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Abstract

We use the framework of Morse theory with differential graded coefficients to study certain operations on the total space of a fibration. More particularly, we focus in this paper on a chain-level description of the Chas-Sullivan product on the homology of the free loop space of an oriented, closed and connected manifold. The idea of "intersecting on the base" and "concatenating on the fiber" are well-adapted to this framework. We also give a Morse theoretical description of other products that follow the same principle. For this purpose, we develop functorial properties with respect to the coefficient in terms of morphisms of $\mathcal{A}_\infty$-modules and morphisms of fibrations. We also build a differential graded version of the Künneth formula and of the Pontryagin-Thom construction.

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1 Introduction and main results

Given an oriented n-dimensional closed manifold X, Chas and Sullivan described in [CS99] a degree $-n$ product

$$H_i(\mathcal{L}X) \otimes H_j(\mathcal{L}X) \to H_{i+j-n}(\mathcal{L}X)$$

on the homology of the free loop space $\mathcal{L}X = C^0(\mathbb{S}^1, X)$ of X as an intersection product on $H_*(X)$ combined with the Pontryagin product on based loops. This multiplication is now a fundamental operation in string topology, the study of the homology of loop spaces.

State of the art.
A geometric realisation of this product has been described by Cohen-Jones [CJ02] in terms of the Thom spectrum $\mathcal{L}X^{-TX} = \Sigma^{-N-d}Th(ev^*\nu^N)$ where $ev : \mathcal{L}X \to X$ is the evaluation at the basepoint and $\nu^N \to X$ is the N-dimensional normal bundle of a fixed embedding $X \to \mathbb{R}^{N+n}$. This approach has been generalized by Gruher-Salvatore [GS08] to define a product on the homology of any total space of a fiber bundle $E \xrightarrow{p} X$ equipped with a "multiplication map" $m : E \,{}_p\!\times_p E \to E$ and a "unit section" $s : X \to E$. A chain-level description of the Chas-Sullivan product on the singular complex of $\mathcal{L}X$ has been given by Hingston-Wahl in [HW23]. This description consists in intersecting two chains $A, B \in C_*(\mathcal{L}X)$ by considering a tubular neighborhood $\mathcal{U}$ of the diagonal $\Delta X \subset X^2$ and taking the cap-product with $A \times B$ of the pullback of a representative τ of the Thom-class of $\mathcal{U} \to \Delta X$ to obtain $(ev \times ev)^*\tau \cap (A \times B) \in C_*((ev \times ev)^{-1}\mathcal{U})$ and then retracting into the **figure-eight space** $\mathcal{L}X \,{}_{ev}\!\times_{ev} \mathcal{L}X$ using "geodesic sticks" before concatenating. Abbondandolo-Schwarz constructed in [AS10] a loop product using the fact that the figure-eight space has a smooth tubular neighborhood in $\mathcal{L}X \times \mathcal{L}X$ and thus use an "Umkehr map" $e_! : H_*(\mathcal{L}X \times \mathcal{L}X) \to H_{*-n}(\mathcal{L}X \,{}_{ev}\!\times_{ev} \mathcal{L}X)$. Laudenbach defines in [Lau11] a loop-product in terms of "transverse bi-simplices" $u \times v$ in $\mathcal{L}X \times \mathcal{L}X$, seeing $\mathcal{L}X$ as a simplicial set instead of a topological space. Chataur introduced in [Cha05] a geometric way to define the loop product from the point of view of geometric homology theory. Abouzaid took in [Abo15] the approach of approximating the loop space $\mathcal{L}X$ by finite dimensional spaces of piecewise geodesic loops in order to use Morse theory.
Most of these constructions either use the Hilbert space of H^1-loops to represent the loop space $\mathcal{L}X$ and use infinite dimensional considerations, or define the loop product only at the homology level. We construct in this paper a chain-level model for Chas-Sullivan products using finite dimensional arguments.

Our construction.
Let $(X^n, \star)$ be a smooth pointed, oriented, closed, connected manifold. The first goal of this paper is to give a finite dimensional Morse theoretic chain-level model for the Chas-Sullivan product and generalize it to any Hurewicz fibration $F \hookrightarrow E \xrightarrow{\pi} X$ equipped with a **morphism of fibrations** $m : E \,{}_\pi\!\times_\pi E \to E$ *i.e*, m is continuous and preserves the fibrations $\pi \circ m = \pi$.
Our construction relies on the fact that there exists a notion of *intersecting on the base* and *multipling on the fiber* in the setting of **enriched Morse homology** defined and studied by Barraud-Damian-Humilière-Oancea [BDHO24] using the original idea of Barraud and Cornea [BC07]. The authors define a Morse homology theory with coefficients in any differential graded (DG) right-module $(\mathcal{F}_*, \partial_\mathcal{F})$ over $C_*(\Omega X)$, the cubical chains on the Moore based loop space over X with algebra structure given by the Pontryagin product. Such a DG-module $\mathcal{F}$ is called a **DG local system**. Given a Morse-Smale pair (f, ξ), they build following [BC07] a Maurer-Cartan element referred to as the **Barraud-Cornea twisting cocycle** $\{m_{x,y} \in C_{|x|-|y|-1}(\Omega X),\ x, y \in \mathrm{Crit}(f)\}$ satisfying

$$\partial m_{x,y} = \sum_{z \in \mathrm{Crit}(f)} (-1)^{|x|-|z|} m_{x,z} \cdot m_{z,y} \tag{1}$$

and use it to define the **enriched Morse complex** or **Morse complex with DG coefficients**

$$C_*(X, \mathcal{F}_*) = \mathcal{F}_* \otimes_\mathbb{Z} \mathbb{Z}\mathrm{Crit}(f)$$

with the twisted differential

$$\partial(\alpha \otimes x) = \partial_\mathcal{F}\alpha \otimes x + (-1)^{|\alpha|} \sum_y \alpha \cdot m_{x,y} \otimes y.$$

One of their main results is what we will refer to as the **Fibration Theorem**.
Given a fibration $F \hookrightarrow E \xrightarrow{\pi} X$, a **transitive lifting function** $\Phi : E \,{}_{\pi}\!\times_{ev_0} \mathcal{P}X \to E$ is map that lifts all paths in X and respects concatenation in the sense that lifting a concatenation of two paths is the same as lifting them one after the other : $\Phi(\Phi(e,\gamma),\delta) = \Phi(e,\gamma\#\delta)$.

Fibration Theorem ([BDHO24] Theorem 7.2.1) : *Let $F \hookrightarrow E \to X$ be a Hurewicz fibration and equip $C_*(F)$ with the $C_*(\Omega X)$-module structure induced by a transitive lifting function associated to this fibration (see [BDHO24, Section 7.1]). Then, there exists a quasi-isomorphism denoted*

$$\Psi_E : C_*(X, C_*(F)) \to C_*(E).$$

We will prove in Corollary 5.13 that the complex $C_*(X, C_*(F))$ does not depend, up to homotopy equivalence, on the choice of transitive lifting function.
This setting therefore provides a description of a chain complex on X that computes the homology of the total space E of a fibration in terms of Morse data on the base X twisted by the cubical chain complex of the fiber F. It is particularly adapted to operations such as the Chas-Sullivan product. This setting also provides an operation that is a model for the *intersection on the base* which is the shriek map

$$\begin{array}{ccc} C_*(X^2, C_*(F^2)) & \xrightarrow{\Delta_!} & C_{*-n}(X, \Delta^* C_*(F^2)) \\ \downarrow{\scriptstyle \Psi_{E^2}} & & \downarrow{\scriptstyle \Psi_{E\,{}_{\pi}\!\times_{\pi} E}} \\ C_*(E \times E) & \dashrightarrow & C_{*-n}(E \,{}_{\pi}\!\times_{\pi} E) \end{array}$$

induced by the diagonal $\Delta : X \to X^2$ (see sections 9 and 10 of [BDHO24]).

We now state our generalization of the Chas-Sullivan product.

Theorem A (7.1) : *Let $F \hookrightarrow E \xrightarrow{\pi} X$ be a fibration and denote $\mathcal{F} = C_*(F)$ with the DG-module structure over $C_*(\Omega X)$ induced by a transitive lifting function $\Phi : E \,{}_{\pi}\!\times_{ev_0} \mathcal{P}X \to E$ associated to the fibration. Let $m : E \,{}_{\pi}\!\times_{\pi} E \to E$ be a morphism of fibrations over X (see definition 5.7). This morphism induces a degree $-n$ product*

$$\mathrm{CS}_{DG} : H_*(X, \mathcal{F})^{\otimes 2} \to H_*(X, \mathcal{F}).$$

The following properties hold :

- **Associativity (7.2) :** *If m_* is associative in homology, then so is CS_{DG}.*
- **Commutativity (7.3) :** *If m_* is commutative in homology, then CS_{DG} is commutative up to sign in homology i.e, $\mathrm{CS}_{DG}(\gamma \otimes \tau) = (-1)^{(n-|\gamma|)(n-|\tau|)}\mathrm{CS}_{DG}(\tau \otimes \gamma)$.*
- **Neutral element (7.5) :** *If π admits a section $s : X \to E$ such that $m(s(\pi(e)), e) = m(e, s(\pi(e))) = e$ for all $e \in E$, then CS_{DG} admits a neutral element.*
- **Functoriality (7.7) :**
  • *For any pointed, oriented, closed and connected manifold Y^k, any continuous map $g : Y \to X$ induces a degree $-k$ product for the fibration $F \hookrightarrow g^*E \to Y$*

  $$\mathrm{CS}^Y_{DG} : H_i(Y, g^*\mathcal{F}) \otimes H_j(Y, g^*\mathcal{F}) \to H_{i+j-k}(Y, g^*\mathcal{F})$$

  such that $g_! : H_*(X, \mathcal{F}) \to H_{*+n-k}(Y, g^*\mathcal{F})$ is a morphism of rings up to sign.
  • *If g is an orientation-preserving homotopy equivalence then, $g_!$ and $g_* : H_*(Y, g^*\mathcal{F}) \to H_*(X, \mathcal{F})$ are isomorphisms of rings inverses of each other.*
- **Spectral sequence (7.10) :** *The canonical filtration*

  $$F_p(C_*(X, \mathcal{F})) = \bigoplus_{\substack{i+j=k \\ i \leq p}} \mathcal{F}_j \otimes \mathbb{Z}\mathrm{Crit}_i(f)$$

  induces a spectral sequence $E^r_{p,q}$ that is endowed with an algebra structure

  $$E^r_{p,q} \otimes E^r_{l,m} \to E^r_{p+l-n,q+m}$$

induced by a chain description $\mathrm{CS}_{DG} : C_*(X,\Xi,\mathcal{F})^{\otimes 2} \to C_*(X,\Xi,\mathcal{F})$ *and converges towards* $H_*(X,\mathcal{F})$ *as algebras. For* $s,t \geq 0$, $E^2_{s,t} = H_s(X, H_t(\mathcal{F}))$ *and the algebra structure is given up to sign by the intersection product on* X *with coefficients in* $H_t(\mathcal{F})$.

Moreover, this product corresponds in homology, via the Fibration Theorem, to the product $\mu_* : H_i(E) \otimes H_j(E) \to H_{i+j-n}(E)$ *defined in [GS08] (see Proposition 7.11). In particular, if the fibration is the loop-loop fibration* $\Omega X \hookrightarrow \mathcal{L}X \to X$, *then* CS_{DG} *corresponds to the Chas-Sullivan product.*

This (re)proves that the product μ_*, and in particular the Chas-Sullivan product, has the properties of Associativity, Commutativity, Neutral element, Functoriality and Spectral sequence.

We will define $\mathrm{CS}_{DG} : H_*(X, C_*(F))^{\otimes 2} \to H_*(X, C_*(F))$ as the composition

$$\begin{array}{ccccc} H_i(X,C_*(F)) \otimes H_j(X,C_*(F)) & \xrightarrow{K^{top}} & H_{i+j}(X^2, C_*(F^2)) & \xrightarrow{\Delta_!} & H_{i+j-n}(X, \Delta^* C_*(F^2)) \\ & & & \xrightarrow{\tilde{m}} & H_{i+j-n}(X, C_*(F)) \end{array}$$

with the Dold sign, *i.e*, $\mathrm{CS}_{DG}(\gamma \otimes \tau) = (-1)^{n(n-|\tau|)} \tilde{m} \circ \Delta_! \circ K(\gamma \otimes \tau)$.
The Dold sign has been first introduced in [Dol95, Chapter VIII, §13.3] in order to make the intersection product in homology Poincaré dual to the unsigned cup-product, and used in [Lau11] in order to make the Chas-Sullivan product graded commutative. It also serves such purposes in our setting.
The map $K^{top} : C_*(X, C_*(F))^{\otimes 2} \to C_*(X^2, C_*(F^2))$ is a version of the cross product in the enriched Morse homology setting. The construction of this map is the object of Section 6.
The map $\tilde{m} : C_*(X, \Delta^* C_*(F^2)) \to C_*(X, C_*(F))$ is the morphism of complexes induced by the morphism of fibrations m. This will be constructed and studied in Section 5. In particular, we will prove that this map is a good notion of *multiplication on the fiber* in our model.

Occurence of $\mathcal{A}_\infty$*-modules and morphisms.*
Let us give an intuition and a motivation behind the use of $\mathcal{A}_\infty$-modules and morphisms in the case of the *loop-loop fibration* $\Omega X \hookrightarrow \mathcal{L}X \xrightarrow{ev} X$. It is quite clear from (1) that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of DG-modules, then $\tilde{\varphi} : C_*(X,\mathcal{F}) \to C_*(X,\mathcal{G})$ defined by $\tilde{\varphi}(\alpha \otimes x) = \varphi(\alpha) \otimes x$ is a morphism of complexes. However, the map $m_* : C_*(\Omega X^2)^{ad} \to C_*(\Omega X)^{ad}$ induced by the concatenation $m : \Omega X^2 \to \Omega X$ is not a morphism of DG-modules over $C_*(\Omega X)$. The notation "ad" indicates that the module structures are induced by the conjugation at the topological level. Indeed,

$$\forall \alpha, \beta \in \Omega X, \forall \gamma \in \Omega X,\ m((\alpha,\beta)\cdot\gamma) = m(\gamma^{-1}\alpha\gamma, \gamma^{-1}\beta\gamma) = \gamma^{-1}\alpha\gamma\gamma^{-1}\beta\gamma \neq \gamma^{-1}\alpha\beta\gamma = m(\alpha,\beta)\cdot\gamma.$$

However, we will prove that there exists a sequence of maps $\{m_n : [0,1]^{n-1} \times \Omega X^2 \times \Omega X^{n-1} \to \Omega X\}_{n\geq 1}$ that successively compensate (as more precisely defined in Definition 5.3) for the fact that the concatenation $m_1 := m$ commutes with the topological module structure induced by the fibrations only up to homotopy. We will refer to such a sequence (m_n) as an **$\mathcal{A}_\infty$-morphism of topological modules**. This yields a morphism of $\mathcal{A}_\infty$-modules $\{m_{n,*}\} : C_*(\Omega X^2)^{ad} \to C_*(\Omega X)^{ad}$ over $C_*(\Omega X)$ (see Proposition 5.5). The concatenation being a core operation for the Chas-Sullivan product, we needed to prove that one is still able to extract a morphism of complexes $C_*(X, C_*(\Omega X^2)^{ad}) \to C_*(X, C_*(\Omega X)^{ad})$ out of it. That is why we will prove

Proposition B (5.1) : *Let* $\mathcal{A}, \mathcal{B}$ *be DG-modules over* $C_*(\Omega X)$. *Any morphism of* $\mathcal{A}_\infty$*-modules* $\boldsymbol{\nu} : \mathcal{A} \to \mathcal{B}$ *induces a morphism of complexes denoted* $\tilde{\nu} : C_*(X,\mathcal{A}) \to C_*(X,\mathcal{B})$.

In the context of a fibration, we will prove the following succession of result in Section 5.

$\varphi : E_1 \to E_2$ is a morphism of fibrations.
$\overset{5.9}{\Longrightarrow}$ There exists a **coherent homotopy** $\{\varphi_{n+1} : I^n \times F_1 \times \Omega X^{n-1} \times \mathcal{P}_{\star \to X} X \to E_2\}$
that induces an $\mathcal{A}_\infty$-morphism of topological modules $\{\varphi_{n+1}^{F_1} : I^n \times F_1 \times \Omega X^n \to F_2\}$.
$\overset{5.5}{\Longrightarrow}$ The map φ induces a morphism $C_*(F_1) \to C_*(F_2)$ of $\mathcal{A}_\infty$-modules over $C_*(\Omega X)$.
$\overset{5.1}{\Longrightarrow}$ The map φ induces a morphism $\tilde{\varphi} : C_*(X, C_*(F_1)) \to C_*(X, C_*(F_2))$.
$\overset{5.8}{\textit{and}}$ A coherent homotopy induces a chain homotopy between $\Psi_2 \circ \tilde{\varphi}$ and $\varphi_* \circ \Psi_1$.

This constitutes the outline of a proof for the following theorem :

Theorem B (5.8): *Let $F_1 \hookrightarrow E_1 \to X$ and $F_2 \hookrightarrow E_2 \to X$ be two fibrations. If $\varphi : E_1 \to E_2$ is a morphism of fibrations, then there exists a morphism of complexes $\tilde{\varphi} : C_*(X, C_*(F_1)) \to C_*(X, C_*(F_2))$ such that the following diagram commutes up to chain homotopy*

$$\begin{array}{ccc} C_*(X, C_*(F_1)) & \xrightarrow{\tilde{\varphi}} & C_*(X, C_*(F_2)) \\ \downarrow{\scriptstyle \Psi_1} & & \downarrow{\scriptstyle \Psi_2} \\ C_*(E_1) & \xrightarrow[\varphi_*]{} & C_*(E_2), \end{array}$$

where the morphisms Ψ_1 and Ψ_2 are the quasi-isomorphisms given by the Fibration Theorem.

We then prove compatibility properties of the morphisms of complexes given by Theorem B to ensure that these morphisms are well-suited to translate operations on the total space of a fibration in an enriched Morse theoretic approach.

This compatibility between $\mathcal{A}_\infty$-structures and DG Morse theory determined us to explore the relation between these theories. We define the enriched Morse complex with coefficients in an $\mathcal{A}_\infty$-module over $C_*(\Omega X)$ (Section 4) and prove invariance properties using the canonical spectral sequences associated to such a complex (section 4.5). We will moreover prove Proposition B for $\mathcal{A}$ and $\mathcal{B}$ any $\mathcal{A}_\infty$-modules over $C_*(\Omega X)$.

Künneth formula.
Let $(X, \star_X)$ and $(Y, \star_Y)$ be pointed, oriented, closed, connected and smooth manifolds with Morse-Smale pairs (f, ξ_X) and (g, ξ_Y). Let $\mathcal{F}$ and $\mathcal{G}$ be DG modules over $C_*(\Omega X)$ and $C_*(\Omega Y)$. We build in Section 6.1 what we will refer to as the **Künneth twisting cocycle** $\{m^K_{z,w} \in C_{|z|-|w|-1}(\Omega X \times \Omega Y),\ z, w \in \mathrm{Crit}(f+g)\}$ adapted to the Cartesian product (see Definition 6.4) such that there exists a homotopy equivalence

$$C_*(X \times Y, \mathcal{H}) \simeq C_*(X \times Y, m^K_{z,w}, \mathcal{H})$$

for any DG-module $\mathcal{H}$ over $C_*(\Omega X \times \Omega Y)$ (see Proposition 6.5).

From there, let us present two versions of cross products on enriched Morse complexes :
• We can use the Serre diagonal $\Delta : C_*(\Omega X \times \Omega Y) \to C_*(\Omega X) \otimes C_*(\Omega Y)$ to define a $C_*(\Omega(X \times Y))$-module structure on the tensor product $\mathcal{F} \otimes_{\mathbb{Z}} \mathcal{G}$ such that

$$K^{alg} : \begin{array}{ccc} C_*(X, \mathcal{F}) \otimes C_*(Y, \mathcal{G}) & \to & C_*(X \times Y, m^K_{z,w}, \mathcal{F} \otimes \mathcal{G}) \\ (\alpha \otimes x) \otimes (\beta \otimes y) & \mapsto & (-1)^{|\beta||x|}(\alpha \otimes \beta) \otimes (x, y) \end{array}$$

is an isomorphism of complexes.
• If $F \hookrightarrow E \to X$ and $G \hookrightarrow E' \to Y$ are two fibrations with respective transitive functions Φ_X and Φ_Y, then the fibration $F \times G \hookrightarrow E \times E' \to X \times Y$ is naturally endowed with the transitive lifting function (Φ_X, Φ_Y). Therefore, to study fibrations, if $\mathcal{F} = C_*(F)$ and $\mathcal{G} = C_*(G)$, it is natural to consider the complex $C_*(X \times Y, m^K_{z,w}, C_*(F \times G))$ where the module structure is defined by (Φ_X, Φ_Y).

That is why if F and G are topological spaces each endowed with a topological module structure over respectively ΩX and ΩY, we define

$$K^{top} : \begin{array}{ccc} C_*(X, C_*(F)) \otimes C_*(Y, C_*(G)) & \to & C_*(X \times Y, m^K_{z,w}, C_*(F \times G)) \\ (\alpha \otimes x) \otimes (\beta \otimes y) & \mapsto & (-1)^{|\beta||x|}(\alpha, \beta) \otimes (x, y) \end{array} .$$

To summarize, we have the following theorem :

Theorem C (6.13): *Let $\mathcal{F}$ and $\mathcal{G}$ be DG modules over $C_*(\Omega X)$ and $C_*(\Omega Y)$. There exists a Künneth twisting cocycle $\{m^K_{z,w} \in C_{|z|-|w|-1}(\Omega X \times \Omega Y),\ z, w \in \mathrm{Crit}(f+g)\}$ which computes the same homology as the Barraud-Cornea cocycle and such that :*

$$K^{alg} : C_*(X, \mathcal{F}) \otimes C_*(Y, \mathcal{G}) \to C_*(X \times Y, m^K_{z,w}, \mathcal{F} \otimes \mathcal{G})$$

is an isomorphism of complexes.
If $\mathcal{F} = C_*(F)$ and $\mathcal{G} = C_*(G)$ are complexes of cubical chains of topological spaces equipped with topological module structures respectively over ΩX and ΩY, then

$$K^{top} : C_*(X, C_*(F)) \otimes C_*(Y, C_*(G)) \to C_*(X \times Y, m^K_{z,w}, C_*(F \times G))$$

is a quasi-isomorphism of complexes.

In particular, this will prove that $H_*(X \times Y, C_*(F \times G))$ is isomorphic to $H_*(X \times Y, C_*(F) \otimes C_*(G))$. In this paper, we will mainly use the cross product $K := K^{top}$ since we will heavily rely on considerations on fibrations to define the product CS_{DG}.

This result is not a consequence of the Künneth formula from the "classical" Morse theory. Consider a Morse-Smale pair (f, ξ_X) on X and a Morse-Smale pair (g, ξ_Y) on Y. Then $F = f \oplus g : X \times Y \to \mathbb{R}$ is a Morse function and $\xi = (\xi_X \oplus \xi_Y)$ is an adapted pseudo-gradient. The identification between the moduli spaces of Morse trajectories

$$\mathcal{L}_F((x, y), (x', y')) \simeq \mathcal{L}_f(x, x') \times \mathcal{L}_g(y, y')$$

is only true if either the moduli spaces are empty, $x = x'$ or $y = y'$. In "classical" Morse theory, this is all we have to consider since the differential only counts trajectories between critical points of consecutive indexes and, if $|x| + |y| = |x'| + |y'| + 1$, then one of the conditions above is true. However, in DG Morse theory, we have to consider all pairs of critical points in order to construct the twisting cocycle $m_{(x,y),(x',y')}$ and this identification no longer holds.

Thom isomorphism in DG Morse theory.
Since the product CS_{DG} is meant to generalize the Chas-Sullivan product, we constructed it so that the diagram

$$\begin{array}{ccccccc} H_i(X, \mathcal{F})^{\otimes 2} & \xrightarrow{K^{top}} & H_{i+j}(X^2, \mathcal{F}^2) & \xrightarrow{\Delta_!} & H_{i+j-n}(X, \mathcal{F}^2) & \xrightarrow{\tilde{m}} & H_{i+j-n}(X, \mathcal{F}) \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ H_i(\mathcal{L}X)^{\otimes 2} & \xrightarrow{\mathrm{EZ}} & H_{i+j}(\mathcal{L}X^2) & \xrightarrow{\tau_*} & \tilde{H}_{i+j}\left((\mathcal{L}X \ {}_{ev}\times_{ev} \mathcal{L}X)^{ev^*TX}\right) & \xrightarrow{u_*} & H_{i+j-n}(\mathcal{L}X \ {}_{ev}\times_{ev} \mathcal{L}X) \xrightarrow{m_*} H_{i+j-n}(\mathcal{L}X) \end{array}$$

commutes, where we denoted $\mathcal{F} = C_*(\Omega X)$ and $\mathcal{F}^2 = C_*(\Omega X^2)$, the vertical arrows are defined by the Fibration Theorem and $\mathrm{EZ} : H_i(\mathcal{L}X)^{\otimes 2} \to H_{i+j}(\mathcal{L}X^2)$ is the Eilenberg-Zilber map.
It is a simple proof (Lemma 6.20) that the first square commutes and Theorem B will conclude that, since the concatenation $m : \mathcal{L}X \ {}_{ev}\times_{ev} \mathcal{L}X \to \mathcal{L}X$ is a morphism of fibrations, the last square also commutes. In order to complete this correspondence, we prove a DG version of the Pontryagin-Thom construction stating that $\Delta_!$ corresponds to the intersection product :
Theorem D (3.2) : *Let $Y^{n-k} \overset{\varphi}{\hookrightarrow} Z^n$ be an embedding of closed manifolds with tubular neighborhood U. Let $F \hookrightarrow E \overset{\pi}{\to} Z$ be a fibration and $F \hookrightarrow E_Y \to Y$ be the pullback fibration by φ. Define a Gysin map*

$\varphi_! : H_*(E) \to H_{*-k}(E_Y)$ as the composition of a Pontryagin-Thom collapse map $\tau_* : H_*(E) \to H_*(E_U, E_{\partial U})$ on $E_U = \pi^{-1}(U)$ with the Thom isomorphism $u_* : H_*(E_U, E_{\partial U}) \to H_{*-k}(E_Y)$.
The following diagram commutes

$$\begin{array}{ccc} H_*(Z, C_*(F)) & \xrightarrow{\varphi_!} & H_{*-k}(Y, \varphi^* C_*(F)) \\ \Psi_E \downarrow \sim & & \sim \downarrow \Psi_{E_Y} \\ H_*(E) & \xrightarrow[\varphi_!]{} & H_{*-k}(E_Y). \end{array}$$

To the best of the author's knowledge, the setting in which the Gysin map $\varphi_! : H_*(E_1) \to H_{*-k}(E_2)$ is classically considered is when $E_2 \hookrightarrow E_1$ are Hilbert manifolds with E_2 of finite codimension in E_1 so that there always exists a tubular neighborhood V of E_2 in E_1, unique up to isotopy, in which to consider the Thom isomorphism (see [CO15, Section 3] and [Lan95]). In a similar fashion as [HW23] did for the particular case where the fibration is $\mathcal{L}Y \times \mathcal{L}Y \to Y^2 = Z$ and $\varphi : Y \to Z$ is the diagonal map, we here avoid infinite dimensional arguments by using the fibration $E \xrightarrow{\pi} Z$ to pull back a tubular neighborhood U of Y in Z which is finite dimensional and prove that there exists a Thom isomorphism $H_*(E_U, E_{\partial U}) \to H_{*-k}(E_Y)$ in this case, where $E_U = \pi^{-1}(U)$.

Path-product.
Theorem B, Theorem C and Theorem D have their own interest independently of Theorem A and may be used to describe other operations involving the idea of combining operations on the base and on the fiber. In this spirit, we also give a Morse-theoretic description of the path product on $H_*(\mathcal{P}_{X\to X}Y)$, the homology of the space of paths in Y starting and ending in X, of degree $-n$ where $(X^n, \star)$ is a pointed, oriented, closed and connected manifold included in a topological space Y. This space and this product have been studied in [Ste23] when Y is a closed manifold. The construction also relies on "intersecting" paths to make them concatenable and then concatenating.
Consider the fibration $\Omega Y \hookrightarrow \mathcal{P}_{X\to X}Y \xrightarrow{(ev_0, ev_1)} X^2$ on the space of paths in Y that start and end on X. We state here a theorem in preparation that will be proved in a future paper :
Theorem E (in preparation) : *The concatenation $m : \mathcal{P}_{X\to X}Y \,{}_{ev_1}\!\times_{ev_0} \mathcal{P}_{X\to X}Y \to \mathcal{P}_{X\to X}Y$ is a morphism of fibrations and defines a product*

$$\mathrm{PP}_{DG} : H_*(X^2, C_*(\Omega Y))^{\otimes 2}) \to H_*(X^2, C_*(\Omega Y))$$

of degree $-n$ that consists in intersecting endpoints of a chain of such paths with the startpoints of another chain, concatenating them and then forgetting the concatenation point. This can be written as the composition

$$H_*(X^2, C_*(\Omega Y))^{\otimes 2} \xrightarrow{K} H_*(X^4, C_*(\Omega Y^2)) \xrightarrow{D_!} H_{*-n}(X^3, D^* C_*(\Omega Y^2)) \xrightarrow{\tilde{m}} H_{*-n}(X^3, p^* C_*(\Omega Y)) \xrightarrow{p_*} H_{*-n}(X^2, C_*(\Omega Y))$$

with the Dold sign. Here $D : X^3 \to X^4$, $D(a,b,c) = (a,b,b,c)$ and $p : X^3 \to X^2$, $p(a,b,c) = (a,c)$,
This product is associative up to the sign $(-1)^n$ and admits a neutral element.

Structure of this paper : Since we use quite often in this paper tools and techniques that have been developed and introduced in [BDHO24], we present in Section 2 a non-exhaustive (and sometimes heuristic) summary of definitions and constructions that we will need in the rest of the paper.
We prove Theorem D in Section 3.
In Section 4, we define the Morse complex with coefficients in an $\mathcal{A}_\infty$-module over $C_*(\Omega X)$ and prove that the homology of this complex is independant of the choices made to define it. We then study in Section 5 the maps induced by an $\mathcal{A}_\infty$-morphism of modules between the coefficients and prove Theorem B.
We prove Theorem C in Section 6. We then use the results of the previous sections to define CS_{DG} and prove Theorem A in Section 7.
We then discuss in Section 8 further directions in which we could expand this paper.

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2 Recollections on Morse homology with DG coefficients

Throughout this paper, X will denote an oriented, closed and connected manifold. Let $\star \in X$ be a preferred point in X and (f, ξ) be a Morse-Smale pair on X. This section is a brief and non-exhaustive summary of [BDHO24] aimed at presenting the main tools used in the present paper.

2.1 Morse homology with DG coefficients

For any topological space V, we can consider its **cubical complex** $C_*(V)$, the $\mathbb{Z}$-module spanned by continuous maps $\sigma : [0,1]^k \to V$ endowed with the differential

$$\partial\sigma = \sum_{i=1}^{k} (-1)^{i-1} (\sigma|_{x_i=1} - \sigma|_{x_i=0}),$$

modded out by the $\mathbb{Z}$-module spanned by the maps $\sigma : [0,1]^k \to V$ that factor through a face of the cube $[0,1]^k$. We call these chains **degenerate**.
Consider $\Omega X = \{\gamma : [0,a] \to X \text{ continuous}, \ a \in [0,+\infty), \ \gamma(0) = \gamma(a)\}$ the **space of Moore loops** based at $\star$. The concatenation is strictly associative on this space. Define the **Pontryagin product** on $C_*(\Omega X)$ by

$$\sigma \otimes \tau \mapsto (\sigma.\tau : (x,y) \mapsto \sigma(x) \# \tau(y))$$

for every continuous maps $\sigma : [0,1]^k \to \Omega X$ and $\tau : [0,1]^l \to \Omega X$, where $\#$ denotes the concatenation in ΩX. It extends to a product $\mu : C_i(\Omega X) \otimes C_j(\Omega X) \to C_{i+j}(\Omega X)$ which defines a DGA structure on $C_*(\Omega X)$.

Definition 2.1. *A **twisting cocycle** is a family $\{m_{x,y} \in C_{|x|-|y|-1}(\Omega X), \ x, y \in \mathrm{Crit}(f)\}$ that satisfies the Maurer-Cartan equation*

$$\partial m_{x,y} = \sum_{|x|>|z|>|y|} (-1)^{|x|-|z|} m_{x,z} m_{z,y}. \tag{2}$$

Definition 2.2. *We call **DG system** a right differential graded module $(\mathcal{F}_*, \partial_{\mathcal{F}})$ on $C_*(\Omega X)$.*

Definition 2.3. *Let $\mathcal{F}$ be a DG system and let $(m_{x,y})$ be a twisting cocycle. The Morse complex of X with coefficients in $\mathcal{F}$ is defined by*

$$C_*(X, m_{x,y}, \mathcal{F}) := \mathcal{F}_* \otimes \mathbb{Z}\mathrm{Crit}(f).$$

with the differential

$$\partial(\alpha \otimes x) = \partial\alpha \otimes x + (-1)^{|\alpha|} \left(\sum_{|y|<|x|} \alpha \cdot m_{x,y} \right) \otimes y.$$

*The resulting homology depends on the choice of twisting cocycle. We denote by $H_*(X, m_{x,y}, \mathcal{F})$ the homology of this complex. We will refer to a complex with coefficient in a DG system as an **enriched Morse complex** and its homology as **enriched Morse homology**.*

In the next section, we describe how the authors define a particular twisting cocycle called **Barraud-Cornea twisting cocycle** as the construction follows the seminal paper [BC07]. It carries homological information on the moduli space of Morse trajectories $\overline{\mathcal{L}}(x,y)$. This construction depend on choices but the resulting homology will not depend on them. We therefore denote $H_*(X, \mathcal{F})$ the homology of an enriched Morse complex $C_*(X, m_{x,y}, \mathcal{F})$ constructed with a Barraud-Cornea twisting cocycle.

2.2 Twisting cocycles defined by DG Morse data

Choose a Morse-Smale pair (f,ξ) on X and o, an orientation of the unstable manifolds $W^u(x)$ for each $x \in \mathrm{Crit}(f)$.

Definition 2.4. *For any pair of critical points $x, y \in \mathrm{Crit}(f)$, define the **moduli space of broken Morse trajectories***

$$\overline{\mathcal{L}}(x,y) := \mathcal{L}(x,y) \cup \bigcup_{\substack{k\geq 1 \\ z_1,\dots,z_k \in \mathrm{Crit}(f)}} \mathcal{L}(x,z_1) \times \mathcal{L}(z_1,z_2) \times \cdots \times \mathcal{L}(z_k,y).$$

We refer to [AD14, Section 3.2] to prove that with its natural topology, $\overline{\mathcal{L}}(x,y)$ is an orientable compact manifold with boundary and corners of dimension $|x|-|y|-1$ whose interior is $\mathcal{L}(x,y)$.
Intuitively, the twisting cocycle $m_{x,y} \in C_{|x|-|y|-1}(\Omega X)$ is constructed by evaluating in the based loop space ΩX a well-suited representative of the fundamental class of $\overline{\mathcal{L}}(x,y)$ for any $x, y \in \mathrm{Crit}(f)$.

Definition 2.5. *A **representing chain system** of the moduli spaces of trajectories is a collection $\{s_{x,y} \in C_{|x|-|y|-1}(\overline{\mathcal{L}}(x,y)),\ x,y \in \mathrm{Crit}(f)\}$ such that*

1. *each $s_{x,y}$ is a cycle relative to the boundary and represents the fundamental class $[\overline{\mathcal{L}}(x,y)]$.*

2. *each $s_{x,y}$ satisfies*

$$\partial s_{x,y} = \sum_z (-1)^{|x|-|z|} s_{x,z} \times s_{z,y}. \tag{3}$$

Such a representing chain system always exists and is inductively constructed. It is nonetheless not unique (although all representing chain system are homologous in a sense defined in [BDHO24, Proposition 5.2.8]).

Definition 2.6. *We say that a family of maps $q_{x,y} : \overline{\mathcal{L}}(x,y) \to \Omega X$ for $x, y \in \mathrm{Crit}(f)$ is a family of **evaluation maps** if it satisfies*

1. *<u>The concatenation relation</u> : For any $(\lambda, \lambda') \in \overline{\mathcal{L}}(x,z) \times \overline{\mathcal{L}}(z,y) \subset \partial\overline{\mathcal{L}}(x,y)$,*

$$q_{x,y}(\lambda,\lambda') = q_{x,z}(\lambda) \# q_{z,y}(\lambda'). \tag{4}$$

2. *<u>Compatibility with lifting</u> : Choose for each critical points x a preferred lift $\tilde{x}$ in $\tilde{X}$, the universal cover of X. For any $|x| = |y|+1$ and $\lambda \in \overline{\mathcal{L}}(x,y)$, denote $g = [q_{x,y}(\lambda)] \in \pi_1(X)$ the homotopy class of the loop $q_{x,y}(\lambda)$. We say that $q_{x,y}$ is compatible with lifting if the lift $\tilde{\lambda}$ of λ starting from $\tilde{x}$ ends in $g\tilde{y}$.*

Here is the construction of a family of evaluation maps given in [BDHO24, Lemma 5.9] that we will use as a blueprint in the rest of the paper when it will come to construct families of evaluation maps:

First, choose

1. A tree $\mathcal{Y}$ in X whose vertices are the critical points of f and whose root is the basepoint $\star$.

2. $\theta : X/\mathcal{Y} \to X$, a homotopy inverse of the canonical projection $p : X \to X/\mathcal{Y}$ such that $\theta([\mathcal{Y}]) = \star$.

Define a parametrization map $\Gamma_{x,y}$ parametrizing a trajectory λ into a path. Here, we parametrize maps using the values of the Morse function f.
The evaluation maps $q_{x,y} : \overline{\mathcal{L}}(x,y) \to \Omega X$ are then defined by $q_{x,y} = \theta \circ p \circ \Gamma_{x,y}$. Here, modding out by the tree $\mathcal{Y}$ turns the path $\Gamma_{x,y}(\lambda)$ into a loop. The twisting cocycle is defined by $m_{x,y} = q_{x,y,*}(s_{x,y}) \in C_{|x|-|y|-1}(\Omega X)$.

To construct the Barraud-Cornea cocycle, we needed six choices :

- A Morse function $f : X \to \mathbb{R}$.
- A pseudo-gradient ξ adapted to f.
- An orientation o of the unstable manifolds of (f, ξ).
- A representing chain system $\{s_{x,y}\}$
- A tree $\mathcal{Y}$
- A homotopy inverse θ of the projection $p : X \to X/\mathcal{Y}$.

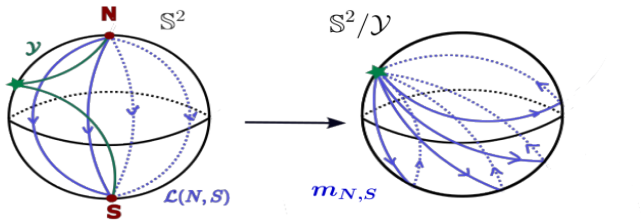


Figure 1 – Example of a Barraud-Cornea twisting cocycle on $\mathbb{S}^2$.

Definition 2.7. *Denote $\Xi = (f, \xi, \mathcal{Y}, o, s_{x,y}, \theta)$ the choices that have been made in order to construct such a twisting cocycle. We will refer to such a set Ξ as **DG Morse data** on $(X, \star_X)$. We will omit to mention the basepoint $\star_X$ if it is not necessary.*

Given a set of DG Morse data Ξ, we will denote $C_*(X, \Xi, \mathcal{F}) := C_*(X, m_{x,y}, \mathcal{F}) = \mathcal{F}_* \otimes \mathbb{Z}\mathrm{Crit}(f)$ the enriched Morse complex with the Barraud-Cornea cocycle constructed with the data Ξ. The homology of this complex does not depend on the set of data Ξ. More precisely :

Theorem 2.8. *[BDHO24, Theorem 6.3.1]*
1) Given two sets Ξ_0 and Ξ_1 of DG Morse data on X, the continuation map $\Psi^{\Xi} : C_*(X, \Xi_0, \mathcal{F}) \to C_*(X, \Xi_1, \mathcal{F})$ is a homotopy equivalence and its chain homotopy type does only depend on Ξ_0 and Ξ_1. The map Ψ^{Ξ} is in particular a quasi-isomorphism.

2) Given another set of data Ξ_2 on X and denoting Ψ_{ij} the continuation map between the data Ξ_i and Ξ_j, then Ψ_{00} is homotopic to the identity and Ψ_{02} is homotopic to $\Psi_{12} \circ \Psi_{01}$. In particular, in homology

$$\Psi_{00} = \mathrm{Id} \text{ and } \Psi_{12} \circ \Psi_{01} = \Psi_{02}.$$

2.3 Fibration Theorem

In this section, we will introduce one of the main result of [BDHO24], the Fibration Theorem, that will connect our construction on the enriched Morse complex of the base X of a fibration with previous constructions that use the singular complex of the total space E of this fibration.

2.3.1 Lifting functions

The notion of lifting function is a tool that we will use all along this paper to study fibrations and enriched Morse complexes.

Notation 2.9.
- *For any topological space A and $x, y \in A$, we will denote*

$$\mathcal{P}_{x \to y} A := \left\{ \gamma \in C^0([0, a], A),\ \gamma(0) = x,\ \gamma(a) = y \right\}$$

the space of Moore paths that start in x and end in y. We will respectively denote ev_0 and ev_1 the evaluation at the startpoint and the endpoint of a Moore path.

- *If $B, C \subset A$, $b \in B$ and $c \in C$ we use the notations*

$$\mathcal{P}_{B\to C}A := \bigcup_{b\in B, c\in C} \mathcal{P}_{b\to c}A, \quad \mathcal{P}_{b\to C}A := \mathcal{P}_{\{b\}\to C}A \text{ and } \mathcal{P}_{B\to c}A := \mathcal{P}_{B\to\{c\}}A.$$

A **Hurewicz fibration** $E \xrightarrow{\pi} X$ is a continuous and surjective map that has the **homotopy lifting property** with respect to all spaces : For any space B and any maps $H : B \times [0,1] \to X$ and $f : B \to E$ such that $\pi \circ f = H(\cdot, 0)$, there exists a lift $\tilde{H}$

$$\begin{array}{ccc} B & \xrightarrow{f} & E \\ \downarrow & \overset{\tilde{H}}{\nearrow} & \downarrow \pi \\ B\times[0,1] & \xrightarrow{H} & X. \end{array}$$

This property is equivalent to the existence of a map called **lifting function** $\phi : E \,{}_{\pi}\!\times_{ev_0} \mathcal{P}X \to \mathcal{P}E$ such that $\pi \circ \phi = pr_2$ and $ev_0 \circ \phi = pr_1$.
A **transitive lifting function** is a map

$$\Phi : E \,{}_{\pi}\!\times_{ev_0} \mathcal{P}X \to E$$

such that $\pi \circ \Phi = ev_1 \circ pr_2$, such that for any $e \in E$, $\Phi(e, \pi(e)) = e$ where $\pi(e)$ is the constant path in $\pi(e) \in X$ and such that for every $\gamma, \delta \in \mathcal{P}X$ such that $ev_1\gamma = ev_0\delta$,

$$\Phi(\Phi(e,\gamma),\delta) = \Phi(e, \gamma\#\delta).$$

It is proven in [DK69, Proposition 5.5] that every fibration is fiber homotopy equivalent to a fibration that admits a transitive lifting function and we will therefore assume that any fibration is equipped with a transitive lifting function.

Denote $F = \pi^{-1}(\star)$ **the fiber** of the fibration. A transitive lifting function induces a right $C_*(\Omega X)$-module structure on $C_*(F)$ given by $\alpha \cdot \sigma = \Phi_*(\alpha \otimes \sigma)$.

2.3.2 Statement

One of the main results of [BDHO24] is the following theorem

Theorem 2.10. *([BDHO24, Theorem 7.2.1]) Let $F \hookrightarrow E \to X$ a (Hurewicz) fibration, $\Phi : E \,{}_{\pi}\!\times_{ev_0} \mathcal{P}X \to E$ a transitive lifting function and Ξ a set of DG Morse data on X. Then there exists a quasi-isomorphism*

$$\Psi_E : C_*(X, \Xi, C_*(F)) \to C_*(E),$$

where $C_*(F)$ is endowed with the $C_*(\Omega X)$-module structure induced by Φ.

Notation 2.11. *For any fibration $F \hookrightarrow E \to X$ over a pointed oriented, closed and connected manifold $(X, \star)$, unless otherwise specified, we will denote $\Psi_E : H_*(X, C_*(F)) \to H_*(E)$ the isomorphism given by the Fibration Theorem.*

Remark 2.12. *([BDHO24, Remark 7.3.5]) This theorem has a version for manifolds with boundary. Let $\partial X = \partial_+ X \cup \partial_- X$, where $\partial_+ X$ is the subset of ∂X where the pseudo-gradient points outwards and $\partial_- X$ the subset where it points inwards. Then, if we denote $E_+ = \pi^{-1}(\partial_+ X)$, there exists a quasi-isomorphism*

$$C_*(X, \partial_+ X, C_*(F)) \to C_*(E, E_+).$$

2.3.3 Tools for the proof

We will need in some proofs of this paper to understand how this quasi-isomorphism is constructed. Therefore, let us lay out the tools necessary for the proof of this theorem.
We first consider $F' \hookrightarrow E' \to X/\mathcal{Y}$ the pullback fibration by $\theta : X/\mathcal{Y} \to X$, the chosen homotopy inverse of the projection $p : X \to X/\mathcal{Y}$. The family of evaluation maps $\left\{q'_{x,y} = p \circ \Gamma_{x,y} : \overline{\mathcal{L}}(x,y) \to \Omega(X/\mathcal{Y})\right\}$ define a twisting cocycle $m'_{x,y} \in C_*(\Omega(X/\mathcal{Y}))$ such that $\theta_*(m'_{x,y}) = m_{x,y}$ and the $C_*(\Omega(X/\mathcal{Y}))$-module structure on $C_*(F')$ is given by $\alpha \cdot \sigma := \alpha \cdot \theta_*(\sigma)$.
The map Ψ_E is then defined as a composition of three maps

$$C_*(X, C_*(F)) \to C_*(X, C_*(F')) \overset{\Psi_{E'}}{\to} C_*(E') \to C_*(E).$$

The only real consideration is on the middle map since the first one is just an identification $\alpha \cdot m'_{x,y} = \alpha \cdot \theta_*(m'_{x,y}) = \alpha \cdot m_{x,y} \in C_*(F') = C_*(F)$ and the last one is a homotopy equivalence given by θ.
The map $\Psi_{E'} : C_*(X, C_*(F')) \to C_*(E')$ is essentially constructed by using the transitive lifting function to lift the cellular decomposition of X given by its unstable manifolds. Let us be more precise.

Definition 2.13. *Define for every $x \in \mathrm{Crit}(f)$, its **Latour cell***

$$\overline{W^u}(x) = W^u(x) \cup \bigcup_{|y|<|x|} \overline{\mathcal{L}}(x,y) \times W^u(y).$$

It has been proven in [Lat94, Proposition 2.11], in [Qin10, Theorem 3] and [AD14, section 4.9.c] that with its natural topology, $\overline{W^u}(x)$ is a compact manifold with boundary and corners of dimension $|x|$ whose interior is $W^u(x)$ and which is homeomorphic to the closed disk $\overline{D}^{|x|}$. Latour cells induce a cellular decomposition of X.

Definition 2.14. *We define the morphism $\Psi_{E'}$ in several steps :*

- *There exists a family of evaluation maps $q_x : \overline{W^u}(x) \to \mathcal{P}_{\star \to X/\mathcal{Y}} X/\mathcal{Y}$ constructed by parametrizing the gradient lines and projecting them into $X/\mathcal{Y}$ (see [BDHO24, Lemma 7.3.3]).*
- *Let $(s_{x,y})$ be a representing chain system for the Morse moduli spaces in B. A **compatible representing chain system of the Latour cells** in B as defined in [BDHO24] is a collection $\{s_x \in C_{|x|}(\overline{W^u}(x)) \mid x \in \mathrm{Crit}(f)\}$ of chains such that :*
  1. *each s_x is a cycle relative to the boundary and represent the fundamental class $[\overline{W^u}(x), \partial\overline{W^u}(x)]$.*
  2. *each s_x satisfies $\partial s_x = \sum_y s_{x,y} \times s_y$ with the product of chains defined via the inclusions $\overline{\mathcal{L}}(x,y) \times \overline{W^u}(y) \subset \partial\overline{W^u}(x) \subset \overline{W^u}(x)$.*

Such a system always exists and is constructed inductively.
- *Define $m_x = q_{x,*}(s_x) \in C_{|x|}(\mathcal{P}_{\star \to B/\mathcal{Y}} B/\mathcal{Y})$ for each $x \in \mathrm{Crit}(f)$. This family satisfies*

$$\partial m_x = \sum_y m'_{x,y} m_y.$$

- *The map $\Psi_{E'} : C_*(X, C_*(F')) \to C_*(E')$ is then defined by $\Psi_{E'}(\alpha \otimes x) = \Phi_*(\alpha \otimes m_x)$.*

To conclude the proof, one remarks that Ψ induces a morphism of spectral sequences between the spectral sequence defined earlier that converges towards $H_*(X, C_*(F'))$ and the Leray-Serre spectral sequence that converges towards $H_*(E')$, and proves that this is an isomorphism between the second pages.

2.4 Direct and shriek maps

Let $\varphi : (X^n, \star_X) \to (Y^m, \star_Y)$ be a continuous map between pointed, oriented, closed and connected manifolds. We will assume $\varphi(\star_X) = \star_Y$.

Let $\mathcal{F}$ be a DG right-module over $C_*(\Omega Y)$ and $\varphi^*\mathcal{F}$ be its pullback by φ, *i.e* $\mathcal{F}$ endowed with the DG right-module over $C_*(\Omega X)$,

$$\forall \alpha \in \mathcal{F},\ \forall \gamma \in C_*(\Omega X),\ \alpha \cdot \gamma := \alpha \cdot \underbrace{\varphi_*(\gamma)}_{\in C_*(\Omega Y)}$$

Theorem 2.15. *[BDHO24, Theorem 8.1.1] There exist two maps respectively called* ***direct map*** *and* ***shriek map****,*

$$\varphi_* : H_*(X, \varphi^*\mathcal{F}) \to H_*(Y, \mathcal{F}) \text{ and } \varphi_! : H_*(Y, \mathcal{F}) \to H_{*+n-m}(X, \varphi^*\mathcal{F})$$

that satisfy the following properties :

1. ***Identity:*** $\mathrm{Id}_* = \mathrm{Id}_! = \mathrm{Id}$ *in homology.*
2. ***Composition:*** *Let $\psi : Z^k \to X$ be a continuous map and $\mathcal{F}$ be a DG right-module over $C_*(\Omega Y)$. Then*
$$(\varphi\psi)_* = \varphi_*\psi_* : H_*(Z, \varphi^*\psi^*\mathcal{F}) \to H_*(X, \varphi^*\mathcal{F}) \to H_*(Y, \mathcal{F})$$
and
$$(\varphi\psi)_! = \psi_!\varphi_! : H_*(Y, \mathcal{F}) \to H_{*+n-m}(X, \varphi^*\mathcal{F}) \to H_{*+k-m}(Z, \varphi^*\psi^*\mathcal{F}).$$
3. ***Homotopy:*** *Two homotopic maps induce the same direct and shriek maps in homology.*
4. ***Spectral sequence :*** *The morphisms φ_* and $\varphi_!$ in homology are limits of morphisms between spectral sequences associated to the corresponding enriched complexes, given at the second page by*
$$\varphi_{p,*} : H_p(X, \varphi^* H_q(\mathcal{F})) \to H_p(Y, H_q(\mathcal{F})) \text{ and } \varphi_{p,!} : H_p(Y, H_q(\mathcal{F})) \to H_{p+n-m}(X, \varphi^* H_q(\mathcal{F})).$$

There are two equivalent definitions for these maps (see [BDHO24, Section 9 and 10]).

3 Thom isomorphism in DG Morse theory. Proof of Theorem D

Let $(Z^n, \star)$ be a pointed, oriented, closed, connected manifold and $F \hookrightarrow E \xrightarrow{\pi} Z$ a fibration. Let $\varphi : Y^{n-k} \hookrightarrow Z^n$ be an embedding of a closed manifold and $i_U : U \to Z$ be an embedding of the total space of the normal bundle $p : U \to Y$ of φ. Denote $[\tau] \in H^k(U, \partial U)$ the Thom class of this disk bundle and choose $\tau \in C^k(U, \partial U)$ a representative. We will see φ as an inclusion since the diffeomorphism $Y \to \varphi(Y)$ will not play a role in this section and we will write Y instead of $\varphi(Y)$ in the arguments that follow. We will therefore see U as a tubular neighborhood of Y in Z and denote $E_Y = \pi^{-1}(Y)$, $E_U = \pi^{-1}(U)$ and $E_U^+ = \pi^{-1}(\partial U)$.
Let (f_U, ξ_U) be a Morse-Smale pair on U constructed by taking a Morse-Smale pair (f_Y, ξ_Y) on Y and extending it on U by $f_U = \chi \cdot f_Y \circ p$ where χ is some cutoff function equals 1 near Y and 0 near ∂U.
Extend it again to a Morse-Smale pair (f, ξ) on Z such that $Crit_j(f) \cap U = Crit_{j-k}(f_Y)$. This is the Morse identification $C_*^{\mathrm{Morse}}(U, \partial U) = C_{*-k}^{\mathrm{Morse}}(Y)$.
The next lemma is a fibered version of the classical Thom isomorphism.

Lemma 3.1. *The cap product with the pullback $\pi^*[\tau] \in H^k(E_U, E_U^+)$ defines an isomorphism*

$$u_* : H_*(E_U, E_U^+) \to H_{*-k}(E_Y).$$

Proof. There exists a projection $p_\pi : E_U \to E_Y$ such that $\pi \circ p_\pi = p \circ \pi$ defined in the following way :
Let $v \in E_U$ and $w = \pi(v) \in U$. Let $\gamma : [0,1] \to p^{-1}(p(w))$ such that $\gamma(0) = w$ and $\gamma(1) = p(w) \in Y$ be the path given by the gradient line passing through w. Let $\tilde{\gamma} : [0,1] \to E_U$ be the lift of γ starting on v. Define $p_\pi(v) = \tilde{\gamma}(1) \in E_Y$.
The fact that $u_* = p_{\pi,*}(\cdot \cap \pi^*\tau)$ is an isomorphism will be a direct consequence of Proposition 3.4. ∎

One can construct a Gysin map $\varphi_! : H_*(E) \to H_{*-k}(E_Y)$ given by the composition of a Pontryagin-Thom collapse map $\tau_{\varphi,E,*} : H_*(E) \to H_*(E, E - E_U) \simeq H_*(E_U, E_U^+)$ with the Thom isomorphism u_* (see [GS08]). We now prove that the shriek map $\varphi_! : H_*(Z, C_*(F)) \to H_{*-k}(Y, \varphi^* C_*(F))$ corresponds via the Fibration Theorem to the Gysin map $\varphi_! : H_*(E) \to H_{*-k}(E_Y)$.

Theorem 3.2. *Let $Y^{n-k} \overset{\varphi}{\hookrightarrow} Z^n$ be an embedding of manifolds, $F \hookrightarrow E \overset{\pi}{\to} Z$ be a fibration endowed with a transitive lifting function $\Phi : E \,{}_{\pi}\!\times_{ev_0} \mathcal{P}Z \to E$ and $F \hookrightarrow E_Y \overset{\pi}{\to} Y$ be the pullback fibration by φ. The shriek map $\varphi_! : H_*(Z, C_*(F)) \to H_{*-k}(Y, \varphi^* C_*(F))$ is compatible with the Gysin map $\varphi_! : H_*(E) \to H_{*-k}(E_Y)$ via the Fibration Theorem [BDHO24, Theorem 7.2.1]. In other words, the following diagram commutes*

$$\begin{array}{ccc} H_*(Z, C_*(F)) & \xrightarrow{\varphi_!} & H_{*-k}(Y, \varphi^* C_*(F)) \\ \Big\downarrow{\scriptstyle \Psi_E}\,\sim & & \sim\,\Big\downarrow{\scriptstyle \Psi_{E_Y}} \\ H_*(E) & \xrightarrow[\varphi_!]{} & H_{*-k}(E_Y). \end{array}$$

Proof. Denote $\mathcal{F} = C_*(F)$. Recall $i_U : U \to Z$ is an embedding of the normal bundle $p : U \to Y$ of φ and that we see $\varphi : Y \hookrightarrow Z$ as an inclusion.
We use the definition of the shriek map in the case of an embedding given in [BDHO24, Section 9] :

$$i_{U,!} : H_*(Z, \mathcal{F}) \to H_*(U, \partial U, i_U^* \mathcal{F}) = H_{*-k}(Y, \varphi^* \mathcal{F}),\ i_{U,!}(\alpha \otimes x) = \begin{cases} \alpha \otimes x & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}.$$

We prove the theorem by showing that $i_{U,!} : H_*(Z, \mathcal{F}) \to H_*(U, \partial U, i_U^* \mathcal{F})$ corresponds to the Pontryagin-Thom collapse $\tau_{\varphi,E,*} : H_*(E) \to H_*(E_U, E_U^+)$ and that the equality $H_*(U, \partial U, i_U^* \mathcal{F}) = H_{*-k}(Y, \varphi^* \mathcal{F})$ corresponds to the Thom isomorphism via the Fibration Theorem.

Proposition 3.3. *The following diagram commutes :*

$$\begin{array}{ccc} H_*(Z, \mathcal{F}) & \xrightarrow{\Psi_E} & H_*(E) \\ \Big\downarrow{\scriptstyle i_{U,!}} & & \Big\downarrow{\scriptstyle \tau_{\varphi,E,*}} \\ H_*(U, \partial U, i_U^* \mathcal{F}) & \xrightarrow{\Psi_{E_U}} & H_*(E_U, E_U^+). \end{array}$$

Proof. Consider $F_{\mathcal{Y}} \hookrightarrow E_{\mathcal{Y}} \overset{\pi_{\mathcal{Y}}}{\to} Z/\mathcal{Y}$ the pullback fibration of $F \hookrightarrow E \to Z$ by $\theta : Z/\mathcal{Y} \to Z$. Denote $E_{U_{\mathcal{Y}}} = \pi_{\mathcal{Y}}^{-1}(U_{\mathcal{Y}})$ where $U_{\mathcal{Y}} := U/\mathcal{Y}$, $E_{U_{\mathcal{Y}}}^+ = \pi_{\mathcal{Y}}^{-1}(\partial U_{\mathcal{Y}})$ and $\mathcal{F}_{\mathcal{Y}} = C_*(F_{\mathcal{Y}})$.
By the definition of the quasi-isomorphisms Ψ_E and Ψ_{E_U} (see Section 2.3 or [BDHO24, Section 7] for more details), the diagram is decomposable into

$$\begin{array}{ccccccc} H_*(Z, \mathcal{F}) & \xrightarrow{\sim} & H_*(Z, \mathcal{F}_{\mathcal{Y}}) & \xrightarrow{\Psi_{E_{\mathcal{Y}}}} & H_*(E_{\mathcal{Y}}) & \xrightarrow{\sim} & H_*(E) \\ \Big\downarrow{\scriptstyle i_{U,!}} & & \Big\downarrow{\scriptstyle i_{U,!}} & & \Big\downarrow{\scriptstyle \tau_{\varphi,E_{\mathcal{Y}},*}} & & \Big\downarrow{\scriptstyle \tau_{\varphi,E,*}} \\ H_*(U, \partial U, i_U^* \mathcal{F}) & \xrightarrow{\sim} & H_*(U, \partial U, i_U^* \mathcal{F}_{\mathcal{Y}}) & \xrightarrow{\Psi_{E_{U,\mathcal{Y}}}} & H_*(E_{U_{\mathcal{Y}}}, E_{U_{\mathcal{Y}}}^+) & \xrightarrow{\sim} & H_*(E_U, E_U^+) \end{array}$$

The rightmost and leftmost squares commute because the left horizontal maps are identifications of DG modules and the commutativity of the square on the right is a property of pullback fibrations. See the proof of Proposition 9.8.1 of [BDHO24] for more details.

Let us now prove that the middle square commutes. Let $x \in \mathrm{Crit}(f)$ and $\alpha \in \mathcal{F}_{\mathcal{Y}}$. If $x \notin U$, then none of the gradient lines coming from x will cross U since the pseudo-gradient points outwards U. Therefore, $\Psi_{E_{\mathcal{Y}}}(\alpha \otimes x) \in C_*(E_{\mathcal{Y}} \setminus E_{U,\mathcal{Y}})$ and

$$\tau_{\varphi,E_{\mathcal{Y}},*} \circ \Psi_{E_{\mathcal{Y}}}(\alpha \otimes x) = 0.$$

Moreover, by definition of $i_{U,!}$,

$$\Psi_{E_{U,\mathcal{Y}}} \circ i_{U,!}(\alpha \otimes x) = 0.$$

If $x \in U$, we decompose the middle square into

$$\begin{array}{ccc} H_*(Z,\mathcal{F}_\mathcal{Y}) & \xrightarrow{\Psi_{E_\mathcal{Y}}} & H_*(E_\mathcal{Y}) \\ \downarrow{\scriptstyle p_{U,*}} & & \downarrow{\scriptstyle p_{E_U,*}} \\ H_*(Z,Z\setminus U,\mathcal{F}_\mathcal{Y}) & & H_*(E_\mathcal{Y},E_\mathcal{Y}\setminus E_{U_\mathcal{Y}}) \\ \| & & \sim\uparrow{\scriptstyle j_{U,*}} \\ H_*(U,\partial U,i_U^*\mathcal{F}_\mathcal{Y}) & \xrightarrow{\Psi_{E_{U_\mathcal{Y}}}} & H_*(E_{U_\mathcal{Y}},E_{U_\mathcal{Y}}^+) \end{array}$$

The map $j_{U,*} : C_*(E_{U_\mathcal{Y}}, E^+_{U_\mathcal{Y}}) \to C_*(E_\mathcal{Y}, E_\mathcal{Y}/E_{U_\mathcal{Y}})$ is the homotopy equivalence given by excision and $p_{E_U,*} : C_*(E_\mathcal{Y}) \to C_*(E_\mathcal{Y}, E_\mathcal{Y}/E_{U_\mathcal{Y}})$, $p_{U,*} : C_*(Z,\mathcal{F}_\mathcal{Y}) \to C_*(Z, Z\setminus U, \mathcal{F}_\mathcal{Y})$ are the canonical projections. Let $\{s_x \in C_{|x|}(\overline{W^u}(x))\}$ be a representing chain system for the Latour cells. We can define evaluation maps (see Definition 2.14) $q_x : \overline{W^u}(x) \to \mathcal{P}_{\star\to X/\mathcal{Y}}X/\mathcal{Y}$ into the space of based paths in $X/\mathcal{Y}$ or $q_x^U : \overline{W^u}(x) \to \mathcal{P}_{\star\to U_\mathcal{Y}}U_\mathcal{Y}$ into $U_\mathcal{Y}$. These two maps respectively evaluate s_x to obtain $m_x \in C_{|x|}(\mathcal{P}_{\star\to X/\mathcal{Y}}X/\mathcal{Y})$, or to obtain $m_x^U \in C_{|x|}(\mathcal{P}_{\star\to U_\mathcal{Y}}U_\mathcal{Y}, \mathcal{P}_{\star\to\partial U_\mathcal{Y}}U_\mathcal{Y})$.
It is then clear that, up to homotopy, for all $\alpha \in \mathcal{F}_\mathcal{Y}$,

$$j_{U,*}(\Psi_{E_{U_\mathcal{Y}}}(\alpha\otimes x)) = j_{U,*}(\Phi(\alpha\otimes m_x^U)) = p_{E_U,*}(\Phi(\alpha\otimes m_x)) = p_{E_U,*}(\Psi_{E_\mathcal{Y}}(\alpha\otimes x)) \in C_*(E_\mathcal{Y}, E_\mathcal{Y}\setminus E_{U_\mathcal{Y}}).$$

■

In order to finish the proof of Theorem 3.2, we prove the following proposition, which is a result that has been referred to in [BDHO24, Remark 9.2.5].

Proposition 3.4. *The Morse identification $H_*(U,\partial U, i_U^*\mathcal{F}) = H_{*-k}(Y, i^*\mathcal{F})$ corresponds to the Thom isomorphism in the sense that the diagram*

$$\begin{array}{ccc} & H_*(U,\partial U,i_U^*\mathcal{F}) = H_{*-k}(Y,i^*\mathcal{F}) & \\ {\scriptstyle\Psi_{E_U}}\swarrow & & \searrow{\scriptstyle\Psi_{E_Y}} \\ H_*(E_U,E_U^+) & \xrightarrow{u_*} & H_{*-k}(E_Y) \end{array}$$

commutes.

Proof. It is enough to consider the pullback fibration by θ and prove that the diagram

$$\begin{array}{ccc} & H_*(U,\partial U,i_U^*\mathcal{F}_\mathcal{Y}) = H_{*-k}(Y,i^*\mathcal{F}_\mathcal{Y}) & \\ {\scriptstyle\Psi_{E_{U,\mathcal{Y}}}}\swarrow & & \searrow{\scriptstyle\Psi_{E_{Y,\mathcal{Y}}}} \\ H_*(E_{U,\mathcal{Y}},E_{U,\mathcal{Y}}^+) & \xrightarrow{u_*} & H_{*-k}(E_{Y,\mathcal{Y}}). \end{array}$$

commutes.
Let $\alpha \in \mathcal{F}$ and $x \in \mathrm{Crit}(f_Y)$. Denote $|x|_U = |x| + k$ the index of x as a critical point of f_U. Let $\{s_{x,y} \in C_{|x|-|y|-1}(\overline{\mathcal{L}}_Y(x,y)),\ x,y \in \mathrm{Crit}(f_Y)\}$ be a representing chain system for the Morse moduli spaces in Y and $\left\{s_x^Y \in C_{|x|_Y}(\overline{W_Y^u}(x)),\ x \in \mathrm{Crit}(f_Y)\right\}$ a compatible representing chain system of the Latour cells in Y as defined in Definition 2.14. Recall that it is a family of chains that satisfies

1. each s_x^Y is a cycle relative to the boundary and represents the fundamental class $[\overline{W_Y^u}(x), \partial\overline{W_Y^u}(x)]$.
2. each s_x satisfies $\partial s_x = \sum_y s_{x,y}\times s_y$ with the product of chains defined via the inclusions $\overline{\mathcal{L}}_Y(x,y)\times \overline{W_Y^u}(y) \subset \partial\overline{W_Y^u}(x) \subset \overline{W_Y^u}(x)$.

We want to build a family $\left\{s_x^U \in C_{|x|_U}\left(\overline{W_U^u}(x), \partial U \cap \overline{W_U^u}(x)\right), x \in \mathrm{Crit}(f_U)\right\}$ such that

1. s_x^U is a cycle relative to the boundary and represents the fundamental class $[\overline{W_U^u}(x), \partial\overline{W_U^u}(x)]$.

2. $s_x^Y = s_x^U|_{[0,1]^{|x|}\times\{0\}} \in C_{|x|}(\overline{W_Y^u}(x))$ is a representing chain system of the unstable manifold of x seen as a critical point in Y.
3. $a = s_x^U|_{\{0\}\times[0,1]^k}$ is a representative of the fundamental class $[\mathbb{D}^k, \partial\mathbb{D}^k]$.

Once such a representing chain system is built, we can define the associated twisting cocycle m_x^Y and m_x^U by evaluating respectively s_x^Y and s_x^U in the space of based paths of $Y/\mathcal{Y}$ and $U/\mathcal{Y}$. Recall that the isomorphisms $\Psi_{E_{U,\mathcal{Y}}}$ and $\Psi_{E_{Y,\mathcal{Y}}}$ are then defined by

$$\Psi_{E_{U,\mathcal{Y}}}(\alpha \otimes x) = \Phi_*(\alpha \otimes m_x^U) \text{ and } \Psi_{E_{Y,\mathcal{Y}}}(\alpha \otimes x) = \Phi_*(\alpha \otimes m_x^Y).$$

We then can just evaluate at the chain level using a representative τ of the Thom class $[\tau] \in H^k(E_{U,\mathcal{Y}}, E^+_{U,\mathcal{Y}})$:

$$u_*\left(\Psi_{E_{U,\mathcal{Y}}}(\alpha \otimes x)\right) = u_*(\Phi_*(\alpha \otimes m_x^U)) = \tau(a)\Phi_*(\alpha \otimes m_x^Y) = \Phi_*(\alpha \otimes m_x^Y) = \Psi_{E_{Y,\mathcal{Y}}}(\alpha \otimes x).$$

It remains to construct such a representing chain system $\{s_x^U,\ x \in \mathrm{Crit}(f_U)\}$. We do it by induction on $|x|_Y$. Let $\{s_x^Y,\ x \in \mathrm{Crit}(f_Y)\}$ be a representing chain system of the Latour cells $\left\{\overline{W}_Y^u(x)\right\}$.
If $|x|_Y = 0$, then $\overline{W}_U^u(x) = \mathbb{D}^k$ and $\partial U \cap \overline{W}_U^u(x) = \partial\overline{W}_U^u(x) = \partial\mathbb{D}^k$. Define $s_x^U = a$ to be a representative of the fundamental class $[\mathbb{D}^k, \partial\mathbb{D}^k] = [\overline{W}_U^u(x), \partial\overline{W}_U^u(x)]$.
Suppose now that s_y^U has been constructed for every $|y|_Y \leq l$ for some $l \in \mathbb{N}$. Let $x \in Crit_{k+1}(f_Y)$. Define $s_x^U = s_x^Y \times a$ and remark that, since a is a cycle,

$$\partial s_x^U = \sum_y s_{x,y} s_y^U$$

and it is a representative of $[\partial\overline{W_U^u}(x)]$ by the induction hypothesis. Indeed,

$$\partial\overline{W_U^u}(x) = \bigcup_{|y|<|x|} \overline{\mathcal{L}}_U(x,y) \times \overline{W_U^u}(y)$$

and the product orientation is equal to the boundary orientation. Therefore, if $s_x'^U$ is a representative of $[\overline{W_U^u}(x)]$, there exists a chain $p_x \in C_{|x|}(\partial\overline{W_U^u}(x))$ such that $\partial(s_x^U - s_x'^U) = \partial p_x$.
It follows that $s_x'^U + p_x - s_x^U \in C_{|x|_U}(\overline{W_U^u}(x))$ is a cycle. However, since $\overline{W_U^u}(x)$ is a $|x|_U$-dimensional, closed and connected manifold with boundary and corners, we have $H_{|x|_U}(\overline{W_U^u}(x)) = 0$ and therefore there exists $b \in C_{|x|_U+1}(\overline{W_U^u}(x))$ such that

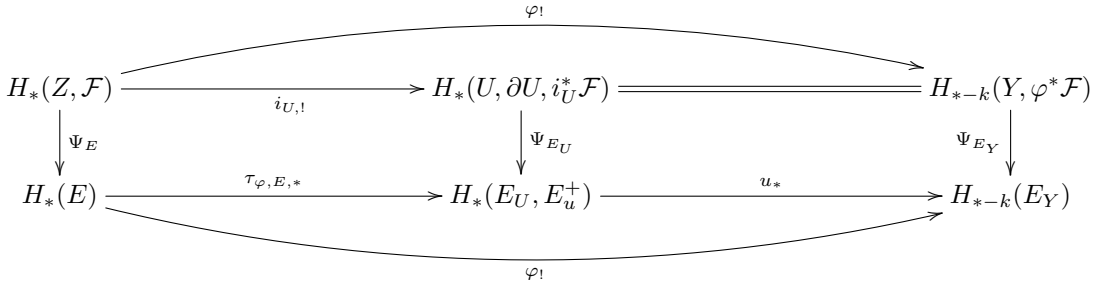
$$s_x'^U + p_x - s_x^U = \partial b$$

and therefore

$$s_x^U = s_x'^U = [\overline{W_U^u}(x), \partial\overline{W_U^u}(x)] \in H_{|x|_U}(\overline{W_U^u}(x), \partial\overline{W_U^u}(x)).$$

■

To summarize, we have proved that the diagram



commutes and therefore Theorem 3.2 is proved.

■

4 Morse homology with coefficients in an $\mathcal{A}_\infty$-module

In this section, we extend the construction of Morse homology with coefficients in a DG module to the more general case of a $\mathcal{A}_\infty$-module structure over $C_*(\Omega X)$. The main motivation to consider this setting is that a morphism of $\mathcal{A}_\infty$-modules $\boldsymbol{\varphi} : \mathcal{A} \to \mathcal{A}'$ induces a morphism of enriched complexes $\tilde{\varphi} : C_*(X, \mathcal{A}) \to C_*(X, \mathcal{A}')$ (Proposition 5.1). This property plays a key role in the definition of the Chas-Sullivan product CS_{DG}.

4.1 $\mathcal{A}_\infty$-modules over $C_*(\Omega X)$

The complex $C_*(\Omega X)$ has an associative algebra structure, which we view as an $\mathcal{A}_\infty$-algebra structure $(\mu_i)_{i\geq 0}$ given by the differential $\mu_1 : C_*(\Omega X) \to C_{*-1}(\Omega X)$, the Pontryagin product $\mu_2 : C_*(\Omega X)^{\otimes 2} \to C_*(\Omega X)$, and $\mu_i = 0$ for $i \geq 3$.

Definition 4.1. *A differential graded $\mathbb{Z}$-module $(\mathcal{A}, \boldsymbol{\nu^A} = \{\nu_n^A\})$ is an $\boldsymbol{\mathcal{A}_\infty}$**-module** over $C_*(\Omega X)$ if for all $n \geq 1$, the operations $\nu_n^A : \mathcal{A} \otimes C_*(\Omega X)^{\otimes n-1} \to \mathcal{A}$ have degrees $n-2$ and satisfy, for all $N \geq 1$,*

$$\sum_{\substack{s+t=N\\ s\geq 1}} (-1)^{st} \nu^A_{t+1}(\nu^A_s \otimes 1^{\otimes t}) + \sum_{\substack{r+s+t=N\\ r,s\geq 1}} (-1)^{r+st} \nu^A_{r+t+1}(1^{\otimes r} \otimes \mu_s \otimes 1^{\otimes t}) = 0.$$

Using the fact that $\mu_i = 0$ for $i \geq 3$, this leads to the equation

$$\sum_{\substack{s+t=n\\ s\geq 1}} (-1)^{st} \nu^A_{t+1}(\nu^A_s \otimes 1^{\otimes t}) + \sum_{\substack{r+t=n-1\\ r\geq 1}} (-1)^{n-1} \nu^A_n(1^{\otimes r} \otimes \mu_1 \otimes 1^{\otimes t}) + \sum_{\substack{r+t=n-2\\ r\geq 1}} (-1)^{r} \nu^A_{n-1}(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes t}) = 0. \quad (5)$$

Note that these functional expressions are subject to the Koszul sign rule when applied to elements: if $f : V_* \to W_*$ and $g : V'_* \to W'_*$ are homogeneous graded linear maps of respective degrees $|f|$ and $|g|$, then for any $x \otimes y \in V \otimes V'$, $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$.

Definition 4.2. *A **morphism** $\boldsymbol{\varphi} : (\mathcal{A}, \boldsymbol{\nu^A}) \to (\mathcal{B}, \boldsymbol{\nu^B})$ **of $\boldsymbol{\mathcal{A}_\infty}$-modules** over $C_*(\Omega X)$ is the data, for all $n \geq 1$, of a chain map $\varphi_n : \mathcal{A} \otimes C_*(\Omega X)^{\otimes n-1} \to \mathcal{B}$ of degree $n-1$ such that for all $N \in \mathbb{N}^*$,*

$$\sum_{r+s+t=N} (-1)^{r+st} \varphi_{r+t+1}(1^{\otimes r} \otimes \nu^A_s \otimes 1^{\otimes t}) = \sum_{s+t=N} (-1)^{(s+1)t} \nu^B_{t+1}(\varphi_s \otimes 1^{\otimes t}).$$

Using that $C_*(\Omega X)$ is an associative algebra, we get

$$\begin{aligned} \sum_{n+k=N} (-1)^{n(k+1)} \varphi_{n+1}(\nu^A_{k+1} \otimes 1^{\otimes n}) + \sum_{r=1}^{N} (-1)^N \varphi_{N+1}(1^{\otimes r} \otimes \mu_1 \otimes 1^{\otimes N-r}) \\ + \sum_{r=1}^{N-1} (-1)^r \varphi_N(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-1-r}) = \sum_{n+k=N} (-1)^{kn} \nu^B_{n+1}(\varphi_{k+1} \otimes 1^{\otimes n}). \end{aligned} \quad (6)$$

4.2 Complex with coefficients in an $\mathcal{A}_\infty$-module

Defining a complex with this larger kind of coefficients is not very different from the case of DG modules. However, in order to prove that there exist $\mathcal{A}_\infty$ analogues for the definitions and tools of [BDHO24], raw computations can be very complicated. That is why, in this paper, we deliberately use as much functional expressions as possible in order to avoid Kozsul signs that quickly make computations unreadable. To this effect, we use the following notation:

Notation 4.3. *Let $\{m_{x,y} \in C_{|x|-|y|-1}(\Omega X),\ x,y \in \mathrm{Crit}(f)\}$ be a twisting cocycle. We denote $\mathbf{m} \in \mathrm{Hom}_{-1}(\mathbb{Z}\mathrm{Crit}(f), C_*(\Omega X) \otimes \mathbb{Z}\mathrm{Crit}(f))$ the morphism given by*

$$\mathbf{m}(x) = \sum_y m_{x,y} \otimes y.$$

For a more compact writing, if $\mathcal{A}$ is an $\mathcal{A}_\infty$-module over $C_*(\Omega X)$, we extend $\mathbf{m}$ to $\tilde{\mathbf{m}} \in \mathrm{End}_{-1}(\mathcal{A} \otimes TC_*(\Omega X) \otimes \mathbb{Z}\mathrm{Crit}(f))$ defined by

$$\tilde{\mathbf{m}}(\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_k \otimes x) = (1^{\otimes k+1} \otimes \mathbf{m})(\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_k \otimes x),$$

where $TC_*(\Omega X) = \bigoplus_{i \geq 0} C_*(\Omega X)^{\otimes i}$ is the tensor algebra of $C_*(\Omega X)$.
We also extend μ_1 and μ_2 to $\mathcal{A} \otimes TC_*(\Omega X) \otimes \mathbb{Z}\mathrm{Crit}(f)$ by

$$\tilde{\mu}_1(\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_k \otimes x) = (1^{\otimes k} \otimes \mu_1 \otimes 1)(\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_k \otimes x),$$
$$\tilde{\mu}_2(\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_{k+1} \otimes x) = (1^{\otimes k} \otimes \mu_2 \otimes 1)(\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_{k+1} \otimes x).$$

Lemma 4.4. *The Maurer-Cartan equation can be written as*

$$\tilde{\mu}_1 \tilde{\mathbf{m}} + \tilde{\mu}_2 \tilde{\mathbf{m}}^2 = 0. \tag{7}$$

Proof. Using the Kozsul sign rule, we check

$$\begin{aligned}
\tilde{\mu}_1 \tilde{\mathbf{m}}(\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_k \otimes x) &= (-1)^{|\alpha|+|\gamma_1|+\cdots+|\gamma_k|} \sum_y \tilde{\mu}_1(\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_k \otimes m_{x,y} \otimes y) \\
&= \sum_{y,z} (-1)^{|x|-|z|} \alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_k \otimes \mu_2(m_{x,z} \otimes m_{z,y}) \otimes y \\
&= \sum_{y,z} (-1)^{|x|-|z|} \tilde{\mu}_2(\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_k \otimes m_{x,z} \otimes m_{z,y} \otimes y) \\
&= -\tilde{\mu_2} \tilde{\mathbf{m}}^2(\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_k \otimes x).
\end{aligned}$$

■

Using this equation, we prove a technical lemma that will be useful in several proofs :

Lemma 4.5. *For all $r \in \{1, \ldots, N\}$,*

$$(-1)^N (1^{\otimes r} \otimes \mu_1 \otimes 1^{\otimes N-r}) \tilde{\mathbf{m}}^N = (-1)^{r+1} (1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-r}) \tilde{\mathbf{m}}^{N+1}.$$

Proof.

$$\begin{aligned}
(-1)^N (1^{\otimes r} \otimes \mu_1 \otimes 1^{\otimes N-r}) \tilde{\mathbf{m}}^N &= (-1)^N (-1)^{N-r} \tilde{\mathbf{m}}^{N-r} \tilde{\mu_1} \tilde{\mathbf{m}}^r \\
&= -(-1)^r \tilde{\mathbf{m}}^{N-r} \tilde{\mu_2} \tilde{\mathbf{m}}^{r+1} \\
&= -(-1)^r (1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-r}) \tilde{\mathbf{m}}^{N+1}.
\end{aligned}$$

■

Definition 4.6. *Let $f : X \to \mathbb{R}$ be a Morse function on X, $\{m_{x,y} \in C_{|x|-|y|-1}(\Omega X),\ x,y \in \mathrm{Crit}(f)\}$ be a twisting cocycle and $(\mathcal{A}, \boldsymbol{\nu} = \{\nu_n\}_n)$ be a $\mathcal{A}_\infty$-module over $C_*(\Omega X)$. We define the complex*

$$C_*(X, m_{x,y}, \mathcal{A}) = C_*(X, \mathcal{A}) := \mathcal{A}_* \otimes \mathbb{Z}\mathrm{Crit}(f),$$

endowed with the differential

$$\partial = \sum_{n \geq 0} (\nu_{n+1} \otimes 1) \tilde{\mathbf{m}}^n.$$

Remark 4.7. *This sum is finite because $\tilde{\mathbf{m}}$ strictly decreases the index of the critical point. We can also remark that if $\mathcal{A}$ is a DG-module (*i.e, *$\nu_{n+1} = 0$ for all $n \geq 2$), then this differential corresponds to the known differential for the DG Morse complex.*

Proposition 4.8. *The application ∂ is a differential on $C_*(X, m_{x,y}, \mathcal{A})$.*

Proof. The application ∂ has degree -1 because $\tilde{\mathbf{m}}^n$ has degree $-n$ and ν_{n+1} has degree $n-1$.
It remains to show that $\partial^2 = 0$.

$$\begin{aligned}
\partial^2 &= \sum_{n,k} (\nu_{n+1} \otimes 1)\tilde{\mathbf{m}}^n(\nu_{k+1} \otimes 1)\tilde{\mathbf{m}}^k \\
&= \sum_{n,k} (-1)^{n(k-1)}(\nu_{n+1}\nu_{k+1} \otimes 1)\tilde{\mathbf{m}}^{n+k} \\
&= -\sum_N \left[\sum_{r=1}^{N} (-1)^N \nu_{N+1}(1^{\otimes r} \otimes \mu_1 \otimes 1^{\otimes N-r}) + \sum_{r=1}^{N-1} (-1)^r \nu_N (1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-1-r}) \right] \tilde{\mathbf{m}}^N.
\end{aligned}$$

We used equation (5) for the last equality. It then follows, using Lemma 4.5, that

$$\begin{aligned}
\partial^2 &= -\sum_N \left[\sum_{r=1}^{N} (-1)^N \nu_{N+1}(1^{\otimes r} \otimes \mu_1 \otimes 1^{\otimes N-r}) + \sum_{r=1}^{N-1} (-1)^r \nu_N (1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-1-r}) \right] \tilde{\mathbf{m}}^N \\
&= \sum_N \left[\sum_{r=1}^{N} (-1)^r \nu_{N+1}(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-r})\tilde{\mathbf{m}}^{N+1} - \sum_{r=1}^{N-1} (-1)^r \nu_N (1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-1-r})\tilde{\mathbf{m}}^N \right] \\
&= 0
\end{aligned}$$

■

We will also denote the associated homology groups by $H_*(X, m_{x,y}, \mathcal{A})$. If $\{m_{x,y}\}$ is a Barraud-Cornea cocycle, we will denote $H_*(X, \mathcal{A})$ the associated homology groups ; we will prove in Theorem 4.15 that this homology does not depend on the Morse data set Ξ used to define the $\{m_{x,y}\}$.

4.3 $\mathcal{A}_\infty$-Morse toolset

The following propositions are $\mathcal{A}_\infty$ analogues to [BDHO24] Proposition 2.3.3 and Proposition 2.3.4. The goal is to prove that a cocycle defined by a homotopy between two Morse-Smale pairs induces a chain map between enriched complexes defined using those pairs. If those cocycles are themselves homotopic (in a sense that will be defined in equation (10)), then it induces an homotopy between the chain maps. However, we will state these propositions in a more general setting, so they can be applied even in situations where the twisting cocycles considered are not Barraud-Cornea cocycles. Let $\mathcal{F}$ be an $\mathcal{A}_\infty$-module over $C_*(\Omega X)$.

Proposition 4.9. *Let $f_0, f_1 : X \to \mathbb{R}$ be two Morse functions on X. Let $\{m^0_{x,y} \in C_{|x|-|y|-1}(\Omega X),\ x, y \in \mathrm{Crit}(f_0)\}$ and $\{m^1_{x',y'} \in C_{|x'|-|y'|-1}(\Omega X),\ x', y' \in \mathrm{Crit}(f_1)\}$ be twisting cocycles.*
Let $\{\tau_{x,y'} \in C_{|x|-|y'|}(\Omega X),\ x \in \mathrm{Crit}(f_0),\ y' \in \mathrm{Crit}(f_1)\}$ be a cocycle satisfying the equation

$$\partial \tau_{x,y'} = \sum_{z \in \mathrm{Crit}(f_0)} m^0_{x,z} \cdot \tau_{z,y'} - \sum_{w' \in \mathrm{Crit}(f_1)} (-1)^{|x|-|w'|} \tau_{x,w'} \cdot m^1_{w',y'}. \tag{8}$$

Then, the map $\Psi : C_*(X, m^0_{x,y}, \mathcal{F}) \to C_*(X, m^1_{x,y}, \mathcal{F})$ defined by

$$\Psi = \sum_{n \geq 1} \sum_{u=1}^{n} (-1)^{u-1} (\nu_{n+1} \otimes 1)\tilde{\mathbf{m}}^{n-u}_{(1)} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}^{u-1}_{(0)}$$

is a morphism of complexes where $\tilde{\boldsymbol{\tau}} \in \mathrm{Hom}_0(\mathcal{F} \otimes TC_*(\Omega X) \otimes \mathbb{Z}\mathrm{Crit}(f_0), \mathcal{F} \otimes TC_*(\Omega X) \otimes \mathbb{Z}\mathrm{Crit}(f_1))$ is defined below.

Remark 4.10. *We used notations similar to those in the previous section : Define the morphism* $\boldsymbol{\tau} \in \mathrm{Hom}_0(\mathbb{Z}\mathrm{Crit}(f_0), C_*(\Omega X) \otimes \mathbb{Z}\mathrm{Crit}(f_1))$ *by*

$$\boldsymbol{\tau}(x) = \sum_{y'} \tau_{x,y'} \otimes y'$$

and $\tilde{\boldsymbol{\tau}} \in \mathrm{Hom}_0(\mathcal{F} \otimes TC_*(\Omega X) \otimes \mathbb{Z}\mathrm{Crit}(f_0), \mathcal{F} \otimes TC_*(\Omega X) \otimes \mathbb{Z}\mathrm{Crit}(f_1))$ *by*

$$\tilde{\boldsymbol{\tau}}(\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_k \otimes x) = (1^{\otimes(k+1)} \otimes \boldsymbol{\tau})(\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_k \otimes x).$$

We can then write equation (8) *as*

$$\tilde{\mu}_1 \tilde{\boldsymbol{\tau}} = \tilde{\mu}_2 \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}_{(0)} - \tilde{\mu}_2 \tilde{\mathbf{m}}_{(1)} \tilde{\boldsymbol{\tau}}. \tag{9}$$

Proof of Proposition 4.9. We compute

$$\begin{aligned}
\partial\Psi &= \sum_{\substack{n\geq 1\\ k\geq 0}} \sum_{u=1}^{n} (-1)^{u-1} (\nu_{k+1} \otimes 1) \tilde{\mathbf{m}}_{(1)}^{k} (\nu_{n+1} \otimes 1) \tilde{\mathbf{m}}_{(1)}^{n-u} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}_{(0)}^{u-1} \\
&= \sum_{\substack{n\geq 1\\ k\geq 0}} \sum_{u=1}^{n} (-1)^{u-1} (-1)^{(n+1)k} (\nu_{k+1}\nu_{n+1} \otimes 1) \tilde{\mathbf{m}}_{(1)}^{n+k-u} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}_{(0)}^{u-1}
\end{aligned}$$

and

$$\begin{aligned}
\Psi\partial &= \sum_{\substack{n\geq 0\\ k\geq 1}} \sum_{u=1}^{k} (-1)^{u-1} (\nu_{k+1} \otimes 1) \tilde{\mathbf{m}}_{(1)}^{k-u} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}_{(0)}^{u-1} (\nu_{n+1} \otimes 1) \tilde{\mathbf{m}}_{(0)}^{n} \\
&= \sum_{\substack{n\geq 0\\ k\geq 1}} \sum_{u=1}^{k} (-1)^{u-1} (-1)^{(k-1)(n+1)} (\nu_{k+1}\nu_{n+1} \otimes 1) \tilde{\mathbf{m}}_{(1)}^{k-u} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}_{(0)}^{n+u-1} \\
&= \sum_{\substack{n\geq 0\\ k\geq 1}} \sum_{u=n+1}^{n+k} (-1)^{u-1} (-1)^{(n+1)k+1} (\nu_{k+1}\nu_{n+1} \otimes 1) \tilde{\mathbf{m}}_{(1)}^{n+k-u} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}_{(0)}^{u-1}.
\end{aligned}$$

Therefore, by taking $N = n + k$,

$$\begin{aligned}
\Psi\partial - \partial\Psi = \sum_{N\geq 1} \sum_{u=1}^{N} (-1)^{u-1} \Bigg[&\sum_{r} (-1)^{N} \nu_{N+1} (1^{\otimes r} \otimes \mu_1 \otimes 1^{\otimes N-r}) \\
&+ \sum_{r} (-1)^{r} \nu_{N} (1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-1-r}) \Bigg] \tilde{\mathbf{m}}_{(1)}^{N-u} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}_{(0)}^{u-1}.
\end{aligned}$$

We decompose the computation of $(1^{\otimes r} \otimes \mu_1 \otimes 1^{\otimes N-r}) \tilde{\mathbf{m}}_{(1)}^{N-u} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}_{(0)}^{u-1}$ into three cases:

1. If $r \leq u - 1$, Lemma 4.5 gives

$$(1^{\otimes r} \otimes \mu_1 \otimes 1^{\otimes N-r}) \tilde{\mathbf{m}}_{(1)}^{N-u} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}_{(0)}^{u-1} = (-1)^{N-r} (1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-r}) \tilde{\mathbf{m}}_{(1)}^{N-u} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}_{(0)}^{u}.$$

2. If $r > u$, the same lemma gives

$$(1^{\otimes r} \otimes \mu_1 \otimes 1^{\otimes N-r})\tilde{\mathbf{m}}_{(1)}^{N-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u-1} = (-1)^{N-r+1}(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-r})\tilde{\mathbf{m}}_{(1)}^{N+1-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u-1}.$$

3. If $r = u$, we use equation (9) to obtain the equality:

$$\begin{aligned}
&(1^{\otimes u} \otimes \mu_1 \otimes 1^{\otimes N-u})\tilde{\mathbf{m}}_{(1)}^{N-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u-1} \\
&= (-1)^{N-u}\tilde{\mathbf{m}}_{(1)}^{N-u}(\tilde{\mu}_1\tilde{\boldsymbol{\tau}})\tilde{\mathbf{m}}_{(0)}^{u-1} \\
&= (-1)^{N-u}\tilde{\mathbf{m}}_{(1)}^{N-u}\left[\tilde{\mu}_2\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)} - \tilde{\mu}_2\tilde{\mathbf{m}}_{(1)}\tilde{\boldsymbol{\tau}}\right]\tilde{\mathbf{m}}_{(0)}^{u-1} \\
&= (-1)^{N-u}(1^{\otimes u} \otimes \mu_2 \otimes 1^{\otimes N-r})\left[\tilde{\mathbf{m}}_{(1)}^{N-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u} - \tilde{\mathbf{m}}_{(1)}^{N+1-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u-1}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_r (1^{\otimes r} \otimes \mu_1 \otimes 1^{\otimes N-r})\tilde{\mathbf{m}}_{(1)}^{N-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u-1} \\
&= \sum_{r=1}^{u}(-1)^{N-r}(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-r})\tilde{\mathbf{m}}_{(1)}^{N-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u} \\
&+ \sum_{r=u}^{N}(-1)^{N-r+1}(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-r})\tilde{\mathbf{m}}_{(1)}^{N+1-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u-1},
\end{aligned}$$

and

$$\begin{aligned}
\Psi\partial - \partial\Psi = \sum_N \sum_{u=1}^{N} \Bigg[&\sum_{r=1}^{u}(-1)^{u+1+r}\nu_{N+1}(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-r})\tilde{\mathbf{m}}_{(1)}^{N+1-(u+1)}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{(u+1)-1} \\
&+ \sum_{r=u}^{N}(-1)^{u+r}\nu_{N+1}(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-r})\tilde{\mathbf{m}}_{(1)}^{N+1-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u-1} \\
&- \sum_{r=1}^{N-1}(-1)^{u+r}\nu_N(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-1-r})\tilde{\mathbf{m}}_{(1)}^{N-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u-1}\Bigg] \\
= 0.&
\end{aligned}$$

Indeed, for every triple (N, u, r) with $N \geq 3$, $N \geq u$, and $N \geq r$, the term

$$(-1)^{u+r}\nu_N(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-r-1})\tilde{\mathbf{m}}_{(1)}^{N-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u-1}$$

appears only once with a positive sign (either in the first sum if $r \leq u-1$ or in the second sum) and always once negatively in the third sum.

■

The next proposition is an $\mathcal{A}_\infty$ analogue to Proposition 2.3.4 of [BDHO24]. It aims to provide homotopies between the maps defined by the previous proposition. We will state this proposition to the same degree of generality as the previous one.

Proposition 4.11. *Let $f_0, f_1 : X \to \mathbb{R}$ be Morse functions. Let $\{m^0_{x,y} \in C_{|x|-|y|-1}(\Omega X),\ x, y \in \mathrm{Crit}(f_0)\}$ and $\{m^1_{x',y'} \in C_{|x'|-|y'|-1}(\Omega X),\ x', y' \in \mathrm{Crit}(f_1)\}$ be twisting cocycles. Let $\{\tau_{x,y'} \in C_{|x|-|y'|}(\Omega X),\ x \in \mathrm{Crit}(f_0),\ y' \in \mathrm{Crit}(f_1)\}$ and $\{\tau'_{x,y'} \in C_{|x|-|y'|}(\Omega X),\ x \in \mathrm{Crit}(f_0),\ y' \in \mathrm{Crit}(f_1)\}$ be cocycles that satisfy (8), and let Ψ and Ψ' be the morphisms associated with $\{\tau_{x,y'}\}$ and $\{\tau'_{x,y'}\}$ respectively (see Proposition 4.9). Suppose that there exists a cocycle $\{h_{x,y'} \in C_{|x|-|y|+1}(\Omega X)\}$ such that*

$$\mu_1 h_{x,y'} = \tau_{x,y'} - \tau'_{x,y'} + \sum_{z \in \mathrm{Crit}(f_0)} (-1)^{|x|-|z|} m^0_{x,z} \cdot h_{z,y'} + \sum_{w' \in \mathrm{Crit}(f_1)} (-1)^{|x|-|w|} h_{x,w'} \cdot m^1_{w',y'}. \tag{10}$$

Then the map $H : C_*(X, m^0_{x,y}, \mathcal{F}) \to C_{*+1}(X, m^1_{x,y}, \mathcal{F})$ *defined by*

$$H = \sum_{n\geq 1}\sum_{u=1}^{n}(\nu_{n+1}\otimes 1)\tilde{\mathbf{m}}^{n-u}_{(1)}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}^{u-1}_{(0)}$$

is a chain homotopy between Ψ *and* Ψ'*.*

Remark 4.12. *We will refer to such a cocycle* $\{h_{x,y}\}$ *as* ***a homotopy cocycle****.*
As for any cocycle, we used the notations

$$\tilde{\boldsymbol{h}} \in \mathrm{Hom}_1(\mathcal{F}\otimes TC_*(\Omega X)\otimes \mathbb{Z}\mathrm{Crit}(f_0), \mathcal{F}\otimes TC_*(\Omega X)\otimes \mathbb{Z}\mathrm{Crit}(f_1))$$

for the morphism associated with a homotopy cocycle $\{h_{x,y}\}$*. We can rewrite equation* (10) *using this morphism :*

$$\tilde{\mu}_1\tilde{\boldsymbol{h}} = \tilde{\boldsymbol{\tau}} - \tilde{\boldsymbol{\tau}}' - \tilde{\mu}_2\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}_{(0)} - \tilde{\mu}_2\tilde{\mathbf{m}}_{(1)}\tilde{\boldsymbol{h}}. \tag{11}$$

Proof of Proposition 4.11. The proof is very similar to the proof of Proposition 4.9. We compute

$$\begin{aligned}\partial H &= \sum_{\substack{n\geq 1\\ k\geq 0}}\sum_{u=1}^{n}(\nu_{k+1}\otimes 1)\tilde{\mathbf{m}}^{k}_{(1)}(\nu_{n+1}\otimes 1)\tilde{\mathbf{m}}^{n-u}_{(1)}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}^{u-1}_{(0)}\\ &= \sum_{\substack{n\geq 1\\ k\geq 0}}\sum_{u=1}^{n}(-1)^{(n+1)k}(\nu_{k+1}\nu_{n+1}\otimes 1)\tilde{\mathbf{m}}^{n+k-u}_{(1)}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}^{u-1}_{(0)}\end{aligned}$$

and

$$\begin{aligned}H\partial &= \sum_{\substack{n\geq 0\\ k\geq 1}}\sum_{u=1}^{k}(\nu_{k+1}\otimes 1)\tilde{\mathbf{m}}^{k-u}_{(1)}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}^{u-1}_{(0)}(\nu_{n+1}\otimes 1)\tilde{\mathbf{m}}^{n}_{(0)}\\ &= \sum_{\substack{n\geq 0\\ k\geq 1}}\sum_{u=1}^{k}(-1)^{k(n+1)}(\nu_{k+1}\nu_{n+1}\otimes 1)\tilde{\mathbf{m}}^{k-u}_{(1)}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}^{n+u-1}_{(0)}\\ &= \sum_{\substack{n\geq 0\\ k\geq 1}}\sum_{u=n+1}^{n+k}(-1)^{(n+1)k}(\nu_{k+1}\nu_{n+1}\otimes 1)\tilde{\mathbf{m}}^{n+k-u}_{(1)}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}^{u-1}_{(0)}.\end{aligned}$$

Therefore, by taking $N = n + k$,

$$\begin{aligned}H\partial + \partial H = -\sum_{N\geq 1}\sum_{u=1}^{N}\Bigg[&\sum_{r}(-1)^{N}\nu_{N+1}(1^{\otimes r}\otimes\mu_1\otimes 1^{\otimes N-r})\\ &+\sum_{r}(-1)^{r}\nu_{N}(1^{\otimes r}\otimes\mu_2\otimes 1^{\otimes N-1-r})\Bigg]\tilde{\mathbf{m}}^{N-u}_{(1)}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}^{u-1}_{(0)}.\end{aligned}$$

We decompose the computation of $(1^{\otimes r}\otimes\mu_1\otimes 1^{\otimes N-r})\tilde{\mathbf{m}}^{N-u}_{(1)}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}^{u-1}_{(0)}$ into three cases:

1. If $r \leq u-1$, Lemma 4.5 gives

$$(1^{\otimes r}\otimes\mu_1\otimes 1^{\otimes N-r})\tilde{\mathbf{m}}^{N-u}_{(1)}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}^{u-1}_{(0)} = (-1)^{N-r+1}(1^{\otimes r}\otimes\mu_2\otimes 1^{\otimes N-r})\tilde{\mathbf{m}}^{N-u}_{(1)}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}^{u}_{(0)}.$$

2. If $r > u$, the same lemma gives

$$(1^{\otimes r} \otimes \mu_1 \otimes 1^{\otimes N-r})\tilde{\mathbf{m}}_{(1)}^{N-u}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}_{(0)}^{u-1} = (-1)^{N-r+1}(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-r})\tilde{\mathbf{m}}_{(1)}^{N+1-u}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}_{(0)}^{u-1}.$$

3. If $r = u$, we use equation (11) to obtain the equality:

$$\begin{aligned}
&(1^{\otimes u} \otimes \mu_1 \otimes 1^{\otimes N-u})\tilde{\mathbf{m}}_{(1)}^{N-u}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}_{(0)}^{u-1} \\
&= (-1)^{N-u}\tilde{\mathbf{m}}_{(1)}^{N-u}(\tilde{\mu}_1\tilde{\boldsymbol{h}})\tilde{\mathbf{m}}_{(0)}^{u-1} \\
&= (-1)^{N-u}\tilde{\mathbf{m}}_{(1)}^{N-u}[\tilde{\boldsymbol{\tau}} - \tilde{\boldsymbol{\tau}}' - \tilde{\mu}_2\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}_{(0)} - \tilde{\mu}_2\tilde{\mathbf{m}}_{(1)}\tilde{\boldsymbol{h}}]\tilde{\mathbf{m}}_{(0)}^{u-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_r (1^{\otimes r} \otimes \mu_1 \otimes 1^{\otimes N-r})\tilde{\mathbf{m}}_{(1)}^{N-u}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}_{(0)}^{u-1} \\
&= \sum_{r=1}^{u}(-1)^{N-r}(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-r+1})\tilde{\mathbf{m}}_{(1)}^{N-u}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}_{(0)}^{u} \\
&+ \sum_{r=u}^{N}(-1)^{N-r+1}(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-r})\tilde{\mathbf{m}}_{(1)}^{N+1-u}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}_{(0)}^{u-1} \\
&+ (-1)^{N-u+1}(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-r})\tilde{\mathbf{m}}_{(1)}^{N-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u-1} - (-1)^{N-u+1}(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-r})\tilde{\mathbf{m}}_{(1)}^{N-u}\tilde{\boldsymbol{\tau}}'\tilde{\mathbf{m}}_{(0)}^{u-1}
\end{aligned}$$

and

$$\begin{aligned}
H\partial + \partial H = \sum_N \sum_{u=1}^{N} \Bigg[&\sum_{r=1}^{u}(-1)^r\nu_{N+1}(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-r})\tilde{\mathbf{m}}_{(1)}^{N+1-(u+1)}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}_{(0)}^{(u+1)-1} \\
&+ \sum_{r=u}^{N}(-1)^r\nu_{N+1}(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-r})\tilde{\mathbf{m}}_{(1)}^{N+1-u}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}_{(0)}^{u-1} \\
&- \sum_{r=1}^{N-1}(-1)^r\nu_N(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-1-r})\tilde{\mathbf{m}}_{(1)}^{N-u}\tilde{\boldsymbol{h}}\tilde{\mathbf{m}}_{(0)}^{u-1}\Bigg] + \Psi - \Psi' \\
= \Psi - \Psi'.&
\end{aligned}$$

■

4.4 Filtration and spectral sequence

If $f : X \to \mathbb{R}$ is a Morse function on X, $\{m_{x,y} \in C_{|x|-|y|-1}(\Omega X),\ x, y \in \mathrm{Crit}(f)\}$ is a twisting cocycle and $\mathcal{F}$ is an $\mathcal{A}_\infty$-module over $C_*(\Omega X)$, then the associated enriched Morse complex has a natural filtration

$$F_p(C_*(X, m_{x,y}, \mathcal{F})) = \bigoplus_{\substack{i+j=k \\ i \leq p}} \mathcal{F}_j \otimes \mathbb{Z}\mathrm{Crit}(f).$$

The spectral sequence $E^r_{p,q}$ associated to this filtration converges to $H_{p+q}(X, \mathcal{F})$ and its first page is

$$E^1_{p,q} = H_q(\mathcal{F}) \otimes \mathbb{Z}\mathrm{Crit}_p(f).$$

The structure of $\mathcal{A}_\infty$ $\mathbb{Z}[\pi_1(X)]$-module on $H_q(\mathcal{F})$ is associative. Therefore, we can define a differential on this first page in the same way as in the case of a DG-module :

$$d^1(\hat{\alpha} \otimes x) = (-1)^q \sum_{|y|=|x|-1} \hat{\alpha} \cdot \hat{m}_{x,y} \otimes y.$$

The second page is given by

$$E^2_{p,q} = H_p(\tilde{C}_p(f); H_q(\mathcal{F})),$$

where $\tilde{C}_p(f) = \mathbb{Z}[\pi_1(X)] \otimes_{\mathbb{Z}} \mathbb{Z}\mathrm{Crit}_p(f)$ endowed with the differential

$$d^2(x) = \sum_{|y|=|x|-1} \hat{m}_{x,y} y.$$

4.5 Invariance and $\mathcal{A}_\infty$ continuation map

Let Ξ_0 and Ξ_1 be enriched Morse data on X and let $\mathcal{F}$ be a $\mathcal{A}_\infty$-module over $C_*(\Omega X)$. We prove that there exists a continuation map $\Psi_{01} : C_*(X, \Xi_0, \mathcal{F}) \to C_*(X, \Xi_1, \mathcal{F})$ which is a quasi-isomorphism. As a consequence, the homology of the enriched complex with coefficients in a $\mathcal{A}_\infty$-module does not depend on any of the choices that have been made to define it.

Following [BDHO24, Lemma 6.2.1], one builds a DG Morse set of **continuation data** Ξ on $X \times [0,1]$ by considering a representing chain system s_F on the moduli spaces of trajectories in $X \times [0,1]$ such that:

- $s_{F,x_0,y_0} = (-1)^{|x|-|y|} s_{(0),x,y}$ for all $x, y \in \mathrm{Crit}(f_0)$.
- $s_{F,x_1,y_1} = s_{(1),x,y}$ for all $x, y \in \mathrm{Crit}(f_1)$.
- $\sigma_{x,y} := s_{F,x_0,y_1}$ for all $x \in \mathrm{Crit}(f_0)$ and $y \in \mathrm{Crit}(f_1)$.

The family $\{\sigma_{x,y}\}$ can then be evaluated into ΩX to build a continuation cocycle $\{\tau_{x,y} \in C_{|x|-|y|}(\Omega X),\ x \in \mathrm{Crit}(f_0),\ y \in \mathrm{Crit}(f_1)\}$ associated to the continuation data Ξ. For $i \in \{0,1\}$, denote $m^i_{x,y}$ the Barraud-Cornea twisting cocycle defined by Ξ_i. The cocycle $\{\tau_{x,y}\}$ satisfies equation (8).

Definition 4.13. *We call* ***$\mathcal{A}_\infty$ continuation map*** *the map induced by $\{\tau_{x,y} \in C_{|x|-|y|}(\Omega X),\ x \in \mathrm{Crit}(f_0),\ y \in \mathrm{Crit}(f_1)\}$ by Proposition 4.9 :*

$$\Psi^\Xi : C_*(X, \Xi_0, \mathcal{F}) \to C_*(X, \Xi_1, \mathcal{F})$$

defined by

$$\Psi^\Xi = \sum_{n \geq 1} \sum_{u=1}^{n} (-1)^{u-1} (\nu_{n+1} \otimes 1) \tilde{\mathbf{m}}^{n-u}_{(1)} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}^{u-1}_{(0)}.$$

Proposition 4.14. *Let $\tilde{\Psi} : \tilde{C}(f_0, \xi_0) \to \tilde{C}(f_1, \xi_1)$ be the morphism between lifted Morse complexes defined by $\tilde{\Psi}(x) = \sum_{|x|=|y|} n_{x,y} y$, where $n_{x,y}$ is the projection of $\tau_{x,y}$ in $H_0(\Omega X) = \mathbb{Z}[\pi_1(X)]$.*
If $\tilde{\Psi}$ is a quasi-isomorphism, then so is Ψ.

Proof. As in [BDHO24, Proposition 4.4.1], the proof comes from the fact that Ψ preserves the canonical filtrations and therefore induces a morphism of spectral sequences $(\Psi^{(r)})_r$ such that $\Psi^{(1)} = \mathrm{Id} \otimes \tilde{\Psi}$. It then follows from the existence of a spectral sequence for the change of coefficients that, if $\tilde{\Psi}$ is a quasi-isomorphism, then so is $\Psi^{(1)}$ and therefore Ψ. ■

The $\mathcal{A}_\infty$ continuation map Ψ^Ξ is therefore a quasi-isomorphism since it is shown in [BDHO24, Proposition 6.2.2] that the cocycle $\{\tau_{x,y}\}$ satisfies the condition of Proposition 4.14. Hence, the homology $H_*(X, \mathcal{F})$ is well-defined if $\mathcal{F}$ is an $\mathcal{A}_\infty$-module over $C_*(\Omega X)$.

In fact more is true : there is an $\mathcal{A}_\infty$ analogue of the invariance Theorem 2.8.

Theorem 4.15. *Let $\mathcal{F}$ be an $\mathcal{A}_\infty$-module over $C_*(\Omega X)$.*
1) Given two sets Ξ_0 and Ξ_1 of enriched Morse data on X and continuation data Ξ on $X \times [0,1]$, the $\mathcal{A}_\infty$ continuation map $\Psi^\Xi : C_*(X, \Xi_0, \mathcal{F}) \to C_*(X, \Xi_1, \mathcal{F})$ is a homotopy equivalence and its chain homotopy type only depends on Ξ_0 and Ξ_1. The map Ψ^Ξ is in particular a quasi-isomorphism.

2) Given another set of data Ξ_2 on X and denoting Ψ_{ij} the $\mathcal{A}_\infty$ continuation map between the data Ξ_i and Ξ_j, then Ψ_{00} is homotopic to the identity and Ψ_{02} is homotopic to $\Psi_{12} \circ \Psi_{01}$. In particular, in homology

$$\Psi_{00} = \mathrm{Id} \text{ and } \Psi_{12} \circ \Psi_{01} = \Psi_{02}.$$

Theorem 2.8 is proven in [BDHO24, Section 6.2.2]. Using Proposition 4.11 to construct homotopies between continuation maps, their proof carries over in our setting.

5 Morphisms of fibrations and $\mathcal{A}_\infty$-morphism of modules. Proof of Theorem B

The main goal of this section is to prove that a morphism of fibrations $\varphi : E_1 \to E_2$ over X induces a morphism of complexes $\tilde{\varphi} : C_*(X, C_*(F_1)) \to C_*(X, C_*(F_2))$ that is compatible with the Fibration theorem, with direct and shriek maps. The main difficulty is that φ does not necessarily respect the transitive lifting functions associated to the fibrations $F_1 \hookrightarrow E_1 \to X$ and $F_2 \hookrightarrow E_2 \to X$ and therefore does not induce a DG morphism of modules over $C_*(\Omega X)$ from $C_*(F_1)$ to $C_*(F_2)$ in general. Hence, the map $\tilde{\varphi} : C_*(X, C_*(F_1)) \to C_*(X, C_*(F_2))$, $\tilde{\varphi}(\alpha \otimes x) = \varphi_*(\alpha) \otimes x$ is *a priori* not a morphism of complexes. We will prove that if there exists a morphism $\boldsymbol{\varphi} : \mathcal{F} \to \mathcal{G}$ of $\mathcal{A}_\infty$-modules over $C_*(\Omega X)$, then there exists a morphism of complexes $\tilde{\varphi} : C_*(X, \mathcal{F}) \to C_*(X, \mathcal{G})$ compatible with direct and shriek maps.
In the context of a fibration, we will prove Theorem B (5.8) that states that a morphism of fibrations $\varphi : E_1 \to E_2$ induces a morphism $\boldsymbol{\varphi} : C_*(F_1) \to C_*(F_2)$ of $\mathcal{A}_\infty$-modules over $C_*(\Omega X)$ and that the map $\tilde{\varphi} : C_*(X, \mathcal{F}) \to C_*(X, \mathcal{G})$ corresponds to the map $\varphi_* : C_*(E_1) \to C_*(E_2)$ via the Fibration Theorem. Therefore, in this context, $\tilde{\varphi}$ inherits the good functorial behavior of singular complexes in our finite dimensional model of enriched Morse complexes.

5.1 Functoriality with respect to the coefficients

Let Ξ be a set of DG Morse data on X with Morse function $f : \mathbb{R} \to X$.

Proposition 5.1. *Let $\boldsymbol{\varphi} : (\mathcal{A}, \nu_i^A) \to (\mathcal{B}, \nu_i^{\mathcal{B}})$ be a morphism of $\mathcal{A}_\infty$-modules over $C_*(\Omega X)$. Then, the map $\tilde{\varphi} : C_*(X, \Xi, \mathcal{A}) \to C_*(X, \Xi, \mathcal{B})$ defined by*

$$\tilde{\varphi} = \sum_{n \geq 0} (\varphi_{n+1} \otimes 1)\tilde{\mathbf{m}}^n$$

is a morphism of complexes.

Proof. The definition of $\tilde{\mathbf{m}} \in \mathrm{End}_{-1}(\mathcal{A} \otimes TC_*(\Omega X) \otimes \mathbb{Z}\mathrm{Crit}(f))$ and is given in 4.3.
We use equation (6) to compute

$$\begin{aligned}
\partial\tilde{\varphi} &= \sum_{n,k} (\nu_{n+1}^B \otimes 1)\tilde{\mathbf{m}}^n(\varphi_{k+1} \otimes 1)\tilde{\mathbf{m}}^k \\
&= \sum_{n,k} (-1)^{nk}(\nu_{n+1}^B(\varphi_{k+1} \otimes 1^{\otimes n}) \otimes 1)\tilde{\mathbf{m}}^{n+k} \\
&= \sum_{n,k} (-1)^{n(k+1)}(\varphi_{n+1}(\nu_{k+1}^A \otimes 1^{\otimes n}) \otimes 1)\tilde{\mathbf{m}}^{n+k} \\
&+ \sum_N \left[\sum_{r=1}^{N} (-1)^N \varphi_{N+1}(1^{\otimes r} \otimes \mu_1 \otimes 1^{\otimes N-r}) + \sum_{r=1}^{N-1} (-1)^r \varphi_N(1^{\otimes r} \otimes \mu_2 \otimes 1^{\otimes N-1-r}) \right] \tilde{\mathbf{m}}^N
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\varphi}\partial &= \sum_{n,k} (\varphi_{n+1} \otimes 1)\tilde{\mathbf{m}}^n(\nu_{k+1}^A \otimes 1)\tilde{\mathbf{m}}^k \\
&= \sum_{n,k} (-1)^{n(k+1)}(\varphi_{n+1}(\nu_{k+1}^A \otimes 1^{\otimes n}) \otimes 1)\tilde{\mathbf{m}}^{n+k}.
\end{aligned}$$

It then follows from Lemma 4.5 that $\partial\tilde{\varphi} - \tilde{\varphi}\partial = 0$, and $\tilde{\varphi}$ is indeed a morphism of complexes.

■

Notation 5.2. *For any $\boldsymbol{\varphi} : (\mathcal{A}, \nu_i^{\mathcal{A}}) \to (\mathcal{B}, \nu_i^{\mathcal{B}})$ morphism of $\mathcal{A}_\infty$-modules over $C_*(\Omega X)$, we will denote*

$$\tilde{\varphi} : C_*(X, \Xi, \mathcal{A}) \to C_*(X, \Xi, \mathcal{B})$$

the associated morphism of complexes.

5.2 Topological modules and morphisms of fibrations. Proof of Theorem B

Since many naturally occuring morphisms of $\mathcal{A}_\infty$-modules are of a topological nature, we define the notion of **$\mathcal{A}_\infty$-morphism of topological modules** over ΩX.
We will denote $\cdot$ the concatenation.

Definition 5.3. *Let (F, ν_F) and (G, ν_G) be two topological spaces endowed with strictly associative topological module structures over ΩX. An **$\mathcal{A}_\infty$-morphism of topological modules** (over ΩX) is the data of a sequence of maps*

$$\varphi_{n+1} : I^n \times F \times \Omega X^n \to G,\ n \geq 0$$

such that for all $t_1, \dots, t_n \in I^n$, $\alpha \in F$ and $\gamma_1, \dots, \gamma_n \in \Omega X$,

$$\varphi_{n+1}(t_1, \dots, t_n, \alpha, \gamma_1, \dots, \gamma_n) = \begin{cases} \varphi_n(t_2, \dots, t_n, \nu_F(\alpha, \gamma_1), \gamma_2, \dots, \gamma_n) & \text{if } t_1 = 1, \\ \varphi_n(\hat{t_j}, \alpha, \gamma_1, \dots, \gamma_{j-1} \cdot \gamma_j, \dots, \gamma_n) & \text{if } t_j = 1,\ j \geq 2, \\ \nu_G\left(\varphi_j(t_1, \dots, t_{j-1}, \alpha, \gamma_1, \dots, \gamma_{j-1}), \gamma_j \cdot \ \dots\ \cdot \gamma_n\right)) & \text{if } t_j = 0. \end{cases} \tag{12}$$

Remark 5.4. *The reader accustomed to $\mathcal{A}_\infty$-theory may expect the maps φ_{n+1} to be defined on the multiplihedra J_n introduced in [Sta70]. Since we will mainly work on fibrations, where we can always assume that there exists a transitive lifting function, we will only work on the case where F and G have strictly associative topological module structures on ΩX itself endowed with a strictly associative multiplication. The next proposition is a proof that cubes are enough to encode coherent homotopies for a map between two strictly associative (topological) modules over a strictly associative (topological) algebra.*

Proposition 5.5. *An $\mathcal{A}_\infty$-morphism of topological modules $\{\varphi_{n+1} : I^n \times F \times \Omega X^n \to G\}$ induces a morphism $\boldsymbol{\phi} : (C_*(F), \partial, \nu_{F,*}) \to (C_*(G), \partial, \nu_{G,*})$ of $\mathcal{A}_\infty$-modules over $C_*(\Omega X)$, where $\phi_{n+1} : C_*(F) \otimes C_*(\Omega X)^{\otimes n} \to C_*(G)$, the map of degree n induced by φ_{n+1}, is defined by*

$$\phi_{n+1}(\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_n) = \varphi_{n+1,*}(\mathrm{Id}_{I^n} \otimes \alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_n)$$

Proof. Let $\mu_* : C_*(\Omega X) \otimes C_*(\Omega X) \to C_*(\Omega X)$ denote the Pontryagin-product. We use equation (12) in order to compute $\partial\phi_{n+1}$ for all $n \in \mathbb{N}$:

$$\begin{aligned} &\partial\phi_{n+1}(\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_n) \\ &= \Bigg[\underbrace{\phi_n(\nu_{F,*} \otimes 1^{\otimes n-1})}_{t_1=1} + \sum_{r=1}^{n-1} (-1)^r \underbrace{\phi_n(1^{\otimes r} \otimes \mu_* \otimes 1^{\otimes n-1-r})}_{t_r=1,\ r\geq 2} - (-1)^{n-1} \underbrace{\nu_{G,*}(\phi_n \otimes 1)}_{t_n=0} + \underbrace{0}_{t_r=0,\ r<n} \\ &+ \sum_{r=0}^{n-1} (-1)^n \phi_{n+1}(1^{\otimes r} \otimes \mu_1 \otimes 1^{\otimes n-r}) \Bigg] (\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_n). \end{aligned}$$

Indeed, for any $\delta \in F$, $\tau_1, \dots, \tau_n \in \Omega X$, if there exists $r < n$ such that $t_r = 0$, then $\varphi_{n+1}(t_1, \dots, t_n, \delta, \tau_1, \dots, \tau_n)$ does not depend on t_{r+1}. Therefore, $\phi_{n+1}(\alpha \otimes \gamma_1 \otimes \cdots \otimes \gamma_n)|_{t_r=0}$ is a degenerate chain.
This shows that $\boldsymbol{\phi} : (C_*(F), \partial, \nu_{F,*}) \to (C_*(G), \partial, \nu_{G,*})$ satisfies equation (6) and finishes the proof.

■

Proposition 5.1 gives a direct Corollary.

Corollary 5.6. *If $\{\varphi_{n+1} : I^n \times F \times \Omega X^n \to G\}$ is an $\mathcal{A}_\infty$-morphism of topological modules, then it induces a morphism of complexes*

$$\tilde{\varphi} : C_*(X, C_*(F)) \to C_*(X, C_*(G)).$$

■

Assume that F and G are fibers of fibrations $F \hookrightarrow E_1 \to X$ and $G \hookrightarrow E_2 \to X$ endowed with the ΩX topological module structures induced by transitive lifting functions. For the morphism of complexes $\tilde{\varphi} : C_*(X, C_*(F)) \to C_*(X, C_*(G))$ to be compatible with the Fibration Theorem, we will require that $\{\varphi_{n+1} : I^n \times F \times \Omega X^n \to G\}$ is induced by a **morphism of fibrations** $\varphi_1 : E_1 \to E_2$.

Definition 5.7. *Let $F \hookrightarrow E_1 \overset{\pi_1}{\to} X$ and $F_2 \hookrightarrow E_2 \overset{\pi_2}{\to} X$ be fibrations over X. A **morphism of fibrations** (over X) is a continuous map $\varphi : E_1 \to E_2$ such that $\pi_2 \circ \varphi = \pi_1$.*

A morphism of fibrations does not in general preserve lifting functions. Indeed, consider $\Omega X \hookrightarrow \mathcal{L}X \overset{ev}{\to} X$ endowed with the transitive lifting function

$$\Phi : \mathcal{L}X \,{}_{ev}\times_{ev_0} \mathcal{P}X \to \mathcal{L}X, \quad \Phi(\alpha, \gamma) = \gamma^{-1}\alpha\gamma$$

and the fibration of the figure-eight space $\Omega X^2 \hookrightarrow \mathcal{L}X \,{}_{ev}\times_{ev} \mathcal{L}X \overset{ev}{\to} X$ endowed with the transitive lifting function

$$\Phi^2 : (\mathcal{L}X \,{}_{ev}\times_{ev} \mathcal{L}X) \,{}_{ev}\times_{ev_0} \mathcal{P}X, \quad \Phi^2((\alpha, \beta), \gamma) = (\gamma^{-1}\alpha\gamma, \gamma^{-1}\beta\gamma).$$

The concatenation $m : (\mathcal{L}X \,{}_{ev}\times_{ev} \mathcal{L}X) \to \mathcal{L}X$ does not preserve those transitive lifting functions :

$$m(\Phi((\alpha, \beta), \gamma)) = \gamma^{-1}\alpha\gamma\gamma^{-1}\beta\gamma \neq \gamma^{-1}\alpha\beta\gamma = \Phi(m(\alpha, \beta), \gamma).$$

This makes it difficult to define a morphism

$$m : \underbrace{H_*(X, C_*(\Omega X^2))}_{\simeq H_*(\mathcal{L}X \,{}_{ev}\times_{ev} \mathcal{L}X)} \to \underbrace{H_*(X, C_*(\Omega X))}_{\simeq H_*(\mathcal{L}X)}$$

at the chain level, since it would have to be compatible with the module structures, which are themselves defined by the respective transitive lifting functions.
However, Theorem B states that every morphism of fibrations yields a morphism between enriched Morse complexes that corresponds to the morphism between the singular complexes of the total spaces. We will now restate Theorem B and prove it.

Theorem 5.8. *Let $F_1 \hookrightarrow E_1 \to X$ and $F_2 \hookrightarrow E_2 \to X$ be two fibrations. Let $\varphi : E_1 \to E_2$ be a morphism of fibrations and Ξ be a set of DG Morse data on X.*

*i) There exists a sequence of maps $\left\{\varphi_{n+1} : I^n \times F_1 \times \Omega X^{n-1} \times \mathcal{P}_{\star \to X} X \to E_2\right\}$ called a **coherent homotopy** for φ that induces a morphism of complexes $\tilde{\varphi} : C_*(X, \Xi, C_*(F_1)) \to C_*(X, \Xi, C_*(F_2))$.*

ii) A coherent homotopy induces a chain homotopy $v : C_*(X, \Xi, C_*(F_1)) \to C_{*+1}(E_2)$ between $\Psi_2 \circ \tilde{\varphi}$ and $\varphi_* \circ \Psi_1$ for any set of DG Morse data Ξ on X where the morphisms Ψ_1 and Ψ_2 are the quasi-isomorphisms given by the Fibration Theorem. In other words, the following diagram commutes up to chain homotopy

$$\begin{array}{ccc} C_*(X, \Xi, C_*(F_1)) & \xrightarrow{\tilde{\varphi}} & C_*(X, \Xi, C_*(F_2)) \\ \downarrow{\scriptstyle \Psi_1} & & {\scriptstyle \Psi_2}\downarrow \\ C_*(E_1) & \xrightarrow[\varphi_*]{} & C_*(E_2). \end{array}$$

Proof of i). We will prove that even if *a priori*

$$\begin{array}{ccccc} F_1 \times \mathcal{P}_{\star\to X}X & \to & E_2 \\ (\alpha,\gamma) & \mapsto & \Phi_2(\varphi(\alpha),\gamma) \end{array} \neq \begin{array}{ccccc} F_1 \times \mathcal{P}_{\star\to X}X & \to & E_2 \\ (\alpha,\gamma) & \mapsto & \varphi(\Phi_1(\alpha,\gamma)) \end{array},$$

these maps are always homotopic and there exists a family of higher homotopies as defined in the following lemma.

Lemma 5.9. *Let $F_1 \hookrightarrow E_1 \overset{\pi_1}{\to} X$ and $F_2 \hookrightarrow E_2 \overset{\pi_2}{\to} X$ be two fibrations over X endowed with transitive lifting functions $\Phi_1 : E_1 \times \mathcal{P}_{\star\to X}X \to E_1$ and $\Phi_2 : E_2 \times \mathcal{P}_{\star\to X}X \to E_2$.*
For any morphism of fibrations $\varphi : E_1 \to E_2$, there exists a sequence of maps $\varphi_{n+1} : I^n \times F_1 \times \Omega X^{n-1} \times \mathcal{P}_{\star\to X}X \to E_2$ for $n \geq 1$ such that $\varphi = \varphi_1$, $\pi_2 \circ \varphi_{n+1}(t_1,\dots,t_n,\alpha,\gamma_1,\dots,\gamma_n) = ev_1(\gamma_n)$ and

$$\varphi_{n+1}(t_1,\dots,t_n,\alpha,\gamma_1,\dots,\gamma_n) = \begin{cases} \varphi_n(t_2,\dots,t_n,\Phi_1(\alpha,\gamma_1),\gamma_2,\dots,\gamma_n) & \text{if } t_1 = 1, \\ \varphi_n(\hat{t}_j,\alpha,\gamma_1,\dots,\gamma_{j-1}\cdot\gamma_j,\dots,\gamma_n) & \text{if } t_j = 1,\ j \geq 2, \\ \Phi_2\left(\varphi_j(t_1,\dots,t_{j-1},\alpha,\gamma_1,\dots,\gamma_{j-1}),\gamma_j\cdot\ \dots\ \cdot\gamma_n\right)\right) & \text{if } t_j = 0. \end{cases} \tag{13}$$

Definition 5.10. *We say that a family*

$$\left\{\varphi_{n+1} : I^n \times F_1 \times \Omega X^{n-1} \times \mathcal{P}_{\star\to X}X \to E_2\right\}$$

satisfying equation (13) *is a* ***coherent homotopy*** *for the morphism of fibrations $\varphi = \varphi_1$.*

Before proving this lemma, we will see how it concludes the proof of i).
If $\varphi : E_1 \to E_2$ is a morphism of fibrations, Lemma 5.9 build a coherent homotopy $\{\varphi_{n+1} : I^n \times F_1 \times \Omega X^{n-1} \times \mathcal{P}_{\star\to X}X \to E_2,\ n \geq 1\}$. It suffices to check that a coherent homotopy induces an $\mathcal{A}_\infty$-morphism of topological modules $\{\varphi_{n+1}^{F_1} : I^n \times F_1 \times \Omega X^{n-1} \times \Omega X \to F_2, n \geq 0\}$ where $\varphi_1^{F_1} = \varphi|_{F_1}\colon F_1 \to F_2$ and $\varphi_{n+1}^{F_1} = \varphi_{n+1}|_{I^n\times F_1\times\Omega X^n}$. Indeed, it would prove that $\varphi|_{F_1}\colon F_1 \to F_2$ induces a morphism

$$\tilde{\varphi} : C_*(X,\Xi,C_*(F_1)) \to C_*(X,\Xi,C_*(F_2))$$

by Corollary 5.6.
First remark that since $\pi_2 \circ \varphi = \pi_1$, $\varphi(F_1) \subset F_2$ and therefore $\varphi|_{F_1}\colon F_1 \to F_2$. Moreover, since for any $t_1,\dots,t_n,\alpha,\gamma_1,\dots,\gamma_{n-1},\tau \in I^n \times F_1 \times \Omega X^{n-1} \times \mathcal{P}_{\star\to X}X$, $\pi_2 \circ \varphi_{n+1}(t_1,\dots,t_n,\alpha,\gamma_1,\dots,\gamma_n) = ev_1(\gamma_n)$, if $\gamma_n \in \Omega X$, then $\varphi_{n+1}(t_1,\dots,t_n,\alpha,\gamma_1,\dots,\gamma_n) \in F_2$.
The module structure on $C_*(F_1)$ and $C_*(F_2)$ are defined by the lifting functions Φ_1 and Φ_2. Therefore since $\{\varphi_{n+1} : I^n \times F_1 \times \Omega X^{n-1} \times \mathcal{P}_{\star\to X}X \to E_2\}$ satisfies equation (13), the family $\{\varphi_{n+1}^{F_1} : I^n \times F_1 \times \Omega X^n \to F_2\}$ satisfies equation (12) and indeed defines an $\mathcal{A}_\infty$-morphism of topological modules over ΩX.

Proof of Lemma 5.9.
Before defining the whole coherent homotopy for the morphism of fibrations $\varphi : E_1 \to E_2$, let us define $\varphi_2 : I \times F_1 \times \mathcal{P}_{\star\to X}X \to E_2$. This map is a homotopy between $\varphi_2(0,\cdot) : (\alpha,\gamma) \mapsto \Phi_2(\varphi(\alpha),\gamma)$ and $\varphi_2(1,\cdot) : (\alpha,\gamma) \mapsto \varphi(\Phi_1(\alpha,\gamma))$.
Let $\alpha \in F_1$ and $\gamma : [0,s] \to X \in \mathcal{P}_{\star\to X}X$. This homotopy is defined by

$$\varphi_2(t,\alpha,\gamma) = \Phi_2(\varphi(\Phi_1(\alpha,\gamma|_{[0,ts]})),\gamma_{[ts,s]}).$$

It lifts a part of γ in E starting from α, sends $\Phi_1(\alpha,\gamma|_{[0,ts]})$ into E_2 by φ and lifts the rest of γ starting from $\varphi(\Phi_1(\alpha,\gamma|_{[0,ts]}))$.

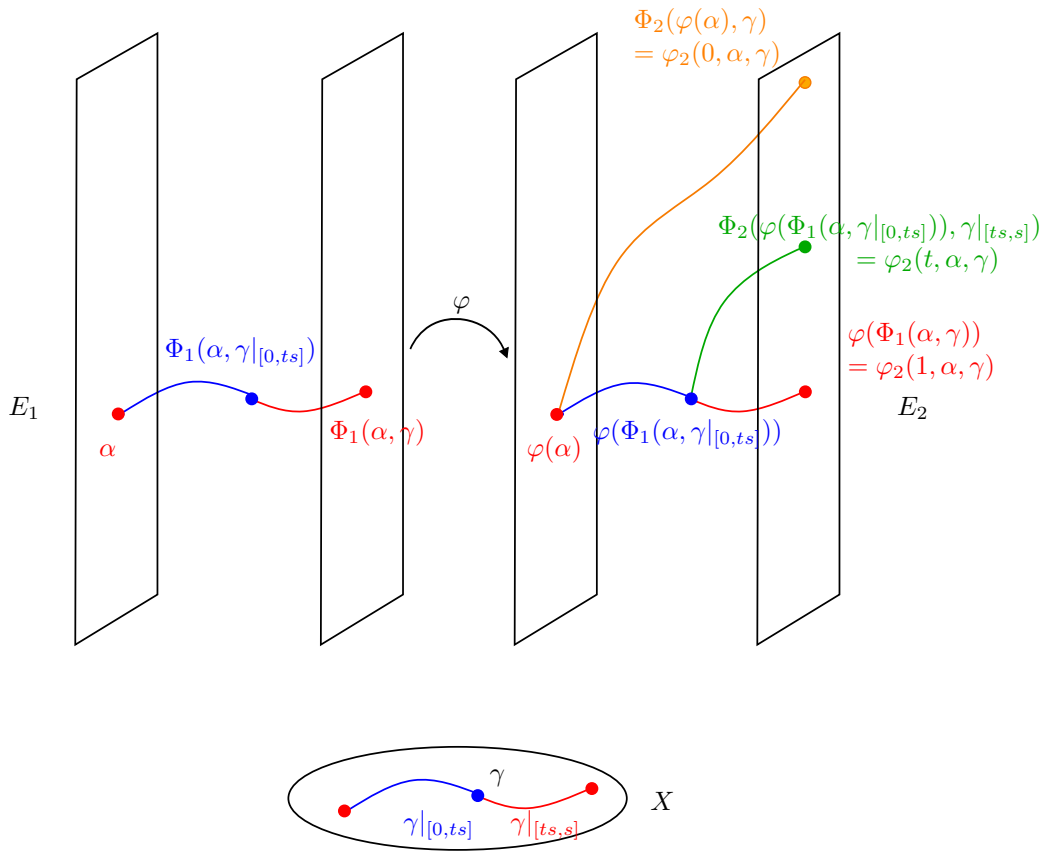


Figure 2 - Homotopy φ_2

We now define $\varphi_{n+1} : I^n \times F_1 \times \Omega X^{n-1} \times \mathcal{P}_{\star\to X}X \to E_2$ for all $n \geq 1$. Let us first fix some notations.
• Let $J_1 : I \times [0,+\infty) \to [0,+\infty)$

$$J_1(t,s) = ts.$$

For any $n \geq 2$, define $J_n : I^n \times [0,+\infty)^n \to [0,+\infty)$ by

$$\begin{aligned} J_n(t_1,\dots,t_n,s_1,\dots s_n) &= J_{n-1}(t_1,\dots,t_{n-1},s_1,\dots,s_{n-1}+t_ns_n) \\ &= t_1(s_1+t_2(s_2+t_3(s_3+\cdots+t_{n-1}(s_{n-1}+t_ns_n)\dots))). \end{aligned}$$

By convention, we set $J_0 = 0$.
For any $n \in \mathbb{N}$, denote

$$I_n^0(t_1,\dots,t_n,s_1,\dots s_n) = [0, J_n(t_1,\dots,t_n,s_1,\dots s_n)]$$

and

$$I_n^1(t_1,\dots,t_n,s_1,\dots s_n) = [J_n(t_1,\dots,t_n,s_1,\dots s_n), s_1+\cdots+s_n].$$

• In the following arguments, for any $\gamma_i \in \mathcal{P}_{\star\to X}X$ we will always denote $a_i \geq 0$ such that $\gamma_i : [0,a_i] \to X$.
Let $r_1 : I \times E_1 \times \mathcal{P}_{\star\to X}X \to E_2$,

$$r_1(t,\alpha,\gamma_1) = \varphi\left(\Phi_1(\alpha,\gamma_1|_{[0,ta_1]})\right).$$

For any $n \geq 2$, define $r_n : I^n \times F_1 \times \Omega X^{n-1} \times \mathcal{P}_{\star\to X}X \to E_2$ by

$$\begin{aligned} r_n(t_1,\dots,t_n,\alpha,\gamma_1,\dots,\gamma_n) &= r_{n-1}\left(t_1,\dots,t_{n-1},\alpha,\gamma_1,\dots,\gamma_{n-1}\cdot\left(\gamma_n|_{[0,t_na_n]}\right)\right) \\ &= \varphi\left(\Phi_1\left(\alpha,(\gamma_1\cdot\ \dots\ \cdot\gamma_n)|_{I_n^0(t_1,\dots,t_n,a_1,\dots,a_n)}\right)\right). \end{aligned}$$

By convention, we set $r_0 = \varphi : E_1 \to E_2$.
• Define the coherent homotopy in the following way :
Let $n \geq 1$. Define $\varphi_{n+1} : I^n \times F_1 \times \Omega X^{n-1} \times \mathcal{P}_{\star \to X} \to E_2$ by

$$\varphi_{n+1}(t_1, \dots, t_n, \alpha, \gamma_1, \dots, \gamma_n) = \Phi_2 \left(r_n(t_1, \dots, t_n, \alpha, \gamma_1, \dots, \gamma_n), (\gamma_1 \cdot \ \dots \ \cdot \gamma_n)|_{I_n^1(t_1, \dots, t_n, a_1, \dots, a_n)} \right).$$

We now prove that this satisfies the three equations of (13).
First, we remark that for any $n \geq 1$, $j \geq 2$, $k \geq 1$, $t_1, t_2, \dots, t_n \in I^n$ and $s_1, \dots, s_n \in [0, +\infty)^n$,

$$J_n(1, t_2, \dots, t_n, s_1, \dots, s_n) = s_1 + J_{n-1}(t_2, \dots, t_n, s_2, \dots, s_n), \tag{14}$$

$$J_n(t_j = 1, s_1, \dots, s_n) = J_{n-1}(\hat{t}_j, s_1, \dots, s_{j-1} + s_j, \dots, s_n), \tag{15}$$

$$J_n(t_k = 0, s_1, \dots, s_n) = J_{k-1}(t_1, \dots, t_{k-1}, s_1, \dots, s_{k-1}), \tag{16}$$

where $\hat{t}_j = (t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_n)$.

First equation : For any $n \geq 1$, $t_1, t_2, \dots, t_n \in I^n$, $\alpha \in F_1$ and $\gamma_1, \dots, \gamma_n \in \Omega X^n \times \mathcal{P}_{\star \to X} X$, using (14),

$$(\gamma_1 \cdot \ \dots \ \cdot \gamma_n)|_{I_n^0(1, t_2, \dots, t_n, a_1, \dots, a_n)} = \gamma_1 \cdot (\gamma_2 \cdot \ \dots \ \cdot \gamma_n)|_{I_{n-1}^0(t_2, \dots, t_n, a_2, \dots, a_n)}$$

and

$$(\gamma_1 \cdot \ \dots \ \cdot \gamma_n)|_{I_n^1(1, t_2, \dots, t_n, a_1, \dots, a_n)} = \begin{cases} (\gamma_2 \cdot \ \dots \ \cdot \gamma_n)|_{I_{n-1}^1(t_2, \dots, t_n, a_2, \dots, a_n)} & \text{if } n \geq 2 \\ ev_1(\gamma_1) & \text{if } n = 1. \end{cases}.$$

Therefore,

$$\begin{aligned} r_n(1, t_2, \dots, t_n, \alpha, \gamma_1, \dots, \gamma_n) &= \varphi \left(\Phi_1 \left(\alpha, \gamma_1 \cdot (\gamma_2 \cdot \ \dots \ \cdot \gamma_n)|_{I_{n-1}^0(t_2, \dots, t_n, a_2, \dots, a_n)} \right) \right) \\ &= \varphi \left(\Phi_1 \left(\Phi_1(\alpha, \gamma_1), (\gamma_2 \cdot \ \dots \ \cdot \gamma_n)|_{I_{n-1}^0(t_2, \dots, t_n, a_2, \dots, a_n)} \right) \right) \\ &= r_{n-1}(t_2, \dots, t_n, \alpha \cdot \gamma_1, \gamma_2, \dots, \gamma_n) \end{aligned}$$

and

$$\varphi_{n+1}(1, t_2, \dots, t_n, \alpha, \gamma_1, \dots, \gamma_n) = \varphi_n(t_2, \dots, t_n, \alpha \cdot \gamma_1, \gamma_2, \dots, \gamma_n).$$

Second equation : For any $n \geq 2$, $j \geq 2$, $t_1, t_2, \dots, t_n \in I^n$, $\alpha \in F_1$ and $\gamma_1, \dots, \gamma_n \in \Omega X^n \times \mathcal{P}_{\star \to X} X$, using (15),

$$(\gamma_1 \cdot \ \dots \ \cdot \gamma_n)|_{I_n^i(t_j = 1, a_1, \dots, a_n)} = (\gamma_1 \cdot \ \dots \ \cdot (\gamma_{j-1} \cdot \gamma_j) \cdot \ \dots \ \cdot \gamma_n)|_{I_{n-1}^i(\hat{t}_j, a_2, \dots, a_{j-1} + a_j, \dots a_n)}$$

for $i \in \{0, 1\}$.
Hence,

$$\varphi_{n+1}(t_j = 1, \alpha, \gamma_1, \dots, \gamma_n) = \varphi_n(\hat{t}_j, \alpha, \gamma_1, \dots, \gamma_{j-1} \cdot \gamma_j, \dots, \gamma_n).$$

Third equation : For any $n \geq 1$, $k \geq 1$, $t_1, t_2, \dots, t_n \in I^n$, $\alpha \in F_1$ and $\gamma_1, \dots, \gamma_n \in \Omega X^n \times \mathcal{P}_{\star \to X} X$, using (16),

$$(\gamma_1 \cdot \ \dots \ \cdot \gamma_n)|_{I_n^0(t_k = 0, a_1, \dots, a_n)} = (\gamma_1 \cdot \ \dots \ \cdot \gamma_{k-1})|_{I_{k-1}^0(t_1, \dots, t_{k-1}, a_1, \dots, a_{k-1})}$$

and

$$(\gamma_1 \cdot \ \dots \ \cdot \gamma_n)|_{I_n^1(t_k = 0, a_1, \dots, a_n)} = (\gamma_1 \cdot \ \dots \ \cdot \gamma_{k-1})|_{I_{k-1}^1(t_1, \dots, t_{k-1}, a_1, \dots, a_{k-1})} \cdot \gamma_k \cdot \ \dots \ \cdot \gamma_n.$$

Therefore,

$$
\begin{aligned}
&\varphi_{n+1}(t_k = 0, \alpha, \gamma_1, \dots, \gamma_n) \\
&= \Phi_2\left(r_{k-1}(t_1, \dots, t_{k-1}, \alpha, \gamma_1, \dots, \gamma_{k-1}), (\gamma_1 \cdot \ \dots \ \cdot \gamma_{k-1})|_{I^1_{k-1}(t_1,\dots,t_{k-1},a_1,\dots,a_{k-1})} \cdot \gamma_k \cdot \ \dots \ \cdot \gamma_n\right) \\
&= \Phi_2\left(\Phi_2\left(r_{k-1}(t_1, \dots, t_{k-1}, \alpha, \gamma_1, \dots, \gamma_{k-1}), (\gamma_1 \cdot \ \dots \ \cdot \gamma_{k-1})|_{I^1_{k-1}(t_1,\dots,t_{k-1},a_1,\dots,a_{k-1})}\right), \gamma_k \cdot \ \dots \ \cdot \gamma_n\right) \\
&= \Phi_2(\varphi_k(t_1, \dots, t_{k-1}, \alpha, \gamma_1, \dots, \gamma_{k-1}), \gamma_k \cdot \ \dots \ \cdot \gamma_n).
\end{aligned}
$$

■

This concludes the proof of i).

■

Proof of ii).
To finish the proof of Theorem 5.8, we now prove that such a coherent homotopy induces a chain homotopy $v : C_*(X, \Xi, C_*(F_1)) \to C_{*+1}(E_2)$ between $\Psi_2 \circ \tilde{\varphi}$ and $\varphi_* \circ \Psi_1$.
For $i \in \{0, 1\}$, we denote $\mathcal{F}_i = C_*(F_i)$ endowed with the module structure induced by a transitive lifting function associated to $F_i \hookrightarrow E_i \to X$. Let Ξ be a set of DG Morse data on X with Morse function f. Let $\{s_{x,y} \in C_{|x|-|y|-1}(\overline{\mathcal{L}}(x, y)),\ x, y \in \mathrm{Crit}(f)\}$ be a representing chain system and for each $x, y \in \mathrm{Crit}(f)$.
Define $\Gamma_{x,y} : \overline{\mathcal{L}}(x, y) \to \mathcal{P}_{x\to y}X$ to be the parametrization map by the values of f. The twisting cocycle $\{m'_{x,y} \in C_{|x|-|y|-1}(\Omega(X/\mathcal{Y})),\ x, y \in \mathrm{Crit}(f)\}$ is defined by $m'_{x,y} = p_* \circ \Gamma_{x,y,*}(s_{x,y})$, so that the Barraud-Cornea twisting cocycle satisfies $m_{x,y} = \theta_* m'_{x,y} \in C_{|x|-|y|-1}(\Omega X)$. The twisting cocycle $\{m'_{x,y}\}$ satisfies the Maurer-Cartan equation (1)

$$\partial m'_{x,y} = \sum_z (-1)^{|x|-|z|} m'_{x,z} m'_{z,y}.$$

We use the definition of the morphisms Ψ_1 and Ψ_2 from in Section 2.3. Each of these morphisms is defined as a composition of three maps :

$$C_*(X, m_{x,y}, \mathcal{F}_1) \tilde{\leftrightarrow} C_*(X, m'_{x,y}, \theta^*\mathcal{F}_1) \xrightarrow{\Psi_1} C_*(\theta^* E_1) \to C_*(E_1),$$

$$C_*(X, m_{x,y}, \mathcal{F}_2) \tilde{\leftrightarrow} C_*(X, m'_{x,y}, \theta^*\mathcal{F}_2) \xrightarrow{\Psi_2} C_*(\theta^* E_2) \to C_*(E_2).$$

We state here a lemma proving that morphisms of fibrations are stable by pullback.

Lemma 5.11. *Let $(Y, \star_Y)$ be a smooth, oriented, pointed, closed and connected manifold. Let $F \hookrightarrow E \xrightarrow{\pi} X$ and $F' \hookrightarrow E' \xrightarrow{\pi'} X$ be fibrations, $F \hookrightarrow E_Y \to Y$, $F' \hookrightarrow E'_Y \to Y$ be the pullback fibrations by a continuous map $\psi : Y \to X$ such that $\psi(\star_Y) = \star$. If $\varphi : E \to E'$ is a morphism of fibrations over X, then*

$$\psi^*\varphi : E_Y \to E'_Y,\ \ \psi^*\varphi(y, \alpha) = (y, \varphi(\alpha))$$

is a morphism of fibrations over Y.

Proof. Recall that

$$E_Y = \{(y, \alpha) \in Y \times E,\ \pi(\alpha) = \psi(y)\}$$
$$E'_Y = \{(y, \alpha') \in Y \times E',\ \pi'(\alpha') = \psi(y)\}$$

and $\pi_Y : E_Y \to Y$, $\pi'_Y : E'_Y \to Y$ are defined by $\pi_Y(y, \alpha) = y$ and $\pi'_Y(y, \alpha') = y$. The statement is therefore clear since for any $(y, \alpha) \in E_Y$, $\pi'(y, \varphi(\alpha)) = y = \pi(y, \alpha)$.

■

We will show that the following diagram is commutative.

$$
\begin{array}{ccccccc}
C_*(X, m_{x,y}, \mathcal{F}_1) & \xrightarrow[\sim]{} & C_*(X, m'_{x,y}, \theta^*\mathcal{F}_1) & \xrightarrow{\Psi_1} & C_*(\theta^* E_1) & \xrightarrow{\theta_*} & C_*(E_1) \\
\downarrow{\scriptstyle \tilde{\varphi}} & & \downarrow{\scriptstyle \theta^*\tilde{\varphi}} & & \downarrow{\scriptstyle \theta^*\varphi} & & \downarrow{\scriptstyle \varphi} \\
C_*(X, m_{x,y}, \mathcal{F}_2) & \xrightarrow{\sim} & C_*(X, m'_{x,y}, \theta^*\mathcal{F}_2) & \xrightarrow{\Psi_2} & C_*(\theta^* E_2) & \xrightarrow{\theta_*} & C_*(E_2).
\end{array}
$$

Since $\mathcal{F}_i \simeq \theta^*\mathcal{F}_i$ is just an identification of DG modules, it is clear that the first square commutes. The third square commutes by definition of the map $\theta^*\varphi : C_*(\theta^*E_1) \to C_*(\theta^*E_2)$ induced by $\varphi : C_*(E_1) \to C_*(E_2)$.

In order to complete this proof, we now have to prove that the middle square commutes in homology :

We use the notation $\mathbf{m} \in \mathrm{Hom}_{-1}(\mathbb{Z}\mathrm{Crit}(f), C_*(\Omega(X/\mathcal{Y})) \otimes \mathbb{Z}\mathrm{Crit}(f))$ for the map $\mathbf{m}(x) = \sum_y m'_{x,y} \otimes y$. We will extend $\mathbf{m}$ in the same manner as in Section 4.2 to

$$\tilde{\mathbf{m}} \in \mathrm{End}_{-1}(C_*(F_1) \otimes TC_*(\Omega(X/\mathcal{Y})) \otimes \mathbb{Z}\mathrm{Crit}(f))$$

defined by

$$\tilde{\mathbf{m}}(\alpha \otimes \sigma_1 \otimes \cdots \otimes \sigma_k \otimes x) = (1^{\otimes k+1} \otimes \mathbf{m})(\alpha \otimes \sigma_1 \otimes \cdots \otimes \sigma_k \otimes x).$$

Given $x \in \mathrm{Crit}(f)$, we can define $m_x \in C_{|x|}(\mathcal{P}_{\star \to X/\mathcal{Y}} X/\mathcal{Y})$ by evaluating on $X/\mathcal{Y}$ a suitable representative of the fundamental class of the Latour cell $\overline{W^u}(x)$ (see [BDHO24, Lemma 7.3.2 and Lemma 7.3.3]). It satisfies the equation

$$\partial m_x = \sum_{y \in \mathrm{Crit}(f)} m'_{x,y} m_y. \tag{17}$$

We denote $\mathbf{m^L} \in \mathrm{Hom}_0(\mathbb{Z}\mathrm{Crit}(f), C_*(\mathcal{P}_{\star \to X/\mathcal{Y}} X/\mathcal{Y}))$ the morphism defined by

$$\mathbf{m^L}(x) = m_x \in C_{|x|}(\mathcal{P}_{\star \to X/\mathcal{Y}} X/\mathcal{Y})$$

for all $x \in \mathrm{Crit}(f)$ and we extend it to

$$\tilde{\mathbf{m}}^{\mathbf{L}} \in \mathrm{Hom}_0(\mathcal{F}_i \otimes TC_*(\Omega(X/\mathcal{Y})) \otimes \mathbb{Z}\mathrm{Crit}(f), \mathcal{F}_i \otimes TC_*(\Omega(X/\mathcal{Y})) \otimes C_*(\mathcal{P}_{\star \to X/\mathcal{Y}} X/\mathcal{Y}))$$

by

$$\tilde{\mathbf{m}}^{\mathbf{L}}(\alpha \otimes \sigma_1 \otimes \cdots \otimes \sigma_k \otimes x) = \alpha \otimes \sigma_1 \otimes \cdots \otimes \sigma_k \otimes m_x.$$

Therefore, for $i \in \{1, 2\}$, the definition of the quasi-isomorphism $\Psi_i : C_*(X, m'_{x,y}, \theta^*\mathcal{F}_i) \to C_*(\theta^*E_1)$ can be written

$$\Psi_i = \Phi_{i,*} \tilde{\mathbf{m}}^{\mathbf{L}}. \tag{18}$$

Let $\left\{\varphi_{n+1} : I^n \times F_1 \times \Omega(X/\mathcal{Y})^{n-1} \times \mathcal{P}_{\star \to X/\mathcal{Y}} X/\mathcal{Y} \to \theta^*E_2\right\}$ be a coherent homotopy for the morphism of fibrations $\theta^*\varphi$.
Denote $\phi_1 = \theta^*\varphi_* : C_*(\theta^*E_1) \to C_*(\theta^*E_2)$ and for all $n \geq 1$, define

$$\phi_{n+1} : \theta^*\mathcal{F}_1 \otimes C_*(\Omega(X/\mathcal{Y}))^{n-1} \otimes C_*(\mathcal{P}_{\star \to X/\mathcal{Y}} X/\mathcal{Y}) \to C_*(\theta^*E_2)$$

by

$$\phi_{n+1}(\alpha \otimes \sigma_1 \otimes \cdots \otimes \sigma_n) = \varphi_{n+1,*}(\mathrm{Id}_{I^n} \otimes \alpha \otimes \sigma_1 \otimes \cdots \otimes \sigma_n).$$

Therefore

$$\theta^*\tilde{\varphi} = \sum_{n \geq 0} \phi_{n+1} \tilde{\mathbf{m}}^n.$$

The proof of Proposition 5.5 also proves the following lemma.

Lemma 5.12. *If we denote ν_1^A and ν_1^B the differentials of $\mathcal{A} = \theta^*C_*(F_1)$ and $\mathcal{B} = C_*(\theta^*E_2)$, $\nu_2^A = \Phi_1$, $\nu_2^B = \Phi_2$ and $\nu_k^A = \nu_k^B = 0$ for $k \geq 3$, then the $\mathcal{A}_\infty$-relation (6) is satisfied as a functional equality on maps $\mathcal{A} \otimes C_*(\Omega(X/\mathcal{Y}))^{n-1} \otimes C_*(\mathcal{P}_{\star \to X/\mathcal{Y}} X/\mathcal{Y}) \to \mathcal{B}$ for the family $\{\phi_{n+1} : \mathcal{A} \otimes C_*(\Omega X)^{n-1} \otimes C_*(\mathcal{P}_{\star \to X} X) \to \mathcal{B}, n \geq 1\}$.*

■

We now define

$$v = \sum_{n\geq 1}(-1)^{n+1}\phi_{n+1}\tilde{\mathbf{m}}^{\mathbf{L}}\tilde{\mathbf{m}}^{n-1} : C_*(X, m'_{x,y}, \theta^*\mathcal{F}_1) \to C_{*+1}(\theta^* E_2)$$

and show that

$$\varphi_*\Psi_1 - \Psi_2\theta^*\tilde{\varphi} = \partial v + v\partial.$$

We will denote μ_1 the differential on $C_*(\Omega(X/\mathcal{Y}))$ as well as the differential on $C_*(\mathcal{P}_{\star\to X/\mathcal{Y}}X/\mathcal{Y})$ and μ_2 will denote the Pontryagin product $\mu_2 : C_*(\Omega(X/\mathcal{Y}))\otimes C_*(\Omega(X/\mathcal{Y})) \to C_*(\Omega(X/\mathcal{Y}))$ as well as the concatenation $\mu_2 : C_*(\Omega(X/\mathcal{Y}))\otimes C_*(\mathcal{P}_{\star\to X/\mathcal{Y}}X/\mathcal{Y}) \to C_*(\mathcal{P}_{\star\to X/\mathcal{Y}}X/\mathcal{Y})$. Let us first remark that the equation (17) can be written

$$\mu_1\mathbf{m}^{\mathbf{L}} = \mu_2\mathbf{m}^{\mathbf{L}}\mathbf{m}. \tag{19}$$

We now use equation (6) to compute

$$\begin{aligned}
\partial v &= \sum_{n\geq 1}(-1)^{n+1}\partial\phi_{n+1}\tilde{\mathbf{m}}^{\mathbf{L}}\tilde{\mathbf{m}}^{n-1}\\
&= \sum_{n\geq 1}(-1)^{n+1}\phi_n\Phi_{1,*}\tilde{\mathbf{m}}^{\mathbf{L}}\tilde{\mathbf{m}}^{n-1} - \sum_n \Phi_{2,*}(\phi_n\otimes 1)\tilde{\mathbf{m}}^{\mathbf{L}}\tilde{\mathbf{m}}^{n-1}\\
&+ \sum_{n\geq 1}(-1)^{n+1}(-1)^n\phi_{n+1}\partial_{\mathcal{F}_1}\tilde{\mathbf{m}}^{\mathbf{L}}\tilde{\mathbf{m}}^{n-1} - \sum_{n\geq 1}\sum_{r\geq 1}\phi_{n+1}(1^{\otimes r}\otimes\mu_1\otimes 1^{\otimes n-r})\tilde{\mathbf{m}}^{\mathbf{L}}\tilde{\mathbf{m}}^{n-1}\\
&- \sum_{n\geq 2}\sum_{r\geq 1}(-1)^{r+n}\phi_n(1^{\otimes r}\otimes\mu_2\otimes 1^{\otimes n-r-1})\tilde{\mathbf{m}}^{\mathbf{L}}\tilde{\mathbf{m}}^{n-1}\\
&= \varphi_*\Psi_1 - \sum_{k\geq 1}(-1)^{k+1}\phi_{k+1}\Phi_{1,*}\tilde{\mathbf{m}}^{\mathbf{L}}\tilde{\mathbf{m}}^{k} - \sum_{n\geq 1}\Phi_{2,*}\tilde{\mathbf{m}}^{\mathbf{L}}\phi_n\tilde{\mathbf{m}}^{n-1}\\
&- \sum_{n\geq 1}(-1)^{n+1}\phi_{n+1}\tilde{\mathbf{m}}^{\mathbf{L}}\tilde{\mathbf{m}}^{n-1}\partial_{\mathcal{F}_1} - \sum_{n\geq 1}\sum_{r\geq 1}\phi_{n+1}(1^{\otimes r}\otimes\mu_1\otimes 1^{\otimes n-r})\tilde{\mathbf{m}}^{\mathbf{L}}\tilde{\mathbf{m}}^{n-1}\\
&+ \sum_{k\geq 1}\sum_{r\geq 1}(-1)^{r+k}\phi_{k+1}(1^{\otimes r}\otimes\mu_2\otimes 1^{\otimes k-r})\tilde{\mathbf{m}}^{\mathbf{L}}\tilde{\mathbf{m}}^{k}.
\end{aligned}$$

Moreover

$$\begin{aligned}
v\partial &= v(\partial_{F_1} + \Phi_{1,*}\tilde{\mathbf{m}})\\
&= \sum_{n\geq 1}(-1)^{n+1}\phi_{n+1}\tilde{\mathbf{m}}^{\mathbf{L}}\tilde{\mathbf{m}}^{n-1}\partial_{\mathcal{F}_1} + \sum_{n\geq 1}(-1)^{n+1}\phi_{n+1}\tilde{\mathbf{m}}^{\mathbf{L}}\tilde{\mathbf{m}}^{n-1}\Phi_{1,*}\tilde{\mathbf{m}}\\
&= \sum_{n\geq 1}(-1)^{n+1}\phi_{n+1}\tilde{\mathbf{m}}^{\mathbf{L}}\tilde{\mathbf{m}}^{n-1}\partial_{\mathcal{F}_1} + \sum_{n\geq 1}(-1)^{n+1}\phi_{n+1}\Phi_{1,*}\tilde{\mathbf{m}}^{\mathbf{L}}\tilde{\mathbf{m}}^{n}.
\end{aligned}$$

Using equation (7) and (19), we get a similar equation as in Lemma 4.5:

$$(1^{\otimes r}\otimes\partial\otimes 1^{\otimes n-r})\tilde{\mathbf{m}}^{\mathbf{L}}\tilde{\mathbf{m}}^{n} = (-1)^{r+n}(1^{\otimes r}\otimes\mu_2\otimes 1^{\otimes n-r})\tilde{\mathbf{m}}^{\mathbf{L}}\tilde{\mathbf{m}}^{n+1}.$$

We therefore get the relation

$$\partial v + v\partial = \varphi_*\Psi_1 - \sum_{n\geq 0}\Phi_{2,*}\tilde{\mathbf{m}}^{\mathbf{L}}\phi_{n+1}\tilde{\mathbf{m}}^{n} = \varphi\Psi_1 - \Psi_2\theta^*\tilde{\varphi}.$$

■

We now state some corollaries of this Theorem.

Corollary 5.13. *Let $\Phi_1 : E \,{}_{\pi}\!\times_{ev_0} \mathcal{P}X \to E$ and $\Phi_2 : E \,{}_{\pi}\!\times_{ev_0} \mathcal{P}X \to E$ be two transitive lifting functions associated to a fibration $F \hookrightarrow E \xrightarrow{\pi} X$. Let Ξ be a set of DG Morse data on X. For $i \in \{1,2\}$, denote $\mathcal{F}_i = C_*(F)$ endowed with the $C_*(\Omega X)$-module structure induced by Φ_i and $\Psi_i : C_*(X,\Xi,\mathcal{F}_i) \to C_*(E)$*

the quasi-isomorphism given by the Fibration Theorem. Then $\mathrm{Id} : E \to E$ *induces a quasi-isomorphism* $\widetilde{\mathrm{Id}} : C_*(X, \Xi, \mathcal{F}_1) \to C_*(X, \Xi, \mathcal{F}_2)$ *such that the following diagram commutes up to homotopy,*

$$\begin{array}{ccc} C_*(X,\Xi,\mathcal{F}_1) & \xrightarrow{\widetilde{\mathrm{Id}}} & C_*(X,\Xi,\mathcal{F}_2) \\ & {\scriptstyle\Psi_1}\searrow \quad \swarrow{\scriptstyle\Psi_2} & \\ & C_*(E). & \end{array}$$

■

This result would be no much smaller task to prove on itself than Theorem 3.2 since one would need to complete a homotopy between Φ_1 and Φ_2 into a whole coherent homotopy for Id and then prove that $\widetilde{\mathrm{Id}}$ corresponds to $\mathrm{Id} : C_*(E) \to C_*(E)$.

Corollary 5.14. *If* $\varphi : E_1 \to E_2$ *is a morphism of fibrations, then the map* $\tilde{\varphi} : H_*(X, C_*(F_1)) \to H_*(X, C_*(F_2))$ *is well-defined.*

Proof. Let Ξ_0 and Ξ_1 be sets of DG Morse data on X. Proving this corollary amounts to prove that the following diagram commutes

$$\begin{array}{ccc} H_*(X,\Xi_0,C_*(F_1)) & \xrightarrow{\tilde{\varphi}} & H_*(X,\Xi_0,C_*(F_1)) \\ \downarrow{\scriptstyle\Psi_{01}} & & \downarrow{\scriptstyle\Psi_{01}} \\ H_*(X,\Xi_1,C_*(F_1)) & \xrightarrow{\tilde{\varphi}} & H_*(X,\Xi_1,C_*(F_1)). \end{array}$$

We decompose this diagram in

$$\begin{array}{ccc} H_*(X,\Xi_0,C_*(F_1)) & \xrightarrow{\tilde{\varphi}} & H_*(X,\Xi_0,C_*(F_1)) \\ \downarrow{\scriptstyle\Psi_{E_1}} & & \downarrow{\scriptstyle\Psi_{E_2}} \\ H_*(E_1) & \xrightarrow{\varphi_*} & H_*(E_2) \\ \downarrow{\scriptstyle\Psi_{E_1}^{-1}} & & \downarrow{\scriptstyle\Psi_{E_2}^{-1}} \\ H_*(X,\Xi_1,C_*(F_1)) & \xrightarrow{\tilde{\varphi}} & H_*(X,\Xi_1,C_*(F_1)). \end{array}$$

(with Ψ_{01} : $H_*(X,\Xi_0,C_*(F_1)) \to H_*(X,\Xi_1,C_*(F_1))$ and Ψ_{01} : $H_*(X,\Xi_0,C_*(F_1)) \to H_*(X,\Xi_1,C_*(F_1))$)

Theorem 5.8 gives that the two squares commutes. It remains to prove that $\Psi_{E_i} \circ \Psi_{01} = \Psi_{E_i}$ in homology for $i \in \{0,1\}$. [BDHO24, Proposition 10.2.1] states that $\Psi_{01} = \mathrm{Id}_* : H_*(X,\Xi_0,C_*(F_1)) \to H_*(X,\Xi_1,C_*(F_1))$ and [BDHO24, Proposition 9.8.1] states that $\Psi_{E_i}\mathrm{Id}_* = \mathrm{Id}_{E_i}\Psi_{E_i}$. This concludes the proof of this Corollary.

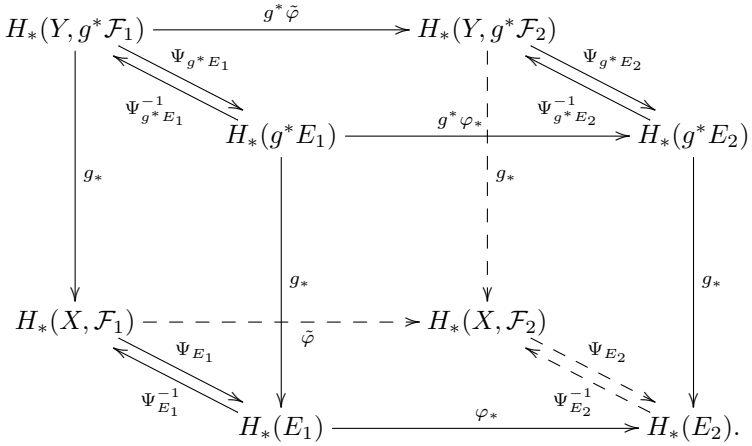
■

Corollary 5.15. *Let* $F_1 \hookrightarrow E_1 \to X$ *and* $F_2 \hookrightarrow E_2 \to X$ *be two fibrations and* $\varphi : E_1 \to E_2$ *a morphism of fibrations. Let* $g : Y^m \to X^n$ *be a continuous map. Denote* $\mathcal{F}_1 = C_*(F_1)$ *and* $\mathcal{F}_2 = C_*(F_2)$. *Then the following diagram commutes*

$$\begin{array}{ccc} H_*(Y,g^*\mathcal{F}_1) & \xrightarrow{g^*\tilde{\varphi}} & H_*(Y,g^*\mathcal{F}_2) \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g_*} \\ H_*(X,\mathcal{F}_1) & \xrightarrow[\tilde{\varphi}]{} & H_*(X,\mathcal{F}_2). \end{array}$$

Remark 5.16. *Using a more algebraic approach for any morphism of complexes* $\tilde{\varphi}$ *induced by a morphism of* $\mathcal{A}_\infty$*-modules over* $C_*(\Omega X)$*, we will reprove this property as well as prove that the map* $\tilde{\varphi}$ *is compatible with shriek maps (Proposition 5.23).*

Proof. We will prove that this diagram is a face of cube where every other faces are commutative.



The front face is commutative by general properties about pullback fibrations. Theorem 5.8 states that the faces on the top and on the bottom are commutative. [BDHO24, Proposition 9.8.1] states that the lateral faces are commutative. From there, a diagram chase concludes the proof.

■

Corollary 5.17. *If $\varphi : E_1 \to E_2$ is a morphism of fibrations, then $\tilde{\varphi} : H_*(X, C_*(F_1)) \to H_*(X, C_*(F_2))$ does not depend on the coherent homotopy but only on $\varphi_* : H_*(E_1) \to H_*(E_2)$. Moreover, if $\varphi_* : H_*(E_1) \to H_*(E_2)$ is an isomorphism, then so is $\tilde{\varphi} : H_*(X, C_*(F_1)) \to H_*(X, C_*(F_2))$.*

■

Corollary 5.18. *If $\varphi : E_0 \to E_1$ and $\psi : E_1 \to E_2$ are morphisms of fibrations, then $\psi \circ \varphi : E_0 \to E_2$ is a morphism of fibrations and*

$$\widetilde{\psi \circ \varphi} = \tilde{\psi} \circ \tilde{\varphi} : H_*(E_0) \to H_*(E_2).$$

Proof. Theorem 5.8 proves that at the homology level,

$$\begin{aligned}
\widetilde{\psi \circ \varphi} &= \Psi_{E_0} \circ (\psi \circ \varphi)_* \circ \Psi^{-1}_{E_2} \\
&= \Psi_{E_0} \circ \psi_* \circ \varphi_* \circ \Psi^{-1}_{E_2} \\
&= \Psi_{E_0} \circ \psi_* \circ \Psi^{-1}_{E_1} \circ \Psi_{E_1} \circ \varphi_* \circ \Psi^{-1}_{E_2} \\
&= \tilde{\psi} \circ \tilde{\varphi}.
\end{aligned}$$

■

5.3 Compatibility with the $\mathcal{A}_\infty$-Morse toolset, direct and shriek maps

The main results of this section are that any morphism $\boldsymbol{\varphi} : \mathcal{F} \to \mathcal{G}$ of $\mathcal{A}_\infty$-modules over $C_*(\Omega X)$ induces a well-defined map $\tilde{\varphi} : H_*(X, \mathcal{F}) \to H_*(X, \mathcal{G})$ in homology and that it is compatible with direct and shriek maps. Therefore, the reader only interested in the case of fibrations will only find here the proof of the compatibility of $\tilde{\varphi}$ with shriek maps. We have provided independent proofs of the compatibility with direct maps (Corollary 5.15) and the fact that $\tilde{\varphi} : H_*(X, \mathcal{F}) \to H_*(X, \mathcal{G})$ is well-defined (Corollary 5.14) using Theorem 5.8 in the case of a fibration.

5.3.1 Compatibility with $\mathcal{A}_\infty$-Morse toolset

We prove here that the map induced by an $\mathcal{A}_\infty$-morphism of modules commutes in homology with the maps defined in Proposition 4.9. The statement and proof are quite technical since the proposition is aimed to be applied in various situations.

Proposition 5.19. *Let $f_0, f_1 : X \to \mathbb{R}$ be two Morse functions and $\mathcal{F}, \mathcal{G}$ two $\mathcal{A}_\infty$-modules over $C_*(\Omega X)$. Let $\{m^0_{x,y} \in C_{|x|-|y|-1}(\Omega X),\ x, y \in \mathrm{Crit}(f_0)\}$ and $\{m^1_{x',y'} \in C_{|x'|-|y'|}(\Omega X), x', y' \in \mathrm{Crit}(f_1)\}$ be two twisting cocycles on X. Let $\{\tau_{x,y'} \in C_{|x|-|y'|}(\Omega X), x \in \mathrm{Crit}(f_0), y' \in \mathrm{Crit}(f_1)\}$ a cocycle satisfying (8). Let $\Psi_{\mathcal{F}} : C_*(X, m^0, \mathcal{F}) \to C_*(X, m^1, \mathcal{F})$, $\Psi_{\mathcal{G}} : C_*(X, m^0, \mathcal{G}) \to C_*(X, m^1, \mathcal{G})$ be the morphisms defined in 4.9 using $\{\tau_{x,y'} \in C_{|x|-|y'|}(\Omega X),\ x \in \mathrm{Crit}(f_0),\ y' \in \mathrm{Crit}(f_1)\}$.*

Let $\boldsymbol{\varphi} : (\mathcal{F}, \boldsymbol{\nu^F}) \to (\mathcal{G}, \boldsymbol{\nu^G})$ be a morphism of $\mathcal{A}_\infty$-modules and $\tilde{\varphi}_0 : C_*(X, m^0, \mathcal{F}) \to C_*(X, m^0, \mathcal{G})$, $\tilde{\varphi}_1 : C_*(X, m^1, \mathcal{F}) \to C_*(X, m^1, \mathcal{G})$ be the induced morphisms of complexes as defined in Proposition 5.1. Then, the following diagram commutes up to chain homotopy :

$$\begin{array}{ccc} C_*(X, m^0, \mathcal{F}) & \xrightarrow{\tilde{\varphi}_0} & C_*(X, m^0, \mathcal{G}) \\ \Big\downarrow{\scriptstyle \Psi_{\mathcal{F}}} & & \Big\downarrow{\scriptstyle \Psi_{\mathcal{G}}} \\ C_*(X, m^1, \mathcal{F}) & \xrightarrow[\tilde{\varphi}_1]{} & C_*(X, m^1, \mathcal{G}). \end{array}$$

Proof. Let $\kappa : C_*(X, m^0, \mathcal{F}) \to C_{*+1}(X, m^1, \mathcal{G})$ defined by

$$\kappa = \sum_{n\geq 1} \sum_{u=1}^{n} (-1)^{u-1} (\varphi_{n+1} \otimes 1) \tilde{\mathbf{m}}^{n-u}_{(1)} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}^{u-1}_{(0)}.$$

We now prove that $\Psi_{\mathcal{G}} \tilde{\varphi}_0 - \tilde{\varphi}_1 \Psi_{\mathcal{F}} = \partial\kappa + \kappa\partial$.
We compute

$$\begin{aligned} \partial\kappa &= \sum_k \sum_{n\geq 1} \sum_{u=1}^{n} (-1)^{u-1} (\nu'_{k+1} \otimes 1) \tilde{\mathbf{m}}^{k}_{(1)} (\varphi_{n+1} \otimes 1) \tilde{\mathbf{m}}^{n-u}_{(1)} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}^{u-1}_{(0)} \\ &= \sum_k \sum_{n\geq 1} \sum_{u=1}^{n} (-1)^{u-1} (-1)^{kn} (\nu'_{k+1} \varphi_{n+1} \otimes 1) \tilde{\mathbf{m}}^{n+k-u}_{(1)} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}^{u-1}_{(0)} \\ &= \sum_k \sum_{n\geq 1} \sum_{u=1}^{n+k} (-1)^{u-1} (-1)^{kn} (\nu'_{k+1} \varphi_{n+1} \otimes 1) \tilde{\mathbf{m}}^{n+k-u}_{(1)} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}^{u-1}_{(0)} \\ &\quad - \sum_k \sum_{n\geq 1} \sum_{u=n+1}^{n+k} (-1)^{u-1} (-1)^{kn} (\nu'_{k+1} \varphi_{n+1} \otimes 1) \tilde{\mathbf{m}}^{n+k-u}_{(1)} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}^{u-1}_{(0)} \end{aligned}$$

and

$$\begin{aligned} \kappa\partial &= \sum_k \sum_{n\geq 1} \sum_{u=1}^{n} (-1)^{u-1} (\varphi_{n+1} \otimes 1) \tilde{\mathbf{m}}^{n-u}_{(1)} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}^{u-1}_{(0)} (\nu_{k+1} \otimes 1) \tilde{\mathbf{m}}^{k+1}_{(0)} \\ &= \sum_k \sum_{n\geq 1} \sum_{u=1}^{n} (-1)^{u-1} (-1)^{(k-1)(n-1)} (\varphi_{n+1} \nu_{k+1} \otimes 1) \tilde{\mathbf{m}}^{n-u}_{(1)} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}^{k+u-1}_{(0)} \\ &= -\sum_k \sum_{n\geq 1} \sum_{u=k+1}^{n+k} (-1)^{u-1} (-1)^{(k-1)n} (\varphi_{n+1} \nu_{k+1} \otimes 1) \tilde{\mathbf{m}}^{n+k-u}_{(1)} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}^{u-1}_{(0)} \\ &= -\sum_k \sum_{n\geq 1} \sum_{u=1}^{n+k} (-1)^{u-1} (-1)^{(k-1)n} (\varphi_{n+1} \nu_{k+1} \otimes 1) \tilde{\mathbf{m}}^{n+k-u}_{(1)} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}^{u-1}_{(0)} \\ &\quad + \sum_k \sum_{n\geq 1} \sum_{u=1}^{k} (-1)^{u-1} (-1)^{(k-1)n} (\varphi_{n+1} \nu_{k+1} \otimes 1) \tilde{\mathbf{m}}^{n+k-u}_{(1)} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{m}}^{u-1}_{(0)}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\partial\kappa + \kappa\partial = &\sum_k \sum_{n\geq 1} \sum_{u=1}^{k} (-1)^{u-1}(-1)^{(k-1)n}(\varphi_{n+1}\nu_{k+1}\otimes 1)\tilde{\mathbf{m}}_{(1)}^{n+k-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u-1} \\
&- \sum_k \sum_{n\geq 1} \sum_{u=n+1}^{n+k} (-1)^{u-1}(-1)^{kn}(\nu'_{k+1}\varphi_{n+1}\otimes 1)\tilde{\mathbf{m}}_{(1)}^{n+k-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u-1} \\
&- \sum_{N\geq 1}\left[\sum_{r=1}^{N}(-1)^N\varphi_{N+1}(1^{\otimes r}\otimes\mu_1\otimes 1^{\otimes N-r})\right. \\
&\left.+\sum_{r=1}^{N-1}(-1)^r\varphi_N(1^{\otimes r}\otimes\mu_2\otimes 1^{\otimes N-1-r})\right]\left[\sum_{u=1}^{N}(-1)^{u-1}\tilde{\mathbf{m}}_{(1)}^{n+k-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u-1}\right]. \\
&= \tilde{\varphi}_D\Psi_{\mathcal{F}} - \Psi_{\mathcal{G}}\tilde{\varphi}_C \\
&- \sum_{N\geq 1}\left[\sum_{r=1}^{N}(-1)^N\varphi_{N+1}(1^{\otimes r}\otimes\mu_1\otimes 1^{\otimes N-r})\right. \\
&\left.+\sum_{r=1}^{N-1}(-1)^r\varphi_N(1^{\otimes r}\otimes\mu_2\otimes 1^{\otimes N-1-r})\right]\left[\sum_{u=1}^{N}(-1)^{u-1}\tilde{\mathbf{m}}_{(1)}^{n+k-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u-1}\right].
\end{aligned}$$

The same arguments as in the proof of Proposition 4.9 concerning the relation between $(1^{\otimes r}\otimes\mu_1\otimes 1^{\otimes N-r})\tilde{\mathbf{m}}_{(1)}^{n+k-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u-1}$ and $(1^{\otimes r}\otimes\mu_2\otimes 1^{\otimes N-1-r})\tilde{\mathbf{m}}_{(1)}^{n+k-u}\tilde{\boldsymbol{\tau}}\tilde{\mathbf{m}}_{(0)}^{u-1}$ depending on u and r show that the last sum is 0 and therefore conclude the proof.

■

Corollary 5.20. *For any data Ξ_0 and Ξ_1 over X and any morphism $\boldsymbol{\varphi}:(\mathcal{G},\nu_n)\to(\mathcal{G}',\nu'_n)$ of $\mathcal{A}_\infty$-module over $C_*(\Omega X)$, the following diagram commutes up to chain homotopy:*

$$\begin{array}{ccc}
C_*(X,\Xi_0,\mathcal{G}) & \xrightarrow{\tilde{\varphi}_0} & C_*(X,\Xi_0,\mathcal{G}') \\
\downarrow{\scriptstyle \Psi_{01}} & & \downarrow{\scriptstyle \Psi_{01}} \\
C_*(X,\Xi_1,\mathcal{G}) & \xrightarrow[\tilde{\varphi}_1]{} & C_*(X,\Xi_1,\mathcal{G}').
\end{array}$$

In particular, the map $\tilde{\varphi}:H_*(X,\mathcal{G})\to H_*(X,\mathcal{G}')$ is then well-defined.

Proof. This is a consequence of Proposition 5.19, with m^0 the Barraud-Cornea twisting cocycle arising from Ξ_0, m^1 the Barraud-Cornea twisting cocycle arising from Ξ_1 and $\Psi_{01}:C_*(X,\Xi_0,\mathcal{G})\to C_*(X,\Xi_1,\mathcal{G})$, $\Psi_{01}:C_*(X,\Xi_0,\mathcal{G}')\to C_*(X,\Xi_1,\mathcal{G}')$ the corresponding $\mathcal{A}_\infty$-continuation maps (see Theorem 4.15).

■

Proposition 5.21. *Let $\boldsymbol{\varphi}:\mathcal{F}\to\mathcal{F}'$ be a morphism of $\mathcal{A}_\infty$-modules over $C_*(\Omega X)$ and denote $\varphi=\varphi_1$. Then, the induced map in homology $\tilde{\varphi}:H_*(X,\mathcal{F})\to H_*(X,\mathcal{F}')$ is a limit of morphisms $\varphi_1^{(r)}:E^r_{p,q}\to E'^r_{p,q}$ between the spectral sequences associated with those homologies and*

$$\varphi^{(1)}=\varphi_*\otimes 1: H_q(\mathcal{F})\otimes C_p(f,\xi)\to H_q(\mathcal{F}')\otimes C_p(f,\xi).$$

Proof. Recall that the enriched complex $C_*(X,\mathcal{A})$ is filtered by

$$F_p(C_k(X,\mathcal{A}))=\bigoplus_{\substack{i+j=k\\ j\leq p}}\mathcal{A}_i\otimes C_j(f,\xi)$$

and that the 0-th page is given by $E^0_{p,q}=\mathcal{A}_q\otimes C_p(f,\xi)$.

The morphism $\tilde{\varphi}$ respects the filtration since $\tilde{\mathbf{m}}$ strictly decreases the index of the critical point. If $\alpha \otimes x \in \mathcal{F}_q \otimes C_p(f,\xi)$, the only term in $\tilde{\varphi}(\alpha \otimes x)$ that belongs to $\mathcal{F}'_q \otimes C_p(f,\xi)$ is $\varphi_*(\alpha) \otimes x$. Therefore

$$\varphi^{(0)}(\alpha \otimes x) = \varphi_*(\alpha) \otimes x$$

and this induces in homology $\varphi^{(1)} = \varphi_* \otimes 1 : H_q(\mathcal{F}) \otimes C_p(f,\xi) \to H_q(\mathcal{F}') \otimes C_p(f,\xi)$.

■

Corollary 5.22. *If $\boldsymbol{\varphi} : \mathcal{F} \to \mathcal{F}'$ is a quasi-isomorphism of $\mathcal{A}_\infty$-modules, then*

$$\tilde{\varphi} : H_*(X,\mathcal{F}) \to H_*(X,\mathcal{F}')$$

is an isomorphism.

■

5.3.2 Compatibility with direct and shriek maps

Let X^n and Y^m be two pointed, oriented, closed and connected manifolds and $(\mathcal{F},\boldsymbol{\nu})$ be an $\mathcal{A}_\infty$-module over $C_*(\Omega Y)$. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be Morse functions that are part of sets of DG Morse data Ξ_X and Ξ_Y on respectively X and Y. Denote $\{m^X_{x,z}\}$ and $\{m^Y_{y',w'}\}$ the Barraud-Cornea cocycles associated to Ξ_X and Ξ_Y.

Let $\varphi : X \to Y$ be a continuous map.
The first definition of direct map $\varphi_* : H_*(X,\varphi^*\mathcal{F}) \to H_*(Y,\mathcal{F})$ and shriek map $\varphi_! : H_*(Y,\mathcal{F}) \to H_{*+n-m}(X,\varphi^*\mathcal{F})$ are exactly the same in the $\mathcal{A}_\infty$ setting as in the DG case (see [BDHO24, Section 9]).

Let $\varphi : X \to Y$ be a smooth map.
The second definition of the direct map $\varphi_* : C_*(X,\Xi_X,\varphi^*\mathcal{F}) \to C_*(Y,\Xi_Y,\mathcal{F})$ is described in [BDHO24, Section 10] by constructing a cocycle $\{\tau_{x,y'} \in C_{|x|-|y'|}(\Omega Y),\ x \in \mathrm{Crit}(f),\ y' \in \mathrm{Crit}(g)\}$ satisfying the equation (8) for the twisting cocycles $\{m^Y_{y',w'} \in C_{|y'|-|w'|-1}(\Omega Y)\}$ and $\{\varphi_*(m^X_{x,z}) \in C_{|x|-|z|-1}(\Omega Y)\}$ on Y.
The second definition of the shriek map $\varphi_! : C_*(Y,\Xi_Y,\mathcal{F}) \to C_{*-m+n}(X,\Xi_X,\varphi^*\mathcal{F})$ is described by constructing a cocycle

$$\{\tau_{y',x} \in C_{[y']-[x]}(\Omega Y),\ y' \in \mathrm{Crit}(g),\ x \in \mathrm{Crit}(f)\}$$

satisfying the equation (8) for the twisting cocycles $\{m^Y_{y',w'} \in C_{[y']-[w']-1}(\Omega Y)\}$ and $\{\varphi_*(m^X_{x,z}) \in C_{[x]-[z]-1}(\Omega Y)\}$ on Y where we used the following grading for each $x \in Crit(f)$, $y' \in \mathrm{Crit}(g)$.

$$[x] = |x| + m \text{ and } [y'] = |y'| + n.$$

In our case, using Proposition 4.9, the direct map $\varphi_* : C_*(X,\varphi^*\mathcal{F}) \to C_*(Y,\mathcal{F})$ is defined by

$$\varphi_* = \sum_{n\geq 1}\sum_{u=1}^{n}(-1)^{u-1}(\nu_{n+1}\otimes 1)\tilde{\mathbf{m}}_Y^{n-u}\tilde{\boldsymbol{\tau}}(\varphi_*\tilde{\mathbf{m}}_X)^{u-1}$$

and the shriek map $\varphi_! : C_*(Y,\mathcal{F}) \to C_{*-m+n}(X,\varphi^*\mathcal{F})$ is defined by

$$\varphi_! = \sum_{n\geq 1}\sum_{u=1}^{n}(-1)^{u-1}(\nu_{n+1}\otimes 1)(\varphi_*\tilde{\mathbf{m}}_X)^{n-u}\tilde{\boldsymbol{\tau}}^{\mathbf{!}}\tilde{\mathbf{m}}_Y^{u-1}.$$

If $\varphi : X \to Y$ is continuous, let $\varphi' : X \to Y$ be a smooth map which is homotopic to φ through a basepoints preserving homotopy H which is C^0-close to the constant homotopy. The identification morphism

$$\Psi^H : C_*(X,\Xi_X,\varphi^*\mathcal{F}) \to C_*(X,\Xi_X,\varphi'^*\mathcal{F})$$

defined in [BDHO24, Proposition 8.2.1] carries over in our setting using again Proposition 4.9 and we also define

$$\varphi_* = \varphi'_* \circ \Psi^H.$$

Using the $\mathcal{A}_\infty$ homotopy property 4.11 instead of the DG one, the property stating in [BDHO24] that the first and second definition of the direct and shriek maps are equivalent in homology still holds.

Proposition 5.23. *Let $h : X^n \to Y^m$ be a continuous map, $\mathcal{G}$ and $\mathcal{G}'$ be two $\mathcal{A}_\infty$-modules over $C_*(\Omega Y)$, and $\{\varphi_n\} : \mathcal{G} \to \mathcal{G}'$ be a morphism of $\mathcal{A}_\infty$-modules.*

Then the following diagrams commute

$$\begin{array}{ccc} H_*(X, h^*\mathcal{G}) & \xrightarrow{h^*\tilde{\varphi}} & H_*(X, h^*\mathcal{G}') \\ \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle h_*} \\ H_*(Y, \mathcal{G}) & \xrightarrow[\tilde{\varphi}]{} & H_*(Y, \mathcal{G}') \end{array} \qquad \begin{array}{ccc} H_{*-m+n}(X, h^*\mathcal{G}) & \xrightarrow{h^*\tilde{\varphi}} & H_{*-m+n}(X, h^*\mathcal{G}') \\ \uparrow{\scriptstyle h_!} & & \uparrow{\scriptstyle h_!} \\ H_*(Y, \mathcal{G}) & \xrightarrow[\tilde{\varphi}]{} & H_*(Y, \mathcal{G}') \end{array}$$

Proof. It is a direct consequence of Proposition 5.19. ∎

6 DG Künneth formula. Proof of Theorem C

Let X, Y be two pointed, oriented, closed and connected manifolds endowed with DG Morse data Ξ_X and Ξ_Y. The goal of this section is to define a cross-product $C_*(X, \Xi_X, \mathcal{F}) \otimes C_*(Y, \Xi_Y, \mathcal{G}) \to C_*(X \times Y, \Xi_{X\times Y}, \mathcal{H})$ where the DG Morse data $\Xi_{X\times Y}$ is defined below and the coefficients $\mathcal{H}$ will be discussed in Section 6.3.

6.1 Morse data on a Cartesian product

Given $\Xi_X = (f, \xi_X, s_{x,x'}, o_X, \mathcal{Y}_X, \theta_X)$ a set of Morse data on $(X, \star_X)$ and $\Xi_Y = (g, \xi_Y, s_{y,y'}, o_Y, \mathcal{Y}_Y, \theta_Y)$ a set of Morse data on $(Y, \star_Y)$, we construct a set of Morse data $\Xi_{X\times Y} = (H, \xi, s_{(x,y),(x',y')}, o, \mathcal{Y}, \theta)$ on $(X \times Y, (\star_X, \star_Y))$ that consists in the following :

1. $H(x, y) = f(x) + g(y)$. Note that H is a Morse function on $X \times Y$ which satisfies $|(x, y)| = |x| + |y|$.

2. $\xi(x, y) = (\xi_X(x), \xi_Y(y))$ is a pseudo-gradient associated to H.

3. There is a canonical identification $\overline{W^u_H}(x, y) \simeq \overline{W^u_f}(x) \times \overline{W^u_g}(y)$. We therefore use the orientation $\mathrm{Or}\ \overline{W^u_H}(x, y) = \Big(\mathrm{Or}\ \overline{W^u_f}(x), \mathrm{Or}\ \overline{W^u_g}(y)\Big)$.

4. The tree $\mathcal{Y} = (\mathcal{Y}_X, \mathcal{Y}_Y)$.

5. The homotopy inverse $\theta = (\theta_X, \theta_Y) : (X \times Y)/\mathcal{Y} \to X \times Y$ of the canonical projection $p : X \times Y \to (X \times Y)/\mathcal{Y}$.

The goal of this section is to define a suitable representing chain system $s_{(x,y),(x',y')}$ of $\overline{\mathcal{L}}_H((x, y), (x', y'))$. First, let us fix our orientation conventions for the spaces of trajectories.

6.1.1 Orientation conventions

We use the same orientation conventions and rules as [BDHO24].
Let (f, ξ) be a Morse-Smale pair and $x \in \mathrm{Crit}(f)$. The orientation o_x of the unstable manifolds $W^u(x)$ is fixed by the set of DG Morse data. This induces a co-orientation of the stable manifolds

$$\mathrm{Coor}\ W^s(x) = \mathrm{Or}\ W^u(x).$$

We orient a tranverse intersection $Z \pitchfork W$ between an oriented manifold Z and a co-oriented manifold W by

$$(\mathrm{Or}\ Z \pitchfork W,\ \mathrm{Coor}\ W) = \mathrm{Or}\ Z.$$

Therefore, for $x, y \in \mathrm{Crit}(f)$, the manifold $\mathcal{M}(x, y) = W^u(x) \pitchfork W^s(y)$ is oriented via the rule

$$(\text{Or } \mathcal{M}(x,y),\ \text{Or } W^u(y)) = \text{Or } W^u(x). \tag{20}$$

Define

$$S^s(y) := W^s(y) \pitchfork f^{-1}(f(y)+\epsilon)$$

for small $\epsilon > 0$ the stable sphere associated to a critical point $y \in \text{Crit}(f)$. We co-orient $S^s(y) \subset W^s(y)$ by

$$\text{Coor } S^s(y) = (-\xi, \text{Coor } W^s(y)) = (-\xi, \text{Or } W^u(y))\,.$$

There is an identification

$$\mathcal{L}(x,y) = W^u(x) \pitchfork S^s(y),$$

which yields the orientation

$$(\text{Or } \mathcal{L}(x,y), -\xi, \text{Or } W^u(y)) = \text{Or } W^u(x).$$

We use the same orientation rules for the compactifications $\overline{\mathcal{L}}(x,y)$ and $\overline{W^u}(x)$.

$$\left(\text{Or } \overline{\mathcal{L}}(x,y), -\xi, \text{Or } \overline{W^u}(y)\right) = \text{Or } \overline{W^u}(x). \tag{21}$$

6.1.2 Representing chain system on a Cartesian product

Lemma 6.1. *If $x = x'$, $\overline{\mathcal{L}}_H((x,y),(x,y')) = \{x\} \times \overline{\mathcal{L}}_g(y,y')$ and the orientations differ by the sign $(-1)^{|x|(|y|-|y'|)}$.*

If $y = y'$, $\overline{\mathcal{L}}_H((x,y),(x',y)) = \overline{\mathcal{L}}_f(x,x') \times \{y\}$ and the orientations coincide.

Proof. The two identifications are clear. It remains to compare orientations.
The rule (20) implies for $x = y$ that a constant trajectory is oriented positively.
For any two critical points $(x,y),(x,y')$ of H,

$$\left(\text{Or } \overline{\mathcal{L}}_H((x,y),(x,y')), -\xi, \text{Or } \overline{W^u_H}(x,y')\right) = \text{Or } \overline{W^u_H}(x,y).$$

If $x = x'$, the equalities

$$\begin{aligned}\left(\text{Or } \{x\}, \text{Or } \overline{\mathcal{L}}_g(y,y'), -\xi, \text{Or } \overline{W^u_H}(x,y')\right) &= (-1)^{|x|(|y|-|y'|)} \left(\text{Or } \overline{W^u_f}(x), \text{Or } \overline{\mathcal{L}}_g(y,y'), -\xi, \text{Or } \overline{W^u_g}(y')\right)\\ &= (-1)^{|x|(|y|-|y'|)} \text{Or } \overline{W^u_H}(x,y).\end{aligned}$$

show the orientation difference between $\overline{\mathcal{L}}_F((x,y),(x,y'))$ and $\{x\} \times \overline{\mathcal{L}}_g(y,y')$.

If $y = y'$, the equalities

$$\left(\text{Or } \overline{\mathcal{L}}_f(x,x'), \text{Or } \{y\}, -\xi, \text{Or } \overline{W^u_H}(x',y)\right) = \left(\text{Or } \overline{W^u_f}(x), \text{Or } \overline{W^u_g}(y')\right) = \text{Or } \overline{W^u_H}(x,y).$$

show that $\overline{\mathcal{L}}_H((x,y),(x',y))$ and $\overline{\mathcal{L}}_f(x,x') \times \{y\}$ have the same orientation.

■

Remark that

$$\begin{aligned}\mathcal{M}_H((x,y),(x',y')) &:= W^u_H(x,y) \pitchfork W^s_H(x',y')\\ &= \left(W^u_f(x) \pitchfork W^s_f(y)\right) \times \left(W^u_f(x) \pitchfork W^s_f(y)\right) =: \mathcal{M}_f(x,x') \times \mathcal{M}_g(y,y'),\end{aligned}$$

and therefore there exist a projection π and a section i,

$$\mathcal{L}_H((x,y),(x',y')) = \mathcal{M}_f(x,x') \times \mathcal{M}_g(y,y') \big/ \mathbb{R}_{diag} \underset{\pi}{\overset{i}{\leftrightarrows}} \mathcal{M}_f(x,x')/\mathbb{R} \times \mathcal{M}_g(y,y')/\mathbb{R} = \mathcal{L}_f(x,x') \times \mathcal{L}_g(y,y')$$

which can be written $\pi([a,b]_{X\times Y}) = ([a]_X, [b]_Y)$ and $i(\lambda_X, \lambda_Y) = [i_X(\lambda_X), i_Y(\lambda_Y)]_{X\times Y}$, where $i_X(\lambda_X) = \lambda_X \cap f^{-1}\left(\frac{f(x)+f(x')}{2}\right)$ and $i_Y(\lambda_Y) = \lambda_Y \cap g^{-1}\left(\frac{g(y)+g(y')}{2}\right)$.
Remark that $\pi = \mathrm{Id}$ if $x = x'$ or $y = y'$.

Lemma 6.2. *Let $\{s^X_{x,x'}\}$ and $\{s^Y_{y,y'}\}$ be representing chain systems for the moduli spaces of trajectories $\overline{\mathcal{L}}_f(x,x')$ and $\overline{\mathcal{L}}_g(y,y')$ respectively. There exists a representing chain system $\{s_{(x,y),(x',y')}\}$ for the moduli spaces $\overline{\mathcal{L}}_H((x,y),(x',y'))$ such that*

1. $s_{(x,y),(x',y')} = (-1)^{|x|(|y|-|y'|)}(\{x\}, s^Y_{y,y'})$ *if* $x = x'$.
2. $s_{(x,y),(x',y')} = (s^X_{x,x'}, \{y\})$ *if* $y = y'$
3. $\pi_*(s_{(x,y),(x',y')}) = 0$ *if* $x \neq x'$ *and* $y \neq y'$.

Proof. • If $x = x'$ or $y = y'$ we can just choose $s_{(x,y),(x,y')} = (-1)^{|x|(|y|-|y'|)}(\{x\}, s^Y_{y,y'})$, $s_{(x,y),(x',y)} = (s^X_{x,x'}, \{y\})$ and complete by induction as in Proposition 5.6 of [BDHO24] in order to obtain $\{s_{(x,y),(x',y')}\}$ a representing chain system for $\overline{\mathcal{L}}_H((x,y),(x',y'))$ that satisfies conditions 1 and 2. We just have to check that

$$\partial s_{(x,y),(x,y')} = \sum_{w\in \mathrm{Crit}(g)} (-1)^{|y|-|w|} s_{(x,y),(x,w)} \times s_{(x,w),(x,y')}$$

and

$$\partial s_{(x,y),(x',y)} = \sum_{z\in \mathrm{Crit}(f)} (-1)^{|x|-|z|} s_{(x,y),(z,y)} \times s'_{(z,y),(x',y)}.$$

We only establish the first equality since the second one is analogous,

$$\begin{aligned}
\partial s_{(x,y),(x,y')} &= (-1)^{|x|(|y|-|y'|)}(\{x\}, \partial s_{y,y'}) \\
&= (-1)^{|x|(|y|-|y'|)} \sum_w (-1)^{|y|-|w|}(\{x\}, s_{y,w} \times s_{w,y'}) \\
&= \sum_w (-1)^{|y|-|w|}(-1)^{|x|(|y|-|w|)}(-1)^{|x|(|w|-|y'|)}(\{x\}, s_{y,w}) \times (\{x\}, s_{w,y'}) \\
&= \sum_w (-1)^{|y|-|w|} s_{(x,y),(x,w)} \times s_{(x,w),(x,y')}.
\end{aligned}$$

• If $x \neq x'$ and $y \neq y'$, we build $\{s_{(x,y),(x',y')}\}$ by induction on $|x| + |y| - |x'| - |y'| = l$. If $x \neq x'$, $y \neq y'$ and $l = 2$, then $|x| - |x'| = 1$ and $|y| - |y'| = 1$ or $\overline{\mathcal{L}}_H((x,y),(x',y'))$ is empty. Any representative $s_{(x,y),(x',y')} \in C_1(\overline{\mathcal{L}}_H((x,y),(x',y')))$ of the fundamental class satisfies condition 3. since $\pi_*(s_{(x,y),(x',y')}) \in C_1(\overline{\mathcal{L}}_f(x,x') \times \overline{\mathcal{L}}_g(y,y'))$ is a one chain in a 0-dimensional space and is thus constant and degenerate.

Suppose a representing chain system $s_{(a,b),(a',b')}$ respecting the conditions 1.,2. and 3. has been constructed for every $|a| + |b| - |a'| - |b'| \leq l$. Let $|x| + |y| - |x'| - |y'| = l + 1$ and $x \neq x'$, $y \neq y'$. Let $\{s'_{(x,y),(x',y')}\}$ be a representative of the fundamental class of $\overline{\mathcal{L}}_H((x,y),(x',y'))$ which satisfies (3). Then $\pi_*(s'_{(x,y),(x',y')}) \in C_l(\overline{\mathcal{L}}_f(x,x') \times \overline{\mathcal{L}}_g(y,y'))$ is a cycle. Indeed, using the induction hypothesis

$$\begin{aligned}
\partial \pi_*(s'_{(x,y),(x',y')}) &= \sum_{(z,w)} (-1)^{|x|+|y|-|z|-|w|} \pi_* s_{(x,y),(z,w)} \times \pi_* s_{(z,w),(x',y')} \\
&= (-1)^{|y|-|y'|} s_{(x,y),(x,y')} \times s_{(x,y'),(x',y')} + (-1)^{|x|-|x'|} s_{(x,y),(x',y)} \times s_{(x',y),(x',y')} \\
&= (-1)^{|x|+|x'|(1+|y|-|y'|)+1} \left((s_{x,x'}, s_{y,y'}) - (s_{x,x'}, s_{y,y'})\right) = 0.
\end{aligned}$$

Since $\overline{\mathcal{L}}_f(x,x') \times \overline{\mathcal{L}}_g(y,y')$ is a manifold of dimension $l-1$ every l-cycle is a boundary. Hence, there exists $b \in C_{l+1}(\overline{\mathcal{L}}_f(x,x') \times \overline{\mathcal{L}}_g(y,y'))$ such that $\partial b = \pi_*(s'_{(x,y),(x',y')})$. We then define $s_{(x,y),(x',y')} = s'_{(x,y),(x',y')} - \partial i_*(b)$, which satisfies condition 3. The resulting representing chain system $\{s_{(x,y),(x',y')}\}$ satisfies all the conditions.

■

6.1.3 Künneth twisting cocycle

We will evaluate the representing chain system constructed in Lemma 6.2 by a family of evaluation maps $q^K_{(x,y),(x',y')} : \overline{\mathcal{L}}((x,y),(x',y')) \to \Omega X \times \Omega Y$ adapted to the Cartesian product to obtain a twisting cocycle $m^K_{(x,y),(x',y')} = q^K_{(x,y),(x',y'),*}(s_{(x,y),(x',y')}) \in C_{|x|+|y|-|x'|-|y'|-1}(\Omega X \times \Omega Y))$. This twisting cocycle is not the Barraud-Cornea cocycle associated to the set of DG Morse data $\Xi_{X\times Y}$ but is well-adapted to chain-level computations and enables to define a chain-level cross product for enriched Morse complexes. We will prove in the next section that the enriched Morse complex constructed with the Barraud-Cornea cocycle is chain homotopy equivalent to the one constructed with m^K.

Lemma 6.3. *The family*

$$q^K_{(x,y),(x',y')} : \overline{\mathcal{L}}((x,y),(x',y')) \xrightarrow{\pi} \overline{\mathcal{L}}(x,x') \times \overline{\mathcal{L}}(y,y') \xrightarrow{(q^X_{x,x'}, q^Y_{y,y'})} \Omega X \times \Omega Y \ .$$

is a family of evaluation maps in the sense of Definition 2.6, i.e, *it satisfies the following two conditions*

1. *If $|x|+|y|-|x'|-|y'|=1$ and $\lambda \in \overline{\mathcal{L}}((x,y),(x',y')) = \mathcal{L}((x,y),(x',y'))$, the class $[q^K_{(x,y),(x',y')}(\lambda)] \in \pi_1(X\times Y)$ is the one associated to λ in the lifted Morse complex.*
2. *If $(\lambda,\lambda') \in \overline{\mathcal{L}}((x,y),(z,w)) \times \overline{\mathcal{L}}((z,w),(x',y')) \subset \partial\overline{\mathcal{L}}((x,y),(x',y'))$, then*

$$q^K_{(x,y),(x',y')}(\lambda,\lambda') = q^K_{(x,y),(z,w)}(\lambda) \# q^K_{(z,w),(x',y')}(\lambda').$$

Proof. Let $(x,y),(z,w),(x',y') \in \mathrm{Crit}(H)$ and $(\lambda,\lambda') \in \overline{\mathcal{L}}((x,y),(z,w)) \times \overline{\mathcal{L}}((z,w),(x',y'))$. If we denote $\pi_X = pr_1 \circ \pi : \overline{\mathcal{L}}((x,y),(x',y')) \to \overline{\mathcal{L}}(x,x')$ and $\pi_Y = pr_2 \circ \pi : \overline{\mathcal{L}}((x,y),(x',y')) \to \overline{\mathcal{L}}(y,y')$, then $\pi_X(\lambda\#\lambda') = (\pi_X(\lambda),\pi_X(\lambda'))$ and $\pi_Y(\lambda\#\lambda') = (\pi_Y(\lambda),\pi_Y(\lambda'))$. Condition 2 is then clear because q^X and q^Y satisfy such concatenation relations. Condition 1 is a consequence of the fact that, if $|x|+|y|-|x'|-|y'|=1$, then $x=x'$ or $y=y'$.

■

Definition 6.4. *The family of evaluation maps q^K yields a twisting cocycle $m^K_{(x,y),(x',y')} = q^K_{(x,y),(x',y'),*}(s_{(x,y),(x',y')}) \in C_{|x|+|y|-|x'|-|y'|-1}(\Omega X \times \Omega Y))$ that satisfies*

$$m^K_{(x,y),(x',y')} = \begin{cases} (-1)^{|x|(|y|-|y'|)}(\star, m_{y,y'}) & \text{if } x = x', \\ (m_{x,x'},\star) & \text{if } y = y', \\ 0 & \text{otherwise} \end{cases} .$$

This is the preferred twisting cocycle associated to the set of DG Morse data $\Xi_{X\times Y}$. We will refer to this cocycle as the ***Künneth twisting cocycle****.*

The evaluation map defining the Barraud-Cornea twisting cocycle $m^0_{(x,y),(x',y')}$ is

$$q^0_{(x,y),(x',y')} = \theta \circ p \circ \Gamma^{X\times Y}_{(x,y),(x',y')} : \overline{\mathcal{L}}((x,y),(x',y')) \to \Omega(X\times Y) \subset \Omega X \times \Omega Y,$$

where the parametrization map $\Gamma^{X\times Y}_{(x,y),(x',y')} : \overline{\mathcal{L}}((x,y),(x',y')) \to \mathcal{P}_{(x,y)\to(x',y')}X \times Y$ is defined by

$$\Gamma^{X\times Y}_{(x,y),(x',y')}(\lambda) : \begin{array}{ccc} [0, H(x,y) - H(x',y')] & \to & X\times Y \\ t & \mapsto & \lambda \cap H^{-1}(H(x,y)-t). \end{array}$$

Proposition 6.5. *For any right $\mathcal{A}_\infty$-module $\mathcal{H}$ over $C_*(\Omega X \times \Omega Y)$, there exists a chain homotopy equivalence*

$$C_*(X\times Y, m^0_{(x,y),(x',y')}, \mathcal{H}) \underset{\Psi_{0K}}{\overset{\Psi_{K0}}{\rightleftharpoons}} C_*(X\times Y, m^K_{(x,y),(x',y')}, \mathcal{H}).$$

6.2 Invariance with respect to the parametrization

The proof that the cocycles m^0 and m^K defined above induce the same homology will be given in the second subsection. We first prove that we can use any Morse function that admits the same pseudo-gradient ξ to parametrize the trajectories. The subsequent complexes will be chain homotopy equivalent.

6.2.1 Parametrizing Morse trajectories by different Morse functions

Let $\Xi = (f, \xi, s_{x,y}, o, \mathcal{Y}, \theta)$ be a set of DG Morse data on a pointed, oriented, closed and connected manifold $(X, \star)$ and let $h, h' : X \to \mathbb{R}$ be Morse functions that admit the vector field ξ as an adapted pseudo-gradient. We prove that the complex built using h for parametrizing the trajectories in the construction of the twisting cocycle (see Section 2.2) is chain homotopy equivalent to the complex built using the Morse function h'. This is equivalent to saying that the enriched complex does not depend on the Morse function used to parametrize the trajectories of a given adapted pseudo-gradient up to chain homotopy equivalence.
This does not prove Proposition 6.5 since there is no Morse function $h : X \times Y \to \mathbb{R}$ that corresponds to parametrizing independently on X and on Y, but this invariance is worth noticing and its proof will inspire the proof of Proposition 6.5.

Proposition 6.6. *Let $h, h' : X \to \mathbb{R}$ be two Morse functions and ξ a pseudo-gradient adapted to both h and h'. Let $\mathcal{F}$ be an $\mathcal{A}_\infty$-module over $C_*(\Omega X)$. Let $C_*(X, \Xi_{h'}, \mathcal{F})$ and $C_*(X, \Xi_h, \mathcal{F})$ be the two complexes built using respectively h' and h for parametrizing the trajectories in the definition of the evaluation maps.*
There exists a quasi-isomorphism $\Phi_{h,h'} : C_*(X, \Xi_h, \mathcal{F}) \to C_*(X, \Xi_{h'}, \mathcal{F})$.

Proof. We use the same construction as in the section 6.2 of [BDHO24] where the authors prove that the constant homotopy between a Morse function and itself induces a continuation map that is homotopic to the identity. Let

$$\pi : [0,1] \times X \to X$$

and

$$p : X \to X/\mathcal{Y}$$

be the canonical projections and let

$$\theta : X/\mathcal{Y} \to X$$

be a homotopy inverse of p.
We follow the proof of invariance for Morse theory [AD14, Section I.3.4] to build $a : [-\epsilon, 1+\epsilon] \to \mathbb{R}$ a strictly decreasing function on $[0,1]$ such that a is a Morse function and $\mathrm{Crit}(a) = \{0,1\}$. We can assume that a is decreasing enough for $F = f + a : [-\epsilon, 1+\epsilon] \times X \to \mathbb{R}$ to be a Morse function whose critical points are, for any $k \in \mathbb{N}$,

$$\mathrm{Crit}_k(F) = \{0\} \times \mathrm{Crit}_{k-1}(f) \cup \{1\} \times \mathrm{Crit}_k(f)$$

and let ξ_F be an adapted pseudo-gradient for F on $[0,1] \times X$.
If $x \in \mathrm{Crit}_k(f)$, we denote $x_0 \in \{0\} \times \mathrm{Crit}_k(f) \subset \mathrm{Crit}_{k+1}(F)$ and $x_1 \in \{1\} \times \mathrm{Crit}_k(f) \subset \mathrm{Crit}_k(F)$ the corresponding critical points of F. Let $\{s_{x,y}\}$ be a representing chain system for the moduli spaces of Morse trajectories. There exists a representing chain system $\{s^F_{x_i,y_j}\}$ for the moduli spaces of Morse trajectories in $[0,1] \times X$ such that for any $x, y \in \mathrm{Crit}(f)$:

- $s^F_{x_0,y_0} = (-1)^{|x|-|y|} s_{x,y} \in C_{|x|-|y|-1}(\overline{\mathcal{L}}(x_0, y_0))$.
- $s^F_{x_1,y_1} = s_{x,y} \in C_{|x|-|y|-1}(\overline{\mathcal{L}}(x_1, y_1))$.
- We denote $\sigma^{\mathrm{Id}}_{x,y} = s^F_{x_0,y_1} \in C_{|x|-|y|}(\overline{\mathcal{L}}(x_0, y_1))$.
- Since a strictly decreases between 0 and 1, then $\overline{\mathcal{L}}(x_1, y_0) = \emptyset$.

Define now an evaluation map $q^{h,h'}_{x_i,y_j} : \overline{\mathcal{L}}(x_i, y_j) \to \Omega X$ such that

- $q^{h,h'}_{x_0,y_0} := q^0_{x,y} := \theta \circ p \circ \pi \circ \Gamma^h_{x_0,y_0}$ where $\Gamma^h_{x_0,y_0} : \overline{\mathcal{L}}(x_0, y_0) \to \mathcal{P}_{x_0 \to y_0}(\{0\} \times X)$ parametrizes the trajectory using the Morse function h. This is the evaluation map that arises from the data Ξ_h and therefore $q^{h,h'}_{x_0,y_0,*}(s_{x,y}) = m^h_{x,y}$ is the twisting cocycle defining $C_*(X, \Xi_h, \mathcal{F})$.

— $q^{h,h'}_{x_1,y_1} := q^1_{x,y} := \theta \circ p \circ \pi \circ \Gamma^{h'}_{x_1,y_1}$ where $\Gamma^{h'}_{x_1,y_1} : \overline{\mathcal{L}}(x_1, y_1) \to \mathcal{P}_{x_1 \to y_1}(\{1\} \times X)$ parametrizes the trajectory using the Morse function h'. This is the evaluation map that arises from the data $\Xi_{h'}$ and therefore $q^{h,h'}_{x_1,y_1,*}(s_{x,y}) = m^{h'}_{x,y}$ is the twisting cocycle defining $C_*(X, \Xi_{h'}, \mathcal{F})$.

It remains to define $q^{h,h'}_{x_0,y_1}$. Let $\lambda \in \overline{\mathcal{L}}(x_0, y_1)$. Since a is strictly decreasing between 0 and 1, the trajectory λ transversely intersects $\{\frac{1}{2}\} \times X$ exactly once. Denote $z \in \lambda \cap X \times \{\frac{1}{2}\}$. We define

$$\Gamma^{h,h'}_{x_0,y_1} : \overline{\mathcal{L}}(x_0, y_1) \to \mathcal{P}_{x_0 \to y_1}([0,1] \times X)$$

by parametrizing the piece of λ going from x_0 to z using h and then parametrizing the other piece of λ going from z to y_1 with h'.

Let us reformulate this idea with some formulas:
We assume w.l.o.g that $A = a(0) > 0$ and $a(1) = 0$.
Define

$$\Gamma^h_{x_0,y_1}(\lambda) : [0, A + h(x) - h(y)] \to [0,1] \times X,$$
$$\Gamma^h_{x_0,y_1}(\lambda)(t) = (h + a)^{-1}(A + h(x) - t)$$

and

$$\Gamma^{h'}_{x_0,y_1}(\lambda) : [0, A + h(x) - h(y)] \to X \times [0,1],$$
$$\Gamma^{h'}_{x_0,y_1}(\lambda)(t) = (h' + a)^{-1}(A + h'(x) - t).$$

There exists a unique $t_h \in [0, A + h(x) - h(y)]$ such that $\Gamma^h_{x_0,y_1}(\lambda) \in X \times \{1/2\}$ and a unique $t_{h'} \in [0, A + h'(x) - h'(y)]$ such that $\Gamma^{h'}_{x_0,y_1}(\lambda) \in X \times \{1/2\}$.
Let,

$$\Gamma^{h,h'}_{x_0,y_1}(\lambda) : [0, A + h'(x) - h'(y) + t_h - t_{h'}] \to X \times [0,1],$$
$$\Gamma^{h,h'}_{x_0,y_1}(\lambda)(t) = \begin{cases} \Gamma^h_{x_0,y_1}(\lambda)(t) & \text{if } t \le t_h \\ \Gamma^{h'}_{x_0,y_1}(\lambda)(t - t_h + t'_h) & \text{if } t \ge t_h \end{cases}.$$

It is clear from the definition that these evaluation maps satisfy :

1. <u>The concatenation relations</u> :
   — for any $(\lambda, \lambda') \in \overline{\mathcal{L}}(x_0, z_0) \times \overline{\mathcal{L}}(z_0, y_1) \subset \partial\overline{\mathcal{L}}(x_0, y_1)$, $q^{h,h'}_{x_0,y_1}(\lambda, \lambda') = q^0_{x,z}(\lambda) \# q^{h,h'}_{z_0,y_1}(\lambda')$.
   — for any $(\lambda, \lambda') \in \overline{\mathcal{L}}(x_0, z_1) \times \overline{\mathcal{L}}(z_1, y_1) \subset \partial\overline{\mathcal{L}}(x_0, y_1)$, $q^{h,h'}_{x_0,y_1}(\lambda, \lambda') = q^{h,h'}_{x_0,z_1}(\lambda) \# q^1_{z,y}(\lambda')$.
2. <u>Compatibility with the lifted Morse complex</u> : Let $\{q^F_{x_0,y_1} : \overline{\mathcal{L}}(x_0, y_1) \to \Omega X, x, y \in \mathrm{Crit}(f)\}$ be the family of evaluation maps defined by parametrizing by the values of F. Since it is only the parametrization of the path that changes between the evaluation maps $q^{h,h'}$ and q^F, then $q^{h,h'}_{x_0,y_1}$ also satisfies the condition that, if x and y have the same Morse index, the homotopy class $[q^{h,h'}_{x_0,y_1}(\lambda)] \in \pi_1(X)$ is the same as the one associated to $\lambda \in \overline{\mathcal{L}}(x_0, y_1)$ in the lifted Morse complex $\tilde{C}_*(F, \xi')$ (see Lemma 6.2.4 of [BDHO24]).

For any $x, y \in \mathrm{Crit}(f)$, define $\nu^{h,h'}_{x,y} = -q^{h,h'}_{x_0,y_1,*}(\sigma^{\mathrm{Id}}_{x,y}) \in C_{|x|-|y|}(\Omega X)$.
Since $\sigma^{\mathrm{Id}}_{x,y}$ satisfies the relation $\partial\sigma^{\mathrm{Id}}_{x,y} = \sum_z s_{x,z}\sigma^{\mathrm{Id}}_{z,y} - \sum_w (-1)^{|x|-|w|}\sigma^{\mathrm{Id}}_{x,w}s_{w,y}$, we have

$$\partial\nu^{h,h'}_{x,y} = \sum_z m^h_{x,z}\nu^{h,h'}_{z,y} - \sum_w (-1)^{|x|-|w|}\nu^{h,h'}_{x,w}m^{h'}_{w,y}.$$

The quasi-isomorphism criterion 4.14 implies that $\{\nu^{h,h'}_{x,y}\}$ yields a morphism of complexes $\Phi_{h,h'} : C_*(X, \Xi_h, \mathcal{F}) \to C_*(X, \Xi_{h'}, \mathcal{F})$ which is a quasi-isomorphism. Indeed, the map between the associated lifted Morse complexes is the continuation map for the lifted Morse complexes (as in the proof [BDHO24, Proposition 6.2.2]) which is therefore a quasi-isomorphism.

■

In fact more is true :

Proposition 6.7. *The map $\Phi_{h,h'} : C_*(X, \Xi_h, \mathcal{F}) \to C_*(X, \Xi_{h'}, \mathcal{F})$ is a chain homotopy equivalence.*

Proof. We prove here that $\Phi_{h,h'} \circ \Phi_{h',h} : C_*(X, \Xi_h, \mathcal{F}) \to C_*(X, \Xi_h, \mathcal{F})$ is homotopic to the composition

$$C_*(X, \Xi_h, \mathcal{F}) \overset{\mathrm{Id}_*}{\to} C_*(X, \Xi_{h'}, \mathcal{F}) \overset{\mathrm{Id}_*}{\to} C_*(X, \Xi_h, \mathcal{F}).$$

The last map being chain homotopic to the identity, the proposition follows.
We will use the notations introduced in the previous proof and follow the idea of the proof of [BDHO24, Proposition 6.3.8] to construct a homotopy cocycle (see Proposition 4.11).
Define $K = f + a_t + a_\tau : [0,1]^2 \times X \to \mathbb{R}$ by $K(t, \tau, x) = f(x) + a(t) + a(\tau)$. This is a Morse function whose critical points are

$$\mathrm{Crit}_{k+1}(K) = \{(0,0)\} \times \mathrm{Crit}_{k-1}(f) \cup \{(0,1)\} \times \mathrm{Crit}_k(f) \cup \{(1,0)\} \times \mathrm{Crit}_k(f) \cup \{(1,1)\} \times \mathrm{Crit}_{k+1}(f).$$

For each critical point $x \in \mathrm{Crit}(K)$, we use the notation x_{ij} if $x \in \{(i,j)\} \times \mathrm{Crit}(f)$ for $i, j \in \{0,1\}$.
Following the proof of [BDHO24, Lemma 6.3.9], we can construct a representing chain system $(s^K_{x,y})$ for the moduli spaces of trajectories $\overline{\mathcal{L}}_K(x,y)$ such that, if we denote $S_{x,y} = s^K_{x_{00},y_{11}} \in C_{|x|-|y|+1}(\overline{\mathcal{L}}_K(x_{00}, y_{11}))$,

$$\begin{aligned}\partial S_{x,y} = &\sum_{z_{00}} (-1)^{|x|-|y|} s^f_{x,z} \cdot S_{z,y} + \sum_{w_{11}} (-1)^{|x|-|w|+2} S_{x,w} \cdot s^f_{w,y} \\ &- \sum_{u_{10}} \sigma^{Id}_{x,u} \cdot \sigma^{Id}_{u,y} + \sum_v \sigma^{Id}_{x,v} \cdot \sigma^{Id}_{v,y}.\end{aligned}$$

We now want to evaluate these chains into $C_*(\Omega X)$.
Define now for each $x, y \in \mathrm{Crit}(K)$ a parametrization map $\Gamma_{x,y} : \overline{\mathcal{L}}_K(x,y) \to \mathcal{P}_{x \to y}([0,1]^2 \times X)$. Let $\lambda \in \overline{\mathcal{L}}_K(x,y)$ and $C = [1/2, 1] \times [0, 1/2] \times X$. The trajectory λ intersects C transversely since a is strictly decreasing and $\Gamma_{x,y}(\lambda)$ is the path parametrizing λ by h outside of C and by h' inside C (see the following figure).

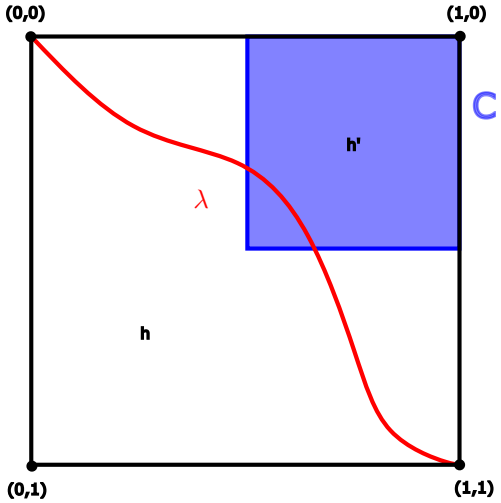


Figure 3 - Parametrization map $\Gamma_{x,y}$

We define for each $x, y \in \mathrm{Crit}(K)$ the evaluation maps

$$Q_{x,y} = \pi \circ (Id_{[0,1]^2} \times \theta) \circ (Id_{[0,1]^2} \times p) \circ \Gamma_{x,y} : \overline{\mathcal{L}}_K(x,y) \to \Omega X,$$

where $\pi : [0,1]^2 \times X \to X$ is the canonical projection. In particular, if $\lambda \in \overline{\mathcal{L}}_K(x_{00}, y_{10})$ and $\lambda' \in \overline{\mathcal{L}}_K(x_{10}, y_{11})$, then $Q_{x,y}(\lambda) = q^{h,h'}_{x,y}(\lambda)$ and $Q_{x,y}(\lambda') = q^{h',h}_{x,y}(\lambda')$.

These evaluation maps again satisfy the concatenation relations and compatibility with the lifted Morse complex.
Denote $H_{x,y} = Q_{x,y,*}(S_{x,y}) \in C_{|x|-|y|+1}(\Omega X)$,

$$\begin{aligned}\partial H_{x,y} = &\sum_z (-1)^{|x|-|z|} m^h_{x,z} \cdot H_{z,y} + \sum_w (-1)^{|x|-|w|} H_{x,w} \cdot m^h_{w,y} \\ &- \sum_u \nu^{h,h'}_{x,u} \cdot \nu^{h',h}_{u,y} + \sum_v \nu^{h,h}_{x,v} \cdot \nu^{h,h}_{v,y}.\end{aligned}$$

Using Proposition 4.11, this proves that the map

$$H : \begin{array}{ccc} C_*(X,\Xi_h,\mathcal{F}) & \to & C_{*+1}(X,\Xi_h,\mathcal{F}) \\ \alpha \otimes x & \mapsto & \sum_y \alpha \cdot H_{x,y} \otimes y \end{array}$$

is a chain homotopy between the map

$$\Phi_{h',h} \circ \Phi_{h,h'} : \begin{array}{ccc} C_*(X,\Xi_h,\mathcal{F}) & \to & C_*(X,\Xi_h,\mathcal{F}) \\ \alpha \otimes x & \mapsto & \sum_y \alpha \cdot \left(\sum_u \nu^{h,h'}_{x,u} \cdot \nu^{h',h}_{u,y} \right) \otimes y \end{array}$$

and the map

$$\mathrm{Id}_* \circ \mathrm{Id}_* : \begin{array}{ccc} C_*(X,\Xi_h,\mathcal{F}) & \to & C_*(X,\Xi_h,\mathcal{F}) \\ \alpha \otimes x & \mapsto & \sum_y \alpha \cdot \left(\sum_v \nu^{h,h}_{x,v} \cdot \nu^{h,h}_{v,y} \right) \otimes y \end{array} .$$

■

6.2.2 Parametrizing independently on a Cartesian product

Let $f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$ and $H = f + g : X \times Y \to \mathbb{R}$ be Morse functions.
Let $\{s_{z,w} \in C_{|z|-|w|-1}(\overline{\mathcal{L}}_H(z,w)),\ z,w \in \mathrm{Crit}(H)\}$ be the twisting cocycle constructed in Section 6.1.
Recall the notation of the previous section : $m^K_{z,w} = q^K_{z,w,*}(s_{z,w})$ is the Künneth twisting cocycle from Definition 6.4 and $m^0_{z,w} = q^0_{z,w,*}(s_{z,w})$ is the Barraud-Cornea twisting cocycle associated to the DG Morse data set $\Xi_{X\times Y}$.

Proof of Proposition 6.5. Proceed as in the proof of the previous proposition to define $W : [0,1]\times X\times Y \to \mathbb{R}$ a Morse function such that $\mathrm{Crit}_k(W) = \{0\} \times \mathrm{Crit}_{k-1}(H) \cup \{1\} \times \mathrm{Crit}_k(H)$, a pseudo-gradient ξ_W and a representing chain system $\{s^W_{z_i,w_j},\ z,w \in \mathrm{Crit}(H),\ i,j \in \{0,1\}\}$ such that

— $s^W_{z_0,w_0} = (-1)^{|z|-|w|} s_{z,w} \in C_{|z|-|w|-1}(\overline{\mathcal{L}}_W(z_0,w_0))$.
— $s^W_{z_1,w_1} = s_{z,w} \in C_{|z|-|w|-1}(\overline{\mathcal{L}}_W(z_1,w_1))$.
— We denote $\sigma_{z,w} = s^W_{z_0,w_1} \in C_{|z|-|w|}(\overline{\mathcal{L}}(z_0,w_1))$.

Denote $\pi^{X\times Y} : [0,1]\times X\times Y \to X\times Y$, $\pi^X : [0,1]\times X \to X$ and $\pi^Y : [0,1]\times Y \to Y$ the canonical projections. We can assume w.l.o.g that $F = f + a : [0,1]\times X \to \mathbb{R}$ and $G = g + a : [0,1]\times Y \to \mathbb{R}$ are Morse functions. Denote $\xi_F = \xi_f - a' \frac{\partial}{\partial t}$ and $\xi_G = \xi_g - a' \frac{\partial}{\partial t}$ adapted pseudo-gradients for F and G respectively.
There is an inclusion

$$I : \mathcal{M}_W((x,y)_0,(x',y')_1) \to \mathcal{M}_F(x_0,x_1) \times \mathcal{M}_G(y_0,y_1), \quad I(\tau,(a,b)) = ((\tau,a),(\tau,b))$$

which induces a map

$$J = (J_X, J_Y) : \overline{\mathcal{L}}_W((x,y)_0,(x',y')_1) \to \overline{\mathcal{L}}_F(x_0,x'_1) \times \overline{\mathcal{L}}_G(y_0,y'_1).$$

Define $\Gamma^K_{(x,y)_0,(x',y')_1} : \overline{\mathcal{L}}_W((x,y)_0,(x',y')_1) \to \mathcal{P}_{x_0\to x'_1}([0,1]\times X) \times \mathcal{P}_{y_0\to y'_1}([0,1]\times Y)$ by

$$\Gamma^K_{(x,y)_0,(x',y')_1}(\lambda) = \left(\Gamma^F_{x_0,x'_1} \circ J_X(\lambda), \Gamma^G_{y_0,y'_1} \circ J_Y(\lambda) \right),$$

where $\Gamma^F_{x_0,x_1'} : \overline{\mathcal{L}}_F(x_0,x_1) \to \mathcal{P}_{x_0\to x_1'}([0,1]\times X)$ and $\Gamma^G_{y_0,y_1'} : \overline{\mathcal{L}}_G(y_0,y_1') \to \mathcal{P}_{y_0\to y_1'}([0,1]\times Y)$ are the parametrization maps by the values of the Morse functions F and G respectively. They are defined in Section 2.2.

Denote

$$\Gamma^0_{(x,y)_0,(x',y')_1} : \overline{\mathcal{L}}_W((x,y)_0,(x',y')_1) \to \mathcal{P}_{x_0\to x_1'}([0,1]\times X)\times\mathcal{P}_{y_0\to y_1'}([0,1]\times Y)$$

the composition of the map

$$\Gamma^W_{(x,y)_0,(x',y')_1} : \overline{\mathcal{L}}_W((x,y)_0,(x',y')_1) \to \mathcal{P}_{(x,y)_0\to(x',y')_1}([0,1]\times X\times Y)$$

which parametrizes by the values of W and the inclusion

$$\mathcal{P}_{(x,y)_0\to(x',y')_1}([0,1]\times X\times Y) \hookrightarrow \mathcal{P}_{x_0\to x_0'}([0,1]\times X)\times\mathcal{P}_{y_0\to y_1'}([0,1]\times Y).$$

Now define

$$\Gamma^{0K}_{(x,y)_0,(x',y')_1} : \overline{\mathcal{L}}_W((x,y)_0,(x',y')_1) \to \mathcal{P}_{x_0\to x_1'}([0,1]\times X)\times\mathcal{P}_{y\to y'}([0,1]\times Y)$$

the parametrization map that shifts the parametrization from Γ^0 to Γ^K when the trajectory crosses $X\times Y\times\{\frac{1}{2}\}$.

Lemma 6.8. *The family $q^{0K}_{(x,y)_0,(x',y')_1} = \theta\circ p\circ(\pi^X,\pi^Y)\circ\Gamma^{0K}_{(x,y)_0,(x',y')_1} : \overline{\mathcal{L}}((x,y)_0,(x',y')_1)\to\Omega X\times\Omega Y$ is a family of evaluation maps, i.e, it satifies*

1. *<u>The concatenation relations</u> : For any $(\lambda,\lambda')\in\overline{\mathcal{L}}((x,y)_0,(z,w)_0)\times\overline{\mathcal{L}}((z,w)_0,(x',y')_1)$,*

$$q^{0K}_{(x,y)_0,(x',y')_1}(\lambda,\lambda') = q^0_{(x,y),(z,w)}(\lambda)\#q^{0K}_{(z,w)_0,(x',y')_1}(\lambda')$$

and for any $(\lambda,\lambda')\in\overline{\mathcal{L}}((x,y)_0,(z,w)_1)\times\overline{\mathcal{L}}((z,w)_1,(x',y')_1)$,

$$q^{0K}_{(x,y)_0,(x',y')_1}(\lambda,\lambda') = q^{0K}_{(x,y)_0,(z,w)_1}(\lambda)\#q^K_{(z,w)_1,(x',y')_1}(\lambda').$$

2. *<u>The compatibility with the lifted Morse complex</u> : if (x,y) and (x',y') have the same index, then the homotopy class $[q^{0K}_{(x,y)_0,(x',y')_1}(\lambda)]\in\pi_1(X\times Y)$ is the same as the one associated to $\lambda\in\overline{\mathcal{L}}((x,y)_0,(x',y')_1)$ in the lifted Morse complex of $X\times Y\times[0,1]$.*

Proof. Condition 1. is clearly satisfied.
Condition 2. is satisfied because, even if $\Gamma^{0K}_{(x,y)_0,(x',y')_1}(\lambda)$ and $\Gamma^0_{(x,y)_0,(x',y')_1}(\lambda)$ do not have the same image, they are homotopic. Indeed, if we denote $(\gamma^{0K}_X,\gamma^{0K}_Y) = \Gamma^{0K}_{(x,y)_0,(x',y')_1}(\lambda)$ and $(\gamma^0_X,\gamma^0_Y) = \Gamma^0_{(x,y)_0,(x',y')_1}(\lambda)$, then $\gamma^{0K}_X = \gamma^0_X$ and $\gamma^{0K}_Y = \gamma^0_Y$ up to reparametrization but $(\gamma^{0K}_X,\gamma^{0K}_Y)$ and (γ^0_X,γ^0_Y) do not have the same image. However, it is enough to reparametrize γ^{0K}_Y in γ'^{0K}_Y so that $(\gamma^{0K}_X,\gamma'^{0K}_Y) = (\gamma^0_X,\gamma^0_Y)$ up to parametrization. Therefore, since q^0 satisfies the condition 2, then so does q^{0K}.
■

From the concatenation relations, it follows that

$$\nu^{0K}_{(x,y),(x',y')} = q^{0K}_{(x,y)_0,(x',y')_1,*}(\sigma_{(x,y),(x',y')}) \in C_{|x|+|y|-|x'|-|y'|}(\Omega X\times\Omega Y)$$

satisfies equation (8)

$$\partial\nu^{0K}_{(x,y),(x',y')} = \sum_{z,w} m^0_{(x,y),(z,w)}\nu^{0K}_{(z,w),(x',y')} - \sum_{z',w'}(-1)^{|x|+|y|-|z'|-|w'|}\nu^{0K}_{(x,y),(z',w')}m^K_{(z',w'),(x',y')}.$$

This concludes the proof because the compatibility with the lifted Morse complex ensures that the continuation cocycle satisfies the quasi-isomorphism criterion 4.14.
Therefore

$$\Psi_{0K}: \begin{array}{ccc} C_*(X\times Y,\Xi_{X\times Y},\mathcal{H}) & \to & C_*(X\times Y,m^K,\mathcal{H}) \\ \alpha\otimes(x,y) & \mapsto & \sum_{(x',y')} \alpha\cdot\nu^{0K}_{(x,y),(x',y')}\otimes(x',y') \end{array}$$

is a quasi-isomorphism for any right DG $C_*(\Omega X\times\Omega Y)$-module $\mathcal{H}$.

■

In fact more is true :

Proposition 6.9. *The map Ψ_{0K} is a chain homotopy equivalence.*

Proof. We will use the notation introduced in the previous proof.
We can define for any $x,x'\in \mathrm{Crit}(f)$, $y,y'\in\mathrm{Crit}(g)$ the parametrization map

$$\Gamma^{K0}_{(x,y),(x',y')}:\overline{\mathcal{L}}_W((x,y)_0,(x',y')_1)\to\mathcal{P}_{x_0\to x'_1}([0,1]\times X)\times\mathcal{P}_{y_0\to y'_1}([0,1]\times Y)$$

that shifts the parametrization from Γ^K to Γ^0 when λ crosses $\{\frac{1}{2}\}\times X\times Y$.

Let us describe this map in more detail. Let $\lambda\in\overline{\mathcal{L}}_W((x,y)_0,(x',y')_1)$ and let $(\frac{1}{2},z_X,z_Y)\in\{\frac{1}{2}\}\times X\times Y$ be the point where λ crosses transversely $\{\frac{1}{2}\}\times X\times Y$. The paths $(\gamma^K_X,\gamma^K_Y):=\Gamma^K_{(x,y)_0,(x',y')_1}(\lambda)$ can be restricted to paths that we still denote

$$(\gamma^K_X,\gamma^K_Y)\in\mathcal{P}_{x_0\to(\frac{1}{2},z_X)}([0,1]\times X)\times\mathcal{P}_{y_0\to(\frac{1}{2},z_Y)}([0,1]\times Y).$$

Denote

$$(\gamma^0_X,\gamma^0_Y)\in\mathcal{P}_{(\frac{1}{2},z_X)\to x'_1}([0,1]\times X)\times\mathcal{P}_{(\frac{1}{2},z_Y)\to y'_1}([0,1]\times Y),$$

the paths obtained by parametrizing the half-infinite gradient line starting at $(\frac{1}{2},z_X,z_Y)$ using Γ^0, or equivalently, obtained by restricting $\Gamma^W_{(x,y)_0,(x',y')_1}(\lambda)$ to start at $(\frac{1}{2},z_X,z_Y)$ and applying the inclusion $\mathcal{P}_{(\frac{1}{2},z_X,z_Y)\to(x',y')_1}([0,1]\times X\times Y)\hookrightarrow\mathcal{P}_{(\frac{1}{2},z_X)\to x'_0}([0,1]\times X)\times\mathcal{P}_{(\frac{1}{2},z_Y)\to y'_1}([0,1]\times Y)$.
We define

$$\Gamma^{K0}_{(x,y),(x',y')}(\lambda)=(\gamma^K_X\cdot\gamma^0_X,\gamma^K_Y\cdot\gamma^0_Y)\in\mathcal{P}_{x_0\to x'_1}([0,1]\times X)\times\mathcal{P}_{y_0\to y'_1}([0,1]\times Y).$$

We can therefore construct $\nu^{K0}_{(x,y),(x',y')}\in C_{|x|+|y|-|x'|-|y'|}(\Omega X\times\Omega Y)$ such that

$$\partial\nu^{K0}_{(x,y),(x',y')}=\sum_{z,w}m^K_{(x,y),(z,w)}\nu^{K0}_{(z,w),(x',y')}-\sum_{z',w'}(-1)^{|x|+|y|-|z'|-|w'|}\nu^{K0}_{(x,y),(z',w')}m^0_{(z',w'),(x',y')}$$

and obtain a quasi-isomorphism

$$\Psi_{K0}: \begin{array}{ccc} C_*(X\times Y,m^K,\mathcal{H}) & \to & C_*(X\times Y,\Xi_{X\times Y},\mathcal{H}) \\ \alpha\otimes(x,y) & \mapsto & \sum_{(x',y')} \alpha\cdot\nu^{K0}_{(x,y),(x',y')}\otimes(x',y') \end{array}$$

for any right $\mathcal{A}_\infty$-module $\mathcal{H}$ over $C_*(\Omega X\times\Omega Y)$.

From there, the same proof as Proposition 6.7 shows that $\Psi_{K0}\circ\Psi_{0K}\simeq\mathrm{Id}$ and $\Psi_{0K}\circ\Psi_{K0}\simeq\mathrm{Id}$.

■

In view of Proposition 6.9, we may and will assume that the complex $C_*(X\times Y,\Xi_{X\times Y},\mathcal{H})$ is constructed using the Künneth twisting cocycle for any right $\mathcal{A}_\infty$-module $\mathcal{H}$ over $C_*(\Omega X\times\Omega Y)$.

6.3 Künneth formula. Proof of Theorem C

The goal of this section is to define an analogue of the Künneth formula for enriched Morse homology using the twisting cocycle defined above. In Section 6.1 and 6.2 we dealt with how sets of DG Morse data Ξ_X and Ξ_Y on X and Y induce a preferred set of DG Morse data $\Xi_{X\times Y}$ on $X\times Y$ and how to construct the preferred Künneth twisting cocycle on $X\times Y$ computing the same homology as the Barraud-Cornea twisting cocycle. It remains to understand how DG systems $\mathcal{F}$ over $C_*(\Omega X)$ and $\mathcal{G}$ over $C_*(\Omega Y)$ induce a DG system over $C_*(\Omega(X\times Y))$.

6.3.1 The cross products K^{alg} and K^{top}

We will start by an algebraic construction $\mathcal{F} \otimes_{\mathbb{Z}} \mathcal{G}$ which can be stated for any DG systems $\mathcal{F}$ and $\mathcal{G}$.

Definition 6.10. *Let $\mathcal{F}$ be a right $C_*(\Omega X)$-module and $\mathcal{G}$ be a right $C_*(\Omega Y)$-module. The tensor product $\mathcal{F} \otimes_{\mathbb{Z}} \mathcal{G}$ has a natural $C_*(\Omega X) \otimes C_*(\Omega Y)$-module structure $(\alpha \otimes \beta) \cdot (\gamma_X \otimes \gamma_Y) = (-1)^{|\beta||\gamma_X|}(\alpha \cdot \gamma_X \otimes \beta \cdot \gamma_Y)$. Consider the Serre diagonal $\Delta : C_*(\Omega X \times \Omega Y) \to C_*(\Omega X) \otimes C_*(\Omega Y)$, i.e the cubic equivalent to the Alexander-Whitney diagonal for singular complexes (see [KS05, Section 2.4] and [Ser51]).*
This can be used to transfer the natural module structure of $\mathcal{F} \otimes \mathcal{G}$ over $C_*(\Omega X) \otimes C_*(\Omega Y)$ described above to a module structure over $C_*(\Omega(X \times Y)) \hookrightarrow C_*(\Omega X \times \Omega Y)$:

$$(\alpha \otimes \beta) \cdot (\omega_X, \omega_Y) := (\alpha \otimes \beta) \cdot \Delta(\omega_X, \omega_Y).$$

On the other hand, if $F \hookrightarrow E \to X$ and $G \hookrightarrow E' \to Y$ are two fibrations with respective transitive functions Φ_X and Φ_Y, then the fibration $F \times G \hookrightarrow E \times E' \to X \times Y$ is naturally endowed with the transitive lifting function (Φ_X, Φ_Y). Therefore, to study fibrations, if $\mathcal{F} = C_*(F)$ and $\mathcal{G} = C_*(G)$, it is natural to consider the complex $C_*(X \times Y, C_*(F \times G))$ where the module structure is defined by (Φ_X, Φ_Y). The main goal of this paper is to define and study the DG Chas-Sullivan product, and this heavily uses considerations on fibrations. Therefore we will prefer the DG system $C_*(F \times G)$ in this case so that we can use morphisms of fibrations to study this product.

Definition 6.11. *Let $\mathcal{F} = C_*(F)$ and $\mathcal{G} = C_*(G)$ be the cubical complexes of some topological spaces F and G, where F has an associative topological module structure $F \times \Omega X \to F$ and G has an associative topological module structure $G \times \Omega Y \to G$. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the canonical projections. For a loop $\gamma \in \Omega(X \times Y)$, we denote $\gamma_X = \pi_X(\gamma)$ and $\gamma_Y = \pi_Y(\gamma)$.*

For any topological space A, B, we use the notation $\mathrm{EZ} : C_*(A) \otimes C_*(B) \to C_*(A \times B)$ for the Eilenberg-Zilber map $\mathrm{EZ}(a \otimes b) = (a, b)$ on cubic complexes.
Topologically, the right $\Omega(X \times Y)$-module $F \times G$ is defined by

$$\begin{array}{ccccc} F \times G \times \Omega(X \times Y) & \to & F \times \Omega X \times G \times \Omega Y & \to & F \times G \\ (\alpha, \beta, \gamma) & \mapsto & (\alpha, \gamma_X, \beta, \gamma_Y) & \mapsto & (\alpha.\gamma_X, \beta.\gamma_Y). \end{array}$$

This gives rise to a module structure

$$C_*(F \times G) \otimes C_*(\Omega(X \times Y)) \overset{\mathrm{EZ}}{\to} C_*(F \times G \times \Omega(X \times Y)) \to C_*(F \times \Omega X \times G \times \Omega Y) \to C_*(F \times G).$$

Remark 6.12. ***1.*** *Let F and G be fibers of fibrations $F \hookrightarrow E_X \to X$ and $G \hookrightarrow E_Y \to Y$ equipped with transitive lifting functions*

$$\Phi_X : E_X \ {}_{\pi}\times_{ev_0} \mathcal{P}X, \text{ and } \Phi_Y : E_Y \ {}_{\pi}\times_{ev_0} \mathcal{P}Y.$$

The module structure described above corresponds to the one that arises from the natural transitive lifting function

$$\Phi_{X \times Y} : \begin{array}{ccc} E_X \times E_Y \ {}_{(\pi_X, \pi_Y)}\times_{ev_0} \mathcal{P}(X \times Y) & \to & E_X \times E_Y \\ (\alpha, \beta), (\gamma, \sigma) & \mapsto & (\Phi_X(\alpha, \gamma), \Phi_Y(\beta, \sigma)) \end{array}$$

of the fibration $F \times G \hookrightarrow E_X \times E_Y \to X \times Y$.

2. *The general definition of this module structure is better understood at the topological level. Indeed, we cannot really write explicit formulas for every chain in $C_*(\Omega(X \times Y))$ due to the fact that the cubical chains on a Cartesian product need not have split parametrizations. Therefore, at the chain level, a sign would appear to keep track of the Kozsul signs corresponding to the switch of parameters.*

However, if $\gamma = \mathrm{EZ}(\gamma_1 \otimes \gamma_2) = (\gamma_1, \gamma_2) \in C_*(\Omega(X \times Y))$, then for every $(\alpha, \beta) = \mathrm{EZ}(\alpha \otimes \beta) \in C_*(F \times G)$,

$$(\alpha, \beta).\gamma = (-1)^{|\beta||\gamma_1|}(\alpha.\gamma_1, \beta.\gamma_2). \tag{22}$$

*In this case, we will say that $\gamma \in C_*(\Omega(X \times Y))$ and $(\alpha, \beta) \in C_*(F \times G)$ are **split**. Remark that if $\{m^X_{x,x'} \in C_{|x|-|x'|-1}(\Omega X)\}$ and $\{m^Y_{y,y'} \in C_{|y|-|y'|-1}(\Omega Y)\}$ are Barraud-Cornea cocycles respectively on X*

and Y, the Künneth twisting cocycle $\{m^K_{(x,y),(x',y')} \in C_{|x|+|y|-|x'|-|y'|-1}(\Omega(X\times Y))\}$ on $X\times Y$ is a family of split chains.

We now restate and prove Theorem C.

Theorem 6.13. *Let Ξ_X, Ξ_Y be sets of DG Morse data on X and Y with respective Morse functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$.*

i) The Künneth twisting cocycle $\{m^K_{z,w} \in C_{|z|-|w|-1}(\Omega X \times \Omega Y),\ z, w \in \mathrm{Crit}(f+g)\}$ associated to Ξ_X and Ξ_Y computes the same homology as the Barraud-Cornea cocycle associated to the set of DG data $\Xi_{X\times Y}$ constructed in Section 6.1.

ii) Let $\mathcal{F}$ and $\mathcal{G}$ be DG modules over $C_*(\Omega X)$ and $C_*(\Omega Y)$. Then,

$$K^{alg} : \begin{array}{ccc} C_*(X,\Xi_X,\mathcal{F}) \otimes C_*(Y,\Xi_Y,\mathcal{G}) & \to & C_*(X\times Y, m^K_{z,w}, \mathcal{F}\otimes\mathcal{G}) \\ (\alpha\otimes x)\otimes(\beta\otimes y) & \mapsto & (-1)^{|\beta||x|}(\alpha\otimes\beta)\otimes(x,y) \end{array}$$

is an isomorphism of complexes.

iii) If $C_*(F)$ and $C_*(G)$ are complexes of cubical chains of topological spaces equipped with topological module structures respectively over ΩX and ΩY, then

$$K^{top} : \begin{array}{ccc} C_*(X,\Xi_X,C_*(F)) \otimes C_*(Y,\Xi_Y,C_*(G)) & \to & C_*(X\times Y, m^K_{z,w}, C_*(F\times G)) \\ (\alpha\otimes x)\otimes(\beta\otimes y) & \mapsto & (-1)^{|\beta||x|}(\alpha,\beta)\otimes(x,y) \end{array}$$

is a quasi-isomorphism of complexes.

Proof. i) We proved in Proposition 6.5 that the complexes constructed with the Barraud-Cornea cocycle and the Künneth cocycle are chain homotopy equivalent.
ii) The map K^{alg} is clearly bijective. We now check that it is a morphism of complexes :

$$\begin{aligned} K\partial\left((\alpha\otimes x)\otimes(\beta\otimes y)\right) &= K\left[\left(\left(\partial\alpha\otimes x + (-1)^{|\alpha|}\sum_{x'}\alpha\cdot m_{x,x'}\otimes x'\right)\otimes(\beta\otimes y)\right)\right. \\ &\quad \left. + (-1)^{|\alpha|+|x|}\left((\alpha\otimes x)\otimes\left(\partial\beta\otimes y + (-1)^{|\beta|}\sum_{y'}\beta\cdot m_{y,y'}\otimes y'\right)\right)\right] \\ &= (-1)^{|\beta||x|}(\partial\alpha\otimes\beta)\otimes(x,x') + (-1)^{|\alpha|+|x||\beta|}(\alpha\otimes\partial\beta)\otimes(x,y) \\ &\quad + \sum_{x'}(-1)^{|\alpha|+|\beta||x'|}(\alpha\cdot m_{x,x'}\otimes\beta)\otimes(x',y) \\ &\quad + \sum_{y'}(-1)^{|\alpha|+|\beta|+|\beta||x|+(|y|-|y'|)|x|}(\alpha\otimes\beta\cdot m_{y,y'})\otimes(x,y'). \end{aligned}$$

$$\begin{aligned} \partial K\left((\alpha\otimes x)\otimes(\beta\otimes y)\right) &= (-1)^{|\beta||x|}(\partial\alpha\otimes\beta)\otimes(x,y) + (-1)^{|\beta||x|}(-1)^{|\alpha|}(\alpha\otimes\partial\beta)\otimes(x,y) \\ &\quad + (-1)^{|\beta||x|}(-1)^{|\alpha|+|\beta|}\sum_{x',y'}(\alpha\otimes\beta)\cdot m^K_{(x,y),(x',y')}\otimes(x',y') \\ &= (-1)^{|\beta||x|}(\partial\alpha\otimes\beta)\otimes(x,y) + (-1)^{|\beta||x|}(-1)^{|\alpha|}(\alpha\otimes\partial\beta)\otimes(x,y) \\ &\quad + \sum_{x'}(-1)^{|\beta||x|+|\beta|(|x|-|x'|-1)}(-1)^{|\alpha|+|\beta|}(\alpha\cdot m_{x,x'}\otimes\beta)\otimes(x',y) \\ &\quad + \sum_{y'}(-1)^{|\alpha|+|\beta|+|\beta||x|+|x|(|y|-|y'|)}(\alpha\otimes\beta\cdot m_{y,y'})\otimes(x,y') \\ &= K\partial\left((\alpha\otimes x)\otimes(\beta\otimes y)\right). \end{aligned}$$

iii) The proof that K^{top} is a morphism of complexes is exactly the same as for K^{alg} since every chain in this computation is split and we can use equation (22) for every multiplication. It will be proven in Corollary 6.19 that K^{top} is a quasi-isomorphism.

■

In particular, this shows the following corollary.

Corollary 6.14. *If $C_*(F)$ and $C_*(G)$ are complexes of cubical chains of topological spaces equipped with topological module structures respectively over ΩX and ΩY, then $H_*(X \times Y, C_*(F \times G))$ and $H_*(X \times Y, C_*(F) \otimes C_*(G))$ are both isomorphic to $H_*\left(C_*(X, \Xi_X, C_*(F)) \otimes C_*(Y, \Xi_Y, C_*(G))\right)$ and therefore there is an isomorphism*

$$H_*(X \times Y, C_*(F \times G)) \simeq H_*(X \times Y, C_*(F) \otimes C_*(G)).$$

■

By coupling Theorem 6.13 with the general Künneth short exact sequence, we obtain a Künneth formula for enriched Morse complexes.

Corollary 6.15. *There are a short exact sequences that split,*

$$0 \to \bigoplus_{k+k'=l} H_k(X, \mathcal{F}) \otimes_{\mathbb{Z}} H_{k'}(Y, \mathcal{G}) \xrightarrow{K} H_l(X \times Y, \mathcal{F} \otimes_{\mathbb{Z}} \mathcal{G}) \to \bigoplus_{k+k'=l-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_k(X, \mathcal{F}), H_{k'}(Y, \mathcal{G})) \to 0.$$

■

6.3.2 Properties of K^{alg} and K^{top}

We now prove that the Künneth maps are compatible with the direct and shriek maps (Lemma 6.16), with maps induced by a morphism of fibrations (Lemma 6.21), and with the Fibration Theorem (Lemma 6.20). We will also prove Corollary 6.19 stating that K^{top} is a quasi-isomorphism, and this will complete the proof of Theorem 6.13 and its Corollary 6.14.
The next Lemmas and Corollary are stated and proved using K^{alg} but also apply for K^{top} if $\mathcal{F}_* = C_*(F)$, $\mathcal{G}_* = C_*(G)$ since the module structure will only be applied to split chains and we can therefore use equation (22). We will therefore use the notation K.

Lemma 6.16. *Let $\varphi : Y^{n_Y} \to X^{n_X}$ et $\psi : Z^{n_Z} \to W^{n_W}$ be two continuous maps. Then the following diagram for the direct maps commutes at the chain level*

$$\begin{array}{ccc} C_*(Y, \varphi^*\mathcal{F}) \otimes C_*(Z, \psi^*\mathcal{G}) & \xrightarrow{K} & C_*(Y \times Z, \varphi^*\mathcal{F} \otimes \psi^*\mathcal{G}) \\ \downarrow{\scriptstyle \varphi_* \otimes \psi_*} & & \downarrow{\scriptstyle (\varphi \times \psi)_*} \\ C_*(X, \mathcal{F}) \otimes C_*(W, \mathcal{G}) & \xrightarrow[K]{} & C_*(X \times W, \mathcal{F} \otimes \mathcal{G}) \end{array}$$

and the following diagram for the shriek maps commutes at the chain level up to sign :

$$\begin{array}{ccc} C_{i+n_Y-n_X}(Y, \varphi^*\mathcal{F}) \otimes C_{j+n_Z-n_W}(Z, \psi^*\mathcal{G}) & \xrightarrow{K} & C_{i+j+n_Y+n_Z-n_X-n_W}(Y \times Z, \varphi^*\mathcal{F} \otimes \psi^*\mathcal{G}) \\ \uparrow{\scriptstyle \varphi_! \otimes \psi_!} & & \uparrow{\scriptstyle (\varphi \times \psi)_!} \\ C_i(X, \mathcal{F}) \otimes C_j(W, \mathcal{G}) & \xrightarrow[K]{} & C_{i+j}(X \times W, \mathcal{F} \otimes \mathcal{G}) \end{array}.$$

More precisely,

$$K(\varphi_!(\alpha \otimes x) \otimes \psi_!(\beta \otimes w)) = (-1)^{(n_Y-n_X)(n_W-|\beta|-|w|)+(n_Z-n_W)(|\alpha|+|x|)}(\varphi_! \times \psi_!)K(\alpha \otimes x \otimes \beta \otimes w).$$

Proof. We prove the result for the shriek map, the same arguments apply for the direct map. Let f_X, f_Y, f_Z, f_W be Morse functions respectively on X, Y, Z, W and x, y, z, w critical points. We start by computing $(\varphi \times \psi)_!$:

$$\begin{aligned}\mathcal{M}^{(\varphi\times\psi)_!}(x,w,y,z) &= \overline{W^s}(y,z) \cap (\varphi\times\psi)^{-1}(\overline{W^u}(x,w)) \\ &= \overline{W^s}(y) \times \overline{W^s}(z) \cap \varphi^{-1}\overline{W^u}(x) \times \psi^{-1}\overline{W^u}(w) \\ &= \mathcal{M}^{\varphi_!}(x,y) \times \mathcal{M}^{g_!}(w,z).\end{aligned}$$

Let's now compare orientations :

$$\Big(\text{Or } \overline{W^s}(x,w), \text{Or } \mathcal{M}^{(\varphi\times\psi)_!}(x,w,y,z)\Big) = \text{Or } \overline{W^s}(y,z).$$

$$\begin{aligned}&\big(\text{Or } \overline{W^s}(x,w), \text{Or } \mathcal{M}^{\varphi_!}(x,y), \text{Or } \mathcal{M}^{\psi_!}(w,z)\big) \\ &= (-1)^{|x|(n_W-|w|)}(-1)^{(n_W-|w|)(|x|-|y|+n_Y-n_X)}\big(\text{Or } \overline{W^s}(y), \text{Or } \overline{W^s}(z)\big) \\ &= (-1)^{(n_W-|w|)(|y|+n_Y-n_X)}(-1)^{|y|(n_Z-|z|)}\text{Or } \overline{W^s}(y,z).\end{aligned}$$

Now, if we choose $\sigma^{\varphi_!}_{x,y}$ and $\sigma^{\psi_!}_{w,z}$ two representing chain systems corrected by a sign (see [BDHO24, section 10.4]) of respectively $\mathcal{M}^{\varphi_!}(x,y)$ and $\mathcal{M}^{\psi_!}(w,z)$, then

$$\sigma^{(\varphi\times\psi)_!}_{(x,w),(y,z)} = (-1)^{(n_W-|w|)(|y|+n_Y-n_X)+1}(-1)^{|y|(n_Z-|z|)}(\sigma^{\varphi_!}_{x,y}, \sigma^{\psi_!}_{w,z})$$

is a corrected representing chain system of $\mathcal{M}^{(\varphi\times\psi)_!}(x,w,y,z)$. Moreover, it is clear from the definition of the evaluation maps [BDHO24, Section 10.4] that for every $x \in \text{Crit}(f_X), y \in \text{Crit}(f_Y), w \in \text{Crit}(f_W), z \in \text{Crit}(f_Z)$, $q^{(\varphi\times\psi)_!}_{(x,w),(y,z)} = (q^{\varphi_!}_{x,y}, q^{\psi_!}_{w,z})$.
Therefore

$$\nu^{(\varphi\times\psi)_!}_{(x,w),(y,z)} = (-1)^{(n_W-|w|)(|y|+n_Y-n_X)}(-1)^{|y|(n_Z-|z|)}(\nu^{\varphi_!}_{x,y}, \nu^{\psi_!}_{w,z})$$

where $\nu^{\varphi_!}_{x,y} := -q^{\varphi_!}_{x,y,*}(\sigma^{\varphi_!}_{x,y})$, $\nu^{\psi_!}_{w,z} := -q^{\psi_!}_{w,z,*}(\sigma^{\psi_!}_{z,w})$ and $\nu^{(\varphi\times\psi)_!}_{(x,w),(y,z)} = -q^{(\varphi\times\psi)_!}_{(x,w),(y,z),*}(\sigma^{(\varphi\times\psi)_!}_{(x,w),(y,z)})$.
We can now compute

$$\begin{aligned}(\varphi\times\psi)_!K(\alpha\otimes x\otimes\beta\otimes w) &= (-1)^{|\beta||x|}\sum_{y,z}(-1)^{(n_W-|w|)(|y|+n_Y-n_X)}(-1)^{|y|(n_Z-|z|)}(\alpha\otimes\beta).(\nu^{\varphi_!}_{x,y}, \nu^{\psi_!}_{w,z})\otimes(y,z) \\ &= (-1)^{|y|(|w|+|z|+|\beta|+n_Z-n_W)}(-1)^{(n_Y-n_X)(n_W-|\beta|-|w|)}\sum_{y,z}(\alpha.\nu^{\varphi_!}_{x,y}\otimes\beta.\nu^{\psi_!}_{z,w})\otimes(y,z).\end{aligned}$$

$$\begin{aligned}K(\varphi_!\otimes\psi_!)(\alpha\otimes x\otimes\beta\otimes w) &= (-1)^{(n_Z-n_W)(|\alpha|+|x|)}\sum_{y,z}K\left((\alpha.\nu^{\varphi_!}_{x,y}\otimes y)\otimes(\beta.\nu^{\psi_!}_{w,z}\otimes z)\right) \\ &= (-1)^{(n_Z-n_W)(|\alpha|+|x|)}(-1)^{|y|(|w|+|z|+|\beta|+n_Z-n_W)}\sum_{y,z}(\alpha.\nu^{\varphi_!}_{x,y}\otimes\beta.\nu^{\psi_!}_{z,w})\otimes(y,z).\end{aligned}$$

■

Corollary 6.17. *The maps K^{alg} and K^{top} are well-defined in homology.*

Proof. Let Ξ^X_0 and Ξ^X_1 be Morse data sets on X and Ξ^Y_0 and Ξ^Y_1 be Morse data sets on Y. Let $\mathcal{F}$ and $\mathcal{G}$ be DG systems on respectively X and Y. [BDHO24, Proposition 10.2.1] states that $\text{Id}_{X,*} : C_*(X, \Xi^X_0, \mathcal{F}) \to C_*(X, \Xi^X_1, \mathcal{F})$, $\text{Id}_{Y,*} : C_*(Y, \Xi^Y_0, \mathcal{G}) \to C_*(Y, \Xi^Y_1, \mathcal{G})$ and $\text{Id}_{X\times Y,*} : C_*(X\times Y, \Xi^{X\times Y}_0, \mathcal{F}\otimes\mathcal{G}) \to C_*(X\times Y, \Xi^{X\times Y}_1, \mathcal{F}\otimes\mathcal{G})$ are respectively chain homotopic to the continuation maps

$$\Psi^X_{01} : C_*(X, \Xi^X_0, \mathcal{F}) \to C_*(X, \Xi^X_1, \mathcal{F}),$$

$$\Psi_{01}^{Y} : C_*(Y, \Xi_0^Y, \mathcal{G}) \to C_*(Y, \Xi_1^Y, \mathcal{G}),$$
$$\Psi_{01}^{X\times Y} : C_*(X\times Y, \Xi_0^{X\times Y}, \mathcal{F}\otimes\mathcal{G}) \to C_*(X\times Y, \Xi_1^{X\times Y}, \mathcal{F}\otimes\mathcal{G})$$

of Theorem 2.8. Since $K \circ (\mathrm{Id}_{X,*} \otimes \mathrm{Id}_{Y,*}) = \mathrm{Id}_{X\times Y,*} \circ K$, it follows that

$$K : H_*(X, \mathcal{F}) \otimes H_*(Y, \mathcal{G}) \to H_*(X\times Y, \mathcal{F}\otimes\mathcal{G})$$

is well-defined in the sense that the diagram

$$\begin{array}{ccc} H_*(X, \Xi_0^X, \mathcal{F}) \otimes H_*(Y, \Xi_0^Y, \mathcal{G}) & \xrightarrow{K} & H_*(X\times Y, \Xi_0^{X\times Y}, \mathcal{F}\otimes\mathcal{G}) \\ \downarrow{\scriptstyle \Psi_{01}^X\otimes\Psi_{01}^Y} & & {\scriptstyle \Psi_{01}^{X\times Y}}\downarrow \\ H_*(X, \Xi_1^X, \mathcal{F}) \otimes H_*(Y, \Xi_1^Y, \mathcal{G}) & \xrightarrow{K} & H_*(X\times Y, \Xi_1^{X\times Y}, \mathcal{F}\otimes\mathcal{G}) \end{array}$$

commutes.

■

Lemma 6.18. *Let Ξ_X and Ξ_Y be sets of DG Morse data respectively on X and Y. The maps*

$$K^{alg} : H_*\left(C_*(X, \Xi_X, \mathcal{F}) \otimes C_*(Y, \Xi_Y, \mathcal{G})\right) \to H_*(X\times Y, \mathcal{F}\otimes\mathcal{G})$$

and

$$K^{top} : H_*\left(C_*(X, \Xi_X, C_*(F)) \otimes C_*(Y, \Xi_Y, C_*(G))\right) \to H_*(X\times Y, C_*(F\times\mathcal{G}))$$

are limits of morphisms of spectral sequences

$$K^{alg,(r)} : E^r_{s,t}((X, \mathcal{F}) \otimes (Y, \mathcal{G})) \to E^r_{s,t}(X\times Y, \mathcal{F}\otimes\mathcal{G})$$

and

$$K^{top,(r)} : E^r_{s,t}((X, F) \otimes (Y, G)) \to E^r_{s,t}(X\times Y, F\times\mathcal{G}).$$

Proof. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be the respective Morse functions of Ξ_X and Ξ_Y. Let $\Xi_{X\times Y}$ be the DG Morse data set on $X\times Y$ defined in Section 6.1.
Recall that the spectral sequence $E^r_{s,t}(X\times Y, \mathcal{F}\otimes\mathcal{G})$ is induced by the filtration

$$F_s(C_k(X\times Y, \Xi_{X\times Y}, \mathcal{F}\otimes\mathcal{G})) = \bigoplus_{\substack{i+j=k \\ i\leq s}} (\mathcal{F}\otimes\mathcal{G})_j \otimes \mathbb{Z}\mathrm{Crit}_i(f+g).$$

Define now define a filtration of the complex $C_*(X, \Xi_X, \mathcal{F}) \otimes C_*(Y, \Xi_Y, \mathcal{G})$. Let $s, k \in \mathbb{N}$,

$$F_s((C_*(X, \Xi_X, \mathcal{F}) \otimes C_*(Y, \Xi_Y, \mathcal{G}))_k) = \bigoplus_{\substack{p+p'+q+q'=k \\ p+p'\leq s}} \left((\mathcal{F}_q \otimes \mathbb{Z}\mathrm{Crit}_p(f)) \otimes (\mathcal{G}_{q'} \otimes \mathbb{Z}\mathrm{Crit}_{p'}(g))\right).$$

This induces a spectral sequence $E^r_{p,q}((X, \mathcal{F})\otimes(Y, \mathcal{G}))$ converging towards $H_{p+q}(C_*(X, \Xi_X, \mathcal{F})\otimes C_*(Y, \Xi_Y, \mathcal{G}))$ whose first page is

$$E^1_{s,t}((X, \mathcal{F}) \otimes (Y, \mathcal{G})) \simeq \bigoplus_{p+p'=s} H_t(\mathcal{F}\otimes\mathcal{G}) \otimes \mathbb{Z}\mathrm{Crit}_p(f) \otimes \mathbb{Z}\mathrm{Crit}_{p'}(g).$$

Moreover, since

$$(\mathcal{F}\otimes\mathcal{G})_j = \bigoplus_{q+q'=j} \mathcal{F}_q \otimes \mathcal{G}_{q'} \text{ and } \mathrm{Crit}_i(f+g) = \bigoplus_{p+p'=i} \mathrm{Crit}_p(f) \times \mathrm{Crit}_{p'}(g),$$

it follows that

$$K^{alg}(F_s((C_*(X, \Xi_X, \mathcal{F}) \otimes C_*(Y, \Xi_Y, \mathcal{G}))_k)) = F_s(C_k(X\times Y, \Xi_{X\times Y}, \mathcal{F}\otimes\mathcal{G})).$$

The same approach applies for K^{top} for $\mathcal{F} = C_*(F)$ and $\mathcal{G} = C_*(G)$. Indeed, since

$$\mathrm{EZ}\left(\bigoplus_{q+q'=j} C_q(F) \otimes C_{q'}(G)\right) \subset C_j(F \times G),$$

$$K^{top}\left(F_s((C_*(X, \Xi_X, C_*(F)) \otimes C_*(Y, \Xi_Y, \mathcal{G}))_k)\right) \subset F_s(C_k(X \times Y, \Xi_{X\times Y}, C_*(F \times G))).$$

■

Corollary 6.19. *For any sets of DG Morse data Ξ_X on X and Ξ_Y on Y, the map $K^{top} : C_*(X, \Xi_X, C_*(F)) \otimes C_*(Y, \Xi_Y, C_*(G)) \to C_*(X \times Y, \Xi_{X\times Y}, C_*(F \times G))$ is a quasi-isomorphism.*

Proof. Denote $f : X \to \mathbb{R}$ the Morse function of Ξ_X and $g : Y \to \mathbb{R}$ the Morse function of Ξ_Y. The map K^{top} induces an isomorphism on the first pages

$$K^{top,(1)} : H_t(C_*(F) \otimes C_*(G)) \otimes \left(\bigoplus_{p+p'=s} \mathbb{Z}\mathrm{Crit}_p(f) \otimes \mathbb{Z}\mathrm{Crit}_{p'}(g)\right) \to H_t(F \times G) \otimes \mathbb{Z}\mathrm{Crit}_s(f+g).$$

Indeed,

$$\mathrm{EZ} : H_t(C_*(F) \otimes C_*(G)) \to H_t(F \times G)$$

is an isomorphism.

■

We finished to prove that $H_*(X\times Y, C_*(F)\otimes C_*(G)) \simeq H_*(X\times Y, C_*(F\times G))$. We will now denote $K = K^{top}$ for the two next lemmas that state compatibility properties of K with the Fibration Theorem and with maps induced by morphisms of fibrations.

Lemma 6.20. *Let $F \hookrightarrow E_X \to X$ and $G \hookrightarrow E_Y \to Y$ be two fibrations over two pointed oriented closed and connected manifolds equipped with transitive lifting functions Φ^X and Φ^Y. Let $\Phi^{X\times Y} = (\Phi^X, \Phi^Y)$ be a transitive lifting function for the fibration $F \times G \hookrightarrow E_X \times E_Y \to X \times Y$. Then the following diagram commutes*

$$\begin{array}{ccc} H_*(X, C_*(F)) \otimes H_*(Y, C_*(G)) & \xrightarrow{K} & H_*(X \times Y, C_*(F \times G)) \\ \downarrow{\scriptstyle \Psi_{E_X} \otimes \Psi_{E_Y}} & & \downarrow{\scriptstyle \Psi_{E_{X\times Y}}} \\ H_*(E_X) \otimes H_*(E_Y) & \xrightarrow{\mathrm{EZ}} & H_*(E_X \times E_Y). \end{array}$$

Proof. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be Morse functions. Consider the Morse function $H = f + g : X \times Y \to \mathbb{R}$. Once again, it suffices to prove the statement for the pullback fibrations by the chosen homotopy inverses θ_X, θ_Y and $\theta_{X\times Y}$ of the canonical projections $p_X : X \to X/\mathcal{Y}_X$, $p_Y : Y \to Y/\mathcal{Y}_Y$ and $p_{X\times Y} : X \times Y \to (X \times Y)/\mathcal{Y}$ (see Section 2.3 for explanation about the Fibration Theorem or [BDHO24, Sections 7.2, 7.3] for more details). We therefore only have to check that, for any $\alpha \in C_*(F)$, $\beta \in C_*(G)$, $x \in \mathrm{Crit}(f)$ and $y \in \mathrm{Crit}(g)$,

$$K(\Phi^X_*(\alpha \otimes m_x) \otimes \Phi^Y_*(\beta \otimes m_y)) = (-1)^{|\beta||x|}\Psi_{E_{X\times Y}}((\alpha, \beta) \otimes (x, y)).$$

Since $\overline{W^u}(x, y) = \overline{W^u}(x) \times \overline{W^u}(y)$, we can choose the representing chain system

$$m_{(x,y)} = (m_x, m_y) \in C_{|x|+|y|}(\mathcal{P}_{\star\to X\times Y/\mathcal{Y}}(X \times Y/\mathcal{Y}))$$

of the Latour cells in $X \times Y$. Moreover, $\Phi^{X\times Y}_*$ is given at the chain-level by

$$\Phi^{X\times Y}_*((\alpha, \beta), (m_x, m_Y)) = (-1)^{|\beta||m_x|}(\Phi^X_*(\alpha, m_x), \Phi^Y_*(\beta, m_y)).$$

Since $|m_x| = |x|$,

$$(\Phi^X_*(\alpha \otimes m_x), \Phi^Y_*(\beta \otimes m_y)) = (-1)^{|\beta||x|}\Phi^{X\times Y}_*((\alpha, \beta) \otimes m_{(x,y)}) = (-1)^{|\beta||x|}\Psi_{E_{X\times Y}}((\alpha, \beta) \otimes (x, y)),$$

and this concludes the proof.

■

Lemma 6.21. *Let $F_i \hookrightarrow E_i^X \to X$ and $G_i \hookrightarrow E_i^Y \to Y$ be fibrations over two pointed oriented closed and connected manifolds $(X, \star)$ and $(Y, \star_Y)$ with respective transitive lifting functions Φ_i^X and Φ_i^Y for $i \in \{1, 2\}$. Let $\varphi : E_1^X \to E_2^X$ and $\psi : E_1^Y \to E_2^Y$ be morphisms of fibrations over X and Y respectively. Endow the fibrations $E_i^X \times E_i^Y \to X \times Y$ with the transitive lifting functions*

$$\Phi_i^{X\times Y} = (\Phi_i^X, \Phi_i^Y) \text{ for } i \in \{1, 2\}.$$

Then $\varphi \times \psi : E_1^X \times E_1^Y \to E_2^X \times E_2^Y$ is a morphism of fibrations and the following diagram commutes

$$\begin{array}{ccc} H_*(X, C_*(F_1)) \otimes H_*(Y, C_*(G_1)) & \xrightarrow{K} & H_*(X \times Y, C_*(F_1 \times G_1)) \\ \downarrow{\scriptstyle \tilde{\varphi}\otimes\tilde{\psi}} & & \downarrow{\scriptstyle \widetilde{\varphi\times\psi}} \\ H_*(X, C_*(F_2)) \otimes H_*(Y, C_*(G_2)) & \xrightarrow{K} & H_*(X \times Y, C_*(F_2 \times G_2)). \end{array}$$

Proof. Using the compatibility between K and the Fibration Theorem (Lemma 6.20) and the compatibility between induced maps and the Fibration Theorem (Theorem 5.8), we obtain the commutativity of the diagram

$$\begin{array}{ccc} H_*(X, C_*(F_1)) \otimes H_*(Y, C_*(G_1)) & \xrightarrow{K} & H_*(X \times Y, C_*(F_1 \times G_1)) \\ \downarrow{\scriptstyle \Psi_{E_1^X}\otimes\Psi_{E_1^Y}} & & \downarrow{\scriptstyle \Psi_{E_1^X\times E_1^Y}} \\ H_*(E_1^X) \otimes H_*(E_1^Y) & \xrightarrow{\text{EZ}} & H_*(E_1^X \times E_1^Y) \\ \downarrow{\scriptstyle \varphi_*\otimes\psi_*} & & \downarrow{\scriptstyle (\varphi\times\psi)_*} \\ H_*(E_2^X) \otimes H_*(E_2^Y) & \xrightarrow{\text{EZ}} & H_*(E_2^X \times E_2^Y) \\ \downarrow{\scriptstyle \Psi^{-1}_{E_2^X}\otimes\Psi^{-1}_{E_2^Y}} & & \downarrow{\scriptstyle \Psi^{-1}_{E_2^X\times E_2^Y}} \\ H_*(X, C_*(F_2)) \otimes H_*(Y, C_*(G_2)) & \xrightarrow{K} & H_*(X \times Y, C_*(F_2 \times G_2)). \end{array}$$

$\tilde{\varphi}\otimes\tilde{\psi}$ (left) and $\widetilde{\varphi\times\psi}$ (right)

■

7 Morse description and generalization of the Chas-Sullivan product. Proof of Theorem A

In this section we will define a product on the homology of the total space E of a fibration $F \hookrightarrow E \xrightarrow{\pi} X^n$ over a n-dimensional closed, connected, oriented and smooth manifold X, endowed with a morphism of fibrations $m : E \,{}_\pi\!\times_\pi E \to E$ using DG Morse theory. For that, we will use :

- The cross-product $K := K^{top} : H_*(X, C_*(F))^{\otimes 2} \to H_*(X^2, C_*(F^2))$ defined in Theorem 6.13.
- The shriek map $\Delta_! : H_*(X^2, C_*(F^2)) \to H_{*-n}(X, \Delta^* C_*(F^2))$ of the diagonal $\Delta : X \to X^2$. As proved by Theorem 3.2, this map is the equivalent in our model to the Pontryagin-Thom construction used in [GS08].
- The morphism $\tilde{m} : H_*(X, \Delta^* C_*(F^2)) \to H_*(X, C_*(F))$ induced by the morphism of fibrations $m : E \,{}_\pi\!\times_\pi E \to E$. We proved in Theorem 5.8 that this morphism is the equivalent in our model to the map $m_* : H_*(E \,{}_\pi\!\times_\pi E) \to H_*(E)$.

7.1 Definition

Let $E \xrightarrow{\pi} X$ a fibration over a closed, connected, oriented and smooth manifold X and let $\star \in X$ a preferred point. Denote $F = \pi^{-1}(\star)$. Let $\Phi : E \,{}_\pi\!\times_{ev_0} \mathcal{P}X \to E$ be a transitive lifting function and consider $\mathcal{F} = C_*(F)$, the cubical complex of the fiber F endowed with the DG module structure over $C_*(\Omega X)$ induced by Φ. Suppose that there exists a morphism of fibrations $m : E \,{}_\pi\!\times_\pi E \to E$ (see Definition 5.7).
We restate Theorem A and we will prove it in the next section.

Theorem 7.1. *The morphism of fibrations $m : E \ {}_{\pi}\times_{\pi} E \to E$ induces a degree $-n$ product*

$$\mathrm{CS}_{DG} : H_*(X,\mathcal{F})^{\otimes 2} \to H_*(X,\mathcal{F}).$$

such that the following properties hold :

— **Associativity :** *If m_* is associative in homology, then so is CS_{DG}.*

— **Commutativity :** *If m_* is commutative in homology, then CS_{DG} is commutative up to sign*

$$\mathrm{CS}_{DG}(\gamma\otimes\tau) = (-1)^{(n-|\gamma|)(n-|\tau|)}\mathrm{CS}_{DG}(\tau\otimes\gamma).$$

— **Neutral element :** *If π admits a section $s : X \to E$ such that $m(s(\pi(e)),e) = m(e,s(\pi(e))) = e$ for all $e \in E$, then CS_{DG} admits a neutral element.*

— **Functoriality :**

• *For any pointed, oriented, closed and connected manifold Y^k, any continuous map $g : Y \to X$ induces a degree $-k$ product for the fibration $F \hookrightarrow g^*E \to Y$*

$$\mathrm{CS}^Y_{DG} : H_i(Y,g^*\mathcal{F}) \otimes H_j(Y,g^*\mathcal{F}) \to H_{i+j-k}(Y,g^*\mathcal{F}),$$

such that $g_! : H_*(X,\mathcal{F}) \to H_{*+n-k}(Y,g^*\mathcal{F})$ is a morphism of rings up to sign.

• *If g is an orientation-preserving homotopy equivalence then, $g_!$ and $g_* : H_*(Y,g^*\mathcal{F}) \to H_*(X,\mathcal{F})$ are isomorphisms of rings inverse of each other.*

— **Spectral sequence :** *Let Ξ be a set of DG Morse data on X. The canonical filtration*

$$F_p(C_*(X,\Xi,\mathcal{F})) = \bigoplus_{\substack{i+j=k\\ i\leq p}} \mathcal{F}_j \otimes \mathbb{Z}\mathrm{Crit}_i(f)$$

induces a spectral sequence $E^r_{p,q}$ that is endowed with an algebra structure

$$E^r_{p,q}\otimes E^r_{l,m} \to E^r_{p+l-n,q+m}$$

induces by a chain-level model $\mathrm{CS}_{DG} : C_*(X,\Xi,\mathcal{F})^{\otimes 2} \to C_*(X,\Xi,\mathcal{F})$ and converges towards $H_*(X,\mathcal{F})$ as algebras. For $s,t \geq 0$ $E^2_{s,t} = H_s(X,H_t(\mathcal{F}))$ and the algebra structure is given up to sign by the intersection product on X with coefficient in $H_t(\mathcal{F})$.

This product corresponds in homology, via the Fibration Theorem, to the product $\mu_* : H_i(E)\otimes H_j(E) \to H_{i+j-n}(E)$ defined in [GS08] and therefore (re)proves that the product μ_* satisfies those properties. In particular, if the considered fibration is the loop-loop fibration $\Omega X \hookrightarrow \mathcal{L}X \to X$ and the morphism of fibration $m : \mathcal{L}X \ {}_{ev}\times_{ev} \mathcal{L}X \to \mathcal{L}X$ is the concatenation, then CS_{DG} corresponds to the Chas-Sullivan product.

We defined in Section 6.3 a $C_*(\Omega X^2)$-module structure on $\mathcal{F}^2 := C_*(F^2)$, which is in this case the module structure defined by the natural holonomy of the fibration $F^2 \hookrightarrow E^2 \to X^2$. We also denote, when the context is clear, $\mathcal{F}^2 = \Delta^*\mathcal{F}^2$ the $C_*(\Omega X)$-module structure on $\mathcal{F}^2$ obtained by pulling back by the diagonal $\Delta : X \to X^2$. It corresponds to the natural holonomy of the pullback fibration $F^2 \hookrightarrow E \ {}_{\pi}\times_{\pi} E \to X$. The morphism of fibrations $m : E \ {}_{\pi}\times_{\pi} E \to E$ induces a morphism of complexes $\tilde{m} : C_*(X,\Xi,\mathcal{F}^2) \to C_*(X,\Xi,\mathcal{F})$ that is compatible in homology with the Fibration Theorem (see Theorem 5.8).
The product

$$\mathrm{CS}_{DG} : H_*(X,\mathcal{F})^{\otimes 2} \to H_*(X,\mathcal{F})$$

is defined in homology by the composition

$$H_i(X,\mathcal{F})\otimes H_j(X,\mathcal{F}) \xrightarrow{K} H_{i+j}(X\times X,\mathcal{F}^2) \xrightarrow{\Delta_!} H_{i+j-n}(X,\Delta^*\mathcal{F}^2) \xrightarrow{\tilde{m}} H_{i+j-n}(X,\mathcal{F})$$

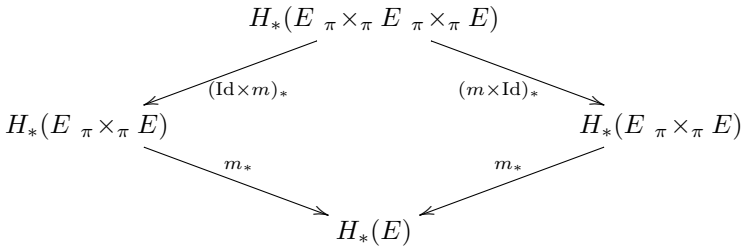
with the Dold sign. In other words,

$$\mathrm{CS}_{DG} : \begin{array}{ccc} H_i(X,\mathcal{F})\otimes H_j(X,\mathcal{F}) & \to & H_{i+j-n}(X,\mathcal{F}) \\ \gamma\otimes\tau & \mapsto & (-1)^{n(n-|\tau|)}\tilde{m}\circ\Delta_!\circ K(\gamma\otimes\tau). \end{array}$$

7.2 Properties of the product

7.2.1 Associativity

Proposition 7.2. *Suppose that $m : E \ {}_\pi\times_\pi E \to E$ is associative in homology,* i.e *the diagram*

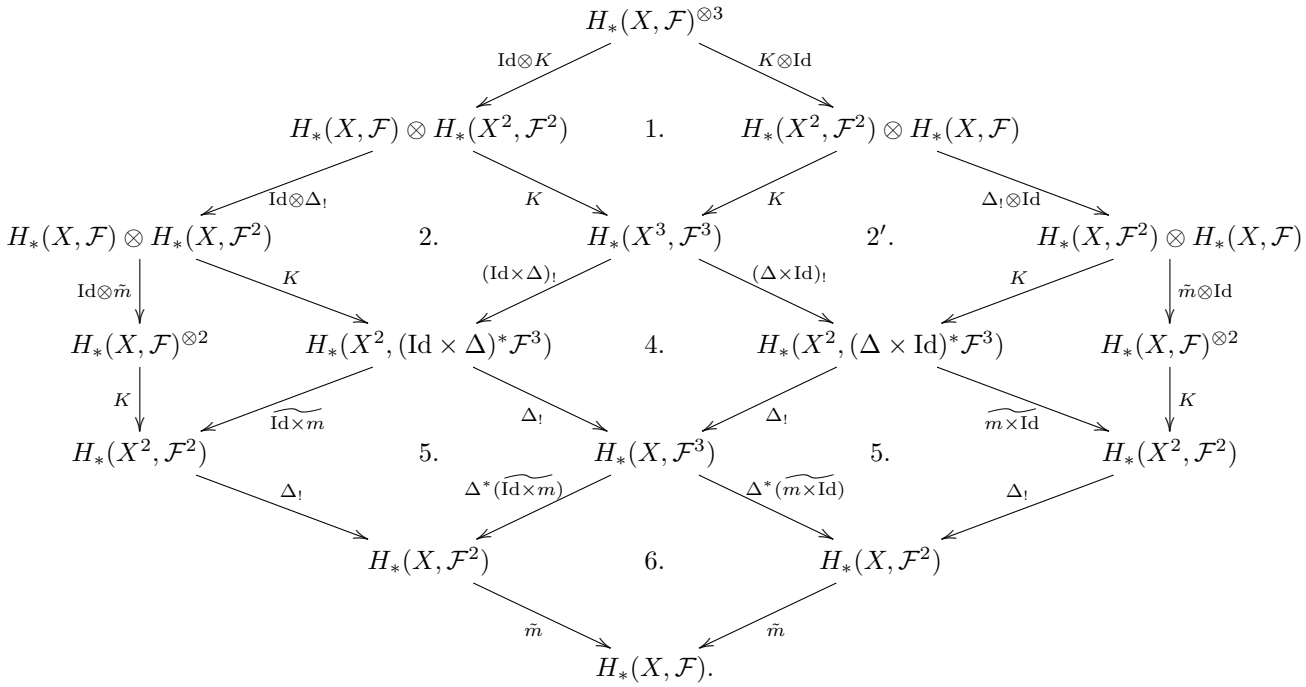


is commutative. Then, the product CS_{DG} *is associative in homology.*

Proof. Let $\sigma, \omega, \delta \in H_*(X, \mathcal{F})$. Recall that if $\varphi : Y \to Z$ is a continuous function and $\mathcal{G}$ is a DG-module over $C_*(\Omega Z)$, then $\varphi^*\mathcal{G}$ is $\mathcal{G}$ endowed with the following DG-module structure over $C_*(\Omega Y)$

$$\forall \alpha \in \mathcal{G},\ \forall \gamma \in C_*(\Omega Y),\ \alpha \cdot \gamma := \alpha \cdot \underbrace{\varphi_*(\gamma)}_{\in C_*(\Omega Z)} .$$

Consider now the diagram



The associativity of CS_{DG} is equivalent to the fact that this diagram commutes up to the Kozsul sign $(-1)^{n(|\sigma|)}$ and the Dold sign $(-1)^{n(n-|\delta|)}$. Indeed, we can compute

$$\begin{aligned}
&\mathrm{CS}_{DG}(\sigma \otimes \mathrm{CS}_{DG}(\omega \otimes \delta)) \\
&= (-1)^{n(n-(|\omega|+|\delta|-n))}(-1)^{n(n-|\delta|)}\tilde{m} \circ \Delta_! \circ K\left(\sigma \otimes (\tilde{m} \circ \Delta_! \circ K(\omega \otimes \delta))\right) \\
&= (-1)^{n|\sigma|}(-1)^{n(n-|\omega|)}\tilde{m} \circ \Delta_! \circ K \circ (\mathrm{Id} \otimes \tilde{m}) \circ (\mathrm{Id} \otimes \Delta_!) \circ (\mathrm{Id} \otimes K)(\sigma \otimes \omega \otimes \delta)
\end{aligned}$$

and,

$$\begin{aligned}
&\mathrm{CS}_{DG}(\mathrm{CS}_{DG}(\sigma\otimes\omega)\otimes\delta)\\
&=(-1)^{n(n-|\omega|)}(-1)^{n(n-|\delta|)}\tilde{m}\circ\Delta_!\circ K\left((\tilde{m}\circ\Delta_!\circ K(\sigma\otimes\omega))\otimes\delta\right)\\
&=(-1)^{n(|\omega|+|\delta|)}\tilde{m}\circ\Delta_!\circ K\circ(\tilde{m}\otimes\mathrm{Id})\circ(\Delta_!\otimes\mathrm{Id})\circ(K\otimes\mathrm{Id})(\sigma\otimes\omega\otimes\delta).
\end{aligned}$$

We then prove the commutativity of this diagram up to the wanted sign in 6 steps :

• ***Step 1 : The square 1. is commutative.***

This is an easy computation at the chain level. Let Ξ_1,Ξ_2,Ξ_3 be sets of DG Morse data on X. Let $\alpha\otimes x\in C_*(X,\Xi_1,\mathcal{F})$, $\beta\otimes y\in C_*(X,\Xi_2,\mathcal{F})$ and $\epsilon\otimes z\in C_*(X,\Xi_3,\mathcal{F})$.

$$\begin{aligned}
K\circ(\mathrm{Id}\otimes K)\left((\alpha\otimes x)\otimes(\beta\otimes y)\otimes(\gamma\otimes z)\right)&=(-1)^{|\gamma||y|}K\left(\alpha\otimes x\otimes((\beta,\gamma)\otimes(y,z))\right)\\
&=(-1)^{(|\beta|+|\gamma|)|x|+|\gamma||y|}(\alpha,\beta,\gamma)\otimes(x,y,z).
\end{aligned}$$

$$\begin{aligned}
K\circ(K\otimes\mathrm{Id})\left((\alpha\otimes x)\otimes(\beta\otimes y)\otimes(\gamma\otimes z)\right)&=(-1)^{|\beta||x|}K\left((\alpha,\beta)\otimes(x,y)\right)\otimes\gamma\otimes z)\\
&=(-1)^{|\beta||x|+|\gamma|(|x|+|y|)}(\alpha,\beta,\gamma)\otimes(x,y,z)\\
&=K\circ(\mathrm{Id}\otimes K)\left((\alpha\otimes x)\otimes(\beta\otimes y)\otimes(\gamma\otimes z)\right)
\end{aligned}$$

• ***Step 2 : The square 2'. is commutative up to the Dold sign and 2. is commutative up to the Kozsul sign.***

This is a direct application of Lemma 6.16.

• ***Step 3 : The rightmost and leftmost triangles are commutative.***

This is a direct application of Lemma 6.21 since $\mathrm{Id}:E\to E$ is a morphism and fibration and $\widetilde{\mathrm{Id}}=\mathrm{Id}:C_*(X,\mathcal{F})\to C_*(X,\mathcal{F})$.

• ***Step 4 : The square 4. is commutative.***

This directly follows from the composition property of shriek maps [BDHO24, Theorem 8.2] since

$$\Delta(\Delta\times\mathrm{Id})=\Delta(\mathrm{Id}\times\Delta).$$

• ***Step 5 : The squares 5. are commutative.***

It is a particular case of Proposition 5.23.

• ***Step 6 : The square 6. is commutative.***

We proved in Lemma 5.11 that $\Delta^*(m\times\mathrm{Id})$ and $\Delta^*(\mathrm{Id}\times m)$ are morphisms of fibrations. This step is therefore a consequence of Corollary 5.18 and Corollary 5.17. Indeed, since we assumed $m(m\times\mathrm{Id})_*=m(\mathrm{Id}\times m)_*:H_*(E\ {}_\pi\times_\pi E\ {}_\pi\times_\pi E)\to H_*(E)$,

$$\begin{aligned}
\tilde{m}\circ\Delta^*\widetilde{m\times\mathrm{Id}}&=m(\widetilde{m\times\mathrm{Id}})\\
&=m(\widetilde{\mathrm{Id}\times m})\\
&=\tilde{m}\circ\Delta^*\widetilde{\mathrm{Id}\times m}.
\end{aligned}$$

■

7.2.2 Commutativity

We will use the following notations for the switches of coordinates :

- $\tau:X^2\to X^2$.

- Consider $\tau^*F^2 \to \tau^*E^2 \to X^2$ the pullback fibration by τ whose natural transitive lifting function is

$$\tau^*\Phi^2((\alpha,\beta)\cdot(\gamma_1,\gamma_2)) = (\Phi(\alpha,\gamma_2),\Phi(\beta,\gamma_1))$$

for all $(\alpha,\beta) \in F^2$ and $(\gamma_1,\gamma_2) \in \mathcal{P}_{\star\to X}X$. The switch of coordinates $\tau_E : \tau^*E^2 \to E^2$ is an isomorphism of fibrations (bijective morphism of fibrations) that commutes with the transitive lifting functions $\tau^*\Phi_2$ and

$$\Phi_2 = (\Phi,\Phi) : E^2 \,{}_{(\pi,\pi)}\!\times_{ev_0} \mathcal{P}X^2 \to E^2$$

Therefore,

$$\tilde{\tau_E} : C_*(X^2,\tau^*C_*(F^2)) \to C_*(X^2,C_*(F^2)), \quad \tilde{\tau_E}((\alpha,\beta)\otimes(x,y)) = \tau_{E,*}(\alpha,\beta)\otimes(x,y).$$

If $(\alpha,\beta) = \mathrm{EZ}(\alpha\otimes\beta)$, then $\tilde{\tau_E}((\alpha,\beta)\otimes(x,y)) = (-1)^{|\alpha||\beta|}(\beta,\alpha)\otimes(x,y)$.

- $\tau_F : E \,{}_{\pi}\!\times_{\pi} E \to E \,{}_{\pi}\!\times_{\pi} E$ is also an isomorphism of fibrations and

$$\tilde{\tau_F} = \Delta^*\tilde{\tau_E} : C_*(X,\Delta^*C_*(F^2)) \to C_*(X,\Delta^*C_*(F^2)).$$

Proposition 7.3. *We suppose that m is commutative in homology, i.e $m_*\tau_{E,*} = m_* : H_*(E \,{}_{\pi}\!\times_{\pi} E) \to H_*(E)$.*
For any $\sigma,\omega \in H_*(X,\mathcal{F})$,

$$\mathrm{CS}_{DG}(\sigma,\omega) = (-1)^{(n-|\sigma|)(n-|\omega|)}\mathrm{CS}_{DG}(\omega,\sigma).$$

In order to prove this property, we have to understand how to switch variables on the manifold X^2. Let Ξ be a set of Morse data on X.
Let $\Xi' = (f',\xi',s'_{x',y'},o',\mathcal{Y}',\theta')$ be another Morse DG data set on X and define Ξ_1, Ξ_2 to be the Morse DG data sets on X^2 described in Section 6.1 with $Y = X$ and respectively $(\Xi_X,\Xi_Y) = (\Xi,\Xi')$ and $(\Xi_X,\Xi_Y) = (\Xi',\Xi)$.

Lemma 7.4. *The shriek map $\tau_! : C_*(X^2,\Xi_1,C_*(F^2)) \to C_*(X^2,\Xi_2,\tau^*C_*(F^2))$ is given up to chain homotopy by*

$$\tau_!((\alpha,\beta)\otimes(x,x')) = (-1)^{|x||x'|+n}(\alpha,\beta)\otimes(x',x).$$

Proof. Let us recall how the direct and shriek maps of a diffeomorphism $\varphi : Y \to Z$ have been defined in [BDHO24, Section 9.1].
Given $\Xi = (f,\xi,\mathcal{Y},o,s_{x,y},\theta)$, define $\varphi^*\Xi = (f\circ\varphi^{-1},\varphi_*\xi,\varphi\mathcal{Y},\varphi(o),\varphi^*s_{\varphi(x),\varphi(y)},\varphi\circ\theta)$ where the orientations $\varphi(o)$ of the unstable manifolds are defined for any $x,y \in \mathrm{Crit}(f)$ by

$$\mathrm{Or}\ \overline{W^u}(\varphi(x)) := \mathrm{Or}\ \overline{W^u}(x),$$

and the representing chain system is

$$\varphi^*s_{\varphi(x),\varphi(y)} = \varphi_*(s_{x,y}) \in C_{|x|-|y|-1}\left(\overline{\mathcal{L}}(\varphi(x),\varphi(y))\right).$$

Then, the direct map and shriek map are defined for any DG system $\mathcal{G}$ on Z by

$$\varphi_* : C_*(Y,\Xi,\varphi^*\mathcal{G}) \to C_*(Z,\varphi^*\Xi,\mathcal{G}),\ \varphi_*(\alpha\otimes x) = \alpha\otimes\varphi(x)$$

and

$$\varphi_! : C_*(Z,\varphi^*\Xi,\mathcal{G}) \to C_*(Y,\Xi,\varphi^*\mathcal{G}),\ \varphi_!(\alpha\otimes x) = (-1)^{deg\ \varphi}\ \alpha\otimes\varphi^{-1}(x).$$

Therefore

$$\tau_! : C_*(X^2,\tau^*\Xi_2,C_*(F^2)) \to C_*(X^2,\Xi_2,\tau^*C_*(F^2)),\ \tau_!((\alpha,\beta)\otimes(x,x')) = (-1)^n(\alpha,\beta)\otimes(x',x).$$

However, $\Xi_1 \neq \tau^*\Xi_2$. Indeed,

$$\begin{aligned} o_1(x,x') := \mathrm{Or}\ \overline{W}^u_{f,f'}(x,x') &= \left(\mathrm{Or}\ \overline{W}^u_f(x),\mathrm{Or}\ \overline{W}^u_{f'}(x')\right) \\ &= (-1)^{|x||x'|}\left(\mathrm{Or}\ \overline{W}^u_{f'}(x'),\mathrm{Or}\ \overline{W}^u_f(x)\right) = (-1)^{|x||x'|}\tau^*o_2(x,x'). \end{aligned}$$

The other data are the same. A corollary of the proof of the first step of [BDHO24, Theorem 6.3.1] shows that the continuation map $\Psi : C_*(X^2, \Xi_1, C_*(F^2)) \to C_*(X^2, \tau^*\Xi_2, C_*(F^2))$ is

$$\Psi((\alpha \otimes \beta) \otimes (x, x')) = (-1)^{|x||x'|}(\alpha \otimes \beta) \otimes (x, x').$$

Therefore, the shriek map $\tau_! : C_*(X^2, \Xi_1, \mathcal{F}^2) \to C_*(X^2, \Xi_2, \tau^*\mathcal{F}^2)$ is given up to chain homotopy by

$$\tau_!((\alpha, \beta) \otimes (x, x')) = (-1)^{|x||x'|+n}(\alpha, \beta) \otimes (x', x).$$

■

Proof of Proposition 7.3. We first remark that $\tau \circ \Delta = \Delta$ and therefore the composition property for the shriek map gives $\Delta_! = \Delta_! \circ \tau_!$ up to chain homotopy.

We compute

$$\begin{aligned}
\mathrm{CS}_{DG}(\alpha \otimes x, \beta \otimes x') &= (-1)^{n(n-|\beta|-|x'|)}(-1)^{|\beta||x|}\tilde{m} \circ \Delta_!((\alpha, \beta) \otimes (x, x')) \\
&= (-1)^{|\beta|(|\alpha|+|x|+n)+(n+|x|)|x'|}\tilde{m} \circ \Delta_! \circ \tilde{\tau_E} \circ \tau_!((\beta, \alpha) \otimes (x', x)) \\
&= (-1)^{|\beta|(|\alpha|+|x|+n)+(n+|x|)|x'|}\tilde{m} \circ \tilde{\tau_F} \circ \Delta_!((\beta, \alpha) \otimes (x', x)) \\
&= (-1)^{|\beta|(|\alpha|+|x|+n)+(n+|x|)|x'|}\tilde{m} \circ \Delta_!((\beta, \alpha) \otimes (x', x)) \\
&= (-1)^{(n-|\alpha|-|x|)(n-|\beta|-|x'|)}\mathrm{CS}_{DG}(\beta \otimes x' \otimes \ \alpha \otimes x).
\end{aligned}$$

The equality $\Delta_! \circ \tilde{\tau_E} = \Delta^*\tilde{\tau} \circ \Delta_! = \tilde{\tau_F} \circ \Delta_!$ is a direct consequence of Proposition 5.23. The equality $\tilde{m} \circ \tilde{\tau_F} = \tilde{m}$ is a direct consequence of Corollary 5.18 and Corollary 5.17.

■

7.2.3 Neutral element

Proposition 7.5. *Suppose that the fibration $F \hookrightarrow E \to X$ admits a section $s : X \to E$ such that*

$$m(s(\pi(e)), e) = m(e, s(\pi(e))) = e$$

for any $e \in E$. Then, $\tilde{s}([X]) \in H_n(X, \mathcal{F})$ is a neutral element for CS_{DG}.

Proof. A section $s : X \to E$ is a morphism of fibrations when one considers X as the total space of the trivial fibration $\star \hookrightarrow X \xrightarrow{\mathrm{Id}} X$. Denote $\mathcal{F} = C_*(F)$ and $\mathcal{F}^2 = C_*(F^2)$.
Consider the following diagram

$$\begin{array}{ccccccc}
H_*(X,\mathcal{F}) \otimes H_*(X,\mathbb{Z}) & \xrightarrow{K} & H_*(X^2, C_*(F \times \{\star\})) & \xrightarrow{\Delta_!} & H_*(X, C_*(F \times \{\star\})) & & \\
\downarrow{\scriptstyle \mathrm{Id}\otimes\tilde{s}} & & \downarrow{\scriptstyle \widetilde{\mathrm{Id}\times s}} & & \downarrow{\scriptstyle \Delta^*\widetilde{\mathrm{Id}\times s}} & \searrow^{\tilde{\sigma}} & \\
H_*(X,\mathcal{F}) \otimes H_*(X,\mathcal{F}) & \xrightarrow{K} & H_*(X^2, \mathcal{F}^2)) & \xrightarrow{\Delta_!} & H_*(X, \Delta^*\mathcal{F}^2)) & \xrightarrow{\tilde{m}} & H_*(X,\mathcal{F}),
\end{array}$$

where $\sigma : E \,{}_{\pi}\!\times_{\mathrm{Id}} X \to E$, $(e, \pi(e)) \mapsto e$. Since $m \circ (\mathrm{Id} \times s) = \sigma$, Corollary 5.18 proves that the last triangle commutes. The first diagram commutes by Lemma 6.21. The second diagram commutes by Proposition 5.23.
It remains to prove that $\tilde{\sigma} \circ \Delta_! \circ K(\gamma \otimes [X]) = \gamma$ for any $\gamma \in H_*(X, \mathcal{F})$. Let Ξ be a set of DG Morse data on X. We can and will assume that the Morse function $f : X \to \mathbb{R}$ has only one local maximum x_{max}. In this case, $[(\star \otimes x_{\max})] = [X] \in H_n(X, \mathbb{Z})$.
Let $\alpha \otimes x \in C_*(X, \Xi, \mathcal{F})$. We start by computing $\Delta_!\left((\alpha, \star) \otimes (x, x_{max})\right)$:

$$\begin{aligned}
\overline{\mathcal{M}}^{\Delta_!}((x, x_{\max}), y) &= \overline{W}^s(y) \cap \Delta^{-1}(\overline{W}^u(x, x_{\max})) \\
&= \overline{W}^s(y) \cap \overline{W}^u(x_{\max}) \cap \overline{W}^u(x) \\
&= \overline{W}^s(y) \cap \overline{W}^u(x) = \overline{\mathcal{M}}^{\mathrm{Id}_!}(x, y).
\end{aligned}$$

We now compare orientations. The orientation rule $(\mathrm{Or}\ \overline{W}^s(x_{\max}), \mathrm{Or}\ \overline{W}^u(x_{\max})) = \mathrm{Or}\ X$ gives $\mathrm{Or}\ \overline{W}^s(x_{\max}) = +$ and

$$\left(\mathrm{Or}\ \overline{W}^s(x, x_{\max}), \mathrm{Or}\ \overline{\mathcal{M}}^{\Delta_!}((x, x_{\max}), y)\right) = \mathrm{Or}\ \overline{W}^s(y) \Leftrightarrow \left(\mathrm{Or}\ \overline{W}^s(x), \mathrm{Or}\ \overline{\mathcal{M}}^{\Delta_!}((x, x_{\max}), y)\right) = \mathrm{Or}\ \overline{W}^s(y).$$

Therefore $\mathrm{Or}\ \overline{\mathcal{M}}^{\Delta_!}((x, x_{max}), y) = \mathrm{Or}\ \overline{\mathcal{M}}^{\mathrm{Id}_!}(x, y)$ and it follows that for any

$$\tilde{\sigma} \circ \Delta_! \circ K(\alpha \otimes x \otimes \star \otimes x_{max}) = \sigma(\alpha, \star) \otimes x = \alpha \otimes x$$

up to chain homotopy. Indeed, $\sigma_* : C_*(F \times \{\star\}) \to C_*(F),\ (\alpha, \star) \mapsto \alpha$ is a morphism of DG-modules and therefore $\tilde{\sigma} : C_*(X, C_*(F \times \{\star\})) \to C_*(X, C_*(F)),\ (\alpha, \star) \otimes x \mapsto \alpha \otimes x$.

■

Remark 7.6. *The inclusion of constant loops $s : X \to \mathcal{L}X$ is a section of the loop-loop fibration $\Omega X \hookrightarrow \mathcal{L}X \overset{ev}{\to} X$.*

7.2.4 Functoriality

Let $(Y^k, \star_Y)$ be a k-dimensional, smooth, pointed, oriented, closed, connected manifold. Let $g : Y \to X$ be a continuous map and $F \hookrightarrow E_Y \overset{\pi_Y}{\to} Y$ be the fibration obtained by pulling back by g the fibration $F \hookrightarrow E \overset{\pi}{\to} X$. This fibration is endowed with the morphism of fibrations

$$g^*m : E_Y \ {}_{\pi_Y}\times_{\pi_Y} E_Y \to E_Y.$$

Proposition 7.7. *The shriek map $g_! : H_*(X, C_*(F)) \to H_{*-n+k}(Y, g^*C_*(F))$ is a morphism of rings up to sign.*
More precisely, up to chain homotopy,

$$\mathrm{CS}^Y_{DG}\left(g_!(\alpha \otimes x) \otimes g_!(\beta \otimes x')\right) = (-1)^{(k-n)(n+|\alpha|+|x|-|\beta|-|x'|)} g_! \left(\mathrm{CS}^X_{DG}(\alpha \otimes x \otimes \beta \otimes x')\right).$$

Proof. We denote $l = k - n \in \mathbb{Z}$.
The proof amounts to show that the following diagram is commutative up to the wanted sign :

$$\begin{array}{ccccccc}
H_{i+l}(Y, g^*\mathcal{F}) \otimes H_{j+l}(Y, g^*\mathcal{F}) & \xrightarrow{K} & H_{i+j+2l}\left(Y^2, (g^2)^*\mathcal{F}^2\right) & \xrightarrow{\Delta_!} & H_{i+j-n+l}(Y, g^*\mathcal{F}^2) & \xrightarrow{g^*\tilde{m}} & H_{i+j-n+l}(Y, g^*\mathcal{F}) \\
\uparrow{\scriptstyle g_!\otimes g_!} & & \uparrow{\scriptstyle (g\times g)_!} & & \uparrow{\scriptstyle g_!} & & \uparrow{\scriptstyle g_!} \\
H_i(X, \mathcal{F}) \otimes H_j(X, \mathcal{F}) & \xrightarrow[K]{} & H_{i+j}\left(X^2, \mathcal{F}^2\right) & \xrightarrow[\Delta_!]{} & H_{i+j-n}(X, \mathcal{F}^2) & \xrightarrow[\tilde{m}]{} & H_{i+j-n}(X, \mathcal{F})
\end{array}$$

The first square commutes up to the wanted sign by Lemma 6.16, the second square commutes by the composition property [BDHO24, Theorem 8.1.1] and the third square commutes according to Proposition 5.23.

■

Remark 7.8. *In particular, if Y is n-dimensional, then $g_! : H_*(X, \mathcal{F}) \to H_*(Y, g^*\mathcal{F})$ is a morphism of rings.*

Corollary 7.9. *If $g : Y \to X$ is an orientation-preserving homotopy equivalence between two manifolds of same dimension, then $g_! : H_*(X, C_*(F)) \to H_*(Y, g^*C_*(F))$ and $g_* : H_*(Y, g^*C_*(F)) \to H_*(X, C_*(F))$ are isomorphisms of rings.*

Proof. Using [BDHO24, Corollary 10.6.4], $g_!$ and g_* are isomorphisms inverse to each other. Therefore $g_!$ is an isomorphism of rings and so is $g_* = (g_!)^{-1}$.

■

7.2.5 Spectral sequence, chain description of the product

We describe a chain-level model $\mathrm{CS}_{DG} : C_i(X,\Xi,\mathcal{F}) \otimes C_j(X,\Xi,\mathcal{F}) \to C_{i+j-n}(X,\Xi,\mathcal{F})$ for the product $\mathrm{CS}_{DG} : H_i(X,\mathcal{F}) \otimes H_i(X,\mathcal{F}) \to H_{i+j-n}(X,\mathcal{F})$.
We will prove that this preserves the canonical filtrations associated to these complexes and therefore prove that our construction also endow the canonical spectral sequence $E^r_{p,q}$ associated to an enriched Morse complex $C_*(X,\Xi_0,\mathcal{F})$ with an algebra structure which converges towards $H_*(X,\mathcal{F})$ as algebras.
This a DG Morse equivalent to [CJY04, Theorem 1] for the Chas-Sullivan product that has been generalized in [GS08, Theorem 3.6].

Proposition 7.10. *Let Ξ_0 be a set of DG Morse data on X. The canonical filtration*

$$F_p(C_*(X,\Xi_0,\mathcal{F})) = \bigoplus_{\substack{i+j=k\\ i\leq p}} \mathcal{F}_j \otimes \mathbb{Z}\mathrm{Crit}_i(f)$$

induces a spectral sequence $E^r_{p,q}$ that is endowed with an algebra structure

$$E^r_{p,q} \otimes E^r_{l,m} \to E^r_{p+l-n,q+m}$$

and converges towards $H_*(X,\mathcal{F})$ as algebras. For $s,t \geq 0$, $E^2_{s,t} = H_s(X, H_t(\mathcal{F}))$ and the algebra structure is given up to sign by the intersection product on X with coefficients in $H_t(\mathcal{F})$.

Proof. We will use for $\Delta_!$ the second definition of the shriek maps given in [BDHO24, Section 10.4] and take Ξ_1 and Ξ_2 generic sets of Morse data on X such that

$$\Delta|_{W^s_{f_2}(y)} \pitchfork W^u_{f_0+f_1}(x,x')$$

for all $y \in \mathrm{Crit}(f_2)$, $x \in \mathrm{Crit}(f_0)$, $x' \in \mathrm{Crit}(f_1)$.
We define a chain-level product

$$\begin{array}{rcl} C_*(X,\Xi_0,\mathcal{F}) \otimes C_*(X,\Xi_0,\mathcal{F}) & \xrightarrow{\mathrm{Id}\otimes\Psi_{01}} & C_*(X,\Xi_0,\mathcal{F}) \otimes C_*(X,\Xi_1,\mathcal{F}) \\ & \xrightarrow{K} & C_*(X^2,\Xi_{01},\mathcal{F}^2) \\ & \xrightarrow{\Delta_!} & C_*(X,\Xi_2,\Delta^*\mathcal{F}^2) \\ & \xrightarrow{\Psi_{20}} & C_*(X,\Xi_0,\Delta^*\mathcal{F}^2) \\ & \xrightarrow{\tilde{m}} & C_*(X,\Xi_0,\mathcal{F}). \end{array}$$

Every map preserves the canonical fibrations associated to an enriched complex. Indeed, [BDHO24, Theorem 8.1.1] states that all direct and shriek maps preserve the filtrations. The continuation maps Ψ_{01} and Ψ_{20} are defined by equation [BDHO24, equation (33)] and clearly preserves filtrations. We proved in Proposition 5.21 that every morphism of complexes induced by a morphism of $\mathcal{A}_\infty$-module preserves filtrations.
It only remains to prove that K preserves filtrations. It suffices to remark that if $\alpha \otimes x \in \mathcal{F}_j \otimes \mathbb{Z}\mathrm{Crit}_i(f)$ and $\beta \otimes y \in \mathcal{G}_l \otimes \mathbb{Z}\mathrm{Crit}_k(g)$, then

$$K^{alg}((\alpha\otimes x)\otimes(\beta\otimes y)) = (-1)^{li}(\alpha\otimes\beta)\otimes(x,y) \in (\mathcal{F}\otimes\mathcal{G})_{j+l} \otimes \mathbb{Z}\mathrm{Crit}_{i+k}(f+g)$$

and in particular

$$K^{alg}(F_p(C_*(X,\Xi_X,\mathcal{F})) \otimes F_q(C_*(Y,\Xi_Y,\mathcal{G}))) = F_{p+q}(C_*(X\times Y, \Xi_{X\times Y}, \mathcal{F}\otimes\mathcal{G})).$$

If $\mathcal{F} = C_*(F)$ and $\mathcal{G} = C_*(G)$, we also have

$$K^{top}((\alpha\otimes x)\otimes(\beta\otimes y)) = (-1)^{li}(\alpha,\beta)\otimes(x,y) \in C_{j+l}(F\times G) \otimes \mathbb{Z}\mathrm{Crit}_{i+k}(f+g)$$

and therefore

$$K^{top}(F_p(C_{i+j}(X,\Xi_X,C_*(F))) \otimes F_q(C_{k+l}(Y,\Xi_Y,C_*(G)))) \subset F_{p+q}(C_{i+j+k+l}(X\times Y, \Xi_{X\times Y}, C_*(F\times G))).$$

It follows that for any $a, b \in \mathbb{N}$,

$$\mathrm{CS}_{DG}(F_p(C_a(X,\Xi_0,\mathcal{F})) \otimes F_q(C_b(X,\Xi_0,\mathcal{F}))) \subset F_{p+q-n}(C_{a+b-n}(X,\Xi_0,\mathcal{F}))$$

and [McC01, Theorem 2.14] proves that CS_{DG} induces an algebra structure

$$\mathrm{CS}_{DG}^{(r)} : E^r_{p,q} \otimes E^r_{l,m} \to E^r_{p+l-n,q+m}$$

such that $E^r_{p,q}$ converges towards $H_*(X,\mathcal{F})$ as algebras.
On the second page, we infer using the description of the cross products above, [BDHO24, Remark 9.2.3] stating that $\Delta_!$ induces on the second page $\Delta_!^{(2)}$ the usual shriek morphism with local coefficients and Proposition 5.21 stating that $\tilde{m}$ is a limit of morphism of spectral sequences, that the algebra structure on $H_s(X, H_t(\mathcal{F}))$ is given up to sign by the intersection product on X with coefficients in $H_t(\mathcal{F})$.

■

7.2.6 Equivalence with the Grüher-Salvatore definition

Proposition 7.11. *The product CS_{DG} corresponds at the homology level via the Fibration Theorem to the product $\mu_* : H_*(E)^{\otimes 2} \to H_*(E)$ defined in [GS08].*

Proof. We prove that the following diagram commutes

$$\begin{array}{ccccccccc}
 & & & \overset{\Delta_!}{\frown} & & & & & \\
H_i(X,\mathcal{F}) \otimes H_j(X,\mathcal{F}) & \xrightarrow{K} & H_{i+j}(X^2,\mathcal{F}^2) & \xrightarrow{i_{\eta_\Delta,!}} & H_{i+j}(\overline{\eta_\Delta}, \partial\overline{\eta_\Delta}, \mathcal{F}^2) & = & H_{i+j-n}(X,\mathcal{F}^2) & \xrightarrow{\tilde{m}} & H_{i+j-n}(X,\mathcal{F}) \\
\downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\
H_i(E) \otimes H_j(E) & \xrightarrow{K} & H_{i+j}(E^2) & \xrightarrow{\tau_*} & \tilde{H}_{i+j}\left((E\ {}_\pi\!\times_\pi E)^{\pi^*TX}\right) & \xrightarrow{u_*} & H_{i+j-n}(E\ {}_\pi\!\times_\pi E) & \xrightarrow{m_*} & H_{i+j-n}(E).
\end{array}$$

The vertical arrows are given by the Fibration Theorem. We used the following notations from [GS08]:
- η_Δ is a tubular open neighborhood of the diagonal in X^2 seen as a disk bundle over X whose normal bundle is denoted by ν_Δ. Then η_Δ is homeomorphic to the total space of ν_Δ.
- $(E\ {}_\pi\!\times_\pi E)^{\pi^*TX}$ is the Thom space of the normal bundle $\pi^*\nu_\Delta \simeq ev^*TX$ associated to the inclusion $E\ {}_\pi\!\times_\pi E \hookrightarrow E^2$. We have an identification $(E\ {}_\pi\!\times_\pi E)^{\pi^*\nu_\Delta} \simeq {}^{E^2}\!/_{E^2 \setminus \pi^*\eta_\Delta}$.
- τ_* is the Pontrjagin-Thom collapse map and u_* is the Thom isomorphism.

• Lemma 6.20 shows that the first square commutes.
• The second and third squares commute. This is a direct application of Theorem 3.2.
• The fourth square commutes by Theorem 5.8 applied to the morphism of fibrations $m : E\ {}_\pi\!\times_\pi E \to E$.

■

Corollary 7.12. *In particular, the product*

$$\mathrm{CS}_{DG} : H_*(X, C_*(\Omega X)^{ad})^{\otimes 2} \to H_*(X, C_*(\Omega X)^{ad})$$

corresponds via the Fibration Theorem to the Cohen-Jones [CJ02] definition of the Chas-Sullivan product.

■

We can reprove using our setting the homotopy invariance of the Chas-Sullivan product.

Proposition 7.13. ***(Homotopy invariance)*** *Let $(Y, \star_Y)$ be a smooth, oriented, pointed, closed and connected manifold. Let $g : Y \to X$ be an orientation-preserving homotopy equivalence such that $\star = g(\star_Y)$. Consider the fibrations*

$$\Omega X \hookrightarrow \mathcal{L}X \to X \text{ and } \Omega Y \hookrightarrow \mathcal{L}Y \to Y.$$

There exists an isomorphism of rings

$$g_\# : H_*(X, C_*(\Omega X)) \to H_*(Y, C_*(\Omega Y)).$$

Proof. We proved in Corollary 7.9 that $g_! : H_*(X, C_*(\Omega X)) \to H_*(Y, g^*C_*(\Omega X))$ is a ring isomorphism. It remains to define a ring isomorphism $H_*(Y, g^*C_*(\Omega X)) \to H_*(Y, C_*(\Omega Y))$.
Consider $f : X \to Y$ a homotopy inverse of g such that $f(\star) = \star_Y$. We see $g^*\mathcal{L}X$ as $g^*\mathcal{L}X = \{(y, \gamma) \in Y \times \mathcal{L}X,\ \gamma(0) = g(y)\}$. Consider the map

$$\mathcal{L}f : g^*\mathcal{L}X \to (f \circ g)^*\mathcal{L}Y,\ \mathcal{L}f(y, \gamma) = (y, f \circ \gamma).$$

The map $\mathcal{L}f$ is a morphism of fibrations over ΩY and induces a isomorphism

$$\mathcal{L}f_* : H_*(g^*\mathcal{L}X) \to H_*((f \circ g)^*\mathcal{L}Y).$$

The map

$$\mathcal{L}(f \circ g) : \mathcal{L}Y \to (f \circ g)^*\mathcal{L}Y,\ \mathcal{L}(f \circ g)(\gamma) = f \circ g \circ \gamma$$

is also a morphism of fibrations and induces an isomorphism

$$\mathcal{L}(f \circ g)_* : H_*(\mathcal{L}Y) \to H_*((f \circ g)^*\mathcal{L}Y).$$

Corollary 5.17 shows that

$$\widetilde{\mathcal{L}(f \circ g)} : H_*(Y, C_*(\Omega Y)) \to H_*(Y, (f \circ g)^*C_*(\Omega Y))$$

and

$$\widetilde{\mathcal{L}f} : H_*(Y, g^*C_*(\Omega X)) \to H_*(Y, (f \circ g)^*C_*(\Omega Y))$$

are isomorphisms.
Denote now

$$g_\# = \widetilde{\mathcal{L}(f \circ g)}^{-1} \circ \widetilde{\mathcal{L}f} \circ g_! : H_*(X, C_*(\Omega X)) \xrightarrow{\sim} H_*(Y, C_*(\Omega Y)).$$

We prove that $g_\#$ is an isomorphism of rings where the products on $H_*(X, \Omega X)$ and $H_*(Y, \Omega Y)$ are respectively the DG Chas-Sullivan products induced by the concatenations $m_X : \mathcal{L}X\ {}_{ev}\times_{ev}\ \mathcal{L}X \to \mathcal{L}X$ and $m_Y : \mathcal{L}Y\ {}_{ev}\times_{ev}\ \mathcal{L}Y \to \mathcal{L}Y$ which are morphisms of fibrations.

We already know from Corollary 7.9 that

$$g_! : H_*(X, C_*(\Omega X)) \to H_*(Y, g^*C_*(\Omega X))$$

is an isomorphism of rings. It remains to prove that

$$\widetilde{\mathcal{L}(f \circ g)}^{-1} \circ \widetilde{\mathcal{L}f} : H_*(X, g^*C_*(\Omega X)) \xrightarrow{\sim} H_*(Y, C_*(\Omega Y))$$

is a morphism of rings.
Denote $\mathcal{F} = C_*(\Omega X)$, $\mathcal{F}^2 = C_*(\Omega X^2)$ and $\mathcal{G} = C_*(\Omega Y)$, $\mathcal{G}^2 = C_*(\Omega Y^2)$. The following diagram commutes

$$\begin{array}{ccccccc}
H_*(Y, g^*\mathcal{F})^{\otimes 2} & \xrightarrow{K} & H_*(Y^2, (g \times g)^*\mathcal{F}^2) & \xrightarrow{\Delta_!} & H_*(Y, \Delta^*(g \times g)^*\mathcal{F}^2) & \xrightarrow{g^*\tilde{m}_X} & H_*(Y, g^*\mathcal{F}) \\
\downarrow{\scriptstyle \widetilde{\mathcal{L}f}^{\otimes 2}} & & \downarrow{\scriptstyle \widetilde{\mathcal{L}f \times \mathcal{L}f}} & & \downarrow{\scriptstyle \Delta^*\widetilde{\mathcal{L}f \times \mathcal{L}f}} & & \downarrow{\scriptstyle \widetilde{\mathcal{L}f}} \\
H_*(Y, (f \circ g)^*\mathcal{G})^{\otimes 2} & \xrightarrow{K} & H_*(Y^2, \left((f \circ g)^{\times 2}\right)^*\mathcal{G}^2) & \xrightarrow{\Delta_!} & H_*(Y, \Delta^*\left((f \circ g)^{\times 2}\right)^*\mathcal{G}^2) & \xrightarrow{(f\circ g)^*\tilde{m}_Y} & H_*(Y, (f \circ g)^*\mathcal{G}) \\
\uparrow{\scriptstyle \widetilde{\mathcal{L}(f \circ g)}^{\otimes 2}} & & \uparrow{\scriptstyle \widetilde{\mathcal{L}(f \circ g)^{\times 2}}} & & \uparrow{\scriptstyle \Delta^*\widetilde{\mathcal{L}(f \circ g)^{\times 2}}} & & \uparrow{\scriptstyle \widetilde{\mathcal{L}(f \circ g)}} \\
H_*(Y, \mathcal{G})^{\otimes 2} & \xrightarrow{K} & H_*(Y^2, \mathcal{G}^2) & \xrightarrow{\Delta_!} & H_*(Y, \Delta^*\mathcal{G}^2) & \xrightarrow{\tilde{m}_Y} & H_*(Y, \mathcal{G}).
\end{array}$$

Indeed, the commutativity of the two squares on
— the first column is a consequence of Lemma 6.21.
— the second column is a consequence of Proposition 5.23.
— the third column is a consequence of Corollary 5.17 and Corollary 5.18 since

$$m_Y \circ (\mathcal{L}f \times \mathcal{L}f) = \mathcal{L}f \circ m_X, \text{ and } m_Y \circ (\mathcal{L}(f \circ g) \times \mathcal{L}(f \circ g)) = \mathcal{L}(f \circ g) \circ m_Y.$$

■

7.2.7 Pullback

Let $(Y, \star_Y)$ be a pointed topological space and $F \hookrightarrow E \xrightarrow{p} Y$ be a fibration. Let $f : X^n \to Y$ be a continuous map such that $f(\star_X) = \star_Y$. Let $m : E \,{}_p\times_p E \to E$ be a morphism of fibrations. Consider the pullback fibrations

$$F \hookrightarrow f^*E \to X \text{ and } F^2 \hookrightarrow f^*(E \,{}_p\times_p E) \to X.$$

It is straightforward that $f^*m_* : f^*\left(E \,{}_p\times_p E\right) \to f^*E$ is also a morphism of fibrations. Therefore, there exists a product of degree $-n$

$$\mathrm{CS}_{DG} : H_*(X, f^*C_*(F))^{\otimes 2} \to H_*(X, f^*C_*(F))$$

For instance :

— Let $X = \{\star\} \overset{i}{\subset} Y$ and consider $\Omega Y \hookrightarrow \mathcal{L}Y \to Y$. Then, $\mathrm{CS}_{DG} : H_*(\star, i^*C_*(\Omega Y))^{\otimes 2} \to H_*(\star, i^*C_*(\Omega Y))$ describes the Pontryagin product on ΩY.

— Let $Y = \mathcal{L}X$ and $f : X \to \mathcal{L}X$ be the inclusion of constant loops. Consider the fibration $\Omega\mathcal{L}X \hookrightarrow \mathcal{L}\mathcal{L}X \to \mathcal{L}X$. Then, $\mathrm{CS}_{DG} : H_*(X, f^*C_*(\Omega\mathcal{L}X))^{\otimes 2} \to H_*(X, f^*C_*(\Omega\mathcal{L}X))$ describes a product degree $-n$ on pinched tori.

7.3 DG Chas-Sullivan product for manifolds with boundary

If X has a boundary, take a Morse function without any critical points on the boundary and a pseudo-gradient pointing outward along $\partial_+X \subset \partial X$ and inward along $\partial_-X \subset \partial X$ so that the boundary decomposes $\partial X = \partial_+X \cup \partial_-X$ (see [AD14, Section 3.5] for further explanations on Morse theory for a manifold with boundary and [BDHO24, Section 5.3] for the construction of Morse homology with DG coefficients for a manifold with boundary).

Let X, Y be pointed, oriented, compact and connected manifolds with boundary. Let $\mathcal{F}, \mathcal{F}'$ be two DG modules over $C_*(\Omega X)$ and $\mathcal{G}$ be a DG module over $C_*(\Omega Y)$. Let $\boldsymbol{\varphi} : \mathcal{F} \to \mathcal{F}'$ be a morphism of $\mathcal{A}_\infty$-modules. The Künneth map

$$K : \begin{array}{ccc} C_*(X, \partial_+X, \mathcal{F}) \otimes C_*(Y, \partial_+Y, \mathcal{G}) & \to & C_*(X \times Y, X \times \partial_+Y \cup \partial_+X \times Y, \mathcal{F} \times \mathcal{G}) \\ \alpha \otimes x \otimes \beta \otimes y & \mapsto & (-1)^{|\beta||x|}(\alpha, \beta) \otimes (x, y) \end{array}$$

and

$$\tilde{\varphi} = \sum_n \varphi_{n+1}\tilde{\mathbf{m}}^n : H_*(X, \partial_+X, \mathcal{F}) \to H_*(X, \partial_+X, \mathcal{F}')$$

are defined in the same way, since all the trajectories between critical points avoid the boundary.

Let $\psi : X \to Y$ be a continuous map. The shriek map $\psi_!$ has target $H_*(X, \partial_+X, \psi^*\mathcal{G})$ and factors through $H_*(Y, \mathcal{G}) \to H_*(Y, \partial_+Y, \mathcal{G}) \to H_*(X, \partial_+X, \psi^*\mathcal{G})$ (see [BDHO24] Remark 10.4.3). Therefore $\Delta_! : H_*(X^2, \partial_+(X^2), \mathcal{F}^2) \to H_*(X, \partial_+X, \Delta^*\mathcal{F}^2)$ is well-defined and has the same properties and the degree $-n$ product

$$\mathrm{CS}_{DG} : H_*(X, \partial_+X, \mathcal{F})^{\otimes 2} \to H_*(X, \partial_+X, \mathcal{F})$$

is defined in the same way and has the same properties under the same assumptions.

In particular, if $F \hookrightarrow E \overset{\pi}{\to} X$ and $m : E \,{}_\pi\times_\pi E \to E$ is a morphism of fibrations, then CS_{DG} induces a degree $-n$ product

$$\mathrm{CS}_{DG} : H_*(E, E_+) \otimes H_*(E, E_+) \to H_*(E, E_+)$$

where we denoted $E_+ = \pi^{-1}(\partial_+X)$.

8 Further directions

In this paper, we laid out the necessary tools to study, using enriched Morse theory, products on total spaces of fibrations that *intersect on the basis* and *multiply on the fiber*. The path product defined and studied by [Ste23] is an example of such a product that does not fall into the category of products studied in this paper.

8.1 Path products

Let Y be a topological space, X^n be a pointed, oriented, closed, connected manifold, and $f : X \to Y$ be a continuous map.

Define

$$P_{X,f}Y = \{(x, x', \alpha) \in X^2 \times \mathcal{P}Y,\ \alpha(0) = f(x), \alpha(1) = f(x')\}$$

and $ev_0, ev_1 : P_{X,f}Y \to X$, $ev_0(x, x', \alpha) = x \in X$, $ev_1(x, x', \alpha) = x' \in X$ the evaluation at the basepoint and endpoint. We will denote $ev = (ev_0, ev_1) : P_{X,f}Y \to X^2$. A degree $-n$ product

$$\Lambda : H_*(P_{X,f}Y)^{\otimes 2} \to H_*(P_{X,f}Y)$$

has been defined and studied by [Ste23] if Y is a closed manifold. This product, as the Chas-Sullivan product, is defined by intersecting on a space where the paths are concatenable and then concatenating.
Denote $\pi : P_{X,f}Y \ {}_{ev_1}\!\times_{ev_0} P_{X,f}Y \to X^3,\ \pi(\gamma, \tau) = (\gamma(0), \gamma(1) = \tau(0), \tau(1))$. Since $P_{X,f}Y$ and $P_{X,f}Y \ {}_{ev_1}\!\times_{ev_0} P_{X,f}$ are the total spaces of the fibrations

$$\Omega Y \hookrightarrow P_{X,f}Y \overset{ev}{\to} X^2$$

and

$$\Omega Y^2 \hookrightarrow P_{X,f}Y \ {}_{ev_1}\!\times_{ev_0} P_{X,f}Y \overset{\pi}{\to} X^3,$$

we interpret this product in our setting by

$$\begin{aligned}
\mathrm{PP}_{DG} :\ & H_i(X^2, C_*(\Omega Y)) \otimes H_j(X^2, C_*(\Omega Y)) & \\
\xrightarrow{K}\ & H_{i+j}(X^4, C_*(\Omega Y^2)) & \simeq H_{i+j}(P_{X,f}Y \times P_{X,f}Y) \\
\xrightarrow{D_!}\ & H_{i+j-n}(X^3, \Delta^* C_*(\Omega Y^2)) & \simeq H_{i+j-n}(P_{X,f}Y \ {}_{ev_1}\!\times_{ev_0} P_{X,f}Y) \\
\xrightarrow{\tilde{m}}\ & H_{i+j-n}(X^3, p_{1,3}^* C_*(\Omega Y)) & \simeq H_{i+j-n}(m(P_{X,f}Y \ {}_{ev_1}\!\times_{ev_0} P_{X,f}Y)) \\
\xrightarrow{p_*}\ & H_{i+j-n}(X^2, C_*(\Omega Y)) & \simeq H_{i+j-n}(P_{X,f}Y)
\end{aligned}$$

where $D : X^3 \to X^4,\ D(a, b, c) = (a, b, b, c)$ and $p : X^3 \to X^2,\ p(a, b, c) = (a, c)$ and

$$m : P_{X,f}Y \ {}_{ev_1}\!\times_{ev_0} P_{X,f}Y \to p^* P_{X,f}Y$$

is the morphism of fibrations over X^3 induced by the concatenation of paths.

Remark 8.1. *Intuitively, given two chains σ, τ in $P_{X,f}Y$, this product will intersect $ev_{1,*}(\sigma)$ with $ev_{0,*}(\tau)$ on X, concatenate at the intersection and forget the concatenation point.*
We can remark that, since we do not intersect along the diagonal $\Delta : X^2 \to X^4$, this product does not fall into the category of products studied in this paper. Nonetheless, the developed tools enable to define it and study it.

Using techniques very similar to those used in Section 7 we will prove in future work

Theorem 8.2. *The product PP_{DG} is associative (up to the sign $(-1)^n$ with the current orientation conventions) and admits a neutral element.*

Using the Fibration Theorem and the fact that all the maps defining PP_{DG} are compatible with it, this product yields a product $\mathrm{PP}_{DG} : H_*(P_{X,f}Y)^{\otimes 2} \to H_*(P_{X,f}Y)$. We will study in further detail the similarities and differences between PP_{DG} and $\Lambda : H_*(P_{X,f}Y)^{\otimes 2} \to H_*(P_{X,f}Y)$.

8.2 Further study on $\mathcal{A}_\infty$-structures

8.2.1 $\mathcal{A}_\infty$-algebra structures on enriched Morse complexes

This introduction of $\mathcal{A}_\infty$-structures to study enriched Morse theory is a clear path to defining an $\mathcal{A}_\infty$-structures on enriched Morse chains as it has been defined in [Abo11] and [Maz22] for the Morse cochains with coefficients in $\mathbb{Z}$. These constructions rely on studying the moduli spaces perturbed Morse gradient trees $\overline{\mathcal{T}}_t^{\mathbb{X}}(y; x_1, \dots, x_n)$ and particularly those of dimension 0 and 1.
The main difficulty in the enriched Morse setting would be to extend this work to define a fundamental class for the manifold with boundary and corners $\overline{\mathcal{T}}_t^{\mathbb{X}}(y; x_1, \dots, x_n)$ in any dimension and understand how the orientations behave with respect to the boundary strata.

8.2.2 Towards a Chas-Sullivan product $\mathbf{CS}_{\mathcal{A}_\infty}$ with coefficients in an $\mathcal{A}_\infty$-module

Whenever $\mathcal{F}$ is an $\mathcal{A}_\infty$-module over $C_*(\Omega X)$ and there exists a morphism of $\mathcal{A}_\infty$-modules $m : \Delta^*(\mathcal{F}\otimes\mathcal{F}) \to \mathcal{F}$, there exists a product

$$CS_{\mathcal{A}_\infty} : H_*(X, \mathcal{F})^{\otimes 2} \to H_*(X, \mathcal{F}).$$

We will study, in future work, the question of associativity, commutativity and the existence of a neutral element for this product.

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