Kähler-Einstein metrics with positive curvature near an isolated log terminal singularity

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Abstract

We analyze the existence of Kähler-Einstein metrics of positive curvature in the neighborhood of a germ of a log terminal singularity (X,p). This boils down to solve a Dirichlet problem for certain complex Monge-Ampère equations. We establish a Moser-Trudinger inequality $(MT)_\gamma$ in subcritical regimes $\gamma < \gamma_{crit}(X,p)$ and show the existence of smooth solutions in that cases. We show that the expected critical exponent $\tilde{\gamma}_{crit}(X,p) = \frac{n+1}{n}\widehat{\mathrm{vol}}(X,p)^{1/n}$ can be expressed in terms of the normalized volume, an important algebraic invariant of the singularity.

Contents

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Introduction

Let (X,p) be a germ of an isolated singularity. We analyze the existence of local Kähler-Einstein metrics of positive curvature in a neighborhood of p. It follows from [BBEGZ19, Proposition 3.8] that the singularity has to be *log terminal*, a relatively mild type of singularity that plays a central role in birational geometry. We refer the reader to Definition 1.8 for a precise formulation and simply indicate here that a prototypical example is the vertex of the affine cone over a Fano manifold. Consider indeed

$$X = \{z \in \mathbb{C}^{n+1},\, P(z) = 0\},$$

P a homogeneous polynomial of degree $d \in \mathbb{N}^*$ so that $H = \{[z] \in \mathbb{CP}^n,\, P(z) = 0\}$ is a smooth hypersurface of the complex projective space. Then $(X,0)$ is log terminal if and only if H is Fano (which is equivalent here to $d < n+1$). Thus log terminal singularities can be seen as a local analogue of Fano varieties.

Given a local embedding $(X,p) \hookrightarrow (\mathbb{C}^N,0)$, constructing such a local Kähler-Einstein metric boils down to solve a complex Monge-Ampère equation

$$(MA)_{\gamma,\phi,\Omega} \begin{cases} (dd^c\varphi)^n = \frac{e^{-\gamma\varphi}d\mu_p}{\int_\Omega e^{-\gamma\varphi}d\mu_p} \\ \varphi_{|\partial\Omega} = \phi, \end{cases}$$

where Ω is a smooth neighborhood of p, ϕ is a smooth boundary data, μ_p is an adapted volume form (see Definition 1.9), and $\gamma > 0$ is a parameter. We seek for a solution $\varphi \in \mathcal{C}^\infty\big(\Omega\setminus\{p\}\big)\cap\mathcal{C}^0(\overline{\Omega})$ which is strictly plurisubharmonic in $\Omega \setminus \{p\}$, so that $\omega_{KE} := dd^c\varphi$ is a Kähler form in $\Omega \setminus \{p\}$ satisfying the Einstein equation

$$\mathrm{Ric}(\omega_{KE}) = \gamma\omega_{KE}.$$

An important motivation comes from the global study of positively curved Kähler-Einstein metrics ω_{KE} on $\mathbb{Q}$-Fano varieties. Such canonical singular metrics have been constructed in [BBEGZ19] and further studied in [BBJ21, LTW21, Li22], extending the resolution of the Yau-Tian-Donaldson conjecture [CDS15] to this singular context. Despite recent important progress [HS17, Dr18, HP19, BGL22], the geometry of these singular metrics remains mysterious and one needs to better understand the asymptotic behaviour of ω_{KE} near the singularities.

We restrict the metric ω_{KE} to a neighborhood of p and wish to analyze the behavior of its local potentials $\omega_{KE} = dd^c\varphi_{KE}$ near p. The latter solve a Monge-Ampère equation $(MA)_{\gamma,\phi,\Omega}$, as can be seen by locally trivializing a representative of the first Chern class (after an appropriate rescaling). The boundary data is thus given by the solution $\varphi_{KE} = \phi$ itself.

Studying the family of equations $(MA)_{\gamma,\phi,\Omega}$ we will give evidence that:

- the possibility of solving $(MA)_{\gamma,\phi,\Omega}$ should be independent of Ω and ϕ;
- the largest exponent $\gamma_{crit}(X,p)$ for which we can solve $(MA)_{\gamma,\phi,\Omega}$ should only depend on the algebraic nature of the log-terminal singularity;

Following earlier works dealing with the case of compact Kähler varieties or the local smooth setting [BBGZ13, GKY13, BB22, BBEGZ19], we develop a variational approach to solve these equations. A crucial role is played by

$$E_\phi(\varphi) = \frac{1}{n+1}\sum_{j=0}^{n}\int_\Omega (\varphi-\phi_0)(dd^c\varphi)^j \wedge (dd^c\phi_0)^{n-j},$$

the Monge-Ampère energy of φ relative to a plurisubharmonic extension ϕ_0 of ϕ. This energy is a primitive of the Monge-Ampère operator and a building block of the functional F_γ whose Euler-Lagrange equation is $(MA)_{\gamma,\phi,\Omega}$,

$$\varphi \in \mathcal{T}_\phi(\Omega) \mapsto F_\gamma(\varphi) = E_\phi(\varphi) + \frac{1}{\gamma}\log \int_\Omega e^{-\gamma\varphi} d\mu_p \in \mathbb{R},$$

Here $\mathcal{T}_\phi(\Omega)$ denotes the set of all plurisubharmonic functions φ in Ω which are continuous on $\overline{\Omega}$ and such that $\varphi_{|\partial\Omega} = \phi$.

In order to solve $(MA)_{\gamma,\phi,\Omega}$ one can try and extremize F_γ by showing that it is a proper functional. Our first main result in this direction (Theorem 4.1) is the following Moser-Trudinger type inequality.

THEOREM A. For any $0 < \gamma < \frac{n+1}{n}\alpha(X,\mu_p)$, there exists $C_\gamma > 0$ such that

$$\left(\int_\Omega e^{-\gamma\varphi} d\mu_p\right)^{\frac{1}{\gamma}} \leqslant C_\gamma \exp\left(-E_\phi(\varphi)\right). \tag{MT_γ}$$

for all $\varphi \in \mathcal{T}_\phi(\Omega)$.

The *alpha invariant* of the singularity (X,p) is defined by

$$\alpha(X,\mu_p) := \sup\left\{\alpha > 0,\ \sup_{\varphi\in\mathcal{F}_1(\Omega)} \int_\Omega e^{-\alpha\varphi} d\mu_p < +\infty\right\}.$$

where $\mathcal{F}_1(\Omega)$ denotes the set of plurisubharmonic functions φ with ϕ-boundary values, whose Monge-Ampère mass is bounded by $\int_\Omega (dd^c\varphi)^n \leqslant 1$.

When (X,p) is smooth, Theorem A has been obtained independently in [BB22, GKY13] with $\alpha(X,\mu_p) = n$ (the normalizations and methods are quite different in these two works, but they eventually produce the same critical exponent).

We introduce

$$\gamma_{crit}(X,p) := \sup\{\gamma > 0 \text{ such that } (MT_\gamma) \text{ holds}\}.$$

While Theorem A provides a lower bound for $\gamma_{crit}(X,p)$, we provide an upper bound in Theorem 3.5, which yields

$$\frac{n+1}{n}\alpha(X,\mu_p) \leqslant \gamma_{crit}(X,p) \leqslant \frac{n+1}{n}\widehat{\mathrm{vol}}(X,p)^{1/n},$$

where $\widehat{\mathrm{vol}}(X,p)$ denotes the *normalized volume* of the singularity (X,p). This is an algebraic invariant of the singularity at p introduced by Chi Li in [Li18], which has recently played a key role in the algebraic understanding of the moduli space of K-stable Fano varieties (see [Blum18, Liu18, LWX21, LXZ22] and the references therein); we refer to Definition 1.15 for a precise definition.

When p is smooth then $\alpha(X,\mu_p) = \widehat{\mathrm{vol}}(X,p)^{1/n} = n$ by [ACKPZ09, Dem09]. It is tempting to conjecture that the equality $\alpha(X,\mu_p) = \widehat{\mathrm{vol}}(X,p)^{1/n}$ always holds. We establish in Section 4 the following partial bounds on $\alpha(X,\mu_p)$.

THEOREM B. The following inequalities hold

$$\frac{n}{\mathrm{mult}(X,p)^{1-1/n}}\frac{\mathrm{lct}(X,p)}{1+\mathrm{lct}(X,p)} \leqslant \alpha(X,\mu_p) \leqslant \widehat{\mathrm{vol}}(X,p)^{1/n}.$$

Moreover $\alpha(X,\mu_p) = \widehat{\mathrm{vol}}(X,p)^{1/n}$ if (X,p) is an admissible singularity.

Here $\mathrm{mult}(X,p)$ denotes the algebraic multiplicity of (X,p), while $\mathrm{lct}(X,p)$ is its log canonical threshold (see Definition 1.12). Bounded $\alpha(X,\mu_p)$ from below is quite involved; we show that $\alpha(X,\mu_p)=\widehat{\mathrm{vol}}(X,p)^{1/n}$ when $n=2$, but our lower-bound is not sharp when $n\geqslant 3$ unless (X,p) is an *admissible singularity*, a notion introduced in [LTW21]. The vertex of the affine cone over a smooth Fano manifold is an example of admissible singularity (see Section 4).

Using analytic Green functions and Demailly's comparison theorem, we provide in Propositions 4.6, 4.8 evidence for the equality $\alpha(X,\mu_p)=\widehat{\mathrm{vol}}(X,p)^{1/n}$. The Appendix, written by S.Boucksom, uses an algebraic approach based on [BdFF12], to establish a stronger result than Proposition 4.8.

We note in Lemma 2.13 that if (MT_γ) holds, then F_γ is *coercive* (a strong quantitative version of properness). When $\gamma<\gamma_{crit}(X,p)$, we then further show the existence of smooth solutions to $(MA)_{\gamma,\phi,\Omega}$.

Theorem C. If $\gamma<\gamma_{crit}(X,p)$ then there exists a plurisubharmonic function $\varphi\in\mathcal{C}^\infty(\Omega\setminus\{p\})$ which is continuous in $\overline{\Omega}$ with $\varphi_{|\partial\Omega}=\phi$, and such that

$$(dd^c\varphi)^n=\frac{e^{-\gamma\varphi}d\mu_p}{\int_\Omega e^{-\gamma\varphi}d\mu_p}\quad\text{in}\quad\Omega.$$

We expect the solution to be unique, at least when Ω is a generic Stein neighborhood of p. We refer the reader to [GKY13, BB22] for partial results in this direction when p is a smooth point.

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1. Preliminaries

1.1 Analysis on singular spaces

Let X be a reduced complex analytic space of pure dimension $n\geqslant 1$. We let X_{reg} denote the complex manifold of regular points of X and $X_{\mathrm{sing}}:=X\setminus X_{\mathrm{reg}}$ be the set of singular points; this is an analytic subset of X of complex codimension $\geqslant 1$. We always assume in this article that:

- $X_{\mathrm{sing}}=\{p\}$ consists of a single isolated point;
- X_{reg} is locally irreducible at p;
- U is a fixed neighborhood of p and $j:U\hookrightarrow\mathbb{C}^N$ is a local embedding onto an analytic subset of $\mathbb{C}^N$ for some $N\geqslant 1$.

As we are interested in the asymptotic behavior of Kähler-Einstein potentials near the singular point p, we shall identify X with $j(U)$ in the sequel.

1.1.1 *Plurisubharmonic functions* Using the local embedding j, it is possible to define the spaces of smooth forms on X as restriction of smooth forms of $\mathbb{C}^N$. The notion of currents on X is defined by duality; the operators ∂ and $\bar\partial$, d, d^c and dd^c are also well defined by duality (see [Dem85] for more details).

Here $d = \partial + \overline{\partial}$ and $d^c = \frac{1}{4i\pi}(\partial - \overline{\partial})$ are real operators and $dd^c = \frac{i}{2\pi}\partial\overline{\partial}$. With this normalization the function $z \in \mathbb{C}^n \mapsto \rho_{FS}(z) = \log[1 + |z|^2] \in \mathbb{R}$ is smooth and plurisubharmonic in $\mathbb{C}^n$, with

$$\int_{\mathbb{C}^n} (dd^c \rho_{FS})^n = 1.$$

Definition 1.1. We say that a function $u : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic (psh for short) on X if it is the restriction of a plurisubharmonic function of $\mathbb{C}^N$.

We let $PSH(X)$ denote the set of all plurisubharmonic functions on X that are not identically $-\infty$.

Recall that u is called weakly plurisubharmonic on X if it is locally bounded from above on X and its restriction to $X_{\rm reg}$ is plurisubharmonic. One can extend it to X by $u^*(p) := \limsup_{X_{\rm reg} \ni y \to p} u(y)$. Since X is locally irreducible, it follows from the work of Fornæss-Narasimhan [FN80] that u is weakly plurisubharmonic if and only if u^* is plurisubharmonic (see [Dem85, Corollary 1.11]).

If $u \in PSH(X)$, then u is upper semi-continuous on X and locally integrable with respect to the volume form

$$dV_X := \omega_{eucl}^n \wedge [X].$$

Here $[X]$ denotes the current of integration along X and $\omega_{eucl} := \sum_{j=1}^N i dz_j \wedge d\overline{z_j}$ is the euclidean Kähler form. In particular $dd^c u$ is a well defined current of bidegree $(1,1)$ which is positive.

1.1.2 *Pseudoconvex domains and boundary data* Following [FN80] we say that X is Stein if it admits a $\mathcal{C}^2$-smooth strongly plurisubharmonic exhaustion.

Definition 1.2. A domain $\Omega \Subset X$ is strongly pseudoconvex if it admits a negative $\mathcal{C}^2$-smooth strongly plurisubharmonic exhaustion, i.e. a function ρ strongly plurisubharmonic in a neighborhood Ω' of $\overline{\Omega}$ such that $\Omega := \{x \in \Omega' \,;\, \rho(x) < 0\}$, $d\rho \neq 0$ on $\partial\Omega$, and for any $c < 0$,

$$\Omega_c := \{x \in \Omega';\, \rho(x) < c\} \Subset \Omega$$

is relatively compact.

We are interested in solving a Dirichlet problem for some complex Monge-Ampère equations in a bounded strongly pseudoconvex domain $\Omega = \{\rho < 0\}$, with given boundary data $\phi \in \mathcal{C}^\infty(\partial\Omega)$.

Definition 1.3. Given $\phi \in \mathcal{C}^\infty(\partial\Omega)$, we fix ϕ_0 a plurisubharmonic function in Ω which is $\mathcal{C}^\infty$-smooth near $\overline{\Omega}$ and such that $\phi_0|\partial\Omega = \phi$.

Such an extension can be obtained as follows: we pick $\tilde{\phi}$ an arbitrary $\mathcal{C}^2$-smooth extension to $\overline{\Omega}$, and then consider $\phi_0 := \tilde{\phi} + A\rho$, for A so large that ϕ_0 is $\mathcal{C}^2$-smooth and plurisubharmonic in $\overline{\Omega}$. All quantities introduced in the remainder of the paper are essentially independent of the particular choice of the extension.

1.1.3 *Monge-Ampère operators* The complex Monge-Ampère operator $(dd^c \cdot)^n$ acts on a smooth psh functions φ. When $X = \mathbb{C}^n$, it boils down to

$$(dd^c\varphi)^n = c_n \det\left(\frac{\partial^2\varphi}{\partial z_j \overline{\partial} z_k}\right) \omega_{eucl}^n,$$

where $c_n > 0$ is a normalizing constant.

Bounded functions Following [BT82] this operator can be extended to the class $PSH(X) \cap L^\infty_{loc}$ by using approximation by smooth psh functions: given $\varphi \in PSH(X) \cap L^\infty_{loc}$, there exists a unique positive Radon measure μ_φ on X such that for *any* sequence (φ_j) of smooth psh functions decreasing to φ, one has

$$\mu_\varphi = \lim (dd^c \varphi_j)^n,$$

where the limit holds in the weak sense. One then sets $(dd^c\varphi)^n := \mu_\varphi$.

DEFINITION 1.4. We set

$$\mathcal{T}^\infty_\phi(\Omega) := \Big\{ \varphi \in SPSH(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega}) \,:\, \varphi_{|\partial\Omega} = \phi \Big\},$$

where $SPSH(\Omega)$ is the set of strictly plurisubharmonic functions, and

$$\mathcal{T}_\phi(\Omega) := \Big\{ \varphi \in PSH(\Omega) \cap \mathcal{C}^0(\overline{\Omega}) \,:\, \varphi_{|\partial\Omega} = \phi,\, \int_\Omega (dd^c\varphi)^n < +\infty \Big\},$$

This latter class has been introduced by Cegrell in [Ceg98]; it can be used as a psh version of test functions (in the sense of distributions), as well as a building block for finite energy classes of mildly unbounded functions.

LEMMA 1.5. *Any $\varphi \in \mathcal{T}_\phi(\Omega)$ is a quasi-decreasing limit of functions in $\mathcal{T}^\infty_\phi(\Omega)$.*

Proof. Fix a local embedding $X \hookrightarrow \mathbb{C}^N$. A function $\varphi \in \mathcal{T}_\phi(\Omega)$ is the restriction of an ambient continuous psh function ψ. We use standard convolution in $\mathbb{C}^N$ to find a sequence of smooth strictly psh functions ψ_j decreasing to ψ. Consider $\varphi_j = \psi_{j|X} - \varepsilon_j$, where $0 < \varepsilon_j$ goes to zero so that $\varphi_j < \phi_0$ near $\partial\Omega$ (the functions $\psi_{j|X}$ uniformly converge to φ by continuity). Set $\tilde{\varphi}_j := \tilde{\max}(\varphi_j, A_j\rho + \phi_0)$, where $\tilde{\max}$ is a regularized maximum, then $\varphi_j \in \mathcal{T}^\infty_\phi(\Omega)$ converges to φ as $A_j \to +\infty$. □

Midly unbounded functions The complex Monge-Ampère operator can be defined for mildly unbounded psh functions. We refer the reader to [Ceg04, Blo06] for the case of smooth domains in $\mathbb{C}^n$; their analysis easily extends to our context.

DEFINITION 1.6. We let $\mathcal{F}(\Omega)$ denote the set of all functions $\varphi \in PSH(\Omega)$ which are decreasing limit of a sequence of functions $\varphi_j \in \mathcal{T}_\phi(\Omega)$ such that

$$\sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty.$$

The operator $(dd^c\cdot)^n$ is well defined on $\mathcal{F}(\Omega)$, continuous along monotonic sequences, and yields Radon measures $(dd^c\varphi)^n$ which have finite mass in Ω. We endow $\mathcal{F}(\Omega)$ with the L^1-topology. Let us stress that the operator $\varphi \mapsto (dd^c\varphi)^n$ is *not* continuous for the L^1-topology, but the class $\mathcal{F}(\Omega)$ enjoys the following useful compactness property.

PROPOSITION 1.7. *The set $\mathcal{F}_1(\Omega) = \big\{\varphi \in \mathcal{F}(\Omega);\, \int_\Omega (dd^c\varphi)^n \leqslant 1\big\}$ is compact.*

This is shown in [Zer09, Observation A.3] for smooth domains, and the same proof applies in our midly singular context. Let us stress that the Monge-Ampère operator cannot be defined for all psh functions: there is e.g. no reasonable way to make sense of $(dd^c \log|z_1|)^n$. A consequence of Proposition 1.7 is that one can not approximate such a function by a decreasing sequence of psh functions with prescribed boundary values and uniformly bounded Monge-Ampère masses.

1.2 Adapted volume form

1.2.1 *Log terminal singularities* Let Y be a connected normal complex variety such that K_Y is $\mathbb{Q}$-Cartier near $p \in Y$. One can consider the dd^c-cohomology class of $-K_Y$, denoted by $c_1(Y)$.

Given a log-resolution $\pi : \tilde{Y} \to Y$ of (Y, p), there exists a unique $\mathbb{Q}$-divisor $\sum_i a_i E_i$ whose push-forward to Y is 0 and with

$$K_{\tilde{Y}} = \pi^*(K_Y) + \sum_i a_i E_i.$$

Definition 1.8. The coefficient $a_i \in \mathbb{Q}$ is the *discrepancy* of Y along E_j. One says that p is a *log terminal* singularity if $a_j > -1$ for all j.

It is classical that this condition is independent of the choice of resolution. In the remainder of this article we assume that

- the singularity $(X, 0)$ is log terminal.
- $Y = \Omega$ is a strongly pseudoconvex neighborhood of $0 = p \in X$;
- the canonical bundle K_Ω is $\mathbb{Q}$-Cartier and $rK_\Omega = 0$ for some $r \in \mathbb{N}$.

Definition 1.9. [EGZ09, Definition 6.5] Fix σ a nowhere vanishing holomorphic section of rK_Ω, and h a smooth hermitian metric of K_Ω, then

$$\mu_p = \lambda \frac{\left(c_n \sigma \wedge \bar{\sigma}\right)^{1/r}}{|\sigma|^{2/r}_{h^r}}$$

is an *adapted measure*, where $\lambda > 0$ is a positive normalizing constant.

Observe that μ_p is independent of the choice of σ, and

$$dd^c \log \mu_p = -\Theta_h(K_\Omega)$$

is the curvature of h, as follows from the Poincaré-Lelong formula.

The measure μ_p has finite mass by [EGZ09, Lemma 6.4]: let $\pi : \tilde{\Omega} \to \Omega$ be a resolution of $(\Omega, 0)$, then

$$\pi^* \mu_p = \lambda \prod_{j=1}^{M} |s_{E_j}|^{2a_j} dV_{\tilde{\Omega}},$$

where $dV_{\tilde{\Omega}}$ is a smooth volume form on $\tilde{\Omega}$, $E_1, \dots, E_M$ are exceptional divisors, the s_{E_j}'s are holomorphic sections such that $E_j = (s_{E_j} = 0)$, and

$$rK_{\tilde{\Omega}} = \pi^*(rK_\Omega) + r \sum_{j=1}^{M} a_j E_j = r \sum_{j=1}^{M} a_j E_j.$$

Thus $\tilde{f} = \prod_{j=1}^{M} |s_{E_j}|^{2a_j}$ belongs to $L^s(dV_{\tilde{\Omega}})$ for some $s > 1$, as p is log terminal.

Definition 1.10. We choose $\lambda = \lambda_\Omega$ so that μ_p is a probability measure in Ω.

The results to follow are independent of this (convenient) normalization.

1.2.2 *Ricci curvature* Let ω be a positive closed current of bidegree $(1, 1)$ in Ω with bounded local potentials. Its top power ω^n is well-defined as explained in Section 1.1.3. If ω^n is absolutely continuous with respect to dV_X, then we set

$$\mathrm{Ric}(\omega) := -dd^c \log \omega^n.$$

Definition 1.11. We say that ω is a Kähler-Einstein metric if it satisfies

$$\mathrm{Ric}(\omega) = \gamma\omega$$

for some $\gamma \in \mathbb{R}$.

In this article we are mainly interested in the case when $\gamma > 0$. We choose the hermitian metric $h \equiv 1$ for K_Ω, so that $\Theta_h = 0$. Since

$$\mathrm{Ric}(\omega) = \mathrm{Ric}(\mu_p) - dd^c \log(\omega^n/\mu_p),$$

the above Kähler-Einstein equation is equivalent, writing $\omega = dd^c\varphi$, to

$$(dd^c\varphi)^n = e^{-\gamma\varphi}e^w\mu_p,$$

where w is a pluriharmonic function in Ω. Changing φ in $\varphi - w/\gamma$ and then φ in $t\varphi$ (observe that $\mathrm{Ric}(t\omega) = \mathrm{Ric}(\omega)$ for any $t > 0$), we can normalize ω by $\int_\Omega \omega^n = 1$ and reduce to

$$(dd^c\varphi)^n = \frac{e^{-\gamma\varphi}\mu_p}{\int_\Omega e^{-\gamma\varphi}\mu_p}.$$

Seeking for a Kähler-Einstein metric thus leads one to solve $(MA)_{\gamma,\phi,\Omega}$.

Conversely solving $(MA)_{\gamma,\phi,\Omega}$ will produce a Kähler-Einstein metric $\omega = dd^c\varphi$, if we can establish enough regularity of the solution φ.

1.2.3 *Log canonical threshold* We consider the density $f = \mu_p/dV_X$. It is related to the density $\tilde{f}$ in a resolution by

$$\pi^*\mu_p = f\circ\pi\cdot\pi^*dV_X = \tilde{f}dV_{\tilde{\Omega}}.$$

An analytic expression for f is obtained as follows. Recall that $dV_X = \omega_{eucl}^n \wedge [X]$, where ω_{eucl} denotes the euclidean Kähler form on $\mathbb{C}^N$. Set $dz_I = dz_{i_1}\wedge\cdots\wedge dz_{i_n}$, where $1 \leqslant i_1 < \cdots < i_n \leqslant N$. There exists germs of holomorphic functions $f_I \in \mathcal{O}_{\Omega,0}$ such that $\big(dz_I\big)^r = f_I\sigma$ since σ is a local generator of rK_X. In particular the volume form $dV_X := \omega_{eucl}^n\wedge[\Omega]$ is comparable to $\Big(\sum_I |f_I|^{\frac{2}{r}}\Big)\mu_p$, i.e.

$$\mu_p = f dV_X, \quad \text{with} \quad f \sim \Big(\sum_I |f_I|^{\frac{2}{r}}\Big)^{-1}.$$

The germs of holomorphic functions f_I generate an ideal $\mathcal{I}_p^r$, where $\mathcal{I}^r$ is an ideal sheaf associated to the singularities of (X,p). In particular

$$\pi^{-1}\mathcal{I}^r\cdot\mathcal{O}_{\tilde{\Omega}} = \mathcal{O}_{\tilde{\Omega}}\big(-r\sum_{j=1}^M b_jE_j\big)$$

for coefficients $b_j \in \mathbb{N}$ such that $f\circ\pi \sim \prod_{j=1}^M |s_{E_j}|^{-2b_j}$.

Definition 1.12. The *log canonical threshold* of (X,p) is given by

$$\mathrm{lct}(X,p) := \inf_{j\in 1,\dots,M}\frac{a_j+1}{b_j}.$$

We let the reader check that the definition is independent of the choice of resolution, and that $\mathrm{lct}(X,\mathcal{I}) \in (0,n]$. One can equivalently use the following point of view: if $\mathcal{I}$ is a general ideal sheaf,

$$\mathrm{lct}(X,\mathcal{I}) := \inf_{E/X}\frac{A_X(E)}{\mathrm{ord}_E(\mathcal{I})} \tag{1.1}$$

where $A_X(E) := 1 + \mathrm{ord}_E(K_{Y/X})$ is the log-discrepancy of E, and the infimum is over all prime divisors E on resolutions Y of X. When $\mathcal{I}$ is supported at p we can restrict in (1.1) to consider prime divisors centered at p.

EXAMPLE 1.13. The ordinary double point (ODP) $X = \left\{ z \in \mathbb{C}^{n+1},\ \sum_{j=0}^n z_j^2 = 0 \right\}$ is the simplest isolated log terminal singularity which is not a quotient singularity when $n \geqslant 3$ (when $n = 2$, log-terminal singularities are precisely the singularities of the form $X = \mathbb{C}^2/G$, $G \subset GL(2,\mathbb{C})$ a finite subgroup).

In this case $\mathcal{I}^2 = (z_1^2, \ldots, z_n^2)$. Indeed the n-forms

$$\sigma_j := \frac{\left(dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n\right)^2}{z_j^2} = -\frac{\left(dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n\right)^2}{\sum_{k \neq j} z_k^2},$$

defined on $U_j := \{z_j \neq 0\}$, glue together to give a local generator σ of $2K_X$ (note that $\sum_{j=0}^n z_j dz_j = 0$). In particular $|f_I|^{2/r} = |z_j|^2$ where $j = [0,n] \setminus I$, $r = 2$ and

$$\mu_p \sim \frac{1}{\sum_{j=0}^n |z_j|^2} dV_X$$

If $\pi : \mathrm{Bl}_0\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ denotes the blow-up at 0, E the exceptional divisor, and F the restriction of E to Y, the strict transform of X, we obtain

$$\pi^{-1}\mathcal{I}^2 \cdot \mathcal{O}_Y = \mathcal{O}_Y(-2F) \quad \text{and} \quad \pi^*\mu_p = |s_F|^{2(n-2)} dV_Y$$

for a smooth volume form dV_Y. Thus $\mathrm{lct}(X,p) = \mathrm{lct}(X,\mathcal{I}) = n-1$.

We will need the following result which connects $\mathrm{lct}(X,p)$ and the integrability properties of the density $f = \mu_p/dV_X$.

LEMMA 1.14. *The density $f = \mu_p/dV_X$ belongs to $L^r(dV_X)$ for $r < 1 + \mathrm{lct}(X,p)$.*

Proof. Let $\pi : \tilde{\Omega} \to \Omega$ be a resolution of the singularity. Recall that

$$f \circ \pi \sim \prod_{j=1}^M |s_{E_j}|^{-2b_j} \quad \text{and} \quad \tilde{f} = \prod_{j=1}^M |s_{E_j}|^{2a_j}, \quad \text{hence} \quad \pi^* dV_X \sim \prod_{j=1}^M |s_{E_j}|^{2(a_j+b_j)} dV_{\tilde{\Omega}}.$$

It follows that $\int_\Omega f^r dV_X \sim \int_{\tilde{\Omega}} \prod_{j=1}^M |s_{E_j}|^{2(a_j+b_j)-2rb_j} dV_{\tilde{\Omega}} < +\infty$ if and only if $r < \frac{1+a_j+b_j}{b_j}$ for all j, which yields the statement since $\mathrm{lct}(X,p) = \inf_j \frac{1+a_j}{b_j}$. □

1.3 Normalized volume

The *(Hilbert-Samuel) multiplicity* of an ideal $\mathcal{I}$ supported at p is defined as

$$\mathrm{e}(X,\mathcal{I}) := \lim_{m \to +\infty} \frac{l(\mathcal{O}_{X,p}/\mathcal{I}^m)}{m^n/n!}$$

where l denotes the length of an Artinian module.

Given a divisor E over X centered at p, the volume of E over $p \in X$ is

$$\mathrm{vol}_{X,p}(E) := \lim_{m \to +\infty} \frac{l(\mathcal{O}_{X,p}/\mathfrak{a}_m(E))}{m^n/n!}$$

where $\mathfrak{a}_m(E) := \left\{ f \in \mathcal{O}_{X,p} \,:\, \mathrm{ord}_E(f) \geqslant m \right\}$ (see [ELS03]).

Definition 1.15. [Li18] The *normalized volume of* $p \in X$ is

$$\widehat{\mathrm{vol}}(X,p) := \inf_{E/X} \widehat{\mathrm{vol}}_{X,p}(E)$$

where the infimum runs over all prime divisors E over X centered at p, and

$$\widehat{\mathrm{vol}}_{X,p}(E) := A_X(E)^n \cdot \mathrm{vol}_{X,p}(E).$$

is the *normalized volume of* E *over* $(x \in X)$.

We shall need the following important result.

Theorem 1.16. *[Liu18, Theorem 27] Let* (X,p) *be a log terminal singularity of complex dimension* $\dim_{\mathbb{C}} X = n$. *Then*

$$\widehat{\mathrm{vol}}(X,p) = \inf_{\mathcal{I} \text{ supported at } p} \mathrm{lct}(X,\mathcal{I})^n \cdot \mathrm{e}(X,\mathcal{I}).$$

Observe that the quantity $\mathrm{lct}(X,\mathcal{I})^n \cdot \mathrm{e}(X,\mathcal{I})$ is invariant under rescaling $\mathcal{I} \to \mathcal{I}^r$, $r \in \mathbb{N}$. One can actually only consider coherent ideal sheaves supported at p. Indeed any ideal $\mathcal{I}$ supported at p is associated to a closed subscheme Z such that $\mathrm{Supp}\, Z = \{p\}$ [Hart77, Corollary II.5.10], while any ideal associated to a closed subscheme is coherent [Hart77, Proposition II.5.9].

Example 1.17. Consider again $X = \left\{ z \in \mathbb{C}^{n+1},\ \sum_{j=0}^n z_j^2 = 0 \right\}$. Recall that $\mathcal{I}^2 = (z_1^2, \ldots, z_n^2)$ is the ideal sheaf associated to the adapted measure, and that the ideal $\mathcal{I}^2$ corresponds to $2F$ where F is the exceptional divisor in the blow up at p. In particular $A_X(F) = n-1$.

We observe here that $e(X,\mathcal{I}^2) = 2^{n+1}$ and $\widehat{\mathrm{vol}}_{X,p}(F) = 2(n-1)^n$ since

$$l\big(\mathcal{O}_{X,p}/\mathcal{I}^{2m}\big) = l\big(\mathcal{O}_{X,p}/\mathfrak{a}_{2m}(F)\big) = 2^{n+1}\frac{m^n}{n!} + O(m^{n-1}).$$

In [Li18, Example 5.3] it is further shown that F is a minimizer for the normalized volume of $p \in X$, i.e. that $\widehat{\mathrm{vol}}(X,p) = 2(n-1)^n$.

2. A variational approach

A variational approach for solving degenerate complex Monge-Ampère equations has been developed in [BBGZ13] in the context of compact Kähler manifolds. It notably applies to the construction of singular Kähler-Einstein metrics of non-positive curvature. This has been partially adapted to smooth pseudoconvex domains of $\mathbb{C}^n$ in [ACC12].

The case of positive curvature is notoriously more difficult, as illustrated by the resolution of the Yau-Tian-Donaldson conjecture by Chen-Donaldson-Sun [CDS15]. It has been treated extensively in [BBEGZ19], and eventually lead to an alternative solution of the Yau-Tian-Donaldson conjecture for Fano varieties [BBJ21, LTW21, Li22]. Adapting [BBEGZ19] to our local singular context, we develop in this section a variational approach for solving the equation

$$(MA)_{\gamma,\phi,\Omega} \begin{cases} (dd^c\varphi)^n = \frac{e^{-\gamma\varphi}d\mu_p}{\int_\Omega e^{-\gamma\varphi}d\mu_p} \\ \varphi_{|\partial\Omega} = \phi. \end{cases} \tag{2.1}$$

2.1 Monge-Ampère energy

2.1.1 *Smooth tests* Fix $\Omega = \{\rho < 0\}$ and ϕ as described previously, and

$$\mathcal{T}_\phi^\infty(\Omega) = \left\{ \varphi \in SPSH(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega}) \,:\, \varphi_{|\partial\Omega} \equiv \phi \right\}.$$

Recall that $\phi_0 \in \mathcal{C}^\infty(\bar\Omega) \cap PSH(\Omega)$ denotes a smooth psh extension of ϕ to $\bar\Omega$. We set $\omega := dd^c\phi_0$. This is a semi-positive form, which can be assumed to be Kähler. However if $\phi \equiv 0$, we can equally well take $\phi_0 \equiv 0$ and get $\omega \equiv 0$.

DEFINITION 2.1. We call $E_\phi(\varphi) := \frac{1}{n+1}\sum_{j=0}^n \int_\Omega (\varphi - \phi_0)(dd^c\varphi)^j \wedge (dd^c\phi_0)^{n-j}$ the ϕ-relative Monge-Ampère energy of $\varphi \in \mathcal{T}^\infty_\phi(\Omega)$.

While the formula depends on the choice of ϕ_0, it follows from Lemma 2.2 that the difference of two such relative energies is constant,

$$E_{\phi_1}(\varphi) - E_{\phi_0}(\varphi) = E_{\phi_1}(\phi_0).$$

For $\phi_0 = 0$, the formula reduces to $E(\varphi) := E_0(\varphi) = \frac{1}{n+1}\int_\Omega \varphi(dd^c\varphi)^n$.

This definition is motivated by the fact the E_ϕ is a primitive of the Monge-Ampère operator for smooth psh functions with ϕ-boundary values.

LEMMA 2.2. *Fix $\varphi \in \mathcal{T}^\infty_\phi(\Omega), v \in \mathcal{D}(\Omega)$. Then $\varphi + tv \in \mathcal{T}^\infty_\phi(\Omega)$ for t small, and*

$$\frac{d}{dt}_{|t=0} E_\phi(\varphi + tv) = \int_\Omega v(dd^c\varphi)^n.$$

In particular $\varphi \mapsto E_\phi(\varphi)$ is increasing.

Here $\mathcal{D}(\Omega)$ denotes the space of smooth functions with compact support in Ω.

Proof. Fix $\varphi \in \mathcal{T}^\infty_\phi(\Omega)$ and $v \in \mathcal{D}(\Omega)$. Since v is smooth with compact support, the function $\pm v + C\rho$ is psh for $C > 0$ large enough, while $\varphi - \varepsilon\rho$ is psh for $\varepsilon > 0$ small enough. It follows that $\varphi + tv$ is psh for t small enough.

Set $\omega = dd^c\phi_0$. The function $\psi_t = \varphi - \phi_0 + tv$ has zero boundary values, and

$$E_\phi(\varphi + tv) = \frac{1}{n+1}\sum_{j=0}^n \int_\Omega \psi_t(\omega + dd^c\psi_t)^j \wedge \omega^{n-j}.$$

It follows from Stokes theorem, as all functions involved in the integration by parts are identically zero on $\partial\Omega$, that

$$\begin{aligned}
&(n+1)\frac{d}{dt}E_\phi(\varphi + tv)\\
&= \sum_{j=0}^n \int_\Omega \dot\psi_t(\omega + dd^c\psi_t)^j \wedge \omega^{n-j} + \sum_{j=1}^n \int_\Omega j\psi_t dd^c\dot\psi_t \wedge (\omega + dd^c\psi_t)^{j-1} \wedge \omega^{n-j}\\
&= \sum_{j=0}^n \int_\Omega \dot\psi_t(\omega + dd^c\psi_t)^j \wedge \omega^{n-j} + \sum_{j=1}^n \int_\Omega j\dot\psi_t dd^c\psi_t \wedge (\omega + dd^c\psi_t)^{j-1} \wedge \omega^{n-j}\\
&= \sum_{j=0}^n \int_\Omega (j+1)\dot\psi_t(\omega + dd^c\psi_t)^j \wedge \omega^{n-j} - \sum_{j=1}^n \int_\Omega j\dot\psi_t(\omega + dd^c\psi_t)^{j-1} \wedge \omega^{n-j+1}\\
&= (n+1)\int_\Omega \dot\psi_t(\omega + dd^c\psi_t)^n
\end{aligned}$$

writing $dd^c\psi_t = (\omega + dd^c\psi_t) - \omega$ in the third line, and then distributing and relabelling so as to obtain a telescopic series. The formula follows for $t = 0$.

In short the derivative of E_ϕ is the complex Monge-Ampère operator $(dd^c\varphi)^n$ which is a positive measure. It follows that $\varphi \mapsto E_\phi(\varphi)$ is increasing. □

2.1.2 *Continuous setting* The previous result extends to the case of continuous plurisubharmonic functions that are not necessarily strictly psh. Recall that

$$\mathcal{T}_\phi(\Omega) := \left\{ \varphi \in PSH(\Omega) \cap C^0(\bar{\Omega}),\ \varphi_{|\partial\Omega} = \phi \text{ and } \int_\Omega (dd^c\varphi)^n < +\infty \right\}.$$

We would like to extend Lemma 2.2 to this less regular setting. As $\varphi + tv$ is not necessarily plurisubharmonic, we need to project it onto the cone of all plurisubharmonic functions. The following result will thus be useful.

Lemma 2.3. *Fix $\varphi \in \mathcal{T}_\phi(\Omega)$ and $f \in \mathcal{D}(\Omega)$. Then $P(\varphi+f) \in \mathcal{T}_\phi(\Omega)$ where*

$$P(\varphi+f) := \sup\left\{\psi \in PSH(\Omega),\ \psi \leqslant \varphi + f\right\}.$$

Moreover $\left(dd^c P(\varphi+f)\right)^n$ is supported on the contact set $\{P(\varphi+f) = \varphi+f\}$.

Proof. Since $\varphi+f$ is bounded and continuous, it is classical to check that the envelope $P(\varphi+f)$ is a well-defined psh function. As f has compact support, one moreover checks that $P(\varphi+f)$ is continuous on $\partial\Omega$ with $P(\varphi+f)_{|\partial\Omega} = \varphi_{|\partial\Omega} = \phi$.

Solving Dirichlet problems in small "balls" not containing the singular point, it follows from a balayage argument that the Monge-Ampère measure of the envelope $\left(dd^c P(\varphi+f)\right)^n$ is supported on the contact set $\{P(\varphi+f) = \varphi+f\}$. □

We extend $E_\phi(\cdot)$ to $\mathcal{T}_\phi(\Omega)$ by monotonicity, setting

$$E_\phi(\varphi) := \inf\left\{E_\phi(\psi),\ \psi \in \mathcal{T}_\phi^\infty(\Omega) \text{ and } \varphi \leqslant \psi\right\}.$$

It has been observed by Berman and Boucksom (in the setting of compact Kähler manifolds [BB10]) that $E_\phi \circ P$ is still differentiable, with $(E_\phi \circ P)' = E_\phi' \circ P$. This result extends to our local singular setting.

Proposition 2.4. *Fix $\varphi \in \mathcal{T}_\phi(\Omega)$ and $f \in \mathcal{D}(\Omega)$. Then $t \to E_\phi\big(P(\varphi+tf)\big)$ is differentiable and*

$$\frac{d}{dt}_{|t=0} E_\phi\big(P(\varphi+tf)\big) = \int_\Omega f(dd^c\varphi)^n.$$

Proof. The proof is very similar to the one in the compact case, we provide it as a courtesy to the reader. Set $\varphi_t := P(\varphi+tf)$. By Lemma 2.5 below we have

$$\int_\Omega (\varphi_t - \varphi)(dd^c\varphi_t)^n \leqslant E_\phi(\varphi_t) - E_\phi(\varphi) \leqslant \int_\Omega (\varphi_t - \varphi)(dd^c\varphi)^n. \tag{2.2}$$

Since $\varphi_t - \varphi \leqslant tf$, the second inequality yields

$$\limsup_{t\to 0^+} \frac{E_\phi(\varphi_t) - E_\phi(\varphi)}{t} \leqslant \int_X f(dd^c\varphi)^n,$$

and $\liminf_{t\to 0^-} \frac{E_\phi(\varphi_t)-E_\phi(\varphi)}{t} \geqslant \int_X f(dd^c\varphi)^n$.

It follows from Lemma 2.3 that $(dd^c\varphi_t)^n$ is supported on $\{\varphi_t = \varphi + tf\}$, hence the first inequality in (2.2) yields

$$\int_\Omega \frac{\varphi_t - \varphi}{t}(dd^c\varphi_t)^n = \int_\Omega f(dd^c\varphi_t)^n.$$

Now $(dd^c\varphi_t)^n \to (dd^c\varphi)^n$ weakly since $\varphi_t \to \varphi$ uniformly, therefore

$$\liminf_{t\to 0^+} \frac{E_\phi(\varphi_t) - E_\phi(\varphi)}{t} \geqslant \liminf_{t\to 0^+} \int_\Omega f(dd^c\varphi_t)^n = \int_\Omega f(dd^c\varphi)^n,$$

and

$$\limsup_{t\to 0^-}\frac{E_\phi(\varphi_t)-E_\phi(\varphi)}{t}\leqslant\limsup_{t\to 0^-}\int_\Omega f(dd^c\varphi_t)^n=\int_\Omega f(dd^c\varphi)^n.$$

□

Lemma 2.5. *For any $\varphi_1,\varphi_2\in\mathcal{T}_\phi(\Omega)$,*

$$\int_\Omega(\varphi_1-\varphi_2)(dd^c\varphi_1)^n\leqslant E_\phi(\varphi_1)-E_\phi(\varphi_2)\leqslant\int_\Omega(\varphi_1-\varphi_2)(dd^c\varphi_2)^n,\tag{2.3}$$

while if $\varphi_1\leqslant\varphi_2$ then

$$E_\phi(\varphi_1)-E_\phi(\varphi_2)\leqslant\frac{1}{n+1}\int_X(\varphi_1-\varphi_2)(dd^c\varphi_1)^n.\tag{2.4}$$

The energy is continuous along decreasing sequence in $\mathcal{T}_\phi(\Omega)$.

Proof. It follows from Stokes theorem that

$$E_\phi(\varphi_1)-E_\phi(\varphi_2)=\frac{1}{n+1}\sum_{j=0}^n\int_\Omega(\varphi_1-\varphi_2)(dd^c\varphi_1)^j\wedge(dd^c\varphi_2)^{n-j}$$

and

$$\int_\Omega(\varphi_1-\varphi_2)(dd^c\varphi_1)^{j+1}\wedge(dd^c\varphi_2)^{n-j-1}\leqslant\int_\Omega(\varphi_1-\varphi_2)(dd^c\varphi_1)^j\wedge(dd^c\varphi_2)^{n-j},$$

for any $j=0,\dots,n-1$. The desired inequalities follow.

Let $\varphi_j\in\mathcal{T}_\phi(\Omega)$ be a decreasing sequence converging to $\varphi\in\mathcal{T}_\phi(\Omega)$. We obtain

$$0\leqslant E_\phi(\varphi_j)-E_\phi(\varphi)\leqslant\int_\Omega(\varphi_j-\varphi)(dd^c\varphi)^n\to 0$$

as $j\to+\infty$ by Monotone Convergence Theorem. □

2.1.3 *Finite energy class* Let $PSH_\phi(\Omega)$ denote the set of decreasing limits of functions in $\mathcal{T}_\phi(\Omega)$. We extend E_ϕ to $PSH_\phi(\Omega)$ by monotonicity, setting

$$E_\phi(\varphi):=\inf\left\{E_\phi(\psi),\ \psi\in\mathcal{T}_\phi(\Omega)\text{ and }\varphi\leqslant\psi\right\}.$$

Definition 2.6. We set $\mathcal{E}^1(\Omega):=\{\varphi\in PSH_\phi(\Omega);\ E_\phi(\varphi)>-\infty\}$.

This "finite energy class" has been introduced and intensively studied by Cegrell for smooth domains of $\mathbb{C}^n$. His analysis extends to our mildly singular context. We summarize here the key facts that we shall need.

Theorem 2.7 Cegrell. *The complex Monge-Ampère operator $(dd^c\cdot)^n$ and the energy E_ϕ are well-defined on the class $\mathcal{E}^1(\Omega)$. Moreover*

- *functions in $\mathcal{E}^1(\Omega)$ have zero Lelong numbers;*
- *the sets $\mathcal{G}_b(\Omega)=\{\varphi\in\mathcal{E}^1(\Omega),\ -b\leqslant E_\phi(\varphi)\}$ are compact for all $b\in\mathbb{R}$;*
- *Lemma 2.5 holds if $\varphi_1,\varphi_2\in\mathcal{E}^1(\Omega)$;*
- *if μ is a non pluripolar probability measure such that $\mathcal{E}^1(\Omega)\subset L^1(\mu)$, then there exists a unique function $v\in\mathcal{E}^1(\Omega)\cap\mathcal{F}_1(\Omega)$ such that $\mu=(dd^cv)^n$.*

We refer the reader to [Ceg98, Theorems 3.8, 7.2 and 8.2] for the proof of these results when Ω is smooth.

2.2 Ding functional

2.2.1 *Euler-Lagrange equation* The Ding functional is

$$F_\gamma(\varphi) := E_\phi(\varphi) + \frac{1}{\gamma}\log\int_\Omega e^{-\gamma\varphi}d\mu_p$$

Proposition 2.8. *If φ maximizes F_γ over $\mathcal{T}_\phi(\Omega)$ then φ solves the complex Monge-Ampère equation (2.1).*

Proof. Assume that φ maximizes F_γ over $\mathcal{T}_\phi(\Omega)$, fix $f \in \mathcal{D}(\Omega)$, and set $\varphi_t := P(\varphi + tf)$. Then

$$E_\phi(\varphi_t) + \frac{1}{\gamma}\log\int_\Omega e^{-\gamma(\varphi+tf)}d\mu_p \leqslant F_\gamma(\varphi_t) \leqslant F_\gamma(\varphi),$$

i.e. the function $t \to E_\phi(\varphi_t) + \frac{1}{\gamma}\log\int_\Omega e^{-\gamma(\varphi+tf)}d\mu_p$ reaches its maximum at $t = 0$. Combining Proposition 2.4 and Lemma 2.9 below, we obtain

$$0 = \frac{d}{dt}\Big(E_\phi(\varphi_t) + \frac{1}{\gamma}\log\int_\Omega e^{-\gamma(\varphi+tf)}d\mu_p\Big) = \int_\Omega f\left((dd^c\varphi)^n - \frac{e^{-\gamma\varphi}d\mu_p}{\int_\Omega e^{-\gamma\varphi}d\mu_p}\right),$$

i.e. φ solves (2.1). □

Lemma 2.9. *Fix $\varphi \in \mathcal{T}_\phi(\Omega)$, $f \in \mathcal{D}(\Omega)$, and set $\psi_t := \varphi + tf$. Then*

$$\frac{d}{dt}\Big(\log\int_\Omega e^{-\gamma\psi_t}d\mu_p\Big)_{|t=0} = -\gamma\frac{\int_\Omega fe^{-\gamma\varphi}d\mu_p}{\int_\Omega e^{-\gamma\varphi}d\mu_p}.$$

Proof. By chain rule, it is enough to observe that

$$\frac{\int_\Omega e^{-\gamma\psi_t}d\mu_p - \int_\Omega e^{-\gamma\varphi}d\mu_p}{t} = -\int_\Omega e^{-\gamma\varphi}\left(\frac{1-e^{-t\gamma f}}{t}\right)d\mu_p$$

and to apply Lebesgue Dominated Convergence Theorem to conclude. □

2.2.2 *Coercivity* In order to solve (2.1), one is lead to try and maximize F_γ. We will show in Section 5 that when F_γ is *coercive*, the complex Monge-Ampère equation (2.1) admits a solution $\varphi \in \mathcal{T}_\phi(\Omega)$ which is smooth away from p.

Definition 2.10. The functional F_γ is coercive if there exists $A, B > 0$ such that

$$F_\gamma(\varphi) \leqslant AE_\phi(\varphi) + B$$

for all $\varphi \in \mathcal{T}_\phi(\Omega)$.

We observe in Lemma 2.13 that $E_\phi(\varphi) \leqslant C(\phi_0)$ is bounded from above, uniformly in $\varphi \in \mathcal{T}_\phi(\Omega)$. In particular if F_γ is coercive with *slope* $A > 0$ then it is coercive for any $A' \in (0, A]$. We can thus assume, without loss of generality, that $A \in (0, 1)$. The coercivity property is then equivalent to

$$\frac{1}{\gamma}\log\int_\Omega e^{-\gamma\varphi}d\mu_p \leqslant (1-A)(-E_\phi(\varphi)) + B,$$

or, equivalently, to the following Moser-Trudinger inequality

$$\left(\int_X e^{-\gamma\varphi}d\mu_p\right)^{\frac{1}{\gamma}} \leqslant Ce^{(1-A)(-E_\phi(\varphi))}.$$

We summarize these observations in the following.

PROPOSITION 2.11. *Fix $\gamma > 0$. The following properties are equivalent:*

i) F_γ is coercive;

ii) there exists $C_\gamma > 0$ and $a \in (0,1)$ such that for all $\varphi \in \mathcal{T}_\phi(\Omega)$,

$$\Big(\int_\Omega e^{-\gamma\varphi} d\mu_p \Big)^{1/\gamma} \leqslant C_\gamma e^{-aE_\phi(\varphi)}.$$

It follows from Hölder inequality (and the normalization $\mu_p(\Omega) = 1$) that if

$$\Big(\int_\Omega e^{-\gamma\varphi} d\mu_p \Big)^{1/\gamma} \leqslant C e^{-E_\phi(\varphi)} \tag{2.5}$$

holds for $\gamma > 0$, then it also holds for $\gamma' < \gamma$. We thus introduce the following critical exponent.

DEFINITION 2.12. We set

$$\gamma_{crit}(X,p) := \sup \Big\{ \gamma > 0,\ (2.5) \text{ holds for all } \varphi \in \mathcal{T}_\phi(\Omega) \Big\}.$$

LEMMA 2.13. *The functional E_ϕ is bounded from above on $\mathcal{T}_\phi(\Omega)$. Moreover*

- *if F_γ is coercive then $\gamma \leqslant \gamma_{crit}(X,p)$;*
- *conversely if $\gamma < \gamma_{crit}(X,p)$ then F_γ is coercive.*

Proof. Consider $\tilde{\phi_0} = P(\phi) := \sup\{\psi, \psi \in \mathcal{T}_\phi(\Omega)\}$. This is the largest psh function in Ω such that $\tilde{\phi_0} = \phi$ on $\partial\Omega$. The reader can check that it is continuous on $\overline{\Omega}$ and satisfies $(dd^c\tilde{\phi_0})^n = 0$ in Ω. If $\varphi \in \mathcal{T}_\phi(\Omega)$ then $\varphi \leqslant \tilde{\phi_0}$ hence $E_{\tilde{\phi_0}}(\varphi) \leqslant 0$. Thus

$$E_{\phi_0}(\varphi) = E_{\tilde{\phi_0}}(\varphi) + E_{\phi_0}(\tilde{\phi_0}) \leqslant E_{\phi_0}(\tilde{\phi_0}),$$

hence $E_{\phi_0}(\varphi)$ is uniformly bounded from above independently of the choice of ϕ_0.

Similarly the coercivity of F_γ or the inequality (2.5) do not depend on the choice of ϕ_0. In the remainder of this proof we thus assume that $\phi_0 = P(\phi)$. Since $E_\phi(\varphi) \leqslant 0$ in this case, it follows from Proposition 2.11 that if F_γ is coercive then (2.5) holds, hence $\gamma \leqslant \gamma_{crit}(X,p)$.

Conversely assume $\gamma < \gamma_{crit}(X,p)$. Fix $\gamma < \gamma' < \gamma_{crit}(X,p)$ and $\lambda = \gamma/\gamma' < 1$. We can assume that λ is close to 1. We assume first that $\phi \equiv 0$. For $\varphi \in \mathcal{T}_0(\Omega)$ we observe that $\lambda\varphi \in \mathcal{T}_0(\Omega)$, with $E_0(\lambda\varphi) = \lambda^{n+1}E_0(\varphi)$. The Moser-Trudinger (2.5) applied to $(\gamma', \lambda\varphi)$ thus yields

$$\Big(\int_\Omega e^{-\gamma\varphi} d\mu_p \Big)^{1/\gamma} \leqslant C_{\gamma'} e^{-\lambda^n E_0(\varphi)},$$

so that F_γ is coercive.

We now treat the general case, replacing the condition $\phi \equiv 0$ by $(dd^c\phi_0)^n \equiv 0$. For $\varphi \in \mathcal{T}_\phi(\Omega)$ we observe that $\varphi_\lambda = \lambda\varphi + (1-\lambda)\phi_0 \in \mathcal{T}_\phi(\Omega)$, with $\varphi_\lambda - \phi_0 = \lambda(\varphi - \phi_0) \leqslant 0$ and

$$\begin{aligned}(n+1)E_{\phi_0}(\varphi_\lambda) &= \lambda \sum_{j=0}^{n-1} \int_\Omega (\varphi - \phi_0)(dd^c\varphi_\lambda)^j \wedge (dd^c\phi_0)^{n-j} \\ &= \lambda \sum_{k=1}^{n} \sum_{j=k}^{n} \binom{j}{k} \lambda^k (1-\lambda)^{j-k} \int_\Omega (\varphi - \phi_0)(dd^c\varphi)^k \wedge (dd^c\phi_0)^{n-k}.\end{aligned}$$

Now $\sum_{j=k}^n \binom{j}{k} \lambda^k(1-\lambda)^{j-k} \leqslant a < 1$ for all $1 \leqslant k \leqslant n$, since $\lambda < 1$ can be chosen arbitrarily close to 1. Thus $E_{\phi_0}(\varphi_\lambda) \geqslant a\lambda E_{\phi_0}(\varphi)$ and the result follows as previously by applying the Moser-Trudinger inequality (2.5) to φ_λ. □

2.3 Invariance

We give here some evidence that that the critical exponent $\gamma_{crit}(X,p)$ should be independent of the domain Ω and the boundary values ϕ.

2.3.1 *Enlarging the domain* We first reduce to the case of zero boundary values.

Proposition 2.14. *Let Ω_2 be a smooth strongly pseudoconvex domain containing $\overline{\Omega}$. If the Moser-Trudinger inequality holds for $(\gamma,\Omega_2,0)$ then it holds for (γ,Ω,ϕ).*

Proof. Consider indeed $\varphi\in\mathcal{T}_\phi(\Omega)$ and set

$$\varphi_2:=\sup\{u\in\mathcal{T}_0(\Omega_2),\ \text{ such that } u\leqslant\varphi \text{ in } \Omega\}.$$

The family $\mathcal{F}$ of such functions is non-empty, as it contains $A\rho_2$ for some large $A>1$, where ρ_2 is a psh defining function for Ω_2. Moreover $\mathcal{F}$ is uniformly bounded from above by 0, so the upper-envelope φ_2 is well-defined and psh, as $\mathcal{F}$ is compact. Finally $\varphi_2\geqslant A\rho_2$, hence φ_2 has zero boundary values, and φ_2 is lower semi-continuous, as an envelope of continuous functions, thus $\varphi_2\in\mathcal{T}_0(\Omega_2)$.

Since $\varphi_2\leqslant\varphi$ in Ω, we observe that

$$\int_\Omega e^{-\gamma\varphi}d\mu_p\leqslant\int_{\Omega_2}e^{-\gamma\varphi_2}d\mu_p.$$

Our claim will follow if we can show that on the other hand $E_0(\varphi_2)\geqslant E_\phi(\varphi)$.

If φ is smooth one can show, by adapting standard techniques, that

- φ_2 is $\mathcal{C}^{1,\overline{1}}$-smooth in $\Omega_2\setminus\{p\}$;
- $(dd^c\varphi_2)^n=0$ in $\Omega_2\setminus\Omega$ and $(dd^c\varphi_2)^n=\mathbf{1}_{\{\varphi_2=\varphi\}}(dd^c\varphi)^n$ in $\overline{\Omega}$.

Assuming $\phi\geqslant 0$ and $\phi_0=\sup\{\psi,\ \psi\in\mathcal{T}_\phi(\Omega)\}$, we infer

$$\begin{aligned}E_0(\varphi_2)&=\frac{1}{n+1}\int_\Omega\varphi_2(dd^c\varphi_2)^n=\frac{1}{n+1}\int_\Omega\mathbf{1}_{\{\varphi_2=\varphi\}}\varphi(dd^c\varphi)^n\\&\geqslant\frac{1}{n+1}\int_\Omega\varphi(dd^c\varphi)^n\geqslant\frac{1}{n+1}\int_\Omega(\varphi-\phi)(dd^c\varphi)^n\geqslant E_\phi(\varphi).\end{aligned}$$

To get rid of the assumption $\phi\geqslant 0$, we observe that the Moser-Trudinger inequality holds for given boundary data ϕ if and only if does so for $\phi+c$, for any $c\in\mathbb{R}$ (by changing φ in $\varphi+c$).

Using Lemma 1.5, one can uniformly approximate φ by a sequence of smooth $\varphi_j\in\mathcal{T}_\phi(\Omega)$. The corresponding sequence $\varphi_{2,j}$ uniformly converges to φ_2, and we obtain the desired inequality by passing to the limit in $E_0(\varphi_{2,j})\geqslant E_\phi(\varphi_j)$. □

2.3.2 *Rescaling* We now assume that $\phi=0$ and reformulate the coercivity property after an appropriate rescaling. Observe that for any $\lambda>0$, the map

$$\varphi\in\mathcal{T}_0(\Omega)\mapsto\lambda\varphi\in\mathcal{T}_0(\Omega)$$

is a homeomorphism. This allows to reformulate the Moser-Trudinger inequality:

Proposition 2.15. *The following statements are equivalent:*

a) F_γ is coercive;

b) $\exists C>0,B\in(0,1)$ such that for all $\varphi\in\mathcal{T}_0(\Omega)$, $\int_\Omega e^{-\varphi}d\mu_p\leqslant Ce^{-\frac{B}{\gamma^n}E_0(\varphi)}$.

In particular we can define the following critical exponent.

Definition 2.16. We set

$$\beta_{crit} := \inf\left\{\beta > 0;\ \sup_{\varphi\in\mathcal{T}_0(\Omega)}\Big(\int_\Omega e^{-\varphi}d\mu_p/e^{-\beta E_0(\varphi)}\Big) < +\infty\right\}.$$

Note that $\gamma_{crit}^n = 1/\beta_{crit}$, hence it follows from the previous analysis that F_γ is coercive if and only if $\gamma < \beta_{crit}^{-1/n}$. When $p \in X$ is smooth, it has been shown in [GKY13, Theorem 9] and independently [BB22, Theorem 1.5] that

$$\beta_{crit}(\Omega) = \frac{1}{(n+1)^n},$$

or equivalently that $\gamma_{crit}(\Omega) = n+1$. In particular it does not depend on Ω.

We extend this independence to the case when p is the vertex of a cone over a Fano manifold.

Proposition 2.17. *Assume that (X,p) is the affine cone over a Fano manifold Z embedded in a projective space by the linear system $|-rK_Z|$ for $r\in\mathbb{N}$ such that $L = rK_Z^*$ is very ample, and fix $\lambda\in\mathbb{C}^*$. The Moser-Trudinger inequality holds for $(\gamma,\Omega,0)$ iff it does so for $(\gamma,\lambda\Omega,0)$.*

Proof. Let $L = rK_Z^*$, let D_λ denote the dilatation $z\mapsto\lambda z$ and set $\Omega_\lambda = D_\lambda(\Omega)$. We blow up p to obtain a resolution $f : Y\to X$, where Y is the total space of L^* and the exceptional divisor E is the zero section of L^*.

Recall that $K_Y = f^*K_X + aE$, where a is the discrepancy of Y along E. The adjunction formula yields $(K_Y+E)_{|E} = K_E$ hence $K_Z^* = (a+1)L$. In particular $a = -1+1/r$ and (X,p) is log terminal. The fibration $\pi : Y = L^*\to Z$ yields $K_Y = \pi^*(K_Z+L)$ hence $f^*K_X = \pi^*(K_Z+L)-aE$.

Since $\pi^*(K_Z+L)$ is $\mathbb{C}^*$-invariant, we can cook up an adapted volume form $\mu_p = \mu_1\cdot\mu_E$ with $D_\lambda^*\mu_1 = \mu_1$ while $D_\lambda^*\mu_E = |\lambda|^{2a}\mu_E$. For $\varphi\in\mathcal{T}_0(\Omega_\lambda)$ we set $\varphi_\lambda = \varphi\circ D_\lambda\in\mathcal{T}_0(\Omega)$ and observe that

$$|\lambda|^{2a}\int_\Omega e^{-\gamma\varphi_\lambda}d\mu_p = \int_{\Omega_\lambda} e^{-\gamma\varphi}d\mu_p,$$

while $E_{\Omega,0}(\varphi_\lambda) = E_{\Omega_\lambda,0}(\varphi)$. The conclusion follows. □

We conjecture in Section 4 that $\gamma_{crit}(X,p)\overset{?}{=}\frac{n+1}{n}\widehat{\mathrm{vol}}(X,p)^{1/n}$ and give partial results towards this equality, which again suggest that $\gamma_{crit}(X,p)$ should be independent of (Ω,ϕ). In the whole article we therefore use the notation $\gamma_{crit}(X,p)$ instead of the more precise -and heavy- $\gamma_{crit}(X,p,\Omega,\phi)$.

3. Upper bound for the coercivity

The purpose of this section is to establish the following upper bound

$$\gamma_{crit}(X,p)\leqslant\frac{n+1}{n}\widehat{\mathrm{vol}}(X,p)^{1/n}.$$

Adapting the proof of [BB17, Theorem 1.6], we will construct approximate Green's functions to test the thresholds in the Moser-Trudinger inequality.

3.1 Functions with algebraic singularities

Let $\mathcal{I}$ be a coherent ideal sheaf, and $f_1,\dots,f_N \in \mathcal{O}_{X,p}$ be local generators of $\mathcal{I}_p$. The plurisubharmonic function

$$\varphi_{\mathcal{I}} := \log\Big(\sum_{i=1}^N |f_i|^2\Big)$$

is well defined near p, with *algebraic singularities* encoded in $\mathcal{I}$.

Proposition 3.1. *Let $\mathcal{I}$ be a coherent ideal sheaf supported at p. Then*

$$e(X,\mathcal{I}) = \int_{\{p\}} \big(dd^c\varphi_{\mathcal{I}}\big)^n$$

and

$$\mathrm{lct}(X,\mathcal{I}) = \sup\left\{\alpha>0 : \int_\Omega e^{-\alpha\varphi_{\mathcal{I}}} d\mu_p < +\infty\right\}$$

where Ω is any (small) neighborhood of $p\in X$.

These algebraic quantities are thus independent of the choice of generators.

Proof. The equality $e(X,\mathcal{I}) = \int_{\{p\}} \big(dd^c\varphi_{\mathcal{I}}\big)^n$ is classical when Ω is smooth (see e.g. [Dem12, Lemma 2.1]), and the proof can be adapted to the singular context (see [Dem85, Chapter 4]).

Let $\pi : \tilde\Omega \to \Omega$ be a local log resolution of the ideal $(X,\mathcal{I})$, i.e. a composition of blow-ups such that $\pi^*\mu_p = \prod_{j=1}^N |s_{E_j}|^{2a_j} dV_{\tilde\Omega}$ and

$$\pi^{-1}\mathcal{I}\cdot\mathcal{O}_{\tilde\Omega} = \mathcal{O}_{\tilde\Omega}\Big(-\sum_{j=1}^M b_j E_j\Big)$$

where $b_j\in\mathbb{N}$, $a_j\in\mathbb{Q}_{>-1}$, and $E_1,\dots,E_M$ have simple normal crossings. Observe that

$$\int_\Omega e^{-\alpha\varphi_{\mathcal{I}}} d\mu_p \sim \int_{\tilde\Omega} \frac{\prod_{j=1}^M |s_{E_j}|^{2a_j}}{\prod_{j=1}^M |s_{E_j}|^{2b_j\alpha}} dV_{\tilde\Omega} = \int_{\tilde\Omega} \prod_{j=1}^M |s_{E_j}|^{2(a_j-\alpha b_j)} dV_{\tilde\Omega},$$

is finite if and only if $a_j-\alpha b_j > -1$ for any $j=1,\dots,M$, i.e. if and only if

$$\alpha < \inf_{j=1,\dots,M} \frac{a_j+1}{b_j} = \mathrm{lct}(X,\mathcal{I}),$$

as recalled in Definition 1.12. □

3.2 Approximate Green functions

The functions $\lambda\varphi_{\mathcal{I}}$ play the role of Green functions adapted to the singularity (X,p). We show here how to approximate them from above by smooth functions with prescribed boundary values.

Lemma 3.2. *Let $\mathcal{I}$ be a coherent ideal sheaf supported at p, and let $f_1,\dots,f_m$ denote local generators of $\mathcal{I}$. Fix an open set $\Omega'\Subset\Omega$. There exists a family $\{\varphi_{\mathcal{I},\lambda,\epsilon}\}_{\lambda>0,\epsilon>0} \in PSH(\Omega)\cap\mathcal{C}^\infty(\bar\Omega)$ such that*

- *i) $\varphi_{\mathcal{I},\lambda,\epsilon|\partial\Omega} = \phi$ for any $\lambda>0, \epsilon\in[0,1]$;*
- *ii) $\varphi_{\mathcal{I},\lambda,\epsilon} = \lambda\log\big(\sum_{j=1}^m |f_j|^2+\epsilon^2\big)+\phi_0$ in Ω',*
- *iii) $\varphi_{\mathcal{I},\lambda,\epsilon} \searrow \varphi_{\mathcal{I},\lambda,0} =: \varphi_{\mathcal{I},\lambda}$ as $\epsilon\searrow 0$ for any $\lambda>0$ fixed.*

Proof. Without loss of generality we can assume that that $\sum_{j=1}^m |f_j|^2 \leqslant 1/e - 1$ in Ω. Let ρ be a smooth psh exhaustion for Ω and fix $0 < r \ll 1$, $0 < \delta \ll 1$ small enough. There exists $A > 0$ big enough and relatively compact open sets $B_r(0) \Subset \Omega' \Subset \tilde{\Omega} \Subset \Omega$ such that

$$\log\big(\sum_{j=1}^m |f_j|^2 + 1\big) + \delta \leqslant A\rho \text{ over } \Omega \setminus \tilde{\Omega},$$

while $\log\big(\sum_{j=1}^m |f_j|^2\big) - \delta \geqslant A\rho$ over $\Omega' \setminus B_r(0)$. We infer that

$$u_{\mathcal{I},\epsilon} := \begin{cases} A\rho \text{ on } \Omega \setminus \tilde{\Omega} \\ \max_\delta\Big(\log\big(\sum_{j=1}^m |f_j|^2 + \epsilon^2\big), A\rho\Big) \text{ on } \tilde{\Omega} \setminus \Omega' \\ \log\big(\sum_{j=1}^m |f_j|^2 + \epsilon^2\big) \text{ on } \Omega' \end{cases}$$

is a decreasing family (in $\epsilon \in [0,1]$) of plurisubharmonic functions which are smooth in $\bar{\Omega} \setminus \{p\}$ (smooth in $\bar{\Omega}$ for $\epsilon > 0$) and which are identically 0 on $\partial\Omega$. Here $\max_\delta(\cdot,\cdot)$ denotes the regularized maximum. The lemma follows by setting $\varphi_{\mathcal{I},\lambda,\epsilon} := \lambda u_{\mathcal{I},\epsilon} + \phi_0$. □

We now compute the asymptotic behavior, as ε decreases to 0, of the quantities involved in the expected Moser-Trudinger inequality.

LEMMA 3.3. *Let $\mathcal{I}$ and $\{\varphi_{\mathcal{I},\lambda,\epsilon}\}_{\epsilon\in(0,1]} \subset \mathcal{T}_\phi(\Omega)$ be as in Lemma 3.2. Then for any $\gamma > 0, \lambda > 0$ fixed there exists a constant $C_{\lambda,\gamma} \in \mathbb{R}$ (independent of ϵ) such that*

$$C_{\lambda,\gamma} + \big(\gamma\lambda - \mathrm{lct}(X,\mathcal{I})\big)\log\epsilon^{-2} \leqslant \log\int_\Omega e^{-\gamma\varphi_{\mathcal{I},\lambda,\epsilon}} d\mu_p \tag{3.1}$$

for all $0 < \epsilon < \epsilon_0$.

Proof. Taking a log resolution $\pi : Y \to X$ we obtain

$$\begin{aligned}\int_\Omega e^{-\gamma\varphi_{\mathcal{I},\lambda,\epsilon}} d\mu_p &\geqslant \int_{\Omega'} \frac{1}{\Big(\sum_{j=1}^m |f_j|^2 + \epsilon^2\Big)^{\gamma\lambda}} d\mu_p \\ &\geqslant C_1 \int_{\pi^{-1}(\Omega')} \frac{\prod_{j=1}^M |s_{E_j}|^{2a_j}}{\Big(\prod_{j=1}^M |s_{E_j}|^{2b_j} + \epsilon^2\Big)^{\gamma\lambda}} dV_{\pi^{-1}(\Omega')},\end{aligned}$$

where C_1 is a uniform constant (independent on ϵ). We set

$$f := \frac{\prod_{j=1}^M |s_{E_j}|^{2a_j}}{\Big(\prod_{j=1}^M |s_{E_j}|^{2b_j} + \epsilon^2\Big)^{\gamma\lambda}}.$$

We can assume without loss of generality that $\mathrm{lct}(X,\mathcal{I}) = \frac{a_1+1}{b_1}$. Pick $x \in E_1, x \notin E_j, j = 2,\dots,M$. We can find $0 < r \ll 1$ so small that $B_r(x) \cap E_j = \emptyset$ for any $j = 2,\dots,M$. We choose holomorphic coordinates $(z_1,\dots,z_n)$ centered at x such that $E_1 = \{z_1 = 0\}$. Thus, setting $a := a_1, b := b_1$ and $c := \gamma\lambda$ we get

$$\int_{\pi^{-1}(\Omega')} f dV_{\pi^{-1}(\Omega')} \geqslant C_2 \int_{B_r(0)} \frac{|z_1|^{2a}}{\big(|z_1|^{2b} + \epsilon^2\big)^c} d\lambda(z) = C_3 \int_0^r \frac{u^{2a+1}}{\big(u^{2b} + \epsilon^2\big)^c} du$$

where C_2, C_3 are uniform constants. If $c \leqslant \frac{a+1}{b}$ (i.e. $\gamma\lambda \leqslant \mathrm{lct}(X,\mathcal{I})$) then

$$\int_0^r \frac{u^{2a+1}}{\big(u^{2b} + \epsilon^2\big)^c} du \geqslant \int_0^1 \frac{u^{2a+1}}{\big(u^{2b} + 1\big)^c} =: C_4$$

and (3.1) trivially follows. If $c > \frac{a+1}{b}$ then the substitution $v := u/\epsilon^{1/b}$ yields

$$\int_0^r \frac{u^{2a+1}}{\left(u^{2b}+\epsilon^2\right)^c} du = \epsilon^{-2\left(c-\frac{a+1}{b}\right)} \int_0^{r/\epsilon^{1/b}} \frac{v^{2a+1}}{\left(v^{2b}+1\right)^c} \geqslant \epsilon^{-2\left(\gamma\lambda-\mathrm{lct}(X,\mathcal{I})\right)} \int_0^r \frac{v^{2a+1}}{\left(v^{2b}+1\right)^c} dv.$$

The lemma follows. □

LEMMA 3.4. *Let $\mathcal{I}$ and $\varphi_{\mathcal{I},\lambda,\epsilon} \in \mathcal{T}_\phi(\Omega)$ be as in Lemma 3.2. There exist positive constants $\{C_{\ell,\lambda}\}_{\ell\in\mathbb{N},\lambda>0}$ and a family of functions $F_\ell : (0,1] \to \mathbb{R}_{>0}$ such that*

$$-E_\phi(\varphi_{\mathcal{I},\lambda,\epsilon}) \leqslant C_{\ell,\lambda} + \frac{\lambda^{n+1}}{n+1} F_\ell(\epsilon) \log \epsilon^{-2},$$

for any $\epsilon \in (0,1]$, where

- *$\{C_{\ell,\lambda}\}_{\ell\in\mathbb{N},\lambda>0}$ is independent of $\epsilon \in (0,1]$;*
- *$F_\ell(\epsilon) \to F_\ell(0) =: e_\ell > 0$ as $\epsilon \searrow 0$;*
- *$e_\ell \searrow e(X,\mathcal{I})$ as $\ell \to +\infty$.*

Proof. We take a sequence $\{\Omega_\ell\}_{\ell\in\mathbb{N}}$ of open sets such that $\Omega_{\ell+1} \Subset \Omega_\ell$ for any $\ell \in \mathbb{N}$ and such that $\bigcap_{\ell\in\mathbb{N}} \Omega_\ell = \{p\}$. Since $\Omega_\ell \subset \Omega'$ (same notation of Lemma 3.2) for $\ell \in \mathbb{N}$ big enough, we obtain

$$\begin{aligned} -E_\phi(\varphi_{\mathcal{I},\lambda,\epsilon}) &= \frac{1}{n+1}\sum_{j=0}^n \int_\Omega \left(\phi_0 - \varphi_{\mathcal{I},\lambda,\epsilon}\right)\left(dd^c\varphi_{\mathcal{I},\lambda,\epsilon}\right)^j \wedge \left(dd^c\phi_0\right)^{n-j} \\ &= \frac{1}{n+1}\sum_{j=0}^n \int_{\Omega\setminus\Omega_\ell} \left(\phi_0 - \varphi_{\mathcal{I},\lambda,\epsilon}\right)\left(dd^c\varphi_{\mathcal{I},\lambda,\epsilon}\right)^j \wedge \left(dd^c\phi_0\right)^{n-j} \\ &\quad - \frac{1}{n+1}\sum_{j=0}^n \int_{\Omega_\ell} \lambda^{j+1} \log\Big(\sum_{k=1}^m |f_k|^2+\epsilon^2\Big)\Big(dd^c \log\Big(\sum_{k=1}^m |f_k|^2+\epsilon^2\Big)\Big)^j \wedge \Big(dd^c\phi_0\Big)^{n-j}. \end{aligned} \tag{3.2}$$

The first term on the right hand side of (3.2) is uniformly bounded in $\epsilon \in [0,1]$, for $\lambda > 0, \ell \in \mathbb{N}$ fixed, since $\{\varphi_{\mathcal{I},\lambda,\epsilon}\}_{\epsilon\in[0,1]}$ is a continuous family of smooth functions on $\Omega\setminus\Omega_\ell$. We let $C_{\ell,\lambda}$ denote a uniform upper bound for this quantity.

The second term on the right hand side of (3.2) is bounded from above by

$$\begin{aligned} &- \frac{1}{n+1}\sum_{j=0}^n \int_{\Omega_\ell} \lambda^{j+1} \log\Big(\sum_{k=1}^m |f_k|^2+\epsilon^2\Big)\Big(dd^c \log\Big(\sum_{k=1}^m |f_k|^2+\epsilon^2\Big)\Big)^j \wedge \Big(dd^c\phi_0\Big)^{n-j} \\ &\qquad \leqslant \frac{\lambda^{n+1}}{n+1} \log\epsilon^{-2} \sum_{j=0}^n \int_{\Omega_\ell} \Big(dd^c \log\Big(\sum_{k=1}^m |f_k|^2+\epsilon^2\Big)\Big)^j \wedge \Big(dd^c\phi_0\Big)^{n-j}. \end{aligned}$$

We set

$$F_\ell(\epsilon) := \sum_{j=0}^n \int_{\Omega_\ell} \Big(dd^c \log\Big(\sum_{k=1}^m |f_k|^2+\epsilon^2\Big)\Big)^j \wedge \Big(dd^c\phi_0\Big)^{n-j}.$$

Observe that for $j = 0, \ldots, n-1$

$$\lim_{\ell\nearrow+\infty} \lim_{\epsilon\searrow 0} \int_{\Omega_\ell} \Big(dd^c \log\Big(\sum_{k=1}^m |f_k|^2+\epsilon^2\Big)\Big)^j \wedge \Big(dd^c\phi_0\Big)^{n-j} = 0$$

since the ideal sheaf $\mathcal{I}$ generated by $f_1, \dots, f_m$ is supported at one point, while

$$\int_{\Omega_\ell} \Big(dd^c \log \big(\sum_{k=1}^m |f_k|^2 + \epsilon^2 \big) \Big)^n \to e_\ell$$

as $\epsilon \searrow 0$, where $e_\ell \geqslant e(X, \mathcal{I})$ and $e_\ell \searrow \int_{\{p\}} \left(dd^c \varphi_{\mathcal{I},1} \right)^n$ as $\ell \nearrow +\infty$.

Proposition 3.1 yields $\int_{\{p\}} \left(dd^c \varphi_{\mathcal{I},1} \right)^n = e(X, \mathcal{I})$, ending the proof. □

3.3 The upper bound

We are now ready for the proof of the following result.

THEOREM 3.5. *Let (X, p) be a an isolated log terminal singularity. Then*

$$\gamma_{crit} \leqslant \frac{n+1}{n} \widehat{\mathrm{vol}}(X, p)^{1/n}.$$

Proof. Fix $\gamma < \gamma_{crit}$ and $C_1 > 0$ such that

$$\frac{1}{\gamma} \log \int_\Omega e^{-\gamma \varphi} d\mu_p \leqslant C_1 - E_\phi(\varphi) \tag{3.3}$$

for any $\varphi \in \mathcal{T}_\phi(\Omega)$.

Fix $\mathcal{I}$ coherent ideal sheaf supported at p, and let $\{\varphi_{\mathcal{I},\lambda,\epsilon}\}_{\lambda>0,\epsilon\in(0,1]} \in \mathcal{T}_\phi(\Omega)$ as defined in Lemma 3.2. Evaluating (3.3) at $\{\varphi_{\mathcal{I},\lambda,\epsilon}\}_{\epsilon\in(0,1]}$ yields

$$\begin{aligned} C_{\gamma,\lambda} + \Big(\lambda - \frac{\mathrm{lct}(X,\mathcal{I})}{\gamma} \Big) \log \epsilon^{-2} &\leqslant \frac{1}{\gamma} \log \int_\Omega e^{-\gamma \varphi_{\mathcal{I},\lambda,\epsilon}} d\mu_p \\ &\leqslant C_1 - E_\phi(\varphi_{\mathcal{I},\lambda,\epsilon}) \\ &\leqslant C_1 + C_{N,\lambda} + \frac{\lambda^{n+1}}{n+1} F_N(\epsilon) \log \epsilon^{-2} \end{aligned}$$

for any $N \in \mathbb{N}, \epsilon \in (0,1]$ thanks to Lemmas 3.3 and 3.4. We infer

$$\Big(\lambda - \frac{\mathrm{lct}(X,\mathcal{I})}{\gamma} - \frac{\lambda^{n+1}}{n+1} F_N(\epsilon) \Big) \log \epsilon^{-2} \leqslant C_1 + C_{N,\lambda} - C_{\gamma,\lambda},$$

hence

$$\lambda - \frac{\lambda^{n+1}}{n+1} e_N \leqslant \frac{\mathrm{lct}(X,\mathcal{I})}{\gamma} \tag{3.4}$$

for any $N \in \mathbb{N}, \lambda > 0$ since $F_N(\epsilon) \to e_N$ as $\epsilon \searrow 0$ (Lemma 3.4).

The function $g_N : \lambda \in (0, +\infty) \mapsto \lambda - \frac{\lambda^{n+1}}{n+1} e_N \in \mathbb{R}$ reaches its maximum at $\lambda_{N,M} := 1/e_N^{1/n}$. It follows therefore from (3.4) that

$$\gamma \leqslant \frac{\mathrm{lct}(X,\mathcal{I})}{g_N(\lambda_{N,M})} = \frac{n+1}{n} \mathrm{lct}(X,\mathcal{I}) e_N^{1/n}.$$

Now $e_N \searrow e(X, \mathcal{I})$ as $N \to +\infty$ by Lemma 3.4, hence

$$\gamma \leqslant \frac{n+1}{n} \mathrm{lct}(X,\mathcal{I}) e(X,\mathcal{I})^{1/n}.$$

Since this holds for any coherent ideal sheaf $\mathcal{I}$ supported at p , we obtain

$$\gamma \leqslant \frac{n+1}{n} \inf_{\mathcal{I}} \mathrm{lct}(X,\mathcal{I}) e(X,\mathcal{I})^{1/n} = \frac{n+1}{n} \widehat{\mathrm{vol}}(X,p)^{1/n}$$

where the equality follows from Theorem 1.16. □

4. Moser-Trudinger inequality

4.1 Uniform integrability vs Moser-Trudinger inequality

Recall that

$$\alpha(X,\mu_p) := \sup\left\{\alpha>0,\ \sup_{\varphi\in\mathcal{F}_1(\Omega)}\int_\Omega e^{-\alpha\varphi}d\mu_p<+\infty\right\}$$

This uniform integrability index is a local counterpart to Tian's celebrated α-invariant, introduced in [Tian87] in the quest for Kähler-Einstein metrics on Fano manifolds. We refer to [DK01, Dem09, Zer09, ACKPZ09, DP14, GZh15, Pham18] for some contributions to the local study of analogous invariants.

In this section we prove Theorem A, which can be seen as a local analog of [BBEGZ19, Proposition 4.13].

Theorem 4.1. *One has* $\gamma_{crit}(X,p)\geqslant\frac{n+1}{n}\alpha(X,\mu_p)$.

When (X,p) is smooth then $\alpha(X,\mu_p)=n$ and this statement is equivalent (after an appropriate rescaling) to [BB22, Theorem 1.5], [GKY13, Theorem 9].

Together with Theorem 3.5, we would obtain the precise value

$$\gamma_{crit}(X,p)\overset{?}{=}\frac{n+1}{n}\widehat{\mathrm{vol}}(X,p)^{1/n}$$

if we knew that $\alpha(X,\mu_p)=\widehat{\mathrm{vol}}(X,p)^{1/n}$. We establish in Section 4.2 the bound $\alpha(X,\mu_p)\leqslant\widehat{\mathrm{vol}}(X,p)^{1/n}$ and analyze the reverse inequality in Section 4.3.

4.1.1 *Entropy* We let $\mathcal{P}(\Omega)$ denote the set of probability measures on Ω. Given two measures $\mu,\nu\in\mathcal{P}(\Omega)$, the *relative entropy of ν with respect to μ* is

$$H_\mu(\nu):=\int_\Omega\frac{d\nu}{d\mu}\log\frac{d\nu}{d\mu}d\mu=\int_\Omega\log\frac{d\nu}{d\mu}d\nu$$

if ν is absolutely continuous with respect to μ, and as $H_\mu(\nu):=+\infty$ otherwise.

Given $\mu\in\mathcal{P}(X)$, the relative entropy $H_\mu(\cdot)$ is the Legendre transform of the convex functional $g\in\mathcal{C}^0(\Omega)\cap L^\infty(\Omega)\mapsto\log\int_\Omega e^g d\mu\in\mathbb{R}$, i.e.

$$H_\mu(\nu)=\sup_{g\in\mathcal{C}^0(\Omega)\cap L^\infty(\Omega)}\Big(\int_\Omega g\,d\nu-\log\int_\Omega e^g d\mu\Big).$$

We shall need the following duality result.

Lemma 4.2. *[BBEGZ19, Lemma 2.11] Fix $\mu\in\mathcal{P}(\Omega)$. Then*

$$\log\int_\Omega e^g d\mu=\sup_{\nu\in\mathcal{P}(\Omega)}\Big(\int_\Omega g\,d\nu-H_\mu(\nu)\Big)$$

for each lower semicontinuous function $g:\Omega\to\mathbb{R}\cup\{+\infty\}$.

Recall that we have normalized the adapted volume form so that $\mu_p\in\mathcal{P}(\Omega)$.

Corollary 4.3. *Fix $0<\alpha<\alpha(X,\mu_p)$. Then there exists $C_\alpha>0$ such that*

$$H_{\mu_p}(\nu)\geqslant-\alpha\int_\Omega\varphi\,d\nu-C_\alpha$$

for all $\varphi\in\mathcal{F}_1(\Omega)$ and for all $\nu\in\mathcal{P}(\Omega)$ such that $H_{\mu_p}(\nu)<+\infty$.

Proof. This follows from Lemma 4.2 applied to $g = -\alpha\varphi$ and $\mu = \mu_p$. By definition of $\alpha(X, \mu_p)$ we obtain $-\log \int_\Omega e^{-\alpha\varphi} d\mu_p \geqslant -C_\alpha$. □

This corollary shows in particular that $\mathcal{F}_1(\Omega) \subset L^1(\nu)$ for any probability measure $\nu \in \mathcal{P}(X)$ with finite μ_p-entropy. Since the measure ν is moreover *non pluripolar*, the following result is a consequence of Theorem 2.7.

Proposition 4.4. *Fix $\nu \in \mathcal{P}(\Omega)$ such that $H_{\mu_p}(\nu) < +\infty$. Then there exists a unique $v \in \mathcal{F}_1(\Omega) \cap \mathcal{E}^1(\Omega)$ such that*

$$\nu = (dd^c v)^n.$$

4.1.2 *Proof of Theorem 4.1* The proof is similar to the derivation of the Moser-Trudinger inequality from Brezis-Merle inequality by Berman-Berndtsson, see [BB22, Section 4.2]. Fix $\varphi \in \mathcal{T}_\phi(\Omega)$ and $0 < \alpha < \alpha(X, \mu_p)$. By Lemma 4.2 for any $\epsilon > 0$ there exists $\nu_\epsilon \in \mathcal{P}(\Omega)$ such that $H_{\mu_p}(\nu_\epsilon) < +\infty$ and

$$\log \int_\Omega e^{-\frac{n+1}{n}\alpha\varphi} d\mu_p \leqslant \epsilon - \frac{n+1}{n}\alpha \int_\Omega \varphi \, d\nu_\epsilon - H_{\mu_p}(\nu_\epsilon). \tag{4.1}$$

Proposition 4.4 ensures the existence of $v_\epsilon \in \mathcal{F}_1(\Omega) \cap \mathcal{E}^1(\Omega)$ such that $\nu_\epsilon = (dd^c v_\epsilon)^n$. It follows moreover from Corollary 4.3 that

$$H_{\mu_p}(\nu_\epsilon) \geqslant -\alpha \int_\Omega v_\epsilon \, d\nu_\epsilon - C_\alpha. \tag{4.2}$$

Combining (4.1) and (4.2) we obtain

$$\log \int_\Omega e^{-\frac{n+1}{n}\alpha\varphi} d\mu_p \leqslant \epsilon + C_\alpha - \frac{n+1}{n}\alpha \int_\Omega \varphi \, d\nu_\epsilon + \alpha \int_\Omega v_\epsilon \, d\nu_\epsilon. \tag{4.3}$$

We observe that

$$\begin{aligned} -\frac{n+1}{n}\alpha \int_\Omega \varphi \, d\nu_\epsilon + \alpha \int_\Omega v_\epsilon \, d\nu_\epsilon &= \frac{n+1}{n}\alpha \int_\Omega (v_\epsilon - \varphi)(dd^c v_\epsilon)^n - \frac{\alpha}{n} \int_\Omega v_\epsilon (dd^c v_\epsilon)^n \\ &\leqslant -\frac{n+1}{n}\alpha E_\phi(\varphi) + \frac{\alpha}{n}\left\{(n+1)E_\phi(v_\varepsilon), -\int_\Omega v_\epsilon (dd^c v_\epsilon)^n\right\} \end{aligned}$$

by using Lemma 2.5 (the latter has been stated for functions in $\mathcal{T}_\phi(\Omega)$, it easily extends to the class $\mathcal{F}_1(\Omega) \cap \mathcal{E}^1(\Omega)$ by approximation). Since $v_\varepsilon \leqslant \phi_0$ and $E_\phi(\phi_0) = 0$, the same Lemma ensures

$$(n+1)E_\phi(v_\varepsilon) - \int_\Omega v_\epsilon (dd^c v_\epsilon)^n \leqslant -\int_\Omega \phi_0 (dd^c v_\varepsilon)^n \leqslant -\inf_\Omega \phi_0,$$

using that $\nu_\epsilon = (dd^c v_\epsilon)^n$ is a probability measure. Altogether this yields

$$\log \int_\Omega e^{-\frac{n+1}{n}\alpha\varphi} d\mu_p \leqslant \epsilon + C_\alpha - \frac{\alpha}{n}\inf_\Omega \phi_0 - \frac{n+1}{n}\alpha E_\phi(\varphi).$$

Letting $\epsilon \searrow 0$ we conclude that

$$\Big(\int_\Omega e^{-\frac{n+1}{n}\alpha\varphi} d\mu_p\Big)^{\frac{n}{(n+1)\alpha}} \leqslant C'_\alpha e^{-E_\phi(\varphi)}$$

for any function $\varphi \in \mathcal{T}_\phi(\Omega)$. Thus $\gamma_{crit}(X, p) \geqslant \frac{n+1}{n}\alpha(X, \mu_p)$.

4.2 Upper bound on the α-invariant

Definition 4.5. We set

$$\tilde{\alpha}(X,\mu_p) := \inf\{c(\varphi),\ \varphi\in\mathcal{F}_1(\Omega)\},$$

where $c(\varphi) := \sup\left\{c>0;\ \int_\Omega e^{-c\varphi}d\mu_p<+\infty\right\}$.

4.2.1 *Bounding the α-invariant by the normalized volume*

Proposition 4.6. *One has $\alpha(X,\mu_p)\leqslant\tilde{\alpha}(X,\mu_p)\leqslant\widehat{\mathrm{vol}}(X,p)^{1/n}$.*

Proof. It follows from the definition that $\alpha(X,\mu_p)\leqslant\tilde{\alpha}(X,\mu_p)$.

For any $\epsilon>0$ and $\mathcal{I}$ coherent ideal sheaf supported at 0, the function

$$\psi_{\mathcal{I},\epsilon} := \psi_{\mathcal{I},\lambda,\epsilon},\quad \text{with}\ \ \lambda=\left(\frac{1-\epsilon}{e(X,\mathcal{I})}\right)^{1/n}$$

given by Lemma 4.7 below, belongs to $\mathcal{F}_1(\Omega)$ and yields

$$\tilde{\alpha}(X,\mu_p)\leqslant c(\psi_{\mathcal{I},\epsilon})=\frac{1}{(1-\epsilon)^{1/n}}\mathrm{lct}(X,\mathcal{I})e(X,\mathcal{I})^{1/n}.$$

The latter equality is a consequence of Proposition 3.1. We conclude the proof by taking the infimum over all $\mathcal{I}$'s and letting $\epsilon\searrow 0$. □

Lemma 4.7. *Let $\mathcal{I}$ be a coherent ideal sheaf supported at p. Then, for any $\lambda,\epsilon>0$ there exists a function $\psi_{\mathcal{I},\lambda,\epsilon}\in\mathcal{F}(\Omega)$ s.t.*

i) $\psi_{\mathcal{I},\lambda,\epsilon}=\lambda\log\left(\sum_{j=1}^m|f_j|^2\right)$ *near 0 for local generators $f_1,\dots,f_m$ of $\mathcal{I}$;*

ii) $\lambda^n e(X,\mathcal{I})\leqslant\int_\Omega\left(dd^c\psi_{\mathcal{I},\lambda,\epsilon}\right)^n\leqslant\lambda^n e(X,\mathcal{I})+\epsilon$.

Proof. Assume that ϕ_0 is the maximal psh extension of ϕ to Ω, i.e. the largest psh function in Ω which lies below ϕ on $\partial\Omega$. It satisfies $(dd^c\phi_0)^n=0$ in Ω.

Fix $f_1,\dots,f_m$ local generators of the ideal $\mathcal{I}$ and set $\psi_\lambda:=\lambda\log\left(\sum_{j=1}^m|f_j|^2\right)$. We can assume without loss of generality that the f_j's are well-defined in Ω and normalized so that $\psi_\lambda\leqslant\phi_0-1$ in Ω. For $r>0$ we consider

$$\varphi_r := \sup\{u\in PSH(\Omega),\ u\leqslant\psi_\lambda\ \text{in}\ B(r)\ \ \text{and}\ \ u\leqslant\phi_0\ \text{in}\ \Omega\}.$$

The corresponding family of psh functions is non empty as it contains ψ_λ. For $A>1$ large enough, the function

$$w_r=\begin{cases}\psi_\lambda & \text{in } B(r)\\ \max(\psi_\lambda,A\rho+\phi_0) & \text{in } \Omega\setminus B(r)\end{cases}$$

is psh and coincides with $A\rho+\phi_0$ near $\partial\Omega$. It follows that

- $\varphi_r\in PSH(\Omega)$ with $\varphi_r=\phi$ on $\partial\Omega$;
- $\varphi_r\equiv\psi_\lambda$ in $B(r)$ hence $\lambda^n e(X,\mathcal{I})\leqslant\int_\Omega(dd^c\varphi_r)^n$;
- $(dd^c\varphi_r)^n=0$ in $\Omega\setminus\overline{B}(r)$ (balayage argument).

The family $r\mapsto\varphi_r$ increases -as $r>0$ decreases to 0- to some psh limit φ whose Monge-Ampère measure $(dd^c\varphi)^n$ is concentrated at the origin. It follows from Bedford-Taylor continuity theorem that $(dd^c\varphi)^n$ is the weak limit of $(dd^c\varphi_r)^n\geqslant\lambda^n e(X,\mathcal{I})\delta_0$, hence $(dd^c\varphi)^n\geqslant\lambda^n e(X,\mathcal{I})\delta_0$. Conversely $\psi_\lambda\leqslant\varphi$ near 0, hence Demailly's comparison theorem ensures that

$$(dd^c\varphi)^n(0)\leqslant(dd^c\psi_\lambda)^n(0)\leqslant\lambda^n e(X,\mathcal{I}),$$

whence equality. Thus $\phi_{\mathcal{I},\lambda,\epsilon} := \varphi_{r_\varepsilon}$ satisfies the required properties. □

4.2.2 *Normalized volume vs uniform integrability*

PROPOSITION 4.8. *One has* $\tilde{\alpha}(X,\mu_p) = \widehat{\mathrm{vol}}(X,p)^{1/n}$.

We refer the reader to the Appendix for a more algebraic approach based on [BdFF12], which moreover provides a slightly stronger result.

When (X,p) is smooth, it follows from [DK01] that $\tilde{\alpha}(X,\mu_p) = \alpha(X,\mu_p)$. The situation is however more subtle in the singular context (see Section 4.3.2).

Proof. By Proposition 4.6 it suffices to show that $\tilde{\alpha}(X,\mu_p) \geqslant \widehat{\mathrm{vol}}(X,p)^{1/n}$, i.e. $\int_\Omega e^{-\alpha\varphi}d\mu_p < +\infty$ for all $\varphi \in \mathcal{F}_1(\Omega)$ and $\alpha < \widehat{\mathrm{vol}}(X,p)^{1/n} = \inf_{\mathcal{I}} \mathrm{lct}(X,\mathcal{I})^n e_p(\mathcal{I})$.

In a log resolution $\pi : \tilde{\Omega} \to \Omega$, this boils down to $\int_{\tilde{\Omega}} e^{-\alpha\varphi\circ\pi}\prod_{i=1}^M |s_i|_{h_i}^{2a_i} dV < +\infty$, where s_i are holomorphic sections defining simple normal crossing exceptional divisors $E_1,\dots,E_M$, $K_{\tilde{\Omega}/\Omega} = \sum_{j=1}^M a_i E_i$ and where dV is a smooth volume form. The log terminal condition ensures that $a_i > -1$ for all $i = 1,\dots,M$.

As $\alpha < \widehat{\mathrm{vol}}(X,p)^{1/n} \leqslant n$, the integrability condition is equivalent to show that for any point $x \in \cup_{i=1}^M E_i$ there exists a small ball $B(x,r)$ such that

$$\int_{B(x,r)} e^{-\alpha\varphi\circ\pi}\prod_{i=1}^M |s_i|_{h_i}^{2a_i} dV < +\infty. \tag{4.4}$$

Set $U := \sum_{i\,:\,a_i\geqslant 0} a_i \log|s_i|_{h_i}^2$, $V := \alpha\varphi\circ\pi$ and $W := -\sum_{i\,:\,a_i<0} a_i \log|s_i|_{h_i}^2$. By [BBJ21, Theorem B.5] the condition (4.4) holds iff there exists $\epsilon > 0$ such that

$$\nu(U\circ g, F) + A_{\tilde{\Omega}}(F) \geqslant (1+\epsilon)\nu(V\circ g, F) + (1+\epsilon)\nu(W\circ g, F) \tag{4.5}$$

for any F prime divisor *over* $\tilde{\Omega}$ with center in a small ball $\overline{B(x,r')} \subset B(x,r)$, i.e. $F \subset \Omega'$ for $g : \Omega' \to \tilde{\Omega}$ modification. Observe that

$$\begin{aligned}\nu(U\circ g, F) + A_{\tilde{\Omega}}(F) - \nu(W\circ g, F) &= \mathrm{ord}_F(g^* K_{\tilde{\Omega}/\Omega}) + 1 + \mathrm{ord}_F(K_{\Omega'/\tilde{\Omega}}) \\ &= 1 + \mathrm{ord}_F(K_{\Omega'/\Omega}) = A_\Omega(F).\end{aligned}$$

Thus (4.5) becomes

$$\alpha(1+\epsilon) \leqslant \frac{A_\Omega(F) - \epsilon\nu(W\circ g, F)}{\nu(\varphi\circ\pi\circ g, F)}. \tag{4.6}$$

As $a_i > -1$ for all i, [BBJ21, Theorem B.5] ensures the existence of $a > 0$ such that $A_{\tilde{\Omega}}(F) \geqslant (1+a)\nu(W\circ g, F)$ for any prime divisor F over $\tilde{\Omega}$ as above. Thus

$$A_{\tilde{\Omega}}(F) \leqslant A_\Omega(F) + \nu(W\circ g, F) \leqslant \frac{1}{1+a} A_{\tilde{\Omega}}(F) + A_\Omega(F),$$

and $\nu(W\circ g, F) \leqslant \frac{1}{1+a} A_{\tilde{\Omega}}(F) \leqslant \frac{1}{a} A_\Omega(F)$. Therefore (4.6) holds if

$$\alpha(1+\epsilon) \leqslant \frac{a-\epsilon}{a}\frac{A_\Omega(F)}{\nu(\varphi\circ\pi\circ g, F)}. \tag{4.7}$$

Since $\varphi \in \mathcal{F}_1(\Omega)$, it follows from the comparison theorem of Demailly [Dem85, Theorem 4.2] that for a coherent ideal sheaf $\mathcal{I}$ supported at $p \in \Omega$,

$$1 \geqslant \int_\Omega (dd^c\varphi)^n \geqslant \nu_{\mathcal{I}}(\varphi,p)^n \int_{\mathbb{C}^N} (dd^c f_{\mathcal{I}})^n \wedge [X] = \nu_{\mathcal{I}}(\varphi,p)^n e_p(\mathcal{I}) \tag{4.8}$$

where $f_{\mathcal{I}} = \log(\sum_i |f_i|^2)$ for generators $\{f_i\}_i$ of $\mathcal{I}$ and

$$\nu_{\mathcal{I}}(\varphi, p) := \sup\big\{s > 0 \,:\, \varphi \leqslant s f_{\mathcal{I}} + O(1)\big\} = \min_G \frac{\nu(\varphi \circ \pi \circ g, G)}{\mathrm{ord}_G \mathcal{I}},$$

where $\pi \circ g : \Omega' \to \Omega$ is a log resolution for $\mathcal{I}$ and the minimum is over all exceptional divisors of $\Omega' \to \Omega$. Lemma 4.9 below ensures that for any prime divisor F and $\delta > 0$ there exists an ideal $\mathcal{I}$ such that

$$\begin{aligned}\frac{A_\Omega(F)}{\nu(\varphi \circ \pi \circ p, F)} &\geqslant (1-\delta)\frac{A_\Omega(F)}{\mathrm{ord}_F \mathcal{I}}\big(\nu_{\mathcal{I}}(\varphi, p)\big)^{-1} \geqslant (1-\delta)\frac{A_\Omega(F)}{\mathrm{ord}_F \mathcal{I}} e_p(\mathcal{I})^{1/n} \\ &\geqslant (1-\delta)\mathrm{lct}(X, \mathcal{I}) e_p(\mathcal{I})^{1/n} \geqslant (1-\delta)\widehat{\mathrm{vol}}(X, p)^{1/n}.\end{aligned}$$

Thus (4.7) holds if $\alpha(1+\epsilon) \leqslant \frac{(a-\epsilon)(1-\delta)}{a}\widehat{\mathrm{vol}}(X,p)^{1/n}$, concluding the proof. □

LEMMA 4.9. *Fix $\varphi \in \mathcal{F}_1(\Omega)$ and $F \subset \Omega'$ prime divisor such that $\pi \circ g(F) = p$. For any $\epsilon > 0$ there exists a coherent ideal sheaf $\mathcal{I}$ supported at p such that*

$$\nu_{\mathcal{I}}(\varphi, p) \geqslant (1-\epsilon)\frac{\nu(\varphi \circ \pi \circ g, F)}{\mathrm{ord}_F(\mathcal{I})}.$$

Proof. Let $c := \nu(\varphi \circ \pi \circ g, F)$ and for $c' \in \mathbb{Q}, c' \leqslant c$, set

$$\mathcal{A}_{mc'}(F) := \big\{f \in \mathcal{O}_{X,p} \,:\, \mathrm{ord}_F(f \circ \pi \circ g) \geqslant mc'\big\}$$

for $m \in \mathbb{N}$ divisible enough. Then $\mathcal{A}_{mc'}(F)$ is an ideal sheaf and

$$\limsup_{m \to +\infty} \frac{\mathrm{ord}_F(\mathcal{A}_{mc'}(F))}{m} = c'. \tag{4.9}$$

In particular if $\varphi_{mc'} \in \mathrm{PSH}\big(B(p,r)\big)$ has algebraic singularities along $\mathcal{A}_{mc'}(F)$, then for any $\epsilon > 0$, $\varphi_{mc'}$ is less singular than $\frac{mc'}{c-\epsilon}\varphi$ around p if $m \geqslant m_1(\epsilon) \gg 1$. For any G exceptional divisor on Ω' and $m \geqslant m_1(\epsilon)$ we infer

$$\frac{\nu(\varphi \circ \pi \circ g, G)}{\mathrm{ord}_G(\mathcal{A}_{mc'}(F))/m} = \frac{\nu(\varphi \circ \pi \circ g, G)}{\nu(\varphi_{mc'} \circ \pi \circ g, G)/m} \geqslant \frac{c-\epsilon}{c'}. \tag{4.10}$$

On the other hand, (4.9) implies that there exists $m_0(\epsilon) \geqslant m_1(\epsilon) \gg 1$ with

$$\frac{\nu(\varphi \circ \pi \circ g, F)}{\mathrm{ord}_F(\mathcal{A}_{m_0 c'}(F))/m_0} \leqslant \frac{c}{c'-\epsilon}. \tag{4.11}$$

Combining (4.10) and (4.11) we obtain

$$\begin{aligned}\min_G \frac{\nu(\varphi \circ \pi \circ g, G)}{\mathrm{ord}_G(\mathcal{A}_{m_0 c'}(F))/m} \geqslant \frac{c-\epsilon}{c'} &\geqslant \Big(1 - \epsilon\frac{c+c'}{cc'}\Big)\frac{c}{c'-\epsilon} \\ &\geqslant \Big(1 - \epsilon\frac{c+c'}{cc'}\Big)\frac{\nu(\varphi \circ \pi \circ g, F)}{\mathrm{ord}_F(\mathcal{A}_{m_0 c'}(F))/m}.\end{aligned}$$

Since c' and ϵ are arbitrary, and $x \mapsto f(x) = \frac{c+x}{cx}$ is decreasing, we deduce that for any $\epsilon > 0$ there exists $c' \in \mathbb{Q}$ and $m_0 = m_0(c, c', \epsilon)$ such that

$$\nu_{\mathcal{A}_{m_0 c'}(F)}(\varphi, p) \geqslant (1-\epsilon)\frac{\nu(\varphi \circ \pi \circ g, F)}{\mathrm{ord}_F(\mathcal{A}_{m_0 c'}(F))}.$$

Setting $\mathcal{A} := \mathcal{A}_{m_0 c'}(F)$ concludes the proof. □

4.3 Lower bounds on the α-invariant

We provide in this section two effective (but not sharp) lower bounds on $\alpha(X, \mu_p)$.

4.3.1 *Using projections on n-planes* A result of Skoda ensures that $e^{-\varphi}$ is integrable if the Lelong numbers of φ are small enough (see [GZ, Theorem 2.50]). This has been largely extended by Demailly and Zeriahi who provided uniform integrability results for functions $\varphi \in \mathcal{F}_1(\Omega)$ [Dem09, ACKPZ09]. In this section we extend these results to our singular setting.

Theorem 4.10. *One has* $\alpha(X, \mu_p) \geqslant \frac{n}{\mathrm{mult}(X,p)^{1-1/n}} \frac{\mathrm{lct}(X,p)}{1+\mathrm{lct}(X,p)}$.

Proof. Recall that $\mu_p = f dV_X$ with $f \in L^r(dV_X)$. The exponent $r > 1$ has been estimated in Lemma 1.14. Using Hölder inequality, we thus obtain

$$\alpha(X, \mu_p) \geqslant \frac{\mathrm{lct}(X,p)}{1+\mathrm{lct}(X,p)} \alpha(\Omega, dV_X).$$

The remainder of the proof consists in establishing the lower bound $\alpha(\Omega, dV_X) \geqslant \frac{n}{\mathrm{mult}(X,p)^{1-1/n}}$.

Recall that $dV_X = \omega_{eucl}^n \wedge [X]$, where ω_{eucl} denotes the euclidean Kähler form. Thus $dV_X = \sum_I (\pi_I)^*(dV_I)$, where $I = (i_1, \ldots, i_n)$ is a n-tuple, $\pi_I : \mathbb{C}^N \to \mathbb{C}^n_I$ denotes the linear projection on $\mathbb{C}^n_I$, and dV_I is the euclidean volume form on $\mathbb{C}^n_I$. We choose coordinates in $\mathbb{C}^N$ so that each projection map $\pi_I : \Omega \to \Omega_I \subset \mathbb{C}^n$ is proper. For $\varphi \in \mathcal{F}_1(\Omega)$ we obtain

$$\int_\Omega e^{-\alpha\varphi} dV_X = \sum_I \int_{\Omega_I} (\pi_I)_*(e^{-\alpha\varphi}) dV_I \leqslant \mathrm{mult}(X,p) \sum_I \int_{\Omega_I} e^{-\alpha(\pi_I)_*\varphi} dV_I.$$

We assume here -without loss of generality- that $\varphi \leqslant 0$, and use the (sub-optimal) inequality $(\pi_I)_*(e^{-\alpha\varphi}) \leqslant \mathrm{mult}(X,p) e^{-\alpha(\pi_I)_*\varphi}$. The function $\varphi_I := (\pi_I)_*\varphi$ is psh in $\Omega_I = \pi_I(\Omega)$, with boundary values $(\pi_I)_*(\phi)$. We claim that

$$\int_{\Omega_I} (dd^c\varphi_I)^n \leqslant \mathrm{mult}(X,p)^{n-1}. \tag{4.12}$$

Once this is established, it follows from the main result of [ACKPZ09] that for all $0 < \varepsilon$ small enough, there exists $C_\varepsilon > 0$ independent of φ such that

$$\int_{\Omega_I} e^{-\frac{n-\varepsilon}{\mathrm{mult}(X,p)^{1-1/n}}\varphi_I} dV_I \leqslant C_\varepsilon,$$

which yields the desired lower bound $\alpha(\Omega, dV_X) \geqslant \frac{n}{\mathrm{mult}(X,p)^{1-1/n}}$.

It remains to check (4.12). We decompose $\varphi_I(z) = \sum_{i=1}^m \varphi(x_i)$, where $m = \mathrm{mult}(X,p)$ and $x_1, \ldots, x_m$ denote the preimages of z counted with multiplicities. The assumption on the Monge-Ampère mass of φ reads

$$\sum_{i=1}^m \int (dd^c\varphi)^n(x_i) \leqslant 1.$$

We set $a_i^n := \int (dd^c\varphi)^n(x_i)$ and use [Ceg04, Corollary 5.6] to estimate

$$\int (dd^c\varphi_I)^n = \sum_{i_1,\ldots,i_n=1}^m \int dd^c\varphi(x_{i_1}) \wedge \cdots \wedge dd^c\varphi(x_{i_n}) \leqslant \sum_{i_1,\ldots,i_n=1}^m a_{i_1} \cdots a_{i_n} = \left(\sum_i^m a_i\right)^n.$$

The latter sum is maximized when $a_1 = \cdots = a_m = m^{-1/n}$, yielding (4.12). □

EXAMPLE 4.11. Let $X = \{z \in \mathbb{C}^{n+1},\ F(z) = 0\}$ be the A_k-singularity, where $F(z) = z_0^{k+1} + z_1^2 + \cdots z_n^2$. Arguing as we have done for the ODP ($k = 1$), one can check that $\mu_p \sim \frac{dV_X}{||F'||^2}$ so that $\mathrm{mult}(X,p) = 2$ and $\mathrm{lct}(X,p) = n - 2 + \frac{2}{k+1}$. Now

$$\widehat{\mathrm{vol}}(A_k,p)^{1/n} = \begin{cases} 2^{1/n}\big(\frac{n-2}{n-1}\big)^{1-1/n} n & \text{if } \frac{k+1}{2} \geqslant \frac{n-1}{n-2}, \\ (k+1)^{1/n}\big(\frac{(n-2)(k+1)+2}{k+1}\big) & \text{if } \frac{k+1}{2} < \frac{n-1}{n-2}, \end{cases}$$

as computed by C.Li in [Li18, Example 5.3]. For $n >> 1$, the lower bound provided by Theorem 4.10 is thus short of a factor $2 = \mathrm{mult}(X,p)$ by comparison with the expected lower bound $\widehat{\mathrm{vol}}(A_k,p)^{1/n}$.

4.3.2 *Using resolutions*

PROPOSITION 4.12. *Let $\pi : \tilde{\Omega} \to \Omega$ be a resolution of singularities with simple normal crossing, and let $\{a_i\}_{i=1,\dots,M}$ be the discrepancies. Then*

$$\alpha(X,\mu_p) \geqslant \frac{\widehat{\mathrm{vol}}(X,p)^{1/n}}{1 + \big(\max_i a_i\big)_+}.$$

In particular if the singularity is "admissible" then $\alpha(X,\mu_p) = \widehat{\mathrm{vol}}(X,p)^{1/n}$.

Following [LTW21, Definition 1.1] we say here that (X,p) is an admissible singularity if there exists a resolution $\pi : \tilde{X} \to X$ (with snc exceptional divisor $E = \sum_j E_j$ and π- ample divisor $-\sum b_j E_j$, $b_j \in \mathbb{Q}^+$) such that the discrepancies $a_i \in (-1, 0]$ are all non-positive. Recall that

- any 2-dimensional log terminal singularity is admissible;
- the vertex of the affine cone over a Fano manifold embedded in a projective space by the linear system associated to a multiple of the anticanonical bundle is admissible (cf. the proof of Proposition 2.17);
- (X,p) is admissible if it is $\mathbb{Q}$-factorial and admits a crepant resolution.

Theorem B from the introduction follows from the combination of Proposition 4.6, Theorem 4.10 and Proposition 4.12.

Proof. We seek $\alpha > 0$ such that

$$\sup_{\varphi \in \mathcal{F}_1(\Omega)} \int_{\tilde{\Omega}} e^{-\alpha\varphi\circ\pi} \prod_{i=1}^M |s_i|_{h_i}^{2a_i} dV < +\infty. \tag{4.13}$$

If all the a_i's are non-positive we can use [DK01, Main Theorem] to show that $\alpha(X,\mu_p) = \tilde{\alpha}(X,\mu_p)$, hence $\alpha(X,\mu_p) = \widehat{\mathrm{vol}}(X,p)^{1/n}$ by Proposition 4.8. Indeed assume that there exists $\gamma > 0$ such that $\alpha(X,\mu_p) < \gamma < \tilde{\alpha}(X,\mu_p)$. By definition we can find $\psi_j \in \mathcal{F}_1(\Omega)$ such that $\int_\Omega e^{-\gamma\psi_j} d\mu_p \to +\infty$. Extracting and relabelling, we can assume that $\psi_j \to \psi$ in L^1 with $c(\psi) > \gamma$. The psh functions $\varphi_j = \psi_j + \gamma^{-1}\sum_{i=1}^M (-a_i)\log|s_i|_{h_i}^2$ converge to $\varphi = \psi + \gamma^{-1}\sum_{i=1}^M (-a_i)\log|s_i|_{h_i}^2$ in $L^1(\tilde{\Omega})$ and $c(\varphi) > \gamma$. It follows therefore from [DK01, Theorem 0.2.2] that

$$\int_\Omega e^{-\gamma\psi_j} d\mu_p = \int_{\tilde{\Omega}} e^{-\gamma\varphi_j} dV \longrightarrow \int_{\tilde{\Omega}} e^{-\gamma\varphi} dV < +\infty,$$

contradicting the assumption $\int_\Omega e^{-\gamma\psi_j} d\mu_p \to +\infty$.

In general we set $U := \sum_{i:a_i>0} a_i \log|s_i|^2_{h_i}$ and $W := -\sum_{i:a_i\leqslant 0} a_i \log|s_i|^2_{h_i}$. Using [DK01, Main Theorem] we obtain

$$\alpha(X,\mu_p) \geqslant \inf_{\varphi\in\mathcal{F}_1(\Omega)} c_W(\varphi\circ\pi). \tag{4.14}$$

where

$$c_W(\varphi\circ\pi) := \sup\big\{\alpha>0 \,:\, \int_{\tilde{\Omega}} e^{-\alpha\varphi\circ\pi - W} dV < +\infty\big\}.$$

is the twisted complex singularity exponent. It then remains to estimate $c_W(\varphi\circ\pi)$ for a fixed $\varphi\in\mathcal{F}_1(\Omega)$. As $\pi^* d\mu_p = e^{U-W}dV$, Hölder inequality yields

$$\int_{\tilde{\Omega}} e^{-\alpha\varphi\circ\pi - W} dV \leqslant \Big(\int_{\tilde{\Omega}} e^{-p'\alpha\varphi\circ\pi}\pi^* d\mu_p\Big)^{1/p'} \Big(\int_{\tilde{\Omega}} e^{(1-q')U-W} dV\Big)^{1/q'}. \tag{4.15}$$

Set $A := (\max_i a_i)_+ > 0$. The second factor on the right-hand side of (4.15) is finite for any $q' < \frac{A+1}{A}$, while the first factor on the right-hand side gives the condition $p'\alpha < \tilde{\alpha} = \widehat{\mathrm{vol}}(X,p)^{1/n}$. We infer $c_W(\varphi\circ\pi) \geqslant \frac{\widehat{\mathrm{vol}}(X,p)^{1/n}}{1+A}$, which concludes the proof. □

As the proof shows, the main obstruction to proving the equality $\alpha(X,\mu_p) = \tilde{\alpha}(X,\mu_p) = \widehat{\mathrm{vol}}(X,p)^{1/n}$ is the lack of a Demailly-Kollàr result on complex spaces. Resolving the singularities, one ends up with a twisted version of Demailly-Kollàr's problem on a smooth manifold. It is known that the general form of such a problem has a negative answer [Pham14, Remark 1.3].

5. Ricci inverse iteration

In this section we prove Theorem C from the introduction. The strategy is similar to that of [GKY13, Theorem 1], with a singular twist.

We fix $\gamma < \gamma_{crit}(X,p)$ and consider, for $j\in\mathbb{N}$, the sequence of functions $\varphi_j \in PSH(\Omega)$ defined by induction as follows: pick $\varphi_0\in\mathcal{T}^\infty_\phi(\Omega)$ a smooth initial data, and let $\varphi_{j+1}\in PSH(\Omega)\cap\mathcal{C}^0(\overline{\Omega})\cap\mathcal{C}^\infty(\overline{\Omega}\setminus\{p\})$ be the unique solution to

$$(dd^c\varphi_{j+1})^n = \frac{e^{-\gamma\varphi_j}\mu_p}{\int_\Omega e^{-\gamma\varphi_j}\mu_p}$$

with boundary values $\varphi_{j+1|\partial\Omega} = \phi$. The existence and regularity of φ_j off the singular locus follows from [Fu23, Theorem 1.4], while the continuity of φ_j near p is a consequence of [GGZ23, Theorem A].

We are going to establish uniform a priori estimates on arbitrary derivatives of the φ_j's in $\overline{\Omega}\setminus\{p\}$, thus (φ_j) admits "smooth" cluster values. We will show that the functional F_γ is constant on the set $\mathcal{K}$ of these cluster points, so that any such ψ is a solution of the Monge-Ampère equation

$$(dd^c\psi)^n = \frac{e^{-\gamma\psi}\mu_p}{\int_\Omega e^{-\gamma\psi}\mu_p}$$

with boundary values $\psi_{|\partial\Omega} = \phi$.

5.1 Uniform estimates

Proposition 5.1. *There exists $C_0>0$ such that $||\varphi_j||_{L^\infty(\Omega)} \leqslant C_0$ for all $j\in\mathbb{N}$.*

This uniform estimate relies crucially on a technique introduced by Kolodziej in [Kol98], which has been extended to this singular setting in [GGZ23].

Proof. We assume without loss of generality that ϕ_0 is the maximal psh extension of ϕ in Ω. In particular $\varphi_j \leqslant \phi_0$ for all $j \in \mathbb{N}$, and $E_\phi(\varphi_j) \leqslant E_\phi(\phi_0) = 0$. Our task is to establish a uniform lower bound $\varphi_j \geqslant -C_0$.

The assumption $\gamma < \gamma_{crit}(X,p)$ ensures, by Lemma 2.13, that the functional F_γ is coercive, in particular there exist $0 < a < 1$ and $0 < b$ such that

$$F_\gamma(\varphi_j) \leqslant aE_\phi(\varphi_j) + b$$

for all $j \in \mathbb{N}$. It follows from [GKY13, Proposition 12] (exactly the same proof applies here) that $j \mapsto F_\gamma(\varphi_j)$ is increasing, hence

$$F_\gamma(\varphi_0) \leqslant F_\gamma(\varphi_j) \leqslant aE_\phi(\varphi_j) + b \leqslant b,$$

showing that the energies $(E_\phi(\varphi_j))$ are uniformly bounded, $-b' \leqslant E_\phi(\varphi_j) \leqslant 0$.

The corresponding family $\mathcal{G}_{b'}$ of psh functions with ϕ-boundary values and energy bounded by b' is compact, and all its members have zero Lelong number at all points in Ω (see Theorem 2.7). Passing through a resolution, one can thus invoke Skoda's uniform integrability theorem [GZ, Theorem 2.50] to conclude that the densities $e^{-\gamma\varphi_j}$ are uniformly in $L^r(dV_X)$ for any $r > 1$.

Now $\mu_p = f dV_X$ with $f \in L^{1+\varepsilon}$ for some $\varepsilon > 0$ since (X,p) is log-terminal. Hölder inequality thus ensures that the densities $g_j := \frac{e^{-\gamma\varphi_j} f}{\int_\Omega e^{-\gamma\varphi_j} d\mu_p}$ are uniformly in $L^{1+\varepsilon'}(dV_X)$ for some $0 < \varepsilon' < \varepsilon$.

It therefore follows from [GGZ23, Proposition 1.8] (an extension of the main result of [Kol98] to the setting of pseudoconvex subsets of a singular complex space) that the φ_j's are uniformly bounded. □

5.2 $\mathcal{C}^2$-estimates

In this section we establish the following a priori estimates.

Proposition 5.2. *For all compact subset K of $\overline{\Omega} \setminus \{p\}$, there exists a constant $C_2(K) > 0$ such that for all $j \in \mathbb{N}$,*

$$0 \leqslant \sup_K \Delta_{\omega_X} \varphi_j \leqslant C_2(K).$$

Here $\Delta_{\omega_X} h := n \frac{dd^c h \wedge \omega_X^{n-1}}{\omega_X^n}$ denotes the Laplace operator with respect to the Kähler form ω_X. Such an estimate goes back to the regularity theory developed in [CKNS85]. The strategy of the proof is similar to that of [GKY13, Theorem 15], with a twist due to the presence of the singular point p.

Proof. To obtain these estimates, one considers a resolution of the singularity $\pi : \tilde{\Omega} \to \Omega$. We let $E = \cup_{\ell=1}^m E_\ell$ denote the exceptional divisor and let

- s_ℓ denote a holomorphic section of $\mathcal{O}(E_\ell)$ such that $E_\ell = (s_\ell = 0)$;
- b_ℓ be positive rational numbers such that $-\sum_\ell b_\ell E_\ell$ is π-ample;
- h_ℓ denote a smooth hermitian metric of $\mathcal{O}(E_\ell)$ and $K >> 1$ such that

$$\beta := K dd^c \rho \circ \pi - \sum_{\ell=1}^m b_\ell \Theta_{h_\ell} \text{ is a Kähler form on } \tilde{\Omega}.$$

Observe that the function $\rho' := K\rho \circ \pi + \sum_{\ell=1}^m b_\ell \log |s_\ell|^2_{h_\ell}$ is strictly psh in $\tilde{\Omega}$, with $dd^c\rho' \geqslant \beta$ and $\rho'(z) \to -\infty$ as $z \to E$.

Recall that $\pi^*\mu_p = \Pi_{\ell=1}^m |s_\ell|^{2a_\ell} dV_{\tilde\Omega}$ with $a_\ell > -1$, and set $|s|^2 = \Pi_{\ell=1}^m |s_\ell|^{2b_\ell}$. We are going to show that there exist uniform constants $C_2 > 0, m \in \mathbb{N}$ such that

$$0 \leqslant |s|^{2m} |\Delta_\beta \varphi_j|(z) \leqslant C_2 \tag{5.1}$$

for all $j \in \mathbb{N}, z \in \overline{\Omega}$, from which Proposition 5.2 follows. Slightly abusing notation, we still denote here by φ_j the function $\varphi_j \circ \pi$.

We approximate φ_j by the smooth solutions $\varphi_{j,\varepsilon}$ of the Dirichlet problem

$$\begin{cases} (\varepsilon\beta + dd^c\varphi_{j+1,\varepsilon})^n = \dfrac{e^{-\gamma\varphi_{j,\varepsilon}} \prod_{l=1}^m (|s_l|_{h_l}^2 + \varepsilon^2)^{a_l}}{c_j} dV_{\tilde\Omega} \\ \varphi_{j+1,\varepsilon|\partial\tilde\Omega} = \phi \end{cases} \tag{5.2}$$

with $\varphi_{0,\varepsilon} = \varphi_0$ and $c_j = \int_\Omega e^{-\gamma\varphi_j} d\mu_p$. We are going to establish a priori estimates on these smooth approximants, whose existence is guaranteed by [GL10, Theorem 1.1]. We then show that $\varphi_{j,\varepsilon}$ converges to φ_j as ε decreases to zero.

Step 1. We first claim that for all j, ε,

$$\sup_{\partial\tilde\Omega} |\nabla\varphi_{j+1,\varepsilon}| \leqslant A_{1,j,\varepsilon}, \tag{5.3}$$

where $A_{1,j,\varepsilon} > 0$ only depends on an upper-bound on $||\varphi_{j,\varepsilon}||_{L^\infty(\tilde\Omega)}$.

Let Φ^- be a smooth psh extension of $-\phi$ to a neighborhood of $\overline{\Omega}$. Observe that $\varphi_{j+1,\varepsilon} + \Phi^- \circ \pi$ is β-psh in $\tilde\Omega$, with zero boundary values. Thus $\varphi_{j+1,\varepsilon} + \Phi^- \circ \pi \leqslant u$, where u is the smooth solution in $\tilde\Omega$ to the Laplace equation $\Delta_\beta u = -n$ with zero boundary values. We infer $\varphi_{j+1,\varepsilon} \leqslant \psi_1 := u - \Phi^- \circ \pi$ in $\tilde\Omega$.

We now construct a psh function $\psi_2 \leqslant \varphi_{j+1,\varepsilon}$ with ϕ-boundary values and such that $\sup_{\partial\tilde\Omega} |\psi_2|$ is controlled from above by $||\varphi_{j,\varepsilon}||_{L^\infty(\tilde\Omega)}$. The upper bound on $\sup_{\partial\tilde\Omega} |\nabla\varphi_{j+1,\varepsilon}|$ thus follows from the inequalities $\psi_2 \leqslant \varphi_{j,\varepsilon} \leqslant \psi_1$.

Recall that $\pi^*\mu_p = \Pi_{\ell=1}^m |s_\ell|^{2a_\ell} dV_{\tilde\Omega}$. We let $P \subset [1, m]$ denote the subset of indices such that $-1 < a_\ell < 0$. For $\delta > 0$ small enough, we observe that $v := \rho' + \delta \sum_{\ell\in P} |s_\ell|^{2\delta}$ is strictly psh in $\tilde\Omega$ and satisfies, in $\tilde\Omega \setminus E$,

$$dd^c v \geqslant c \left\{ \beta + \sum_{\ell\in P} \frac{i ds_\ell \wedge d\overline{s_\ell}}{|s_\ell|^{2(1-\delta)}} \right\}$$

for some $c > 0$, hence $(dd^c v)^n \geqslant c'\pi^*\mu_p$. Replacing v by $\lambda_{j,\varepsilon} v$, we obtain

$$(\varepsilon\beta + dd^c\lambda_{j,\varepsilon} v)^n \geqslant \lambda_{j,\varepsilon}^n (dd^c v)^n \geqslant \frac{e^{-\gamma\varphi_{j,\varepsilon}} \prod_{l=1}^m (|s_l|_{h_l}^2 + \varepsilon^2)^{a_l}}{c_j} dV_{\tilde\Omega},$$

for some $\lambda_{j,\varepsilon} > 0$ which only depends on an upper-bound on $||\varphi_{j,\varepsilon}||_{L^\infty(\tilde\Omega)}$. In other words $\lambda_{j,\varepsilon} v$ is a subsolution to the Monge-Ampère equation in $\tilde\Omega \setminus E$.

We modify $\lambda_{j,\varepsilon} v$ near $\partial\tilde\Omega$ to produce a subsolution with the right boundary values. Let χ be a cut-off function which is 1 near E and has compact support in $\tilde\Omega$. The function $\psi_2 = \chi\lambda_{j,\varepsilon} v + (1-\chi)\phi_0 + A\rho \circ \pi$ satisfies all our requirements for $A > 0$ large enough. Note however that it is only locally bounded in $\tilde\Omega \setminus E$.

Consider finally $\max(\psi_2, \varphi_{j,\varepsilon})$. This is a subsolution of the Dirichlet problem which is globally bounded in $\tilde\Omega$. It follows from the maximum principle that $\max(\psi_2, \varphi_{j,\varepsilon}) \leqslant \varphi_{j,\varepsilon}$, hence $\psi_2 \leqslant \max(\psi_2, \varphi_{j,\varepsilon}) \leqslant \varphi_{j,\varepsilon}$.

Step 2. We next claim that there exist constants $A_2, A_{3,j+1,\varepsilon} > 0$ such that

$$\sup_{\tilde{\Omega}}[|s|_h^{2A_2}|\nabla\varphi_{j+1,\varepsilon}|_\beta^2] \leqslant A_{3,j+1,\varepsilon} \tag{5.4}$$

where $A_{3,j+1,\varepsilon}$ only depends on an upper-bound on $\|\varphi_{k,\varepsilon}\|_{L^\infty(\tilde{\Omega})}$ for $k \leqslant j+1$.

Proof. The proof is a variant of [DFS23, Proposition 2.2], which itself relies on previous estimates due to Blocki and Phong-Sturm.

As we work in $\tilde{\Omega} \setminus \mathrm{Supp}(E)$, we identify β with $dd^c\big(K\rho\circ\pi + \log|s|_h^2\big)$. Replacing $\varphi_{j+1,\epsilon}$ by $\tilde{\varphi}_{j+1,\epsilon} := \varphi_{j+1,\epsilon} - (K\rho\circ\pi + \log|s|_h^2)$ the equation (5.2) becomes

$$\begin{cases} \big((1+\epsilon)\beta + dd^c\tilde{\varphi}_{j+1,\epsilon}\big)^n = c_j^{-1}e^{-\gamma\varphi_{j,\epsilon}}\prod_{l=1}^m(|s_l|_{h_l}^2+\epsilon^2)^{a_l}dV_{\tilde{\Omega}}, \\ \tilde{\varphi}_{j+1,\epsilon|\partial\tilde{\Omega}} = \phi - \log|s|_h^2. \end{cases} \tag{5.5}$$

As

$$\Big||\nabla\varphi_{j,\epsilon}|_\beta - |\nabla\tilde{\varphi}_{j,\epsilon}|_\beta\Big| \leqslant \frac{|\nabla|s|_h^2|_\beta}{|s|_h} + C,$$

to get the estimate (5.4) for $\tilde{\varphi}_{j,\epsilon}$ it is enough to prove by induction that there exists positive constants $B_2, B_{3,j+1,\varepsilon}$ such that

$$\sup_{\tilde{\Omega}}[|s|_h^{2B_2}|\nabla\tilde{\varphi}_{j+1,\varepsilon}|_\beta^2] \leqslant \max\Big\{\sup_{\tilde{\Omega}}[|s|_h^{2B_2}|\nabla\tilde{\varphi}_{j,\varepsilon}|_\beta^2], B_{3,j+1,\varepsilon}\Big\} \tag{5.6}$$

where B_2 is uniform in j,ε while $B_{3,j+1,\varepsilon}$ only depends on upper bounds on $\|\varphi_{j+1,\varepsilon}\|_{L^\infty(\tilde{\Omega})}, \|\varphi_{j,\varepsilon}\|_{L^\infty(\tilde{\Omega})}$, and where $\tilde{\varphi}_{0,\varepsilon} := -\big(K\rho\circ\pi + \log|s|_h^2\big)$. To lighten notations we rewrite the equation

$$\begin{cases} \big(\beta_\epsilon + dd^c u\big)^n = e^{-v}\prod_{l=1}^m(|s_l|_{h_l}^2+\epsilon^2)^{a_l}\beta_\epsilon^n \\ u_{|\partial\tilde{\Omega}} = \tilde{\phi} \end{cases} \tag{5.7}$$

where $\beta_\epsilon := (1+\epsilon)\beta$ is a non-degenerate smooth family of Kähler forms. Note that (5.6) becomes

$$\sup_{\tilde{\Omega}}[|s|_h^{2B_2}|\nabla u|_\beta^2] \leqslant \max\Big\{\sup_{\tilde{\Omega}}[|s|_h^{2B_2}|\nabla(v/\gamma - \log|s|_h^2 - f_\epsilon)|_\beta^2], B_{3,j+1,\varepsilon}\Big\} \tag{5.8}$$

where $\{f_\epsilon\}_{\epsilon>0}$ is a non-degenerate smooth family. In the estimates that follows we indicate with C_i all the constants *under control*, i.e. that depend on a upper bounds on $\|\varphi_{j+1,\epsilon}\|_{L^\infty(\tilde{\Omega})}, \|\varphi_{j,\epsilon}\|_{L^\infty(\Omega)}$. Observe that $\|u+\log|s|_h^2\|_{L^\infty(\tilde{\Omega})}, \|v\|_{L^\infty(\tilde{\Omega})}$ and $\sup_{\partial\tilde{\Omega}}|\nabla u|$ are under control. The constant $B_{3,j+1,\varepsilon}$ in (5.8) will clearly depend on the C_i's. We indicate with D_i all the constants uniform in j,ε, which will be used to determine the uniform constant B_2 in (5.8).

We denote by $\Delta_\epsilon, \Delta'_\epsilon$ respectively the Laplacian operators with respect to β_ϵ and to $\eta_\epsilon := \beta_\epsilon + dd^c u$. Consider

$$H := \log|\nabla u|_{\beta_\epsilon}^2 + \log|s|_h^{2k} - G(u)$$

where $G(x) = Ax - \frac{B}{x+C+1}$ for C chosen so that $u \geqslant -C$, while $A>0, B>0$ to be determined later. The constants A,k are chosen to be uniform in j,ε while B is under control. If H reaches its maximum at x_M, then

$$|\nabla u|_{\beta_\epsilon}^2|s|_h^{2(k+A)} \leqslant C_1\big(|\nabla u|_{\beta_\epsilon}^2|s|_h^{2(k+A)}\big)(x_M) \tag{5.9}$$

for a constant C_1 under control.

As $u + \log|s|_h^2$ is smooth on $\tilde{\Omega}$, we ensure that $H(x) \simeq (k+A-1)\log|s|_h^2 \to -\infty$ as $x \to \mathrm{Supp}(E_j)$ by imposing $k \geqslant 1$. If H reaches its maximum on $\partial\tilde{\Omega}$ then we are done since $\sup_{\partial\tilde{\Omega}}|\nabla u|$

is under control. From now on we thus suppose that H reaches its maximum in $\tilde{\Omega}\setminus\{s=0\}$. A direct computation [PSS12, eq. (5.11), (5.20)] yields

$$\Delta'_\epsilon \log|\nabla u|^2_{\beta_\epsilon} \geqslant \frac{2\mathrm{Re}\langle \nabla v + \sum_{l=1}^m a_l \nabla \log\big(|s_l|^2_{h_l}+\epsilon^2\big), \nabla u\rangle_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}} - \Lambda \mathrm{tr}_{\eta_\epsilon}\beta_\epsilon + 2\mathrm{Re}\langle \frac{\nabla|\nabla u|^2_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}}, \frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\rangle_{\eta_\epsilon} - 2\mathrm{Re}\langle \frac{\nabla|\nabla u|^2_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}}, \frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\rangle_{\beta_\epsilon} \quad (5.10)$$

where Λ denotes a (uniform in ε) lower bound on the holomorphic bisectional curvature of β_ϵ. At the point where H reaches its maximum we obtain

$$\frac{\nabla|\nabla u|^2_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}} = \nabla \log|\nabla u|^2_{\beta_\epsilon} = -\nabla\big(\log|s|^{2k}_h - G(u)\big) = -\frac{k\nabla|s|^2_h}{|s|^2_h} + G'(u)\nabla u,$$

hence

$$\begin{aligned}
&2\mathrm{Re}\langle \frac{\nabla|\nabla u|^2_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}}, \frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\rangle_{\eta_\epsilon} - 2\mathrm{Re}\langle \frac{\nabla|\nabla u|^2_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}}, \frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\rangle_{\beta_\epsilon}\\
&= 2k\mathrm{Re}\langle \frac{\nabla|s|^2_h}{|s|^2_h}, \frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\rangle_{\beta_\epsilon} - 2k\mathrm{Re}\langle \frac{\nabla|s|^2_h}{|s|^2_h}, \frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\rangle_{\eta_\epsilon} + 2G'(u)\frac{|\nabla u|^2_{\eta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}} - 2G'(u)\\
&\geqslant 2k\mathrm{Re}\langle \frac{\nabla|s|^2_h}{|s|^2_h}, \frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\rangle_{\beta_\epsilon} - 2k\mathrm{Re}\langle \frac{\nabla|s|^2_h}{|s|^2_h}, \frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\rangle_{\eta_\epsilon} - 2G'(u),
\end{aligned}$$

using the monotonicity of $G(x)$ in the last inequality. By (5.9) and asking $k\geqslant 2$, we can assume that $|s|^2_h|\nabla u|_{\beta_\epsilon}\geqslant 1$ at x_M. Thus,

$$|2\mathrm{Re}\langle \frac{\nabla|s|^2_h}{|s|^2_h}, \frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\rangle_{\beta_\epsilon}| \leqslant 2|\mathrm{Re}\langle \nabla|s|^2_h, \frac{\nabla u}{|\nabla u|_{\beta_\epsilon}}\rangle_{\beta_\epsilon}| \leqslant D_1$$

and

$$\begin{aligned}
|2\mathrm{Re}\langle \frac{\nabla|s|^2_h}{|s|^2_h}, \frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\rangle_{\eta_\epsilon}| &\leqslant 2|\mathrm{Re}\langle \nabla|s|^2_h, \frac{\nabla u}{|\nabla u|_{\beta_\epsilon}}\rangle_{\eta_\epsilon}| \leqslant |\nabla|s|^2_h|^2_{\eta_\epsilon} + \frac{|s|^4_h|\nabla u|^2_{\eta_\epsilon}}{|s|^4_h|\nabla u|^2_{\beta_\epsilon}}\\
&\leqslant |\nabla|s|^2_h|^2_{\beta_\epsilon}\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon + |s|^4_h|\nabla u|^2_{\eta_\epsilon}.
\end{aligned}$$

We infer that at $x=x_M$,

$$\begin{aligned}
&2\mathrm{Re}\langle \frac{\nabla|\nabla u|^2_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}}, \frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\rangle_{\eta_\epsilon} - 2\mathrm{Re}\langle \frac{\nabla|\nabla u|^2_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}}, \frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\rangle_{\beta_\epsilon}\\
&\geqslant -kD_1 - k|\nabla|s|^2_h|^2_{\beta_\epsilon}\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon - k|s|^4_h|\nabla u|^2_{\eta_\epsilon} - 2G'(u)\\
&\geqslant -kD_1 - kD_2\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon - k|s|^4_h|\nabla u|^2_{\eta_\epsilon} - 2G'(u),
\end{aligned}$$

which is the first estimate of the right-hand side in (5.10).

Next, as we want to prove (5.8), as a consequence of (5.9) and of $|\nabla u|^2_\beta\leqslant D_3|\nabla u|^2_{\beta_\varepsilon}$ in the estimate that follows we can assume that

$$D_3C_1|\nabla u|^2_{\beta_\varepsilon}\geqslant \max\big\{|\nabla(v/\gamma-\log|s|^2_h-f_\varepsilon)|^2_\beta, 1\big\}$$

at the point x_M. We deduce

$$\Big|\frac{2\mathrm{Re}\langle\nabla v+\sum_{l=1}^m\nabla a_l\log(|s_l|^2_{h_l}+\epsilon^2),\nabla u\rangle_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}}\Big|$$
$$\leqslant D_4+\frac{|\nabla(v-\gamma\log|s|^2_h-\gamma f_\varepsilon)|^2_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}}+\frac{D_5}{|\nabla u|^2_{\beta_\varepsilon}}|s|^{-2}_h+\frac{D_6}{|\nabla u|^2_{\beta_\epsilon}}\sum_{l=1}^m|s_l|^{-2}_{h_l}\leqslant C_2+C_3|s|^{-2M}_h$$

for $M:=\frac{1}{\min_l b_l}$ so that $Mb_l\geqslant 1$ for any l. The previous inequalities yield

$$\Delta'_\epsilon\log|\nabla u|^2_{\beta_\epsilon}\geqslant -kD_1-C_2-C_3|s|^{-2M}_h-\big(kD_2+\Lambda\big)\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon-k|s|^4_h|\nabla u|^2_{\eta_\epsilon}-2G'(u). \tag{5.11}$$

Moreover

$$-\Delta'_\epsilon G(u)=-G'(u)\Delta'_\epsilon u-G''(u)|\nabla u|^2_{\eta_\epsilon}=G'(u)\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon-nG'(u)-G''(u)|\nabla u|^2_{\eta_\epsilon}$$

and $\Delta'_\epsilon\log|s|^{2k}_h\geqslant -kD_7\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon$. Together with (5.11) we obtain

$$\Delta'_\epsilon H\geqslant\big(G'-kD_2-\Lambda-kD_7\big)\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon-(n+2)G'-\big(G''+k|s|^4_h\big)|\nabla u|^2_{\eta_\epsilon}-kD_1-C_2-C_3|s|^{-2M}_h$$

Taking $k=M(n+1)+1$, this can be rewritten

$$\Delta'_\epsilon H\geqslant(G'-D_8)\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon-(n+2)G'-(G''+D_9|s|^4_h)|\nabla u|^2_{\eta_\epsilon}-C_4|s|^{-2M}_h. \tag{5.12}$$

We now define $G(x):=(D_8+1)x-\frac{B}{x+C+1}$ where $B>0$ is so large that

$$\frac{2B}{(u+C+1)^3}-D_9|s|^4_h\geqslant|s|^2_h$$

at x_M. Note that B can be chosen such that it only depends on C,D_9 and on $\|u+\log|s|^2_h\|_{L^\infty(\tilde\Omega)}$, i.e. it is under control. From (5.12) we deduce at x_M

$$0\geqslant\Delta'_\epsilon H\geqslant\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon+|s|^2_h|\nabla u|^2_{\eta_\epsilon}-C_5|s|^{-2M}_h.$$

This yields $\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon\leqslant C_5|s|^{-2M}_h$ and $|\nabla u|^2_{\eta_\epsilon}\leqslant C_5|s|^{-2M-2}_h$, hence

$$|\nabla u|^2_{\beta_\epsilon}\leqslant|\nabla u|^2_{\eta_\epsilon}\mathrm{tr}_{\beta_\epsilon}\eta_\epsilon\leqslant|\nabla u|^2_{\eta_\epsilon}\big(\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon\big)^{n-1}\big(\frac{\eta^n_\varepsilon}{\beta^n_\varepsilon}\big)\leqslant C_6|s|^{-2M}_h|\nabla u|^2_{\eta_\varepsilon}\big(\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon\big)^{n-1}\leqslant C_7|s|^{-2k}_h$$

where we also used [GZ, Lemma 14.4], the Monge-Ampère equation (5.7) and the fact that $\prod_{l=1}^m\big(|s_l|^2_{h_l}+\varepsilon^2\big)^{a_l}\leqslant D_{10}|s|^{-2M}_h$ as $a_l>-1$. From (5.9) we deduce $|\nabla u|^2_{\beta_\epsilon}|s|^{2(k+D_8+1)}_h\leqslant C_8$. As $\{\beta_\epsilon\}_{\epsilon>0}$ is a non-degenerate continuous family of Kähler forms converging to β as $\epsilon\to 0$, we get

$$|s|^{2(k+D_8+1)}_h|\nabla u|^2_\beta\leqslant\max\big\{\sup_{\tilde\Omega}[|s|^{2(k+D_8+1)}_h|\nabla(v/\gamma-\log|s|^2-f_\varepsilon)|^2_\beta],C_9\big\},$$

i.e. (5.8), which concludes the proof by setting $B_2:=k+D_8+1, B_{3,j+1,\varepsilon}:=C_9$. □

Step 3. Fix V a small neighborhood of $\partial\tilde\Omega$ (intersected with $\overline{\tilde\Omega}$). We claim that

$$\sup_{\partial\tilde\Omega}|\Delta_\beta\varphi_{j,\varepsilon}|\leqslant C_V[1+\sup_V|\nabla\varphi_{j,\varepsilon}|^2], \tag{5.13}$$

for some uniform constant C_V independent of j,ε. This follows from a long series of estimates established in [GKY13, Lemma 18] (which itself was adapting the technique developed by [CKNS85]) when μ_p and Ω are smooth. The statement of [GKY13, Lemma 18] mentions $\sup_{\tilde\Omega}|\nabla\varphi_j|^2$, however the arguments only involve

– local reasonings in a small fixed neighborhood of the boundary;

– smoothness of μ_p in this neighborhood and pseudoconvexity of $\partial\tilde{\Omega}$.

Step 4. We now show that there exist constants $m, B_{3,j,\varepsilon} > 0$ such that

$$\sup_{\tilde{\Omega}} |s|^{2m} |\Delta_\beta \varphi_{j,\varepsilon}| \leqslant B_{3,j,\varepsilon} \left[1 + \sup_{\partial\tilde{\Omega}} |\Delta_\beta \varphi_{j,\varepsilon}| \right], \tag{5.14}$$

where $B_{3,j,\varepsilon}$ only depends on an upper-bound on $||\varphi_{k,\varepsilon}||_{L^\infty(\tilde{\Omega})}$, for $k \leqslant j$. This is a variant of [GKY13, Lemma 17], for which we provide a detailed proof.

We set $\omega_j := \varepsilon\beta + dd^c\varphi_{j,\varepsilon}$ and observe that

$$\omega_j^n = e^{\psi_\varepsilon - \varphi_{j-1,\varepsilon} - c'_{j-1}} \beta^n,$$

where ψ_ε is a difference of quasi-psh functions in $\tilde{\Omega}$ such that $e^{\psi_\varepsilon} \leqslant c_1 |s|^{-2a}$ and $dd^c\psi_\varepsilon \geqslant -c_1|s|^{-2}\beta$ in $\tilde{\Omega}$, for some uniform constants $a, c_1 > 0$. We consider

$$H_j := \log \mathrm{Tr}_\beta(\omega_j) + \varphi_{j-1,\varepsilon} - A\varphi_{j,\varepsilon} + A\rho',$$

where $A > 0$ is chosen below. We use here the classical notations

$$\mathrm{Tr}_\eta(\omega) := n\frac{\omega \wedge \eta^{n-1}}{\eta^n} \quad \text{and} \quad \Delta_\eta(h) := n\frac{dd^c h \wedge \eta^{n-1}}{\eta^n}.$$

Either H_j reaches its maximum on $\partial\tilde{\Omega}$ and we are done, or it reaches its maximum at some point $x_j \in \tilde{\Omega} \setminus E$ since $\rho \to -\infty$ along E. We are going to estimate $\Delta_{\omega_j} H_j$ from below and use the fact that $0 \geqslant \Delta_{\omega_j} H_j(x_j)$.

It follows from [Siu87] that

$$\Delta_{\omega_j} \log \mathrm{Tr}_\beta(\omega_j) \geqslant -\frac{\mathrm{Tr}_\beta(\mathrm{Ric}(\omega_j))}{\mathrm{Tr}_\beta(\omega_j)} - B\mathrm{Tr}_{\omega_j}(\beta),$$

where $-B$ is a lower bound on the holomorphic bisectional curvature of β. Now

$$-\mathrm{Ric}(\omega_j) = -\mathrm{Ric}(\beta) + dd^c(\psi_\varepsilon - \varphi_{j-1,\varepsilon}) \geqslant -\omega_{j-1} - \frac{A_1}{|s|^2}\beta$$

in $\tilde{\Omega} \setminus E$. Moreover $\mathrm{Tr}_\beta(\omega_{j-1}) \leqslant \mathrm{Tr}_\beta(\omega_j)\mathrm{Tr}_{\omega_j}(\omega_{j-1})$ hence

$$\Delta_{\omega_j} \log \mathrm{Tr}_\beta(\omega_j) \geqslant -\mathrm{Tr}_{\omega_j}(\omega_{j-1}) - \frac{nA_1}{|s|^2\mathrm{Tr}_\beta(\omega_j)} - B\mathrm{Tr}_{\omega_j}(\beta).$$

Using that $dd^c\rho' \geqslant \beta$, we obtain

$$\Delta_{\omega_j} H_j \geqslant -An + (A-B)\mathrm{Tr}_{\omega_j}(\beta) - \frac{nA_1}{|s|^2\mathrm{Tr}_\beta(\omega_j)}.$$

Using the classical inequality $n[\mathrm{Tr}_{\omega_j}(\beta)]^{n-1} \geqslant (\beta^n/\omega_j^n)\mathrm{Tr}_\beta(\omega_j)$, we infer

$$\Delta_{\omega_j} H_j \geqslant -An + c(A-B)e^{\frac{-\psi_\varepsilon}{n-1}}[\mathrm{Tr}_\beta(\omega_j)]^{\frac{1}{n-1}} - \frac{nA_1}{|s|^2\mathrm{Tr}_\beta(\omega_j)}. \tag{5.15}$$

Let us stress that the constant c depends here on an upper bound on $||\varphi_{j-1,\varepsilon}||_{L^\infty(\tilde{\Omega})}$.

We fix A so large that $A > B$ and $\psi_\varepsilon + A\rho' \leqslant c'_1 - a\log|s|^2 + A\rho'$ is bounded from above. At the point x_j we obtain $0 \geqslant \Delta_{\omega_j} H_j$, therefore

– either $|s|^2\mathrm{Tr}_\beta(\omega_j) \leqslant 1$ hence $H_j(x_j) \leqslant (\varphi_{j-1,\varepsilon} - A\varphi_{j,\varepsilon} + A\rho')(x_j) \leqslant C$;

– or $|s|^2\mathrm{Tr}_\beta(\omega_j) \geqslant 1$ and (5.15) yields $\mathrm{Tr}_\beta(\omega_j) \leqslant C' e^{\psi_\varepsilon(x_j)}$ hence

$$H_j(x_j) \leqslant \psi_\varepsilon(x_j) + A\rho'(x_j) + C'' \leqslant C'''.$$

Thus H_j is uniformly bounded from above in both cases, and (5.14) follows (we use here an upper bound on $||\varphi_{j-1,\varepsilon}||_{L^\infty(\tilde\Omega)}$ and $||\varphi_{j,\varepsilon}||_{L^\infty(\tilde\Omega)}$).

Step 5. We finally show by induction on j that $\varphi_{j,\varepsilon}$ uniformly converges towards φ_j as ε decreases to 0. There is nothing to prove for $j = 0$ since $\varphi_{0,\varepsilon} = \varphi_0$.

For $j = 1$, it follows from (a slight generalization of) [GGZ23, Proposition 1.8] that $||\varphi_{1,\varepsilon}||_{L^\infty(\tilde\Omega)} \leqslant C_1$ is bounded uniformly in $\varepsilon > 0$. Proceeding by induction we similarly obtain that for all $j \in \mathbb{N}$,

$$||\varphi_{j,\varepsilon}||_{L^\infty(\tilde\Omega)} \leqslant C_j$$

is bounded uniformly in $\varepsilon > 0$. By previous steps, the family $(\varphi_{j,\varepsilon})_\varepsilon$ is relatively compact in $\mathcal{C}^{1,\alpha}$ for all $0 < \alpha < 1$. Any cluster point ψ_j, as $\varepsilon \to 0$, is a solution of

$$(dd^c\psi_{j+1})^n = \frac{e^{-\gamma\psi_j}\mu_p}{c_j}$$

with boundary values $\psi_{j+1|\partial\Omega} = \phi$, hence $\psi_j = \varphi_j$ by uniqueness. Thus $\varphi_{j,\varepsilon}$ converges to φ_j as ε decreases to zero, and the convergence is moreover uniform on $\tilde\Omega$ by [GGZ23, Proposition 1.8].

We can thus let ε tend to zero in previous inequalities. Now $||\varphi_{j,\varepsilon}||_{L^\infty(\tilde\Omega)} \to ||\varphi_j||_{L^\infty(\Omega)}$, and the latter is uniformly bounded in j by Proposition 5.1. For $\varepsilon = 0$, (5.3), (5.4), (5.13) and (5.14) thus provide uniform bounds in j, and conclude the proof of (5.1). The proof of Proposition 5.2 is thus complete. □

5.3 Higher order estimates and convergence

Once the uniform $\mathcal{C}^2$-estimate is established (Proposition 5.2), one can then linearize the complex Monge-Ampère equation and apply standard elliptic theory (Evans-Krylov method and Schauder bootstrapping) to derive higher order estimates:

Proposition 5.3. *Given K a compact subset of $\Omega \setminus \{p\}$ and $\alpha > 0, \ell \in \mathbb{N}$, there exists $C(K,\ell,\alpha) > 0$ such that for all $j \in \mathbb{N}$, $||\varphi_j||_{\mathcal{C}^{\ell,\alpha}(K)} \leqslant C(K,\ell,\alpha)$.*

It follows that the sequence (φ_j) is relatively compact in $\mathcal{C}^\infty(\Omega \setminus \{p\})$ We let $\mathcal{K}$ denote the set of cluster values of the sequence (φ_j). Any function $\psi \in \mathcal{K}$ is

– psh in Ω and smooth in $\Omega \setminus \{p\}$, with $\psi_{|\partial\Omega} = \phi$;
– uniformly bounded in $\overline{\Omega}$ (Proposition 5.1);
– continuous on $\overline{\Omega}$, as the uniform limit of (φ_{j_k}) [GGZ23, Proposition 1.8];

The set $\mathcal{K}$ is invariant under the action of $T_\gamma : \varphi \in \mathcal{T}_\phi(\Omega) \mapsto \psi \in \mathcal{T}_\phi(\Omega)$, which associates, to a given $\varphi \in \mathcal{T}_\phi(\Omega)$, the unique solution $\psi \in \mathcal{T}_\phi(\Omega)$ to the complex Monge-Ampère equation

$$(dd^c\psi)^n = \frac{e^{-\gamma\varphi}\mu_p}{\int_\Omega e^{-\gamma\varphi}d\mu_p}.$$

It follows from [GKY13, Proposition 12] that the functional F_γ is constant on $\mathcal{K}$ and that $\mathcal{K}$ is pointwise invariant under the action of T_γ. Thus a cluster value of (φ_j) provides a desired solution to Theorem C.

Appendix A. Appendix by S.Boucksom

The purpose of this appendix is to provide an alternative approach to Proposition 4.8, emphasizing the role of b-divisors. We use [BdFF12] as a main reference for what follows.

A.1 Nef b-divisors over a point

Consider for the moment any normal singularity (X, p), and set $n := \dim X$.

In what follows, a *birational model* means a projective birational morphism $\pi\colon X_\pi \to X$ with X_π normal. A *b-divisor over p* is defined as a collection $B = (B_\pi)_\pi$ of $\mathbb{R}$-divisors B_π on X_π for all birational models π, compatible under push-forward, and such that each B_π has support in $\pi^{-1}(p)$. The $\mathbb{R}$-vector space of b-divisors over p can thus be written as the projective limit

$$\mathrm{Div}_{\mathrm{b}}(X, p) := \varprojlim_{\pi} \mathrm{Div}_p(X_\pi),$$

where $\mathrm{Div}_p(X_\pi)$ denotes the (finite dimensional) $\mathbb{R}$-vector space of divisors on X_π with support in $\pi^{-1}(p)$, and we endow $\mathrm{Div}_{\mathrm{b}}(X, p)$ with the projective limit topology.

A b-divisor $B \in \mathrm{Div}_{\mathrm{b}}(X, p)$ is said to be *Cartier* if it is determined by some birational model π, in the sense that $B_{\pi'}$ is the pullback of B_π for any higher birational model π'. There is a symmetric, multilinear *intersection pairing*

$$(B_1, \dots, B_n) \mapsto (B_1 \cdot \ldots \cdot B_n) \in \mathbb{R} \tag{A.1}$$

for Cartier b-divisors B_i, defined as the intersection number $(B_{1,\pi} \cdot \ldots \cdot B_{n,\pi})$ computed on X_π for any choice of common determination π of the B_i (the result being independent of the choice of π, by the projection formula).

A *valuation centered at p* is a valuation $v\colon \mathcal{O}_{X,p} \to \mathbb{R}_{\geqslant 0}$ such that $v(\mathfrak{m}_p) > 0$ on the maximal ideal $\mathfrak{m}_p \subset \mathcal{O}_{X,p}$. It is further *divisorial* if it can be written as $v = c\,\mathrm{ord}_E$ for a prime divisor $E \subset \pi^{-1}(p)$ on some birational model X_π and $c \in \mathbb{Q}_{>0}$. Given a b-divisor B over $p \in X$, we then set $v(B) := c\,\mathrm{ord}_E(B_\pi)$. The function $v \mapsto v(B)$ so defined on the space $\mathrm{DivVal}(X, p)$ of divisorial valuations centered at p is homogeneous with respect to the scaling of $\mathbb{Q}_{>0}$, and this yields a topological vector space isomorphism between $\mathrm{Div}_{\mathrm{b}}(X, p)$ and the space of homogeneous functions on $\mathrm{DivVal}(X, p)$, endowed with the topology of pointwise convergence.

Pick a b-divisor B over p. If B is Cartier, we say that B is *(relatively) nef* if B_π is π-nef for some (hence any) determination π. In the general case, we say that B is nef if it can be written as a limit of nef Cartier b-divisors. By the Negativity Lemma, any nef b-divisor $B \in \mathrm{Div}_{\mathrm{b}}(X, p)$ is automatically antieffective, i.e. $v(B) \leqslant 0$ for all $v \in \mathrm{DivVal}(X, p)$. By [BdFF12, Lemma 2.10], we further have:

Lemma A.1. *A b-divisor B over p is nef iff, for each birational model π, the numerical class of B_π in $\mathrm{N}^1(X_\pi/X)$ is* nef in codimension 1 *(aka* movable*).*

Example A.2. Consider an ideal $\mathfrak{a} \subset \mathcal{O}_{X,p}$, and assume that $\mathfrak{a}$ is *primary*, i.e. containing some power of the maximal ideal. Then $\mathfrak{a}$ determines a nef Cartier b-divisor $Z(\mathfrak{a})$, defined by $v(Z(\mathfrak{a})) = -v(\mathfrak{a})$ for each $v \in \mathrm{DivVal}(X, p)$, and determined on the normalized blowup of $\mathfrak{a}$. For any tuple of primary ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n$,

$$e(\mathfrak{a}_1, \dots, \mathfrak{a}_n) = -(Z(\mathfrak{a}_1) \cdot \ldots \cdot Z(\mathfrak{a}_n))$$

further coincides with the mixed multiplicity of the $\mathfrak{a}_i$.

EXAMPLE A.3. For any valuation v centered at p, the valuation ideals

$$\mathfrak{a}_m(v) := \{f \in \mathcal{O}_{X,p} \mid v(f) \geqslant m\}$$

define a graded sequence of primary ideals $\mathfrak{a}_\bullet(v)$, and hence a nef b-divisor over p

$$Z(v) := Z(\mathfrak{a}_\bullet(v)) = \lim_m m^{-1} Z(\mathfrak{a}_m(v)),$$

(see [BdFF12, Lemma 2.11]), which is not Cartier in general.

LEMMA A.4. *If $B \in \mathrm{Div}_{\mathrm{b}}(X,p)$ is nef, then $B \leqslant -v(B)Z(v)$ for all $v \in \mathrm{DivVal}(X,p)$,*

Proof. Write $v = c\,\mathrm{ord}_E$ for a prime divisor E on X_π and $c \in \mathbb{Q}_{>0}$. Then $Z(v)$ coincides with $\mathrm{Env}_\pi(-c^{-1}E)$ (see [BdFF12, Definition 2.3]), and the result thus follows from [BdFF12, Proposition 2.12]. □

A.2 Normalized volume and b-divisors

From now on, we assume that the normal singularity $p \in X$ is further *isolated*.

By [BdFF12, Theorem 4.14], the intersection pairing (A.1) then extends to arbitrary tuples of nef b-divisors over p. This extended pairing takes values in $\mathbb{R} \cup \{-\infty\}$, and is symmetric, additive and non-decreasing in each variable, and continuous along decreasing nets.

DEFINITION A.5. For any nef b-divisor B over p, we define the *Hilbert–Samuel multiplicity* of B as

$$e(B) := -B^n \in [0, +\infty].$$

When (X,p) is further klt, we define the *log canonical threshold* of B as

$$\mathrm{lct}(B) := \inf_{v \in \mathrm{DivVal}(X,p)} \frac{\mathrm{A}_X(v)}{-v(B)} \in [0, +\infty),$$

where $\mathrm{A}_X(v) \geqslant 0$ denotes the log discrepancy of v.

EXAMPLE A.6. For any primary ideal $\mathfrak{a} \subset \mathcal{O}_{X,p}$, the associated nef Cartier b-divisor $B := Z(\mathfrak{a})$ (see Example A.2) satisfies $e(B) = e(\mathfrak{a})$, and $\mathrm{lct}(B) = \mathrm{lct}(\mathfrak{a})$ when (X,p) is klt.

EXAMPLE A.7. Pick any valuation v centered at p, with associated nef b-divisor $Z(v)$ (see Example A.3). Then it follows from [BdFF12, Remark 4.17] that the volume $\mathrm{Vol}(v) := \lim_{m\to\infty} \frac{n!}{m^n} \dim \mathcal{O}_{X,p}/\mathfrak{a}_m(v)$ satisfies

$$\mathrm{Vol}(v) = e(Z(v)). \tag{A.2}$$

LEMMA A.8. *For each nef b-divisor B over p, we have $e(B) = \sup_{C \geqslant B} e(C)$, where C ranges over all nef Cartier b-divisors of the form $C = m^{-1}Z(\mathfrak{a})$ for a primary ideal $\mathfrak{a} \subset \mathcal{O}_{X,p}$ and $m \in \mathbb{Z}_{>0}$, and such that $C \geqslant B$.*

Proof. Since B is the limit of the decreasing net $(\mathrm{Env}_\pi(B_\pi))$ (see [BdFF12, Remark 2.17]), it is enough to prove the result when $B = \mathrm{Env}_\pi(B_\pi)$, by continuity of the intersection pairing along decreasing nets. By [BdFF12, Theorem 4.11], we can then write B as the limit of a decreasing sequence (C_i) of nef Cartier b-divisors of the desired form, and we are done since $e(C_i) \to e(B)$. □

Consider now a psh function φ on X. The collection of its Lelong numbers on all birational models defines a homogeneous function $v \mapsto v(\varphi)$ on $\mathrm{DivVal}(X,p)$, and hence an antieffective b-divisor $Z(\varphi)$ over p, such that $v(Z(\varphi)) = -v(\varphi)$.

PROPOSITION A.9. *The b-divisor $Z(\varphi)$ is nef. Further:*

(i) if φ is locally bounded outside p, then

$$e(Z(\varphi)) \leqslant e(\varphi) := (dd^c\varphi)^n(\{p\});$$

(ii) if (X,p) is klt, then $\mathrm{lct}(Z(\varphi)) = \mathrm{lct}(\varphi)$.

Proof. Consider the closed positive $(1,1)$-current $T := dd^c\varphi$, and pick a log resolution $\pi\colon X_\pi \to X$ of (X,p). The Siu decomposition of $\pi^\star T = dd^c\pi^\star\varphi$ shows that $\pi^\star T + [Z(\varphi)_\pi]$ is a positive current with zero generic Lelong numbers along each component of $\pi^{-1}(p)$. By Demailly regularization, it follows that the class of $Z(\varphi)_\pi$ in $\mathrm{N}^1(X_\pi/X)$ is nef in codimension 1, and hence that $Z(\varphi)$ is nef (see Lemma A.1).

Assume next that φ is locally bounded outside p, and pick a primary ideal $\mathfrak{a} \subset \mathcal{O}_{X,p}$ and $m \in \mathbb{Z}_{>0}$ such that $C := m^{-1}Z(\mathfrak{a}) \geqslant Z(\varphi)$. Choose a finite set of local generators (f_i) of $\mathfrak{a}$, and consider the psh function $\psi := m^{-1}\log\sum_i |f_i|$. Then $Z(\varphi) \leqslant C = Z(\psi)$, and hence $\varphi \leqslant \psi + O(1)$ (to see this, pull back φ and ψ to a log resolution of $\mathfrak{a}$, and use the Siu decomposition). By Demailly's comparison theorem, it follows that $e(C) = e(\psi) \leqslant e(\varphi)$, and taking the supremum over C yields (i), by Lemma A.8.

Finally, (ii) is a rather simple consequence of [BBJ21, Theorem B.5] applied to the pullback of φ to a log resolution of (X,p). □

We can now state the following variant of Proposition 4.8.

THEOREM A.10. *Let (X,p) be an isolated klt singularity. Then*

$$\widehat{\mathrm{vol}}(X,p) = \inf_B e(B)\,\mathrm{lct}(B)^n = \inf_\varphi e(\varphi)\,\mathrm{lct}(\varphi)^n,$$

where B runs over all nef b-divisors over p, and φ runs over all psh functions on X that are locally bounded outside p.

Proof. By Theorem 1.16 we have $\widehat{\mathrm{vol}}(X,p) = \inf_{\mathfrak{a}} e(\mathfrak{a})\,\mathrm{lct}(\mathfrak{a})^n$, where $\mathfrak{a} \subset \mathcal{O}_{X,p}$ runs over all primary divisors, and hence $\widehat{\mathrm{vol}}(X,p) \geqslant \inf_B e(B)\,\mathrm{lct}(B)^n$, by Example A.6. Conversely, pick a nef b-divisor B over p. For any $v \in \mathrm{DivVal}(X,p)$, Lemma A.4 yields $B \leqslant -v(B)Z(v)$. By monotonicity and homogeneity of the intersection pairing, this yields $B^n \leqslant (-v(B))^n Z(v)^n$, i.e. $e(B) \geqslant (-v(B))^n\mathrm{Vol}(v)$, by (A.2). Thus

$$e(B)\left(\frac{\mathrm{A}_X(v)}{-v(B)}\right)^n \geqslant \mathrm{A}_X(v)^n\mathrm{Vol}(v) \geqslant \widehat{\mathrm{vol}}(X,p).$$

Taking the infimum over v yields $e(B)\,\mathrm{lct}(B)^n \geqslant \widehat{\mathrm{vol}}(X,p)$ for any nef b-divisor B over p, and hence also $e(\varphi)\,\mathrm{lct}(\varphi)^n \geqslant \widehat{\mathrm{vol}}(X,p)$ for any psh function φ locally bounded outside p, by Proposition A.9. □

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