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Quadratic obstructions to small-time local controllability for multi-input systems

Théo Gherdaoui*

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Abstract

We present a necessary condition for the small-time local controllability of multi-input control-affine systems on $\mathbb{R}^d$. This condition is formulated on the vectors of $\mathbb{R}^d$ resulting from the evaluation at zero of the Lie brackets of the vector fields: it involves both their direction and their amplitude.

The proof is an adaptation to the multi-input case of a general method introduced by Beauchard and Marbach in the single-input case - see [5]. It relies on a Magnus-type representation formula: the state is approximated by a linear combination of the evaluation at zero of the Lie brackets of the vector fields, whose coefficients are functionals of the time and the controls. Finally, obstructions to small-time local controllability result from interpolation inequalities.

Keywords: Control theory, ODEs, Obstruction for controllability, Lie Brackets, Small-time local exact controllability, control-affine systems.

Contents

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1 Introduction

1.1 Multi-input control-affine systems

One considers the control-affine system

$$x'(t) = f_0(x(t)) + u(t)f_1(x(t)) + v(t)f_2(x(t)), \tag{1.1}$$

where the state is $x(t) \in \mathbb{R}^d$, the controls are scalar functions $u(t), v(t) \in \mathbb{R}$, and f_0, f_1 and f_2 are real-analytic vector fields on a neighborhood of 0 in $\mathbb{R}^d$ such that $f_0(0) = 0$. The last hypothesis ensures that 0 is an equilibrium of the free system (i.e. with $(u, v) \equiv 0$).

For each $t > 0$, $u, v \in L^1((0,t), \mathbb{R})$, there exists a unique maximal mild solution to (1.1) with initial data 0, which we will denote by $x(\cdot; (u, v))$. As we are interested in small time, and small controls, the solution is well-defined up to time t.

In this article, we study the small-time local controllability of system (1.1) in the sense of Definition 1.1 below, which requires the following notion. For $t > 0$, $m \in \mathbb{N}$ and $p \in [1, +\infty]$, we consider the usual Sobolev space $W^{m,p}((0,t), \mathbb{R})$ equipped with the usual norm

$$\|u\|_{W^{m,p}} := \|u\|_{L^p} + \cdots + \|u^{(m)}\|_{L^p}.$$

The following concept was introduced by Beauchard and Marbach in [4] for scalar-input systems.

Definition 1.1 ($W^{m,p} \times W^{m',p'}$-STLC)**.** *Let $m, m' \in \mathbb{N}$, $p, p' \in [1, +\infty]$. We say that system* (1.1) *is $W^{m,p} \times W^{m',p'}$-STLC when, for every $t, \rho > 0$, there exists $\delta > 0$, such that, for every $x^* \in B(0, \delta)$, there exist $(u, v) \in W^{m,p}((0,t), \mathbb{R}) \times W^{m',p'}((0,t), \mathbb{R})$ with $\|(u, v)\|_{W^{m,p} \times W^{m',p'}} \leqslant \rho$ and $x(t; (u, v)) = x^*$.*

We say that system (1.1) *is $W^{m,p}$-STLC when* (1.1) *is $W^{m,p} \times W^{m,p}$-STLC.*

Remark 1.2. *The historical notion of STLC corresponds to $m = 0$, $p = \infty$ - see [8, 24, 22].*

1.2 Notations

We use the definitions and notations of Beauchard and Marbach in [5]. Let $X := \{X_0, X_1, X_2\}$ be a set of 3 non-commutative indeterminates.

Definition 1.3 (Free algebra)**.** *We consider $\mathcal{A}(X)$ the free algebra generated by X over the field $\mathbb{R}$, i.e. the unital associative algebra of polynomials of the indeterminates X_0, X_1 and X_2. The space $\mathcal{A}(X)$ can be seen as a graded algebra*

$$\mathcal{A}(X) = \bigoplus_{n\in\mathbb{N}} \mathcal{A}_n(X),$$

where $\mathcal{A}_n(X)$ is the finite-dimensional $\mathbb{R}$-vector space spanned by monomials of degree n over X.

Definition 1.4 (Free Lie algebra)**.** *For $a, b \in \mathcal{A}(X)$, one defines the Lie bracket of a and b as $[a, b] := ab - ba$, also called $ad_a(b)$. One defines by induction on $n \in \mathbb{N}$, $ad_a^{n+1}(b) = [a, ad_a^n(b)]$. This operation is anti-symmetric and satisfies the Jacobi's identity: for all $a, b, c \in \mathcal{A}(X)$,*

$$[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0. \tag{1.2}$$

Let $\mathcal{L}(X)$ be the free Lie algebra generated by X over the field $\mathbb{R}$, i.e. the smallest linear subspace of $\mathcal{A}(X)$ containing X and stable by the Lie bracket $[\cdot, \cdot]$. The space $\mathcal{L}(X)$ is a graded Lie algebra

$$\mathcal{L}(X) = \bigoplus_{n\in\mathbb{N}} \mathcal{L}_n(X), \qquad [\mathcal{L}_n(X), \mathcal{L}_m(X)] \subset \mathcal{L}_{n+m}(X),$$

where we define $\mathcal{L}_n(X) := \mathcal{L}(X) \cap \mathcal{A}_n(X)$, for any integer $n \in \mathbb{N}$.

Definition 1.5 (Iterated brackets)**.** *Let $\mathrm{Br}(X)$ be the free magma over X, i.e. the set of iterated brackets of elements of X, defined by induction as: $X_0, X_1, X_2 \in \mathrm{Br}(X)$ and if $a, b \in \mathrm{Br}(X)$, then the ordered pair (a, b) belongs to $\mathrm{Br}(X)$.*

There is a natural evaluation mapping E from $\mathrm{Br}(X)$ to $\mathcal{L}(X)$ defined by induction by $\mathrm{E}(X_i) := X_i$ for $i = 0, 1, 2$ and $\mathrm{E}((a, b)) := [\mathrm{E}(a), \mathrm{E}(b)]$.

Definition 1.6 (Length and homogeneous layers within $\mathcal{L}(X)$)**.** *For $b \in \mathrm{Br}(X)$, $|b|$ denotes the length of b. For $i \in [\![0, 2]\!]$, for $b \in \mathrm{Br}(X)$, $n_i(b)$ denotes the number of occurrences of the indeterminate X_i in b. We will use the notation: $n(b) := n_1(b) + n_2(b) = |b| - n_0(b)$. For $A_1, A_2 \subset \mathbb{N}$, $S_{A_1,A_2}(X)$ is the vector subspace of $\mathcal{L}(X)$ defined by*

$$S_{A_1,A_2}(X) := Span\{\mathrm{E}(b),\ b \in \mathrm{Br}(X),\ n_1(b) \in A_1, n_2(b) \in A_2\}.$$

For $A \subset \mathbb{N}$, $S_A(X)$ is defined by

$$S_A(X) := Span\{\mathrm{E}(b),\ b \in \mathrm{Br}(X),\ n(b) \in A\}.$$

For $i \in \mathbb{N}$, we write $S_i(X)$ instead of $S_{\{i\}}(X)$.

For example, with $b = (((X_1, (X_0, X_2)), X_2), (X_1, X_2))$, $|b| = 6$, $n_0(b) = 1$, $n_1(b) = 2$, $n_2(b) = 3$ and $n(b) = 5$.

Definition 1.7 (Left and right factors)**.** *For $b \in \mathrm{Br}(X)$ with $|b| > 1$, b can be written in a unique way as $b = (b_1, b_2)$, with $b_1, b_2 \in \mathrm{Br}(X)$. We use the notations $\lambda(b) = b_1$ and $\mu(b) = b_2$, which define maps $\lambda, \mu : \mathrm{Br}(X) \setminus X \to \mathrm{Br}(X)$.*

For example, with $b = ((X_1, X_2), ((X_2, X_0), X_0))$, one has $\lambda(b) = (X_1, X_2)$ and $\mu(b) = ((X_2, X_0), X_0)$.

Definition 1.8 (Bracket integration $b0^\nu$)**.** *For $b \in \mathrm{Br}(X)$ and $\nu \in \mathbb{N}$, we use the unconventional short-hand $b0^\nu$ to denote the right-iterated bracket $(\cdots(b, X_0), \ldots, X_0)$, where X_0 appears ν times.*

For example, if $b = ((X_1, X_2), (X_0, X_2))$, then $b0^2 = ((((X_1, X_2), (X_0, X_2)), X_0), X_0)$.

Definition 1.9 (Lie bracket of vector fields)**.** *Let $f, g : \Omega \to \mathbb{R}^d$ be two smooth (at least $\mathcal{C}^1$) vector fields on an open subset Ω of $\mathbb{R}^d$. One defines*

$$[f, g] : x \in \Omega \mapsto \mathrm{D}g_x f(x) - \mathrm{D}f_x g(x). \tag{1.3}$$

Definition 1.10 (Evaluated Lie bracket)**.** *Let f_0, f_1, f_2 be $\mathcal{C}^\infty(\Omega, \mathbb{R}^d)$ vector fields on an open subset Ω of $\mathbb{R}^d$ and $f = \{f_0, f_1, f_2\}$.*

For $b \in \mathcal{L}(X)$, we define $f_b := \Lambda(b)$, where $\Lambda : \mathcal{L}(X) \to \mathcal{C}^\infty(\Omega, \mathbb{R}^d)$ is the unique homomorphism of Lie algebras such that $\Lambda(X_i) = f_i$, for $i \in [\![0, 2]\!]$.

We will write f_b instead of $f_{E(b)}$ when $b \in \mathrm{Br}(X)$. Finally, for $\mathcal{N} \subset \mathrm{Br}(X)$, we use the notation

$$\mathcal{N}(f)(0) := Span\{f_b(0),\ b \in \mathcal{N}\} \subseteq \mathbb{R}^d. \tag{1.4}$$

For example, if $b = (((X_1, X_2), X_0), (X_1, X_0))$, one has $f_b := [[[f_1, f_2], f_0], [f_1, f_0]]$.

1.3 A Hall basis

The purpose of this section is to introduce tools in order to construct a basis of the free Lie algebra $\mathcal{L}(X)$.

Definition 1.11 (Hall set)**.** *A* Hall set *is a subset $\mathcal{B}$ of $\mathrm{Br}(X)$ endowed with a total order $<$ such that*

- *$X \subset \mathcal{B}$,*
- *for all $b_1, b_2 \in \mathrm{Br}(X)$, $(b_1, b_2) \in \mathcal{B}$ iff $b_1, b_2 \in \mathcal{B}$, $b_1 < b_2$ and either $b_2 \in X$ or $\lambda(b_2) \leqslant b_1$,*
- *for all $b_1, b_2 \in \mathcal{B}$ such that $(b_1, b_2) \in \mathcal{B}$ then $b_1 < (b_1, b_2)$.*

Theorem 1.12 (Viennot, [18])**.** *Let $\mathcal{B} \subset \mathrm{Br}(X)$ be a Hall set. Then $E(\mathcal{B})$ is a basis of $\mathcal{L}(X)$.*

For a Hall set $\mathcal{B}$ and $A \subset \mathbb{N}$, we denote by $\mathcal{B}_A$ the subset of $\mathcal{B}$ defined by $\mathcal{B}_A := \{b \in \mathcal{B};\ n(b) \in A\}$. When A is a singleton, we write $\mathcal{B}_N$ instead of $\mathcal{B}_{\{N\}}$.
The definition of a Hall set is also an algorithm for its construction. Indeed, the subsets $\mathcal{B}_N$ of a Hall set $\mathcal{B}$ can be constructed by induction on N. One may start, for example, with $\mathcal{B}_0 = \{X_0\}$ and $\mathcal{B}_1 = \{X_1 0^{\nu_1}, X_2 0^{\nu_2};\ \nu_1, \nu_2 \in \mathbb{N}\}$ with the following order

$$\forall k \in \mathbb{N}, \qquad X_1 0^k < X_2 0^k < X_1 0^{k+1} < X_2 0^{k+1} < \cdots < X_0.$$

which is compatible with the 3 axioms above. For $N \geqslant 2$, to find all Hall elements $b \in \mathcal{B}_N$ given $\mathcal{B}_{[\![1,N-1]\!]}$, one adds first all (a, b) with $a \in \mathcal{B}_{N-1}$, $b \in X$ and $a < b$. Then for each bracket $b = (b_1, b_2) \in \mathcal{B}_{[\![1,N-1]\!]}$, one adds all the (a, b) with $a \in \mathcal{B}_{N-n(b)}$ and $b_1 \leqslant a < b$. Finally, one inserts the newly generated elements of $\mathcal{B}_N$ into an ordering, maintaining the condition that $a < (a, b)$. With this construction, one obtains the following statement, already used in [10, Proposition 2.13].

Proposition 1.13. *There exists a Hall set $\mathcal{B}$ such that X_0 is maximal,*

$$\mathcal{B}_1 = \left\{M_j^1 := X_1 0^j, \quad M_j^2 := X_2 0^j; \quad j \in \mathbb{N}\right\}, \tag{1.5}$$

and $\mathcal{B}_2 = \mathcal{B}_{2,good} \cup \mathcal{B}_{2,bad}$ with

$$\mathcal{B}_{2,bad} = \left\{W_{j,l}^1 := (M_{j-1}^1, M_j^1)0^l, \quad W_{j,l}^2 := (M_{j-1}^2, M_j^2)0^l; \quad j \in \mathbb{N}^*, l \in \mathbb{N}\right\}, \tag{1.6}$$

and

$$\mathcal{B}_{2,good} = \left\{C_{j,l} := (-1)^j \left(M^1_{\lfloor \frac{j+1}{2} \rfloor}, M^2_{\lfloor \frac{j}{2} \rfloor}\right) 0^l; \quad j, l \in \mathbb{N}\right\}. \tag{1.7}$$

We used the following notation.

Notation. *For $b = (b_1, b_2) \in \mathrm{Br}(X)$ and $j \in \mathbb{N}$, we use the notation $(-1)^j b$ as $(-1)^j b = (b_2, b_1)$ if j is odd and $(-1)^j b = b$ if j is even. This notation is unconventional, it is used to condense the writing.*

Remark 1.14. *When $l = 0$, we will write W_j^1, W_j^2, C_j instead of $W_{j,0}^1, W_{j,0}^2$ and $C_{j,0}$.*

1.4 Main result

The purpose of this article is to prove the following theorem.

Theorem 1.15. *Let f_0, f_1, f_2 be analytic vector fields over $\mathbb{R}^d$ such that $f_0(0) = 0$. Let $k, m \in \mathbb{N}^*$. We define the integer*

$$\pi(k,m) := 1 + \left\lceil \frac{2k}{m} \right\rceil, \tag{1.8}$$

and the set

$$\begin{aligned} \mathcal{N}_k^m &:= S_{[\![1,\pi(k,m)]\!]\setminus\{2\}}(X) \\ &\cup \{C_{j,l};\ j \in [\![0, 2k-2]\!], l \in \mathbb{N}\} \cup \left\{W_{j,l}^1, W_{j,l}^2;\ j \in [\![1, k-1]\!], l \in \mathbb{N}\right\}, \end{aligned}$$

where the last set in the right-hand side is empty if $k = 1$. Let $\sigma : \mathbb{R}^d \to \mathbb{R}^d/\mathcal{N}_k^m(f)(0)$ be the canonical surjection, $\tilde{e_1} := \sigma\left(f_{W_k^1}(0)\right)$, $\tilde{e_2} := \sigma\left(f_{W_k^2}(0)\right)$ and $\tilde{e_3} := \sigma\left(f_{C_{2k-1}}(0)\right)$. If the system (1.1) is $W^{m,1}$-STLC, then one of the following conditions is satisfied

- *$\tilde{e_1} = 0$ or $\tilde{e_2} = 0$,*
- *$(\tilde{e_1}, \tilde{e_2})$ is a linearly independent family and $\tilde{e_3} = a\tilde{e_1} + b\tilde{e_2}$ with $ab \geqslant \frac{1}{4}$,*
- *$\tilde{e_2} \in \mathbb{R}_-^* \tilde{e_1}$,*
- *$\tilde{e_2} = \beta\tilde{e_1}$, $\tilde{e_3} = \gamma\tilde{e_1}$ with $\beta \leqslant \gamma^2$ and $\beta \neq 0$.*

The parameter k is associated with the drift order, m with the regularity of the controls. The historical case $m = 0$ can be studied using this theorem, for systems with an integrator, *i.e.* satisfying $x_1' = u$. In general, the restriction to the case $m \geqslant 1$ is technical and could be improved in future work. A heuristic explanation of this theorem is given in the following sections.

Remark 1.16. *For any $m \in \mathbb{N}^*$, $p \in [1, +\infty]$, $t \in (0,1)$ and $u \in W^{m,p}(0,t)$, one has $\|u\|_{W^{m,1}} \leqslant \|u\|_{W^{m,p}}$. Consequently, we more generally prove a necessary condition for $W^{m,p}$-STLC.*

Remark 1.17. *Note that, it may be possible to deal with techniques used in [5, Section 10] to remove the assumption of analyticity of the vector fields f_0, f_1 and f_2. This will not be explored in the article.*

Example 1: an elementary example to which the previous theorem applies is the following one

$$\begin{cases} x' &= & u \\ y' &= & v \\ z' &= & x^2 + y^2 + \alpha xy \end{cases}, \tag{1.9}$$

with a parameter $\alpha \in \mathbb{R}$. This is a control-affine system of the form (1.1) that satisfies

$$f_{W_1^1}(0) = f_{W_1^2}(0) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \quad f_{C_1}(0) = \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}, \quad \mathcal{N}_1^m(f)(0) = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right),$$

for any integer $m \in \mathbb{N}^*$. Let $\sigma : \mathbb{R}^3 \to \mathbb{R}^3/\mathcal{N}_1^m(f)(0)$ be the canonical surjection, $\tilde{e_1} = \sigma\left(f_{W_1^1}(0)\right)$, $\tilde{e_2} = \sigma\left(f_{W_1^2}(0)\right)$ and $\tilde{e_3} = \sigma\left(f_{C_1}(0)\right)$. Thus, $\tilde{e_1} = \tilde{e_2} = 2$ and $\tilde{e_3} = \alpha$. The first three points of Theorem 1.15 are not satisfied. The last one is not verified if and only if $|\alpha| < 2$ and then, by contraposition, Theorem 1.15 proves that, for every $m \in \mathbb{N}^*$, the system (1.9) is not $W^{m,1}$-STLC. Indeed,

$$x^2 + y^2 + \alpha xy = \left(x + \frac{\alpha}{2} y\right)^2 + \frac{4 - \alpha^2}{4} y^2,$$

thus, if $|\alpha| \leqslant 2$, then $z' \geqslant 0$. The use of Lie brackets formalizes the fact that such a subsystem is hidden, in some sense, inside the system (1.1). Note that, when $|\alpha| > 2$, for every $m \in \mathbb{N}$, the system is $W^{m,\infty} - STLC$. Indeed, we can choose $u : s \in (0,t) \mapsto \varphi'(\frac{s}{t})$, and $v : s \in (0,t) \mapsto \psi'(\frac{s}{t})$ with $\varphi, \psi \in \mathcal{C}_c^\infty((0,1),\mathbb{R})$. If $\psi \equiv 0$, the state moves along $+e_3$. If $\varphi = -\frac{\alpha}{2}\psi$, the state moves along $-e_3$, as

$$z(t) = -\frac{\alpha^2 - 4}{2} t^3 \int_0^1 \psi^2.$$

Example 2: let us focus on the following system

$$\begin{cases} x_1' &= & u \\ x_2' &= & v \\ x_3' &= & x_1 v \\ x_4' &= & x_3 \\ x_5' &= & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + x_4 \end{cases}. \tag{1.10}$$

We can write (1.10) as $x' = f_0(x) + uf_1(x) + vf_2(x)$. Let $\varepsilon > 0$ and $z \in \mathbb{R}$. Using the following controls,

$$u_z(t) := \sqrt{|z|}\left(\mathbb{1}_{(0,\varepsilon)} - \mathbb{1}_{(2\varepsilon,3\varepsilon)}\right)(t), \qquad v_z(t) := \operatorname{sgn}(z)\sqrt{|z|}\left(\mathbb{1}_{(\varepsilon,2\varepsilon)} - \mathbb{1}_{(3\varepsilon,4\varepsilon)}\right)(t),$$

one has

$$\begin{gathered} \|(u_z, v_z)\|_{L^\infty} \leqslant |z|^{\frac{1}{2}}, \\ x_1(4\varepsilon) = x_2(4\varepsilon) = 0, \qquad x_3(4\varepsilon) = z\varepsilon^2, \\ x_4(4\varepsilon), x_5(4\varepsilon) = \mathcal{O}\left(z\varepsilon^3\right). \end{gathered}$$

Thus, $e_3 = [f_1, f_2](0)$ is a second-order tangent-vector, in the sense of [17]. Noticing that $e_4 = [[f_1, f_2], f_0](0)$, $e_5 = [[[f_1, f_2], f_0], f_0](0)$, one deduces from [12, Theorem 6] that e_4 and e_5 are also tangent-vectors. Using [17, Corollary 2.5], one obtains the L^∞-STLC[1] of system (1.10). This example emphasizes the fact that we need to put not only the bracket (X_1, X_2) but also $\{(X_1, X_2)0^\nu;\ \nu \in \mathbb{N}\}$, in the set $\mathcal{N}_k^m$.

Example 3: the necessary condition for STLC given by Theorem 1.15 is not sufficient, i.e. there are systems that are not $W^{m,1}$-STLC, but that verify at least one of the four points, for a fixed $m \in \mathbb{N}^*$. For instance, consider the following system

$$\begin{cases} x' &= & u \\ y' &= & v \\ z' &= & x^4 + y^4 \end{cases}. \tag{1.11}$$

For this system, for every $k \in \mathbb{N}^*$, $f_{W_k^1}(0) = 0$. Thus, for all $m \in \mathbb{N}^*$, this system satisfies the first point of Theorem 1.15. However, for every $m \in \mathbb{N}^*$, this system is not $W^{m,1}$-STLC because $z' \geqslant 0$. In this situation, quadratic terms do not prevent the system from being controllable. Theorem 1.15 doesn't allow us to conclude for this example, since the drift arises from quartic terms. An opening to the case of quartic drifts is given in Section 4.6.

Example 4: another counter-example is given by the following system

$$\begin{cases} x_1' &= & u \\ x_2' &= & x_1 \\ y_1' &= & v \\ z_1' &= & x_2^2 + y_1^2 \end{cases}. \tag{1.12}$$

Then, $f_{W_1^1}(0) = 0$ and for all $k \in \mathbb{N}_{\geqslant 2}$, $f_{W_k^2}(0) = 0$. Consequently, for all $k, m \in \mathbb{N}^*$, the first point is satisfied by this system. However, for every $m \in \mathbb{N}^*$, the system is not $W^{m,1}$-STLC

[1] In fact, we can prove that this system is $W_0^{m,\infty}$-STLC for every $m \in \mathbb{N}$, using the same strategy as in [10, Proposition 2.25].

because $z_1' \geqslant 0$. Here, the obstacle to controllability is created by the bracket $W_2^1 + W_1^2$. Indeed, Theorem 1.15 is designed to study competitions with quadratic brackets associated with controls of the same homogeneity in time. For this reason, we prove a generalization of Theorem 1.15 that allows us to deal with asymmetrical cases, this is the purpose of the following statement.

Theorem 1.18. *Let f_0, f_1, f_2 be analytic vector fields over $\mathbb{R}^d$ such that $f_0(0) = 0$. Let $k, k', m, m' \in \mathbb{N}^*$ be such that $k' \leqslant k$. One recalls that π is given by* (1.8)*. We define*

$$\mathcal{N}_{k,k'}^{m,m'} = S_{[\![1,\pi(k,m)]\!]\setminus\{2\}}(X) \cap S_{[\![0,\pi(k,m)]\!],[\![0,\pi(k',m')]\!]}(X)$$
$$\cup \{C_{j,l};\ j \in [\![0, k+k'-2]\!], l \in \mathbb{N}\} \cup \left\{W_{i,l}^1, W_{j,l}^2;\ (i,j) \in [\![1,k-1]\!] \times [\![1,k'-1]\!], l \in \mathbb{N}\right\},$$

where the last set in the right-hand side is empty if $k = 1$ or $k' = 1$. Let $\sigma : \mathbb{R}^d \to \mathbb{R}^d/\mathcal{N}_{k,k'}^{m,m'}(f)(0)$ be the canonical surjection, $\tilde{e_1} := \sigma\left(f_{W_k^1}(0)\right)$, $\tilde{e_2} := \sigma\left(f_{W_{k'}^2}(0)\right)$ and $\tilde{e_3} := \sigma\left(f_{C_{k+k'-1}}(0)\right)$. If the system (1.1) *is $W^{m,1} \times W^{m',1}$-STLC, then one of the following conditions is satisfied*

- *$\tilde{e_1} = 0$ or $\tilde{e_2} = 0$,*
- *$(\tilde{e_1}, \tilde{e_2})$ is a linearly independent family and $\tilde{e_3} = a\tilde{e_1} + b\tilde{e_2}$ with $ab \geqslant \frac{1}{4}$,*
- *$\tilde{e_2} \in \mathbb{R}_-^* \tilde{e_1}$,*
- *$\tilde{e_2} = \beta\tilde{e_1}$, $\tilde{e_3} = \gamma\tilde{e_1}$ with $\beta \leqslant \gamma^2$ and $\beta \neq 0$.*

Remark 1.19. *This theorem gives a necessary condition for $W^{m,p} \times W^{m,p'}$-STLC, for every $p, p' \in [1, +\infty]$.*

Remark 1.20. *The case when $k \leqslant k'$ can be proved in the same way.*

Theorem 1.18 can be used to study the system (1.12) with $k = 2$, $k' = 1$, $m, m' \in \mathbb{N}^*$.

1.5 Drift for obstructions

Our strategy is to prove that system (1.1) has a drift, to deny $W^{m,p} \times W^{m',p'}$-STLC - see Section 3.1. More precisely,

Definition 1.21 (Drift)**.** *Let $e \in \mathbb{R}^d$, $N \subset \mathbb{R}^d$ be a vector subspace, $m, m' \in \mathbb{N}^*$, $p, p' \in [1, +\infty]$ and $\Delta : L^1((0,1), \mathbb{R})^2 \to \mathbb{R}^+$. We say that system* (1.1) *has a drift along e parallel to N with strength Δ as $\left(t, \|(u,v)\|_{W^{m,p} \times W^{m',p'}}\right) \to 0$ when there exist $C > 0$, $\beta > 1$ and $0 < \rho < 1$ such that, for every $t \in (0, \rho)$, and $(u,v) \in W^{m,p}((0,t), \mathbb{R}) \times W^{m',p'}((0,t), \mathbb{R})$ with $\|(u,v)\|_{W^{m,p} \times W^{m',p'}} \leqslant \rho$,*

$$\mathbb{P}x(t;(u,v)) \geqslant C\Delta(u,v) - C\left\|x(t;(u,v))\right\|^\beta, \tag{1.13}$$

where $\mathbb{P}$ is a linear form, satisfying $\mathbb{P}(e) > 0$ and $\mathbb{P}_{|N} \equiv 0$.

Lemma 1.22. *With the notations of Definition 1.21, if the system* (1.1) *has a drift along e parallel to N with strength Δ as $\left(t, \|(u,v)\|_{W^{m,p} \times W^{m',p'}}\right) \to 0$, then, the system* (1.1) *is not $W^{m,p} \times W^{m',p'}$-STLC.*

Proof. The solution $x(t;(u,v))$ cannot reach targets of the form $x^* = -ae$ with $a > 0$ small because this would entail

$$-a\mathbb{P}(e) \geqslant C\Delta(u,v) - C\left\|x^*\right\|^\beta \geqslant -C\|e\|^\beta a^\beta,$$

which is impossible for a small since $\beta > 1$ and $\mathbb{P}(e) > 0$. □

1.6 The strategy to prove drifts

To explain the strategy, we need the following definition.

Definition 1.23 (Iterated primitives)**.** *For $j \in \mathbb{N}$, $t > 0$, we define by induction the iterated primitives of $u \in L^1((0,t),\mathbb{R})$, denoted $u_j : (0,t) \to \mathbb{R}$ as: $u_0 := u$ and $u_{j+1}(t) := \displaystyle\int_0^t u_j(s)\mathrm{d}s$.*

The proof of Theorem 1.15 is based on the Magnus-type representation formula and a strategy developed by Beauchard and Marbach in [5] to prove quadratic obstructions to STLC for single-input systems. Here are the main points to adapt their strategy to the case of multi-input systems. We fix $m \in \mathbb{N}^*$ and $p \in [1,+\infty]$. The purpose is to create a drift - see Definition 1.21 - as $(t, \|(u,v)\|_{W^{m,p}}) \to 0$. Let $M \in \mathbb{N}^*$. The solution to (1.1) is given by the following formula (see Proposition 2.8), as $\|(u,v)\|_{L^1} \to 0$,

$$x(t;(u,v)) = \mathcal{Z}_M(t;f,(u,v))(0) + \mathcal{O}\left(\|(u,v)\|_{L^1}^{M+1} + \|x(t;(u,v))\|^{1+\frac{1}{M}}\right),$$

where $\mathcal{Z}_M(t;f,(u,v))$ is an analytic vector field belonging to $S_{[\![1,M]\!]}(f)$ and given by

$$\mathcal{Z}_M(t;f,(u,v)) = \sum_{b\in\mathcal{B}_{[\![1,M]\!]}} \eta_b(t,(u,v))f_b,$$

where η_b are functionals, called coordinates of the pseudo-first kind (see [3, Proposition 44] for more details). These functionals are not easy to compute. However, the coordinates of the second kind $(\xi_b)_{b\in\mathcal{B}}$ - see Definition 2.1 and [3, Section 2.5.3.] - are easy to compute, and there is a link between η_b and ξ_b. Heuristically, we can think that $\eta_b \approx \xi_b$. Then, the Magnus formula becomes

$$\begin{aligned} x(t;(u,v)) = &\sum_{b\in\mathcal{B}_{[\![1,M]\!]}} \xi_b(t,(u,v))f_b(0) + \sum_{b\in\mathcal{B}_{[\![1,M]\!]}} (\eta_b-\xi_b)(t,(u,v))f_b(0) \\ &+\mathcal{O}\left(\|(u,v)\|_{L^1}^{M+1} + \|x(t;(u,v))\|^{1+\frac{1}{M}}\right), \end{aligned} \tag{1.14}$$

where the dominant part is the first sum. We now consider $\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{b}_3 \in \mathcal{B}_{[\![1,M]\!]}$, brackets that will be used to create a drift and $\mathcal{N} \subset \mathcal{B}_{[\![1,M]\!]} \setminus \{\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{b}_3\}$ the set defined as

$$\forall b \in \mathcal{B}_{[\![1,M]\!]} \setminus \{\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{b}_3\}, \quad (\forall i \in [\![1,3]\!],\ \xi_b \neq o(\xi_{\mathfrak{b}_i}) \text{ as } (t,\|(u,v)\|_{W^{m,p}}) \to 0) \Rightarrow b \in \mathcal{N}.$$

The elements of $\mathcal{N}$ are the brackets whose second-kind coordinates are not negligible compared to those of the brackets generating the drift. We don't know how to deal with these terms, so we've decided to neutralize them. Then, we consider a linear form $\mathbb{P}$ on $\mathbb{R}^d$, so that

$$\mathbb{P}|_{\mathcal{N}(f)(0)} \equiv 0 \quad \text{and} \quad \Delta(u,v) := \sum_{i=1}^{3} \xi_{\mathfrak{b}_i}(t,(u,v))\mathbb{P}\left(f_{\mathfrak{b}_i}(0)\right) \geqslant 0. \tag{1.15}$$

The existence of such a $\mathbb{P}$ is given by the framework of Theorem 1.15 - see Section 3.1. Finally, we fix M (in terms of the parameters $m,p,\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{b}_3$) and use interpolation inequalities to absorb the remainder term $\|(u,v)\|_{L^1}^{M+1}$ by the drift $\Delta(u,v)$ and obtain

$$\|(u,v)\|_{L^1}^{M+1} \lesssim \|(u,v)\|_{W^{m,p}}^{\alpha}\Delta(u,v), \tag{1.16}$$

for some $\alpha > 0$. Now, the formula (1.14) becomes

$$\begin{aligned} \mathbb{P}x(t;(u,v)) = \Delta(u,v) &+ \sum_{\substack{b\in\mathcal{B}_{[\![1,M]\!]},\\ b\notin\mathcal{N}\cup\{\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{b}_3\}}} \xi_b(t,(u,v))\mathbb{P}\left(f_b(0)\right) \\ + \sum_{\substack{b\in\mathcal{B}_{[\![1,M]\!]},\\ b\notin\mathcal{N}}} &(\eta_b-\xi_b)(t,(u,v))\mathbb{P}\left(f_b(0)\right) + \mathcal{O}\left(\|(u,v)\|_{W^{m,p}}^{\alpha}\Delta(u,v) + \|x(t;(u,v))\|^{1+\frac{1}{M}}\right). \end{aligned} \tag{1.17}$$

Intuitively, as $(t,\|(u,v)\|_{W^{m,p}}) \to 0$, in the right-hand side of (1.17),

1. the second term is bounded by $\varepsilon\Delta(u,v)$, thanks to the choice of $\mathcal{N}$,
2. the third term is small, as $\eta_b \approx \xi_b$,
3. the forth term is bounded by $\varepsilon\Delta(u,v)$, thanks to the asymptotic $\|(u,v)\|_{W^{m,p}} \to 0$,
4. finally, the last term is part of the definition of a drift (see Definition 1.21).

Here, we pretend that a series of "small" terms keeps the same asymptotic, this fact is true in a precise framework defined in Section 2.5. The brackets in competition here are $\mathfrak{b}_1 := W_k^1$, $\mathfrak{b}_2 := W_k^2$ and $\mathfrak{b}_3 := C_{2k-1}$, for $k \in \mathbb{N}^*$. In the asymmetrical case (Theorem 1.18), we use the same strategy, with a different truncation in the Magnus-type representation formula of the state - see Proposition 4.1.

1.7 State of the art

The first known statement linking Lie brackets and small-time local controllability is proved by Hermann [13] and Nagano [19]. These articles assert that the Lie Algebra Rank Condition is a necessary condition, i.e. if (1.1) is L^∞-STLC, then, $\mathrm{Lie}(f_0,f_1,f_2)(0) = \mathbb{R}^d$. This is the case for a control-affine system with an arbitrary number of controls. This condition is sufficient for systems with $f_0 \equiv 0$; this is proved by Chow in [7], and Rashevski in [21] in $1938-39$. However, this condition is not sufficient in general when $f_0 \not\equiv 0$.

1.7.1 For single-input systems

Hermes and Sussmann proved in [24, 14] in 1983 the following theorem.

Theorem 1.24. *Let f_0, f_1 be analytic vector fields over $\mathbb{R}^d$ with $f_0(0) = 0$. Assume that the LARC is verified, i.e. $Lie(f_0,f_1)(0) = \mathbb{R}^d$ and that,*

$$\forall k \in \mathbb{N}^*, \quad S_{2k}(f)(0) \subseteq S_{2k-1}(f)(0). \tag{1.18}$$

Then, system $x' = f_0(x) + uf_1(x)$ is L^∞-STLC.

Sussmann was interested in the reciprocal of condition (1.18): is it necessary? Let us focus first on the case $k = 1$. Assume that there exists a bracket $b \in \mathcal{B}$ such that $f_b(0) \in S_2(f)(0) \setminus S_1(f)(0)$. The easiest bracket possible in the basis is $b = W_1^1$. Then, in [24], Sussmann proves the first necessary condition, which is as follows.

Theorem 1.25. *Let f_0, f_1 be analytic vector fields over $\mathbb{R}^d$ with $f_0(0) = 0$. If $x' = f_0(x) + uf_1(x)$ is L^∞-STLC, then $f_{W_1^1}(0) = [f_1,[f_1,f_0]](0) \in S_1(f)(0)$.*

If $f_{W_1^1}(0) \in S_1(f)(0)$, we can now ask what about the bracket $W_2^1 = (X_1 0, X_1 0^2)$. One considers the system

$$\left\{ \begin{array}{rcl} x_1' &=& u \\ x_2' &=& x_1 \\ x_3' &=& x_1^3 + x_2^2 \end{array} \right. . \tag{1.19}$$

For this system,

$$S_1(f)(0) = \mathrm{Span}(e_1,e_2), \qquad f_{W_1^1}(0) = 0, \qquad f_{W_2^1}(0) = 2e_3.$$

Thus, $f_{W_1^1}(0) \in S_1(f)(0)$ and $f_{W_2^1}(0) \notin S_1(f)(0)$. However, Sussmann proved in [24] that this system is L^∞-STLC. The study of all quadratic drifts is therefore not obvious. Nevertheless, in [5], Beauchard and Marbach propose a general method for demonstrating obstructions to the controllability of affine systems. This method is based in an adaptation of the Magnus formula - see [3] - which gives the expression of the solution to a control-affine system in the form of a series of Lie brackets. They use this method to study all quadratic drifts, proving the following statement.

Theorem 1.26. *Let f_0, f_1 be analytic vector fields over $\mathbb{R}^d$ with $f_0(0) = 0$ Let $m \in [\![-1, +\infty[\![$. If system $x' = f_0(x) + uf_1(x)$ is $W^{m,\infty}$-STLC, then,*

$$\forall k \in \mathbb{N}^*, \quad f_{W_k^1}(0) \in S_{[\![1,\pi(k,m)]\!]\setminus\{2\}}(f)(0). \tag{1.20}$$

where $\pi(k,m) := 1 + \left\lceil \dfrac{2k-2}{m+1} \right\rceil$ with the convention $\pi(k,-1) = +\infty$ and $\pi(1,-1) = 1$.

This is an extension of Theorem 1.25 ($k = 1$, $m = 0$). In particular, system (1.19) is not $W^{1,\infty}$-STLC (apply Theorem 1.26 with $k = 2$, $m = 1$).

Another necessary condition for controllability is Stefani's, in [22], which is concerned with the drift of the k^{th} order term, when k is an even integer. The statement is the following one.

Proposition 1.27. *Let f_0, f_1 be analytic vector fields over $\mathbb{R}^d$ with $f_0(0) = 0$. If system $x' = f_0(x) + uf_1(x)$ is L^∞-STLC, then,*

$$\forall k \in \mathbb{N}^*, \quad \mathrm{ad}_{f_1}^{2k}(f_0)(0) \in S_{2k-1}(f)(0). \tag{1.21}$$

1.7.2 For multi-input systems: the necessary condition of [11]

In [11], Giraldi, Lissy, Moreau and Pomet consider affine systems (1.1) with $f_0(0) = f_2(0) = 0$ (this class of systems is different from those studied in this article, because our assumptions imply that $f_2(0) \neq 0$ - see Lemma 3.12). They prove a necessary condition for STLC; it is formulated on the bracket W_1^1. Their statement can be formulated as follow.

Theorem 1.28. *Let f_0, f_1, f_2 be analytic vector fields over $\mathbb{R}^d$ with $f_0(0) = f_2(0) = 0$. If (1.1) is L^∞-STLC, then $f_{W_1^1}(0) \in \mathcal{R}_1(f)(0)$ with*

$$\mathcal{R}_1 := \{b \in \mathrm{Br}(X);\ n_1(b) \leqslant 1\}.$$

In this framework, $f_{W_1^2}(0) = 0$ and the equivalent of set $\mathcal{N}_k^m$ is $\mathcal{R}_1$. They prove a drift thanks to the coercivity of $\displaystyle\int u_1^2$, the drift is not a combination as in our situation. As a consequence, there is no bound on $n_2(b)$ is $\mathcal{R}_1$, they put brackets with associated coordinates of the second kind of any order on v in the set $\mathcal{R}_1$. For the proof, they use the Chen-Fliess formula and re-organize its terms to form Lie brackets (as Stefani did in [22]).

In the same paper [11], the authors prove another necessary condition for STLC of multi-input control-affine systems. In simpler special cases, the theorem can be stated as follows.

Theorem 1.29. *Let f_0, f_1, f_2 be analytic vector fields over $\mathbb{R}^d$ with $f_0(0) = f_2(0) = 0$. Assume that $f_{W_1^1}(0), f_{(X_1,(X_2,X_1))}(0), f_{((X_2,X_1),(X_0,X_1))}(0) \in \mathcal{R}_1(f)(0)$. Let*

$$q : (x,y) \in \mathbb{R}^2 \mapsto -x^2 f_{W_2^1}(0) - y^2 f_{((X_2,X_1),(X_2,X_1)0)}(0)$$
$$-xy\left(f_{((X_2,X_1),M_2^1)} + f_{(M_1^1,(X_2,X_1)0)}\right)(0).$$

If there exists a linear form $\varphi : \mathbb{R}^d \to \mathbb{R}$, whose restriction to $\mathcal{R}_1(f)(0)$ is zero and such that the quadratic form $(a_1,a_2) \in \mathbb{R}^2 \mapsto \varphi\left(q(a_1,a_2)\right)$ is positive definite, then system (1.1) is not $W^{1,\infty} \times L^\infty$-STLC.

In this article, Theorem 1.15 is proved as a consequence of the more precise Theorem 3.7. The hypothesis of Theorem 3.7 is (3.1), where (BC) is defined in Definition 3.1. Finally, Corollary 3.4 ensures that (BC) can be exactly formulated as the assumption of Theorem 1.29. Consequently,we use the same types of assumptions.

1.7.3 For multi-input systems: link with Sussmann's sufficient $\mathcal{S}(\theta)$-condition

Now, let us compare our necessary condition with Sussmann's sufficient $\mathcal{S}(\theta)$-condition (see [23, Theorem 7.3]), recalled in Proposition 1.31 below (see [8, Theorem 3.29]).

Definition 1.30. *The map $\sigma : \mathrm{Br}(X) \mapsto \mathcal{L}(X)$ is defined by $\sigma(b) = E(b) + \pi(E(b))$, where $\pi : \mathcal{L}(X) \mapsto \mathcal{L}(X)$ is the unique morphism of Lie algebra such that $\pi(X_0) = X_0$, $\pi(X_1) = X_2$ and $\pi(X_2) = X_1$.*

For instance, $\sigma(W_k^1) = W_k^1 + W_k^2$.

Proposition 1.31 (Sussmann's $\mathcal{S}(\theta)$-condition)**.** *Let f_0, f_1, f_2 be analytic vector fields over $\mathbb{R}^d$ that satisfy the Lie Algebra Rank Condition, $Lie(f_0, f_1, f_2)(0) = \mathbb{R}^d$. If there exists $\theta \in [0, 1]$ such that, for every $\mathfrak{b} \in \mathrm{Br}(X)$ with $n_0(\mathfrak{b})$ odd and both $n_1(\mathfrak{b})$ and $n_2(\mathfrak{b})$ even, we have*

$$f_{\sigma(\mathfrak{b})}(0) \in Span\{f_b(0);\ b \in \mathrm{Br}(X),\ n(b) + \theta n_0(b) < n(\mathfrak{b}) + \theta n_0(\mathfrak{b})\} \tag{1.22}$$

then the system $x' = f_0(x) + uf_1(x) + vf_2(x)$ is L^∞-STLC.

Sussmann's $\mathcal{S}(\theta)$-condition is a very popular sufficient condition for controllability of affine systems because it is quite simple to apply. Nevertheless, more powerful conditions are known, for example Agrachev and Gamkrelidze's - see [2, Theorem 4].

For $b = W_k^1$, we have $\sigma(b) = W_k^1 + W_k^2$ and our necessary condition also involves the Lie brackets W_k^1 and W_k^2. If $m = 1$ and the four points of Theorem 1.15 are not verified, then, for every $\theta \in [0, 1]$, (1.22) does not hold for $\mathfrak{b} = W_k^1$ (see Appendix A.4 for a proof).

Moreover, for the system (1.9), one has $f_{\sigma(W_1^1)}(0) = 4e_3$, and $\mathcal{N}_1^1(f)(0) = \mathrm{Span}(e_1, e_2)$. Furthermore, for every $\theta \in [0, 1]$,

$$\mathrm{Span}(f_b(0),\ b \in \mathrm{Br}(X),\ n(b) + \theta n_0(b) < 2 + \theta) \subseteq \mathcal{N}_1^1(f)(0).$$

Consequently, W_1^1 doesn't satisfy the condition (1.22). However, as already explained in Section 1.4, the $W^{1,1}$-STLC of (1.9) is function of the value of α. The Sussmann's $\mathcal{S}(\theta)$-condition is sensitive to the direction of the Lie brackets, but not to their amplitude.

1.7.4 For multi-input systems: the necessary condition of [15]

In [15], Lewis and Hirschorn use the Chen–Fliess series to prove the following result, which we restate using our slightly different context and our own notations.

Theorem 1.32. *Let f_0, f_1, f_2 be analytic vector fields with $f_0(0) = 0$ and $(f_1(0), f_2(0))$ linearly independent. Assume that 0 is a regular point for $Lie(f_1, f_2)(0)$[2]. Let $N := S_1(f)(0) + Span\{\mathrm{ad}_{f_0}^\mu([f_1, f_2])(0);\ \mu \in \mathbb{N}\} + Lie(f_1, f_2)(0)$. Assume that the system (1.1) is L^∞-STLC. Let $\sigma : \mathbb{R}^d \to \mathbb{R}^d/N$ be the canonical surjection. Let us define the vector-valued quadratic form*

$$B_N : (a_1, a_2) \mapsto a_1^2\sigma(f_{W_1^1}(0)) + a_2^2\sigma(f_{W_1^2}(0)) + 2a_1a_2\sigma(f_{C_1}(0)).$$

Then, there doesn't exist a linear form $\mathbb{P} : \mathbb{R}^2/N \to \mathbb{R}$ such that the quadratic form $(a_1, a_2) \mapsto \mathbb{P}(B_N(a_1, a_2))$ is a positive definite quadratic form.

In [15], the authors focus on the case where $k = 1$, whereas we deal with the case of $k \in \mathbb{N}^*$. They prove a necessary condition for L^∞-STLC of control-affine systems without any assumption of smallness on the controls. In this article, we work in the functional framework $W^{m,p}$-STLC, with $m \in \mathbb{N}^*$. They require the compensation of all the elements of $\mathrm{Lie}(f_1, f_2)(0)$, while we can use the smallness of the controls. However in our low regularity regime $k = m = 1$, we require compensation for all brackets of S_3, which is not the case in Theorem 1.32. Enlightened by [15], it might be possible te refine our result.

Other authors have studied obstructions to controllability linked with quadratic phenomenons (see e.g. [1, 6]).

[2] *i.e.* $\dim\{g(x); g \in \mathrm{Lie}(f_1, f_2)\}$ doesn't depend on x on a neighborhood of 0

1.8 Structure of the article

The paper is organized as follow: in section 2, we present some tools and properties that will be used in the next sections. In section 3, we give the proof of the main theorem of this article by contraposition - see Theorem 1.15. Finally, in section 4, we prove a generalization of this result in the asymmetrical case - see Theorem 1.18. Some elements of proof are developed in the appendix.

2 Requirements for the proof

2.1 Expression of the coordinates of the second kind

Definition 2.1 (Coordinates of the second kind)**.** *Let $\mathcal{B} \subset \mathrm{Br}(X)$ be a Hall set. The coordinates of the second kind associated with $\mathcal{B}$ is the unique family $(\xi_b)_{b\in\mathcal{B}}$ of functionals $\xi_b : \mathbb{R}_+ \times L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})^2 \to \mathbb{R}$ defined by induction as: for every $t > 0$ and $u, v \in L^1((0,t), \mathbb{R})$,*

- $\xi_{X_0}(t,(u,v)) := t, \quad \xi_{X_1}(t,(u,v)) := u_1(t) \quad$ *and* $\quad \xi_{X_2}(t,(u,v)) := v_1(t)$,
- *for $b \in \mathcal{B} \setminus X$, there exists a unique couple (b_1, b_2) of elements of $\mathcal{B}$ such that $b_1 < b_2$ and a unique maximal integer $m \geqslant 1$ with $b = \mathrm{ad}^m_{b_1}(b_2)$, and then*

$$\xi_b(t,(u,v)) := \frac{1}{m!}\int_0^t \xi^m_{b_1}(s,(u,v))\xi'_{b_2}(s,(u,v))\,\mathrm{d}s. \tag{2.1}$$

The following proposition is taken from [10, Proposition 2.18].

Proposition 2.2. *The following equalities hold.*

1. *For $b \in \mathcal{B}$, and $\nu \in \mathbb{N}$,*

$$\xi_{b0^\nu}(t,(u,v)) = \int_0^t \frac{(t-s)^\nu}{\nu!}\xi'_b(s,(u,v))\mathrm{d}s. \tag{2.2}$$

2. *For every $j \in \mathbb{N}$,*

$$\xi_{M^1_j}(t,(u,v)) = u_{j+1}(t), \tag{2.3}$$

$$\xi_{M^2_j}(t,(u,v)) = v_{j+1}(t). \tag{2.4}$$

3. *For every $j \in \mathbb{N}^*$, $l \in \mathbb{N}$,*

$$\xi_{W^1_{j,l}}(t,(u,v)) = \frac{1}{2}\int_0^t \frac{(t-s)^l}{l!}u_j^2(s)\mathrm{d}s, \tag{2.5}$$

$$\xi_{W^2_{j,l}}(t,(u,v)) = \frac{1}{2}\int_0^t \frac{(t-s)^l}{l!}v_j^2(s)\mathrm{d}s. \tag{2.6}$$

4. *For every $j, l \in \mathbb{N}$,*

$$\xi_{C_{j,l}}(t,(u,v)) = \int_0^t \frac{(t-s)^l}{l!}u_{\lfloor \frac{j}{2}\rfloor+1}(s)v_{\lfloor \frac{j+1}{2}\rfloor}(s)\mathrm{d}s. \tag{2.7}$$

Proof. The first three points are proved in [5, Lemma 3.6 and Proposition 3.7]. Let us prove the last one: let $j, l \in \mathbb{N}$. First, assume that $j = 2j_0$ is even. Thus, using the first point,

$$\xi_{C_{j,l}}(t,(u,v)) = \int_0^t \frac{(t-s)^l}{l!}\xi'_{C_j}(s,(u,v))\mathrm{d}s.$$

Moreover, $C_j = \mathrm{ad}^1_{X_1 0^{j_0}}(X_2 0^{j_0})$, thus using Definition 2.1 and (2.3), (2.4), one obtains

$$\xi_{C_j}(t,(u,v)) = \int_0^t u_{j_0+1}(s)v_{j_0}(s)\mathrm{d}s = \int_0^t u_{\lfloor \frac{j}{2}\rfloor+1}(s)v_{\lfloor \frac{j+1}{2}\rfloor}(s)\mathrm{d}s.$$

This is the same proof when j is odd. □

Lemma 2.3. *Let $j, l, N \in \mathbb{N}$, $t > 0$ and $u, v \in L^1((0,t),\mathbb{R})$. Then,*

$$\begin{aligned}\xi_{C_{j,l}}(t,(u,v)) = \sum_{\mu=0}^{N}(-1)^{\mu} u_{\lfloor \frac{j}{2}\rfloor+\mu+2}(t)\left(v_{\lfloor \frac{j+1}{2}\rfloor}\frac{(t-\cdot)^l}{l!}\right)^{(\mu)}(t)+\\ (-1)^{N+1}\int_0^t u_{\lfloor \frac{j}{2}\rfloor+N+2}(s)\frac{\mathrm{d}^{N+1}}{\mathrm{d}s^{N+1}}\left(v_{\lfloor \frac{j+1}{2}\rfloor}(s)\frac{(t-s)^l}{l!}\right)\mathrm{d}s.\end{aligned} \tag{2.8}$$

Proof. We prove the lemma by induction on N, using an integration by parts. □

2.2 Estimates on the coordinates of the second kind

Lemma 2.4. *For every $p \in [1,+\infty]$, $j \geqslant j_0 \geqslant 1$, $t > 0$, and $u \in L^1((0,t),\mathbb{R})$,*

$$\|u_j\|_{L^p} \leqslant \frac{t^{j-j_0}}{(j-j_0)!}\|u_{j_0}\|_{L^p}. \tag{2.9}$$

Proof. This lemma is proved in [5, Lemma A.6]. □

Proposition 2.5. *The following inequalities hold.*

1. *Let $p \in [1,+\infty]$, and $j_0 \in \mathbb{N}^*$. There exists $c > 0$ such that, for every $j \geqslant j_0$, $t > 0$, $u, v \in L^1((0,t),\mathbb{R})$,*

$$\left|\xi_{M_j^1}(t,(u,v))\right| \leqslant \frac{(ct)^{|M_j^1|}}{|M_j^1|!}t^{-(j_0+1)}t^{1-\frac{1}{p}}\|u_{j_0}\|_{L^p}. \tag{2.10}$$

$$\left|\xi_{M_j^2}(t,(u,v))\right| \leqslant \frac{(ct)^{|M_j^2|}}{|M_j^2|!}t^{-(j_0+1)}t^{1-\frac{1}{p}}\|v_{j_0}\|_{L^p}. \tag{2.11}$$

2. *Let $p \in [1,+\infty]$, and $j_0 \in \mathbb{N}^*$. There exists $c > 0$ such that, for every $j \geqslant j_0$, $l \in \mathbb{N}$, $t > 0$, $u, v \in L^1((0,t),\mathbb{R})$,*

$$\left|\xi_{W_{j,l}^1}(t,(u,v))\right| \leqslant \frac{(ct)^{|W_{j,l}^1|}}{|W_{j,l}^1|!}t^{-(2j_0+1)}t^{1-\frac{1}{p}}\|u_{j_0}\|_{L^{2p}}^2, \tag{2.12}$$

$$\left|\xi_{W_{j,l}^2}(t,(u,v))\right| \leqslant \frac{(ct)^{|W_{j,l}^2|}}{|W_{j,l}^2|!}t^{-(2j_0+1)}t^{1-\frac{1}{p}}\|v_{j_0}\|_{L^{2p}}^2. \tag{2.13}$$

3. *Let $p, q \in [1,+\infty]$ such that $\frac{1}{p}+\frac{1}{q} \leqslant 1$, and $k, k' \in \mathbb{N}^*$ with $k' \leqslant k$. There exists $c > 0$ such that, for every $j \geqslant k+k'-1$, $l \in \mathbb{N}$, $0 < t < 1$, $u, v \in L^1((0,t),\mathbb{R})$,*

$$\begin{aligned}\left|\xi_{C_{j,l}}(t,(u,v))\right| \leqslant \frac{(ct)^{|C_{j,l}|}}{|C_{j,l}|!}t^{-(1+k+k')}t^{1-\left(\frac{1}{p}+\frac{1}{q}\right)}\|u_k\|_{L^p}\|v_{k'}\|_{L^q}\\ +\mathbb{1}_{l\leqslant k-2-\lfloor\frac{j}{2}\rfloor}K\left(\sum_{\mu=1}^{k}|u_\mu(t)|^2+t\|v_{k'}\|_{L^2}^2\right),\end{aligned} \tag{2.14}$$

   where K only depends on k.

Proof. The first two points are proved in [5, Proposition 3.10]. We prove the last one: let $j \geqslant k+k'-1$, $l \in \mathbb{N}$, $t > 0$, $u, v \in L^1((0,t),\mathbb{R})$. *First, assume that $\lfloor\frac{j}{2}\rfloor + 1 \geqslant k$ (this is always the case when $k = k'$).* Using (2.7) and Hölder's inequality, we obtain

$$\left|\xi_{C_{j,l}}(t,(u,v))\right| \leqslant \frac{t^l}{l!}t^{1-\left(\frac{1}{p}+\frac{1}{q}\right)}\left\|u_{\lfloor\frac{j}{2}\rfloor+1}\right\|_{L^p}\left\|v_{\lfloor\frac{j+1}{2}\rfloor}\right\|_{L^q}.$$

Notice that $\lfloor \frac{j+1}{2} \rfloor \geqslant \lfloor \frac{k+k'}{2} \rfloor \geqslant \lfloor \frac{2k'}{2} \rfloor = k'$. Using two times Lemma 2.4 with $j_0 = k$ and with $j_0 = k'$, we get

$$\left|\xi_{C_{j,l}}(t,(u,v))\right| \leqslant \frac{t^{l+j+1-k-k'}}{l!(\lfloor \frac{j}{2} \rfloor + 1 - k)!(\lfloor \frac{j+1}{2} \rfloor - k')!} t^{1-\left(\frac{1}{p}+\frac{1}{q}\right)} \left\|u_k\right\|_{L^p} \left\|v_{k'}\right\|_{L^q}.$$

We obtain the result because, for all $j \geqslant k + k' - 1$, $l \in \mathbb{N}$,

$$\frac{1}{l!(\lfloor \frac{j}{2} \rfloor + 1 - k)!(\lfloor \frac{j+1}{2} \rfloor - k')!} \leqslant \frac{2^{3|C_{j,l}|}(k+k'+1)!}{(l+j+2)!}.$$

Now, we assume that $\lfloor \frac{j}{2} \rfloor \leqslant k-2$. We use Lemma 2.3 with $N = k - 2 - \lfloor \frac{j}{2} \rfloor \geqslant 0$ to obtain $\xi_{C_{j,l}}(t,(u,v) = A + B$ with A the boundary terms and B the integral part. Using Leibniz formula, one gets

$$\left(v_{\lfloor \frac{j+1}{2} \rfloor} \frac{(t-\cdot)^l}{l!}\right)^{(\mu)}(t) = \begin{cases} 0 & \text{if } \mu < l \\ \binom{\mu}{l}(-1)^l v_{\lfloor \frac{j+1}{2} \rfloor + l - \mu}(t) & \text{otherwise} \end{cases}.$$

Consequently, the following inequality holds

$$|A| \leqslant \mathbb{1}_{l \leqslant N} \sum_{\mu=l}^{N} \binom{\mu}{l} \left| u_{\lfloor \frac{j}{2} \rfloor + \mu + 2}(t) v_{\lfloor \frac{j+1}{2} \rfloor + l - \mu}(t) \right|.$$

Then, using Young's and Cauchy–Schwarz's inequality,

$$|A| \leqslant \mathbb{1}_{l \leqslant N} 2^{N-1} \left(\sum_{\mu=1}^{k} |u_\mu(t)|^2 + t \max_{\mu \in [\![l,N]\!]} \left\| v_{\lfloor \frac{j+1}{2} \rfloor + l - \mu - 1} \right\|_{L^2}^2 \right),$$

as $1 \leqslant \lfloor \frac{j}{2} \rfloor + \mu + 2 \leqslant k$, for $\mu \in [\![1, N]\!]$. Note that $\lfloor \frac{j+1}{2} \rfloor + l - \mu - 1 \geqslant k'$. Finally, applying Lemma 2.4 with $p = 2$ and $j_0 = k'$, we obtain the following estimate

$$|A| \leqslant \mathbb{1}_{l \leqslant N} 2^{k-3} \left(\sum_{\mu=1}^{k} |u_\mu(t)|^2 + t \left\| v_{k'} \right\|_{L^2}^2 \right). \tag{2.15}$$

We finally estimate B. Using Leibniz formula, and Hölder's inequality, one gets

$$|B| \leqslant \sum_{\mu=0}^{\min(N+1,l)} \binom{N+1}{\mu} \frac{t^{l-\mu}}{(l-\mu)!} t^{1-\left(\frac{1}{p}+\frac{1}{q}\right)} \left\|u_k\right\|_{L^p} \left\|v_{j+\mu+1-k}\right\|_{L^q}.$$

Using Lemma 2.4 with $p = q$ and $j_0 = k'$, one has

$$|B| \leqslant \frac{(2^{N+2}t)^{|C_{j,l}|}}{l!} t^{-(1+k+k')} t^{1-\left(\frac{1}{p}+\frac{1}{q}\right)} \left\|u_k\right\|_{L^p} \left\|v_{k'}\right\|_{L^q}. \tag{2.16}$$

Finally, for all $l \geqslant 0$, for all $j \in [\![0, 2k-3]\!]$,

$$\frac{1}{l!} = \frac{(j+2)!}{(j+l+2)!} \binom{j+l+2}{l} \leqslant \frac{(2k-1)!}{(j+l+2)!} 2^{j+l+2} \leqslant \frac{(2(2k-1)!)^{|C_{j,l}|}}{(j+l+2)!}. \tag{2.17}$$

Thus, equations (2.15), (2.16) and (2.17) lead to the desired inequality. □

2.3 Analytic norms

The following paragraph is inspired by [5, Section 4.1]. We introduce some basic notions about analytic vector fields and norms of analytic vector fields. These will be useful for ensuring the convergence of the series that we will consider in the following sections.

Definition 2.6 (Length and factorial of a multi-index, partial derivative)**.** *Let $d \in \mathbb{N}^*$ be a positive integer and $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$ be a multi-index. We define*

1. *the length of α as: $|\alpha| := \alpha_1 + \cdots + \alpha_d$,*
2. *the factorial of α as: $\alpha! := \alpha_1! \times \cdots \times \alpha_d!$,*
3. *the partial derivative: $\partial_\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$.*

Definition 2.7 (Analytic vector fields, analytic norms)**.** *Let $\delta > 0$ and $\overline{B}_\delta$ be the closed ball of radius δ, centered at $0 \in \mathbb{R}^d$. For $r > 0$, we define $C^{\omega,r}\left(\overline{B}_\delta; \mathbb{R}^d\right)$ as the subspace of analytic vector fields on an open neighborhood of $\overline{B}_\delta$, for which the following norm is finite*

$$|||f|||_r := \sum_{i=1}^{d} \sum_{\alpha \in \mathbb{N}^d} \frac{r^{|\alpha|}}{\alpha!} \left\|\partial^\alpha f_i\right\|_{L^\infty\left(\overline{B}_\delta\right)}.$$

2.4 An approximate formula for the state, of Magnus-type

This article relies on an approximate representation formula for the state, involving Lie brackets and coordinates $\left(\eta_b : \mathbb{R}^+ \times L^1_{loc}(\mathbb{R}^+, \mathbb{R})^2 \to \mathbb{R}\right)_{b \in \mathcal{B}}$ defined in [3, Proposition 44]. The goal of the following proposition is to introduce this formula.

Proposition 2.8 (Magnus formula)**.** *Let $M \in \mathbb{N}^*$, $\delta, T > 0$, $f_0, f_1, f_2 : B(0, 2\delta) \to \mathbb{R}^d$ be analytic vector fields with $f_0(0) = 0$ and $T \left\|f_0\right\|_\infty \leqslant \delta$. For $u, v \in L^1((0,T), \mathbb{R})$, as $\|(u,v)\|_{L^1} \to 0$,*

$$x(t;(u,v)) = \mathcal{Z}_M(t; f, (u,v))(0) + \mathcal{O}\left(\|(u,v)\|^{M+1}_{L^1(0,t)} + \|x(t;u,v)\|^{1+\frac{1}{M}}\right), \tag{2.18}$$

where

$$\mathcal{Z}_M(t; f, (u,v))(0) = \sum_{b \in \mathcal{B}_{[\![1,M]\!]}} \eta_b(t, (u,v)) f_b(0). \tag{2.19}$$

Proof. This proposition stems from [3, Proposition 161]. The ideas of proof are recalled in [10, Appendix A.3.2] □

2.5 A black-box estimate

Here are a few definitions and notations.

Definition 2.9 (Support)**.** *Let $a \in \mathcal{L}(X)$. For $b \in \mathcal{B}$, we denote by $\langle a, b \rangle$ the coefficient of $\mathrm{E}(b)$ in the expansion of a on the basis $\mathrm{E}(\mathcal{B})$. We define*

$$supp(a) := \{b \in \mathcal{B},\ \langle a, b \rangle \neq 0\}.$$

If $A \subset \mathrm{Br}(X)$, we let $supp(A) := \bigcup_{a \in A} supp(a)$.

With this definition, one has, for $a \in \mathcal{L}(X)$, $a = \sum_{b \in \mathrm{supp}(a)} \langle a, b \rangle \mathrm{E}(b)$.

Definition 2.10 ($\mathcal{F}$)**.** *Given $q \geqslant 2$ and $b_1, \cdots, b_q \in \mathrm{Br}(X)$, we define $\mathcal{F}(b_1, \cdots, b_q)$ as the subset of $\mathrm{Br}(X)$ of brackets of $b_1, \cdots, b_q$ involving each of these elements exactly once.*

To put the strategy (described in Section 1.6) into practice i.e. to extract the dominant terms from $\mathcal{Z}_M(t; f, (u,v))(0)$, we will use the following propositions. They guarantee the convergence of the series involved and legitimize the heuristic. This part of the article is based on [5, Section 4.4]. The following proposition allows us to estimate the second term in the right-hand side of (1.17).

Proposition 2.11 (Estimate of main terms)**.** *Let $M, L \in \mathbb{N}^*$. Let $\mathcal{E} \subset \mathcal{B}_{[\![1,M]\!]}$. Assume that there exist $c > 0$, and a functional $\Xi : \mathbb{R}_+^* \times L^1_{loc}(\mathbb{R}^+)^2 \to \mathbb{R}^+$ such that the following holds: for all $b \in \mathcal{E}$, there exists an exponent $\sigma \leqslant \min(L, |b|)$, such that, for all $t > 0$ and $u, v \in L^1((0,t), \mathbb{R})$,*

$$|\xi_b(t,(u,v))| \leqslant \frac{(ct)^{|b|}}{|b|!} t^{-\sigma} \Xi(t,(u,v)). \tag{2.20}$$

Let $\delta, r > 0$ and $f_0, f_1, f_2 \in \mathcal{C}^{\omega,r}\left(\overline{B}_\delta, \mathbb{R}^d\right)$ be analytic vector fields. Then, for any $r' \in [r/e, r)$, as $(t, \|(u,v)\|_{L^1}) \to 0$,

$$\sum_{b \in \mathcal{E}} |||\xi_b(t,(u,v)) f_b|||_{r'} = \mathcal{O}\left(\Xi(t,(u,v))\right).$$

The following proposition allows us to estimate the third term in the right-hand side of (1.17).

Proposition 2.12 (Estimate of cross terms)**.** *Let $M, L \in \mathbb{N}^*$. Let $\mathcal{E} \subset \mathcal{B}_{[\![1,M]\!]}$. Assume that there exist $c > 0$, and a functional $\Xi : \mathbb{R}_+^* \times L^1_{loc}(\mathbb{R}^+)^2 \to \mathbb{R}^+$ with $\Xi(t,(u,v)) = \mathcal{O}(1)$ such that the following holds: for all $q \geqslant 2$, $b_1 \geqslant \cdots \geqslant b_q \in \mathcal{B} \setminus \{X_0\}$ such that $supp\mathcal{F}(b_1, \cdots, b_q) \cap \mathcal{E} \neq \emptyset$, there exist $\sigma_1, \cdots, \sigma_q \leqslant L$ with $\sigma_i \leqslant |b_i|$ and $(\alpha_1, \cdots, \alpha_q) \in [0,1]^q$ with $\alpha_1 + \cdots + \alpha_q \geqslant 1$ such that, for all $t > 0$ and $u, v \in L^1((0,t), \mathbb{R})$,*

$$|\xi_{b_i}(t,(u,v))| \leqslant \frac{(ct)^{|b_i|}}{|b_i|!} t^{-\sigma_i} \left(\Xi(t,(u,v))\right)^{\alpha_i}. \tag{2.21}$$

Let $\delta, r > 0$ and $f_0, f_1, f_2 \in \mathcal{C}^{\omega,r}\left(\overline{B}_\delta, \mathbb{R}^d\right)$ be analytic vector fields. Then, for any $r' \in [r/e, r)$, as $(t, \|(u,v)\|_{L^1}) \to 0$,

$$\sum_{b \in \mathcal{E}} |||\left(\eta_b - \xi_b\right)(t,(u,v)) f_b|||_{r'} = \mathcal{O}\left(\Xi(t,(u,v))\right).$$

The two previous propositions are proved in [5, Appendix A.5]. The following corollary is a direct consequence of (2.19) and the two previous propositions. This clarifies several steps of the heuristic (Section 1.6).

Corollary 2.13. *Let $M, L, r \in \mathbb{N}^*$. Let $\mathfrak{b}_1, \cdots, \mathfrak{b}_r \in \mathcal{B}_{[\![1,M]\!]}$ and $\mathcal{N} \subset \mathcal{B}_{[\![1,M]\!]}$. Assume that there exist $c > 0$ and a functional $\Xi : \mathbb{R}_+^* \times L^1_{loc}(\mathbb{R}^+)^2 \to \mathbb{R}^+$ with $\Xi(t,(u,v)) = \mathcal{O}(1)$ such that*

1. *the assumptions of Proposition 2.11 hold for $\mathcal{E} = \mathcal{B}_{[\![1,M]\!]} \setminus \mathcal{N} \cup \{\mathfrak{b}_1, \cdots, \mathfrak{b}_r\}$,*
2. *the assumptions of Proposition 2.12 hold for $\mathcal{E} = \mathcal{B}_{[\![1,M]\!]} \setminus \mathcal{N}$.*

Let f_0, f_1, f_2 be analytic vector fields over $\mathbb{R}^d$. If $\mathcal{P}$ is linear form such that $\mathcal{P}_{|\mathcal{N}(f)(0)} \equiv 0$, as $(t, \|(u,v)\|_{L^1}) \to 0$,

$$\mathcal{P}\mathcal{Z}_M(t; f, (u,v))(0) = \sum_{i=1}^{r} \xi_{\mathfrak{b}_i}(t,(u,v)) \mathcal{P}\left(f_{\mathfrak{b}_i}(0)\right) + \mathcal{O}\left(\Xi(t,(u,v))\right). \tag{2.22}$$

2.6 Interpolation inequalities

We recall the Gagliardo–Nirenberg interpolation inequalities used in this article and proved in [9, 20].

Proposition 2.14 (Gagliardo–Nirenberg inequalities)**.** *Let $P, q, r, s \in [1, +\infty]$, $j < l \in \mathbb{N}$, and $\alpha \in (0,1)$ such that*

$$\frac{j}{l} \leqslant \alpha \quad \text{and} \quad \frac{1}{P} = j + \left(\frac{1}{r} - l\right)\alpha + \frac{1-\alpha}{q}.$$

There exists $C > 0$ such that, for every $t > 0$ and $\varphi \in \mathcal{C}^\infty([0,t], \mathbb{R})$,

$$\left\|D^j \varphi\right\|_{L^P} \leqslant C \left\|D^l \varphi\right\|_{L^r}^{\alpha} \|\varphi\|_{L^q}^{1-\alpha} + C t^{\frac{1}{P} - j - \frac{1}{s}} \|\varphi\|_{L^s}. \tag{2.23}$$

3 Necessary conditions for STLC in the symmetrical case

3.1 Bracket condition and drift result

To prove Theorem 1.15, we use the following definition.

Definition 3.1 (BC). *Let $e_1, e_2, e_3 \in \mathbb{R}^d$ be three vectors and $N \subset \mathbb{R}^d$ a vector subspace. We say that*

$$e_1, e_2, e_3, N \text{ satisfy (BC) if there exists a linear form } \mathbb{P} : \mathbb{R}^d \to \mathbb{R} \text{ such that}$$
$$\mathbb{P}_{|N} = 0, \quad \text{and} \quad \mathbb{P}(e_3)^2 < \mathbb{P}(e_1)\mathbb{P}(e_2).$$

Remark 3.2. *The hypothesis $\mathbb{P}(e_3)^2 < \mathbb{P}(e_1)\mathbb{P}(e_2)$ ensures that $\mathbb{P}(e_1)$ and $\mathbb{P}(e_2)$ have the same sign. Even if it means replacing $\mathbb{P}$ by $-\mathbb{P}$, we can assume that $\mathbb{P}(e_1) > 0$, $\mathbb{P}(e_2) > 0$. This is the case in the rest of the article.*

Lemma 3.3. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ and $q : (a_1, a_1) \in \mathbb{R}^2 \mapsto \frac{1}{2}\alpha a_1^2 + \frac{1}{2}\beta a_2^2 + \gamma a_1 a_2$. Then, q is a positive definite quadratic form iff $\alpha > 0$ and $\gamma^2 < \alpha\beta$.*

Proof. If q is positive definite, then $q(1,0) > 0$ so $\alpha > 0$. Moreover, for all $a_1 \in \mathbb{R}$, $q(a_1, 1) > 0$. Thus $\Delta = \gamma^2 - \alpha\beta < 0$. Conversely, the result follows from the equality: for all $a_1, a_2 \in \mathbb{R}$,

$$q(a_1, a_2) = \frac{1}{2}\alpha\left(\left(a_1 + \frac{\gamma}{\alpha}a_2\right)^2 + \frac{\alpha\beta - \gamma^2}{\alpha^2}a_2^2\right).$$

□

Corollary 3.4. *Let $e_1, e_2, e_3 \in \mathbb{R}^d$ be three vectors and $N \subset \mathbb{R}^d$ a vector subspace. Let $\sigma : \mathbb{R}^d \to \mathbb{R}^d/N$ be the canonical surjection and $q : (a_1, a_2) \in \mathbb{R}^2 \mapsto \frac{1}{2}a_1^2 e_1 + \frac{1}{2}a_2^2 e_2 + a_1 a_2 e_3 \in \mathbb{R}^d$. Then, the following are equivalent*

1. *e_1, e_2, e_3, N satisfy (BC),*
2. *there exists a linear form $\mathbb{P} : \mathbb{R}^d \to \mathbb{R}$ such that $N \subset \ker(\mathbb{P})$ and $(a_1, a_2) \in \mathbb{R}^2 \mapsto \mathbb{P}(q(a_1, a_2))$ is a positive definite quadratic form,*
3. *there exists a linear form $\tilde{\mathbb{P}} : \mathbb{R}^d/N \to \mathbb{R}$ such that $(a_1, a_2) \in \mathbb{R}^2 \mapsto \tilde{\mathbb{P}}(\sigma(q(a_1, a_2)))$ is a positive definite quadratic form.*

Remark 3.5. *Let $k \in \mathbb{N}^*$. We fix $\mathfrak{b}_1 = W_k^1$, $\mathfrak{b}_2 = W_k^2$ and $\mathfrak{b}_3 = C_{2k-1}$. With the linear form $\mathbb{P} : \mathbb{R}^d \to \mathbb{R}$ given by (BC), the quantity Δ introduced in* (1.15) *is a positive definite quadratic form (the expressions of $\xi_{\mathfrak{b}_i}$ are given in* (2.5), (2.6), (2.7)*).*

The following proposition, proved in Appendix, makes a direct link between this definition and the problem under study.

Proposition 3.6. *Let $e_1, e_2, e_3 \in \mathbb{R}^d$ be three vectors and $N \subset \mathbb{R}^d$ a vector subspace. Let $\sigma : \mathbb{R}^d \to \mathbb{R}^d/N$ be the canonical surjection, and $\tilde{e_i} := \sigma(e_i)$ for $i \in [\![1, 3]\!]$. Then, e_1, e_2, e_3, N **don't** satisfy (BC) iff one of the following conditions is satisfied*

- *$\tilde{e_1} = 0$ or $\tilde{e_2} = 0$,*
- *$(\tilde{e_1}, \tilde{e_2})$ is a linearly independent family and $\tilde{e_3} = a\tilde{e_1} + b\tilde{e_2}$ with $ab \geqslant \frac{1}{4}$,*
- *$\tilde{e_2} \in \mathbb{R}_-^* \tilde{e_1}$,*
- *$\tilde{e_2} = \beta\tilde{e_1}$, $\tilde{e_3} = \gamma\tilde{e_1}$ with $\beta \leqslant \gamma^2$ and $\beta \neq 0$.*

Let $k, m \in \mathbb{N}^*$. We reason by contraposition and we assume that the four point of Theorem 1.15 are not satisfied, i.e.

$$f_{W_k^1}(0), \quad f_{W_k^2}(0), \quad f_{C_{2k-1}}(0), \quad \mathcal{N}_k^m(f)(0) \quad \text{satisfy} \quad \text{(BC)}, \tag{3.1}$$

thanks to Proposition 3.6. The purpose of this section is to prove Theorem 1.15, as a consequence of the following more precise statement.

Theorem 3.7. *Let $k, m \in \mathbb{N}^*$ and $p \in [1, +\infty]$. Assume that (3.1) holds. Then, system (1.1) has a drift along $f_{W_k^1}(0) + f_{W_k^2}(0)$ parallel to $\mathcal{N}_k^m(f)(0)$ with strength $\Delta : (u, v) \in L^1((0,1), \mathbb{R})^2 \mapsto \int_0^t \left(u_k^2 + v_k^2\right) \in \mathbb{R}^+$ as $(t, t^\alpha \|(u,v)\|_{W^{m,p}}) \to 0$ where $\alpha = \frac{\pi(k,m)-2k}{\pi(k,m)-1}$.*

Remark 3.8. *For example, if $k \in \mathbb{N}^*$ and $m = 1$, $\alpha = \frac{1}{2k} > 0$. Thus, the controls u, v need only to be bounded in $W^{1,p}$.*

If this theorem holds, we immediately obtain Theorem 1.15 thanks to Lemma 1.22.

3.2 Dominant part of the logarithm

We use Corollary 2.13 to extract the main terms from the dynamics. This is the goal of the following statement.

Lemma 3.9. *Let $k, m \in \mathbb{N}^*$. Let $\mathcal{P}$ be a linear form satisfying $\mathcal{P}_{|\mathcal{N}_k^m(f)(0)} \equiv 0$. Then, as $(t, \|(u,v)\|_{L^1}) \to 0$,*

$$\begin{aligned} \mathcal{P}\mathcal{Z}_{\pi(k,m)}(t; f, (u,v))(0) = \mathcal{P}\left(f_{W_k^1}(0)\right)\xi_{W_k^1}(t,(u,v)) + \mathcal{P}\left(f_{W_k^2}(0)\right)\xi_{W_k^2}(t,(u,v)) \\ + \mathcal{P}\left(f_{C_{2k-1}}(0)\right)\xi_{C_{2k-1}}(t,(u,v)) + \mathcal{O}\left(t\,\|(u_k,v_k)\|_{L^2}^2 + |(u_1,\cdots,u_k,v_1,\cdots,v_k)(t)|^2\right). \end{aligned} \tag{3.2}$$

Proof. We apply Corollary 2.13 with $M = \pi(k,m)$, $\mathcal{N} = \mathcal{N}_k^m$, $L = 2k+2$, $r = 3$, $\sigma = 2k+2$, $\mathfrak{b}_1 = W_k^1$, $\mathfrak{b}_2 = W_k^2$, and $\mathfrak{b}_3 = C_{2k-1}$.

1. *Estimates on the main terms:* let $b \in \mathcal{B}_{[\![1,\pi(k,m)]\!]}$ be such that $b \notin \mathcal{N}_k^m \cup \{\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3\}$. Then, $n(b) = 2$ and
   (a) If $b \in \mathcal{B}_{2,bad}$, $b = W_{j,l}^1$ or $b = W_{j,l}^2$ with $j > k$ or ($j = k$ and $l \geqslant 1$). Consequently, $|b| \geqslant 2k+2$ and the estimates (2.12) and (2.13) with $p = 1$ and $j_0 = k$ give (2.20) with $\Xi(t,(u,v)) = t\,\|(u_k,v_k)\|_{L^2}^2$.
   (b) If $b \in \mathcal{B}_{2,good}$, then $b = C_{j,l}$ with $j > 2k-1$ or ($j = 2k-1$ and $l \geqslant 1$). Similarly, $|b| \geqslant 2k+2$ and the estimate (2.14) with $p = q = 2$ and $k' = k$ gives (2.20) with $\Xi(t,(u,v)) = t\,\|(u_k,v_k)\|_{L^2}^2$ as $k - 2 - \lfloor \frac{j}{2} \rfloor < 0$.
2. *Estimates of cross terms:* let $b_1 \geqslant \cdots \geqslant b_q \in \mathcal{B}\backslash\{X_0\}$ be such that $n(b_1)+\cdots+n(b_q) \leqslant \pi(k,m)$, and $\mathrm{supp}\mathcal{F}(b_1,\cdots,b_q) \not\subset \mathcal{N}_k^m$. For $i \in [\![1,q]\!]$,
   (a) if $b_i = M_j^1$ or M_j^2 with $j \in [\![0, k-1]\!]$, then by (2.3) and (2.4),
   $$|\xi_{b_i}(t,(u,v))| \leqslant |(u_{j+1}, v_{j+1})(t)|.$$
   Then, the estimate (2.21) is verified with $\sigma_i = j+1$, $\alpha_i = \frac{1}{2}$ and $\Xi(t,(u,v)) = |(u_1,\cdots,u_k,v_1,\cdots,v_k)(t)|^2$.
   (b) If $b_i = M_j^1$ or M_j^2, with $j \geqslant k$, $|b_i| \geqslant k+1$, and the estimates (2.10) and (2.11) with $j_0 = k$ and $p = 2$ give
   $$|\xi_{b_i}(t,(u,v))| \leqslant \frac{(ct)^{|b_i|}}{|b_i|!} t^{-(k+1)} t^{\frac{1}{2}} \|(u_k,v_k)\|_{L^2} = \frac{(ct)^{|b_i|}}{|b_i|!} t^{-(k+1)} \left(t\,\|(u_k,v_k)\|_{L^2}^2\right)^{\frac{1}{2}}.$$
   We obtain (2.21) with $\sigma_i = k+1$, and $\alpha_i = \frac{1}{2}$.

Since $\mathrm{supp}\mathcal{F}(b_1,\cdots,b_q) \not\subset \mathcal{N}_k^m$, one has $q = 2$, and $b_1, b_2 \in \mathcal{B}_1$. Then, as $\alpha_1 + \alpha_2 = 1$, we can apply Corollary 2.13 and we obtain the desired equality.

□

3.3 Vectorial relations

The purpose of this section is to prove that the condition (BC) implies algebraic properties on the Lie brackets. Using this fact, we will be able to estimate one by one - in the next paragraph - the terms $|(u_1, \cdots, u_k, v_1, \cdots, v_k)(t)|$ which appear in the previous proposition.

Lemma 3.10 (A bracket relation)**.** *Let $k, m \in \mathbb{N}^*$. For all $l \in [\![0, k-1]\!]$, for all $(\alpha_{j,1})_{j\in[\![0,l]\!]} \in \mathbb{R}^{l+1}, (\alpha_{j,2})_{j\in[\![0,l]\!]} \in \mathbb{R}^{l+1}$, one considers the bracket*

$$B := \sum_{j=0}^{l} \alpha_{j,1} M_j^1 + \sum_{j=0}^{l} \alpha_{j,2} M_j^2.$$

Then, the following expansion holds

$$\left[B0^{k-l-1}, B0^{k-l}\right] \in \alpha_{l,1}^2 W_k^1 + \alpha_{l,2}^2 W_k^2 + 2\alpha_{l,1}\alpha_{l,2} C_{2k-1} + \mathcal{N}_k^m.$$

This lemma is proved in Appendix A.2.

Lemma 3.11. *Let $e_1, e_2, e_3 \in \mathbb{R}^d$ be three vectors and $N \subset \mathbb{R}^d$ a vector subspace. If e_1, e_2, e_3, N satisfy (BC), then, there doesn't exist $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ such that*

$$a^2 e_1 + b^2 e_2 \pm 2ab e_3 \in N. \tag{3.3}$$

Proof. By contradiction, assume that there exists $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ satisfying (3.3). If $a = 0$, then (3.3) gives $b^2 e_2 \in N$. As, $b \neq 0$, one obtains $e_2 \in N$. This is a contradiction with (BC). Thus, $a \neq 0$. Similarly, $b \neq 0$. Hence, using $\mathbb{P}$ given by (BC), one has

$$a^2 \mathbb{P}(e_1) + b^2 \mathbb{P}(e_2) \pm 2ab \mathbb{P}(e_3) = 0. \tag{3.4}$$

Nevertheless, by hypothesis (BC) and Young's inequality,

$$|2ab\mathbb{P}(e_3)| < 2|ab| \sqrt{\mathbb{P}(e_1)\mathbb{P}(e_2)} < a^2 \mathbb{P}(e_1) + b^2 \mathbb{P}(e_2).$$

This is a contradiction with (3.4). □

Lemma 3.12. *Let $k, m \in \mathbb{N}^*$, and $\nu(k, m) := \left\lfloor \frac{\pi(k,m)}{2} \right\rfloor$. Assume that (3.1) is verified. Then,*

1. *the family $\left(f_{M_0^1}(0), \cdots f_{M_{k-1}^1}(0), f_{M_0^2}(0), \cdots, f_{M_{k-1}^2}(0)\right)$ is linearly independent.*
2. *if $\nu(k, m) \geqslant 2$,*

$$Span\left(f_{M_0^1}(0), \cdots, f_{M_{k-1}^1}(0), f_{M_0^2}(0), \cdots, f_{M_{k-1}^2}(0)\right) \cap S_{[\![2,\nu(k,m)]\!]}(f)(0) = \{0\}.$$

In particular, $f_1(0) \neq 0$ and $f_2(0) \neq 0$.

Proof. We prove the second point: assume by contradiction that there exist $(\alpha_{i,1}), (\alpha_{i,2}) \in \mathbb{R}^k$ not all zero, and $B \in S_{[\![2,\nu(k,m)]\!]}(X)$ such that $f_{B_1}(0) = 0$, with

$$B_1 := \sum_{j=0}^{k-1} \left(\alpha_{j,1} M_j^1 + \alpha_{j,2} M_j^2\right) + B.$$

Let $K = \max\{j \in [\![0, k-1]\!];\ (\alpha_{j,1}, \alpha_{j,2}) \neq (0, 0)\}$. As $f_0(0) = 0$, $f_{B_2}(0) = 0$, with

$$B_2 := [B_1 0^{k-1-K}, B_1 0^{k-K}] \in \alpha_{K,1}^2 W_k^1 + \alpha_{K,2}^2 W_k^2 + 2\alpha_{K,1}\alpha_{K,2} C_{2k-1} + \mathcal{N}_k^m + S_{[\![3,2\nu(k,m)]\!]}(X),$$

the expansion is given by Lemma 3.10 with $l = K$. As $\pi(k, m) \geqslant 2\nu(k, m)$ and $\nu(k, m) \geqslant 2$, one has $S_{[\![3,2\nu(k,m)]\!]}(X) \subseteq S_{[\![1,\pi(k,m)]\!]\setminus\{2\}}(X) \subseteq \mathcal{N}_k^m$. Thus,

$$\alpha_{K,1}^2 f_{W_k^1}(0) + \alpha_{K,2}^2 f_{W_k^2}(0) + 2\alpha_{K,1}\alpha_{K,2} f_{C_{2k-1}}(0) \in \mathcal{N}_k^m(f)(0).$$

We use Lemma 3.11 with $e_1 = f_{W_k^1}(0)$, $e_2 = f_{W_k^2}(0)$, $e_3 = f_{C_{2k-1}}(0)$ and $N = \mathcal{N}_k^m(f)(0)$ to obtain a contradiction. We obtain the first point in the same way, taking $B = 0$. □

3.4 Closed-loop estimate

Using the algebraic properties proved in the previous section, we can now estimate the terms $|(u_1, \cdots, u_k, v_1, \cdots, v_k)(t)|$, using the representation formula of the state of Magnus-type - see Proposition 2.8.

Lemma 3.13. *Let $k, m \in \mathbb{N}^*$, and $\nu(k,m) := \left\lfloor \frac{\pi(k,m)}{2} \right\rfloor$. Assume that (3.1) holds. Then, as $(t, \|(u,v)\|_{L^1}) \to 0$,*

$$|(u_1, \cdots, u_k, v_1, \cdots, v_k)(t)| = \mathcal{O}\left(t^{\frac{1}{2}} \|(u_k, v_k)\|_{L^2} + \|(u,v)\|_{L^1}^{\nu(k,m)+1} + \|x(t;(u,v))\|\right). \tag{3.5}$$

Proof. Let $i \in [\![0, k-1]\!]$, By Lemma 3.12, one can consider a linear form $\mathcal{P}$ such that $\mathcal{P}(f_{M_i^1}(0)) = 1$ and $\mathcal{P}_{|\mathcal{N}(f)(0)} \equiv 0$, with $\mathcal{N} := \mathcal{B}_{[\![2,\nu(k,m)]\!]} \cup \{M_0^1, \cdots, M_{k-1}^1, M_0^2, \cdots, M_{k-1}^2\} \setminus \{M_i^1\}$. Now, we use Corollary 2.13 with $M = \nu(k,m)$, $L = k+1$, $r = 1$, and $\mathfrak{b}_1 = M_i^1$.

Indeed, for all $b \in \mathcal{B}_{[\![1,\nu(k,m)]\!]}$ such that $b \notin \mathcal{N} \cup \{\mathfrak{b}\}$, necessarily, $n(b) = 1$, and $b = M_j^l$ for $j \geqslant k$ and $l \in \{1,2\}$. Thus, estimates (2.10) and (2.11), with $j_0 = k$ and $p = 2$ give

$$|\xi_b(t,(u,v))| \leqslant \frac{(ct)^{|b|}}{|b|!} t^{-(k+1)} \left(t^{\frac{1}{2}} \|(u_k, v_k)\|_{L^2}\right).$$

Then, (2.20) holds with $\Xi(t,(u,v)) = t^{\frac{1}{2}} \|(u_k, v_k)\|_{L^2}$ and $\sigma = k+1$. Moreover, there is no cross terms. Then, Corollary 2.13 leads to the equality

$$\mathcal{P}\mathcal{Z}_{\nu(k,m)}(t; f, (u,v))(0) = u_{i+1}(t) + \mathcal{O}\left(t^{\frac{1}{2}} \|(u_k, v_k)\|_{L^2}\right).$$

Using the Magnus formula given by Proposition 2.8 with $M = \nu(k,m)$, we finally get

$$\mathcal{P}x(t;(u,v)) = u_{i+1}(t) + \mathcal{O}\left(t^{\frac{1}{2}} \|(u_k, v_k)\|_{L^2} + \|(u,v)\|_{L^1}^{\nu(k,m)+1} + \|x(t;(u,v))\|^{1+\frac{1}{\nu(k,m)}}\right).$$

We obtain the result. We can obtain the same estimate for $|v_{i+1}(t)|$, $i \in [\![0, k-1]\!]$. □

3.5 Interpolation inequality

The representation formula of the state (Proposition 2.8 with $M = \pi(k,m)$) makes a strong link between $x(t;(u,v))$ and $\mathcal{Z}_{\pi(k,m)}(t; f, (u,v))(0)$. Lemma 3.9 gives an expansion of $\mathbb{P}\mathcal{Z}_{\pi(k,m)}(t; f, (u,v))(0)$. Furthermore, the edge terms $|(u_1, \cdots, u_k, v_1, \cdots, v_k)(t)|$ are estimated by Lemma 3.13. However, there is an error term in the Magnus-type formula shaped as $\mathcal{O}\left(\|(u,v)\|_{L^1}^{\pi(k,m)+1}\right)$. We then relate this quantity to the size of the drift $\|(u_k, v_k)\|_{L^2}^2$, thanks to the Gagliardo–Nirenberg interpolation inequalities. This is the purpose of the following lemma.

Lemma 3.14. *Let $k, m \in \mathbb{N}^*$, and $p \in [1, +\infty]$. There exists $C > 0$ such that, for every $t > 0$, and $u \in L^1((0,t), \mathbb{R})$*

$$\|u\|_{L^1}^{\pi(k,m)+1} \leqslant C t^{\pi(k,m)-2k} \|u\|_{W^{m,p}}^{\pi(k,m)-1} \|u_k\|_{L^2}^2. \tag{3.6}$$

Proof. For simplicity, we write π instead of $\pi(k,m)$. We use the Gagliardo–Nirenberg interpolation inequalities (2.23) with $P = \frac{2(m+k)p}{2k+mp}$, $q = 2$, $r = p$, $s = 2$, $j = k$, $l = m+k$, $\alpha = \frac{k}{k+m}$, and $\varphi = u_k$, and we obtain

$$\|u\|_{L^P} \leqslant C \left\|u^{(m)}\right\|_{L^p}^{\alpha} \|u_k\|_{L^2}^{1-\alpha} + C t^{\frac{1}{P}-(k+\frac{1}{2})} \|u_k\|_{L^2}. \tag{3.7}$$

Moreover, using Hölder's inequality,

$$\|u\|_{L^1} \leqslant t^{1-\frac{1}{P}} \|u\|_{L^P}. \tag{3.8}$$

Using (3.7) and (3.8), we obtain

$$\|u\|_{L^1}^{\pi+1} \leqslant Ct^{(\pi+1)\left(1-\frac{1}{P}\right)} \left(\left\|u^{(m)}\right\|_{L^p}^{\alpha(\pi+1)} \|u_k\|_{L^2}^{(1-\alpha)(\pi+1)} + t^{(\pi+1)\left(\frac{1}{P}-(k+\frac{1}{2})\right)} \|u_k\|_{L^2}^{\pi+1} \right)$$

Thus, if we define $\beta := 1 + \frac{2k}{m}$, then, $(1-\alpha)(1+\beta) = 2$. We get

$$\|u\|_{L^1}^{\pi+1} \leqslant Ct^{(\pi+1)\left(1-\frac{1}{P}\right)} \left(\|u\|_{W^{m,p}}^{\alpha(\pi+1)} \|u_k\|_{L^2}^{(1-\alpha)(\pi-\beta)} + t^{(\pi+1)\left(\frac{1}{P}-(k+\frac{1}{2})\right)} \|u_k\|_{L^2}^{\pi-1} \right) \|u_k\|_{L^2}^2 . \tag{3.9}$$

Moreover,

$$\|u_k\|_{L^2} \leqslant t^{k+\frac{1}{2}} \|u\|_{L^\infty} \leqslant Ct^{k+\frac{1}{2}} \|u\|_{W^{m,p}} . \tag{3.10}$$

Using (3.10) in (3.9), we obtain

$$\begin{aligned} \|u\|_{L^1}^{\pi+1} \leqslant Ct^{(\pi+1)\left(1-\frac{1}{P}\right)} \Big(& \|u\|_{W^{m,p}}^{\alpha(\pi+1)+(1-\alpha)(\pi-\beta)} t^{(k+\frac{1}{2})(1-\alpha)(\pi-\beta)} \\ & + t^{(\pi+1)\left(\frac{1}{P}-(k+\frac{1}{2})\right)+(\pi-1)\left(k+\frac{1}{2}\right)} \|u\|_{W^{m,p}}^{\pi-1} \Big) \|u_k\|_{L^2}^2 . \end{aligned}$$

As $\alpha(\pi+1) + (1-\alpha)(\pi-\beta) = \pi - 1$, one obtains

$$\|u\|_{L^1}^{\pi+1} \leqslant Ct^{(\pi+1)\left(1-\frac{1}{P}\right)} \left(t^{(k+\frac{1}{2})(1-\alpha)(\pi+1)-(2k+1)} + t^{\frac{\pi+1}{P}-(2k+1)} \right) \|u\|_{W^{m,p}}^{\pi-1} \|u_k\|_{L^2}^2 .$$

Finally,

$$\left(k+\frac{1}{2}\right)(1-\alpha) = \frac{(2k+1)m}{2(k+m)} \geqslant \frac{1}{P},$$

Thus,

$$\|u\|_{L^1}^{\pi+1} \leqslant Ct^{(\pi+1)\left(1-\frac{1}{P}\right)} t^{\frac{\pi+1}{P}-(2k+1)} \|u\|_{W^{m,p}}^{\pi-1} \|u_k\|_{L^2}^2 .$$

□

3.6 Proof of the drift

We can now use the Magnus-type representation formula given by Proposition 2.8, the expansion of $\mathbb{P}\mathcal{Z}_{\pi(k,m)}(t,f,(u,v))(0)$ given by Lemma 3.9, the estimate of Lemma 3.13 and the interpolation inequality given by Lemma 3.14 to prove Theorem 3.7.

Proof of Theorem 3.7. Let $k, m \in \mathbb{N}^*$ and $p \in [1, +\infty]$. We will write π instead of $\pi(k,m)$. Let $e_1 := f_{W_k^1}(0)$, $e_2 := f_{W_k^2}(0)$ and $e_3 := f_{C_{2k-1}}(0)$. Let $\mathbb{P}$ be a linear form given by (BC). The Magnus-type expansion formula given by Proposition 2.8 with $M = \pi$, the equalities (2.5), (2.6) and (2.7) and (3.2) give, as $(t, \|(u,v)\|_{L^1}) \to 0$,

$$\begin{aligned} \mathbb{P}x(t;(u,v)) = \int_0^t \left(\mathbb{P}(e_1)\frac{u_k^2}{2} + \mathbb{P}(e_2)\frac{v_k^2}{2} + \mathbb{P}(e_3)u_k v_k \right) + \mathcal{O}\Big(t\|(u_k,v_k)\|_{L^2}^2 \\ + |(u_1,\cdots,u_k,v_1,\cdots,v_k)(t)|^2 + \|(u,v)\|_{L^1}^{\pi+1} + \|x(t;(u,v))\|^{1+\frac{1}{\pi}} \Big) . \end{aligned} \tag{3.11}$$

The closed-loop estimates (3.5) gives, with $\nu := \left\lfloor \frac{\pi}{2} \right\rfloor$,

$$|(u_1,\cdots,u_k,v_1,\cdots,v_k)(t)|^2 = \mathcal{O}\left(t\|(u_k,v_k)\|_{L^2}^2 + \|(u,v)\|_{L^1}^{2\nu+2} + \|x(t;(u,v))\|^2 \right) . \tag{3.12}$$

By definition of ν, one has $2(\nu+1) \geqslant \pi + 1$. In particular, as $\|(u,v)\|_{L^1} \to 0$,

$$\|(u,v)\|_{L^1}^{2\nu+2} = \mathcal{O}\left(\|(u,v)\|_{L^1}^{\pi+1} \right) .$$

Using (3.12) in (3.11) and the interpolation inequality (3.6), one gets

$$\begin{aligned} \mathbb{P}x(t;(u,v)) = \int_0^t \left(\mathbb{P}(e_1)\frac{u_k^2}{2} + \mathbb{P}(e_2)\frac{v_k^2}{2} + \mathbb{P}(e_3)u_k v_k \right) \\ + \mathcal{O}\left(\left(t + t^{\pi-2k} \|(u,v)\|_{W^{m,p}}^{\pi-1} \right) \|(u_k,v_k)\|_{L^2}^2 + \|x(t;(u,v))\|^{1+\frac{1}{\pi}} \right) . \end{aligned}$$

We prove that the system (1.1) has a drift in the regime $(t, t^\alpha \|(u,v)\|_{W^{m,p}}) \to 0$, with $\alpha = \frac{\pi-2k}{\pi-1}$: by definition, there exist $C, \rho > 0$ such that, for every $t \in (0,\rho)$, there exists $\eta > 0$ s.t. for every $u, v \in W^{m,p}((0,t),\mathbb{R})$ with $\|(u,v)\|_{W^{m,p}} \leqslant \eta$,

$$\begin{aligned} &\left|\mathbb{P}x(t;(u,v)) - \int_0^t \left(\mathbb{P}(e_1)\frac{u_k^2}{2} + \mathbb{P}(e_2)\frac{v_k^2}{2} + \mathbb{P}(e_3)u_k v_k\right)\right| \\ &\leqslant C\left(\left(t + t^{\pi-2k}\|(u,v)\|_{W^{m,p}}^{\pi-1}\right)\|(u_k,v_k)\|_{L^2}^2 + \|x(t;(u,v))\|^{1+\frac{1}{\pi}}\right). \end{aligned} \tag{3.13}$$

Let $\gamma := \frac{|\mathbb{P}(e_3)|}{\sqrt{\mathbb{P}(e_1)\mathbb{P}(e_2)}} < 1$, by hypothesis (BC). Using Young's inequality, one obtains

$$\int_0^t \left(\mathbb{P}(e_1)\frac{u_k^2}{2} + \mathbb{P}(e_2)\frac{v_k^2}{2} + \mathbb{P}(e_3)u_k v_k\right) \geqslant K\int_0^t \left(u_k^2 + v_k^2\right), \tag{3.14}$$

with $K := \frac{1}{2}(1-\gamma)\min(\mathbb{P}(e_1),\mathbb{P}(e_2))$. Thus, for all $t \in \left(0, \min\left(\rho, \frac{K}{4C}\right)\right)$, for all $u, v \in W^{m,p}((0,t),\mathbb{R})$, with $\|(u,v)\|_{W^{m,p}} \leqslant \min\left(\eta, \left(t^{2k-\pi}\frac{K}{4C}\right)^{\frac{1}{\pi-1}}\right)$, the equalities (3.13) and (3.14) lead to

$$\mathbb{P}x(t;(u,v)) \geqslant \frac{K}{2}\Delta(u,v) - C\|x(t;(u,v))\|^{1+\frac{1}{\pi}},$$

with $\Delta : (u,v) \in L^1((0,1),\mathbb{R})^2 \mapsto \int_0^t \left(u_k^2 + v_k^2\right) \in \mathbb{R}^+$. Then, the system (1.1) has a drift along $e_1 + e_2$ parallel to $\mathcal{N}_k^m(f)(0)$ with strength Δ as $(t, t^\alpha\|(u,v)\|_{W^{m,p}}) \to 0$. This concludes the proof of Theorem 3.7. □

4 Necessary conditions for STLC in the asymmetrical case

4.1 A new truncation in the Magnus-type representation formula

In order to prove Theorem 1.18, we first give an asymmetrical Magnus-type representation formula. This is the purpose of the following statement.

Proposition 4.1 (Asymmetrical Magnus expansion)**.** *Let $M, N \in \mathbb{N}^*$ with $N \leqslant M$, let $\delta, T > 0$ and $f_0, f_1, f_2 : B(0,2\delta) \to \mathbb{R}^d$ be analytic vector fields with $f_0(0) = 0$ and $T\|f_0\|_\infty \leqslant \delta$. For $u, v \in L^1((0,T),\mathbb{R})$, as $\|(u,v)\|_{L^1} \to 0$,*

$$x(t;(u,v)) = \mathcal{Z}_{M,N}(t;f,(u,v))(0) + \mathcal{O}\left(\|u\|_{L^1(0,t)}^{M+1} + \|v\|_{L^1(0,t)}^{N+1} + \|x(t;(u,v))\|^{1+\frac{1}{M}}\right), \tag{4.1}$$

where $\mathcal{Z}_{M,N}$ is defined as

$$\mathcal{Z}_{M,N}(t;f,(u,v)) := \sum_{\substack{b\in\mathcal{B}_{[\![1,M]\!]} \\ n_2(b)\leqslant N}} \eta_b(t,(u,v))f_b. \tag{4.2}$$

Proof. By definition

$$\mathcal{Z}_M(t;f,(u,v))(0) = \mathcal{Z}_{M,N}(t;f,(u,v))(0) + \sum_{\substack{b\in\mathcal{B}_{[\![1,M]\!]}, \\ n_2(b)\geqslant N+1}} \eta_b(t,(u,v))f_b(0).$$

We use analytic estimates, as for Propositions 2.11 and 2.12, and the estimates on the coordinates of pseudo-first kind given by [3, Proposition 52] to obtain, as $\|(u,v)\|_{L^1} \to 0$,

$$\sum_{\substack{b\in\mathcal{B}_{[\![1,M]\!]}, \\ n_2(b)\geqslant N+1}} |||\eta_b(t,(u,v))f_b|||_{r'} = \mathcal{O}\left(\|v\|_{L^1(0,t)}^{N+1}\right),$$

for a given $r' > 0$. The Magnus-type representation formula (2.18) leads to the conclusion. □

We are now in a position to prove Theorem 1.18. We adopt the approach presented in Section 3.1. Let $k, k', m, m' \in \mathbb{N}^*$. We reason by contraposition and we assume that

$$f_{W^1_k}(0), \quad f_{W^2_{k'}}(0), \quad f_{C_{k+k'-1}}(0), \quad \mathcal{N}^{m,m'}_{k,k'}(f)(0) \quad \text{satisfy (BC).} \tag{4.3}$$

We prove Theorem 1.18 as a consequence of the following more precise statement.

Theorem 4.2. *Let $k, k', m, m' \in \mathbb{N}^*$ and $p, p' \in [1, +\infty]$. Assume that (4.3) holds, then, system (1.1) has a drift along $f_{W^1_k}(0) + f_{W^2_{k'}}(0)$ parallel to $\mathcal{N}^{m,m'}_{k,k'}(f)(0)$ with strength $\Delta : (u,v) \in L^1((0,1),\mathbb{R})^2 \mapsto \int_0^t \left(u_k^2 + v_{k'}^2\right) \in \mathbb{R}^+$ as $\left(t, \left(t^\alpha \|u\|_{W^{m,p}}, t^{\alpha'} \|v\|_{W^{m',p'}}\right)\right) \to 0$, where $\alpha = \frac{\pi(k,m)-2k}{\pi(k,m)-1}$ and $\alpha' = \frac{\pi(k',m')-2k'}{\pi(k',m')-1}$.*

Thus, we obtain Theorem 1.18 thanks to Lemma 1.22. From now on, we will sometimes refer to $\pi(k,m)$ as π, $\pi(k',m')$ as π' and $\mathcal{N}^{m,m'}_{k,k'}$ as $\mathcal{N}$.

4.2 Dominant part of the logarithm

Lemma 4.3. *Let $k, k', m, m' \in \mathbb{N}^*$ be such that $k' \leqslant k$. Let $\mathcal{P}$ be a linear form satisfying $\mathcal{P}_{|\mathcal{N}(f)(0)} \equiv 0$. Then, as $(t, \|(u,v)\|_{L^1}) \to 0$,*

$$\begin{aligned}\mathcal{P}\mathcal{Z}_{\pi,\pi'}(t; f, (u,v))(0) = \mathcal{P}\left(f_{W^1_k}(0)\right)\xi_{W^1_k}(t,(u,v)) + \mathcal{P}\left(f_{W^2_{k'}}(0)\right)\xi_{W^2_{k'}}(t,(u,v))\\ +\mathcal{P}\left(f_{C_{k+k'-1}}(0)\right)\xi_{C_{k+k'-1}}(t,(u,v)) + \mathcal{O}\left(t\,\|(u_k,v_{k'})\|^2_{L^2} + |(u_1,\cdots,u_k,v_1,\cdots,v_{k'})(t)|^2\right).\end{aligned} \tag{4.4}$$

Proof. As for Lemma 3.9, we fix $M = \pi$, $\mathcal{N} = \mathcal{N}^{m,m'}_{k,k'}$, $r = 3$, $\mathfrak{b}_1 = W^1_k$, $\mathfrak{b}_2 = W^2_{k'}$, and $\mathfrak{b}_3 = C_{k+k'-1}$. Let $N = \pi'$.

1. *Estimates of the main terms:* let $b \in \mathcal{E} := \mathcal{B}_{[\![1,\pi]\!]} \cap \{b \in \mathrm{Br}(X),\ n_2(b) \leqslant \pi'\} \setminus \mathcal{N} \cup \{\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3\} \cup \{C_{j,l};\ l + \lfloor \frac{j}{2} \rfloor \leqslant k-2\}$. Then, $n(b) = 2$ and
   (a) If $b \in \mathcal{B}_{2,bad} \cap \{b \in \mathrm{Br}(X),\ n_1(b) = 2\}$, $b = W^1_{j,l}$ with $j > k$ or ($j = k$ and $l \geqslant 1$). Thus $|b| \geqslant 2k+2$ and the estimate (2.12) with $j_0 = k$ and $p = 1$ gives (2.20) with $\Xi(t,(u,v)) := t\,\|u_k\|^2_{L^2}$.
   (b) If $b \in \mathcal{B}_{2,bad} \cap \{b \in \mathrm{Br}(X),\ n_2(b) = 2\}$, $b = W^2_{j,l}$ with $j > k'$ or ($j = k'$ and $l \geqslant 1$). Thus $|b| \geqslant 2k'+2$ and the estimate (2.13) with $j_0 = k'$ and $p = 1$ gives (2.20) with $\Xi(t,(u,v)) := t\,\|v_{k'}\|^2_{L^2}$.
   (c) Else, $b \in \mathcal{B}_{2,good}$ and $b = C_{j,l}$ with $j \geqslant k + k'$ or ($j \geqslant k + k' - 1$ and $l \geqslant 1$). Then, $|b| \geqslant k + k' + 2$ and the estimate (2.14) with $p = q = 2$ gives (2.20) with $\Xi(t,(u,v)) := t\,\|(u_k,v_{k'})\|^2_{L^2}$.

   Thus, one can apply Proposition 2.11 to obtain as $\|(u,v)\|_{L^1} \to 0$,

$$\sum_{b\in\mathcal{E}} |||\xi_b(t,(u,v))f_b|||_{r'} = \mathcal{O}\left(t\,\|(u_k,v_{k'})\|^2_{L^2}\right),$$

   for a given $r' > 0$. We finally need to examine the brackets $b \in \{C_{j,l};\ l + \lfloor \frac{j}{2} \rfloor \leqslant k-2\}$, where the set is finite. Then, estimate (2.14) with $p = q = 2$ gives, as $\|(u,v)\|_{L^1} \to 0$,

$$\sum_{l+\lfloor \frac{j}{2} \rfloor \leqslant k-2} |||\xi_{C_{j,l}}(t,(u,v))f_{C_{j,l}}|||_{r'} = \mathcal{O}\left(t\,\|(u_k,v_{k'})\|^2_{L^2} + |(u_1,\cdots,u_k)(t)|^2\right).$$

   Finally,

$$\sum_{\substack{b\in\mathcal{B}_{[\![1,\pi]\!]},\ n_2(b)\leqslant\pi',\\ b\notin\mathcal{N}\cup\{\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{b}_3\}}} |||\xi_b(t,(u,v))f_b|||_{r'} = \mathcal{O}\left(t\,\|(u_k,v_{k'})\|^2_{L^2} + |(u_1,\cdots,u_k)(t)|^2\right).$$

2. *Estimates of cross terms:* we apply Proposition 2.12 with the set $\mathcal{E} := \mathcal{B}_{[\![1,\pi]\!]} \cap \{b \in \mathrm{Br}(X),\ n_2(b) \leqslant \pi'\} \setminus \mathcal{N}$. Let $b_1 \geqslant \cdots \geqslant b_q \in \mathcal{B} \setminus \{X_0\}$ be such that $n_1(b_1) + \cdots + n_1(b_q) \leqslant \pi$, $n_2(b_1) + \cdots + n_2(b_q) \leqslant \pi'$ and $\mathrm{supp}\mathcal{F}(b_1, \cdots, b_q) \not\subset \mathcal{N}$. For $i \in [\![1, q]\!]$,
   (a) If $b_i = M_j^1$ with $j \in [\![0, k-1]\!]$ or $b_i = M_l^2$ with $l \in [\![0, k'-1]\!]$, then by (2.3) and (2.4),
   $$|\xi_{b_i}(t,(u,v))| \leqslant |(u_{j+1}, v_{l+1})(t)|.$$
   Then, the estimate (2.21) is verified with $\sigma_i = j+1$ or $\sigma_i = l+1$, $\alpha_i = \frac{1}{2}$ and $\Xi(t,(u,v)) = |(u_1, \cdots, u_k, v_1, \cdots, v_k')(t)|^2$.
   (b) If $b_i = M_j^1$ with $j \geqslant k$, $|b_i| \geqslant k+1$, and the estimates (2.10) with $j_0 = k$ and $p = 2$ gives
   $$|\xi_{b_i}(t,(u,v))| \leqslant \frac{(ct)^{|b_i|}}{|b_i|!} t^{-(k+1)} t^{\frac{1}{2}} \|u_k\|_{L^2} = \frac{(ct)^{|b_i|}}{|b_i|!} t^{-(k+1)} \left(t \|u_k\|_{L^2}^2\right)^{\frac{1}{2}}.$$
   We obtain (2.21) with $\sigma_i = k+1$, and $\alpha_i = \frac{1}{2}$.
   (c) If $b_i = M_l^2$ with $l \geqslant k'$, $|b_i| \geqslant k'+1$, and the estimates (2.11) with $j_0 = k'$ and $p = 2$ gives
   $$|\xi_{b_i}(t,(u,v))| \leqslant \frac{(ct)^{|b_i|}}{|b_i|!} t^{-(k'+1)} t^{\frac{1}{2}} \|v_{k'}\|_{L^2} = \frac{(ct)^{|b_i|}}{|b_i|!} t^{-(k'+1)} \left(t \|v_{k'}\|_{L^2}^2\right)^{\frac{1}{2}}.$$
   We obtain (2.21) with $\sigma_i = k'+1$, and $\alpha_i = \frac{1}{2}$.

   Since $\mathrm{supp}\mathcal{F}(b_1, \cdots, b_q) \not\subset \mathcal{N}$, one has $q = 2$, and $b_1, b_2 \in \mathcal{B}_1$. Moreover, $\alpha_1 + \alpha_2 = 1$.

Thus, the definition of $\mathcal{Z}_{\pi,\pi'}$ - see (4.2) - leads to the result. □

4.3 Vectorial relations

Lemma 4.4 (A bracket relation)**.** *Let $k, k', m, m' \in \mathbb{N}^*$ be such that $k' \leqslant k$. For all $l \in [\![0, k'-1]\!]$, for all $(\alpha_j)_{j\in[\![0,l+k-k']\!]} \in \mathbb{R}^{l+k-k'+1}, (\beta_j)_{j\in[\![0,l]\!]} \in \mathbb{R}^{l+1}$, one considers the bracket*
$$B := \sum_{j=0}^{l+k-k'} \alpha_j M_j^1 + \sum_{j=0}^{l} \beta_j M_j^2.$$
Then, the following expansion holds
$$\left[B0^{k'-l-1}, B0^{k'-l}\right] \in \alpha_{l+k-k'}^2 W_k^1 + \beta_l^2 W_{k'}^2 + 2(-1)^{\lfloor \frac{k-k'}{2} \rfloor} \alpha_{l+k-k'} \beta_l C_{k+k'-1} + \mathcal{N}_{k,k'}^{m,m'}.$$

This lemma is proved in Appendix A.3. This is a generalization of Lemma 3.10, in the asymmetrical case.

Lemma 4.5. *Let $k, k', m, m' \in \mathbb{N}^*$ be such that $k' \leqslant k$ and $\nu := \lfloor \frac{\pi}{2} \rfloor$, $\nu' := \left\lfloor \frac{\pi'}{2} \right\rfloor$. Assume that (4.3) is verified. Then,*

1. *the family $\left(f_{M_0^1}(0), \cdots f_{M_{k-1}^1}(0), f_{M_0^2}(0), \cdots, f_{M_{k'-1}^2}(0)\right)$ is linearly independent.*
2. *if $\nu \geqslant 2$,*
$$\mathrm{Span}\left(f_{M_0^1}(0), \cdots, f_{M_{k-1}^1}(0), f_{M_0^2}(0), \cdots, f_{M_{k'-1}^2}(0)\right) \cap S_{[\![2,\nu]\!]}(f)(0) \cap S_{[\![0,\nu]\!],[\![0,\nu']\!]}(f)(0) = \{0\}.$$

Proof. We prove the result in the same way as Lemma 3.12, we start with the second point. Assume by contradiction that there exist $(\alpha_i), (\beta_i)$, reals, not all zero, and $B \in S_{[\![2,\nu]\!]}(X) \cap S_{[\![0,\nu]\!],[\![0,\nu']\!]}(X)$ such that $f_{B_1}(0) = 0$, with
$$B_1 := \sum_{j=0}^{k-1} \alpha_j M_j^1 + \sum_{j=0}^{k'-1} \beta_j M_j^2 + B.$$

1. Firstly, assume that $\alpha_{k-k'} = \cdots = \alpha_{k-1} = \beta_0 = \cdots = \beta_{k'-1} = 0$. One considers $K := \max\{j \in [\![0, k-k'-1]\!],\ \alpha_j \neq 0\}$. As $f_0(0) = 0$, one has $f_{B_2}(0) = 0$ with

$$B_2 := [B_1 0^{k-1-K}, B_1 0^{k-K}] \in \alpha_K^2 W_k^1 + \mathcal{N} + S_{[\![3,2\nu]\!]} \cap S_{[\![0,2\nu]\!],[\![0,2\nu']\!]}(X).$$

Moreover, by definition of ν and ν',

$$S_{[\![3,2\nu]\!]} \cap S_{[\![0,2\nu]\!],[\![0,2\nu']\!]}(X) \subseteq S_{[\![1,\pi]\!]} \cap S_{[\![0,\pi]\!]\setminus\{2\},[\![0,\pi']\!]}(X) \subseteq \mathcal{N}. \tag{4.5}$$

As $\alpha_K \neq 0$, we obtain a contradiction with (4.3), as $f_{W_k^1}(0) \notin \mathcal{N}(f)(0)$.

2. Else, $K := \max\{j \in [\![0, k'-1]\!],\ (\alpha_{k-k'+j}, \beta_j) \neq (0,0)\}$ is well defined. As $f_0(0) = 0$, $f_{B_2}(0) = 0$, with

$$B_2 := [B_1 0^{k'-1-K}, B_1 0^{k'-K}] \in \alpha_{k-k'+K}^2 W_k^1 + \beta_K^2 W_{k'}^2$$
$$+2(-1)^{\lfloor \frac{k-k'}{2} \rfloor} \alpha_{k-k'+K}\beta_K C_{k+k'-1} + \mathcal{N} + S_{[\![3,2\nu]\!]} \cap S_{[\![0,2\nu]\!],[\![0,2\nu']\!]}(X),$$

the expansion is given by Lemma 4.4 with $l = K$. Using (4.5), one finally obtains

$$\alpha_{k-k'+K}^2 f_{W_k^1}(0) + \beta_K^2 f_{W_{k'}^2}(0) + 2(-1)^{\lfloor \frac{k-k'}{2} \rfloor} \alpha_{k-k'+K}\beta_K f_{C_{k+k'-1}}(0) \in \mathcal{N}(f)(0).$$

We use Lemma 3.11 to obtain a contradiction. The first point is obtained with $B = 0$.

□

4.4 Closed-loop estimate

Lemma 4.6. *Let $k, k', m, m' \in \mathbb{N}^*$ be such that $k' \leqslant k$ and $\nu := \left\lfloor \frac{\pi}{2} \right\rfloor$, $\nu' := \left\lfloor \frac{\pi'}{2} \right\rfloor$. Assume that (4.3) holds. Then, as $(t, \|(u,v)\|_{L^1}) \to 0$,*

$$\begin{aligned} |\left(u_1, \cdots, u_k, v_1, \cdots, v_{k'}\right)(t)| = \mathcal{O}\Big(t^{\frac{1}{2}} \|(u_k, v_{k'})\|_{L^2} + \|u\|_{L^1}^{\nu+1} \\ + \|v\|_{L^1}^{\nu'+1} + \|x(t;(u,v))\|\Big). \end{aligned} \tag{4.6}$$

Proof. We prove the estimate as Lemma 3.13. For that, we use the asymmetrical Magnus-type representation formula given by Proposition 4.1 with $M = \nu$, $N = \nu'$ instead of Proposition 2.8. □

4.5 Proof of the drift

Proof of Theorem 4.2. Let $k, k', m, m' \in \mathbb{N}^*$ and $p, p' \in [1, +\infty]$ be such that $k' \leqslant k$. Let $e_1 := f_{W_k^1}(0)$, $e_2 := f_{W_{k'}^2}(0)$ and $e_3 := f_{C_{k+k'-1}}(0)$. Let $\mathbb{P}$ be a linear form given by (BC). The asymmetrical Magnus expansion formula given by Proposition 4.1 with $M = \pi$, $N = \pi'$, the equalities (2.5), (2.6) and (4.4) give, as $(t, \|(u,v)\|_{L^1}) \to 0$,

$$\begin{aligned} \mathbb{P}x(t;(u,v)) = \int_0^t \left(\mathbb{P}(e_1)\frac{u_k^2}{2} + \mathbb{P}(e_2)\frac{v_{k'}^2}{2}\right) + \mathbb{P}(e_3)\xi_{C_{k+k'-1}}(t,(u,v)) + \mathcal{O}\Big(t\,\|(u_k, v_{k'})\|_{L^2}^2 \\ + |(u_1, \cdots, u_k, v_1, \cdots, v_{k'})(t)|^2 + \|u\|_{L^1}^{\pi+1} + \|v\|_{L^1}^{\pi'+1} + \|x(t;(u,v))\|^{1+\frac{1}{\pi}}\Big). \end{aligned}$$

1. If $k' \leqslant k-2$, then, one can apply Lemma 2.3 with $j = k+k'-1$, $l = 0$, and $N = k-2-\lfloor \frac{k+k'-1}{2} \rfloor$ to obtain

$$\xi_{C_{k+k'-1}}(t,(u,v)) = \sum_{\mu=0}^{N} (-1)^\mu u_{\lfloor \frac{k+k'-1}{2} \rfloor + 2 + \mu}(t) v_{\lfloor \frac{k+k'}{2} \rfloor - \mu}(t) + (-1)^{N+1} \int_0^t u_k v_{k'}.$$

2. If $k' \in \{k-1, k\}$, the equation (2.7) lead to $\xi_{C_{k+k'-1}}(t,(u,v)) = \int_0^t u_k v_{k'}$, and the writing is already convenient. In these cases, $N+1=0$.

In all cases, the following equality holds

$$\xi_{C_{k+k'-1}}(t,(u,v)) = (-1)^{N+1}\int_0^t u_k v_{k'} + \mathcal{O}\left(|(u_1,\cdots,u_k)(t)|^2 + t\,\|v_{k'}\|_{L^2}^2\right). \tag{4.7}$$

The fact that $2(\nu+1) \geqslant \pi+1$, $2(\nu'+1)\geqslant \pi'+1$, the closed-loop estimates (4.6) and the equality (4.7) give, as $\|(u,v)\|_{L^1}\to 0$,

$$\begin{aligned}\mathbb{P}x(t;(u,v)) = \int_0^t \left(\mathbb{P}(e_1)\frac{u_k^2}{2} + \mathbb{P}(e_2)\frac{v_{k'}^2}{2} + (-1)^{N+1}\mathbb{P}(e_3)u_k v_{k'}\right)\\ +\mathcal{O}\left(t\,\|(u_k,v_{k'})\|_{L^2}^2 + \|u\|_{L^1}^{\pi+1} + \|v\|_{L^1}^{\pi'+1} + \|x(t;(u,v))\|^{1+\frac{1}{\pi}}\right).\end{aligned}$$

Finally, we use the interpolation inequality (3.6) to obtain

$$\begin{aligned}\mathbb{P}x(t;(u,v)) = \int_0^t \left(\mathbb{P}(e_1)\frac{u_k^2}{2} + \mathbb{P}(e_2)\frac{v_{k'}^2}{2} + (-1)^{N+1}\mathbb{P}(e_3)u_k v_{k'}\right)\\ +\mathcal{O}\left(\left(t + t^{\pi-2k}\,\|u\|_{W^{m,p}}^{\pi-1}\right)\|u_k\|_{L^2}^2 + \left(t + t^{\pi'-2k'}\,\|v\|_{W^{m',p'}}^{\pi'-1}\right)\|v_{k'}\|_{L^2}^2 + \|x(t;(u,v))\|^{1+\frac{1}{\pi}}\right).\end{aligned}$$

Let $\alpha := \frac{\pi-2k}{\pi-1}$, $\alpha' := \frac{\pi'-2k'}{\pi'-1}$ and $\Delta : (u,v)\in L^1((0,1),\mathbb{R})^2 \mapsto \int_0^t \left(u_k^2 + v_{k'}^2\right) \in \mathbb{R}^+$. We prove that the system (1.1) has a drift along e_1+e_2, parallel to $\mathcal{N}(f)(0)$ with strength Δ in regime $\left(t, \left(t^{\alpha}\,\|u\|_{W^{m,p}}, t^{\alpha'}\,\|v\|_{W^{m',p'}}\right)\right)\to 0$, as before. This concludes the proof of Theorem 4.2. □

4.6 Towards a quartic obstruction result

Having dealt with the case of quadratic drifts at any order, a natural question is to ask what happens for the case of quartic drifts, and more generally, for any even-order drift. We recall that this situation has been done in the case of single-input control-affine systems by Stefani and Sussmann in [22] - see Proposition 1.27. Here, we have to study a quartic competition between 5 brackets,

$$\mathfrak{b}_i := \mathrm{ad}_{X_2}^i\left(\mathrm{ad}_{X_1}^{4-i}(X_0)\right), \quad i\in[\![0,4]\!].$$

The associated functionals are the following ones: $\xi_{\mathfrak{b}_i}(t,(u,v)) = \dfrac{1}{i!(4-i)!}\displaystyle\int_0^t u_1^{4-i}v_1^i$, for $i\in[\![0,4]\!]$. If we choose $\mathcal{N}$, a vector subspace of $\mathbb{R}^d$ and $\mathbb{P}$, a linear form verifying $\mathbb{P}_{|\mathcal{N}}\equiv 0$ so that the dominant term of $\mathbb{P}\mathcal{Z}_4$ are theses terms, then

$$\mathbb{P}\mathcal{Z}_4(t;f,(u,v))(0) \simeq \sum_{i=0}^4 \frac{\mathbb{P}(e_{i+1})}{i!(4-i)!}\int_0^t u_1^{4-i}v_1^i(s)\mathrm{d}s, \tag{4.8}$$

with $e_{i+1} = f_{\mathfrak{b}_i}(0)$ for $i\in[\![0,4]\!]$. Using Young's inequality in (4.8), one has

$$\begin{aligned}\mathbb{P}\mathcal{Z}_4(t;f,(u,v))(0) \gtrsim \left(\frac{\mathbb{P}(e_1)}{24} - \frac{|\mathbb{P}(e_2)|}{8} - \frac{|\mathbb{P}(e_4)|}{24}\right)\|u_1\|_{L^4}^4 + \frac{\mathbb{P}(e_3)}{4}\int_0^t u_1^2v_1^2\\ +\left(\frac{\mathbb{P}(e_5)}{24} - \frac{|\mathbb{P}(e_4)|}{8} - \frac{|\mathbb{P}(e_2)|}{24}\right)\|v_1\|_{L^4}^4,\end{aligned} \tag{4.9}$$

Thus, the condition (BC) can be adapted in this case as: let $e_1,e_2,e_3,e_4,e_5\in\mathbb{R}^d$ be five vectors and $N\subset\mathbb{R}^d$ a vector subspace. We say that e_1,e_2,e_3,e_4,e_5,N verify (BC) if there exists $\mathbb{P}:\mathbb{R}^d\to\mathbb{R}$ a linear form such that:

$$3\,|\mathbb{P}(e_2)| + |\mathbb{P}(e_4)| < \mathbb{P}(e_1), \qquad |\mathbb{P}(e_2)| + 3\,|\mathbb{P}(e_4)| < \mathbb{P}(e_5), \qquad \mathbb{P}(e_3)\geqslant 0, \qquad \mathbb{P}_{|N}\equiv 0.$$

With this condition and a good choice of $\mathcal{N}$, we can prove that $(f_1(0), f_2(0))$ is a linearly independent family. Thus, we can use the same strategy for the closed-loop estimates. Moreover, the remainder term of the Magnus-type representation formula, shaped as $\|(u,v)\|_{L^1}^M$, can be estimated by Gagliardo–Nirenberg interpolation inequalities. The major difficulty lies in extracting the dominant part of $\mathcal{Z}_4$. This requires the ability to estimate the coordinates of the pseudo-first kind. However, this is very time-consuming: $\mathcal{B}_2$ is made up of 3 families, $\mathcal{B}_3$ of 8 families, and $\mathcal{B}_4$ is made up of 36 different families! Thus, this is very tedious to extend Proposition 2.5 is the case of $\mathcal{B}_4$. This strategy would make it possible to treat systems as (1.11).

A Postponed proofs

A.1 Geometric conditions

Proof of Proposition 3.6. Assume that (BC) is not satisfied, and that the points 1, 3 and 4 are not verified. The purpose is to show that the second one is. Then, one of the three following possibilities holds

a. $(\tilde{e_1}, \tilde{e_2})$ is a linearly independent family,

b. $\tilde{e_2} = \beta \tilde{e_1}$ with $\beta > 0$ and $(\tilde{e_1}, \tilde{e_3})$ is a linearly independent family,

c. $\tilde{e_1} \neq 0$, $\tilde{e_2} = \beta \tilde{e_1}$ and $\tilde{e_3} = \gamma \tilde{e_1}$ with $\gamma^2 < \beta$.

If the point *b.* holds, then $\mathrm{Span}(e_1) \oplus \mathrm{Span}(e_3) \oplus N$. In this situation, one can define $\mathbb{P}$ as

$$\mathbb{P}(e_1) = 1, \qquad \mathbb{P}(e_3) = 0, \qquad \mathbb{P}_{|N} = 0.$$

Then, $\mathbb{P}$ satisfies (BC). This is a contradiction. If the point *c.* holds, then $\mathrm{Span}(e_1) \oplus N$. Thus, one can define $\mathbb{P}$ as

$$\mathbb{P}(e_1) = 1, \qquad \mathbb{P}_{|N} = 0.$$

Then, $\mathbb{P}(e_3)^2 - \mathbb{P}(e_1)\mathbb{P}(e_2) = \gamma^2 - \beta < 0$ and $\mathbb{P}$ satisfies (BC), this is a contradiction. Necessarily, *a.* holds, i.e. $\mathbf{(\tilde{e_1}, \tilde{e_2})}$ **is a linearly independent family**. If $\dim\left(\mathrm{Span}(\tilde{e_1}, \tilde{e_2}, \tilde{e_3})\right) = 3$, then $\mathrm{Span}(e_1) \oplus \mathrm{Span}(e_2) \oplus \mathrm{Span}(e_3) \oplus N$. Thus, one can define $\mathbb{P}$ as

$$\mathbb{P}(e_1) = 1, \qquad \mathbb{P}(e_2) = 1, \qquad \mathbb{P}(e_3) = 0, \qquad \mathbb{P}_{|N} = 0.$$

Once again, $\mathbb{P}$ satisfies (BC), this is a contradiction. Consequently, **there exists** $\mathbf{a, b} \in \mathbb{R}$ **such that** $\mathbf{\tilde{e_3} = a\tilde{e_1} + b\tilde{e_2}}$. Finally, assume that $ab < \frac{1}{4}$.

1. If $a = 0$, then, one can define $\mathbb{P}$ as

   $$\mathbb{P}(e_1) = b^2 + 1, \qquad \mathbb{P}(e_2) = 1, \qquad \mathbb{P}_{|N} = 0.$$

   Then, $\mathbb{P}$ satisfies (BC).

2. Else, $Q := a^2x^2 + (2ab - 1)x + b^2$ verifies $\Delta = 1 - 4ab > 0$. Then $x^* := \frac{1-2ab}{2a^2} > 0$ satisfies $Q(x^*) < 0$. One can define $\mathbb{P}$ as

   $$\mathbb{P}(e_1) = x^*, \qquad \mathbb{P}(e_2) = 1, \qquad \mathbb{P}_{|N} = 0.$$

   Then, $\mathbb{P}$ satisfies (BC).

This is a contradiction. Consequently, $\mathbf{ab \geqslant \frac{1}{4}}$. This is the desired property.

Conversely, we reason by contraposition. Assume that (BC) holds, and let $\mathbb{P}$ be such a linear form.

1. If the point 1 is satisfied, $e_1 \in N$ and $\mathbb{P}(e_1) = 0$ or $e_2 \in N$ and $\mathbb{P}(e_2) = 0$.

2. If the point 2 holds, $a \neq 0$ and

$$\mathbb{P}(e_3)^2 - \mathbb{P}(e_1)\mathbb{P}(e_2) = a^2 \left(\mathbb{P}(e_1) + \left(\frac{b}{a} - \frac{1}{2a^2} \right) \mathbb{P}(e_2) \right)^2 + \frac{4ab-1}{4a^2} \mathbb{P}(e_2)^2 \geqslant 0.$$

3. If the point 3 is satisfied, $\mathbb{P}(e_1)\mathbb{P}(e_2) = \beta \mathbb{P}(e_1)^2 \leqslant 0$.

4. If the point 4 holds, $\mathbb{P}(e_3)^2 - \mathbb{P}(e_1)\mathbb{P}(e_2) = (\gamma^2 - \beta)\mathbb{P}(e_1)^2 \geqslant 0$.

All these points are in contradiction with (BC). □

A.2 A bracket expansion in the symmetrical case

Proof of Lemma 3.10. Let $l \in [\![0, k-1]\!]$, $(\alpha_{j,1})$, $(\alpha_{j,2}) \in \mathbb{R}^{l+1}$. To prove the desired relation, we compute in the quotient space $\mathcal{L}(X)/\{\mathrm{E}(b), b \in \mathrm{Br}(X),\ n(b) = 2,\ n_0(b) < 2k-1\}$. We note $\bar{a}$ the class of $a \in \mathcal{L}(X)$ in this quotient. Expending the bracket, one has

$$[B0^{k-l-1}, B0^{k-l}] = \sum_{\substack{i,i' \in \{1,2\} \\ j,j' \in [\![0,l]\!]}} \alpha_{j,i}\alpha_{j',i'} [M^i_{j+k-l-1}, M^{i'}_{j'+k-l}].$$

We note that, for all $(i, i') \in \{1,2\}^2$,

$$\forall j, j' \in [\![0, l]\!] \text{ such that } j + j' < 2l, \quad n_0\left(\left[M^i_{j+k-l-1}, M^{i'}_{j'+k-l} \right] \right) < 2k-1.$$

Using this remark,

$$\overline{[B0^{k-l-1}, B0^{k-l}]} = \sum_{i,i' \in \{1,2\}} \alpha_{l,i}\alpha_{l,i'} \overline{[M^i_{k-1}, M^{i'}_k]}.$$

Finally,

$$\overline{[B0^{k-l-1}, B0^{k-l}]} = \alpha^2_{l,1} \overline{W^1_k} + \alpha^2_{l,2} \overline{W^2_k} + \alpha_{l,1}\alpha_{l,2} \left(2\overline{C_{2k-1}} + \overline{C_{2k-2,1}} \right).$$

As $\mathcal{L}(X)$ is a graded Lie algebra, $\mathcal{B}_{2,2k-2} := \mathcal{B}_2 \cap \{\mathrm{E}(b);\ b \in \mathrm{Br}(X),\ n(b) = 2,\ n_0(b) < 2k-1\}$ generates all the elements $\mathrm{E}(b)$ with $n(b) = 2$ and $n_0(b) < 2k-1$. The elements of $\mathcal{B}_{2,2k-2}$ are in $\mathcal{N}^m_k$. Finally, as $C_{2k-2,1} \in \mathcal{N}^m_k$, the desired result follows. □

A.3 A bracket expansion in the asymmetrical case

The purpose of this subsection is to prove the expansion of Lemma 4.4. The proof of this lemma is quite different from the case $k = k'$ studied in Lemma 3.10, and is based on the following lemma.

Lemma A.1. *The following expansions hold.*

1. *For any $\nu \in \mathbb{N}$, and $a, b \in \mathcal{L}(X)$,*

$$[a, b0^\nu] = \sum_{\nu'=0}^{\nu} \binom{\nu}{\nu'} (-1)^{\nu'} [a0^{\nu'}, b] 0^{\nu-\nu'}. \tag{A.1}$$

2. *For any $\nu \in \mathbb{N}^*$, there exist coefficients $\alpha^\nu_r \in \mathbb{Z}$ for $1 \leqslant 2r+1 \leqslant \nu$ such that, for every $b \in \mathcal{L}(X)$,*

$$[b, b0^\nu] = \sum_{1 \leqslant 2r+1 \leqslant \nu} \alpha^\nu_r [b0^r, b0^{r+1}] 0^{\nu-2r-1}. \tag{A.2}$$

3. *For any $\nu \in \mathbb{N}$, there exist coefficients $\beta^\nu_r \in \mathbb{Z}$ for $0 \leqslant r \leqslant \nu$ such that, for every $p \in \mathbb{N}$,*

$$[M^1_p, M^2_p 0^\nu] = \sum_{r=0}^{\nu} \beta^\nu_r C_{2p+r,\nu-r}. \tag{A.3}$$

4. For any $\nu \in \mathbb{N}$, there exist coefficients $\gamma^\nu \in \mathbb{Z}$ for $0 \leqslant r \leqslant \nu$ such that, for every $p \in \mathbb{N}$,

$$[M_p^2, M_p^1 0^\nu] = \sum_{r=0}^{\nu} \gamma_r^\nu C_{2p+r,\nu-r}. \tag{A.4}$$

Moreover, $\gamma_\nu^\nu = (-1)^{1+\lfloor \frac{\nu+1}{2} \rfloor}$.

Proof. The first two points are proved in [5, Lemma 4.11]. We prove the last point by induction on ν (the proof of 3. is very similar): the equality is true for $\nu = 0$ with $\gamma_0^0 = -1$. The equality if true for $\nu = 1$ with $\gamma_0^1 = 0$ and $\gamma_1^1 = 1$. We assume that the formula holds for $\nu, \nu+1$, with $\nu \geqslant 0$. Then, for every $p \in \mathbb{N}$,

$$[M_p^2, M_p^1 0^{\nu+2}] = [M_p^2, M_p^1 0^{\nu+1}]0 - [M_{p+1}^2, M_{p+1}^1 0^\nu],$$

thanks to the Jacobi's equality (1.2). Using the induction hypothesis and a change of variable, we get

$$[M_p^2, M_p^1 0^{\nu+2}] = \sum_{r=0}^{\nu+1} \gamma_r^{\nu+1} C_{2p+r,(\nu+2)-r} - \sum_{r=2}^{\nu+2} \gamma_{r-2}^\nu C_{2p+r,(\nu+2)-r}.$$

Thus, $[M_p^2, M_p^1 0^{\nu+2}] = \sum_{r=0}^{\nu+2} \gamma_r^{\nu+2} C_{2p+r,\nu+2-r}$, with

$$\forall r \in [\![2, \nu+1]\!], \quad \gamma_r^{\nu+2} = \gamma_r^{\nu+1} - \gamma_{r-2}^\nu, \qquad \gamma_{\nu+2}^{\nu+2} = -\gamma_\nu^\nu, \qquad \gamma_0^{\nu+2} = \gamma_0^{\nu+1}, \qquad \gamma_1^{\nu+2} = \gamma_1^{\nu+1}.$$

We obtain the desired equality, as $\gamma_{\nu+2}^{\nu+2} = -\gamma_\nu^\nu = -(-1)^{1+\lfloor \frac{\nu+1}{2} \rfloor} = (-1)^{1+\lfloor \frac{(\nu+2)+1}{2} \rfloor}$. □

We are now in a position to prove Lemma 4.4.

Proof of Lemma 4.4. By definition,

$$\left[B0^{k'-l-1}, B0^{k'-l}\right] = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}), \tag{A.5}$$

with

$$(\mathrm{I}) = \sum_{i,j=0}^{l+k-k'} \alpha_i \alpha_j [M_{i+k'-l-1}^1, M_{j+k'-l}^1], \qquad (\mathrm{II}) = \sum_{i=0}^{l+k-k'} \sum_{j=0}^{l} \alpha_i \beta_j [M_{i+k'-l-1}^1, M_{j+k'-l}^2],$$
$$(\mathrm{III}) = \sum_{i=0}^{l} \sum_{j=0}^{l+k-k'} \beta_i \alpha_j [M_{i+k'-l-1}^2, M_{j+k'-l}^1], \qquad (\mathrm{IV}) = \sum_{i,j=0}^{l} \beta_i \beta_j [M_{i+k'-l-1}^2, M_{j+k'-l}^2].$$

Then,

$$(\mathrm{I}) = \sum_{i=k'-l}^{k} \alpha_{i+l-k'}^2 W_i^1 + \left(\sum_{i=2}^{l+k-k'} \sum_{j=0}^{i-2} + \sum_{i=0}^{l+k-k'-1} \sum_{j=i+1}^{l+k-k'} \right) \alpha_i \alpha_j [M_{i+k'-l-1}^1, M_{j+k'-l}^1], \tag{A.6}$$

as the bracket is zero if $j = i-1$. Moreover, the equation (A.2), applied with $b = M_{j+k'-l}^1$ and $\nu = i-j-1 \geqslant 1$ gives: for all $2 \leqslant i \leqslant l+k-k'$, $0 \leqslant j \leqslant i-2$,

$$[M_{i+k'-l-1}^1, M_{j+k'-l}^1] = - \sum_{1 \leqslant 2r+1 \leqslant i-j-1} \alpha_r^{i-j-1} W_{j+k'-l+r+1,i-j-2r-2}^1.$$

As $j + k' - l + r + 1 \leqslant k-1$, we obtain

$$\text{for all } 2 \leqslant i \leqslant l+k-k',\ 0 \leqslant j \leqslant i-2, \quad [M_{i+k'-l-1}^1, M_{j+k'-l}^1] \in \mathcal{N}_{k,k'}^{m,m'}. \tag{A.7}$$

Similarly, we obtain:

$$\text{for all } 0 \leqslant i \leqslant l+k-k'-1,\ i+1 \leqslant j \leqslant l+k-k', \quad [M^1_{i+k'-l-1}, M^1_{j+k'-l}] \in \mathcal{N}^{m,m'}_{k,k'}. \tag{A.8}$$

Thus, the equations (A.6), (A.7) and (A.8) give

$$(\mathrm{I}) - \alpha^2_{l+k-k'} W^1_k \in \mathcal{N}^{m,m'}_{k,k'}. \tag{A.9}$$

We can manipulate the term (IV) in the same way and obtain

$$(\mathrm{IV}) - \beta^2_l W^2_{k'} \in \mathcal{N}^{m,m'}_{k,k'}. \tag{A.10}$$

Finally, we need to examine the cross terms (II) and (III).

$$(\mathrm{II}) = \alpha_{l+k-k'}\beta_l [M^1_{k-1}, M^2_{k'}] + \sum_{\substack{(i,j)\in[\![0,l+k-k']\!]\times[\![0,l]\!] \\ (i,j)\neq(l+k-k',l)}} \alpha_i\beta_j [M^1_{i+k'-l-1}, M^2_{j+k'-l}]. \tag{A.11}$$

Assume temporarily that $k \neq k'$. The equation (A.4), applied with $p = k'$ and $\nu = k-k'-1 \geqslant 0$ gives

$$[M^1_{k-1}, M^2_{k'}] = (-1)^{\lfloor \frac{k-k'}{2} \rfloor} C_{k+k'-1} - \sum_{r=0}^{k-k'-2} \gamma_r^{k-k'-1} C_{2k'+r,k-k'-1-r}.$$

As, in the sum, $2k' + r \leqslant k + k' - 2$, one obtain

$$[M^1_{k-1}, M^2_{k'}] - (-1)^{\lfloor \frac{k-k'}{2} \rfloor} C_{k+k'-1} \in \mathcal{N}^{m,m'}_{k,k'}. \tag{A.12}$$

Using the Jacobi's formula, this equality is also true when $k = k'$. We expand on the basis the second term of the right-hand side (A.11). We split the space of subscripts as

$$[\![0, l+k-k']\!] \times [\![0,l]\!] \setminus \{(l+k-k', l)\} = A \sqcup B \sqcup C \sqcup D \sqcup E \sqcup F,$$

with

$$A = \{(i,j),\ 1 \leqslant i \leqslant l-1,\ 0 \leqslant j \leqslant i-1\} \quad B = \{(i,j),\ 0 \leqslant i \leqslant l-1,\ i \leqslant j \leqslant l-1\},$$

$$C = \{(i,j),\ l \leqslant i \leqslant l+k-k',\ 0 \leqslant j \leqslant l-1\} \quad D = [\![0, l-1]\!] \times \{l\}$$

$$E = [\![l+1, l+k-k'-1]\!] \times \{l\}, \quad F = \{(l,l)\}.$$

Note that the spaces E and F are empty if $k = k'$. For all $(i,j) \in A$, one can apply (A.4) with $p = j + k' - l$ and $\nu = i - j - 1$ to have

$$[M^1_{i+k'-l-1}, M^2_{j+k'-l}] = -\sum_{r=0}^{i-j-1} \gamma_r^{i-j-1} C_{2(j+k'-l)+r,i-j-1-r}.$$

As $2(j + k' - l) + r \leqslant k + k' - 4$, one has

$$\forall (i,j) \in A, \quad [M^1_{i+k'-l-1}, M^2_{j+k'-l}] \in \mathcal{N}^{m,m'}_{k,k'}.$$

We can do the same manipulations for $(i,j) \in B$, C, D, E, and F, thanks to the equations (A.3) and (A.4). Using, (A.12) in (A.11), one finally gets

$$(\mathrm{II}) - (-1)^{\lfloor \frac{k-k'}{2} \rfloor} \alpha_{l+k-k'}\beta_l C_{k+k'-1} \in \mathcal{N}^{m,m'}_{k,k'}. \tag{A.13}$$

With the same manipulations, one has

$$(\mathrm{III}) - (-1)^{\lfloor \frac{k-k'}{2} \rfloor} \alpha_{l+k-k'}\beta_l C_{k+k'-1} \in \mathcal{N}^{m,m'}_{k,k'}. \tag{A.14}$$

The equations (A.5), (A.9), (A.10), (A.13), and (A.14) lead to the conclusion. □

A.4 Relation between Sussmann's $\mathcal{S}(\theta)$-condition and Theorem 1.15

Let $k \in \mathbb{N}^*$, $m = 1$. We assume that

$$f_{W_k^1}(0), \quad f_{W_k^2}(0), \quad f_{C_{2k-1}}(0), \quad \mathcal{N}_k^1(f)(0) \quad \text{satisfy (BC)}.$$

Let us show that, for every $\theta \in [0,1]$, (1.22) is not verified for $\mathfrak{b} := W_k^1$. We assume the opposite: there exists $\theta \in [0,1]$ such that (1.22) holds for $\mathfrak{b} = W_k^1$. Then, $n_0(\mathfrak{b}) = 2k-1$ is odd, $n_1(\mathfrak{b}) = 2$, $n_2(\mathfrak{b}) = 0$ are even and $f_{\sigma(\mathfrak{b})}(0) = f_{W_k^1}(0) + f_{W_k^2}(0)$. Let $b \in \mathrm{Br}(X)$ be such $n(b) + \theta n_0(b) < n(\mathfrak{b}) + \theta n_0(\mathfrak{b}) = 2 + (2k-1)\theta$. Then,

$$n(b) < 2 + (2k-1)\theta \leqslant 2k+1.$$

Moreover, if $n(b) = 2$, then

$$2 + \theta n_0(b) < 2 + \theta(2k-1) \qquad \text{so} \qquad n_0(b) < 2k-1.$$

Let $\mathcal{E} := S_{[\![1,\pi(k,1)]\!]\setminus\{2\}}(X) \cup \{b \in \mathrm{Br}(X);\ n(b) = 2,\ n_0(b) < 2k-1\}$. As $\pi(k,1) = 2k+1$, the previous inequalities lead to $f_{\sigma(\mathfrak{b})}(0) \in \mathcal{E}(f)(0)$. As $\mathcal{E}(f)(0) \subset \mathcal{N}_k^1(f)(0)$, one obtains

$$\mathbb{P}\left(f_{W_k^1}(0)\right) + \mathbb{P}\left(f_{W_k^2}(0)\right) = 0,$$

where $\mathbb{P}$ is a linear form given by (BC). Thus,

$$0 \leqslant \mathbb{P}\left(f_{C_{2k-1}}(0)\right)^2 < \mathbb{P}\left(f_{W_k^1}(0)\right)\mathbb{P}\left(f_{W_k^2}(0)\right) = -\mathbb{P}\left(f_{W_k^1}(0)\right)^2 \leqslant 0.$$

This is a contradiction. Consequently, (1.22) is not verified for $\mathfrak{b} = W_k^1$.

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References

[1] Andrei Agrachev. Quadratic mappings in geometric control theory. *Journal of Mathematical Sciences*, 51(6):2667–2734, 1990.

[2] Andrei Agrachev and Revaz Gamkrelidze. *Local controllability and semigroups of diffeomorphisms*, volume 32. Acta Applicandae Mathematica, 1993.

[3] Karine Beauchard, Jérémy Le Borgne, and Frédéric Marbach. On expansions for nonlinear systems error estimates and convergence issues. *Comptes Rendus. Mathématique*, 361(G1):97–189, January 2023.

[4] Karine Beauchard and Frédéric Marbach. Quadratic obstructions to small-time local controllability for scalar-input differential systems, 2017.

[5] Karine Beauchard and Frédéric Marbach. A unified approach of obstructions to small-time local controllability for scalar-input systems, 2024.

[6] Roger Brockett. Controllability with quadratic drift. *Mathematical Control and Related Fields*, 3(4):433–446, 2013.

[7] W.L. Chow. Über systeme von linearen partiellen differentialgleichungen erster ordnung. *Mathematische Annalen*, 117:98–105, 1940/1941.

[8] Jean-Michel Coron. *Control and Nonlinearity.* Mathematical Surveys and Monographs, vol. 136, 2007.

[9] E. Gagliardo. Ulteriori proprieta di alcune classi di funzioni in piu variabili. *Ric. Mat.*, 8:24–51, 1959.

[10] Théo Gherdaoui. Small-time local controllability of the multi-input bilinear schrödinger equation thanks to a quadratic term. 2024.

[11] Giraldi, Laetitia, Lissy, Pierre, Moreau, Clément, and Pomet, Jean-Baptiste. Necessary conditions for local controllability of a particular class of systems with two scalar controls. *ESAIM: COCV*, 30:4, 2024.

[12] Matthias Kawski Henry Hermes. Local controllability of a single input, affine system. *Nonlinear Controllability and Optimal Control*, 109:235–248, 1987.

[13] Robert Hermann. On the accessibility problem in control theory. In Joseph P. LaSalle and Solomon Lefschetz, editors, *International Symposium on Nonlinear Differential Equations and Nonlinear Mechanics*, pages 325–332. Academic Press, 1963.

[14] Henry Hermes. Control systems which generate decomposable lie algebras. *Journal of Differential Equations*, 44(2):166–187, 1982.

[15] R.M. Hirschorn and A.D. Lewis. Geometric local controllability: second-order conditions. In *Proceedings of the 41st IEEE Conference on Decision and Control, 2002.*, volume 1, pages 368–369 vol.1, 2002.

[16] Matthias Kawski. Nilpotent lie algebras of vectorfields and local controllability of nonlinear systems. *ProQuest LLC, Ann Arbor, MI*, page 167–179, 1986.

[17] Matthias Kawski. High-order small-time local controllability. *Nonlinear Controllability and Optimal Control*, 133, 03 2000.

[18] Daniel Krob. Codes limites et factorisations finies du monoïde libre. *RAIRO - Theoretical Informatics and Applications - Informatique Théorique et Applications*, 21(4):437–467, 1987.

[19] Tadashi Nagano. Linear differential systems with singularities and an application to transitive Lie algebras. *Journal of the Mathematical Society of Japan*, 18(4):398 – 404, 1966.

[20] L. Nirenberg. On elliptic partial differential equations. *Annali della Scuola Normale Superiore di Pisa - Scienze Fisiche e Matematiche*, Ser. 3, 13(2):115–162, 1959.

[21] Petr K. Rashevski. About connecting two points of complete non-holonomic space by admissible curve. *Uch. Zapiski ped. inst. Libknexta*, 2:83–94, 1938.

[22] Gianna Stefani. On the local controllability of a scalar-input control system. *Theory and applications of nonlinear control systems (Stockholm, 1985)*, page 167–179, 1986.

[23] H. J. Sussmann. A general theorem on local controllability. *SIAM Journal on Control and Optimization*, 25(1):158–194, 1987.

[24] Héctor J. Sussmann. Lie brackets and local controllability: A sufficient condition for scalar-input systems. *SIAM Journal on Control and Optimization*, 21(5):686–713, 1983.