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► **To cite this version:**

Abraham Sylla, Mauro Garavello, Elena Rossi. Differential Games for a Mixed ODE-PDE System. 2024. hal-04844842

HAL Id: hal-04844842

<https://hal.science/hal-04844842v1>

Preprint submitted on 18 Dec 2024

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Differential Games for a Mixed ODE-PDE System

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Abstract

Motivated by a vaccination coverage problem, we consider here a zero-sum differential game governed by a differential system consisting of a hyperbolic partial differential equation (PDE) and an ordinary differential equation (ODE). Two players act through their respective controls to influence the evolution of the system with the aim of minimizing their objective functionals \mathcal{F}_1 and \mathcal{F}_2 , under the assumption that $\mathcal{F}_1 + \mathcal{F}_2 = 0$.

First we prove a well posedness and a stability result for the differential system, once the control functions are fixed. Then we introduce the concept of non-anticipating strategies for both players and we consider the associated value functions, which solve two infinite-dimensional Hamilton-Jacobi-Isaacs equations in the viscosity sense.

Key Words: Differential games; value functions; infinite-dimensional Hamilton-Jacobi-Isaacs equation; zero-sum game; vaccination coverage

AMS Subject Classifications: 35Q91, 91A23, 91A80, 35L65

1 Introduction

We investigate a two-person zero-sum differential game governed by a system consisting of a hyperbolic partial differential equation (PDE) and an ordinary differential equation (ODE). In this game, two players use their respective controls to influence the system's evolution, aiming to optimize their objective functionals, \mathcal{F}_1 and \mathcal{F}_2 . Notably, we assume $\mathcal{F}_1 + \mathcal{F}_2 = 0$ in this *zero-sum* game, implying that one player's gain is the other's loss.

Here we focus on a specific system, mainly motivated by epidemiology, which aims to consider the dispute about the vaccination policies between the scientific community and groups of people (no-vax) against these practices. More precisely, inspired by the classical SIR model and by the paper [16] dealing with optimal vaccination policies, we introduce a simplified model incorporating vaccination. Let $S = S(t, \xi)$ denote the density at time t of susceptible individuals, where $\xi \in [0, 1]$ represents the leaning towards vaccination: namely, a value of $\xi = 0$ indicates no vaccination, while $\xi = 1$ signifies guaranteed vaccination. Additionally, $I = I(t)$ denotes the density of infected individuals at time t . The evolution for the S and I populations is described by the following coupled ODE-PDE system

$$\begin{cases} \partial_t S + \partial_\xi (g(\xi, u_1, u_2) S) &= -(f(\xi) + \alpha(I))S, & t > 0, \xi \in (0, 1), \\ \dot{I} &= -\beta I + \alpha(I) \int_0^1 S(t, \xi) d\xi, & t > 0, \end{cases} \quad (1.1)$$

where $\alpha : [0, \infty) \rightarrow [0, \infty)$ is the infection rate function, $\beta > 0$ is a coefficient taking into account both the death and the recovery rate of the infected individuals, $f = f(\xi)$ is the vaccination rate

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for individuals based on their vaccination leaning ξ , the flux function $g = g(\xi, u_1, u_2)$ describes the change of individuals' leaning towards vaccinations, and $u_1 \in [0, M_1]$ and $u_2 \in [0, M_2]$ are the control variables, respectively for the first and for the second player. Here the first player can represent the government, which, supported by the scientific community, implements policies to increase the number of vaccinated people, while the second player is made up of no-vax groups with the aim of disadvantaging such policies, for example through the use of social media [17]. Game theory for similar topics was already considered in the literature; see for example [8, 25, 27]. As pointed out in [1], the influence of anti-vaccine groups is one of several factors which affects the herd immunity and has a non trivial economical and healthy cost for the society; see [19] for the case of the pertussis control.

We consider the Cauchy problem for system (1.1) without explicitly prescribing any boundary conditions, although the PDE is defined on a bounded domain. This is because the chosen flux function g inherently satisfies zero-flux boundary conditions. First we prove the well posedness of the Cauchy problem (1.1), both in \mathbf{L}^1 and in \mathbf{L}^2 , for any given choice of the control functions. Then we pass to a game perspective and we construct the value functions for both players, using non anticipating strategies, which, roughly speaking, are strategies based on the past and present status, but not on the future one. The main result consists in the proof that each value function satisfies a suitable Hamilton-Jacobi-Bellman (HJB) infinite-dimensional equation in the viscosity sense. The key tool here is a *dynamical programming principle*, satisfied by each value function. The Hamiltonian function is strictly related to the choice of a duality, which, in the present infinite-dimensional setting, is not straightforward. Indeed, the natural space for the unknown S in (1.1) is \mathbf{L}^1 in the variable ξ , as usual for hyperbolic conservation laws. However, the Banach space \mathbf{L}^1 lacks reflexivity, since the bi-dual space strictly contains \mathbf{L}^1 . Hence we use here the \mathbf{L}^2 duality. This has the consequence that we need to prove the well posedness of the PDE equation in (1.1) in the \mathbf{H}^1 setting, since, otherwise, the term $\partial_\xi (g(\xi, u_1, u_2) S)$ is not well defined in \mathbf{L}^2 .

As intermediate results we also prove that the solution to the PDE is stable with respect to the flux and that, once the control for one player is fixed, there exists an optimal control for the other player. The proof of the existence of optimal controls is based on the classical method of the calculus of variations. We remark that no total variation estimates are needed for the stability result and that no Lipschitz continuity hypotheses on the flux function with respect to the control variable are needed in the proof of existence of optimal controls.

A natural question for a two-person zero-sum differential game is whether the value functions coincide, briefly said as the game has a value. Under the so called Isaacs condition [4, Equation (2.2)], it is possible to prove that the value functions coincide, provided that the uniqueness property for viscosity solutions of a HJB equation holds; see [12]. Unfortunately, this is not the case in the infinite-dimensional setting, see for example [12, 13]. As a consequence, the question whether the game has a value is still open in general in this context.

The works of Isaacs [22, 23] and of Pontryagin [26] initiated the theory of two-person zero-sum differential games governed by ordinary differential equations; see also the monographs [18, 23] for a complete introduction. The original motivation was the study of military problems and the well known pursuit-evasion game [20] represents the most simple example of a zero-sum differential game in this subject. The definitions of strategies and of the value were introduced by Varaiya in [30], see also [14, 28]. The paper [15] contains the first proof that the value function satisfies a Hamilton-Jacobi-Isaacs equation in viscosity sense. In the context of infinite dimensional Isaacs equation, the literature contains very few references. We cite here the recent paper [3], where a zero-sum differential game between a single player and a mass of agents is considered; see also [9].

The paper is structured as follows. In Section 2 we give the basic definitions and hypotheses and we state the analytic results. More precisely, we introduce the notions of non-anticipating strategies and of value functions, we state the well posedness and stability for system (1.1). We also provide results about existence of optimal controls in the case of a single player, and about the fact that the value functions satisfy a dynamical programming principle and consequently a Hamilton-Jacobi-Isaacs equation in viscosity sense. Finally, Section 3 contains the proofs of these results.

2 General framework

This section is devoted to present both the general framework of the problem, in Section 2.1 and the analytical results, in Section 2.2. Throughout, the notation $\mathbb{R}^+ = [0, \infty)$ is used.

2.1 Basic definitions and assumptions

Here the positive constants M_1 and M_2 describe the maximum strength for the two controls u_1 and u_2 . Therefore the sets

$$\mathcal{U}_1 := \mathbf{L}^\infty((0, \infty); [0, M_1]) \quad \text{and} \quad \mathcal{U}_2 := \mathbf{L}^\infty((0, \infty); [0, M_2]) \quad (2.1)$$

denote the admissible controls for the first and the second player, respectively. The first equation in (1.1) is a hyperbolic partial differential equation, while the second one is an ordinary differential equation.

System (1.1) is supplemented with the initial conditions

$$\begin{cases} S(0, \xi) = S_o(\xi), & \xi \in (0, 1), \\ I(0) = I_o, \end{cases} \quad (2.2)$$

where $S_o : [0, 1] \rightarrow \mathbb{R}^+$ and $I_o \in \mathbb{R}^+$ are given.

For the functions α , f , and g in (1.1) we introduce the following assumptions.

(**α**) $\alpha \in \mathbf{Lip}(\mathbb{R}^+; \mathbb{R}^+)$ satisfies $\alpha(0) = 0$.

(**f**) $f \in \mathbf{Lip}([0, 1]; [0, 1])$ is non decreasing and satisfies $f(0) = 0$, $f(1) = 1$.

(**g**) $g \in \mathbf{C}^0([0, 1] \times [0, M_1] \times [0, M_2]; \mathbb{R})$ satisfies:

- (a) for all $(u, v) \in [0, M_1] \times [0, M_2]$ the map $\xi \mapsto g(\xi, u, v)$ belongs to $\mathbf{C}^2([0, 1]; \mathbb{R})$;
- (b) for every $(u, v) \in [0, M_1] \times [0, M_2]$, $g(0, u, v) = g(1, u, v) = 0$;
- (c) there exists $\gamma > 0$ such that $|\partial_\xi g(\xi, u, v)| \leq \gamma$ for every $\xi \in [0, 1]$, $u \in [0, M_1]$, and $v \in [0, M_2]$.

(**$G-1$**) g satisfies (**g**) and $g(\xi, u, v) = g_1(\xi) + g_2(\xi)u + g_3(\xi)v$ for some $g_1, g_2, g_3 \in \mathbf{C}_c^2([0, 1]; \mathbb{R})$.

Remark 2.1. The infection rate function α , also known as the force of infection, models the rate susceptible individuals become infected. In classical SIR-type models, the simple linear law

$$\alpha(I) = \bar{\alpha}I, \quad (2.3)$$

with $\bar{\alpha} > 0$, is considered. As pointed out in [31], a complex phenomenon like the process of infection hardly can be represented by a simple law as (2.3). The first nonlinear models, of type $\bar{\alpha}I^p$ with $p > 1$, for the rate of infections were proposed in [6, 29, 31].

More realistic rate functions, also called Holling type functional responses, of the type

$$\alpha(I) = \frac{\bar{\alpha}I^p}{1 + \beta I^q}, \quad (2.4)$$

with $p, q > 0$ and $\beta > 0$, were proposed in [7, 21]; see Figure 1. For more details see also [2].

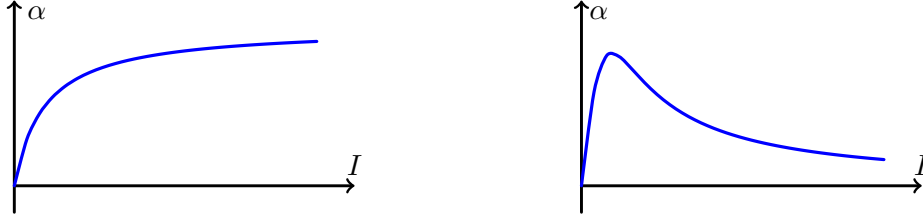


Figure 1: Left: the graph of the infection rate function (2.4) with $p = q = 1$. Right: the graph of the infection rate function (2.4) with $p = 1$ and $q = 2$.

Remark 2.2. Note that g depends on the tendency variable ξ and also on the two controls u_1 and u_2 , which describe the efforts of two competitive players aiming at guiding the choices of individuals regarding the vaccination. As an example, one player can represent the national health system, which is worried about the possible spread of an epidemic, while anti-vaccination activist groups can be identified with the other player. A possible choice of the flux function g is given by a regularization of the map

$$(\xi, u_1, u_2) \mapsto \xi(1 - \xi) \left(\xi - \frac{1}{2} - \lambda \frac{u_2}{M_2} \mathbf{1}_{A_2}(\xi) + (1 - \lambda) \frac{u_1}{M_1} \mathbf{1}_{A_1}(\xi) \right), \quad (2.5)$$

where the controls $u_1 \in [0, M_1]$ and $u_2 \in [0, M_2]$ act respectively on the sets $A_1 \subseteq [0, 1]$ and $A_2 \subseteq [0, 1]$; see Figure 2. In formula (2.5), $\mathbf{1}_{A_1}$ and $\mathbf{1}_{A_2}$ denote the characteristic functions of the sets A_1 and A_2 respectively. The coefficient $\lambda \in]0, 1[$ acts as a weight between the two controls. Clearly, formula (2.5) represents in general a discontinuous function of ξ . In order to avoid difficulties arising in conservation laws with discontinuous flux, we prefer to consider the flux function g in (1.1) as a regularization of (2.5).

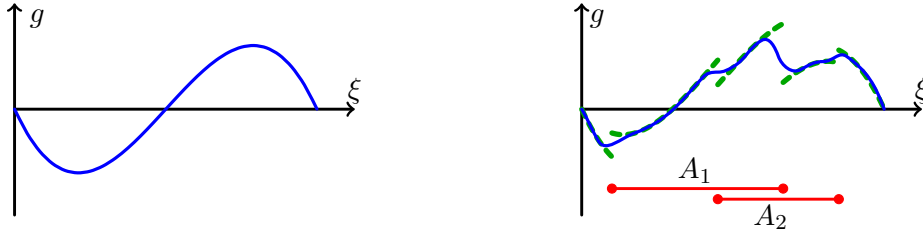


Figure 2: In the left picture the plot of the function in (2.5) with controls $u_1 = u_2 = 0$. In this case no regularization is needed.

In the right picture the solid-line plot represents a possible regularization of the map in (2.5), represented with a dashed line, with controls $u_1 = M_1$ and $u_2 = M_2$ acting on the intervals $A_1 = [0.1, 0.666]$ and $A_2 = [0.45, 0.85]$.

2.2 Analytical Results

First we introduce the definition of solution for system (1.1)–(2.2) and we state an existence and uniqueness result for such system.

Definition 2.1. Fix $T > 0$, $u_1 \in \mathbf{L}^\infty((0, T); [0, M_1])$, and $u_2 \in \mathbf{L}^\infty((0, T); [0, M_2])$. We say that the couple (S, I) is a solution to (1.1) on $[0, T]$ with initial condition $(S_o, I_o) \in \mathbf{L}^2((0, 1); \mathbb{R}) \times \mathbb{R}$ if the following conditions are satisfied.

1. $S \in \mathbf{C}^0([0, T]; \mathbf{L}^2((0, 1); \mathbb{R}^+))$ and $I \in \mathbf{C}^0([0, T]; \mathbb{R})$.
2. For all test functions $\phi \in \mathbf{C}_c^\infty((-\infty, T) \times (0, 1); \mathbb{R})$, it holds

$$\int_0^{+\infty} \int_0^1 S(\partial_t \phi + g(\xi, u_1(t), u_2(t)) \partial_\xi \phi) - (f + \alpha(I)) \phi \, d\xi \, dt + \int_0^1 S_o(\xi) \phi(0, \xi) \, d\xi = 0. \quad (2.6)$$

3. For every $t \in [0, T]$

$$I(t) = I_o - \int_0^t \beta I(\tau) d\tau + \int_0^t \alpha(I(\tau)) \left(\int_0^1 S(\tau, \xi) d\xi \right) d\tau. \quad (2.7)$$

Definition 2.2. Fix $u_1 \in \mathcal{U}_1$ and $u_2 \in \mathcal{U}_2$. We say that the couple (S, I) is a solution to (1.1) on \mathbb{R}^+ with initial condition $(S_o, I_o) \in \mathbf{L}^2((0, 1); \mathbb{R}^+) \times \mathbb{R}^+$ if it is a solution to (1.1) on $[0, T[$ for every $T > 0$.

Remark 2.3. With similar reasoning as in [24, Theorem 3.2], if (S, I) is a solution to (1.1) with initial condition $(S_o, I_o) \in \mathbf{L}^2((0, 1); \mathbb{R}^+) \times \mathbb{R}^+$ according to Theorem 2.2, then

$$S(t, \xi) = S_o(X(0; t, \xi)) \exp \left(- \int_0^t [f(X(s; t, \xi)) + \alpha(I(s)) + \partial_\xi g(X(s; t, \xi), u_1(s), u_2(s))] ds \right), \quad (2.8)$$

where $X(s; t, \xi)$ solves the Cauchy problem

$$\begin{cases} \frac{d}{ds} X(s; t, \xi) = g(X(s; t, \xi), u_1(s), u_2(s)), \\ X(t; t, \xi) = \xi. \end{cases} \quad (2.9)$$

The next result deals with existence and uniqueness of solution to (1.1) together with growth estimates.

Theorem 2.3. Assume (α) -(F)-(G). Fix $u_1 \in \mathcal{U}_1$ and $u_2 \in \mathcal{U}_2$. Then for all $S_o \in \mathbf{L}^2((0, 1); \mathbb{R}^+)$ and $I_o \in \mathbb{R}^+$, there exists a unique solution to (1.1) on \mathbb{R}^+ with initial condition (S_o, I_o) in the sense of Theorem 2.2. Moreover, for all $t > 0$,

$$\|S(t)\|_{\mathbf{L}^1((0,1);\mathbb{R})} \leq \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})}, \quad \|S(t)\|_{\mathbf{L}^2((0,1);\mathbb{R})} \leq e^{t\gamma/2} \|S_o\|_{\mathbf{L}^2((0,1);\mathbb{R})} \quad (2.10)$$

and

$$I(t) \leq \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} + I_o. \quad (2.11)$$

If moreover $S_o \in \mathbf{H}^1((0, 1); \mathbb{R}^+)$, then for all $t > 0$, $S(t) \in \mathbf{H}^1((0, 1); \mathbb{R}^+)$.

We state now the well posedness result for (1.1).

Theorem 2.4. Assume (α) -(F)-(G). Fix $u_1 \in \mathcal{U}_1$, $u_2 \in \mathcal{U}_2$, $S_o, \bar{S}_o \in \mathbf{L}^2((0, 1); \mathbb{R}^+)$, and $I_o, \bar{I}_o \in \mathbb{R}^+$. Denote by (S, I) , resp. (\bar{S}, \bar{I}) , the global solution to (1.1) with initial condition (S_o, I_o) , resp. (\bar{S}_o, \bar{I}_o) . Then for all $t > 0$,

$$\begin{aligned} & \|S(t) - \bar{S}(t)\|_{\mathbf{L}^1((0,1);\mathbb{R})} + |I(t) - \bar{I}(t)| \\ & \leq \left[\|S_o - \bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} + e^{K_1 t} |I_o - \bar{I}_o| \right] (1 + te^{K_2 t} \mathcal{K}_t \exp(te^{K_2 t} \mathcal{K}_t)), \end{aligned} \quad (2.12)$$

where

$$K_1 := \beta + \mathbf{Lip}(\alpha) \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})}, \quad K_2 := \mathbf{Lip}(\alpha) \|\bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \quad (2.13)$$

depend on β , on $\mathbf{Lip}(\alpha)$, and on the \mathbf{L}^1 -norm of S_o and \bar{S}_o respectively, while

$$\mathcal{K}_t := \mathbf{Lip}(\alpha) \left(\|\bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} + e^{K_1 t} |\bar{I}_o| \right) = K_2 + \mathbf{Lip}(\alpha) |\bar{I}_o| e^{K_1 t} \quad (2.14)$$

depends on K_1 , on $\mathbf{Lip}(\alpha)$, and on \bar{I}_o .

The proofs of Theorem 2.3 and of Theorem 2.4 are deferred to Section 3.1. They are based on the separate study of the two equations in (1.1) and an application of Banach Fixed Point Theorem.

Remark 2.4. When dealing with the unique solution to (1.1) with initial condition (S_o, I_o) and controls u_1, u_2 , it may be relevant to emphasize the dependence on those values. As far as it concerns the controls, we write

$$S(t) = S_{u_1, u_2}(t) \qquad I(t) = I_{u_1, u_2}(t) \qquad (2.15)$$

and we omit to specify the control functions u_1, u_2 when they are clear from the context.

Remark 2.5. Theorem 2.4 provides only estimates about stability with respect to the initial datum. With similar techniques, it is possible to consider also stability estimates with respect to the flux function g and therefore to the controls. However it would require a bound on the total variation of the solution to the PDE; see [10] for similar results in this direction.

We now pass to optimal control problems for (1.1). To this aim, for positive numbers κ, θ , we introduce the cost functional $\mathcal{F} : \mathbf{L}^2((0, 1); \mathbb{R}^+) \times \mathbb{R}^+ \times \mathcal{U}_1 \times \mathcal{U}_2 \rightarrow \mathbb{R}$, defined as

$$\begin{aligned} \mathcal{F}(S_o, I_o, u_1, u_2) := & \kappa \int_0^{+\infty} e^{-\theta t} (I(t) + u_1(t)) dt \\ & - \int_0^{+\infty} e^{-\theta t} \left[u_2(t) + \int_0^1 g(\xi, u_1(t), u_2(t)) d\xi \right] dt, \end{aligned} \qquad (2.16)$$

where (S, I) solves (1.1) with initial datum (S_o, I_o) and controls u_1 and u_2 . The first term in (2.16) is proportional to the number of infected individuals and to the efforts of the first player, while the second term is proportional to the efforts of the second player and to the area of the region where g is positive. Roughly speaking, this area is related to the change in the leaning towards vaccination. So the aim of the first, respectively of the second, player is to minimize, respectively maximize, the functional \mathcal{F} in (2.16). The first result deals with the existence of optimal controls.

Theorem 2.5. *Assume (α) -**(F)**-**(G)** and fix $(S_o, I_o) \in \mathbf{L}^2((0, 1); \mathbb{R}^+) \times \mathbb{R}^+$. Then:*

- (i) *for all $u_2 \in \mathcal{U}_2$, the map $u_1 \mapsto \mathcal{F}(S_o, I_o, u_1, u_2)$ admits a minimizer in \mathcal{U}_1 ;*
- (ii) *for all $u_1 \in \mathcal{U}_1$, the map $u_2 \mapsto \mathcal{F}(S_o, I_o, u_1, u_2)$ admits a maximizer in \mathcal{U}_2 .*

The proof is deferred to Section 3.2.

Following [4, Chapter VIII], we give the definitions of non-anticipating strategies for both players.

Definition 2.6. A strategy for the first player is a map $\mathcal{S}_1 : \mathcal{U}_2 \rightarrow \mathcal{U}_1$. Moreover, the strategy \mathcal{S}_1 is said to be non-anticipating if, for all $t > 0$ and for all $u_2, \bar{u}_2 \in \mathcal{U}_2$, such that $u_2(s) = \bar{u}_2(s)$ for a.e. $s \in [0, t]$, then

$$\mathcal{S}_1(u_2)(s) = \mathcal{S}_1(\bar{u}_2)(s)$$

for a.e. $s \in [0, t]$. With the symbol Γ we denote the set of all the non-anticipating strategies for the first player.

Definition 2.7. A strategy for the second player is a map $\mathcal{S}_2 : \mathcal{U}_1 \rightarrow \mathcal{U}_2$. Moreover, the strategy \mathcal{S}_2 is said to be non-anticipating if, for all $t > 0$ and for all $u_1, \bar{u}_1 \in \mathcal{U}_1$, such that $u_1(s) = \bar{u}_1(s)$ for a.e. $s \in [0, t]$, then

$$\mathcal{S}_2(u_1)(s) = \mathcal{S}_2(\bar{u}_1)(s)$$

for a.e. $s \in [0, t]$. With the symbol Δ we denote the set of all the non-anticipating strategies for the second player.

For readers' convenience, define

$$\ell(I, u_1, u_2) := \kappa(I + u_1) - u_2 - \int_0^1 g(\xi, u_1, u_2) d\xi, \quad (2.17)$$

so that the cost functional \mathcal{F} , defined in (2.16), can be compactly written as

$$\mathcal{F}(S_o, I_o, u_1, u_2) = \int_0^{+\infty} e^{-\theta t} \ell(I(t), u_1(t), u_2(t)) dt. \quad (2.18)$$

Now we introduce the lower and upper values for the game.

Definition 2.8. Let \mathcal{F} be defined by (2.16). The lower value V and upper value U of the game are respectively

$$\begin{aligned} V(S_o, I_o) &:= \inf_{\mathcal{S}_1 \in \Gamma} \sup_{u_2 \in \mathcal{U}_2} \mathcal{F}(S_o, I_o, \mathcal{S}_1(u_2), u_2), \\ U(S_o, I_o) &:= \sup_{\mathcal{S}_2 \in \Delta} \inf_{u_1 \in \mathcal{U}_1} \mathcal{F}(S_o, I_o, u_1, \mathcal{S}_2(u_1)). \end{aligned} \quad (2.19)$$

The lower and upper values satisfy a dynamic programming principle.

Theorem 2.9. Assume (α) -**(F)**-**(G)**. Let ℓ be as in (2.18). For all $(S_o, I_o) \in \mathbf{L}^2((0, 1); \mathbb{R}) \times \mathbb{R}^+$ and for all $T > 0$, we have

$$V(S_o, I_o) = \inf_{\mathcal{S}_1 \in \Gamma} \sup_{u_2 \in \mathcal{U}_2} \left\{ \int_0^T e^{-\theta t} \ell(I_{\mathcal{S}_1, u_2}(t), \mathcal{S}_1(u_2)(t), u_2(t)) dt + V(S_{\mathcal{S}_1, u_2}(T), I_{\mathcal{S}_1, u_2}(T)) e^{-\theta T} \right\}, \quad (2.20)$$

where, with the notation (2.15), (S, I) is the solution to (1.1) with controls $\mathcal{S}_1(u_2), u_2$, and

$$U(S_o, I_o) = \sup_{\mathcal{S}_2 \in \Delta} \inf_{u_1 \in \mathcal{U}_1} \left\{ \int_0^T e^{-\theta t} \ell(I_{u_1, \mathcal{S}_2}(t), u_1(t), \mathcal{S}_2(u_1)(t)) dt + U(S_{u_1, \mathcal{S}_2}(T), I_{u_1, \mathcal{S}_2}(T)) e^{-\theta T} \right\}, \quad (2.21)$$

where, with the notation (2.15), (S, I) is the solution to (1.1) with controls $u_1, \mathcal{S}_2(u_1)$.

The proof is deferred to Section 3.3. Above, we make a slightly abuse of notation writing only \mathcal{S}_1 or \mathcal{S}_2 instead of $\mathcal{S}_1(u_2)$ or $\mathcal{S}_2(u_1)$.

Taking advantage of Theorem 2.9, we prove that V and U both solve an Hamilton-Jacobi equation. Define, for all $\mathbf{Y} = (S, I) \in \mathbf{H}^1((0, 1); \mathbb{R}) \times \mathbb{R}$ and $\mathbf{P} = (p, q) \in \mathbf{L}^2((0, 1); \mathbb{R}) \times \mathbb{R}$, the pre-Hamiltonian

$$\mathcal{H}(\mathbf{Y}, \mathbf{P}, u_1, u_2) := \langle \partial_\xi(gS) + (f + \alpha(I))S, p \rangle_{\mathbf{L}^2} + \left(\beta I - \alpha(I) \int_0^1 S(\xi) d\xi \right) q - \ell(I, u_1, u_2) \quad (2.22)$$

and the Hamiltonians

$$\underline{\mathcal{H}}(\mathbf{Y}, \mathbf{P}) := \min_{u_2 \in [0, M_2]} \max_{u_1 \in [0, M_1]} \mathcal{H}(\mathbf{Y}, \mathbf{P}, u_1, u_2) \quad (2.23)$$

and

$$\overline{\mathcal{H}}(\mathbf{Y}, \mathbf{P}) := \max_{u_1 \in [0, M_1]} \min_{u_2 \in [0, M_2]} \mathcal{H}(\mathbf{Y}, \mathbf{P}, u_1, u_2). \quad (2.24)$$

Remark 2.6. Note that in the definition of the Hamiltonian functions (2.23) and (2.24) we decided to consider $S \in \mathbf{H}^1$ and the dual variable $p \in \mathbf{L}^2$. Indeed, there is not any natural duality working in our setting, because if $S \in \mathbf{L}^2$ then the derivative $\partial_\xi(gS)$ is not well defined in \mathbf{L}^2 . Similarly, choosing the scalar product of \mathbf{H}^1 , and thus $p \in \mathbf{H}^1$, would require higher regularity for S to ensure $\partial_\xi(gS) \in \mathbf{H}^1$.

We give now the definition of viscosity solution for the Hamilton-Jacobi equation $\theta U + \underline{\mathcal{H}}(\mathbf{Y}, \nabla U) = 0$. The definition with the Hamiltonian $\overline{\mathcal{H}}$, instead of $\underline{\mathcal{H}}$, is completely identical.

Definition 2.10. Fix $\theta > 0$ and let $\underline{\mathcal{H}}$ be defined by (2.23). Consider the equation

$$\theta U + \underline{\mathcal{H}}(\mathbf{Y}, \nabla U) = 0, \quad \mathbf{Y} \in \mathbf{H}^1((0, 1); \mathbb{R}) \times \mathbb{R}. \quad (2.25)$$

We say that $U \in \mathbf{C}^0(\mathbf{H}^1((0, 1); \mathbb{R}) \times \mathbb{R}; \mathbb{R})$ is a viscosity sub-solution of (2.25) if, for every $\phi \in \mathbf{C}^1(\mathbf{L}^2((0, 1); \mathbb{R}) \times \mathbb{R}; \mathbb{R})$,

$$\mathbf{Y}_o \in \mathbf{H}^1((0, 1); \mathbb{R}) \times \mathbb{R} \text{ is a local maximum of } U - \phi \implies \theta U(\mathbf{Y}_o) + \underline{\mathcal{H}}(\mathbf{Y}_o, \nabla \phi(\mathbf{Y}_o)) \leq 0.$$

Similarly, $U \in \mathbf{C}^0(\mathbf{H}^1((0, 1); \mathbb{R}) \times \mathbb{R}; \mathbb{R})$ is a viscosity super-solution of (2.25) if, for every $\phi \in \mathbf{C}^1(\mathbf{L}^2((0, 1); \mathbb{R}) \times \mathbb{R}; \mathbb{R})$,

$$\mathbf{Y}_o \in \mathbf{H}^1((0, 1); \mathbb{R}) \times \mathbb{R} \text{ is a local minimum of } U - \phi \implies \theta U(\mathbf{Y}_o) + \underline{\mathcal{H}}(\mathbf{Y}_o, \nabla \phi(\mathbf{Y}_o)) \geq 0.$$

Finally, $U \in \mathbf{C}^0(\mathbf{H}^1((0, 1); \mathbb{R}) \times \mathbb{R}; \mathbb{R})$ is viscosity solution of (2.25) if it is both a viscosity sub-solution and a viscosity super-solution.

The value functions U and V , defined in (2.19), are viscosity solutions to some Hamilton-Jacobi equations.

Theorem 2.11. Assume (α) -(F)-(G). Let $\underline{\mathcal{H}}$ and $\overline{\mathcal{H}}$ be defined by (2.23) and by (2.24). Then

(i) V , defined by (2.19), is a viscosity solution to (2.25) with Hamiltonian $\underline{\mathcal{H}}$.

(ii) U , defined by (2.19), is a viscosity solution to (2.25) with Hamiltonian $\overline{\mathcal{H}}$.

3 Proofs

This section contains the proofs of the analytic results presented in Section 2.2.

3.1 Proof of Theorem 2.3

The proof of Theorem 2.3 consists in studying the two equations of (1.1) separately and then applying a Fixed Point Theorem.

3.1.1 The Controlled Partial Differential Equation

Here, let us consider the Cauchy Problem for the partial differential equation:

$$\begin{cases} \partial_t S + \partial_\xi(g(\xi, u_1(t), u_2(t))S) = -(f + \alpha(I))S, & t > 0, \xi \in (0, 1), \\ S(0, \xi) = S_o(\xi), & \xi \in (0, 1), \end{cases} \quad (3.1)$$

where $S_o \in \mathbf{L}^2((0, 1); \mathbb{R})$, α satisfies (α) , f satisfies (F), $I \in \mathbf{C}^0(\mathbb{R}^+; \mathbb{R}^+)$, and u_1 and u_2 are control functions. Let us introduce the definition of solution for (3.1).

Definition 3.1. We say that the function S is a solution to (3.1) with initial condition $S_o \in \mathbf{L}^2((0, 1); \mathbb{R})$ if the following conditions are satisfied.

1. $S \in \mathbf{C}^0(\mathbb{R}^+; \mathbf{L}^2((0, 1); \mathbb{R}))$.
2. For all test functions $\phi \in \mathbf{C}_c^\infty(\mathbb{R} \times (0, 1); \mathbb{R})$, it holds

$$\int_0^{+\infty} \int_0^1 S(\partial_t \phi + g(\xi, u_1(t), u_2(t))\partial_\xi \phi) - (f + \alpha(I))\phi \, d\xi \, dt + \int_0^1 S_o(\xi)\phi(0, \xi) \, d\xi = 0. \quad (3.2)$$

Here we state the well posedness for system (3.1).

Lemma 3.2. Assume (α) , (\mathbf{F}) and (\mathbf{G}) . Fix $I \in \mathbf{C}^0(\mathbb{R}^+; \mathbb{R}^+)$, and the control functions $u_1 \in \mathcal{U}_1$ and $u_2 \in \mathcal{U}_2$. Then, for all $S_o \in \mathbf{L}^2((0, 1); \mathbb{R})$, there exists a unique solution S to (3.1), in the sense of Theorem 3.1. Moreover:

(i) if $S_o \geq 0$, then for all $t > 0$, $S(t) \geq 0$;

(ii) for all $t > 0$, we have that

$$\|S(t)\|_{\mathbf{L}^1((0,1);\mathbb{R})} \leq \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})}, \quad (3.3)$$

and

$$\|S(t)\|_{\mathbf{L}^2((0,1);\mathbb{R})} \leq e^{t\gamma/2} \|S_o\|_{\mathbf{L}^2((0,1);\mathbb{R})}, \quad (3.4)$$

where γ is defined in (\mathbf{G}) .

Proof. For readers' convenience, set the flux function

$$G(\xi, t) := g(\xi, u_1(t), u_2(t)). \quad (3.5)$$

We split the proof into several parts.

Existence of \mathbf{L}^∞ solutions. Let $(S_o^{(n)})_n$ be a sequence of functions in $\mathbf{C}_c^\infty((0, 1); \mathbb{R})$ such that

$$\lim_{n \rightarrow +\infty} \left\| S_o^{(n)} - S_o \right\|_{\mathbf{L}^2((0,1);\mathbb{R})} = 0.$$

For all $n \in \mathbb{N}$, the Cauchy problem (3.1), with initial condition $S_o^{(n)}$, admits a unique classical solution S_n , which can be obtained using the method of characteristics, for instance. Here (\mathbf{G}) , together with the regularity of the controls u_1 and u_2 , permits to have globally defined characteristics curves. Since, for every $n \in \mathbb{N}$, S_n is a classical solution, then S_n satisfies condition (3.2) for every test function ϕ .

Multiplying the first equation in (3.1) by S_n and integrating, we get that, for all $t > 0$,

$$\begin{aligned} \frac{d}{dt} \|S_n(t)\|_{\mathbf{L}^2((0,1);\mathbb{R})}^2 &= 2 \int_0^1 S_n \partial_t S_n \, d\xi = -2 \int_0^1 (f + \alpha(I)) S_n^2 \, d\xi - 2 \int_0^1 \partial_\xi (G(\xi, t) S_n) S_n \, d\xi \\ &\leq -2 \int_0^1 \partial_\xi (G(\xi, t) S_n) S_n \, d\xi = 2 \int_0^1 G(\xi, t) S_n \partial_\xi S_n \, d\xi = \int_0^1 G(\xi, t) \partial_\xi S_n^2 \, d\xi \\ &= - \int_0^1 \partial_\xi G(\xi, t) S_n^2 \, d\xi \leq \gamma \|S_n(t)\|_{\mathbf{L}^2((0,1);\mathbb{R})}^2, \end{aligned}$$

where we used (α) , (\mathbf{F}) , and (\mathbf{G}) . Here the constant γ is defined in (\mathbf{G}) . Gronwall Lemma ensures that, for every $n \in \mathbb{N}$ and $t > 0$,

$$\|S_n(t)\|_{\mathbf{L}^2((0,1);\mathbb{R})} \leq e^{t\gamma/2} \|S_o^{(n)}\|_{\mathbf{L}^2((0,1);\mathbb{R})}. \quad (3.6)$$

We deduce that, for all $T > 0$, $(S_n)_n$ is bounded in $\mathbf{L}^\infty((0, T); \mathbf{L}^2((0, 1); \mathbb{R}))$. Therefore, for all $T > 0$, there exists $S_T \in \mathbf{L}^\infty((0, T); \mathbf{L}^2((0, 1); \mathbb{R}))$ such that a subsequence of $(S_n)_n$ converges to S_T in the \mathbf{L}^∞ -weak* sense. By a standard diagonal process, we can construct S such that there exists a subsequence of $(S_n)_n$ that weakly* converges to S in $\mathbf{L}^\infty((0, \infty); \mathbf{L}^2((0, 1); \mathbb{R}))$. By linearity, S satisfies (3.2) for every test function. Passing to the limit as $n \rightarrow +\infty$ in (3.6), we deduce (3.4).

Uniqueness. By linearity, it is sufficient to prove that if $S_o \equiv 0$, then any weak solution is also zero. We follow here a similar technique as in [11]. Fix $\psi \in \mathbf{C}_c^\infty(\mathbb{R}^+ \times (0, 1); \mathbb{R})$. We claim that

$$\int_0^{+\infty} \int_0^1 \partial_t S(t, \xi) \psi(t, \xi) \, d\xi \, dt = 0$$

so that $S(t, \xi) = 0$ for a.e. (t, ξ) by the arbitrariness of ψ ; hence uniqueness holds.

To this aim, there exists $\phi \in \mathbf{C}_c^\infty(\mathbb{R}^+ \times (0, 1); \mathbb{R})$ such that

$$\partial_t \phi + G(\xi, t) \partial_\xi \phi - (f + \alpha(I)) \phi = \psi. \quad (3.7)$$

Therefore, using (3.7) and (3.2), we deduce that

$$\begin{aligned} \int_0^{+\infty} \int_0^1 S(t, \xi) \psi(t, \xi) \, d\xi \, dt &= \int_0^{+\infty} \int_0^1 S(\partial_t \phi + G(\xi, t) \partial_\xi \phi) - (f + \alpha(I)) \phi \, d\xi \, dt \\ &= - \int_0^1 S_o(\xi) \phi(0, \xi) \, d\xi = 0, \end{aligned}$$

since $S_o \equiv 0$. Thus uniqueness holds.

Positivity. The method of characteristics implies that the unique solution to the Cauchy problem (3.1) can be written in the form

$$S(t, \xi) = S_o(X(0; t, \xi)) \exp \left(- \int_0^t [f(X(s; t, \xi)) + \alpha(I(s)) + \partial_\xi G(X(s; t, \xi), s)] \, ds \right), \quad (3.8)$$

where $X(s; t, \xi)$ solves the Cauchy problem

$$\begin{cases} X'(s; t, \xi) = G(X(s; t, \xi), s), \\ X(t; t, \xi) = \xi. \end{cases}$$

In the case $S_o(\xi) \geq 0$ for a.e. $\xi \in (0, 1)$, formula (3.8) implies that, for every $t > 0$, $S(t, \xi) \geq 0$ for a.e. $\xi \in (0, 1)$.

L¹ estimate. For all $t > 0$, by assumptions (α) and (F), formula (3.8) and the change of variable $y = X(0; t, \xi)$, we get

$$\begin{aligned} &\|S(t)\|_{\mathbf{L}^1((0,1);\mathbb{R})} \\ &= \int_0^1 |S_o(X(0; t, \xi))| \exp \left(- \int_0^t [f(X(s; t, \xi)) + \alpha(I(s)) + \partial_1 G(X(s; t, \xi), s)] \, ds \right) \, d\xi \\ &= \int_0^1 |S_o(y)| \exp \left(- \int_0^t [f(X(s; 0, y)) + \alpha(I(s))] \, ds \right) \, dy \leq \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})}, \end{aligned}$$

proving (3.3).

Continuity in time. We exploit an idea from [24, Lemma 2.11]. Fix $t_1 \geq 0$ and consider $t_2 \geq 0$. Without loss of generality we assume that $t_1 < t_2$. By the method of characteristics we have:

$$S(t_2, \xi) - S(t_1, \xi) = S(t_1, X(t_1; t_2, \xi)) \mathcal{E}(t_1, t_2, \xi) - S(t_1, \xi),$$

where

$$\mathcal{E}(t_1, t_2, \xi) = \exp \left(- \int_{t_1}^{t_2} [f(X(s; t_2, \xi)) + \alpha(I(s)) + \partial_\xi G(X(s; t_2, \xi), s)] \, ds \right). \quad (3.9)$$

In particular, adding and subtracting $S(t_1, \xi) \mathcal{E}(t_1, t_2, \xi)$, it yields

$$\begin{aligned} |S(t_2, \xi) - S(t_1, \xi)|^2 &= |(S(t_1, X(t_1; t_2, \xi)) - S(t_1, \xi)) \mathcal{E}(t_1, t_2, \xi) + S(t_1, \xi) (\mathcal{E}(t_1, t_2, \xi) - 1)|^2 \\ &\leq |S(t_1, X(t_1; t_2, \xi)) - S(t_1, \xi)|^2 (\mathcal{E}(t_1, t_2, \xi))^2 \\ &\quad + (S(t_1, \xi))^2 |\mathcal{E}(t_1, t_2, \xi) - 1|^2 \\ &\quad + 2 |S(t_1, X(t_1; t_2, \xi)) - S(t_1, \xi)| \cdot |S(t_1, \xi)| \mathcal{E}(t_1, t_2, \xi) |\mathcal{E}(t_1, t_2, \xi) - 1|. \end{aligned} \quad (3.10)$$

By the positivity of f and α , and exploiting also (G), we have that

$$\begin{aligned} &\int_0^1 |S(t_1, X(t_1; t_2, \xi)) - S(t_1, \xi)|^2 (\mathcal{E}(t_1, t_2, \xi))^2 \, d\xi \\ &\leq e^{2\gamma(t_2-t_1)} \int_0^1 |S(t_1, X(t_1; t_2, \xi)) - S(t_1, \xi)|^2 \, d\xi, \end{aligned} \quad (3.11)$$

and the right hand side goes to zero as $t_2 \rightarrow t_1$ by a similar reasoning as in [5, Lemma 4.3].

The definition (3.9) of \mathcal{E} , together with **(α)**, **(F)** and **(G)**, yields

$$\|\mathcal{E}(t_1, t_2, \cdot) - 1\|_{\mathbf{L}^\infty((0,1);\mathbb{R})} \leq (t_2 - t_1)e^{\gamma(t_2-t_1)} \left(\|f\|_{\mathbf{L}^\infty((0,1);\mathbb{R})} + \mathbf{Lip}(\alpha) \|I\|_{\mathbf{L}^\infty([t_1,t_2];\mathbb{R})} + \gamma \right), \quad (3.12)$$

so that

$$\begin{aligned} \int_0^1 (S(t_1, \xi))^2 |\mathcal{E}(t_1, t_2, \xi) - 1|^2 d\xi &\leq \|S(t_1)\|_{\mathbf{L}^2((0,1);\mathbb{R})}^2 \|\mathcal{E}(t_1, t_2, \cdot) - 1\|_{\mathbf{L}^\infty((0,1);\mathbb{R})}^2 \\ &\leq (t_2 - t_1)^2 e^{2\gamma(t_2-t_1)} \|S(t_1)\|_{\mathbf{L}^2((0,1);\mathbb{R})}^2 \\ &\quad \times \left(\|f\|_{\mathbf{L}^\infty((0,1);\mathbb{R})} + \mathbf{Lip}(\alpha) \|I\|_{\mathbf{L}^\infty([t_1,t_2];\mathbb{R})} + \gamma \right)^2 \end{aligned} \quad (3.13)$$

and the right hand side goes to zero as $t_2 \rightarrow t_1$, since $\|S(t_1)\|_{\mathbf{L}^2((0,1);\mathbb{R})}$ is bounded by (3.4).

Finally, using Young's inequality, the positivity of f and α , and exploiting also **(G)**, we have

$$\begin{aligned} &2 \int_0^1 |S(t_1, X(t_1; t_2, \xi)) - S(t_1, \xi)| \cdot |S(t_1, \xi)| \mathcal{E}(t_1, t_2, \xi) |\mathcal{E}(t_1, t_2, \xi) - 1| d\xi \\ &\leq \|\mathcal{E}(t_1, t_2, \cdot) - 1\|_{\mathbf{L}^\infty((0,1);\mathbb{R})} \left(\int_0^1 |S(t_1, X(t_1; t_2, \xi)) - S(t_1, \xi)|^2 d\xi + \int_0^1 |S(t_1, \xi)|^2 (\mathcal{E}(t_1, t_2, \xi))^2 d\xi \right) \\ &\leq \|\mathcal{E}(t_1, t_2, \cdot) - 1\|_{\mathbf{L}^\infty((0,1);\mathbb{R})} \left(\int_0^1 |S(t_1, X(t_1; t_2, \xi)) - S(t_1, \xi)|^2 d\xi + e^{2\gamma(t_2-t_1)} \|S(t_1)\|_{\mathbf{L}^2((0,1);\mathbb{R})}^2 \right), \end{aligned} \quad (3.14)$$

and the right hand side goes to zero as $t_2 \rightarrow t_1$ by (3.12), [5, Lemma 4.3], and the boundedness of $\|S(t_1)\|_{\mathbf{L}^2((0,1);\mathbb{R})}$ due to (3.4).

Integrating (3.10) over $\xi \in [0, 1]$ and using (3.11), (3.13), and (3.14) we deduce that

$$\lim_{t_2 \rightarrow t_1} \|S(t_2) - S(t_1)\|_{\mathbf{L}^2((0,1);\mathbb{R})} = 0,$$

proving that $S \in \mathbf{C}^0(\mathbb{R}^+; \mathbf{L}^2((0, 1); \mathbb{R}))$. \square

We provide now a stability result for (3.1). For simplicity, we call ‘‘weak solution to (3.1) with data (S_o, I) ’’ a weak solution to (3.1) with initial datum S_o and given function I .

Lemma 3.3. *Assume **(α)**, **(F)** and **(G)**. Fix $u_1 \in \mathcal{U}_1$, $u_2 \in \mathcal{U}_2$, $I, \bar{I} \in \mathbf{C}^0(\mathbb{R}^+; \mathbb{R}^+)$ and $S_o, \bar{S}_o \in \mathbf{L}^2((0, 1); \mathbb{R})$. Denote by S , resp. \bar{S} , the weak solution to (3.1) with data (S_o, I) , resp. (\bar{S}_o, \bar{I}) . Then for all $t > 0$,*

$$\|S(t) - \bar{S}(t)\|_{\mathbf{L}^1((0,1);\mathbb{R})} \leq \|S_o - \bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} + \mathbf{Lip}(\alpha) \|\bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \|I - \bar{I}\|_{\mathbf{L}^1((0,t);\mathbb{R})} \quad (3.15)$$

and

$$\|S(t) - \bar{S}(t)\|_{\mathbf{L}^2((0,1);\mathbb{R})} \leq e^{t\gamma/2} \left(\|S_o - \bar{S}_o\|_{\mathbf{L}^2((0,1);\mathbb{R})} + \mathbf{Lip}(\alpha) \sqrt{t} \|\bar{S}_o\|_{\mathbf{L}^2((0,1);\mathbb{R})} \|I - \bar{I}\|_{\mathbf{L}^2((0,t);\mathbb{R})} \right), \quad (3.16)$$

where γ is defined in **(G)**.

Proof. Let Σ be the unique weak solution to (3.1) with data (\bar{S}_o, I) . Since $S - \Sigma$ solves (3.1) with initial data $S_o - \bar{S}_o$, we deduce from Theorem 3.2 that, for all $t > 0$,

$$\|S(t) - \Sigma(t)\|_{\mathbf{L}^1((0,1);\mathbb{R})} \leq \|S_o - \bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \quad (3.17)$$

and

$$\|S(t) - \Sigma(t)\|_{\mathbf{L}^2((0,1);\mathbb{R})} \leq e^{t\gamma/2} \|S_o - \bar{S}_o\|_{\mathbf{L}^2((0,1);\mathbb{R})}. \quad (3.18)$$

The function $\delta := \Sigma - \bar{S}$ is the unique weak solution to

$$\begin{cases} \partial_t \delta + \partial_\xi(G(\xi, t)\delta) = -(f + \alpha(I))\delta - (\alpha(I) - \alpha(\bar{I}))\bar{S} \\ \delta(0, \xi) = 0, \end{cases}$$

and therefore, can be written using characteristics as

$$\begin{aligned} \delta(t, \xi) &= - \int_0^t (\alpha(I(\tau)) - \alpha(\bar{I}(\tau))) \bar{S}(\tau, X(\tau; t, \xi)) \\ &\quad \times \exp\left(- \int_\tau^t [f(X(s; t, \xi)) + \alpha(I(s)) + \partial_1 G(X(s; t, \xi), s)] ds\right) d\tau, \end{aligned} \quad (3.19)$$

where $X = X(\tau; t, \xi)$ and G are defined as in the proof of Theorem 3.2. Using **(F)** and **(α)**, the change of variable $y = X(\tau; t, \xi)$, and formula (3.3), for all $t > 0$, we obtain that

$$\begin{aligned} \|\delta(t)\|_{\mathbf{L}^1((0,1);\mathbb{R})} &\leq \int_0^1 \int_0^t |\alpha(I(\tau)) - \alpha(\bar{I}(\tau))| \cdot |\bar{S}(\tau, X(\tau, t, \xi))| \exp\left(- \int_\tau^t \partial_1 G(X(s; t, \xi), s) ds\right) d\tau d\xi \\ &\leq \mathbf{Lip}(\alpha) \int_0^t |I(\tau) - \bar{I}(\tau)| \int_0^1 |\bar{S}(\tau, y)| dy d\tau \\ &= \mathbf{Lip}(\alpha) \int_0^t |I(\tau) - \bar{I}(\tau)| \|\bar{S}(\tau)\|_{\mathbf{L}^1((0,1);\mathbb{R})} d\tau \\ &\leq \mathbf{Lip}(\alpha) \|\bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \int_0^t |I(\tau) - \bar{I}(\tau)| d\tau. \end{aligned}$$

Hence, using (3.17), we conclude that, for all $t > 0$,

$$\begin{aligned} \|S(t) - \bar{S}(t)\|_{\mathbf{L}^1((0,1);\mathbb{R})} &\leq \|S(t) - \Sigma(t)\|_{\mathbf{L}^1((0,1);\mathbb{R})} + \|\Sigma(t) - \bar{S}(t)\|_{\mathbf{L}^1((0,1);\mathbb{R})} \\ &\leq \|S_o - \bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} + \mathbf{Lip}(\alpha) \|\bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \int_0^t |I(\tau) - \bar{I}(\tau)| d\tau, \end{aligned}$$

proving (3.15).

Moreover, using **(α)** and **(F)**, from (3.19) we deduce that

$$\begin{aligned} |\delta(t, \xi)| &\leq \int_0^t |\alpha(I(\tau)) - \alpha(\bar{I}(\tau))| \cdot |\bar{S}(\tau, X(\tau; t, \xi))| \exp\left(- \int_\tau^t \partial_1 G(X(s; t, \xi), s) ds\right) d\tau \\ &\leq \mathbf{Lip}(\alpha) \int_0^t |I(\tau) - \bar{I}(\tau)| \cdot |\bar{S}(\tau, X(\tau; t, \xi))| \exp\left(- \int_\tau^t \partial_1 G(X(s; t, \xi), s) ds\right) d\tau, \end{aligned}$$

so that, by Hölder inequality,

$$|\delta(t, \xi)|^2 \leq (\mathbf{Lip}(\alpha))^2 t \int_0^t |I(\tau) - \bar{I}(\tau)|^2 \cdot |\bar{S}(\tau, X(\tau; t, \xi))|^2 \exp\left(- \int_\tau^t 2\partial_1 G(X(s; t, \xi), s) ds\right) d\tau.$$

Integrating over $\xi \in [0, 1]$ and using the change of variable $y = X(\tau; t, \xi)$ and (3.4), we obtain that, for all $t > 0$,

$$\begin{aligned} \|\delta(t)\|_{\mathbf{L}^2((0,1);\mathbb{R})}^2 &\leq (\mathbf{Lip}(\alpha))^2 t \int_0^t |I(\tau) - \bar{I}(\tau)|^2 \int_0^1 |\bar{S}(\tau, y)|^2 e^{(t-\tau)\gamma} dy d\tau \\ &\leq (\mathbf{Lip}(\alpha))^2 t \int_0^t |I(\tau) - \bar{I}(\tau)|^2 e^{(t-\tau)\gamma} \|\bar{S}(\tau)\|_{\mathbf{L}^2([0,1];\mathbb{R})}^2 d\tau \\ &\leq (\mathbf{Lip}(\alpha))^2 t \|\bar{S}_o\|_{\mathbf{L}^2((0,1);\mathbb{R})}^2 \int_0^t |I(\tau) - \bar{I}(\tau)|^2 e^{\tau\gamma} d\tau \\ &\leq (\mathbf{Lip}(\alpha))^2 t \|\bar{S}_o\|_{\mathbf{L}^2((0,1);\mathbb{R})}^2 e^{t\gamma} \|I - \bar{I}\|_{\mathbf{L}^2((0,t);\mathbb{R})}^2. \end{aligned}$$

Therefore, for all $t > 0$, using (3.18), we conclude that

$$\begin{aligned} \|S(t) - \bar{S}(t)\|_{\mathbf{L}^2((0,1);\mathbb{R})} &\leq \|S(t) - \Sigma(t)\|_{\mathbf{L}^2((0,1);\mathbb{R})} + \|\Sigma(t) - \bar{S}(t)\|_{\mathbf{L}^2((0,1);\mathbb{R})} \\ &\leq e^{t\gamma/2} \|S_o - \bar{S}_o\|_{\mathbf{L}^2((0,1);\mathbb{R})} + (\mathbf{Lip}(\alpha)) \sqrt{t} \|\bar{S}_o\|_{\mathbf{L}^2((0,1);\mathbb{R})} e^{t\gamma/2} \|I - \bar{I}\|_{\mathbf{L}^2((0,t);\mathbb{R})}, \end{aligned}$$

proving (3.16). \square

Remark 3.1. Note that the estimates (3.15) and (3.16) in the statement of Theorem 3.3 are symmetric with respect to the initial data S_o or \bar{S}_o . Hence (3.15) can be replaced by

$$\begin{aligned} \|S(t) - \bar{S}(t)\|_{\mathbf{L}^1((0,1);\mathbb{R})} &\leq \|S_o - \bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \\ &\quad + \mathbf{Lip}(\alpha) \min \left\{ \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})}, \|\bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \right\} \|I - \bar{I}\|_{\mathbf{L}^1((0,t);\mathbb{R})} \end{aligned}$$

while (3.16) can be replaced by

$$\begin{aligned} \|S(t) - \bar{S}(t)\|_{\mathbf{L}^2((0,1);\mathbb{R})} &\leq e^{t\gamma/2} \|S_o - \bar{S}_o\|_{\mathbf{L}^2((0,1);\mathbb{R})} \\ &\quad + e^{t\gamma/2} \mathbf{Lip}(\alpha) \sqrt{t} \min \left\{ \|S_o\|_{\mathbf{L}^2((0,1);\mathbb{R})}, \|\bar{S}_o\|_{\mathbf{L}^2((0,1);\mathbb{R})} \right\} \|I - \bar{I}\|_{\mathbf{L}^2((0,t);\mathbb{R})}. \end{aligned}$$

3.1.2 The Ordinary Differential Equation

In this part we recall classical results about the Cauchy problem

$$\begin{cases} \dot{I}(t) = -\beta I(t) + \alpha(I(t)) \int_0^1 S(t, \xi) \, d\xi, \\ I(0) = I_o, \end{cases} \quad (3.20)$$

where $\beta > 0$ and $I_o \in \mathbb{R}$.

Lemma 3.4. Assume (α) . Let $S \in \mathbf{L}^\infty((0, \infty); \mathbf{L}^2((0, 1); \mathbb{R}))$. Then, for all $I_o \in \mathbb{R}$, the Cauchy problem (3.20) admits a unique solution $I \in \mathbf{C}^0(\mathbb{R}^+; \mathbb{R})$. Moreover,

(i) if $I_o \geq 0$, then $I(t) \geq 0$ for all $t > 0$;

(ii) for all $t > 0$,

$$|I(t)| \leq |I_o| e^{K_t t}, \quad (3.21)$$

where $K_t := \beta + \mathbf{Lip}(\alpha) \|S\|_{\mathbf{L}^\infty((0,t);\mathbf{L}^1((0,1);\mathbb{R}))}$.

Proof. The proof of the well-posedness as well as Item (i) are standard, hence we omit them. Let us precise here the importance of the fact that $\alpha(0) = 0$. Inequality (3.21) comes from an application of Gronwall Lemma since for all $t > 0$,

$$|I(t)| \leq |I_o| + \left(\beta + \mathbf{Lip}(\alpha) \|S\|_{\mathbf{L}^\infty((0,t);\mathbf{L}^1((0,1);\mathbb{R}))} \right) \int_0^t |I(\tau)| \, d\tau.$$

\square

We continue with a stability result for the Cauchy problem (3.20). For simplicity, we call ‘‘solution to (3.20) with data (S, I_o) ’’ a solution to (3.20) with initial datum I_o and given function S .

Lemma 3.5. Assume (α) . Fix $S, \bar{S} \in \mathbf{L}^\infty((0, \infty); \mathbf{L}^2((0, 1); \mathbb{R}))$ and the initial conditions $I_o, \bar{I}_o \in \mathbb{R}$. Denote by I , resp. \bar{I} , the solution to the Cauchy problem (3.20) with data (S, I_o) , resp. (\bar{S}, \bar{I}_o) . Then for all $t > 0$,

$$|I(t) - \bar{I}(t)| \leq e^{K_t t} \left(|I_o - \bar{I}_o| + \mathbf{Lip}(\alpha) |\bar{I}_o| \int_0^t e^{(\bar{K}_\tau - K_\tau) \tau} \|S(\tau) - \bar{S}(\tau)\|_{\mathbf{L}^1((0,1);\mathbb{R})} \, d\tau \right) \quad (3.22)$$

where $K_t := \beta + \mathbf{Lip}(\alpha) \|S\|_{\mathbf{L}^\infty((0,t);\mathbf{L}^1((0,1);\mathbb{R}))}$ and $\bar{K}_t := \beta + \mathbf{Lip}(\alpha) \|\bar{S}\|_{\mathbf{L}^\infty((0,t);\mathbf{L}^1((0,1);\mathbb{R}))}$.

Proof. We have

$$I(t) = I_o + \int_0^t \left(-\beta I(\tau) + \alpha(I(\tau)) \int_0^1 S(\tau, \xi) d\xi \right) d\tau$$

and similarly for \bar{I} . Thus

$$\begin{aligned} |I(t) - \bar{I}(t)| &\leq |I_o - \bar{I}_o| + \beta \int_0^t |I(\tau) - \bar{I}(\tau)| d\tau \\ &\quad + \int_0^t \left| \alpha(I(\tau)) \int_0^1 S(\tau, \xi) d\xi - \alpha(\bar{I}(\tau)) \int_0^1 \bar{S}(\tau, \xi) d\xi \right| d\tau. \end{aligned}$$

Now we can add and subtract $\alpha(I(\tau)) \int_0^1 \bar{S}(\tau, \xi) d\xi$ or $\alpha(\bar{I}(\tau)) \int_0^1 S(\tau, \xi) d\xi$. In the first case we get

$$\begin{aligned} |I(t) - \bar{I}(t)| &\leq |I_o - \bar{I}_o| + \beta \int_0^t |I(\tau) - \bar{I}(\tau)| d\tau + \int_0^t |\alpha(I(\tau))| \cdot \|S(\tau) - \bar{S}(\tau)\|_{\mathbf{L}^1((0,1);\mathbb{R})} d\tau \\ &\quad + \int_0^t |\alpha(I(\tau)) - \alpha(\bar{I}(\tau))| \int_0^1 |\bar{S}(\tau, \xi)| d\xi d\tau \\ &\leq |I_o - \bar{I}_o| + \beta \int_0^t |I(\tau) - \bar{I}(\tau)| d\tau + \mathbf{Lip}(\alpha) |I_o| \int_0^t e^{K\tau} \|S(\tau) - \bar{S}(\tau)\|_{\mathbf{L}^1((0,1);\mathbb{R})} d\tau \\ &\quad + \mathbf{Lip}(\alpha) \|\bar{S}\|_{\mathbf{L}^\infty((0,t);\mathbf{L}^1((0,1);\mathbb{R}))} \int_0^t |I(\tau) - \bar{I}(\tau)| d\tau, \end{aligned}$$

where we used also (3.21) with $K_t := \beta + \mathbf{Lip}(\alpha) \|S\|_{\mathbf{L}^\infty((0,t);\mathbf{L}^1((0,1);\mathbb{R}))}$. Setting also $\bar{K}_t := \beta + \mathbf{Lip}(\alpha) \|\bar{S}\|_{\mathbf{L}^\infty((0,t);\mathbf{L}^1((0,1);\mathbb{R}))}$, by Gronwall Lemma we get

$$|I(t) - \bar{I}(t)| \leq e^{\bar{K}_t t} \left(|I_o - \bar{I}_o| + \mathbf{Lip}(\alpha) |I_o| \int_0^t e^{(K_\tau - \bar{K}_\tau)\tau} \|S(\tau) - \bar{S}(\tau)\|_{\mathbf{L}^1((0,1);\mathbb{R})} d\tau \right).$$

In the second case, analogous computations lead to

$$|I(t) - \bar{I}(t)| \leq e^{K_t t} \left(|I_o - \bar{I}_o| + \mathbf{Lip}(\alpha) |\bar{I}_o| \int_0^t e^{(\bar{K}_\tau - K_\tau)\tau} \|S(\tau) - \bar{S}(\tau)\|_{\mathbf{L}^1((0,1);\mathbb{R})} d\tau \right).$$

Therefore we deduce (3.22). \square

Remark 3.2. Clearly, in Theorem 3.5, the roles of S and \bar{S} are symmetric. Therefore (3.22) can be changed into

$$|I(t) - \bar{I}(t)| \leq e^{\bar{K}_t t} \left(|I_o - \bar{I}_o| + \mathbf{Lip}(\alpha) |\bar{I}_o| \int_0^t e^{(K_\tau - \bar{K}_\tau)\tau} \|S(\tau) - \bar{S}(\tau)\|_{\mathbf{L}^1((0,1);\mathbb{R})} d\tau \right).$$

3.1.3 The Mixed System

This part contains the proofs of Theorem 2.3 and of Theorem 2.4.

Proof of Theorem 2.3. We divide the proof into various steps.

Step 1: Local in time existence. Choose $T > 0$ such that

$$(\mathbf{Lip}(\alpha))^2 \left(|I_o| + \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \right) \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} T^2 e^{2KT} \leq 1, \quad (3.23)$$

where the constant K is defined by

$$K := \beta + \mathbf{Lip}(\alpha) \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})}.$$

Consider the mappings

$$\Lambda : \begin{cases} \mathbf{C}^0([0, T]; \mathbb{R}^+) & \longrightarrow & \mathbf{C}^0([0, T]; \mathbf{L}^2((0, 1); \mathbb{R}^+)) \\ I & \longmapsto & S, \end{cases}$$

where S is the unique weak solution to (3.1) with data (S_o, I) according to Theorem 3.2, and

$$\Upsilon : \begin{cases} \mathbf{C}^0([0, T]; \mathbf{L}^2((0, 1); \mathbb{R}^+)) & \longrightarrow & \mathbf{C}^0([0, T]; \mathbb{R}^+) \\ S & \longmapsto & I, \end{cases}$$

where I is the unique solution to (3.20) with data (S, I_o) , according to Theorem 3.4. The positivity of $\Lambda(I)$ and $\Upsilon(S)$ are ensured by Theorem 3.2 and Theorem 3.4 respectively.

Fix $I, \bar{I} \in \mathbf{C}^0([0, T]; \mathbb{R}^+)$. By (3.22), defining $K_T := \beta + \mathbf{Lip}(\alpha) \|\Lambda(I)\|_{\mathbf{L}^\infty((0, T); \mathbf{L}^1((0, 1); \mathbb{R}))}$ and $\bar{K}_T := \beta + \mathbf{Lip}(\alpha) \|\Lambda(\bar{I})\|_{\mathbf{L}^\infty((0, T); \mathbf{L}^1((0, 1); \mathbb{R}))}$, we have that, for every $t \in [0, T]$,

$$\begin{aligned} |(\Upsilon \circ \Lambda(I))(t) - (\Upsilon \circ \Lambda(\bar{I}))(t)| &\leq e^{K_T t} \mathbf{Lip}(\alpha) |I_o| \int_0^t e^{\bar{K}_T \tau} \|\Lambda(I)(\tau) - \Lambda(\bar{I})(\tau)\|_{\mathbf{L}^1((0, 1); \mathbb{R})} d\tau \\ &\leq e^{2K t} \mathbf{Lip}(\alpha) |I_o| \int_0^t \|\Lambda(I)(\tau) - \Lambda(\bar{I})(\tau)\|_{\mathbf{L}^1((0, 1); \mathbb{R})} d\tau, \end{aligned} \quad (3.24)$$

where we used the fact that $K_T \leq K$ and $\bar{K}_T \leq K$ by (3.3). Using now (3.15) and (3.23), by (3.24) we deduce that, for all $t \in [0, T]$,

$$\begin{aligned} |(\Upsilon \circ \Lambda(I))(t) - (\Upsilon \circ \Lambda(\bar{I}))(t)| &\leq e^{2K t} (\mathbf{Lip}(\alpha))^2 |I_o| \cdot \|S_o\|_{\mathbf{L}^1((0, 1); \mathbb{R})} \int_0^t \int_0^\tau |I(\sigma) - \bar{I}(\sigma)| d\sigma d\tau \\ &\leq \frac{(\mathbf{Lip}(\alpha))^2 |I_o| \cdot \|S_o\|_{\mathbf{L}^1((0, 1); \mathbb{R})} T^2 e^{2K T}}{2} \|I - \bar{I}\|_{\mathbf{C}^0([0, T]; \mathbb{R})} \\ &\leq \frac{1}{2} \|I - \bar{I}\|_{\mathbf{C}^0([0, T]; \mathbb{R})}. \end{aligned}$$

Therefore

$$\|\Upsilon \circ \Lambda(I) - \Upsilon \circ \Lambda(\bar{I})\|_{\mathbf{C}^0([0, T]; \mathbb{R})} \leq \frac{1}{2} \|I - \bar{I}\|_{\mathbf{C}^0([0, T]; \mathbb{R})},$$

ensuring that $\Upsilon \circ \Lambda$ admits a unique fixed point $I^* \in \mathbf{C}^0([0, T]; \mathbb{R})$. By construction and Theorem 3.2, the couple (S^*, I^*) , where $S^* := \Lambda(I^*) \in \mathbf{C}^0([0, T]; \mathbf{L}^2((0, 1); \mathbb{R}))$, is a solution to (1.1) on $[0, T]$ with initial data (S_o, I_o) , according to Theorem 2.1.

Step 2: A priori estimate. Let (S, I) be a solution to (1.1) on the time interval $[0, T]$ for some $T > 0$, according to Theorem 2.1.

Clearly, by Theorem 3.2, the bounds (2.10) for S hold for every $t \in [0, T]$.

We now prove the bound (2.11) for $t \in [0, T]$, related to I . Let $(S_o^{(n)})_n$ be a sequence of nonnegative \mathbf{C}^1 functions that converges to S_o in $\mathbf{L}^1((0, 1); \mathbb{R})$. For all $n \in \mathbb{N}$, call S_n the distributional solution to

$$\begin{cases} \partial_t S_n + \partial_\xi(g(\xi, u_1(t), u_2(t))S_n) = -(f + \alpha(I))S_n \\ S_n(0, \xi) = S_o^{(n)}(\xi), \end{cases}$$

which is also a classical solution since $S_o^{(n)}$ is smooth. Let $\varphi \in \mathbf{C}_c^\infty((0, +\infty); \mathbb{R}^+)$. For all $n \in \mathbb{N}$ and

$t \in [0, T]$, we have

$$\begin{aligned}
& \int_0^{+\infty} \varphi'(t) \left(\int_0^1 S_n(t, \xi) d\xi + I(t) \right) dt = \int_0^{+\infty} \varphi'(t) \int_0^1 S_n(t, \xi) d\xi dt + \int_0^{+\infty} \varphi'(t) I(t) dt \\
&= \int_0^{+\infty} \int_0^1 \varphi'(t) S_n(t, \xi) d\xi dt + \beta \int_0^{+\infty} \varphi(t) I(t) dt - \int_0^{+\infty} \varphi(t) \alpha(I(t)) \int_0^1 S(t, \xi) d\xi dt \\
&= \int_0^{+\infty} \int_0^1 \partial_\xi (S_n(t, \xi) g(\xi, u_1(t), u_2(t))) \varphi(t) d\xi dt + \int_0^{+\infty} \int_0^1 S_n(t, \xi) f(\xi) \varphi(t) d\xi dt \\
&\quad + \int_0^{+\infty} \int_0^1 S_n(t, \xi) \alpha(I(t)) \varphi(t) d\xi dt - \int_0^1 S_o^{(n)}(\xi) \varphi(0) d\xi dt \\
&\quad + \beta \int_0^{+\infty} \varphi(t) I(t) dt - \int_0^{+\infty} \varphi(t) \alpha(I(t)) \int_0^1 S(t, \xi) d\xi dt \\
&= \int_0^{+\infty} \int_0^1 S_n(t, \xi) f(\xi) \varphi(t) d\xi dt + \int_0^{+\infty} \int_0^1 S_n(t, \xi) \alpha(I(t)) \varphi(t) d\xi dt \\
&\quad + \beta \int_0^{+\infty} \varphi(t) I(t) dt - \int_0^{+\infty} \varphi(t) \alpha(I(t)) \int_0^1 S(t, \xi) d\xi dt \\
&\geq \int_0^{+\infty} \int_0^1 S_n(t, \xi) \alpha(I(t)) \varphi(t) d\xi dt - \int_0^{+\infty} \varphi(t) \alpha(I(t)) \int_0^1 S(t, \xi) d\xi dt.
\end{aligned}$$

Using Theorem 3.3, the bound (2.10), and passing to the limit as $n \rightarrow +\infty$ in the previous estimates, we deduce that

$$\int_0^{+\infty} \varphi'(t) \left(\int_0^1 S(t, \xi) d\xi + I(t) \right) dt \geq 0,$$

so that the distributional derivative of

$$t \mapsto \int_0^1 S(t, \xi) d\xi + I(t)$$

is a positive measure. Thus, for every $t \in [0, T]$, we obtain that

$$I(t) \leq \int_0^1 S(t, \xi) d\xi + I(t) \leq \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} + I_o,$$

proving (2.11) for $t \in [0, T]$.

Step 3: Global in time existence. Introduce the time

$$T^* := \sup \left\{ \tau > 0 : (1.1) \text{ admits a solution on } [0, t] \text{ for all } t \in [0, \tau] \right\}.$$

Step 1 ensures that T^* is well-defined and $T^* > 0$. Suppose, by contradiction, that $T^* < +\infty$ and choose $0 < \tau < \frac{T^*}{2}$ such that

$$2(\mathbf{Lip}(\alpha))^2 \left(|I_o| + \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \right) \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \tau^2 e^{2K\tau} \leq 1. \quad (3.25)$$

Set $T := T^* - \tau/2$. Since $T < T^*$, system (1.1) admits a solution on $[0, T]$, say (S_1, I_1) . By (2.10), (2.11), and (3.25) we deduce that

$$\begin{aligned}
& (\mathbf{Lip}(\alpha))^2 \left(|I_1(T)| + \|S_1(T)\|_{\mathbf{L}^1((0,1);\mathbb{R})} \right) \|S_1(T)\|_{\mathbf{L}^1((0,1);\mathbb{R})} \tau^2 e^{2K\tau} \\
&\leq (\mathbf{Lip}(\alpha))^2 \left(|I_o| + 2\|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \right) \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \tau^2 e^{2K\tau} \\
&\leq 2(\mathbf{Lip}(\alpha))^2 \left(|I_o| + \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \right) \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \tau^2 e^{2K\tau} \\
&\leq 1.
\end{aligned}$$

Therefore, by Step 1, we can construct a solution (S_2, I_2) to (1.1) on $[T, T + \tau]$ with initial condition $(S_1(T), I_1(T))$. Clearly the concatenation (S, I) , patching together (S_1, I_1) and (S_2, I_2) , is a solution to (1.1) on $[0, T + \tau]$, with $T + \tau = T^* + \frac{\tau}{2} > T^*$, contradicting the choice of T^* . Therefore, $T^* = +\infty$, concluding the proof of the global in time existence.

Step 4: Uniqueness of solution. Assume, by contradiction, that there are two distinct solutions, namely (S, I) and (\tilde{S}, \tilde{I}) , to (1.1) with the same initial condition (2.2). For every $t > 0$ and $\xi \in (0, 1)$, a similar reasoning as in [24, Theorem 3.2] implies that

$$S(t, \xi) = S_o(X(0; t, \xi)) \exp \left(- \int_0^t [f(X(s; t, \xi)) + \alpha(I(s)) + \partial_\xi G(X(s; t, \xi), s)] ds \right) \quad (3.26)$$

$$\tilde{S}(t, \xi) = S_o(X(0; t, \xi)) \exp \left(- \int_0^t \left[f(X(s; t, \xi)) + \alpha(\tilde{I}(s)) + \partial_\xi G(X(s; t, \xi), s) \right] ds \right) \quad (3.27)$$

$$I(t) = I_o - \int_0^t \beta I(\tau) d\tau + \int_0^t \alpha(I(\tau)) \left(\int_0^1 S(\tau, \xi) d\xi \right) d\tau \quad (3.28)$$

$$\tilde{I}(t) = I_o - \int_0^t \beta \tilde{I}(\tau) d\tau + \int_0^t \alpha(\tilde{I}(\tau)) \left(\int_0^1 \tilde{S}(\tau, \xi) d\xi \right) d\tau, \quad (3.29)$$

where G is defined in (3.5).

Using (3.26), (3.27), the change of variable $y = X(0; t, \xi)$, (α) , and (\mathbf{F}) , we deduce that, for every $t > 0$,

$$\begin{aligned} \left\| S(t) - \tilde{S}(t) \right\|_{\mathbf{L}^1((0,1);\mathbb{R})} &\leq \int_0^1 |S_o(X(0; t, \xi))| \exp \left(- \int_0^t [f(X(s; t, \xi)) + \partial_\xi G(X(s; t, \xi), s)] ds \right) \\ &\quad \times \left| \exp \left(- \int_0^t \alpha(I(s)) ds \right) - \exp \left(- \int_0^t \alpha(\tilde{I}(s)) ds \right) \right| d\xi \\ &\leq \int_0^1 |S_o(y)| \exp \left(- \int_0^t f(y) ds \right) dy \int_0^t \left| \alpha(I(s)) - \alpha(\tilde{I}(s)) \right| ds \\ &\leq \mathbf{Lip}(\alpha) \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \left\| I - \tilde{I} \right\|_{\mathbf{L}^1((0,t);\mathbb{R})}. \end{aligned} \quad (3.30)$$

Using (3.28), (3.29), (α) , (2.10), (2.11), and (3.30), we deduce that, for every $t > 0$,

$$\begin{aligned} \left| I(t) - \tilde{I}(t) \right| &\leq \beta \int_0^t \left| I(\tau) - \tilde{I}(\tau) \right| d\tau + \int_0^t \alpha(I(\tau)) \left| \int_0^1 S(\tau, \xi) d\xi - \int_0^1 \tilde{S}(\tau, \xi) d\xi \right| d\tau \\ &\quad + \int_0^t \left| \int_0^1 \tilde{S}(\tau, \xi) d\xi \right| \cdot \left| \alpha(I(\tau)) - \alpha(\tilde{I}(\tau)) \right| d\tau \\ &\leq \beta \int_0^t \left| I(\tau) - \tilde{I}(\tau) \right| d\tau + \int_0^t \alpha(I(\tau)) \left\| S(\tau) - \tilde{S}(\tau) \right\|_{\mathbf{L}^1((0,1);\mathbb{R})} d\tau \\ &\quad + \mathbf{Lip}(\alpha) \int_0^t \left\| \tilde{S}(\tau) \right\|_{\mathbf{L}^1((0,1);\mathbb{R})} \left| I(\tau) - \tilde{I}(\tau) \right| d\tau \\ &\leq \beta \int_0^t \left| I(\tau) - \tilde{I}(\tau) \right| d\tau + (\mathbf{Lip}(\alpha))^2 \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \int_0^t |I(\tau)| \left\| I - \tilde{I} \right\|_{\mathbf{L}^1((0,\tau);\mathbb{R})} d\tau \\ &\quad + \mathbf{Lip}(\alpha) \left\| \tilde{S}_o \right\|_{\mathbf{L}^1((0,1);\mathbb{R})} \int_0^t \left| I(\tau) - \tilde{I}(\tau) \right| d\tau \\ &\leq \left(\beta + (\mathbf{Lip}(\alpha))^2 \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \left(I_o + \|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \right) t + \mathbf{Lip}(\alpha) \left\| \tilde{S}_o \right\|_{\mathbf{L}^1((0,1);\mathbb{R})} \right) \\ &\quad \times \int_0^t \left| I(\tau) - \tilde{I}(\tau) \right| d\tau. \end{aligned}$$

Gronwall Lemma implies that $I(t) = \tilde{I}(t)$ for every $t > 0$. Finally (3.30) implies that $S(t) = \tilde{S}(t)$ in $\mathbf{L}^1((0, 1); \mathbb{R})$ for all $t > 0$.

In the case $S_o \in \mathbf{H}^1((0, 1); \mathbb{R}^+)$, the required higher regularity of the solution follows by the representation formula (2.9). This concludes the proof. \square

Proof of Theorem 2.4. By Theorem 3.3, for all $t > 0$ we have

$$\|S(t) - \bar{S}(t)\|_{\mathbf{L}^1((0,1);\mathbb{R})} \leq \|S_o - \bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} + \mathbf{Lip}(\alpha) \|\bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} \|I - \bar{I}\|_{\mathbf{L}^1((0,t);\mathbb{R})}.$$

On the other hand, by Theorem 3.5, for all $t > 0$ we have

$$|I(t) - \bar{I}(t)| \leq e^{K_1 t} \left(|I_o - \bar{I}_o| + \mathbf{Lip}(\alpha) |\bar{I}_o| \int_0^t e^{K_2 \tau} \|S(\tau) - \bar{S}(\tau)\|_{\mathbf{L}^1((0,1);\mathbb{R})} d\tau \right),$$

where K_1 and K_2 are defined in (2.13). Hence, for all $t > 0$

$$\begin{aligned} & \|S(t) - \bar{S}(t)\|_{\mathbf{L}^1((0,1);\mathbb{R})} + |I(t) - \bar{I}(t)| \\ & \leq \|S_o - \bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} + e^{K_1 t} |I_o - \bar{I}_o| \\ & \quad + \mathbf{Lip}(\alpha) \left(\|\bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} + e^{K_1 t} |\bar{I}_o| \right) \int_0^t e^{K_2 \tau} \left(\|S(\tau) - \bar{S}(\tau)\|_{\mathbf{L}^1((0,1);\mathbb{R})} + |I(\tau) - \bar{I}(\tau)| \right) d\tau. \end{aligned}$$

Set \mathcal{K}_t as in (2.14). An application of Gronwall Lemma yields

$$\begin{aligned} & \|S(t) - \bar{S}(t)\|_{\mathbf{L}^1((0,1);\mathbb{R})} + |I(t) - \bar{I}(t)| \\ & \leq \|S_o - \bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} + e^{K_1 t} |I_o - \bar{I}_o| \\ & \quad + \mathcal{K}_t \int_0^t \left(\|S_o - \bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} + e^{K_1 \tau} |I_o - \bar{I}_o| \right) e^{K_2 \tau} \exp \left(\int_\tau^t e^{K_2 s} \mathcal{K}_s ds \right) d\tau \\ & \leq \left[\|S_o - \bar{S}_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} + e^{K_1 t} |I_o - \bar{I}_o| \right] (1 + te^{K_2 t} \mathcal{K}_t \exp(te^{K_2 t} \mathcal{K}_t)). \end{aligned}$$

concluding the proof. \square

3.2 Decoupled optimization

Proof of Theorem 2.5. Observe first that, since (G) holds, for all $u_1 \in \mathcal{U}_1$ and $u_2 \in \mathcal{U}_2$, the functional \mathcal{F} , defined in (2.16), satisfies the estimate

$$|\mathcal{F}(S_o, I_o, u_1, u_2)| \leq \frac{\kappa(\|S_o\|_{\mathbf{L}^1((0,1);\mathbb{R})} + I_o + M_1) + M_2 + \gamma}{\theta}, \quad (3.31)$$

where γ is defined in (G).

Consider Item (i). Fix $u_2 \in \mathcal{U}_2$ and consider a minimizing sequence of controls $(u_1^{(n)})_n$, taking values in \mathcal{U}_1 , such that

$$\lim_{n \rightarrow +\infty} \mathcal{F}(S_o, I_o, u_1^{(n)}, u_2) = \inf_{u_1 \in \mathcal{U}_1} \mathcal{F}(S_o, I_o, u_1, u_2). \quad (3.32)$$

The sequence $(u_1^{(n)})_n$ is bounded in \mathbf{L}^∞ , therefore it admits a subsequence, which we do not relabel, that converges in the weak* sense: there exists $\bar{u}_1 \in \mathcal{U}_1$ such that, for every $\psi \in \mathbf{L}^1(\mathbb{R}^+; \mathbb{R})$,

$$\lim_{n \rightarrow +\infty} \int_0^{+\infty} u_1^{(n)}(t) \psi(t) dt = \int_0^{+\infty} \bar{u}_1(t) \psi(t) dt.$$

For all $n \in \mathbb{N}$, call (S_n, I_n) the solution to (1.1) with controls $(u_1^{(n)}, u_2)$ and initial data (S_o, I_o) . By Remark 2.3, we have that

$$S_n(t, \xi) = S_o(X_n(0; t, \xi)) \exp \left(- \int_0^t \left[c_n(s, X_n(s; t, \xi)) + \partial_1 g(X_n(s; t, \xi), u_1^{(n)}(s), u_2(s)) \right] ds \right), \quad (3.33)$$

where $c_n(s, \zeta) = f(\zeta) + \alpha(I_n(s))$ and $X_n = X_n(s; t, \xi)$ solves (2.9) with u_1 replaced by $u_1^{(n)}$. Hence, by (G-1), for every $s \in \mathbb{R}^+$

$$\begin{aligned} X_n(s; t, \xi) &= \xi + \int_t^s g \left(X_n(\tau; t, \xi), u_1^{(n)}(\tau), u_2(\tau) \right) d\tau \\ &= \xi + \int_t^s g_1(X_n(\tau; t, \xi)) d\tau + \int_t^s g_2(X_n(\tau; t, \xi)) u_1^{(n)}(\tau) d\tau + \int_t^s g_3(X_n(\tau; t, \xi)) u_2(\tau) d\tau. \end{aligned} \quad (3.34)$$

For all $(t, \xi) \in (0, \infty) \times [0, 1]$, the sequence $s \mapsto X_n(s; t, \xi)$ is uniformly equicontinuous and bounded on any compact subset of \mathbb{R}^+ , since $|\partial_1 X_n(s; t, \xi)| \leq \gamma$, by (G). Then, Ascoli-Arzelà Theorem implies that, up to a subsequence, it converges to a function $s \mapsto \Psi_{t, \xi}(s)$.

Using (G-1), the Lebesgue Theorem, and the weak* convergence of $u_1^{(n)}$, we deduce that, for all $s \in \mathbb{R}^+$,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_t^s g_1(X_n(\tau; t, \xi)) d\tau &= \int_t^s g_1(\Psi_{t, \xi}(\tau)) d\tau \\ \lim_{n \rightarrow +\infty} \int_t^s g_3(X_n(\tau; t, \xi)) u_2(\tau) d\tau &= \int_t^s g_3(\Psi_{t, \xi}(\tau)) u_2(\tau) d\tau \\ \lim_{n \rightarrow +\infty} \int_t^s g_2(\Psi_{t, \xi}(\tau)) u_1^{(n)}(\tau) d\tau &= \int_t^s g_2(\Psi_{t, \xi}(\tau)) \bar{u}_1(\tau) d\tau. \end{aligned} \quad (3.35)$$

Note that

$$\begin{aligned} \left| \int_t^s (g_2(X_n(\tau; t, \xi)) - g_2(\Psi_{t, \xi}(\tau))) u_1^{(n)}(\tau) d\tau \right| &\leq \int_{\min\{t, s\}}^{\max\{t, s\}} |g_2(X_n(\tau; t, \xi)) - g_2(\Psi_{t, \xi}(\tau))| |u_1^{(n)}(\tau)| d\tau \\ &\leq M_1 \int_{\min\{t, s\}}^{\max\{t, s\}} |g_2(X_n(\tau; t, \xi)) - g_2(\Psi_{t, \xi}(\tau))| d\tau \\ &\leq M_1 \sup_{\tau} |g_2(X_n(\tau; t, \xi)) - g_2(\Psi_{t, \xi}(\tau))| |t - s| \end{aligned}$$

and so

$$\lim_{n \rightarrow +\infty} \int_t^s (g_2(X_n(\tau; t, \xi)) - g_2(\Psi_{t, \xi}(\tau))) u_1^{(n)}(\tau) d\tau = 0. \quad (3.36)$$

Using (3.35) and (3.36) and passing to the limit as $n \rightarrow +\infty$ in (3.34), we deduce that

$$\begin{aligned} \Psi_{t, \xi}(s) &= \xi + \int_t^s g_1(\Psi_{t, \xi}(\tau)) d\tau + \int_t^s g_2(\Psi_{t, \xi}(\tau)) \bar{u}_1(\tau) d\tau + \int_t^s g_3(\Psi_{t, \xi}(\tau)) u_2(\tau) d\tau \\ &= \xi + \int_t^s g(\Psi_{t, \xi}(\tau), \bar{u}_1(\tau), u_2(\tau)) d\tau \end{aligned} \quad (3.37)$$

for every $s \in \mathbb{R}^+$. Hence $\Psi_{t, \xi}$ solves the Cauchy problem (2.9) with the controls \bar{u}_1 and u_2 .

The sequence $t \mapsto I_n(t)$ is uniformly equicontinuous and bounded on any compact subset \mathbb{R}^+ . Indeed, the boundedness follows directly from (2.11), while by (α), (2.10) and (2.11), for all $t \in \mathbb{R}^+$

$$\left| \dot{I}_n(t) \right| \leq \left(\beta + \mathbf{Lip}(\alpha) \|S_o\|_{\mathbf{L}^1((0,1); \mathbb{R})} \right) \left(\|S_o\|_{\mathbf{L}^1((0,1); \mathbb{R})} + I_o \right).$$

Combining Ascoli-Arzelà Theorem and a standard diagonal procedure, we can find $\bar{I} \in \mathbf{C}^0(\mathbb{R}^+; \mathbb{R}^+)$ and a subsequence of $(I_n)_n$ that converges uniformly to \bar{I} on any compact subset of \mathbb{R}^+ .

Define \bar{S} , for $t \in \mathbb{R}^+$ and $\xi \in (0, 1)$, as

$$\bar{S}(t, \xi) := S_o(\Psi_{t,\xi}(0)) \exp \left(- \int_0^t [f(\Psi_{t,\xi}(s)) + \alpha(\bar{I}(s)) + \partial_\xi g(\Psi_{t,\xi}(s), \bar{u}_1(s), u_2(s))] ds \right), \quad (3.38)$$

that is, \bar{S} is a solution to the partial differential equation (3.1) with data (S_o, \bar{I}) and controls (\bar{u}_1, u_2) . We claim that, for every $t > 0$,

$$\lim_{n \rightarrow +\infty} \|S_n(t) - \bar{S}(t)\|_{\mathbf{L}^2((0,1);\mathbb{R})} = 0. \quad (3.39)$$

Fix $t > 0$, $\varepsilon > 0$, and choose $S_{o,\varepsilon} \in \mathbf{C}^\infty((0, 1); \mathbb{R}^+)$ such that $\|S_o - S_{o,\varepsilon}\|_{\mathbf{L}^2((0,1);\mathbb{R})} < \varepsilon$. Therefore

$$\begin{aligned} \left(\int_0^1 |S_o(\Psi_{t,\xi}(0)) - S_o(X_n(0; t, \xi))|^2 d\xi \right)^{\frac{1}{2}} &\leq \left(\int_0^1 |S_o(\Psi_{t,\xi}(0)) - S_{o,\varepsilon}(\Psi_{t,\xi}(0))|^2 d\xi \right)^{\frac{1}{2}} \\ &\quad + \left(\int_0^1 |S_{o,\varepsilon}(\Psi_{t,\xi}(0)) - S_{o,\varepsilon}(X_n(0; t, \xi))|^2 d\xi \right)^{\frac{1}{2}} \\ &\quad + \left(\int_0^1 |S_{o,\varepsilon}(X_n(0; t, \xi)) - S_o(X_n(0; t, \xi))|^2 d\xi \right)^{\frac{1}{2}}. \end{aligned}$$

Using the change of variable $y = \Psi_{t,\xi}(0)$ and **(G)**, we deduce that

$$\begin{aligned} \int_0^1 |S_o(\Psi_{t,\xi}(0)) - S_{o,\varepsilon}(\Psi_{t,\xi}(0))|^2 d\xi &= \int_0^1 |S_o(y) - S_{o,\varepsilon}(y)|^2 e^{\int_0^t \partial_1 g(\Psi_{0,y}(s), \bar{u}_1(s), u_2(s)) ds} dy \\ &\leq e^{\gamma t} \int_0^1 |S_o(y) - S_{o,\varepsilon}(y)|^2 dy < e^{\gamma t} \varepsilon^2. \end{aligned}$$

Similarly, we obtain that

$$\int_0^1 |S_{o,\varepsilon}(X_n(0; t, \xi)) - S_o(X_n(0; t, \xi))|^2 d\xi < e^{\gamma t} \varepsilon^2$$

and so

$$\left(\int_0^1 |S_o(\Psi_{t,\xi}(0)) - S_o(X_n(0; t, \xi))|^2 d\xi \right)^{\frac{1}{2}} \leq 2e^{\frac{\gamma t}{2}} \varepsilon + \left(\int_0^1 |S_{o,\varepsilon}(\Psi_{t,\xi}(0)) - S_{o,\varepsilon}(X_n(0; t, \xi))|^2 d\xi \right)^{\frac{1}{2}}.$$

Moreover, Lebesgue Theorem implies that

$$\lim_{n \rightarrow +\infty} \int_0^1 |S_{o,\varepsilon}(\Psi_{t,\xi}(0)) - S_{o,\varepsilon}(X_n(0; t, \xi))|^2 d\xi = 0.$$

Hence we conclude that

$$\lim_{n \rightarrow +\infty} \left(\int_0^1 |S_o(\Psi_{t,\xi}(0)) - S_o(X_n(0; t, \xi))|^2 d\xi \right)^{\frac{1}{2}} \leq 2e^{\frac{\gamma t}{2}} \varepsilon.$$

Since the previous inequality holds for every $\varepsilon > 0$, we conclude that

$$\lim_{n \rightarrow +\infty} \left(\int_0^1 |S_o(\Psi_{t,\xi}(0)) - S_o(X_n(0; t, \xi))|^2 d\xi \right)^{\frac{1}{2}} = 0. \quad (3.40)$$

By **(\alpha)**, **(F)**, **(G-1)**, (2.11), the uniform convergence of X_n and of I_n implies that the sequence of functions

$$\xi \mapsto \exp \left(- \int_0^t [f(X_n(s; t, \xi)) + \alpha(I_n(s)) + \partial_1 g(X_n(s; t, \xi), u_1^{(n)}(s), u_2(s))] ds \right)$$

converges in $\mathbf{L}^2((0, 1); \mathbb{R})$ to the function

$$\xi \mapsto \exp \left(- \int_0^t [f(\Psi_{t,\xi}(s)) + \alpha(\bar{I}(s)) + \partial_1 g(\Psi_{t,\xi}(s), \bar{u}_1(s), u_2(s))] ds \right).$$

Thus, using (3.40), we conclude that (3.39) holds, so that the claim is proved.

Since

$$I_n(t) = I_o + \int_0^t \left(-\beta I_n(\tau) + \alpha(I_n(\tau)) \int_0^1 S_n(\tau, \xi) d\xi \right) d\tau,$$

passing to the limit we obtain that

$$\bar{I}(t) = I_o + \int_0^t \left(-\beta \bar{I}(\tau) + \alpha(\bar{I}(\tau)) \int_0^1 \bar{S}(\tau, \xi) d\xi \right) d\tau,$$

that is, \bar{I} is a solution to (3.20) with data (\bar{S}, I_o) .

Therefore, the couple (\bar{S}, \bar{I}) is the solution to (1.1) with controls (\bar{u}_1, u_2) and initial data (S_o, I_o) .

Finally, using (G-1), (3.32), and (2.16), we deduce that

$$\begin{aligned} \inf_{u_1 \in \mathcal{U}_1} \mathcal{F}(S_o, I_o, u_1, u_2) &= \liminf_{n \rightarrow +\infty} \mathcal{F}(S_o, I_o, u_1^{(n)}, u_2) \\ &\geq \liminf_{n \rightarrow +\infty} \kappa \int_0^{+\infty} e^{-\theta t} (I_n(t) + u_1^{(n)}(t)) dt \\ &\quad - \lim_{n \rightarrow +\infty} \int_0^{+\infty} e^{-\theta t} \left[u_2(t) + \int_0^1 g(\xi, u_1^{(n)}(t), u_2(t)) d\xi \right] dt \\ &= \kappa \int_0^{+\infty} e^{-\theta t} (\bar{I}(t) + \bar{u}_1(t)) dt \\ &\quad - \int_0^{+\infty} e^{-\theta t} \left[u_2(t) + \int_0^1 g(\xi, \bar{u}_1(t), u_2(t)) d\xi \right] dt \\ &= \mathcal{F}(S_o, I_o, \bar{u}_1, u_2) \end{aligned}$$

ensuring that \bar{u}_1 is a minimizer of $u_1 \mapsto \mathcal{F}(S_o, I_o, u_1, u_2)$ in \mathcal{U}_1 .

The proof of Item (ii) in Theorem 2.5 is similar, thus omitted. \square

3.3 Two-player game

3.3.1 Dynamic Programming Principle

Proof of Theorem 2.9. We only give the details for V since the proof for U is similar.

Fix $T > 0$ and let us call $\tilde{V}(S_o, I_o)$ the right hand side of (2.20), i.e.

$$\tilde{V}(S_o, I_o) := \inf_{S_1 \in \Gamma} \sup_{u_2 \in \mathcal{U}_2} \left\{ \int_0^T e^{-\theta t} \ell(I_{S_1, u_2}(t), S_1(u_2)(t), u_2(t)) dt + V(S_{S_1, u_2}(T), I_{S_1, u_2}(T)) e^{-\theta T} \right\},$$

where the couple $(S_{S_1, u_2}, I_{S_1, u_2})$ denotes the solution to (1.1) with initial datum (S_o, I_o) and controls $S_1(u_2)$ and u_2 .

First we prove that $V(S_o, I_o) \leq \tilde{V}(S_o, I_o)$.

Let $\varepsilon > 0$ be fixed. Choose a strategy $\hat{S}_1 \in \Gamma$ such that

$$\tilde{V}(S_o, I_o) + \varepsilon \geq \sup_{u_2 \in \mathcal{U}_2} \left\{ \int_0^T e^{-\theta t} \ell(I_{\hat{S}_1, u_2}(t), \hat{S}_1(u_2)(t), u_2(t)) dt + e^{-\theta T} V(S_{\hat{S}_1, u_2}(T), I_{\hat{S}_1, u_2}(T)) \right\}, \quad (3.41)$$

where the couple $(S_{\hat{S}_1, u_2}, I_{\hat{S}_1, u_2})$ denotes the solution to (1.1) with initial datum (S_o, I_o) and controls $\hat{S}_1(u_2)$ and u_2 .

Fix now $u_2 \in \mathcal{U}_2$ arbitrarily. Using (3.41), we obtain that

$$\tilde{V}(S_o, I_o) \geq \int_0^T e^{-\theta t} \ell \left(I_{\hat{\mathcal{S}}_1, u_2}(t), \hat{\mathcal{S}}_1(u_2)(t), u_2(t) \right) dt + e^{-\theta T} V \left(S_{\hat{\mathcal{S}}_1, u_2}(T), I_{\hat{\mathcal{S}}_1, u_2}(T) \right) - \varepsilon. \quad (3.42)$$

Since, by Theorem 2.8,

$$V \left(S_{\hat{\mathcal{S}}_1, u_2}(T), I_{\hat{\mathcal{S}}_1, u_2}(T) \right) = \inf_{\mathcal{S}_1 \in \Gamma} \sup_{\tilde{u}_2 \in \mathcal{U}_2} \mathcal{F} \left(S_{\hat{\mathcal{S}}_1, u_2}(T), I_{\hat{\mathcal{S}}_1, u_2}(T), \mathcal{S}_1(\tilde{u}_2), \tilde{u}_2 \right),$$

then there exists $\check{\mathcal{S}}_1 \in \Gamma$ such that

$$V \left(S_{\hat{\mathcal{S}}_1, u_2}(T), I_{\hat{\mathcal{S}}_1, u_2}(T) \right) + \varepsilon \geq \mathcal{F} \left(S_{\hat{\mathcal{S}}_1, u_2}(T), I_{\hat{\mathcal{S}}_1, u_2}(T), \check{\mathcal{S}}_1(u_2(\cdot + T)), u_2(\cdot + T) \right). \quad (3.43)$$

Therefore, by (2.16), (3.42), and (3.43), we get

$$\begin{aligned} \tilde{V}(S_o, I_o) &\geq \int_0^T e^{-\theta t} \ell \left(I_{\hat{\mathcal{S}}_1, u_2}(t), \hat{\mathcal{S}}_1(u_2)(t), u_2(t) \right) dt \\ &\quad + e^{-\theta T} \mathcal{F} \left(S_{\hat{\mathcal{S}}_1, u_2}(T), I_{\hat{\mathcal{S}}_1, u_2}(T), \check{\mathcal{S}}_1(u_2(\cdot + T)), u_2(\cdot + T) \right) - \varepsilon \left(1 + e^{\theta T} \right) \\ &= \int_0^{+\infty} e^{-\theta t} \ell \left(I_{\bar{\mathcal{S}}_1, u_2}(t), \bar{\mathcal{S}}_1(u_2)(t), u_2(t) \right) dt - \varepsilon \left(1 + e^{\theta T} \right), \end{aligned} \quad (3.44)$$

where we use the non-anticipating strategy $\bar{\mathcal{S}}_1 \in \Gamma$, defined as

$$\begin{aligned} \bar{\mathcal{S}}_1 : \mathcal{U}_2 &\longrightarrow \mathcal{U}_2 \\ u_2 &\longmapsto \bar{\mathcal{S}}_1(u_2)(t) := \begin{cases} \hat{\mathcal{S}}(u_2)(t), & \text{if } t \in [0, T], \\ \check{\mathcal{S}}(u_2(\cdot + T))(t - T), & \text{if } t > T. \end{cases} \end{aligned} \quad (3.45)$$

Since u_2 is arbitrary, by (3.44) we deduce that

$$\begin{aligned} \tilde{V}(S_o, I_o) &\geq \sup_{u_2 \in \mathcal{U}_2} \left\{ \int_0^{+\infty} e^{-\theta t} \ell \left(I_{\bar{\mathcal{S}}_1, u_2}(t), \bar{\mathcal{S}}_1(u_2)(t), u_2(t) \right) dt \right\} - \varepsilon \left(1 + e^{\theta T} \right) \\ &\geq V(S_o, I_o) - \varepsilon(1 + e^{-\theta T}), \end{aligned}$$

proving that $\tilde{V}(S_o, I_o) \geq V(S_o, I_o)$ by the arbitrariness of $\varepsilon > 0$.

We now prove that $V(S_o, I_o) \geq \tilde{V}(S_o, I_o)$.

Let $\varepsilon > 0$ be fixed. Choose a strategy $\hat{\mathcal{S}}_1 \in \Gamma$ such that

$$V(S_o, I_o) \geq \sup_{u_2 \in \mathcal{U}_2} \int_0^{+\infty} e^{-\theta t} \ell \left(I_{\hat{\mathcal{S}}_1, u_2}(t), \hat{\mathcal{S}}_1(u_2)(t), u_2(t) \right) dt - \varepsilon, \quad (3.46)$$

and then a control $\hat{u}_2 \in \mathcal{U}_2$ such that

$$\tilde{V}(S_o, I_o) \leq \int_0^T e^{-\theta t} \ell \left(I_{\hat{\mathcal{S}}_1, \hat{u}_2}(t), \hat{\mathcal{S}}_1(\hat{u}_2)(t), \hat{u}_2(t) \right) dt + e^{-\theta T} V \left(S_{\hat{\mathcal{S}}_1, \hat{u}_2}(T), I_{\hat{\mathcal{S}}_1, \hat{u}_2}(T) \right) + \varepsilon, \quad (3.47)$$

where the couple $(S_{\hat{\mathcal{S}}_1, \hat{u}_2}, I_{\hat{\mathcal{S}}_1, \hat{u}_2})$ denotes the solution to (1.1) with initial datum (S_o, I_o) and controls $\hat{\mathcal{S}}_1(u_2), \hat{u}_2$. Define the map $\mathcal{C} : \mathcal{U}_2 \rightarrow \mathcal{U}_2$ such that, for all $u_2 \in \mathcal{U}_2$,

$$\mathcal{C}(u_2)(t) := \begin{cases} \hat{u}_2(t), & \text{if } t \leq T, \\ u_2(t - T), & \text{if } t > T, \end{cases} \quad (3.48)$$

and define the non-anticipating strategy $\bar{\mathcal{S}}_1 \in \Gamma$ by

$$\bar{\mathcal{S}}_1(u_2)(t) := \hat{\mathcal{S}}_1(\mathcal{C}(u_2))(t + T), \quad (3.49)$$

where $u_2 \in \mathcal{U}_2$ and $t > 0$. Moreover, fix a control $\check{u}_2 \in \mathcal{U}_2$ such that

$$V \left(S_{\hat{S}_1, \hat{u}_2}(T), I_{\hat{S}_1, \hat{u}_2}(T) \right) \leq \int_0^{+\infty} e^{-\theta t} \ell \left(\tilde{I}(t), \bar{\mathcal{S}}_1(\check{u}_2)(t), \check{u}_2(t) \right) dt + \varepsilon, \quad (3.50)$$

where (\tilde{S}, \tilde{I}) is the solution to (1.1) with initial datum $(S_{\hat{S}_1, \hat{u}_2}(T), I_{\hat{S}_1, \hat{u}_2}(T))$ and controls $\bar{\mathcal{S}}_1(\check{u}_2)$ and \check{u}_2 . By construction, we easily deduce that

$$\int_0^{+\infty} e^{-\theta t} \ell \left(\tilde{I}(t), \bar{\mathcal{S}}_1(\check{u}_2)(t), \check{u}_2(t) \right) dt = \int_T^{+\infty} e^{-\theta(t-T)} \ell \left(I_{\hat{S}_1, \mathcal{C}(\check{u}_2)}(t), \hat{\mathcal{S}}_1(\mathcal{C}(\check{u}_2))(t), \mathcal{C}(\check{u}_2)(t) \right) dt. \quad (3.51)$$

Therefore, using (3.46), (3.47), (3.48), (3.49), (3.50), and (3.51) we deduce that

$$\begin{aligned} \tilde{V}(S_o, I_o) &\leq \int_0^T e^{-\theta t} \ell \left(I_{\hat{S}_1, \hat{u}_2}(t), \hat{\mathcal{S}}_1(\hat{u}_2)(t), \hat{u}_2(t) \right) dt + e^{-\theta T} V \left(S_{\hat{S}_1, \hat{u}_2}(T), I_{\hat{S}_1, \hat{u}_2}(T) \right) + \varepsilon \\ &\leq \int_0^T e^{-\theta t} \ell \left(I_{\hat{S}_1, \hat{u}_2}(t), \hat{\mathcal{S}}_1(\hat{u}_2)(t), \hat{u}_2(t) \right) dt \\ &\quad + e^{-\theta T} \int_0^{+\infty} e^{-\theta t} \ell \left(\tilde{I}(t), \bar{\mathcal{S}}_1(\check{u}_2)(t), \check{u}_2(t) \right) dt + \varepsilon \left(1 + e^{-\theta T} \right) \\ &\leq \int_0^T e^{-\theta t} \ell \left(I_{\hat{S}_1, \mathcal{C}(\check{u}_2)}(t), \hat{\mathcal{S}}_1(\mathcal{C}(\check{u}_2))(t), \mathcal{C}(\check{u}_2)(t) \right) dt \\ &\quad + e^{-\theta T} \int_T^{+\infty} e^{-\theta(t-T)} \ell \left(I_{\hat{S}_1, \mathcal{C}(\check{u}_2)}(t), \hat{\mathcal{S}}_1(\mathcal{C}(\check{u}_2))(t), \mathcal{C}(\check{u}_2)(t) \right) dt + \varepsilon \left(1 + e^{-\theta T} \right) \\ &= \int_0^{+\infty} e^{-\theta t} \ell \left(I_{\hat{S}_1, \mathcal{C}(\check{u}_2)}(t), \hat{\mathcal{S}}_1(\mathcal{C}(\check{u}_2))(t), \mathcal{C}(\check{u}_2)(t) \right) dt + \varepsilon \left(1 + e^{-\theta T} \right) \\ &\leq \sup_{u_2 \in \mathcal{U}_2} \left\{ \int_0^{+\infty} e^{-\theta t} \ell \left(I_{\hat{S}_1, u_2}(t), \hat{\mathcal{S}}_1(u_2)(t), u_2(t) \right) dt \right\} + \varepsilon \left(1 + e^{-\theta T} \right) \\ &\leq V(S_o, I_o) + \varepsilon \left(2 + e^{-\theta T} \right). \end{aligned}$$

Take the limit as $\varepsilon \rightarrow 0^+$ to obtain the second inequality, which concludes the proof. \square

3.3.2 Hamilton-Jacobi equations

First we deduce the regularity of the Hamiltonian functions.

Lemma 3.6. *Set $\mathbf{E} := \mathbf{H}^1((0, 1); \mathbb{R}) \times \mathbb{R} \times \mathbf{L}^2((0, 1); \mathbb{R}) \times \mathbb{R}$. Assume (α) -**(F)**-**(G)**. Let $\underline{\mathcal{H}}$ and $\overline{\mathcal{H}}$ be defined by (2.23) and (2.24). Then $\underline{\mathcal{H}}, \overline{\mathcal{H}} \in \mathbf{C}^0(\mathbf{E}; \mathbb{R})$.*

Proof. It suffices to check that for all $u_1 \in [0, M_1]$ and $u_2 \in [0, M_2]$, the pre-Hamiltonian \mathcal{H} , defined by (2.22), is continuous with respect to its first two variables. To show it, we prove that \mathcal{H} is Lipschitz continuous on bounded subsets of \mathbf{E} . Let Ω be a bounded subset of \mathbf{E} and let $c \in \mathbb{R}^+$ such that

$$\forall (S, I, p, q) \in \Omega, \quad \max\{\|S\|_{\mathbf{H}^1}, |I|, \|p\|_{\mathbf{L}^2}, |q|\} \leq c.$$

Let $u_1 \in [0, M_1]$ and $u_2 \in [0, M_2]$. For all $(\mathbf{Y}_1, \mathbf{P}_1), (\mathbf{Y}_2, \mathbf{P}_2) \in \Omega$, we have

$$\begin{aligned}
& |\mathcal{H}(\mathbf{Y}_2, \mathbf{P}_2, u_1, u_2) - \mathcal{H}(\mathbf{Y}_1, \mathbf{P}_1, u_1, u_2)| \\
& \leq |\langle \partial_\xi(gS_2) + (f + \alpha(I_2))S_2, p_2 - p_1 \rangle_{\mathbf{L}^2}| + |\langle \partial_\xi(gS_2 - gS_1) + f(S_2 - S_1), p_1 \rangle_{\mathbf{L}^2}| \\
& \quad + |\langle \alpha(I_2)(S_2 - S_1), p_1 \rangle_{\mathbf{L}^2}| + |\langle (\alpha(I_2) - \alpha(I_1))S_1, p_1 \rangle_{\mathbf{L}^2}| \\
& \quad + \beta|I_2 - I_1| \cdot |q_1| + \beta|I_1| \cdot |q_2 - q_1| + \left| \alpha(I_2)q_2 \int_0^1 (S_2(\xi) - S_1(\xi)) \, d\xi \right| \\
& \quad + \alpha(I_2)|q_2 - q_1| \cdot \left| \int_0^1 S_1(\xi) \, d\xi \right| + |\alpha(I_2) - \alpha(I_1)| \cdot \left| q_1 \int_0^1 S_1(\xi) \, d\xi \right| + |\ell(I_2, u_1, u_2) - \ell(I_1, u_1, u_2)| \\
& \leq \left(\gamma + \sup_{\xi \in [0,1]} |g(\xi, u_1, u_2)| + 1 + \sup_{I \leq c} \alpha(I) \right) \|S_2\|_{\mathbf{H}^1} \|p_2 - p_1\|_{\mathbf{L}^2} \\
& \quad + \left(\gamma + \sup_{\xi \in [0,1]} |g(\xi, u_1, u_2)| + 1 \right) \|S_2 - S_1\|_{\mathbf{H}^1} \|p_1\|_{\mathbf{L}^2} + \sup_{I \leq c} \alpha(I) \|S_2 - S_1\|_{\mathbf{L}^2} \|p_1\|_{\mathbf{L}^2} \\
& \quad + \mathbf{Lip}(\alpha)|I_2 - I_1| \cdot \|S_1\|_{\mathbf{L}^2} \|p_1\|_{\mathbf{L}^2} + \beta|I_2 - I_1| \cdot |q_1| + \beta|I_1| \cdot |q_2 - q_1| + \sup_{I \leq c} \alpha(I)|q_2| \cdot \|S_2 - S_1\|_{\mathbf{L}^2} \\
& \quad + \sup_{I \leq c} \alpha(I)|q_2 - q_1| \cdot \|S_1\|_{\mathbf{L}^2} + \mathbf{Lip}(\alpha)|I_2 - I_1| \cdot |q_1| \cdot \|S_1\|_{\mathbf{L}^2} + \kappa|I_2 - I_1| \\
& \leq C \max\{\|S_2 - S_1\|_{\mathbf{H}^1}, |I_2 - I_1|, \|p_2 - p_1\|_{\mathbf{L}^2}, |q_2 - q_1|\},
\end{aligned}$$

concluding the proof. \square

We now tackle the proof of Theorem 2.11, which is a direct adaptation of [4, Theorem 1.10, Chapter VIII] to our infinite dimensional setting. The following two results are of use below.

Lemma 3.7. *Assume (α) -**(F)**-**(G)**. Let $u_1 \in \mathcal{U}_1$, $u_2 \in \mathcal{U}_2$ and $\mathbf{Y}_o = (S_o, I_o) \in \mathbf{H}^1((0, 1); \mathbb{R}) \times \mathbb{R}$. Denote by $\mathbf{Y} = (S, I)$ be the solution of (1.1) with initial datum \mathbf{Y}_o and controls u_1, u_2 . Then, for all $\phi \in \mathbf{C}^1(\mathbf{L}^2((0, 1); \mathbb{R}) \times \mathbb{R}; \mathbb{R})$ and for all $t > 0$,*

$$\begin{aligned}
\frac{d}{dt} \phi(\mathbf{Y}(t)) &= - \langle \nabla_1 \phi(\mathbf{Y}(t)), \partial_\xi(gS(t)) + (f(\xi) + \alpha(I(t)))S(t) \rangle_{\mathbf{L}^2} \\
& \quad + \nabla_2 \phi(\mathbf{Y}(t)) \left(-\beta I(t) + \alpha(I(t)) \int_0^1 S(t, \xi) \, d\xi \right),
\end{aligned}$$

where we denoted by $\nabla_1 \phi$ and $\nabla_2 \phi$ the gradient w.r.t. the first variable of ϕ and, respectively, w.r.t. the second variable of ϕ .

The proof is standard and so we omit it.

Lemma 3.8. [4, Lemma 1.11, Chapter VIII]. *Let $\underline{\mathcal{H}}$ be defined by (2.23). Let $\mathbf{Y}_o \in \mathbf{H}^1((0, 1); \mathbb{R}) \times \mathbb{R}$ and $\phi \in \mathbf{C}^1(\mathbf{L}^2((0, 1); \mathbb{R}) \times \mathbb{R})$ be such that*

$$\nu := \theta \phi(\mathbf{Y}_o) + \underline{\mathcal{H}}(\mathbf{Y}_o, \nabla \phi(\mathbf{Y}_o)) > 0.$$

Then, there exists $\mathcal{S}_1^* \in \Gamma$ and $\tau > 0$ such that for all $u_2 \in \mathcal{U}_2$ and for all $t \in (0, \tau)$,

$$\int_0^t \ell(I(s), \mathcal{S}_1^*(u_2), u_2) e^{-\theta s} \, ds + \phi(\mathbf{Y}(t)) e^{-\theta t} - \phi(\mathbf{Y}_o) \leq -\frac{\nu t}{4},$$

where $\mathbf{Y} = (S, I)$ is the solution of (1.1) with the initial datum \mathbf{Y}_o and controls $(\mathcal{S}_1^*(u_2), u_2)$.

The proof is exactly the same of that of [4, Lemma 1.11, Chapter VIII] even if the domain here is infinite dimensional.

Proof of Theorem 2.11. We only prove the first item of Theorem 2.11, the second one being completely similar.

Regularity. $V \in \mathbf{C}^0(\mathbf{H}^1((0, 1); \mathbb{R}) \times \mathbb{R}; \mathbb{R})$. It suffices to prove that for all $\mathcal{S}_1 \in \Gamma$ and for all $u_2 \in \mathcal{U}_2$, \mathcal{F} defined by (2.16) is continuous with respect to its first two arguments. Fix $(\bar{S}_o, \bar{I}_o) \in \mathbf{L}^2((0, 1); \mathbb{R}) \times \mathbb{R}^+$ and consider a sequence (S_o^n, I_o^n) converging to (\bar{S}_o, \bar{I}_o) in $\mathbf{L}^2((0, 1); \mathbb{R}) \times \mathbb{R}^+$. We claim that

$$\lim_{n \rightarrow +\infty} \mathcal{F}(S_o^n, I_o^n, \mathcal{S}_1(u_2), u_2) = \mathcal{F}(\bar{S}_o, \bar{I}_o, \mathcal{S}_1(u_2), u_2).$$

Denote by (S^n, I^n) , resp. (\bar{S}, \bar{I}) , the solution to (1.1) with initial datum (S_o^n, I_o^n) , resp. (\bar{S}_o, \bar{I}_o) , and controls $\mathcal{S}_1(u_2), u_2$. By (2.17), it is sufficient to prove that

$$\lim_{n \rightarrow +\infty} \int_0^{+\infty} e^{-\theta t} I^n(t) dt = \int_0^{+\infty} e^{-\theta t} \bar{I}(t) dt. \quad (3.52)$$

First note that Theorem 2.4 implies that the sequence I^n converges pointwise to \bar{I} .

Fix now an arbitrary subsequence I^{n_h} . By Theorem 2.3 we deduce that there exists $\bar{n} \in \mathbb{N}$ such that, for every $n \geq \bar{n}$ and $t > 0$,

$$0 \leq I^n(t) \leq \|S_o^n\|_{\mathbf{L}^1((0,1); \mathbb{R})} + I_o^n \leq \|S_o^n\|_{\mathbf{L}^2((0,1); \mathbb{R})} + I_o^n \leq \|\bar{S}_o\|_{\mathbf{L}^2((0,1); \mathbb{R})} + \bar{I}_o + 1.$$

Therefore we deduce the existence of an additional subsequence, say $I^{n_{h_k}}$, and a function \tilde{I} such that $I^{n_{h_k}}$ converges to \tilde{I} in the weak* topology of $\mathbf{L}^\infty((0, +\infty); \mathbb{R})$. Let $\varphi \in \mathbf{L}^1((0, +\infty); \mathbb{R})$; then

$$\int_0^{+\infty} \varphi(t) \tilde{I}(t) dt = \lim_{k \rightarrow +\infty} \int_0^{+\infty} \varphi(t) I^{n_{h_k}}(t) dt = \int_0^{+\infty} \varphi(t) \bar{I}(t) dt,$$

where the last equality is due to the Lebesgue Dominated Convergence Theorem. This implies, by the arbitrariness of φ , that $\bar{I}(t) = \tilde{I}(t)$ for a.e. $t > 0$. The Uryshon subsequence principle permits to deduce that the whole sequence $I^n(t)$ weakly* converges to $\bar{I}(t)$ in $\mathbf{L}^\infty((0, +\infty); \mathbb{R})$. Therefore (3.52) holds, concluding the proof of the continuity of V .

Sub-solution. Let $\phi \in \mathbf{C}^1(\mathbf{L}^2((0, 1); \mathbb{R}) \times \mathbb{R}; \mathbb{R})$ and $\mathbf{Y}_o \in \mathbf{H}^1((0, 1); \mathbb{R}) \times \mathbb{R}$ a local maximum point of $V - \phi$. Without loss of generality, we can assume that $V(\mathbf{Y}_o) - \phi(\mathbf{Y}_o) = 0$. We want to show that

$$\theta V(\mathbf{Y}_o) + \underline{\mathcal{H}}(\mathbf{Y}_o, \nabla \phi(\mathbf{Y}_o)) \leq 0.$$

Suppose instead that $\nu := \theta V(\mathbf{Y}_o) + \underline{\mathcal{H}}(\mathbf{Y}_o, \nabla \phi(\mathbf{Y}_o)) > 0$. Use Theorem 3.8 to find $\mathcal{S}_1^* \in \Gamma$ and $\tau > 0$ such that for all controls $u_2 \in \mathcal{U}_2$ and for all $t \in (0, \tau)$,

$$\int_0^t \ell(I(s), \mathcal{S}_1^*(u_2), u_2) e^{-\theta s} ds + \phi(\mathbf{Y}(t)) e^{-\theta t} - \phi(\mathbf{Y}_o) \leq -\frac{\nu t}{4}, \quad (3.53)$$

where $\mathbf{Y} = (S, I)$ is the solution of (1.1) with initial datum \mathbf{Y}_o and controls $(\mathcal{S}_1^*(u_2), u_2)$.

Notice that since \mathbf{Y}_o is a local maximum of $V - \phi$, for all $t \in]0, \tau[$, we have

$$V(\mathbf{Y}(t)) - \phi(\mathbf{Y}(t)) \leq 0 \implies V(\mathbf{Y}(t)) e^{-\theta t} \leq \phi(\mathbf{Y}(t)) e^{-\theta t},$$

which inserted in (3.53) leads to

$$\int_0^t \ell(I(s), \mathcal{S}_1^*(u_2), u_2) e^{-\theta s} ds + V(\mathbf{Y}(t)) e^{-\theta t} - V(\mathbf{Y}_o) \leq -\frac{\nu t}{4}.$$

This contradicts (2.20), therefore, V is a viscosity sub-solution.

Super-solution. Let $\phi \in \mathbf{C}^1(\mathbf{L}^2((0, 1); \mathbb{R}) \times \mathbb{R}; \mathbb{R})$ and $\mathbf{Y}_o = (S_o, I_o) \in \mathbf{H}^1((0, 1); \mathbb{R}) \times \mathbb{R}$ a local minimum point of $V - \phi$. Without loss of generality, we can assume that $V(\mathbf{Y}_o) - \phi(\mathbf{Y}_o) = 0$. We want to show that

$$\theta V(\mathbf{Y}_o) + \underline{\mathcal{H}}(\mathbf{Y}_o, \nabla \phi(\mathbf{Y}_o)) \geq 0.$$

Suppose instead that $\theta V(\mathbf{Y}_o) + \underline{H}(\mathbf{Y}_o, \nabla \phi(\mathbf{Y}_o)) =: -\nu < 0$. Consequently, there exists $u_2^* \in [0, M_2]$ such that for all $u_1 \in [0, M_1]$,

$$\begin{aligned} & \theta \phi(\mathbf{Y}_o) + \langle \partial_\xi (g(\cdot, u_1, u_2^*) S_o) + (f(\xi) + \alpha(I_o)) S_o, \nabla_1 \phi(\mathbf{Y}_o) \rangle_{\mathbf{L}^2} \\ & + \left(\beta I_o - \alpha(I_o) \int_0^1 S_o \right) \nabla_2 \phi(\mathbf{Y}_o) - \ell(I_o, u_1, u_2^*) \leq -\nu. \end{aligned}$$

For all strategies $\mathcal{S}_1 \in \Gamma$, for all $t > 0$ small enough, we have

$$\begin{aligned} & \theta \phi(\mathbf{Y}(t)) + \langle \partial_\xi (g(\cdot, \mathcal{S}_1(u_2^*), u_2^*) S(t)) + (f(\xi) + \alpha(I(t))) S(t), \nabla_1 \phi(\mathbf{Y}(t)) \rangle_{\mathbf{L}^2} \\ & + \left(\beta I_o - \alpha(I(t)) \int_0^1 S(t) \right) \nabla_2 \phi(\mathbf{Y}(t)) - \ell(I(t), \mathcal{S}_1(u_2^*), u_2^*) \leq -\nu, \end{aligned}$$

where $\mathbf{Y} = (S, I)$ is the solution to (1.1) with initial datum \mathbf{Y}_o and controls $(\mathcal{S}_1^*(u_2), u_2)$. In view of Theorem 3.7, the previous inequality can be re-written, for all $t > 0$ small enough, as

$$\theta \phi(\mathbf{Y}(t)) - \frac{d}{dt} \phi(\mathbf{Y}(t)) - \ell(I(t), \mathcal{S}_1(u_2^*), u_2^*) \leq -\nu.$$

Multiply by $e^{-\theta t}$ and integrate:

$$\phi(\mathbf{Y}(t)) e^{-\theta t} - \phi(\mathbf{Y}_o) + \int_0^t \ell(I(s), \mathcal{S}_1(u_2^*), u_2^*) e^{-\theta s} ds \geq \frac{\nu}{\theta} (1 - e^{-\theta t}). \quad (3.54)$$

Notice that since \mathbf{Y}_o is a local minimum of $V - \phi$, we have for all $t > 0$ small enough,

$$V(\mathbf{Y}(t)) - \phi(\mathbf{Y}(t)) \geq 0 \implies V(\mathbf{Y}(t)) e^{-\theta t} \geq \phi(\mathbf{Y}(t)) e^{-\theta t}.$$

We insert it in (3.54) to obtain, for all $t > 0$ small enough,

$$V(\mathbf{Y}(t)) e^{-\theta t} + \int_0^t \ell(I(s), \mathcal{S}_1(u_2^*), u_2^*) e^{-\theta s} ds \geq V(\mathbf{Y}_o) + \frac{\nu}{\theta} (1 - e^{-\theta t}) > V(\mathbf{Y}_o),$$

which contradicts (2.20). This shows that V is a viscosity super-solution. \square

Acknowledgments

MG and ER were partially supported by the 2024 INdAM GNAMPA project *Modelling and Analysis through Conservation Laws*. The authors acknowledge the PRIN 2022 project *Modeling, Control and Games through Partial Differential Equations* D53D23005620006, funded by the European Union-Next Generation EU.

MG was partially supported by the Project funded under the National Recovery and Resilience Plan (NRRP) of Italian Ministry of University and Research funded by the European Union (NextGenerationEU Award, No. CN000023, Concession Decree No. 1033 of 17/06/2022 adopted by the Italian Ministry of University and Research, CUP: H43C22000510001, Centro Nazionale per la Mobilità Sostenibile).

MG was also partially supported by the Project funded under the National Recovery and Resilience Plan (NRRP) of Italian Ministry of University and Research funded by the European Union (NextGenerationEU Award, No. ECS_00000037, CUP H43C22000510001, MUSA – Multilayered Urban Sustainability Action).

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