling the static setting of Ballester et al. (2010) in Section 2.1 and then by introducing our dynamic framework in Section 2.2. Section 3 is devoted to the Markov Feedback Equilibrium and the vector of Bonacich centralities. In Section 4, we conduct a comparative dynamic analysis with respect to the network size, the network density, the marginal expected punishment, and the implicit growth rate. In Section 5, we address the issue of identifying the key player in the network. Section 6 concludes. All the proofs are presented in the Appendices A till H.

2. Model

2.1. Static setting

The static crime network game Our point of departure is the framework of Ballester et al. (2010) as recalled below. We consider a criminal network game with a set $N = \{1, \dots, n\}$ of players (criminals) embedded in a network g of social connections.⁵ Let $\mathbf{G} = [g_{ij}]$ denote the n -square adjacency matrix of network g , keeping track of the (direct) connections in the network. Criminals i and j are connected in g if and only if $g_{ij} = 1$, and $g_{ij} = 0$ otherwise. By convention, $g_{ii} = 0$. We denote by g_N the complete network, i.e., the network where each node is connected to every other node with a direct link, and by g_\emptyset the empty network, i.e., the network with no connections between nodes. The criminals decide how much crime effort to exert. Let $\mathbf{x} = (x_1, \dots, x_n)$ denote the population crime profile, with $x_i \geq 0$ being crime effort exerted by criminal $i \in N$.

Following Becker (1968), Ballester et al. (2010) assume that criminals trade off the costs and benefits of criminal activities when deciding about their crime efforts. The expected gains to criminal i are given by

$$u_i(\mathbf{x}, g) = \underbrace{z_i(\mathbf{x})}_{\text{proceeds}} - \underbrace{p_i(\mathbf{x}, g)}_{\text{apprehension}} \underbrace{f}_{\text{fine}}. \tag{1}$$

The proceeds $z_i(\mathbf{x})$ correspond to the gross crime payoffs of criminal i . Ballester et al. (2010) assume that the higher the criminal connections to a criminal and/or the higher the involvement in criminal activities of these connections, the lower i 's probability to be caught $p_i(\mathbf{x}, g)$. Furthermore, for the sake of tractability, they restrict attention to the following expressions:

$$z_i(\mathbf{x}) = x_i \max \left\{ 1 - \delta \sum_{j=1}^n x_j, 0 \right\}, \tag{2}$$

$$p_i(\mathbf{x}, g) = p_0 x_i \max \left\{ 1 - \phi \sum_{j=1}^n g_{ij} x_j, 0 \right\}, \tag{3}$$

where $\delta > 0$ is the global substitutability parameter, $\phi > 0$ is the local complementarity parameter, and p_0 is the marginal probability of being caught for an isolated criminal.⁶ It is assumed that, at an equilibrium \mathbf{x}^* ,

$$1 - \delta \sum_{j=1}^n x_j^* \geq 0 \quad \text{and} \quad 1 - \phi \sum_{j=1}^n g_{ij} x_j^* \geq 0. \tag{4}$$

Then, by substituting $z_i(\mathbf{x})$ and $p_i(\mathbf{x}, g)$ given in (2) and (3) into $u_i(\mathbf{x}, g)$ given in (1), we get the following utility function⁷ of criminal i :

$$u_i(\mathbf{x}, g) = (1 - \pi) x_i - \delta \sum_{j=1}^n x_i x_j + \pi \phi \sum_{j=1}^n g_{ij} x_i x_j, \tag{5}$$

where $\pi = p_0 f$ is the marginal expected punishment cost for an isolated criminal. We assume (following Ballester et al., 2010) that $\pi < 1$.

The Bonacich centrality and Nash equilibrium Let $\mathbf{G}^k = [g_{ij}^{[k]}]$ denote the k th power of \mathbf{G} , where $k \in N$, keeping track of the indirect connections in the network. In particular, $\mathbf{G}^0 = \mathbf{I}$. Every coefficient $g_{ij}^{[k]} \geq 0$ measures the number of walks of length $k \geq 1$ in g between

⁵ Ballester et al. (2010) focus on petty crimes and therefore consider delinquents rather than criminals. In the present paper, we consider criminal networks.

⁶ Note that $p_i(\mathbf{x}, g)$ given in (3) is constant. This will be different in our dynamic setting, where the probability of being caught is going to depend on the entire history of the game (summarized by the evolving stock of total wealth in the economy).

⁷ The crime network game of Ballester et al. (2010) is developed by using the network model of Ballester et al. (2006) to the case of criminal networks. Ballester et al. (2006) consider the utility function $u_i(x_1, \dots, x_n) = \alpha x_i - \frac{1}{2}(\beta - \gamma) x_i^2 - \gamma \sum_{j=1}^n x_i x_j + \lambda \sum_{j=1}^n g_{ij} x_i x_j$. Hence, we have the following parameterization: $\alpha = 1 - \pi$, $\lambda = \pi \phi$, $\gamma = \beta - \delta$.

i and j , where a walk of length $k \geq 1$ between i and j is a sequence (i_0, \dots, i_k) of players such that $i_0 = i, i_k = j, i_p \neq i_{p+1}$ and $g_{i_p i_{p+1}} = 1$ for all $0 \leq p \leq k - 1$.

Definition 1. Consider a network g with adjacency n -square matrix \mathbf{G} and a scalar $a \geq 0$ such that the matrix

$$\mathbf{M}(g, a) = [m_{ij}(g, a)] = [\mathbf{I} - a\mathbf{G}]^{-1} = \sum_{k=0}^{+\infty} a^k \mathbf{G}^k$$

is well defined and nonnegative.⁸ Hence, the coefficients $m_{ij}(g, a) = \sum_{k=0}^{+\infty} a^k g_{ij}^{[k]}$ count the number of walks in g that start at i and end in j , where walks of length k are weighted by a^k .

(i) The vector of Bonacich centralities of parameter a in g is

$$\mathbf{b}(g, a) = [\mathbf{I} - a\mathbf{G}]^{-1} \cdot \mathbf{1},$$

where $\mathbf{1}$ is the n -dimensional vector of ones.⁹ Hence, the Bonacich centrality of node i given by

$$b_i(g, a) = \sum_{j=1}^n m_{ij}(g, a)$$

counts the total number of walks in g that start at i , where walks of length k are weighted by a^k . Note that $b_i(g, a) \geq 1$, with equality when $a = 0$.

(ii) The vector of weighted Bonacich centralities of parameter a in g is

$$\mathbf{b}_w(g, a) = [\mathbf{I} - a\mathbf{G}]^{-1} \cdot \mathbf{w}$$

with $\mathbf{w} = (w_1, \dots, w_n)^T$.

Let $b(g, a)$ denote the sum of the Bonacich centralities of all criminals, i.e.,

$$b(g, a) = \sum_{i=1}^n b_i(g, a).$$

Define $\theta = \pi\phi/\delta$. The numerator of θ is the weight given to *local* interactions whereas the denominator of θ is the weight given to *global* interactions (see (5)). Therefore, θ measures the relative importance of local interactions in terms of global interactions. In the right-hand-side sum of $m_{ij}(g, \theta) = \sum_{k=0}^{+\infty} \theta^k g_{ij}^{[k]}$, θ is a decay (or attenuating) factor that reduces the weight of longer walks; a generic coefficient $m_{ij}(g, \theta)$ is the sum of the direct and indirect influence of i on j , where indirect influence is via all arbitrarily long walks, with longer walks discounted according to the decay parameter θ . Note that an increase in θ leads to an increase in the Bonacich centralities since the weight given to $g_{ij}^{[k]}$ increases. Ballester et al. (2006) show that if $\theta\rho(g) < 1$, where $\rho(g)$ is the spectral radius of the adjacency matrix \mathbf{G} , i.e., the largest eigenvalue of \mathbf{G} , there exists a unique Nash equilibrium $\mathbf{x}_S^* = (x_{S,1}^*, \dots, x_{S,n}^*)^T$, which is interior, and given by

$$\mathbf{x}_S^* = \frac{(1 - \pi) \mathbf{b}(g, \theta)}{\delta [1 + b(g, \theta)]}.$$

Hence, the aggregate crime level $x_S^* = \sum_{i=1}^n x_{S,i}^*$ is equal to

$$x_S^* = \frac{(1 - \pi) b(g, \theta)}{\delta [1 + b(g, \theta)]}.$$

2.2. Dynamic setting

We extend the static game previously described to a dynamic setting.¹⁰ Time is continuous and denoted by $t \in [0, \infty)$. Let $y(t) \geq 0$ denote the aggregate stock of wealth which is legally produced and $x(t) = \sum_{i=1}^n x_i(t)$ the aggregate crime rate in the economy at t , with $x_i(t) \geq 0$ denoting the rate of criminal activities by criminal i at t . We assume that crime is constrained by wealth. While in the static model crime is unconstrained, or it is implicitly assumed that the wealth constraint is not binding, we explicitly consider the intertemporal relationship between $y(t)$ and $x(t)$, which is captured by the following differential equation:

$$\dot{y}(t) = \mu y(t) - x(t), \quad y(0) = y_0 \geq 0, \tag{6}$$

⁸ For any matrix \mathbf{G} , the matrix $\mathbf{M} = [\mathbf{I} - a\mathbf{G}]^{-1}$ is called the Leontief inverse of \mathbf{G} with parameter a .

⁹ More precisely, $\mathbf{b}(g, a)$ is obtained from Bonacich centrality (Bonacich, 1987) by an affine transformation and $\mathbf{b}(g, a) = \mathbf{1} + \mathbf{k}(g, a)$ with $\mathbf{k}(g, a)$ being Katz prestige measure (Katz, 1953). In the literature, Bonacich centrality is also called Katz-Bonacich centrality, as the measure is due to both authors.

¹⁰ Henceforth, we use players and criminals interchangeably and interpret effort levels in Ballester et al. (2010) as crime rates.

with $\mu > 0$ denoting the *implicit* rate of growth of total wealth and y_0 the initial level of total wealth in the economy. Crime is assumed to be wealth-reducing: one unit of criminal activities corresponds to one unit of wealth expropriation. The idea behind (6) is that criminal activities such as robberies and tax evasion have a negative impact on the accumulation of total wealth in the economy. Clearly, in the absence of crime, the growth rate of total wealth in the economy is strictly positive; otherwise, it can be negative (or nil).

Criminal i 's objective functional is given by

$$J_i = \int_0^{\infty} e^{-rt} u_i(x_1(t), \dots, x_n(t), g) dt, \quad (7)$$

with $u_i(x_1(t), \dots, x_n(t), g)$ being the dynamic counterpart of (5) and $r > 0$ the discount rate. Criminal i seeks to maximize J_i w.r.t. $x_i(t)$ subject to (6). Note that, in the maximization problem, g is time-invariant.

We consider a closed-loop information structure: at every point in time, criminal i chooses a level of the control variable $x_i(t)$ after having observed the current state variable, $y(t)$, which summarizes the entire history of the game.¹¹ As such, strategies are of the Markovian type. We define criminal i 's strategy space as follows.

Definition 2. (Stationary) Markov Strategy Space. Let L be a domain of the state space. Player i 's (stationary) Markov strategy space LS_i is such that $LS_i \equiv \{x_i \mid x_i \equiv \psi_i(y)$ is continuous in $y \in L$, $\psi_i(y) \geq 0$, and $|\psi_i(y) - \psi_i(y')| \leq K |y - y'|$ for some x_i -independent constant $K \geq 0\}$.

Admissible strategies are of the stationary type due to the structure of the game: the equation of motion is autonomous, and the instantaneous payoffs as well as the feasible sets do not explicitly depend on time. Stationarity implies that the payoff for player i of the game that starts at y at $t = \tau$ is equal to the payoff for player i of the game that starts at y at $t = 0$.¹² Given our definition of admissible strategies, from (6), we have

$$\dot{y} = \mu y - \sum_{i=1}^n \psi_i(y), \quad y(0) = y_0 \in L. \quad (8)$$

For each n -tuple of (stationary) Markov strategies $(\psi_1, \dots, \psi_n) \in LS_1 \times \dots \times LS_n$, Cauchy's theorem guarantees the existence of a solution to (8), $y^*(t)$.

Definition 3. (Stationary) Markov Feedback Equilibrium. Let L be a domain of the state space. An n -tuple of (stationary) Markov strategies $\psi^* = (\psi_1^*, \dots, \psi_n^*) \in LS_1 \times \dots \times LS_n$ is a (stationary) Markov Feedback Equilibrium if for every $y \in L$,

$$J_i(y; \psi^*) \geq J_i(y; \psi_i, \psi_{-i}^*)$$

for every $\psi_i \in LS_i$, ψ_{-i}^* being the $(n-1)$ -tuple of (stationary) Markov strategies ψ_j^* , with $j = 1, \dots, n$, $j \neq i$.

Markov Feedback Equilibrium strategies are computed via dynamic programming, which implies that the solution is Markov perfect by construction and satisfies the perfectness property introduced by Selten (1975) for extensive form games. It is common in the differential game literature to refer to the equilibrium computed via dynamic programming as a feedback (or closed-loop) equilibrium, whereas such equilibrium is usually referred to as Markov-perfect equilibrium in economic applications. Markov Feedback Equilibrium strategies are perfect state-space equilibria because the necessary optimality conditions are required to hold for all admissible values of the state variable (in the case of a unique state variable), and not just the values that lie on the optimal state-space trajectories. Therefore, the solutions obtained continue to remain optimal at each point in time after the game has begun. As is well-known in the differential game literature, the Markov perfect property is not lost when the domain of the state space is explicitly taken into account (see, e.g., Tsutsui and Mino, 1990; Dockner and Long, 1993; Dockner et al., 2000; Dockner and Wagener, 2014).¹³ In contrast to the open-loop equilibrium, where controls depend only on time and, consequently, the Markov perfect property does not hold in general, in a feedback equilibrium, controls are allowed to depend on past history through the current value of the state variable. Moreover, in a feedback equilibrium, players consider how changes in controls affect the state variable and, in turn, rivals' strategies. This feedback effect is by definition nil in an open-loop equilibrium, where players precommit to future actions at the beginning of the game and stick to them over the entire planning horizon.

¹¹ This restriction captures the notion that bygones are bygones (see Başar and Olsder, 1995; Dockner et al., 2000; Maskin and Tirole, 2001). By definition, history dependent strategies, such as trigger strategies, are ruled out.

¹² Note that stationarity alone is not sufficient to rule out equilibria involving non stationary strategies. However, as pointed out in Dockner et al. (2000), non stationary equilibria are of less interest and therefore they are generally not considered.

¹³ Dockner and Wagener (2014), in particular, make a distinction between local and global Markov Perfect Equilibria.

3. Markov feedback equilibrium and Bonacich centrality

In this section, we look for a Markov Feedback Equilibrium and, in the same spirit as Ballester et al. (2006), study the relationship between the equilibrium of the dynamic game and the vector of Bonacich centralities for general networks. In order to derive the equilibrium strategies, we adopt the value function approach. Let $V_i(y)$ denote criminal i 's value function, representing the discounted value of the stream of utilities for a game that starts at y in network g . By standard arguments (see Starr and Ho, 1969), Markov Feedback Equilibrium strategies must satisfy the following Hamilton-Jacobi-Bellman (HJB) equations ($i = 1, \dots, n$):

$$rV_i(y) = \max_{x_i \geq 0} \left\{ u_i(x_i, \psi_{-i}^*, g) + V_i'(y) \left[\mu y - x_i - \sum_{j=1, j \neq i}^n \psi_j^*(y) \right] \right\}, \tag{9}$$

where $V_i'(y) = \partial V_i(y)/\partial y$ denotes the nonnegative shadow price of total wealth for criminal i . $V_i'(y)$ can be interpreted as criminal i 's intertemporal marginal cost of committing crime. Suppose that $V_i'(y) \leq \partial u_i(x_i, \psi_{-i}^*, g)/\partial x_i$ (to ensure nonnegative criminal rates). Then, maximization of the RHS of (9) yields the following necessary and sufficient condition (given the concavity of u_i):

$$\frac{\partial u_i(x_i, \psi_{-i}^*, g)}{\partial x_i} - V_i'(y) = 0. \tag{10}$$

Given the linear-quadratic structure of the game, we guess a value function of the form

$$V_i(y) = A_i \frac{y^2}{2} + B_i y + C_i \Rightarrow \frac{\partial V_i(y)}{\partial y} = A_i y + B_i \geq 0,$$

and consider (stationary) linear Markov strategies

$$\psi_i(y(t)) = \alpha_i y(t) + \beta_i \geq 0,$$

where α_i and β_i are constants that depend on the parameters of the model. The linearity of $\psi_i(y(t))$ ensures that, given α_i and β_i , the initial value problem $\dot{y}(t) = \mu y(t) - \sum_{i=1}^n \psi_i(y(t)) = (\mu - \sum_{i=1}^n \alpha_i)y(t) - \sum_{i=1}^n \beta_i$, $y(0) = y_0 \in L$ has a unique solution. Let $\hat{y} = \sum_{i=1}^n \beta_i / (\mu - \sum_{i=1}^n \alpha_i)$ denote the steady-state equilibrium. We assume that the domain of the state space L is such that $\hat{y} \in L$. This guarantees that $y^*(t) \in L$ for all $t \in [0, \infty)$. Moreover, for analytical tractability, we focus on the domain of the state space L such that interior solutions prevail: the nonnegativity constraints on the control variables and the nonnegativity constraints on the shadow prices are not binding (implying that criminals play nondegenerate Markovian strategies). A sufficient but not necessary condition for the shadow price of total wealth in criminal i 's maximization problem to be nonnegative is $\psi_i(y) \leq x_{S,i}^*$ (see (10)). In addition, $\psi_i(y) \leq x_{S,i}^*$ implies that the assumptions specified in (4), which have to hold in the static game, also hold in our dynamic game. Constructing an equilibrium that involves corner solutions with active and inactive players would require the use of nonlinear strategies (for value functions to be continuously differentiable). Also, considering levels of y for which the nonnegativity constraint on the shadow price of total wealth in the economy is binding for some players but not for others could lead to possible discontinuities in the value functions. In such a case, one would need to check that there are no incentives for players to deviate from their linear strategies.¹⁴

In the next theorem, we provide a characterization of the equilibrium for interior solutions.

Theorem 1. Assume that $\mathbf{M}(g, \theta) = [\mathbf{I} - \theta \mathbf{G}]^{-1}$ satisfies the condition $\theta \rho(g) < 1$, and let

$$\psi^* = \mathbf{x}_S^* - \frac{\mathbf{b}_{\mathbf{V}'}(g, \theta)}{\delta [1 + b(g, \theta)]} \text{ for } y \in L$$

where

$$\mathbf{x}_S^* = \frac{(1 - \pi) \mathbf{b}(g, \theta)}{\delta [1 + b(g, \theta)]},$$

and $\mathbf{b}(g, \theta)$ and $\mathbf{b}_{\mathbf{V}'}(g, \theta)$ are the vector of Bonacich centralities of parameter θ in g and the vector of weighted Bonacich centralities of parameter θ in g with weights $\mathbf{V}' = (V'_1, \dots, V'_n)^T = (A_1 y + B_1, \dots, A_n y + B_n)^T$, respectively. The n -tuple of (stationary) Markov strategies $\psi^* = (\psi_1^*, \dots, \psi_n^*) \in LS_1 \times \dots \times LS_n$ constitutes a (stationary) Markov Feedback Equilibrium.

The condition $\theta \rho(g) < 1$ is needed for the matrix $\mathbf{M}(g, \theta) = [\mathbf{I} - \theta \mathbf{G}]^{-1}$ to be well-defined and nonnegative. This is the same condition as in the static game. As argued in Ballester et al. (2010), it relates the payoff function to the network topology and guarantees that local complementarities are not too large compared to own concavity. The condition $\theta \rho(g) < 1$ implies that instantaneous best responses are downward sloping, i.e., criminal rates are instantaneous “strategic substitutes”. Although criminals learn from their peers, there exists competition among them: for any given level of total wealth in the economy, an increase in crime by criminal j leads to a decrease in crime by criminal i , with $j \neq i$.

¹⁴ For the characterization of an asymmetric equilibrium involving corner solutions in the context of a two-player linear quadratic differential game of exploitation of a common-pool renewable resource see Benchekroun et al. (2014).

As we will show later in the paper, an explicit characterization of the upper and the lower bound of y such that interior solutions prevail can be obtained analytically in the case of a regular network, i.e., a network where each node has the same degree, or, equivalently, the same number of connections. Regular networks encompass the extreme cases of an empty and a complete network. For nonregular networks, instead, such thresholds can be computed numerically (once parameter values have been assigned).

The next two remarks are about the relationship between the Markov Feedback Equilibrium and the vector of Bonacich centralities.

Remark 1. Unlike in the static game studied in Ballester et al. (2006, 2010), the vector of equilibrium crime rates, ψ^* , is not necessarily proportional to the vector of Bonacich centralities, $\mathbf{b}(g, \theta)$.

Remark 2. When $V'_i = V'$ for all $i = 1, \dots, n$, since $\mathbf{b}_{V'}(g, \theta) = V' \mathbf{b}(g, \theta)$, ψ^* given in Theorem 1 becomes

$$\psi^* = \frac{(1 - \pi - V') \mathbf{b}(g, \theta)}{\delta [1 + b(g, \theta)]}.$$

In this case, as in the static game studied in Ballester et al. (2006, 2010), the vector of equilibrium crime rates is proportional to the vector of Bonacich centralities; if $V'_i = V' = 0$ for all $i = 1, \dots, n$ then ψ^* corresponds to x_S^* given in Theorem 1.

Corollary 1 below characterizes aggregate crime as a function of total wealth in the economy.

Corollary 1. *The Markov Feedback Equilibrium aggregate crime rate is given by*

$$x^*(y) = x_S^* \left[1 - \frac{1}{1 - \pi} \frac{b_{V'}(g, \theta)}{b(g, \theta)} \right] \quad \text{for } y \in L$$

where

$$x_S^* = \frac{(1 - \pi) b(g, \theta)}{\delta [1 + b(g, \theta)]} \tag{11}$$

and $b_{V'}(g, \theta)$ and $b(g, \theta)$ are the sum of the coordinates of the vector of weighted Bonacich centralities of parameter θ in g , with weights $V' = (V'_1, \dots, V'_n)^T = (A_1 y + B_1, \dots, A_n y + B_n)^T$, and the sum of the coordinates of the vector of Bonacich centralities of parameter θ in g , respectively.

Define

$$\omega(g, \theta) = \frac{b_{V'}(g, \theta)}{b(g, \theta)} \geq 0. \tag{12}$$

$x^*(y)$ given in Corollary 1 can then be rewritten as

$$x^*(y) = x_S^* \left[1 - \frac{\omega(g, \theta)}{1 - \pi} \right] \quad \text{for } y \in L. \tag{13}$$

For regular networks, $\omega(g, \theta)$ given in (12) corresponds to the shadow price of total wealth in the economy, V' . Hence, it can be interpreted as the intertemporal marginal cost of committing crime, the same for each criminal. For nonregular networks, instead, the intertemporal marginal cost of committing crime will be different for players with different Bonacich centralities. In this case, $\omega(g, \theta) = \sum_{i=1}^n b_i(g, \theta) V'_i / b(g, \theta)$ can be interpreted as a *weighted average* of intertemporal marginal costs of committing crime, with weights given by $(b_1(g, \theta) / b(g, \theta), \dots, b_n(g, \theta) / b(g, \theta))$. Clearly, for criminal i , the intertemporal marginal cost of committing crime is weighted by the *relative* influence that criminal i exerts over the other criminals embedded in the network.

Corollary 2 below characterizes aggregate crime as a function of time.

Corollary 2. *Take $y_0 \in L$. The Markov Feedback Equilibrium trajectory of aggregate crime is given by*

$$x^*(t) = x_S^* \left[1 - \frac{b_A(g, \theta) y^*(t) + b_B(g, \theta)}{(1 - \pi) b(g, \theta)} \right],$$

where

$$y^*(t) = \hat{y} + (y_0 - \hat{y}) \exp \left[t \left(\mu + \frac{b_A(g, \theta)}{\delta [1 + b(g, \theta)]} \right) \right],$$

and

$$\hat{y} = \frac{(1 - \pi) b(g, \theta) - b_B(g, \theta)}{\mu \delta [1 + b(g, \theta)] + b_A(g, \theta)} \in L,$$

and $b_A(g, \theta)$ and $b_B(g, \theta)$ are the sum of the coordinates of the vector of weighted Bonacich centralities of parameter θ in g with weights $\mathbf{A} = (A_1, \dots, A_n)^T$ and the sum of the coordinates of the vector of weighted Bonacich centralities of parameter θ in g with weights $\mathbf{B} = (B_1, \dots, B_n)^T$, respectively. The Markov Feedback Equilibrium trajectory of aggregate crime converges to $\hat{x} = \mu \hat{y}$ as $t \rightarrow \infty$ provided that

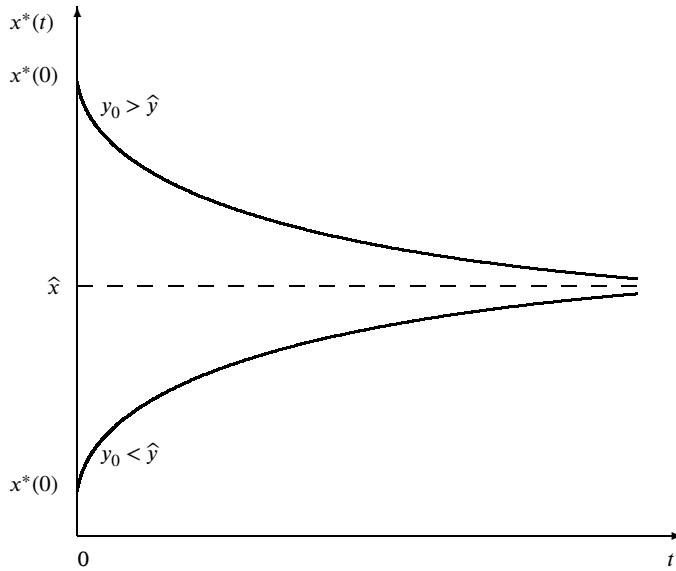


Fig. 1. Equilibrium trajectory of aggregate crime.

$$\mu + \frac{b_A(g, \theta)}{\delta [1 + b(g, \theta)]} < 0.$$

A necessary condition for the steady-state equilibrium to be stable is that $b_A(g, \theta) < 0$, which implies that $b_{V'}(g, \theta) = b_A(g, \theta)y + b_B(g, \theta)$ is decreasing in y . Hence, recalling from Corollary 1 that x^* is decreasing in $b_{V'}(g, \theta)$, steady-state stability implies that aggregate crime be increasing in y . Our theory then predicts that, *ceteris paribus*, higher (resp. lower) levels of aggregate crime should be observed in richer (resp. poorer) economies. A sufficient condition for this to occur is that $\mathbf{A} < \mathbf{0}$, which also guarantees that each criminal’s quadratic value function be concave, and therefore bounded on any bounded interval of the state space. $\mathbf{A} < \mathbf{0}$ implies that there exists “intertemporal strategic substitutability”: an increase in crime by criminal j at time t_0 leads to a decrease in crime by criminal i at time $t_1 > t_0$, with $j \neq i$. Each criminal realizes that an increase in own crime today will cause a reduction in total wealth in the economy in the future. Intuitively, given that equilibrium strategies are increasing in y , there exists a dynamic strategic incentive for criminal i to commit more crime today in order to reduce crime committed by the other criminals in the future.

Aggregate crime as a function of time is depicted in Fig. 1.

As an illustration of Corollaries 1 and 2, let us consider a regular network with $n \geq 2$ criminals, each having the same degree d .¹⁵ Regular networks encompass two important extreme cases, namely, the empty and the complete network. The former occurs when $d = 0$ and the latter when $d = n - 1$. Routine calculations lead to

$$b_i(g, \theta) = \frac{1}{1 - d\theta},$$

which implies that

$$b(g, \theta) = \frac{n}{1 - d\theta}.$$

From (11) and given that $\theta = \pi\phi/\delta$, we obtain

$$x_S^* = \frac{n(1 - \pi)}{\delta(1 + n) - d\pi\phi}.$$

Next, we verify that, for regular networks, $b_{V'}(g, \theta) = V' b(g, \theta)$ and therefore (from Corollary 1)

$$x^* = x_S^* \left(1 - \frac{V'}{1 - \pi} \right).$$

By definition, $b_{V',i}(g, \theta) = \sum_{j=1}^n m_{ij}(g, a)V'_j$, where $[m_{ij}(g, a)] = [\mathbf{I} - a\mathbf{G}]^{-1}$. Regular networks imply that $V'_j = V'$ for all $j = 1, \dots, n$. Hence, $b_{V',i}(g, \theta) = V' \sum_{j=1}^n m_{ij}(g, a)$. Given that $b_i(g, \theta) = \sum_{j=1}^n m_{ij}(g, a)$, it follows that $b_{V',i}(g, \theta) = V' b_i(g, \theta)$ and that $b_{V'}(g, \theta) = V' b(g, \theta)$.

¹⁵ It can be easily checked that, for regular networks, the condition $\theta\rho(g) < 1$ is satisfied when $\theta < 1/d$.

Maximization of the RHS of (9) yields

$$\psi^* = \frac{1 - \pi - V'}{\delta(1+n) - d\pi\phi}. \quad (14)$$

For each criminal, we guess a value function of the form $V = Ay^2/2 + By + C$. Using (9) and (14), the coefficients of V can be obtained by identification:

$$A = \frac{(r - 2\mu)[\delta(1+n) - d\pi\phi]^2}{2[\delta n^2 - (n-1)d\pi\phi]}, \quad (15)$$

$$B = \frac{(\pi - 1)(r - 2\mu)[\delta(n^2 + 1) - (n-1)d\pi\phi]}{2\mu[\delta n^2 - (n-1)d\pi\phi]}, \quad (16)$$

and

$$C = \frac{(B + \pi - 1)[\delta(Bn^2 + \pi - 1) - B(n-1)d\pi\phi]}{r[\delta(1+n) - d\pi\phi]^2}.$$

We know that a necessary condition for the steady-state equilibrium to be stable is that $b_A(g, \theta) < 0$, therefore $A < 0$. Since $\theta\rho(g) < 1$ implies $\delta - d\pi\phi > 0$ it follows that $\delta n^2 - (n-1)d\pi\phi > 0$ and that $\delta(n^2 + 1) - (n-1)d\pi\phi > 0$. Hence, for $r < 2\mu$, we have $A < 0$ and $B > 0$ (since $\pi - 1 < 0$). Note that $A < 0$ implies that ψ_i^* be increasing in y .

The Markov Feedback Equilibrium aggregate crime can be written as

$$x^*(y) = x_S^* \left(1 - \frac{Ay + B}{1 - \pi} \right), \quad (17)$$

with A and B given in (15) and (16), respectively. The admissible range for y is $L = \{y | \underline{y} \leq y \leq \bar{y}\}$, where $\underline{y} = (1 - \pi - B)/A > 0$ and $\bar{y} = -B/A > 0$, with $\bar{y} - \underline{y} = (\pi - 1)/A > 0$.

The Markov Feedback Equilibrium trajectory of aggregate crime is given by

$$x^*(t) = x_S^* \left[1 - \frac{Ay^*(t) + B}{1 - \pi} \right], \quad (18)$$

where

$$y^*(t) = \hat{y} + (y_0 - \hat{y}) \exp \left[t \left(\mu + \frac{An}{\delta(1+n) - d\pi\phi} \right) \right], \quad (19)$$

and

$$\hat{y} = \frac{n(1 - \pi - B)}{An + \mu[\delta(1+n) - d\pi\phi]},$$

with A and B previously defined. Aggregate crime increases (resp. decreases) over time if $y_0 < (\text{resp. } >) \hat{y}$. (19) corresponds to $y^*(t)$ given in Corollary 2. Indeed,

$$\frac{b_A(g, \theta)}{\delta[1 + b(g, \theta)]} = \frac{An}{\delta(1+n) - d\pi\phi},$$

since $b_A(g, \theta) = An/(1 - d\theta)$, with $\theta = \pi\phi/\delta$. Moreover, we have

$$\hat{x} = x_S^* \left(1 - \frac{A\hat{y} + B}{1 - \pi} \right),$$

which corresponds to \hat{x} given in Corollary 2 since, for regular networks, $b_A(g, \theta) = Ab(g, \theta)$ and $b_B(g, \theta) = Bb(g, \theta)$.

We now provide two numerical examples of Corollaries 1 and 2.¹⁶ The first example is based on the regular network case discussed above. The second is based on the simplest possible nonregular network, which is the star network with one central criminal and two periphery criminals. In these numerical examples, and also in all the numerical examples presented in the subsequent sections, we are going to use the following baseline parameter values: $\pi = 0.05$, $\phi = 0.1$, $\delta = 0.025$, $\mu = 0.03$ and $r = 0.0001$. Parameters ϕ , δ and r will be kept constant, whereas parameters π and μ will be changed in Sections 4.2 and 4.3, respectively. Note that the implied value of $\theta = \pi\phi/\delta$ is 0.2, which was also used in the numerical examples in Ballester et al. (2006, Table I) and Ballester et al. (2010, Table I).¹⁷ A value of μ equal to 0.03 means that the implicit growth rate of total wealth in the economy is 3%, which is a plausible growth rate, and a value of r equal to 0.0001 means that criminals slightly discount future payoffs (i.e., they are forward-looking).¹⁸ Note also that the considered parameter values are for illustration only. With reference to the comparative dynamics in Section 4, as long

¹⁶ In all numerical examples in the paper, we obtain exact values which we report rounded to four decimal places.

¹⁷ $\theta = 0.2$ guarantees that $\mathbf{M}(g, \theta) = [\mathbf{I} - \theta\mathbf{G}]^{-1}$ satisfies the condition $\theta\rho(g) < 1$ for all the network structures considered in this paper.

¹⁸ In numerical simulations of discrete-time models with forward-looking agents, the discount factor $1/(1+r)^t$ is usually taken to be very close to unity, implying a very low r . The continuous-time discount factor counterpart is $e^{-\delta t}$ with $\delta = \ln(1+r)$.

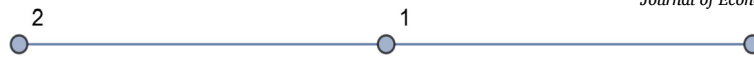


Fig. 2. Star network with three criminals.

as the general conditions stated in the relevant propositions are satisfied, the qualitative results of the numerical examples (meant to illustrate those propositions) remain unchanged. Indeed, we have run several simulations using different parameter values and found the same qualitative results as those in Section 4, and similarly for the numerical example about the key player in Section 5.¹⁹

Example 1. Consider a regular network with $n = 4$ and $d = 2$. From (17) and (18), we have, respectively,

$$x^*(y) = -2.1739 + 0.0372y,$$

for $y \in L = [58.3823, 945.8097]$ and

$$x^*(t) = -2.1739 + 0.0372y^*(t),$$

for $t \in [0, \infty)$, where

$$y^*(t) = 300.4607 + (y_0 - 300.4607)\exp(-0.0072t), \tag{20}$$

for $y_0 \in L = [58.3823, 945.8097]$. It can be easily checked that (20) corresponds to $y^*(t)$ in Corollary 2. If $y_0 <(\text{resp. } >)$ 300.4607, aggregate crime increases (resp. decreases) over time; as $t \rightarrow \infty$, aggregate crime converges to 9.0138.

Example 2. Consider network g in Fig. 2.

Routine calculations lead to $b_1(g, \theta) = 1.5217$ and $b_i(g, \theta) = 1.3043$ with $i = 2, 3$. It follows that $b(g, \theta) = 4.1304$ and that $x_S^* = 30.5932$.

In order to compute x^* we need $b_{V'}(g, \theta)$. Performing the maximization indicated in (9) leads to the following value functions:

$$V_1 = A_1 y^2 / 2 + B_1 y + C_1,$$

and

$$V_i = A_i y^2 / 2 + B_i y + C_i,$$

with $i = 2, 3$, where $A_1 = -0.0012$, $B_1 = 1.0431$, $C_1 = 29811.8260$, $A_i = -0.0013$, $B_i = 1.0712$, and $C_i = 23498.1967$. Hence, $V'_1 = 1.0431 - 0.0012y$, $V'_i = 1.0712 - 0.0013y$, and $b_{V'}(g, \theta) = 4.3817 - 0.0051y$. It can be easily checked that $\omega(g, \theta) = 1.0608 - 0.0012y$. From Corollary 1, we then get

$$x^*(y) = -3.5692 + 0.0396y,$$

for $y \in L = [108.7184, 856.7616]$. The resulting equilibrium trajectory of aggregate crime is given by

$$x^*(t) = -3.5692 + 0.0396y^*(t),$$

where

$$y^*(t) = 371.3222 + (y_0 - 371.3222)\exp(-0.0096t), \tag{21}$$

for $y_0 \in L$. It can be easily checked that (21) corresponds to $y^*(t)$ in Corollary 2. If $y_0 <(\text{resp. } >)$ 371.3222, aggregate crime increases (resp. decreases) over time; as $t \rightarrow \infty$, aggregate crime converges to 11.1397.

4. Comparative dynamics

In this section, we conduct a comparative dynamic analysis with respect to the network size, the network density, the marginal expected punishment, and the implicit growth rate of total wealth in the economy.

4.1. Network size and density

In the static game studied in Ballester et al. (2010), aggregate crime is increasing in either network size or density or both, a feature often referred to as *social multiplier effect*. In the absence of dynamic considerations, policies aimed at reducing aggregate crime should be designed so as to reduce the number of criminals or the number of links or both. In our dynamic game, instead, things are more involved, and a social multiplier effect does not necessarily exist.

¹⁹ The results of these simulations are available upon request from the authors.

Consider two networks, g with associated adjacency matrix \mathbf{G} and g' with associated adjacency matrix \mathbf{G}' , with $g \subset g'$ (meaning that g' contains either more criminals or more links or both). Formally: for all i, j , $g'_{ij} = 1$ if $g_{ij} = 1$. Recall that $b(g, \theta)$ counts the total number of weighted walks in g . Hence, $b(g, \theta)$ is an increasing function in g (for the inclusion ordering), as more links imply more such walks. Call $\Delta x_S^* = x_S^*(g') - x_S^*(g)$.²⁰ We have

$$\begin{aligned} \Delta x_S^* &= \frac{(1 - \pi) b(g', \theta)}{\delta [1 + b(g', \theta)]} - \frac{(1 - \pi) b(g, \theta)}{\delta [1 + b(g, \theta)]} \\ &= \frac{(1 - \pi)}{\delta} \left[\frac{[1 + b(g, \theta)] b(g', \theta) - [1 + b(g', \theta)] b(g, \theta)}{[1 + b(g', \theta)] [1 + b(g, \theta)]} \right], \end{aligned}$$

implying that

$$\Delta x_S^* \stackrel{s}{=} [1 + b(g, \theta)] b(g', \theta) - [1 + b(g', \theta)] b(g, \theta) = b(g', \theta) - b(g, \theta) > 0,$$

where $\stackrel{s}{=}$ means same sign as.

4.1.1. Short-run impact

First, we evaluate how aggregate crime responds to a change in the number of criminals, or the number of links, or both, in the neighborhood of a given initial stock of total wealth (short-run impact). Henceforth, for the sake of consistency, we redefine $x^*(y)$ given in (13) as follows

$$x^*(g) = x_S^*(g) \left[1 - \frac{\omega(g, \theta)}{1 - \pi} \right],$$

and denote $\Delta x^* = x^*(g') - x^*(g)$. Note that $x^*(g)$ remains a function of the state y via $\omega(g, \theta)$. The state is simply suppressed for brevity. The total effect of an increase in the number of criminals, or the number of links, or both, can be decomposed into the sum of two effects, a static effect, which is positive, and a dynamic effect, the sign of which is a priori ambiguous:

$$\Delta x^* = \underbrace{\Delta x_S^*}_{\text{static effect}} + \underbrace{\frac{\omega(g, \theta) x_S^*(g) - \omega(g', \theta) x_S^*(g')}{1 - \pi}}_{\text{dynamic effect}}.$$

Δx^* can be rewritten as

$$\Delta x^* = x_S^*(g') \left[1 - \frac{\omega(g', \theta)}{1 - \pi} \right] - x_S^*(g) \left[1 - \frac{\omega(g, \theta)}{1 - \pi} \right].$$

When ω is decreasing in g , therefore an increase in the number of criminals, or the number of links, or both, leads to a decrease in the weighted average of intertemporal marginal costs of committing crime, $\Delta x^* \stackrel{s}{=} \Delta x_S^* > 0$ since $x_S^*(g') > x_S^*(g)$ and $1 - \omega(g', \theta) / (1 - \pi) > 1 - \omega(g, \theta) / (1 - \pi)$. When instead ω is increasing in g , it is possible that $\Delta x^* < 0$. Take, for instance, $\omega(g', \theta) \rightarrow 1 - \pi$. It follows that $\lim_{\omega(g', \theta) \rightarrow 1 - \pi} \Delta x^* = -x_S^*(g) [1 - \omega(g, \theta) / (1 - \pi)] < 0$.

The above discussion leads to the following proposition.

Proposition 1. Take $y = y_0 \in L$. If $\omega(g', \theta) < \omega(g, \theta)$ then $\Delta x^* > 0$ (as in the static game). However, if $\omega(g', \theta) > \omega(g, \theta)$ then it is possible that $\Delta x^* = x^*(g') - x^*(g) < 0$.

$\omega(g', \theta) > \omega(g, \theta)$ means that, on average, a marginal increase in the stock of total wealth in the economy is more valuable (in terms of value functions) in network g' than in network g . Consequently, given that crime is wealth reducing, committing crime is more costly, on average, in the former than in the latter. Observe that a necessary condition for $\omega(g', \theta) > \omega(g, \theta)$ is $b_{V'}(g', \theta) > b_{V'}(g, \theta)$, since $b(g', \theta) > b(g, \theta)$. Then, aggregate crime may decrease and therefore a social multiplier effect may not exist when $\sum_{i=1}^n b_i(g', \theta) V'_i(y) > \sum_{i=1}^n b_i(g, \theta) V'_i(y)$.

In order to illustrate such a possibility result, we now consider the two extreme cases of an empty and a complete network, and show analytically that, in our dynamic game, in contrast with the static game, when the number of links is increased from 0 to $n - 1$ for each criminal a social multiplier effect does not arise.

Proposition 2. Take $y = y_0 \in L$. $\Delta x^* = x^*(g_N) - x^*(g_\emptyset) < 0$.

The intuitive explanation is as follows. The shadow price of total wealth in the economy can be written as:

$$V'_i(y) = V'(y)|_{g_\emptyset} + \Omega_i(y), \tag{22}$$

²⁰ In this subsection and Section 5, we add explicitly g and g' to the notation of aggregate crime, as we compare the aggregate crime in different networks.

where $V'(y)|_{g_\emptyset}$ is the shadow price of total wealth in the empty network (the same for all criminals given the symmetry of u_i), and $\Omega_i(y)$ is the residual, i.e., what is left in the explanation of $V'_i(y)$ after accounting for $V'(y)|_{g_\emptyset}$. $\Omega_i(y)$ can be interpreted as the “network effect”. Such effect can be either positive or negative (or nil). From (22), the shadow price of total wealth in the economy in the complete network can be decomposed as follows:

$$V'(y)|_{g_N} = V'(y)|_{g_\emptyset} + \Omega(y). \tag{23}$$

In Appendix E, we show that the network effect $\Omega(y)$ in (23) is positive, implying that, for each criminal, the intertemporal marginal cost of committing crime is higher (and therefore the level of criminal activities is lower) in the complete than in the empty network.

We now evaluate the impact of an increase in n in the two extreme cases previously considered. Let g'_\emptyset and g'_N denote the enlarged empty and complete network, respectively. In the next proposition, we show that, both in the empty and the complete network, if the stock of total wealth in the economy is relatively large, aggregate crime decreases as the number of criminals increases.

Proposition 3. Take $y = y_0 \in L$. (i) There exists y_{EN} such that $\Delta x^* = x^*(g'_\emptyset) - x^*(g_\emptyset) <(\text{resp. } >)0$ for $y >(\text{resp. } <)y_{EN}$. (ii) There exists y_{CN} such that $\Delta x^* = x^*(g'_N) - x^*(g_N) <(\text{resp. } >)0$ for $y >(\text{resp. } <)y_{CN}$.

Intuitively, both in the empty and the complete network, when the stock of total wealth in the economy is relatively large, an increase in the number of criminals leads to an increase in the shadow price of total wealth in the economy. As a consequence, for each criminal, the intertemporal marginal cost of committing crime increases. Each incumbent criminal then decreases her/his criminal activities. Such a reduction outweighs the increase in criminal activities due to the added criminals, and therefore, aggregate crime drops.

In the next two numerical examples, we show that an increase in the number of criminals, or the number of links, can lead to a decrease in aggregate crime (in the neighborhood of a given y_0) beyond the two extreme cases considered in Propositions 2 and 3.

Example 1.1. Consider the regular network g' resulting from an increase in d from 2 to 3. All the other parameters of the model remain unchanged with respect to Example 1 (i.e., $\pi = 0.05$, $\phi = 0.1$, $\delta = 0.025$, $\mu = 0.03$ and $r = 0.0001$). Routine calculations lead to $V'(y)|_{g'} = 1.0152 - 0.001y$. From Corollary 1, it follows that $x^*(g') = -2.3711 + 0.0371y$ for $y \in L = [63.8760, 994.4904]$. By setting $y_0 = 100$, at $t = 0$, we get

$$x^*(g'') = \begin{cases} 1.5496 & \text{for } g'' = g \text{ with } d = 2 \\ 1.3410 & \text{for } g'' = g' \text{ with } d = 3 \end{cases}$$

In the static game, instead,

$$x^*_S(g'') = \begin{cases} 33.0435 & \text{for } g'' = g \text{ with } d = 2 \\ 34.5455 & \text{for } g'' = g' \text{ with } d = 3 \end{cases}$$

In this example, unlike in the static game, more connected criminals are associated with lower aggregate crime. The reason is that an increase in d leads to an increase in the shadow price of total wealth in the economy ($V'(y)|_{g', y=y_0} = 0.9131 > V'(y)|_{g, y=y_0} = 0.9054$), thus making criminal activities more costly for all criminals. We can use (22) to see how much of the difference in shadow prices is due to the network effect. From (22), we have

$$\Delta V'(y) = \Delta V'(y)|_{g_\emptyset} + \Delta \Omega(y).$$

Since, in this example, $\Delta V'(y)|_{g_\emptyset} = 0$, the difference in shadow prices is solely attributed to the network effect.

Example 2.1. Consider the network g' resulting from adding one periphery criminal to the network g depicted in Fig. 2, so that the total number of criminals increases from 3 to 4. All the other parameters of the model remain unchanged with respect to Example 2. Routine calculations lead to $b_1(g', \theta) = 1.8182$ and $b_i(g', \theta) = 1.3636$ with $i = 2, 3, 4$. It follows that $b(g', \theta) = 5.9091$ and $x^*_S(g') = 32.5000$. Performing the maximization indicated in (9) leads to the following value functions:

$$V_1 = A_1 y^2 / 2 + B_1 y + C_1,$$

and

$$V_i = A_i y^2 / 2 + B_i y + C_i,$$

with $i = 2, 3, 4$, where $A_1 = -0.0010$, $B_1 = 0.9939$, $C_1 = 20347.5012$, $A_i = -0.0011$, $B_i = 1.0194$, and $C_i = 14623.8491$. Hence, $V'_1(y)|_{g'} = 0.9939 - 0.0010y$, $V'_i(y)|_{g'} = 1.0194 - 0.0011y$, and $b_{V'}(g', \theta) = 5.9773 - 0.0064y$. From Corollary 1, we then get $x^*(g') = -2.1055 + 0.0373y$ for $y \in L = [73.7826, 913.7767]$. Recall that $x^*(g) = -3.5692 + 0.0396y$. Let $y_0 = 800$. It follows that aggregate crime at $t = 0$ is given by

$$x^*(g'') = \begin{cases} 28.1205 & \text{for } g'' = g \text{ with } n = 3 \\ 27.7200 & \text{for } g'' = g' \text{ with } n = 4 \end{cases}$$

and

$$x_S^*(g'') = \begin{cases} 30.5932 & \text{for } g'' = g \text{ with } n = 3 \\ 32.5000 & \text{for } g'' = g' \text{ with } n = 4 \end{cases}$$

In this example, unlike in the static game, more criminals are associated with lower aggregate crime. For each criminal, the shadow price is higher and the weight associated with the shadow price is lower in network g' than in network g , with the increase in the shadow price exceeding the decrease in the respective weight. As a consequence, the increase in the number of criminals leads to an increase in the (weighted) average intertemporal marginal cost of committing crime. Indeed, it can be easily checked that $b_{V'}(g', \theta)|_{y=y_0} = 0.8256 > b_{V'}(g, \theta)|_{y=y_0} = 0.3172$ and that $\omega(g', \theta) = 0.1397 > \omega(g, \theta) = 0.0768$.²¹ Each incumbent criminal decreases her/his criminal activities. The sum of the reductions in crime by the incumbent criminals outweighs the increase in crime by the added criminal, causing a reduction in aggregate crime. An intertemporal business stealing effect occurs, which is responsible for $\Delta x^* < 0$.

We conclude this example with the decomposition of $\Delta V'_k(y)$:

$$\Delta V'_k(y) = \Delta V'(y)|_{g_\theta} + \Delta \Omega_k(y),$$

with $k = 1, 2$. For $k = 1$, we have $\Delta V'_1(y) = 0.0818$ and $\Delta \Omega_1(y) = -0.0011$; for $k = 2$, we have $\Delta V'_2(y) = 0.0560$ and $\Delta \Omega_2(y) = -0.0269$. Moreover, $\Delta V'(y)|_{g_\theta} = 0.0828$. For both types of players, the network effect is negative, stronger for a periphery criminal than for criminal 1, and the vast majority of the difference in shadow prices, especially for criminal 1, is explained by $\Delta V'(y)|_{g_\theta}$. Unlike in Example 1.2, here the network effect goes in the opposite direction to $\Delta V'_k(y)$.

4.1.2. Long-run impact

Next, we evaluate how aggregate crime responds at the steady-state equilibrium (long-run impact). Denote $\Delta \hat{y} = \hat{y}(g') - \hat{y}(g)$, where $\hat{y}(g'')$ for $g'' = g, g'$ is implicitly given by

$$\mu \hat{y}(g'') = x_S^*(g'') \left[1 - \frac{\omega(g'', \theta)}{1 - \pi} \right]. \tag{24}$$

The RHS of (24) is increasing in y (since $\omega(g'', \theta)$ is decreasing in y) and intersects the LHS of (24) from below (for steady-state stability). Take $y_0 = \hat{y}(g)$. Hence, $\Delta x^* >$ (resp. $<$) 0 implies that $\Delta \hat{y} <$ (resp. $>$) 0 . Clearly, if $\Delta x^* > 0$, which occurs, for instance, when $\omega(g', \theta) < \omega(g, \theta)$, then $x^*(g') > \mu y$. Consequently, $x^*(g')$ intersects μy at a point to the left of $\hat{y}(g)$. From Corollary 2, we know that \hat{y} and \hat{x} are positively correlated. Hence, we can write the following proposition.

Proposition 4. Take $y_0 = \hat{y}(g)$. If $\Delta x^* >$ (resp. $<$) 0 then $\Delta \hat{x} <$ (resp. $>$) 0 .

The case $\Delta \hat{x} < 0$ in Proposition 4 is illustrated in Fig. 3.

As can be seen in Fig. 3, there exists a trade-off between short and long run: aggregate crime increases in the short run and decreases in the long run. Starting from point A, aggregate crime moves up to point B, then down to point C (moving along $x^*(g')$). Indeed, when $\Delta \hat{x} < 0$ a social multiplier effect exists only in the short run.

A numerical example of the case $\Delta \hat{x} > 0$ is provided below.

Example 1.1 (continued). For the regular network, the trajectories of aggregate crime are given by

$$x^*(g'', t) = \begin{cases} 9.0138 + (0.0372y_0 - 11.1877) \exp(-0.0072t) & \text{for } g'' = g \text{ with } d = 2 \\ 9.9892 + (0.0371y_0 - 12.3604) \exp(-0.0071t) & \text{for } g'' = g' \text{ with } d = 3 \end{cases}$$

Clearly, as $t \rightarrow \infty$, aggregate crime converges to 9.0138 with $d = 2$ and to 9.9892 with $d = 3$. Therefore, in line with the static game, long-run aggregate crime is lower with $d = 2$ than with $d = 3$. The reason is that the steady-state shadow price of total wealth in the economy is higher with $d = 2$ than with $d = 3$ ($\hat{\omega}(g, \theta) = 0.6909 > \hat{\omega}(g', \theta) = 0.6753$). At the steady state, it is more costly, for each criminal, to commit crime in the network with fewer links. However, as previously shown in Example 1.1, for $t = 0$ and $y_0 = 100$, in contrast with the static game, aggregate crime is higher with $d = 2$ than with $d = 3$. This implies that, for the given parameter values, the trajectories of aggregate crime in g and g' intersect.

A final remark is in order. Proposition 4 establishes that if $y_0 = \hat{y}(g)$ then the long-run response of an increase in the number of criminals, or the number of links, or both, is the opposite in sign to the short-run response. However, if $y_0 \neq \hat{y}(g)$ we can have either an increase or a decrease in aggregate crime, not only in the short run but also at the steady state. The case of a decrease in aggregate crime is illustrated in Fig. 4 as well as in Example 2.1 below.

Starting from point A, aggregate crime moves down first to point B, then to point C (moving along $x^*(g')$).

²¹ Recall that $b_{V'}(g', \theta) > b_{V'}(g, \theta)$ is a necessary condition for $\omega(g', \theta) > \omega(g, \theta)$.

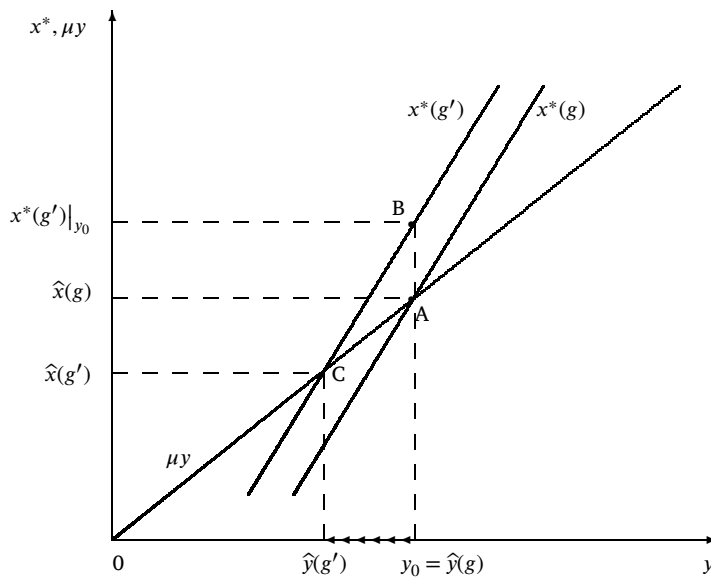


Fig. 3. Comparative steady-state analysis with $y_0 = \hat{y}(g)$.

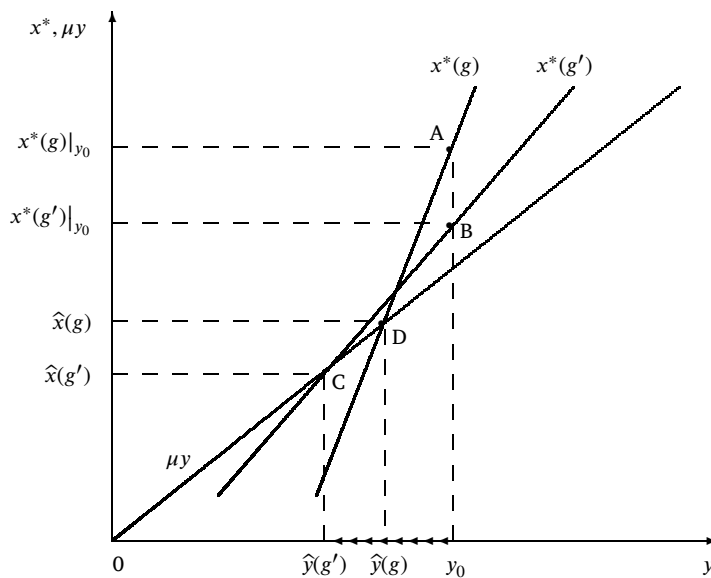


Fig. 4. Comparative steady-state analysis with $y_0 \neq \hat{y}(g)$.

Example 2.1 (continued). For the star network, the trajectories of aggregate crime are given by

$$x^*(g'', t) = \begin{cases} 11.1397 + (0.0396y_0 - 14.7088) \exp(-0.0096t) & \text{for } g'' = g \text{ with } n = 3 \\ 8.6743 + (0.0373y_0 - 10.7798) \exp(-0.0073t) & \text{for } g'' = g' \text{ with } n = 4 \end{cases}$$

Clearly, as $t \rightarrow \infty$, aggregate crime converges to 11.1397 with $n = 3$ and to 8.6743 with $n = 4$. Therefore, in contrast with the static game, long-run aggregate crime is higher with $n = 3$ than with $n = 4$. It can be checked that, at the steady state, $\hat{\omega}(g, \theta) = 0.6041 < \hat{\omega}(g', \theta) = 0.6964$. Hence, the (weighted) average intertemporal marginal cost of committing crime is lower with $n = 3$ than with $n = 4$. The reduction in criminal activities by the three incumbent criminals more than compensate for the criminal activities by the added criminal. Moreover, in contrast with the static game, as previously shown in Example 2.1, for $t = 0$ and $y_0 = 800$ aggregate crime is higher with $n = 3$ than with $n = 4$. This implies that, in this example, more criminals are associated with a lower aggregate crime both in the short and the long run.

We conclude this example with the decomposition of $\Delta V'_k(\hat{y}) = V'_k(\hat{y}(g')) - V'_k(\hat{y}(g))$:

$$\Delta V'_k(\hat{y}) = \Delta V'(\hat{y}) \Big|_{g_0} + \Delta \Omega_k(\hat{y}),$$

with $k = 1, 2$, where $\Delta V'(\hat{y}) \Big|_{g_0} = V'(\hat{y}(g')) \Big|_{g_0} - V'(\hat{y}(g)) \Big|_{g_0}$ and $\Delta \Omega_k(\hat{y}) = \Omega_k(\hat{y}(g')) - \Omega_k(\hat{y}(g))$. For $k = 1$, we have $\Delta V'_1(\hat{y}) = 0.0963$ and $\Delta \Omega_1(\hat{y}) = -0.0135$; for $k = 2$, we have $\Delta V'_2(\hat{y}) = 0.0899$ and $\Delta \Omega_2(\hat{y}) = -0.0200$. Moreover, $\Delta V'(\hat{y}) \Big|_{g_0} = 0.1099$. For both types of players, the steady-state network effect is negative, stronger for a periphery criminal than for criminal 1 (as in the short run), and the vast majority of the difference in steady-state shadow prices, especially for criminal 1, is explained by $\Delta V'(\hat{y}) \Big|_{g_0}$.

4.2. Marginal expected punishment

In the static game studied in Ballester et al. (2010), the impact of π on x^*_S can be decomposed into the sum of a direct and an indirect effect as follows:

$$\frac{\partial x^*_S}{\partial \pi} = \underbrace{\left(-\frac{x^*_S}{1-\pi}\right)}_{\text{direct effect}} + \underbrace{\left(\frac{\partial x^*_S}{\partial \theta} \frac{\phi}{\delta}\right)}_{\text{(static) indirect effect}}$$

where

$$\frac{\partial x^*_S}{\partial \theta} = \frac{(1-\pi)}{\delta [1 + b(g, \theta)]^2} \frac{\partial b(g, \theta)}{\partial \theta} > 0.$$

The direct effect is negative, whereas the (static) indirect effect is positive, implying that the impact of π on x^*_S is ambiguous.

4.2.1. Short-run impact

In our dynamic setting, a change in π has an impact not only on the Bonacich centralities, as in the static game, but also on the weighted Bonacich centralities with weights given by the vector of shadow prices. The short-run impact of π on x^* is given by

$$\begin{aligned} \frac{\partial x^*}{\partial \pi} &= \underbrace{\left(-\frac{x^*_S}{1-\pi}\right)}_{\text{direct effect}} + \underbrace{\left(\frac{\partial x^*_S}{\partial \theta} \frac{\phi}{\delta}\right)}_{\text{static indirect effect}} + \underbrace{\left(\frac{\partial x^*_S}{\partial \theta} \frac{\phi}{\delta} \frac{\omega(g, \theta)}{\pi-1} + \frac{x^*_S}{\pi-1} \frac{\partial \omega}{\partial \pi}\right)}_{\text{dynamic indirect effect}} \\ &= \underbrace{\left(-\frac{x^*_S}{1-\pi}\right)}_{\text{direct effect}} + \underbrace{\frac{\partial x^*_S}{\partial \theta} \frac{\phi}{\delta} \left(1 - \frac{\omega(g, \theta)}{1-\pi}\right) + \frac{x^*_S}{\pi-1} \frac{\partial \omega}{\partial \pi}}_{\text{total indirect effect}} \end{aligned} \tag{25}$$

While the static indirect effect is always positive, the total indirect effect can be negative. Indeed, it is easy to verify that the total indirect effect is negative if

$$\frac{\partial x^*_S}{\partial \theta} \frac{\phi}{\delta} [1 - \pi - \omega(g, \theta)] < x^*_S \frac{\partial \omega}{\partial \pi}. \tag{26}$$

Letting $E_k f(z, k) = [k/f(z, k)] \partial f(z, k) / \partial k$ denote the elasticity of $f(z, k)$ with respect to k , and recalling that $\theta = \pi \phi / \delta$, (26) can be rewritten as

$$\frac{E_\pi \omega}{E_\theta x^*_S} > \frac{1 - \pi - \omega(g, \theta)}{\omega(g, \theta)}. \tag{27}$$

Clearly, a necessary condition for (27) to hold is that $\omega(g, \theta)$ be increasing in π (since the RHS of (27) and $E_\theta x^*_S$ are both positive). When $\omega(g, \theta)$ is decreasing in π , the total indirect effect has the same sign as the static indirect effect, i.e., it is positive. In this case, as in the static game analyzed in Ballester et al. (2010), the impact of π on aggregate crime is ambiguous. When instead $\omega(g, \theta)$ is increasing in π , the total indirect effect is negative. In this case, an increase in π unambiguously leads to a decrease in aggregate crime.

The above discussion leads to the following proposition, which provides sufficient conditions for short-run aggregate crime to be decreasing in the marginal expected punishment.

Proposition 5. *If $E_\pi \omega / E_\theta x^*_S > [1 - \pi - \omega(g, \theta)] / \omega(g, \theta)$ then $\partial x^* / \partial \pi < 0$ for $y = y_0 \in L$.*

When the dynamic indirect effect is positive (resp. negative), we have $\partial x^* / \partial \pi >$ (resp. $<$) $\partial x^*_S / \partial \pi$. Assume that $\partial x^* / \partial \pi > \partial x^*_S / \partial \pi$. There are two cases in which the sign of one derivative implies the sign of the other. If $\partial x^* / \partial \pi < 0$ then $\partial x^*_S / \partial \pi < 0$, or if $\partial x^*_S / \partial \pi > 0$ then $\partial x^* / \partial \pi > 0$. Assume now that $\partial x^* / \partial \pi < \partial x^*_S / \partial \pi$. There are two cases in which the sign of one derivative implies the sign of the other. If $\partial x^*_S / \partial \pi < 0$ then $\partial x^* / \partial \pi < 0$, or if $\partial x^*_S / \partial \pi > 0$ then $\partial x^* / \partial \pi > 0$. Interestingly, the qualitative impact of π on aggregate

crime in the static and the dynamic model may differ: when $\partial x^*/\partial\pi > \partial x_S^*/\partial\pi$ we can have $\partial x^*/\partial\pi > 0$ together with $\partial x_S^*/\partial\pi < 0$; when, instead, $\partial x^*/\partial\pi < \partial x_S^*/\partial\pi$ we can have $\partial x^*/\partial\pi < 0$ together with $\partial x_S^*/\partial\pi > 0$.

In order to illustrate the possibility of a divergence between the static and the dynamic model in terms of impact of π on aggregate crime, we now consider the two extreme cases of an empty and a complete network.

Proposition 6. Take $y = y_0 \in L$. (i) In the empty network, $\partial x^*/\partial\pi > 0$. (ii) In the complete network, there exists $y_{C\Pi}$ such that $\partial x^*/\partial\pi < (\text{resp. } >) 0$ for $y > (\text{resp. } <) y_{C\Pi}$.

In the empty network, aggregate crime decreases in the static model but increases in the dynamic model. In the static model, the total indirect effect is nil (since $\partial b(g, \theta)/\partial\theta = 0$), and therefore an increase in the marginal expected punishment leads to a decrease in aggregate crime. In the dynamic model, instead, from (25), the total indirect effect is given by $x_S^*(g_\theta)/(\pi - 1)\partial V'(y)|_{g_\theta}/\partial\pi$. It can be verified that $\partial V'(y)|_{g_\theta}/\partial\pi < 0$, implying that $x_S^*(g_\theta)/(\pi - 1)\partial V'(y)|_{g_\theta}/\partial\pi > 0$. Aggregate crime increases since the (positive) total indirect effect outweighs the (negative) direct effect.

In the complete network, the total indirect effect depends on y not only through the impact of the marginal expected punishment on the shadow price of total wealth in the economy but also through the shadow price of total wealth in the economy itself. The total indirect effect is given by

$$\frac{\partial x_S^*(g_N)}{\partial\theta} \frac{\phi}{\delta} \left(1 - \frac{V'(y)|_{g_N}}{1 - \pi} \right) + \frac{x_S^*(g_N)}{\pi - 1} \frac{\partial V'(y)|_{g_N}}{\partial\pi}. \tag{28}$$

As in the static game, criminals react strategically to an increase in the marginal expected punishment by increasing their involvement in criminal activities (since $\partial b(g, \theta)/\partial\theta > 0$). In addition to this static indirect effect, in our dynamic game, there exists another indirect effect, which depends on how the shadow price of total wealth in the economy changes as π changes. It can be checked that $\partial V'(y)|_{g_N}/\partial\pi > (\text{resp. } <) 0$ if the stock of total wealth in the economy is relatively large (resp. small). This implies that the total indirect effect in (28) is positive for relatively small y and ambiguous otherwise. Proposition 6 establishes that when the stock of total wealth in the economy is relatively large (resp. small), aggregate crime decreases (resp. increases). $\partial x^*/\partial\pi$ can then take either sign depending on y . For parameter values such that $\partial x_S^*/\partial\pi > 0$, a relatively large y would lead to $\partial x^*/\partial\pi < 0$. Vice versa, for parameter values such that $\partial x_S^*/\partial\pi < 0$, a relatively small y would lead to $\partial x^*/\partial\pi > 0$.

In what follows, we provide two numerical examples illustrating the possible divergence between the static and the dynamic model in terms of impact of π on aggregate crime beyond the two extreme cases of an empty and a complete network considered in Proposition 6.

Example 1.2. Consider an increase in parameter π from 0.05 to 0.1 in the regular network g of Example 1. Let $y_0 = 100$.²² At $t = 0$, aggregate crime is given by

$$x^* = \begin{cases} 1.5496 & \text{for } \pi = 0.05 \\ 1.2400 & \text{for } \pi = 0.1 \end{cases}$$

In the static game, instead,

$$x_S^* = \begin{cases} 33.0435 & \text{for } \pi = 0.05 \\ 34.2857 & \text{for } \pi = 0.1 \end{cases}$$

In this example, an increase in the expected marginal punishment leads to a decrease in aggregate crime in the dynamic game and to an increase in aggregate crime in the static game. More precisely, in the former, the total indirect effect is positive but outweighed by the negative direct effect. In the latter, the opposite holds true.

Example 2.2. Consider an increase in parameter π from 0.05 to 0.1 in the star network g of Example 2. Let $y_0 = 800$.²³ At $t = 0$, aggregate crime is given by

$$x^* = \begin{cases} 28.1205 & \text{for } \pi = 0.05 \\ 27.4152 & \text{for } \pi = 0.1 \end{cases}$$

In the static game, instead,

$$x_S^* = \begin{cases} 30.5932 & \text{for } \pi = 0.05 \\ 31.3636 & \text{for } \pi = 0.1 \end{cases}$$

As in Example 1.2, in this example, an increase in the expected marginal punishment leads to a decrease in aggregate crime in the dynamic game and to an increase in aggregate crime in the static game. In the dynamic (resp. static) game, the negative direct effect outweighs (resp. is outweighed by) the positive total indirect effect.

²² Admissible values for y_0 are [66.4827, 945.8097].

²³ Admissible values for y_0 are [148.6540, 897.7852].

4.2.2. Long-run impact

Next, we look at the steady-state equilibrium. Denote

$$F = \mu y - x_S^* \left[1 - \frac{\omega(g, \theta)}{1 - \pi} \right].$$

By implicit differentiation, we have

$$\frac{d\hat{y}}{d\pi} = - \frac{\partial F / \partial \pi}{\partial F / \partial y} = \frac{- \frac{x_S^*}{1 - \pi} \left(1 + \frac{\partial \omega}{\partial \pi} \right) + \frac{\partial x_S^*}{\partial \theta} \frac{\phi}{\delta} \left(1 - \frac{\omega(g, \theta)}{1 - \pi} \right)}{\mu + \frac{x_S^*}{1 - \pi} \frac{\partial \omega}{\partial y}}.$$

Since

$$\mu + \frac{x_S^*}{1 - \pi} \frac{\partial \omega}{\partial y} < 0$$

is required for steady-state stability, it follows that

$$\frac{d\hat{y}}{d\pi} > 0.$$

From Corollary 2, we know that \hat{y} and \hat{x} are positively correlated. Hence, we can write the following proposition.

Proposition 7. Take $y_0 = \hat{y}(g)$. If $\partial x^* / \partial \pi > (\text{resp. } <) 0$ then $d\hat{x} / d\pi < (\text{resp. } >) 0$.

Proposition 7 is illustrated in the two examples below.

Example 1.2 (continued). For the regular network, the trajectories of aggregate crime are given by

$$x^*(t) = \begin{cases} 9.0138 + (0.0372y_0 - 11.1877) \exp(-0.0072t) & \text{for } \pi = 0.05 \\ 10.5458 + (0.0370y_0 - 13.0055) \exp(-0.0070t) & \text{for } \pi = 0.1 \end{cases}$$

Clearly, as $t \rightarrow \infty$, aggregate crime converges to 9.0138 with $\pi = 0.05$ and to 10.5458 with $\pi = 0.1$. Therefore, in line with the static game, long-run aggregate crime is higher with $\pi = 0.1$ than with $\pi = 0.05$. However, for $t = 0$ and $y_0 = \hat{y}(g) = 300.4607$, in contrast with the static game, aggregate crime is lower with $\pi = 0.1$ than with $\pi = 0.05$. This implies that, for the given parameter values, the trajectories of aggregate crime with $\pi = 0.05$ and $\pi = 0.1$ intersect.

Example 2.2 (continued). For the star network, the trajectories of aggregate crime are given by

$$x^*(t) = \begin{cases} 11.1397 + (0.0396y_0 - 14.7088) \exp(-0.0096t) & \text{for } \pi = 0.05 \\ 12.8970 + (0.0392y_0 - 16.8641) \exp(-0.0092t) & \text{for } \pi = 0.1 \end{cases}$$

Clearly, as $t \rightarrow \infty$, aggregate crime converges to 11.1397 with $\pi = 0.05$ and to 12.8970 with $\pi = 0.1$. It is immediate to check that, for $y_0 = \hat{y}(g) = 371.3222$, short run aggregate crime is higher with $\pi = 0.05$ than with $\pi = 0.1$. This implies that, for the given parameter values, the trajectories of aggregate crime intersect.

4.3. Implicit growth rate and voracity effect

In this subsection, we investigate the possibility that an increase in the implicit growth rate of total wealth in the economy such as a productivity gain lowers economic growth, i.e., whether a voracity effect (see Tornell and Lane, 1999) arises. Formally, a voracity effect exists when $\partial(\dot{y}^*(t) / y^*(t)) / \partial \mu < 0$. Note that, in the absence of crime, $\dot{y}^*(t) / y^*(t) = \mu$, therefore a voracity effect never arises. The question addressed here is new in the network theory literature. In this respect, the focus of this subsection is different from that of the previous subsections, which was on the comparison between the static and the dynamic impact of an increase in the number of criminals or links (or both), or an increase in the marginal expected punishment.

Let $\eta^*(t) = \dot{y}^*(t) / y^*(t)$. Take $y_0 \in L$. From Corollary 2, we obtain:

$$\dot{y}^*(t) = (y_0 - \hat{y}) \left(\mu + \frac{b_A(g, \theta)}{\delta [1 + b(g, \theta)]} \right) \exp \left[t \left(\mu + \frac{b_A(g, \theta)}{\delta [1 + b(g, \theta)]} \right) \right].$$

It follows that the rate of growth of $y^*(t)$ can be written as

$$\eta^*(t) = \frac{(y_0 - \hat{y}) \left(\mu + \frac{b_A(g, \theta)}{\delta [1 + b(g, \theta)]} \right) \exp \left[t \left(\mu + \frac{b_A(g, \theta)}{\delta [1 + b(g, \theta)]} \right) \right]}{\hat{y} + (y_0 - \hat{y}) \exp \left[t \left(\mu + \frac{b_A(g, \theta)}{\delta [1 + b(g, \theta)]} \right) \right]}.$$

Differentiating $\eta^*(t)$ with respect to μ and evaluating the derivative at $t = 0$ gives

$$\left. \frac{\partial \eta^*(t)}{\partial \mu} \right|_{t=0} = \delta y_0 [1 + b(g, \theta)] (y_0 - \hat{y}) \left\{ \delta [1 + b(g, \theta)] + \frac{\partial b_A(g, \theta)}{\partial \mu} \right\} - \{ \delta \mu [1 + b(g, \theta)] + b_A(g, \theta) \} \frac{\partial \hat{y}}{\partial \mu}. \tag{29}$$

Observe that (29) is quadratic in y_0 and concave (resp. convex) in y_0 if $\delta [1 + b(g, \theta)] + \partial b_A(g, \theta) / \partial \mu < (\text{resp. } >) 0$. Call $y_{0,1}$ and $y_{0,2}$ the real roots of the RHS of (29), with $y_{0,1} < y_{0,2}$. We can then state the following proposition.

Proposition 8. Take $y_0 \in L$ and t sufficiently close to zero. If $\delta [1 + b(g, \theta)] + \partial b_A(g, \theta) / \partial \mu < (\text{resp. } >) 0$ then $\partial \eta^*(t) / \partial \mu < 0$ (i.e., a voracity effect exists) for $y_0 \notin (\text{resp. } \in) (y_{0,1}, y_{0,2})$.

Proposition 8 establishes that the likelihood of a voracity effect crucially depends on initial conditions as well as the network structure.²⁴ In general, depending on the position of $y_{0,1}$ and $y_{0,2}$ and the sign of $\delta [1 + b(g, \theta)] + \partial b_A(g, \theta) / \partial \mu$, several cases can arise. For regular networks, which encompass the two extreme cases of an empty and a complete network, we can show analytically that $\delta [1 + b(g, \theta)] + \partial b_A(g, \theta) / \partial \mu < 0$, implying that a voracity effect arises for values of y_0 outside the interval $(y_{0,1}, y_{0,2})$.

In order to shed some light on the impact of a change in the network structure on the likelihood of a voracity effect, defined as the ratio between the interval of y_0 where a voracity effect occurs and the length of the domain L , we consider a change in the network size and density with the baseline parameter values of Examples 1 and 2. We take the following approach: for each network size, we start from player 1 in the empty network and keep adding one link at the time connecting player 1 to the other players (following the numbering of players) until exhausting all possible links in the network of the given size. Then, for $n \geq 3$, we move to player 2 and keep adding one link at the time connecting player 2 to the other players (starting from player 3, given that a link between players 1 and 2 was established before) until exhausting all possible links, and so on so forth. Table T1 below reports the likelihood of a voracity effect in the several network structures considered²⁵:

	0 links	1 link	2 links	3 links	4 links	5 links	6 links	10 links
$n = 2$	0.5259	0.4649**	N/A	N/A	N/A	N/A	N/A	N/A
$n = 3$	0.6843	0.6883	0.6594*	0.6270**	N/A	N/A	N/A	N/A
$n = 4$	0.7649	0.7658	0.7667	0.7505*	0.7517	0.7342	0.7147**	N/A
$n = 5$	0.8142	0.8143	0.8145	0.8146	0.8043*	0.8047	0.8050	0.7700**

(T1)

where the starred and the double starred numbers refer to the likelihood of a voracity effect in the star and the complete networks, respectively. From Table T1, we can see that, for any number of links considered from 0 to 6, an increase in the network size leads to an increase in the likelihood of a voracity effect. A positive relationship between network size and likelihood of a voracity effect can also be inferred for the complete and the star networks. As to the impact of the number of links on the likelihood of a voracity effect for a given network size, except for the case $n = 2$, we can observe a nonmonotonic relationship. Finally, for any given $n > 2$, it is possible to compare the empty, the complete, and the star network, and conclude that the likelihood of a voracity effect is the highest in the empty and the lowest in the complete network.

The intuitive explanation for the occurrence of a voracity effect is that the indirect effect of an increase in total crime, which, given the wealth-reducing nature of crime, is negative, outweighs the direct positive effect of an increase in μ . Consequently, an increase in the implicit growth rate of total wealth in the economy depresses economic growth.²⁶ We conclude this subsection with two additional numerical examples illustrating Proposition 8.

Example 1.3. Consider the regular network with the parameter values of Example 1. We now increase parameter μ marginally (starting from 0.03). Recall that the admissible values for y are [58.3823, 945.8097]. We have $\delta [1 + b(g, \theta)] + \partial b_A(g, \theta) / \partial \mu = -0.0466$, implying that the rate of growth of total wealth in the economy (evaluated at $t = 0$) is negative for $y_0 \notin (y_{0,1}, y_{0,2})$. Since $y_{0,1} = 5.1663 < 58.3823$ and $y_{0,2} = 295.2944 < 945.8097$ we can conclude that a voracity effect exists for $y_0 \in (y_{0,2}, 945.8097]$. Take for instance $y_0 = \hat{y} = 300.4607$. It is immediate to verify that $\partial \eta^*(t) / \partial \mu|_{t=0} = -13.6323$. If we consider $y_0 < y_{0,2}$, for instance, $y_0 = 290$, we get $\partial \eta^*(t) / \partial \mu|_{t=0} = 13.4754$.

Example 2.3. Consider the star network with the parameter values of Example 2. We now increase parameter μ marginally (starting from 0.03). Recall that the admissible values for y are [108.7184, 856.7616]. We have $\delta [1 + b(g, \theta)] + \partial b_A(g, \theta) / \partial \mu = -0.0414$, implying that the rate of growth of total wealth in the economy (evaluated at $t = 0$) is negative for $y_0 \notin (y_{0,1}, y_{0,2})$. Since $y_{0,1} = 7.8384 < 108.7184$ and $y_{0,2} = 363.4839 < 856.7616$ we can conclude that a voracity effect exists for $y_0 \in (y_{0,2}, 856.7616]$. Take for instance

²⁴ We thank an anonymous referee for suggesting the analysis of the relationship between the network structure and the voracity effect.

²⁵ $n < 6$ ensures that the condition $\theta \rho(g) < 1$ is satisfied for all the network structures considered in T1.

²⁶ A similar result can be found in a number of related papers (e.g., Tornell and Lane, 1999; Long and Sorger, 2006; Van der Ploeg, 2011). However, to our knowledge, it has never been derived in the context of criminal networks.

$y_0 = \hat{y} = 371.3222$. It is immediate to verify that $\partial \eta^*(t)/\partial \mu|_{t=0} = -15.1207$. If we consider $y_0 < y_{02}$, for instance, $y_0 = 350$, we get $\partial \eta^*(t)/\partial \mu|_{t=0} = 24.4853$.

5. Key player

In this section, building on Ballester et al. (2006, 2010), we study the problem of identifying the optimal target in the population of criminals when the planner's objective is to minimize aggregate crime at each point in time. Let g_{-i} denote the network resulting from removing criminal i from network g , and let $x^*(g_{-i}, t)$ denote the level of aggregate crime associated with network g_{-i} at time $t \in [0, \infty)$. Take y_0 such that all criminals in g and g_{-i} are active and play nondegenerate Markovian strategies for all $i = 1, \dots, n$ and all $t \in [0, \infty)$.

From Corollary 2, the trajectory of aggregate crime in g_{-i} for $t \in [0, \infty)$ is given by

$$x^*(g_{-i}, t) = x_S^*(g_{-i}) \left[1 - \frac{b_A(g_{-i}, \theta) y^*(g_{-i}, t) + b_B(g_{-i}, \theta)}{(1 - \pi) b(g_{-i}, \theta)} \right],$$

where

$$x_S^*(g_{-i}) = \frac{(1 - \pi) b(g_{-i}, \theta)}{\delta [1 + b(g_{-i}, \theta)]},$$

and

$$y^*(g_{-i}, t) = \hat{y}(g_{-i}) + [y_0 - \hat{y}(g_{-i})] \exp \left[t \left(\mu + \frac{b_A(g_{-i}, \theta)}{\delta [1 + b(g_{-i}, \theta)]} \right) \right],$$

with

$$\hat{y}(g_{-i}) = \frac{(1 - \pi) b(g_{-i}, \theta) - b_B(g_{-i}, \theta)}{\mu \delta [1 + b(g_{-i}, \theta)] + b_A(g_{-i}, \theta)}$$

being the corresponding (locally stable) steady-state level of y .

The planner's problem is to remove the criminal who is associated with the largest drop in aggregate crime at each t . Formally:

$$\min \{ x^*(g_{-i}, t) \mid i = 1, \dots, n \} = \min \left\{ x_S^*(g_{-i}) \left[1 - \frac{1}{1 - \pi} \frac{b_{V'}^*(g_{-i}, \theta)}{b(g_{-i}, \theta)} \right] \mid i = 1, \dots, n \right\}, \forall t \in [0, \infty),$$

with $b_{V'}^*(g_{-i}, \theta) = b_A(g_{-i}, \theta) y^*(g_{-i}, t) + b_B(g_{-i}, \theta)$. We denote with i^* the solution to the above problem.

In the static game, where $\mathbf{A} = \mathbf{B} = \mathbf{0}$, the planner's problem becomes

$$\min \{ x_S^*(g_{-i}) \mid i = 1, \dots, n \},$$

which is equivalent to

$$\min \{ b(g_{-i}, \theta) \mid i = 1, \dots, n \},$$

since $x_S^*(g_{-i})$ is increasing in $b(g_{-i}, \theta)$. We denote with i_S^* the solution to the static problem. From Ballester et al. (2006, 2010), we know that i_S^* is the criminal with the highest intercentrality of parameter θ in g , defined as

$$c_i(g, \theta) = b(g, \theta) - b(g_{-i}, \theta) = \frac{b_i(g, \theta)^2}{m_{ii}(g, \theta)},$$

where $m_{ij}(g, \theta)$ are the coefficients of $\mathbf{M}(g, \theta) = [\mathbf{I} - \theta \mathbf{G}]^{-1} = \sum_{p=0}^{\infty} \theta^p \mathbf{G}^p$ counting the number of walks from i to j with walks of length p being discounted by θ^p . As pointed out in Ballester et al. (2006, 2010), the intercentrality measure $c_i(g, \theta)$ is equal to the sum of i 's Bonacich centrality and i 's contribution to every other player's Bonacich centrality. Keeping $b_i(g, \theta)$ fixed, $c_i(g, \theta)$ decreases with the proportion of i 's Bonacich centrality due to self-loops, $m_{ii}(g, \theta)/b_i(g, \theta)$.

In our dynamic game, things are more involved, and maximizing $c_i(g, \theta)$ (or, equivalently, minimizing $b(g_{-i}, \theta)$) does not necessarily lead to the largest drop in aggregate crime. Keeping $b_{V'}^*(g_{-i}, \theta)$ fixed, $x^*(g_{-i}, t)$ is still increasing in $b(g_{-i}, \theta)$. However, the removal of player i from g is not only captured by $b(g_{-i}, \theta)$, but also by $b_{V'}^*(g_{-i}, \theta)$, which negatively impacts $x^*(g_{-i}, t)$.

Theorem 2. (i) A necessary (but not sufficient) condition for $i^* \neq i_S^*$ is

$$\frac{b_{V'}^*(g_{-i_S^*}, \theta)}{b(g_{-i_S^*}, \theta)} < \frac{b_{V'}^*(g_{-i^*}, \theta)}{b(g_{-i^*}, \theta)},$$

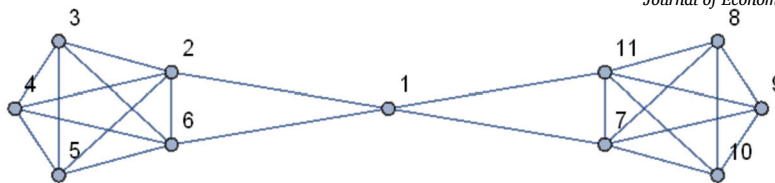


Fig. 5. Bridge network with eleven criminals.

for some $i \neq i_S^*$. (ii) A sufficient (but not necessary) condition for $i^* = i_S^*$ is

$$b_{\mathbf{V}'}^*(g_{-i_S^*}, \theta) > b_{\mathbf{V}'}^*(g_{-i}, \theta),$$

for all $i \neq i_S^*$.

Theorem 2 establishes that the key player in the static and the dynamic game may differ, either temporarily or *ad infinitum* (since $b_{\mathbf{V}'}^*$ changes over time). Moreover, in the dynamic game, the key player in the short and the long run are not necessarily the same. In the remainder of this section, we provide an illustrative example of the possible divergence between the key player in the static and the dynamic game and between the key player in the short and the long run.

Example 3. Consider the network g in Fig. 5 (see Ballester et al., 2006, 2010).

As can be seen in Fig. 5, there are three types of players, Type 1 (Player 1), Type 2 (Players 2,6,7,11) and Type 3 (Players 3,4,5,8,9,10). We consider the baseline parameter values of Examples 1 and 2. Note that the implied value of θ ($\theta = 0.2$) is also considered in the static analysis of the key player in Ballester et al. (2006, Table I) and Ballester et al. (2010, Table I). T2 below gives the Bonacich and the intercentrality measures together with the sum of the coordinates of the vector of weighted Bonacich centralities of parameter θ in g_{-i} , with weights $\mathbf{V}' = (A_1 y^*(g_{-i}, t) + B_1, \dots, A_n y^*(g_{-i}, t) + B_n)^T$ (i.e., evaluated along the equilibrium trajectories of y resulting from permanently removing criminal i from g) for the three types of players.

Player Type	b_i	c_i	$b_{\mathbf{V}'}^*(g_{-i}, \theta)$
1	8.3333	41.6667	$43.0621 + e^{-0.0029t} (4.8697 - 0.0420y_0)$
2	9.1667	40.3333	$44.2484 + e^{-0.0029t} (4.9581 - 0.0431y_0)$
3	7.7778	32.6667	$50.8248 + e^{-0.0029t} (5.7336 - 0.0494y_0)$

(T2)

The trajectories of aggregate crime associated with g_{-1} , g_{-2} , and g_{-3} are given by

$$x^*(g_{-1}, t) = 3.4807 + (0.0329y_0 - 3.8194) \exp(-0.0029t),$$

$$x^*(g_{-2}, t) = 3.4534 + (0.0329y_0 - 3.7896) \exp(-0.0029t),$$

and

$$x^*(g_{-3}, t) = 3.4835 + (0.0329y_0 - 3.8224) \exp(-0.0029t),$$

respectively, with $y_0 \in L = [15.7521, 1120.5136]$. Aggregate crime converges to 3.4807 in g_{-1} , to 3.4534 in g_{-2} , and to 3.4835 in g_{-3} as $t \rightarrow \infty$. Clearly, in the long run, the key player is Type 2. This is in contrast with Ballester et al. (2006, 2010), in which the key player is Type 1, the one with the highest intercentrality.²⁷ It can be checked that for some y_0 the trajectories of aggregate crime intersect. Take, for instance, $y_0 = 115.5$. Aggregate crime in g_{-1} and g_{-3} is increasing, whereas aggregate crime in g_{-2} is decreasing over time. For $t \in [0, 31.8000]$, we have $x^*(g_{-3}, t) < x^*(g_{-1}, t) < x^*(g_{-2}, t)$. This implies that, initially, the key player is Type 3. At $t = 31.8000$, we have $x^*(g_{-3}, t) = x^*(g_{-1}, t) = x^*(g_{-2}, t) = 3.4650$. For $t \in (31.8000, \infty)$, instead, we have $x^*(g_{-2}, t) < x^*(g_{-1}, t) < x^*(g_{-3}, t)$. Hence, after the initial phase where the key player is Type 3, the key player becomes Type 2. Interestingly, Type 1, who is the key player in the static game, is never the key player in the dynamic game (see Figs. 6a and 6b, where $x^*(g_{-1}, t)$, $x^*(g_{-2}, t)$, and $x^*(g_{-3}, t)$ are indicated with x_{without1} , x_{without2} , and x_{without3} , respectively).

It can be checked that the necessary condition for $i^* \neq i_S^*$ given in Theorem 2 is satisfied. Indeed, for $t \in [0, \infty)$, we have

$$\frac{b_{\mathbf{V}'}^*(g_{-1}, \theta)}{b(g_{-1}, \theta)} < \frac{b_{\mathbf{V}'}^*(g_{-3}, \theta)}{b(g_{-3}, \theta)},$$

where $b_{\mathbf{V}'}^*(g_{-i}, \theta)$ is given in Table T2 and $b(g_{-i}, \theta) = b(g, \theta) - c_i(g, \theta)$, with $i = 1, 3$.

The intuitive explanation for having different key players in the static and the dynamic model is as follows. In the dynamic model, for each criminal, there exists an additional intertemporal marginal cost captured by the shadow price of total wealth in the

²⁷ For $\pi = 0.05$, $\phi = 0.1$ and $\delta = 0.025$, aggregate crime levels in the static game are given by $x_S^*(g_{-1}) = 37.2549$, $x_S^*(g_{-2}) = 37.2739$, and $x_S^*(g_{-3}) = 37.3667$.

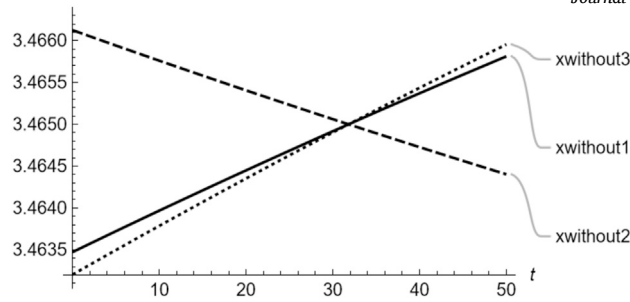


Fig. 6a. Key player over time with $t \in [0, 50]$.

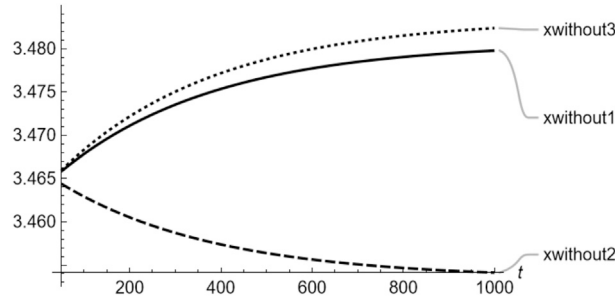


Fig. 6b. Key player over time with $t \in [50, 1000]$.

economy. Aggregate crime is decreasing in $\omega(g, \theta) = b_{\mathcal{V}'}(g, \theta) / b(g, \theta)$, which, as previously mentioned, corresponds to the weighted average of shadow prices in network g with weights given by $(b_1(g, \theta) / b(g, \theta), \dots, b_n(g, \theta) / b(g, \theta))$. In the bridge network with eleven criminals depicted in Fig. 5, we have $\omega(g, \theta) = 0.9569 - 0.0008y$. Interestingly, the average shadow price of total wealth in the economy, $(V'_1 + 4V'_2 + 6V'_3) / 11$, is approximately equal to $\omega(g, \theta)$. It can be checked that for $y_0 = 115.5$ previously considered, the highest $\omega(g, \theta)$, or, equivalently, the highest average shadow price of total wealth in the economy, is that associated with g_{-3} . Hence, in the short run, committing crime is more costly, on average, in the network without a Type 3 criminal than in the other networks resulting from removing either a Type 1 or a Type 2 criminal. In contrast with the static analysis, short-run aggregate crime is the lowest in g_{-3} , implying that, in the short run, the key player is Type 3. At the steady state, we have $\omega(g_{-2}, \theta) > \omega(g_{-1}, \theta) > \omega(g_{-3}, \theta)$ (and similarly for the order relationship between average shadow prices of total wealth in the economy). It follows that, in the long run, it is more costly, on average, to commit crime in g_{-2} than in g_{-1} than in g_{-3} . The steady-state aggregate crime ranking follows the reversed $\omega(g, \theta)$ ranking: steady-state aggregate crime is the lowest in g_{-2} and the highest in g_{-3} . By contrast, in the static model, $\omega(g, \theta)$ is always nil, and the identification of the key player relies on the intercentrality measure ranking, exclusively.

6. Concluding remarks

In this paper, we have taken a novel approach, namely, a differential game approach, to the study of criminal networks, with the aim to reconsider some results derived in the static literature, and to answer a new set of questions related to the network structure and its impact on the evolution of crime.

The existing literature on criminal networks abstracts from dynamic intertemporal considerations. Both the benefits and the costs of crime for criminals are assumed to be static, thus precluding the analysis of important topics such as the impact of network structure on the evolution of crime and the relationship between productivity shocks, crime and growth. Besides theoretical interest, these topics have real-world relevance and their understanding is of paramount importance for designing effective policies.

An established result in the static literature is that the vector of Nash equilibrium crime efforts is proportional to the vector of Bonacich centralities. We have challenged this result by showing that such an established proportionality between the Nash equilibrium and the Bonacich centrality does not hold in general in a dynamic setting.

One of the key lessons that can be drawn from the static literature on criminal networks is the existence of a social multiplier effect: networks with a higher number of criminals or links or both are associated with higher levels of aggregate crime. This lesson is valid as long as time does not play any role. Indeed, our dynamic analysis, which, to our knowledge, is novel in the network theory literature, has shown that more criminals or more connected criminals or both may lead to the counterintuitive opposite result, i.e., a decrease in aggregate crime. This holds true not only in the short run, but also at the steady state. Intuitively, the intertemporal cost of committing crime, which our dynamic framework is able to capture, may increase as a result of an increase in either network size or density or both to such an extent that aggregate crime is reduced. Conditions exist under which forward-looking criminals anticipate that an increase in network size or density or both will lead to an increase in crime by all the other criminals, and, therefore, to a

decrease in total wealth in the economy. Consequently, each criminal will find it optimal to decrease their own criminal activities (since equilibrium crime levels are increasing in total wealth), leading to an equilibrium in which aggregate crime is lower.

Another lesson that can be drawn from the static literature on criminal networks is that the impact of an increase in the marginal expected punishment on aggregate crime can be either positive or negative (or nil) depending on the interplay between two effects, namely, a direct and an indirect effect. Our dynamic analysis has shown that, together with these (static) effects, there exists also a dynamic effect, which, in some cases, outweighs the static effects, thus profoundly changing policy recommendations aimed at reducing aggregate crime.

In this paper, we have also highlighted the presence of a voracity effect, occurring when the implicit growth rate of total wealth in the economy is increased and, as a consequence of that, economic growth is reduced. This finding points to the counterintuitive conclusion that, in the presence of crime, positive productivity shocks may have a detrimental effect on economic growth.

Finally, we have reconsidered the problem of identifying the key player in the network, i.e., the player who, if removed, leads to the largest drop in aggregate crime. A well-known result in the static literature is that the key player is the player with the highest intercentrality measure, defined as the difference between the sum of Bonacich centralities in the original network and the sum of Bonacich centralities in the network without the removed player. We have shown that conditions exist under which the key player in the static and the dynamic setting differ: the key player in the dynamic setting is not necessarily the player with the highest intercentrality measure, due to the presence of an intertemporal marginal cost of committing crime. We have also shown that the key player in the dynamic setting may change over time. The policy implication of this finding is that it might be optimal for a planner seeking to minimize aggregate crime at each point in time to remove (through imprisonment) some criminals up to a certain point, at which they should be reintegrated into society. From this point onwards, other criminals should be removed, either temporarily or ad infinitum, depending on the specific network structure and the parameter values.

Given the complexities of the problems under study, it is not surprising that our model is based on several simplifying assumptions, which could be addressed in future works. First, we have assumed that the network structure is exogenously given. As such, we don't allow for the possibility that criminals choose who to commit crime with. Second, in our model, the network structure is time invariant. One could consider a network which expands or shrinks over time. Third, we have assumed that total wealth in the economy evolves in a deterministic manner. One could model the evolution of the state variable by means of a stochastic process and study how uncertainty impacts on crime. Fourth, but this is in common with the static analysis, it would be interesting to investigate the extent to which the assumed quadratic utility function drives the results.

CRediT authorship contribution statement

Luca Colombo: Writing – review & editing, Writing – original draft, Validation, Software, Resources, Project administration, Methodology, Investigation, Formal analysis, Conceptualization. **Paola Labrecciosa:** Writing – review & editing, Writing – original draft, Validation, Software, Resources, Project administration, Methodology, Investigation, Formal analysis, Conceptualization. **Agnieszka Rusinowska:** Writing – review & editing, Writing – original draft, Validation, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

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Appendix A. Proof of Theorem 1

For the proof of the necessary and sufficient condition for $[\mathbf{I} - \theta\mathbf{G}]^{-1}$ to be well-defined and nonnegative, see the proof of Theorem 1 in Ballester et al. (2006).

By standard arguments (see Starr and Ho, 1969), Markov Feedback Equilibrium strategies must satisfy the following HJB equations ($i = 1, \dots, n$):

$$rV'_i(y) = \max_{x_i \geq 0} \left\{ u_i(x_i, \boldsymbol{\psi}_{-i}^*, g) + V'_i(y) \left[\mu y - x_i - \sum_{j=1, j \neq i}^n \psi_j^*(y) \right] \right\},$$

where $V'_i(y) = \partial V_i(y) / \partial y$ denotes the shadow price of total wealth for criminal i . Assuming that $\theta \rho(g) < 1$, maximization of the RHS of the above HJB implies that²⁸

$$[\delta \mathbf{I} + \delta \mathbf{U} - \pi \phi \mathbf{G}] \boldsymbol{\psi}^* = (1 - \pi) \mathbf{1} - \mathbf{V}',$$

where \mathbf{U} is the n -square matrix of ones, or equivalently,

$$\boldsymbol{\psi}^* = [\delta \mathbf{I} + \delta \mathbf{U} - \pi \phi \mathbf{G}]^{-1} [\mathbf{1} - \pi - \mathbf{V}'],$$

where $\mathbf{V}' = (V'_1, \dots, V'_n)^T$.

Recall that $\theta = \pi \phi / \delta$. Since $\mathbf{U} \boldsymbol{\psi}^* = \boldsymbol{\psi}^* \mathbf{1}$, where $\boldsymbol{\psi}^* = \sum_{i=1}^n \psi_i^* \mathbf{e}_i$, then

$$\delta [\mathbf{I} - \theta \mathbf{G}] \boldsymbol{\psi}^* = [1 - \pi - \delta \boldsymbol{\psi}^*] \mathbf{1} - \mathbf{V}',$$

and

$$\delta \boldsymbol{\psi}^* = [1 - \pi - \delta \boldsymbol{\psi}^*] [\mathbf{I} - \theta \mathbf{G}]^{-1} \mathbf{1} - [\mathbf{I} - \theta \mathbf{G}]^{-1} \mathbf{V}'.$$

Using the definitions of $\mathbf{b}(g, \theta)$ and $\mathbf{b}_{V'}(g, \theta)$, we obtain

$$\delta \boldsymbol{\psi}^* = [1 - \pi - \delta \boldsymbol{\psi}^*] \mathbf{b}(g, \theta) - \mathbf{b}_{V'}(g, \theta),$$

and since $\boldsymbol{\psi}^* = \mathbf{1}^T \boldsymbol{\psi}^*$ it follows that, at an interior solution,

$$\boldsymbol{\psi}^* = \frac{(1 - \pi) \mathbf{b}(g, \theta) - \mathbf{b}_{V'}(g, \theta)}{\delta [1 + b(g, \theta)]}.$$

Hence, we have

$$\psi_i^* = \frac{(1 - \pi) b_i(g, \theta) - b_{V',i}(g, \theta)}{\delta [1 + b(g, \theta)]},$$

which, using $x_{S,i}^*$ given in Theorem 1, can be rewritten as

$$\psi_i^* = x_{S,i}^* \left[1 - \frac{1}{1 - \pi} \frac{b_{V',i}(g, \theta)}{b_i(g, \theta)} \right],$$

where $b_{V',i}(g, \theta)$ is the i -th coordinate of the vector $\mathbf{b}_{V'}(g, \theta)$, with $\mathbf{V}' = (V'_1, \dots, V'_n)^T = (A_1 y + B_1, \dots, A_n y + B_n)^T$.

Appendix B. Proof of Corollary 1

From $\boldsymbol{\psi}^*$ in Theorem 1, for interior solutions, we get

$$\mathbf{1}^T \boldsymbol{\psi}^* = \frac{(1 - \pi) \mathbf{1}^T \mathbf{b}(g, \theta) - \mathbf{1}^T \mathbf{b}_{V'}(g, \theta)}{\delta [1 + b(g, \theta)]},$$

implying that

$$x^* = \frac{(1 - \pi) b(g, \theta) - b_{V'}(g, \theta)}{\delta [1 + b(g, \theta)]} = x_S^* \left[1 - \frac{1}{1 - \pi} \frac{b_{V'}(g, \theta)}{b(g, \theta)} \right],$$

where

$$x_S^* = \frac{(1 - \pi) b(g, \theta)}{\delta [1 + b(g, \theta)]}.$$

Appendix C. Proof of Corollary 2

The trajectory of y is the solution to the following first-order linear differential equation

$$\dot{y}(t) = \mu y(t) - x_S^* \left[1 - \frac{b_A(g, \theta) y(t) + b_B(g, \theta)}{(1 - \pi) b(g, \theta)} \right],$$

²⁸ This solution represents a maximum since the expression in curly brackets in (9) is concave in x_i .

with initial condition $y(0) = y_0 \in L$. Routine calculations lead to $y^*(t)$ given in Corollary 2. We have $\lim_{t \rightarrow \infty} y^*(t) = \hat{y}$ provided that $\mu + b_A(g, \theta) / \{\delta[1 + b(g, \theta)]\} < 0$. The trajectory of x , $x^*(t)$, can be computed from x^* given in Corollary 1 by evaluating $b_{V'}(g, \theta) = b_A(g, \theta)y + b_B(g, \theta)$ at $y = y^*(t)$.

Appendix D. Proof of Proposition 2

We need to show that $\Delta x^* = x^*(g_N) - x^*(g_\theta) < 0$. From Corollary 1, we have

$$\Delta x^* = \frac{1 - \pi - V'(y)|_{g_N}}{\delta(1+n) - (n-1)\pi\phi} - \frac{1 - \pi - V'(y)|_{g_\theta}}{\delta(1+n)},$$

where $V'(y) = Ay + B$, with A and B given in (15) and (16), respectively. Δx^* is linear in y and the coefficient of y is given by

$$\frac{(n-1)(r-2\mu)\pi\phi}{2n[n^2\delta - \pi\phi(n-1)^2]}. \quad (\text{D.1})$$

The numerator of (D.1) is negative since it must be that $r < 2\mu$ for steady-state stability. The denominator of (D.1) is positive if $\delta > \pi\phi(n-1)^2/n^2$, which is the case since it must be that $\delta > \pi\phi(n-1)$ for the condition $\theta\rho(g) < 1$ to be satisfied and $\pi\phi(n-1) > \pi\phi(n-1)^2/n^2$. Hence, the denominator of (D.1) is positive, implying that Δx^* is decreasing in y . Call y_{CE} the root of Δx^* . It follows that $\Delta x^* > (\text{resp. } <) 0$ for $y < (\text{resp. } >) y_{CE}$. Define

$$\underline{y} = \frac{(1-\pi)[-r\delta(1+n^2) + 2\delta\mu + d\pi r\phi(n-1)]}{\mu(2\mu-r)(\delta+n\delta-d\pi\phi)^2},$$

$$\bar{y} = \frac{(1-\pi)[\delta(1+n^2) - d\pi\phi(n-1)]}{\mu(\delta+n\delta-d\pi\phi)^2}.$$

For y_{CE} to be admissible and for steady-state convergence in both the empty and the complete network, we must have $0 < \max\{\underline{y}|_{g_\theta}, \underline{y}|_{g_N}\} < y_{CE} < \min\{\bar{y}|_{g_\theta}, \bar{y}|_{g_N}\}$, where $\underline{y}|_{g_\theta}$ and $\underline{y}|_{g_N}$ are equal to \underline{y} for $d=0$ and $d=n-1$, respectively; $\bar{y}|_{g_\theta}$ and $\bar{y}|_{g_N}$ are equal to \bar{y} for $d=0$ and $d=n-1$, respectively. We have $\underline{y}|_{g_\theta} > 0$ if $r < 2\mu/(1+n^2)$. We are going to show that $r < 2\mu/(1+n^2)$ implies $y_{CE} < 0$ and therefore $\Delta x^* < 0$. The sign of y_{CE} is given by

$$r\delta(n^4 + 2n^2 - 1) + 2\delta\mu(1 - 2n^2) - \pi\phi(n-1)^2(r + rn^2 - 2\mu), \quad (\text{D.2})$$

which is linear in r . The coefficient of r in (D.2) is equal to

$$\delta(n^4 + 2n^2 - 1) - \pi\phi(n-1)^2(1+n^2). \quad (\text{D.3})$$

(D.3) is increasing in δ and positive if $\delta > \pi\phi(n-1)^2(1+n^2)/(n^4+2n^2-1)$. It is easy to check that $\pi\phi(n-1)^2(1+n^2)/(n^4+2n^2-1) < \pi\phi(n-1)$, hence (D.3) is positive, implying that (D.2) is increasing in r . By equating (D.2) to zero and solving for r we get

$$r_{CE} = \frac{2\mu[\delta - 2\delta n^2 + \pi\phi(n-1)^2]}{\pi\phi(n-1)^2(1+n^2) - \delta(n^4+2n^2-1)}.$$

A direct comparison between r_{CE} and $2\mu/(1+n^2)$ gives

$$r_{CE} - \frac{2\mu}{1+n^2} = \frac{2n^2\delta\mu(n^2-1)}{\delta(n^6+3n^4+n^2-1) - \pi\phi(n-1)^2(1+n^2)^2},$$

revealing that $r_{CE} > 2\mu/(1+n^2)$ if $\delta > \pi\phi(n-1)^2(1+n^2)/(n^4+2n^2-1)$. Since $\pi\phi(n-1)^2(1+n^2)/(n^4+2n^2-1) < (n-1)\pi\phi$ then $\delta > \pi\phi(n-1)^2(1+n^2)/(n^4+2n^2-1)$, implying that $r_{CE} > 2\mu/(1+n^2)$ and that $y_{CE} < 0$. We can then conclude that $\Delta x^* < 0$.

Appendix E. The network effect

We aim to show that $\Omega(y) = V'(y)|_{g_N} - V'(y)|_{g_\theta} > 0$. By direct computation, we have

$$\Omega(y) = \frac{\pi\phi(r-2\mu)\{(n-1)^2[\mu n^2\pi y\phi + (\pi-1)] - y\delta\mu(n^4-1)\}}{2\mu n^2[\delta n^2 - \pi\phi(n-1)^2]}. \quad (\text{E.1})$$

The denominator of the RHS of (E.1) is positive given that $\delta > \pi\phi(n-1)$ is required for $\theta\rho(g) < 1$. Since it must be that $r < 2\mu$ for steady-state stability, the sign of the numerator of the RHS of (E.1) is given by the sign of $y\delta\mu(n^4-1) - (n-1)^2[\mu n^2\pi y\phi + (\pi-1)]$. Collecting for y gives $[(n^4-1)\delta\mu - (n-1)^2n^2\pi\mu\phi]y + (n-1)^2(1-\pi)$. By solving the coefficient of y for δ we obtain the following

threshold $\delta_c = [n^2\pi\phi(n-1)]/(1+n+n^2+n^3)$. It is immediate to verify that $\delta_c < \pi\phi(n-1)$. It follows that the coefficient of y is positive, and that the sign of $y\delta\mu(n^4-1) - (n-1)^2[\mu n^2\pi y\phi + (\pi-1)]$ is positive if $y > y_c$, with

$$y_c = \frac{(n-1)(1-\pi)}{(n-1)n^2\pi\mu\phi - (n+1)(1+n^2)\delta\mu}. \tag{E.2}$$

The denominator of the RHS of (E.2) is negative since $\delta > \pi\phi(n-1) > (n-1)n^2\pi\phi/[(n+1)(1+n^2)]$, implying that $y_c < 0$ and that $y > y_c$. We can then conclude that $\Omega(y) > 0$.

Appendix F. Proof of Proposition 3

(i) First, we evaluate the impact of an increase in n on an empty network. Let $\Delta x^* = x^*(g'_\emptyset) - x^*(g_\emptyset)$, where g_\emptyset and g'_\emptyset denote the original empty network with n criminals and the enlarged empty network with n' criminals ($n' > n$), respectively. By direct computation, we have

$$\Delta x^* = \frac{(n'-n)(1-\pi)\{r[n(n'-1) - n' - 1] + 2\mu(1+n+n')\}}{2nn'\delta\mu(1+n)(1+n')} + \frac{(n'-n)(r-2\mu)}{2nn'}y.$$

The coefficient of y in Δx^* is negative since $r - 2\mu < 0$ for steady-state stability. Therefore, Δx^* is decreasing in y . By equating Δx^* to zero and solving for y we get

$$y_{EN} = \frac{(1-\pi)\{r[n(n'-1) - n' - 1] + 2\mu(1+n+n')\}}{\delta\mu(1+n)(1+n')(2\mu-r)}.$$

Hence, $\Delta x^* >$ (resp. $<$) 0 for $y <$ (resp. $>$) y_{EN} . For y_{EN} to be admissible it must be that $\underline{y}|_{g_\emptyset} < y_{EN} < \bar{y}|_{g_\emptyset}$. We now verify that y_{EN} is admissible by means of a numerical example. Let $\pi = 0.05$, $n = 9$, $n' = 10$, $\mu = 0.1$, $r = 0.001$, $\phi = 7$, $\delta = 5$. For these parameter values, we have $y_{EN} = 0.3533$, $\underline{y}|_{g_\emptyset} = 0.0113$, $\bar{y}|_{g_\emptyset} = 1.5580$. We take $y = 1.55 > y_{EN}$. In this case, $x^*(g_\emptyset) = 0.1701 > x^*(g'_\emptyset) = 0.1688$. Now we take $y = 0.1 < y_{EN}$. In this case, $x^*(g_\emptyset) = 0.0098 < x^*(g'_\emptyset) = 0.0101$.

(ii) Next, we evaluate the impact of an increase in n on a complete network. Let $\Delta x^* = x^*(g'_N) - x^*(g_N)$, where g_N and g'_N denote the original complete network with n criminals and the enlarged complete network with n' criminals ($n' > n$), respectively. By direct computation, we obtain an expression for Δx^* (omitted for brevity) which is linear in y . The sign of the coefficient of y in Δx^* is given by

$$-\pi\phi[n'(\delta - \pi\phi) + \delta + \pi\phi] - n(\delta - \pi\phi)[n'(\delta - \pi\phi) + \pi\phi],$$

which is negative (since $\delta > \pi\phi$ for the condition $\theta\rho(g) < 1$ to be satisfied). Therefore, Δx^* is decreasing in y . Call y_{CN} the value of y that solves $\Delta x^* = 0$. Hence, $\Delta x^* >$ (resp. $<$) 0 for $y <$ (resp. $>$) y_{CN} . For y_{CN} to be admissible it must be that $\underline{y}|_{g_N} < y_{CN} < \bar{y}|_{g_N}$. We now verify that y_{CN} is admissible by means of a numerical example. As in part (i), let $\pi = 0.05$, $n = 9$, $n' = 10$, $\mu = 0.1$, $r = 0.001$, $\phi = 7$, $\delta = 5$. For these parameter values, we have $y_{CN} = 0.3970$, $\underline{y}|_{g_N} = 0.0131$, $\bar{y}|_{g_N} = 1.6528$. We take $y = 1.55 > y_{CN}$. In this case, $x^*(g) = 0.1698 > x^*(g') = 0.1685$. Now we take $y = 0.1 < y_{CN}$. In this case, $x^*(g) = 0.0096 < x^*(g') = 0.0099$.

Appendix G. Proof of Proposition 6

(i) In the empty network, we have

$$\frac{\partial x^*}{\partial \pi} = -\frac{r(1+n^2) - 2\mu}{2n\delta\mu(1+n)} > 0,$$

given that $r < 2\mu/(1+n^2)$ (required for $\underline{y}|_{g_\emptyset} > 0$, which is needed for steady-state stability).

(ii) Consider now the complete network. By direct computation, we obtain an expression for $\partial x^*/\partial \pi$ (omitted for brevity) which is linear in y . The coefficient of y in $\partial x^*/\partial \pi$ is given by $\delta\phi n(n-1)(r-2\mu)/\{2[n^2\delta - \pi\phi(n-1)^2]\}$, which is negative (since $r < 2\mu$ for steady-state stability). Therefore, $\partial x^*/\partial \pi$ is decreasing in y . Call $y_{C\Pi}$ the value of y that solves $\partial x^*/\partial \pi = 0$. Hence, $\partial x^*/\partial \pi >$ (resp. $<$) 0 for $y <$ (resp. $>$) $y_{C\Pi}$. We now verify that $y_{C\Pi}$ is admissible by means of a numerical example. Let $\pi = 0.1$, $n = 4$, $\mu = 0.1$, $r = 0.001$, $\phi = 7$, $\delta = 9$. For these parameter values, we have $y_{C\Pi} = 0.1600$, $\underline{y}|_{g_N} = 0.0406$, $\bar{y}|_{g_N} = 0.7174$. We take $y = 0.1 < y_{C\Pi}$. In this case, $\partial x^*/\partial \pi = 0.0002$. Now we take $y = 0.2 > y_{C\Pi}$. In this case, $\partial x^*/\partial \pi = -0.0002$.

Appendix H. Proof of Theorem 2

By definition, l_S^* is such that $x_S^*(g_{-i_S^*}) \leq x_S^*(g_{-i})$ and $b(g_{-i_S^*}, \theta) \leq b(g_{-i}, \theta)$, for all $i \neq i_S^*$. Moreover, we have

$$x^*(g_{-i_S^*}) = x_S^*(g_{-i_S^*}) \left[1 - \frac{1}{1-\pi} \frac{b_{V'}^*(g_{-i_S^*}, \theta)}{b(g_{-i_S^*}, \theta)} \right].$$

(i) A necessary (but not sufficient) condition for $x^*(g_{-i_S^*}) > x^*(g_{-i})$ is

$$1 - \frac{1}{1-\pi} \frac{b_{V'}^*(g_{-i_S^*}, \theta)}{b(g_{-i_S^*}, \theta)} > 1 - \frac{1}{1-\pi} \frac{b_{V'}^*(g_{-i}, \theta)}{b(g_{-i}, \theta)}$$

which simplifies to

$$\frac{b_{V'}^*(g_{-i_S^*}, \theta)}{b(g_{-i_S^*}, \theta)} < \frac{b_{V'}^*(g_{-i}, \theta)}{b(g_{-i}, \theta)}.$$

(ii) $x^*(g_{-i_S^*}) < x^*(g_{-i})$ if

$$\frac{b_{V'}^*(g_{-i_S^*}, \theta)}{b(g_{-i_S^*}, \theta)} > \frac{b_{V'}^*(g_{-i}, \theta)}{b(g_{-i}, \theta)}.$$

A sufficient (but not necessary) condition for the above inequality to hold is

$$b_{V'}^*(g_{-i_S^*}, \theta) > b_{V'}^*(g_{-i}, \theta).$$

Data availability

No data was used for the research described in the article.

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