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Quantum Codes from Group Ring Codes *

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Abstract

This article examines group ring codes over finite fields and finite groups. We also present a section on two-dimensional cyclic codes in the quotient ring $\mathbb{F}_q[x, y]/\langle x^l - 1, y^m - 1 \rangle$. These two-dimensional cyclic codes can be analyzed using the group ring $\mathbb{F}_q(C_l \times C_m)$, where C_l and C_m represent cyclic groups of orders l and m , respectively. The aim is to show that studying group ring codes provides a more compact approach compared to the quotient ring method. We further extend this group ring framework to study codes over other group structures, such as the dihedral group, direct products of cyclic and dihedral groups, direct products of two cyclic groups, and semidirect products of two groups. Additionally, we explore necessary and sufficient conditions for such group ring codes to be self-orthogonal under Euclidean, Hermitian, and symplectic inner products and propose a construction for quantum codes.

Keywords: Group rings, two-dimensional cyclic codes, quantum error-correcting codes (QECCs).

Mathematics Subject Classifications(2010): 94B05, 94B15, 94B60.

1 Introduction

Group ring codes were first introduced by Berman [3, 4], with the initial focus on abelian and semisimple abelian codes. Significant advancements in this area were made in subsequent works [13, 15, 18, 19]. Hurley [14] discovered an isomorphism between a group ring and a matrix ring, which simplified the encoding of group ring codes. This paper primarily utilizes this matrix representation of group codes in our study. More recently, group ring codes following this matrix approach have been explored by Dougherty et al. [10, 9, 11]. In [5], it was shown that group codes over fields are asymptotically good in any characteristic. Additionally, the family of group codes has been used to construct new linear codes through group ring codes [16].

Quantum error-correcting codes (QECCs) are vital for preserving quantum information from the detrimental effects of decoherence and quantum noise, making them crucial in both quantum computing and quantum communication. While quantum computers hold the potential to solve certain problems much faster than their classical counterparts, they are highly susceptible to errors caused by environmental interactions. Addressing these errors is a major challenge, and QECCs have become a key strategy for protecting quantum information in both computational and communications contexts. The concept of QECCs was initially introduced in foundational works such as [8, 20, 21]. The Calderbank-Shor-Steane (CSS) framework [7] has provided a cornerstone for much of the subsequent research in the field. Techniques for constructing non-binary quantum codes have been further investigated in works like [1, 2, 17].

The primary motivations for studying linear codes using group rings and constructing quantum codes stem from two key factors. First, group rings provide multiple ways to generate a generator matrix for

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a linear code of fixed length. For example, to construct a code of length 12 over \mathbb{F}_q , we can utilize the group ring $\mathbb{F}_q G$, where G is a group of order 12. There are five distinct small groups of order 12: $C_{12}, D_6, C_2 \times C_6, A_4, C_3 \times C_4$; here, C_n denotes the cyclic group of order n , D_n is the dihedral group of order $2n$, A_4 is the alternating group of order 12, \times represents the direct product, and \rtimes represents the semidirect product. For a fixed code length of 12, we can generate different types of generator matrices based on these five group structures. By exploring these diverse structures, we gain flexibility in designing codes and may discover better code parameters. Studying codes via group rings thus offers more options for generating linear codes and can lead to the construction of codes with improved performance.

Second, group rings simplify the analysis of these codes compared to quotient rings. For instance, in the study of one-dimensional cyclic codes, we often use the quotient polynomial ring $\mathbb{F}_q[x]/\langle x^{12} - 1 \rangle$, which corresponds to the group ring $\mathbb{F}_q C_{12}$. Similarly, for two-dimensional cyclic codes, we work within the quotient polynomial ring $\mathbb{F}_q[x, y]/\langle x^2 - 1, y^6 - 1, xy - yx \rangle$, which can be compactly represented using the group ring $\mathbb{F}_q(C_2 \times C_6)$. Similarly, codes in the polynomial ring $\mathbb{F}_q[x, y]/\langle x^6 - 1, y^2 - 1, yx - x^{-1}y \rangle$ can be compactly studied using the group ring $\mathbb{F}_q D_6$. This group ring framework can also be extended to study codes based on other group structures, such as direct products of cyclic and dihedral groups, or semidirect products of two groups.

This paper is organized as follows. Section 2 covers the basics of group ring codes. In Section 3, we examine two-dimensional cyclic codes in the quotient ring $\mathbb{F}_q[x, y]/\langle x^l - 1, y^m - 1 \rangle$, establish conditions for such codes to be self-orthogonal, which present a framework for constructing quantum codes from them. This approach highlights a comparative perspective, relating these codes to group ring codes viewed as a direct product of two finite groups. We further demonstrate that this framework can be compactly represented using the group ring $\mathbb{F}_q(C_l \times C_m)$, where C_l and C_m represent cyclic groups of orders l and m , respectively, as shown in the next section. In Section 4, we present the formulation of group ring codes derived from the group rings $\mathbb{F}_q G$ for various groups G , including $D_n, C_l \times D_m$, and $C_l \times C_m$. Section 5 provides necessary and sufficient conditions for these codes to be self-orthogonal under the Euclidean, Hermitian, and Symplectic inner products. We also construct quantum codes based on these findings. Section 6 concludes our study.

2 Preliminaries

Let G be a group and R a ring. We denote by RG the set of all formal linear combinations of the form

$$a = \sum_{g \in G} \alpha_g g,$$

where $\alpha_g \in R$ and $\alpha_g = 0$ for all but finitely many $g \in G$, that is only a finite number of the coefficients α_g are nonzero. Since our focus is primarily on codes of finite length, we restrict our attention to finite groups.

Given two elements $a = \sum_{g \in G} \alpha_g g$ and $b = \sum_{g \in G} \beta_g g \in RG$, we have $a = b$ if and only if $\alpha_g = \beta_g$ for all $g \in G$. To define an algebra structure on the set RG , we first show that RG forms a ring and then introduce a scalar multiplication.

We define the sum of two elements in RG as:

$$\sum_{g \in G} \alpha_g g + \sum_{g \in G} \beta_g g = \sum_{g \in G} (\alpha_g + \beta_g) g.$$

Additionally, for two elements $a = \sum_{g \in G} \alpha_g g$ and $b = \sum_{h \in G} \beta_h h$ in RG , their product is defined as:

$$ab = \sum_{g, h \in G} \alpha_g \beta_h gh = \sum_{g' \in G} \gamma_{g'} g', \quad \text{where } \gamma_{g'} = \sum_{gh=g'} \alpha_g \beta_h.$$

It is straightforward to verify that with these operations, RG is a ring with a unity element, specifically $1 = \sum_{g \in G} u_g g$, where the coefficient corresponding to the identity element of G is 1, and $u_g = 0$ for all other $g \in G$. Suppose R is a ring with m elements and G is a group with n elements. Then RG is a finite group ring with $|R|^{|G|} = m^n$ elements.

We can further define scalar multiplication for $\lambda \in R$ and $a \in RG$ as:

$$\lambda \left(\sum_{g \in G} \alpha_g g \right) = \sum_{g \in G} (\lambda \alpha_g) g.$$

Once again, it is easy to verify that RG is an R -module. In fact, if R is a field, then RG is a (group) algebra over R . Let \mathbb{F}_q be a finite field with q elements, where q is a power of a prime. A cyclic code of length n over \mathbb{F}_q can be described as an ideal in the quotient ring $\mathbb{F}_q[x]/\langle x^n - 1 \rangle$. Alternatively, this code can be understood as an ideal in the group ring $\mathbb{F}_q G$, where G is a cyclic group of order n . From this point onward, we assume R is the finite field \mathbb{F}_q , where q is a prime power and G is a finite group of order n . We express any element $a \in \mathbb{F}_q G$ in a compact way as $a = \sum_{g \in G} \alpha_g g$. The support of a is defined as the set

$$\text{supp}(a) = \{g \in G \mid \alpha_g \neq 0\}.$$

The Hamming distance between two elements $a = \sum_{g \in G} \alpha_g g$ and $b = \sum_{g \in G} \beta_g g$ in $\mathbb{F}_q G$ is given by

$$d(a, b) = |\{g \in G \mid \alpha_g \neq \beta_g\}|.$$

The weight of an element a , denoted $w(a)$, is the Hamming distance between a and the zero element:

$$w(a) = d(a, 0) = |\text{supp}(a)| = |\{g \in G \mid \alpha_g \neq 0\}|.$$

Explicitly, we write the elements of the finite group G as $\{g_1, g_2, \dots, g_n\}$ in some order, and take $a = \alpha_1 g_1 + \alpha_2 g_2 + \dots + \alpha_n g_n \in \mathbb{F}_q G$, where $\alpha_i \in \mathbb{F}_q$ for $i = 1, 2, \dots, n$. Then the matrix $\sigma(a) \in M_n(\mathbb{F}_q)$ is defined [14] as follows

$$\sigma(a) = \begin{pmatrix} \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \alpha_{g_1^{-1}g_3} & \cdots & \alpha_{g_1^{-1}g_n} \\ \alpha_{g_2^{-1}g_1} & \alpha_{g_2^{-1}g_2} & \alpha_{g_2^{-1}g_3} & \cdots & \alpha_{g_2^{-1}g_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{g_n^{-1}g_1} & \alpha_{g_n^{-1}g_2} & \alpha_{g_n^{-1}g_3} & \cdots & \alpha_{g_n^{-1}g_n} \end{pmatrix}.$$

Theorem 2.1. [10, 14] *The map $\sigma : \mathbb{F}_q G \rightarrow M_n(\mathbb{F}_q)$ is an injective ring homomorphism.*

We define a linear code $C(a)$ whose generator matrix is $\sigma(a)$. We note that $C(a)$ is a linear code of length n over \mathbb{F}_q with dimension $n - \text{rank}(\sigma(a))$.

It appears that for each element of $\mathbb{F}_q G$, there is a corresponding matrix as described above, and from each matrix, a code is derived. However, this is not the scenario we are dealing with here. We observe that each row of the matrix σ represents the coefficients of an element of $\mathbb{F}_q G$, and no two rows are identical. If we choose any row of the matrix as the first row, corresponding to an element of $\mathbb{F}_q G$, we will end up with the same code. This is because the code is generated only by the linearly independent rows of the matrix. Intuitively, this suggests that multiple elements can generate the same code. Our goal now is to identify which elements produce the same code. To do this, we look at the following facts.

We consider the quotient polynomial ring $\frac{\mathbb{F}_q[x]}{\langle x^n - 1 \rangle}$, where any polynomial in this ring takes the form $a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1}$, with $a_i \in \mathbb{F}_q$ for $i = 0, 1, \dots, n-1$. Now, if we examine an element of $\mathbb{F}_q C_n$, where C_n is a cyclic group of order n , any such element can be written as $\alpha_0 g_0 + \alpha_1 g_1 + \dots + \alpha_{n-1} g_{n-1}$, where $\{g_0 = 1, g_1, \dots, g_{n-1}\}$ are the elements of C_n with g_0 being the identity element, and $\alpha_i \in \mathbb{F}_q$. It is easy to observe that

$$\frac{\mathbb{F}_q[x]}{\langle x^n - 1 \rangle} \cong \mathbb{F}_q C_n.$$

We also know that an ideal in $\frac{\mathbb{F}_q[x]}{\langle x^n - 1 \rangle}$ corresponds to a cyclic code of length n over \mathbb{F}_q , which is principally generated by an element of the quotient ring. This intuitive connection suggests that the structure of codes in $\mathbb{F}_q C_n$ relates to the ideal structure of $\mathbb{F}_q C_n$.

From this discussion, we can relate the code $C(a)$ to the ideal generated by $a \in \mathbb{F}_q G$. Thus, we arrive at the following result.

Theorem 2.2. [10] *Let $a \in \mathbb{F}_q G$ and $C(a)$ be the corresponding code over \mathbb{F}_q . Suppose $I(a)$ is the set of elements of $\mathbb{F}_q G$ such that $\sum \alpha_i g_i \in I(a)$ if and only if $(\alpha_1, \alpha_2, \dots, \alpha_n) \in C(a)$. Then $I(a)$ is a left ideal in $\mathbb{F}_q G$.*

A group code or, more precisely, a G -code over \mathbb{F}_q is an ideal of the group algebra $\mathbb{F}_q G$.

Any element $a = \alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_n g_n \in \mathbb{F}_q G$, where G is a finite group of order n with elements $\{g_1, g_2, \dots, g_n\}$ in a fixed order, can be represented as the n -tuple $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{F}_q^n$. The Euclidean inner product of two elements $a = \sum_{i=1}^n \alpha_i g_i \in \mathbb{F}_q G$ and $b = \sum_{i=1}^n \beta_i g_i \in \mathbb{F}_q G$ is defined as

$$\langle a, b \rangle_e = \sum_{i=1}^n \alpha_i \beta_i.$$

Two elements a and b are orthogonal if $\langle a, b \rangle_e = 0$. The dual code C_e^\perp is defined as

$$C_e^\perp = \{a \in \mathbb{F}_q G \mid \langle a, b \rangle_e = 0 \text{ for all } b \in C\}.$$

If $C \subseteq C_e^\perp$, then C is Euclidean self-orthogonal, and Euclidean dual-containing if $C_e^\perp \subseteq C$.

Now, for $n = 2m$, the symplectic inner product of two elements $a = \sum_{i=1}^{2m} \alpha_i g_i \in \mathbb{F}_q G$ and $b = \sum_{i=1}^{2m} \beta_i g_i \in \mathbb{F}_q G$ is defined as

$$\langle a, b \rangle_s = (\alpha_1 \beta_{m+1} + \cdots + \alpha_m \beta_{2m}) - (\alpha_{m+1} \beta_1 + \cdots + \alpha_{2m} \beta_m).$$

This inner product can also be written as

$$\langle a, b \rangle_s = (\alpha_1, \alpha_2, \dots, \alpha_{2m}) \Omega (\beta_1, \beta_2, \dots, \beta_{2m})^t, \quad \text{where } \Omega = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix},$$

with I_n denoting the $n \times n$ identity matrix and O_n the $n \times n$ zero matrix.

The symplectic weight $wt_s(a)$ is defined as the number of indices i for which $(\alpha_i, \alpha_{m+i}) \neq (0, 0)$, i.e.,

$$wt_s(a) = |\{i \mid (\alpha_i, \alpha_{m+i}) \neq (0, 0)\}|.$$

The symplectic distance $d_s(C)$ of a linear code C is the minimum nonzero symplectic weight of any codeword in C .

The dual code of C with respect to the symplectic inner product is given by

$$C^{\perp_s} = \{a \in \mathbb{F}_q^{2m} \mid \langle a, b \rangle_s = 0 \text{ for all } b \in C\}.$$

If $C \subseteq C^{\perp_s}$, then C is symplectic self-orthogonal, and symplectic dual-containing if $C^{\perp_s} \subseteq C$.

3 Two-dimensional cyclic codes

In this section, we study two-dimensional cyclic codes using approaches similar to standard one-dimensional ones.

Definition 3.1. *Suppose C is a linear code of length lm over \mathbb{F}_q . Consider a codeword $\mathbf{c} = (c_{0,0}, c_{0,1}, \dots, c_{0,l-1}, c_{1,0}, \dots, c_{1,l-1}, \dots, c_{m-1,0}, \dots, c_{m-1,l-1}) \in C$. We can express this codeword as*

$$\mathbf{c} = \begin{pmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,l-1} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,l-1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m-1,0} & c_{m-1,1} & \cdots & c_{m-1,l-1} \end{pmatrix}.$$

If C is closed under row shift and column shift of codewords, then C is a two-dimensional cyclic code of length lm that is,

$$\begin{pmatrix} c_{0,l-1} & c_{0,1} & \cdots & c_{0,l-2} \\ c_{1,l-1} & c_{1,1} & \cdots & c_{1,l-2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m-1,l-1} & c_{m-1,1} & \cdots & c_{m-1,l-2} \end{pmatrix} \in C, \quad \text{and} \quad \begin{pmatrix} c_{m-1,0} & c_{m-1,1} & \cdots & c_{m-1,l-1} \\ c_{0,0} & c_{0,1} & \cdots & c_{0,l-1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m-2,0} & c_{m-2,1} & \cdots & c_{m-2,l-1} \end{pmatrix} \in C.$$

We consider a bivariate polynomial ring $\mathbb{F}_q[x, y]$; any element of this ring is of the form

$$\mathbb{F}_q[x, y] = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} r_{i,j} x^i y^j; \quad r_{i,j} \in \mathbb{F}_q, n_1, n_2 \in \mathbb{N}.$$

Let \mathbb{Z} be the set of integers, and $\mathbb{Z}_+^2 = \{(i, j) : i, j \geq 0, i, j \in \mathbb{Z}\}$. We consider a lexicographic order on \mathbb{Z}_+^2 as $(i, j) \succeq (k, l)$ if and only if $i > k$ or both $i = k$ and $j \geq l$. If $(i, j) \succeq (k, l)$ and $(i, j) \neq (k, l)$, then $(i, j) \succ (k, l)$. We note that $(\mathbb{Z}_+^2, \succeq)$ is a totally ordered set.

A nonzero polynomial $f(x, y)$ in $\mathbb{F}_2[x, y]$ is said to have a degree (l_1, l_2) if there exists a nonzero term $r_{i,j} x^i y^j$ such that $(l_1, l_2) \succ (m_1, m_2)$ for any other nonzero term $s_{i,j} x^i y^j$ within $f(x, y)$. In such a case, $r_{i,j}$ is the leading coefficient of $f(x, y)$.

We define two relations over $\mathbb{F}_q[x, y]$ as follows; for any two polynomial $f(x, y), g(x, y) \in \mathbb{F}_q[x, y]$, $f(x, y) \succeq g(x, y)$ if and only if $\deg(f(x, y)) \succeq \deg(g(x, y))$, and $f(x, y) \geq g(x, y)$ if and only if $\deg(f(x, y)) \geq \deg(g(x, y))$, where we take $(\mathbb{Z}_+^2, \succeq)$ as a partial ordered set with $(i, j) \geq (k, l)$ if and only if $i \geq k$ and $j \geq l$.

We denote $\mathfrak{R} = \mathbb{F}_q[x, y]/\langle x^l - 1, y^m - 1 \rangle$ and $R = \mathbb{F}_q[x]/\langle x^l - 1 \rangle$. We have that $\mathfrak{R} \cong R[y]/\langle y^m - 1 \rangle$. In polynomial representation, we can write the corresponding polynomial of the codeword \mathbf{c} as $c(x, y) = c_0(x, y) + y c_1(x, y) + y^2 c_2(x, y) + \dots + y^{m-1} c_{m-1}(x, y) \in \mathfrak{R}$, where $c_i(x, y) = c_{i,0}(x) + x c_{i,1}(x) + x^2 c_{i,2}(x) + \dots + x^{l-1} c_{i,l-1}(x) \in R$. With this correspondence, it can be shown that there is a one-to-one correspondence between a two-dimensional cyclic code of length lm over \mathbb{F}_q and ideals of \mathfrak{R} . This type of generator is not general; it is specifically considered when a two-dimensional cyclic code is principally generated.

Theorem 3.2. *Let C be a two-dimensional cyclic code of length lm over \mathbb{F}_q . Suppose $g(x, y)$ is the monic bivariate polynomial with a minimal degree in \mathfrak{R} with respect to \succeq . The following holds*

1. C is an ideal of \mathfrak{R} , and $C = \langle g(x, y) \rangle$;
2. $g(x, y) \mid (x^l - 1)(y^m - 1)$; and
3. the dimension of C is $k = (l - a)(m - b)$ with S as a basis, where

$$S = \{g, xg, \dots, x^{l-a-1}g, yg, xyg, \dots, x^{l-a-1}yg, \dots, y^{m-b-1}g, xy^{m-b-1}g, \dots, x^{l-a-1}y^{m-b-1}g\},$$

$$\text{and } \deg(g(x, y)) = (a, b).$$

Proposition 3.3. *The dual of a 2D-cyclic code of length lm over \mathbb{F}_q is also a 2D-cyclic code of length lm over \mathbb{F}_q .*

Proposition 3.4. *If $f(x, y) \in \mathbb{F}_q[x, y]$, its reciprocal polynomial is defined as $f^*(x, y)$, where*

$$f(x, y) = \sum_{i=0}^{l_1} \sum_{j=0}^{l_2} f_{i,j} x^i y^j, \quad \text{and } f^*(x, y) = \sum_{i=0}^{l_1} \sum_{j=0}^{l_2} f_{i,j} x^{l_1-i} y^{l_2-j}.$$

Then, for $f_1(x, y), f_2(x, y) \in \mathbb{F}_q[x, y]$, $(f_1(x, y)f_2(x, y))^* = f_1^*(x, y)f_2^*(x, y)$.

Proof. We note that for any $f(x, y) \in \mathbb{F}_q[x, y]$, $f^*(x, y) = x^{l_1} y^{l_2} f_{i,j}(x^{-1}, y^{-1})$. Suppose

$$f_1(x, y) = \sum_{i=0}^a \sum_{j=0}^b f'_{i,j} x^i y^j \quad \text{and } f_2(x, y) = \sum_{i=0}^c \sum_{j=0}^d f''_{i,j} x^i y^j.$$

Then

$$f_1^*(x, y) = x^a y^b f_1(x^{-1}, y^{-1}) \quad \text{and } f_2^*(x, y) = x^c y^d f_2(x^{-1}, y^{-1}).$$

We also note that $f_1(x, y)f_2(x, y)$ is a bivariate polynomial of x degree ac and y degree bd . Then

$$\begin{aligned} (f_1(x, y)f_2(x, y))^* &= (f_1 f_2(x, y))^* \\ &= x^{a+c} y^{b+d} f_1 f_2(x^{-1}, y^{-1}) \\ &= x^{a+c} y^{b+d} f_1(x^{-1}, y^{-1}) f_2(x^{-1}, y^{-1}) \\ &= f_1^*(x, y) f_2^*(x, y). \end{aligned}$$

□

Remark 3.5. We emphasize the fact that Proposition 3.4 holds true for any two polynomials over $\mathbb{F}_q[x, y]$. However, the same proposition does not hold over for any two polynomials over $\mathfrak{R} = \mathbb{F}_q[x, y]/\langle x^l - 1, y^m - 1 \rangle$. Proposition 3.4 remains valid for any two polynomials, denoted as $f_1(x, y)$ and $f_2(x, y)$, in \mathfrak{R} if the x -degree of $f_1(x, y)f_2(x, y)$ is less than l , that is, $a + c < l$ and the y -degree of $f_1(x, y)f_2(x, y)$ is less than m , that is, $b + d < m$. We illustrate this discussion with the following example.

We consider $l = 8, m = 6, f_1(x, y) = x^5y^4 + x^5y^2 + x^5$ and $f_2(x, y) = x^6y^5 + x^6y^4 + y^4$. Then,

$$\begin{aligned} f_1(x, y)f_2(x, y) &= x^{11}y^9 + x^{11}y^8 + x^{11}y^7 + x^{11}y^6 + x^{11}y^5 + x^{11}y^4 + x^5y^8 + x^5y^6 + x^5y^4, \\ f_1(x, y)f_2(x, y) \bmod(x^l - 1, y^m - 1) &= x^5y^4 + x^5y^2 + x^5 + x^3y^5 + x^3y^4 + x^3y^3 + x^3y^2 + x^3y + x^3; \\ f_1^*(x, y) &= y^4 + y^2 + 1, \quad f_2^*(x, y) = x^6y + y + 1, \\ f_1^*(x, y)f_2^*(x, y) &= x^6y^5 + x^6y^3 + x^6y + y^5 + y^4 + y^3 + y^2 + y + 1, \\ (f_1(x, y)f_2(x, y))^* &= x^2y^5 + x^2y^4 + x^2y^3 + x^2y^2 + x^2y + x^2 + y^5 + y^3 + y. \end{aligned}$$

We have that $f_1^*(x, y)f_2^*(x, y) \neq (f_1(x, y)f_2(x, y))^*$.

On the other hand, if we consider the above polynomials over $\mathbb{F}_q[x, y]$, we get the following

$$\begin{aligned} f_1^*(x, y) &= y^4 + y^2 + 1, \quad f_2^*(x, y) = x^6y + y + 1, \\ f_1^*(x, y)f_2^*(x, y) &= x^6y^5 + x^6y^3 + x^6y + y^5 + y^4 + y^3 + y^2 + y + 1, \\ f_1(x, y)f_2(x, y) &= x^{11}y^9 + x^{11}y^8 + x^{11}y^7 + x^{11}y^6 + x^{11}y^5 + x^{11}y^4 + x^5y^8 + x^5y^6 + x^5y^4, \\ (f_1(x, y)f_2(x, y))^* &= x^6y^5 + x^6y^3 + x^6y + y^5 + y^4 + y^3 + y^2 + y + 1. \end{aligned}$$

Therefore, $f_1^*(x, y)f_2^*(x, y) = (f_1(x, y)f_2(x, y))^*$. The above discussion is also applicable for any two polynomials over $\mathbb{F}_q[x, y]$ and over $\mathbb{F}_q[x, y]/\langle x^l - 1, y^m - 1 \rangle$.

Proposition 3.6. Given $(x^l - 1)(y^m - 1) = g(x, y)h(x, y)$, we have $h^*(x, y) \mid (x^l - 1)(y^m - 1)$. We define $C^* = \langle h^*(x, y) \rangle$, where $\deg(h^*(x, y)) = (a', b')$. Then, $\dim(C^*) = (l - a')(m - b')$, and $C^* \subseteq C^\perp$.

Proof. We observe that $(x^l - 1)(y^m - 1) = g(x, y)h(x, y)$, then

$$\begin{aligned} (x^{-l} - 1)(y^{-m} - 1) &= g(x^{-1}, y^{-1})h(x^{-1}, y^{-1}) \\ \implies (1 - x^l)(1 - y^m) &= x^l y^m g(x^{-1}, y^{-1})h(x^{-1}, y^{-1}) \\ \implies (x^l - 1)(y^m - 1) &= x^l y^m h(x^{-1}, y^{-1})g(x^{-1}, y^{-1}) \\ &= h^*(x, y)g^*(x, y). \end{aligned}$$

Therefore $h^*(x, y) \mid (x^l - 1)(y^m - 1)$. □

Proposition 3.7. $C^* \subseteq C$ if and only if $g(x, y)g^*(x, y) \mid (x^l - 1)(y^m - 1)$.

Proof. Let $C^* \subseteq C$. As $C^* = \langle h^*(x, y) \rangle$ and $C = \langle g(x, y) \rangle$, we get $h^*(x, y) = g(x, y)k(x, y)$ for some $k(x, y) \in \mathfrak{R}$. From the definition of the reciprocal polynomial we note that $\deg(h(x, y)) = \deg(h^*(x, y))$, also $\deg(h^*(x, y)) < (l, m)$. Therefore, by Proposition 3.4, we get $h(x, y) = g^*(x, y)k^*(x, y)$. Then,

$$(x^l - 1)(y^m - 1) = g(x, y)h(x, y) = g(x, y)g^*(x, y)k^*(x, y).$$

Thus, $g(x, y)g^*(x, y) \mid (x^l - 1)(y^m - 1)$.

On the other hand, suppose $r(x, y) \in C^* = \langle h^*(x, y) \rangle$, then $r(x, y) = h^*(x, y)s(x, y)$ for some $s(x, y) \in \mathfrak{R}$. As $g(x, y)g^*(x, y) \mid (x^l - 1)(y^m - 1)$, we get $(x^l - 1)(y^m - 1) = g(x, y)g^*(x, y)t(x, y)$, for some $t(x, y) \in \mathfrak{R}$. Then by Proposition 3.4,

$$h(x, y) = g^*(x, y)t(x, y) \text{ implying } h^*(x, y) = g(x, y)t^*(x, y).$$

Therefore, $r(x, y) = h^*(x, y)s(x, y) = g(x, y)t^*(x, y)s(x, y) \in \langle g(x, y) \rangle$, and we conclude that, $C^* \subseteq C$. □

Example 3.8. We take $l = 15, m = 12$, and consider $g(x, y)$ such that $g(x, y) \mid (x^{15} - 1)(y^{12} - 1)$, where

$$g(x, y) = x^4y^4 + x^4y^2 + x^4 + x^3y^4 + x^3y^2 + x^3 + y^4 + y^2 + 1,$$

$$\begin{aligned} h(x, y) = & x^{11}y^8 + x^{11}y^6 + x^{11}y^2 + x^{11} + x^{10}y^8 + x^{10}y^6 + x^{10}y^2 + x^{10} + x^9y^8 + x^9y^6 + x^9y^2 + \\ & x^9 + x^8y^8 + x^8y^6 + x^8y^2 + x^8 + x^6y^8 + x^6y^6 + x^6y^2 + x^6 + x^4y^8 + x^4y^6 + x^4y^2 + x^4 + \\ & x^3y^8 + x^3y^6 + x^3y^2 + x^3 + y^8 + y^6 + y^2 + 1. \end{aligned}$$

Then $C = \langle g(x, y) \rangle$ is a 2D-cyclic code with parameters $[180, 88, 6]$, and $C^* = \langle h^*(x, y) \rangle$ is a 2D-cyclic code with parameters $[180, 16, 32]$, where

$$\begin{aligned} h^*(x, y) = & x^{11}y^8 + x^{11}y^6 + x^{11}y^2 + x^{11} + x^8y^8 + x^8y^6 + x^8y^2 + x^8 + \\ & x^7y^8 + x^7y^6 + x^7y^2 + x^7 + x^5y^8 + x^5y^6 + x^5y^2 + x^5 + x^3y^8 + x^3y^6 + \\ & x^3y^2 + x^3 + x^2y^8 + x^2y^6 + x^2y^2 + x^2 + xy^8 + xy^6 + xy^2 + x + y^8 + y^6 + y^2 + 1. \end{aligned}$$

We also note that, $g(x, y)g^*(x, y) \mid (x^{15} - 1)(y^{12} - 1)$, where $g^*(x, y) = x^4y^4 + x^4y^2 + x^4 + xy^4 + xy^2 + x + y^4 + y^2 + 1$, and

$$\begin{aligned} g(x, y)g^*(x, y) = & x^8y^8 + x^8y^4 + x^8 + x^7y^8 + x^7y^4 + x^7 + x^5y^8 + x^5y^4 + \\ & x^5 + x^4y^8 + x^4y^4 + x^4 + x^3y^8 + x^3y^4 + x^3 + xy^8 + xy^4 + x + y^8 + y^4 + 1, \end{aligned}$$

Therefore, $C^* \subseteq C$.

Proposition 3.9. $C \subseteq C^*$ if and only if $(x^l - 1)(y^m - 1) \mid g(x, y)g^*(x, y)$.

Proof. Let $C \subseteq C^*$. As $C^* = \langle h^*(x, y) \rangle$ and $C = \langle g(x, y) \rangle$, we get $g(x, y) = h^*(x, y)k(x, y)$ for some $k(x, y) \in \mathfrak{A}$. As $\deg(g(x, y)) < (l, m)$, by Proposition 3.4, we get $g^*(x, y) = h(x, y)k^*(x, y)$. Then,

$$g(x, y)g^*(x, y) = g(x, y)h(x, y)k^*(x, y) = (x^l - 1)(y^m - 1)k^*(x, y),$$

and we have $(x^l - 1)(y^m - 1) \mid g(x, y)g^*(x, y)$.

On the other hand, suppose $c(x, y) \in C = \langle g(x, y) \rangle$, then $c(x, y) = g(x, y)s(x, y)$ for some $s(x, y) \in \mathfrak{A}$. As $(x^l - 1)(y^m - 1) \mid g(x, y)g^*(x, y)$, we get $g(x, y)g^*(x, y) = (x^l - 1)(y^m - 1)t(x, y)$, for some $t(x, y) \in \mathfrak{A}$, and

$$g^*(x, y) = h(x, y)t(x, y) \text{ implying } g(x, y) = h^*(x, y)t^*(x, y).$$

Therefore, $c(x, y) = h^*(x, y)t^*(x, y)s(x, y) \in \langle h^*(x, y) \rangle$, implying $c(x, y) \in \langle h^*(x, y) \rangle$, and, $C \subseteq C^*$. \square

Proposition 3.10. $C \subseteq C^\perp$ if and only if $(x^l - 1)(y^m - 1) \mid g(x, y)g^*(x, y)$.

Proof. We observe that $C \subseteq C^*$ if and only if $(x^l - 1)(y^m - 1) \mid g(x, y)g^*(x, y)$, and $C^* \subseteq C^\perp$. Thus $C \subseteq C^\perp$ if and only if $(x^l - 1)(y^m - 1) \mid g(x, y)g^*(x, y)$. \square

Remark 3.11. Propositions 3.7 and 3.9 are true only when $\dim(C) > 0$ and $\dim(C^*) > 0$.

4 Group ring codes and their generator matrices

The generator matrix for a cyclic code can be represented in two ways. First, it can be presented in standard form, where the rows are independent of each other. Alternatively, the same matrix can be expressed in circulant form, which has more rows than the standard form, and the rows may not be linearly independent. We primarily use the circulant form of the generator matrix when constructing generator matrices for group rings. To do this, we define the following two matrix forms: the circulant matrix and the reverse circulant matrix, as follows:

$$\text{Circ}(a_1, a_2, \dots, a_n) = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}, \quad \text{RevCirc}(a_1, a_2, \dots, a_n) = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_1 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_1 & \cdots & a_{n-1} \end{pmatrix}.$$

We start with the group ring $\mathbb{F}_q D_n$; this will help us implement the idea in the other case easily.

4.1 Group ring $\mathbb{F}_q D_n$:

Consider the dihedral group $D_n = \langle a, b \mid a^n = b^2 = e, bab^{-1} = a^{-1} \rangle$, which has order $2n$. Since this is a non-abelian group and $ba = a^{-1}b$, the elements of the group can be expressed in two different forms:

1. $D_n = \{e, a, a^2, \dots, a^{n-1}, b, ba, ba^2, \dots, ba^{n-1}\}$,
2. $D_n = \{e, a, a^2, \dots, a^{n-1}, b, ab, a^2b, \dots, a^{n-1}b\}$.

Remark 4.1. It is important to note that the two listings of elements above differ only in the order of their arrangement. For instance, in the first form, ba appears in the $(n+2)$ -th position, while in the second form, it is in the $2n$ -th position, where the element at that position is $a^{n-1}b = a^{-1}b = ba$. Thus, the elements are simply presented in two distinct arrangements. This arrangement of elements is crucial for identifying a generator matrix. \square

Depending on how the elements are listed, the corresponding generator matrix will differ. We present both forms as follows.

4.1.1 Form 1:

Suppose the elements of D_n are listed as $\{1, a, a^2, \dots, a^{n-1}, b, ba, ba^2, \dots, ba^{n-1}\}$. Then, any element of the group ring $\mathbb{F}_q D_n$ is of the form:

$$c = \alpha_0 + \alpha_1 a + \alpha_2 a^2 + \dots + \alpha_{n-1} a^{n-1} + \beta_0 b + \beta_1 ba + \dots + \beta_{n-1} ba^{n-1},$$

where $\alpha_i, \beta_i \in \mathbb{F}_q$ for $i = 0, 1, \dots, n-1$.

If the coefficients of c form the first row of the generator matrix, the second row is given by the coefficients of ac , which is:

$$ac = \alpha_0 a + \alpha_1 a^2 + \dots + \alpha_{n-2} a^{n-1} + \alpha_{n-1} a^n + \beta_0 ab + \beta_1 aba + \dots + \beta_{n-1} aba^{n-1}.$$

Using the relation $ba = a^{-1}b$, i.e., $aba = b$, this simplifies to:

$$ac = \alpha_{n-1} + \alpha_0 a + \alpha_1 a^2 + \dots + \alpha_{n-2} a^{n-1} + \beta_1 b + \beta_2 ba + \dots + \beta_{n-1} ba^{n-2} + \beta_0 ba^{n-1}.$$

Similarly, the third row consists of the coefficients of a^2c , the $(n+1)$ -th row contains the coefficients of bc , and the $(n+2)$ -th row has the coefficients of bac , and so on. After applying the relation $ba = a^{-1}b$, the generator matrix can be written as

$$\mathcal{G}_1 = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} \\ \alpha_{n-1} & \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} & \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_0 & \beta_{n-1} & \beta_0 & \beta_1 & \cdots & \beta_{n-2} \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_n & \alpha_{n-1} & \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{n-1} & \beta_0 & \beta_1 & \cdots & \beta_{n-2} & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_0 \end{pmatrix}.$$

In short we can write it as,

$$\mathcal{G}_1 = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where

$$A = \text{Circ}(\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \quad \text{and} \quad B = \text{RevCirc}(\beta_0, \beta_1, \dots, \beta_{n-1}).$$

4.1.2 Form 2:

Suppose the elements of D_n are listed as $\{e, a, a^2, \dots, a^{n-1}, b, ab, a^2b, \dots, a^{n-1}b\}$. Then, any element of the group ring $\mathbb{F}_q D_n$ is of the form:

$$c = \alpha_0 + \alpha_1 a + \alpha_2 a^2 + \dots + \alpha_{n-1} a^{n-1} + \beta_0 b + \beta_1 ab + \dots + \beta_{n-1} a^{n-1}b.$$

Following the approach described above, we obtain a generator matrix of the form:

$$\mathcal{G}_2 = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} \\ \alpha_{n-1} & \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} & \beta_{n-1} & \beta_0 & \beta_1 & \cdots & \beta_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_0 & \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_0 \\ \beta_0 & \beta_{n-1} & \beta_{n-2} & \cdots & \beta_1 & \alpha_0 & \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 \\ \beta_1 & \beta_0 & \beta_{n-1} & \cdots & \beta_2 & \alpha_1 & \alpha_0 & \alpha_{n-1} & \cdots & \alpha_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{n-1} & \beta_{n-2} & \beta_{n-3} & \cdots & \beta_0 & \alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} & \cdots & \alpha_0 \end{pmatrix},$$

that is,

$$\mathcal{G}_2 = \begin{pmatrix} A & D \\ D^T & A^T \end{pmatrix},$$

where A^T and D^T represent the transpose of the matrices A and D , respectively. These matrices are defined as:

$$A = \text{Circ}(\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \quad \text{and} \quad D = \text{Circ}(\beta_0, \beta_1, \dots, \beta_{n-1}).$$

Remark 4.2. The elements can also be listed in other ways. For example, from Form 1, we derive a new listing, Form 3: $\{e, b, a, ba, a^2, ba^2, \dots, a^{n-1}, ba^{n-1}\}$. Similarly, another listing, Form 4 can be derived from Form 2: $\{e, b, a, ab, a^2, a^2b, \dots, a^{n-1}, a^{n-1}b\}$. However, we focus primarily on Form 1 and Form 2, as the generator matrices based on these listings can be represented by two matrices of order $n \times n$, specifically A and B for Form 1, and A and D for Form 2. If we were to consider Forms 3 and 4, then each matrix within the block circulant generator matrix would have order 2×2 , with n such matrices, which can sometimes make it more complex to express them explicitly within the generator matrix. \square

Remark 4.3. The generalized quaternion group of order $4n$ shares a structural similarity with the dihedral group D_n and is defined by the presentation:

$$Q_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle,$$

where $n \geq 2$, with the usual quaternion group Q_8 corresponding to the case $n = 2$. The elements of Q_{4n} can be listed in either of the following two forms:

1. $Q_{4n} = \{1, a, a^2, \dots, a^{2n-1}, b, ab, a^2b, \dots, a^{2n-1}b\}$,
2. $Q_{4n} = \{1, a, a^2, \dots, a^{2n-1}, b, ba, ba^2, \dots, ba^{2n-1}\}$.

Using similar reasoning as in the case of D_n , the generator matrices for the group ring $\mathbb{F}_q Q_{4n}$ can be determined. \square

Now, we turn our attention to the group rings $\mathbb{F}_q G$, where G is a direct product of two finite groups. From the previous discussion, we have observed that element listing plays a crucial role in the construction of the generator matrix. In the case of a direct product of two groups, there are many more ways to list the elements. Before going into that, we highlight the following point:

Suppose the elements of G are of the form

$$\{a_i b_j c_k : i = 1, 2, \dots, l; j = 1, 2, \dots, m; k = 1, 2, \dots, n\}.$$

We note that i represents the inner loop, j represents the middle loop, and k represents the outer loop. These can be thought of as three nested loops. From now on, we denote $i = 1, 2, \dots, l; j = 1, 2, \dots, m; k = 1, 2, \dots, n$ collectively as (i, j, k) .

Here, both the form $a_i b_j c_k$ and the order in which (i, j, k) are considered are important. Fixing the form $a_j b_i c_k$, we can vary (i, j, k) in six distinct ways:

$$(i, j, k), (i, k, j), (j, i, k), (j, k, i), (k, i, j), (k, j, i).$$

We can observe that each permutation of the order yields a different element listing.

For example, consider an element listing of a direct product group G , denoted by

$$\{a_i b_j c_k : i = 1, 2; j = 1, 2, 3; k = 1, 2\},$$

where we vary the loops in the six different orders: $(i, j, k), (i, k, j), (j, i, k), (j, k, i), (k, i, j), (k, j, i)$. We have for, example,

$$\begin{aligned} & \{a_i b_j c_k : i = 1, 2; j = 1, 2, 3; k = 1, 2\} \quad (\text{we denote this loop as } (i, j, k)) \\ &= \left\{ \left\{ \{a_i b_j c_k : i = 1, 2\} : j = 1, 2, 3\} : k = 1, 2 \right\} \right. \\ &= \left\{ \left\{ \{a_1 b_j c_k, a_2 b_j c_k\} : j = 1, 2, 3\} : k = 1, 2 \right\} \right. \\ &= \left\{ \left\{ \{a_1 b_1 c_k, a_2 b_1 c_k\}, \{a_1 b_2 c_k, a_2 b_2 c_k\}, \{a_1 b_3 c_k, a_2 b_3 c_k\} \right\} : k = 1, 2 \right\} \\ &= \left\{ \{a_1 b_1 c_1, a_2 b_1 c_1, a_1 b_2 c_1, a_2 b_2 c_1, a_1 b_3 c_1, a_2 b_3 c_1\}, \{a_1 b_1 c_2, a_2 b_1 c_2, a_1 b_2 c_2, a_2 b_2 c_2, a_1 b_3 c_2, a_2 b_3 c_2\} \right\} \\ &= \left\{ a_1 b_1 c_1, a_2 b_1 c_1, a_1 b_2 c_1, a_2 b_2 c_1, a_1 b_3 c_1, a_2 b_3 c_1, a_1 b_1 c_2, a_2 b_1 c_2, a_1 b_2 c_2, a_2 b_2 c_2, a_1 b_3 c_2, a_2 b_3 c_2 \right\}. \end{aligned}$$

Similarly,

$$\begin{aligned} & \{a_i b_j c_k : j = 1, 2, 3; i = 1, 2; k = 1, 2\} \quad (\text{we denote this loop as } (j, i, k)) \\ &= \left\{ a_1 b_1 c_1, a_1 b_2 c_1, a_1 b_3 c_1, a_2 b_1 c_1, a_2 b_2 c_1, a_2 b_3 c_1, a_1 b_1 c_2, a_1 b_2 c_2, a_1 b_3 c_2, a_2 b_1 c_2, a_2 b_2 c_2, a_2 b_3 c_2 \right\}, \end{aligned}$$

and

$$\begin{aligned} & \{a_i b_j c_k : j = 1, 2, 3; k = 1, 2; i = 1, 2\} \quad (\text{we denote this loop as } (j, k, i)) \\ &= \left\{ a_1 b_1 c_1, a_1 b_2 c_1, a_1 b_3 c_1, a_1 b_1 c_2, a_1 b_2 c_2, a_1 b_3 c_2, a_2 b_1 c_1, a_2 b_2 c_1, a_2 b_3 c_1, a_2 b_1 c_2, a_2 b_2 c_2, a_2 b_3 c_2 \right\}. \end{aligned}$$

We have shown only three forms above, there are three more forms. Our main objective is to explain that by fixing the element form $a_i b_j c_k$ and varying the order of (i, j, k) , we obtain six different element listings for $(i, j, k), (i, k, j), (j, i, k), (j, k, i), (k, i, j), (k, j, i)$.

Now, if we fix the order of the loops as (i, j, k) and vary the element form from $a_i b_j c_k$ to $a_i c_k b_j, b_j a_i c_k, b_j c_k a_i, c_k a_i b_j, c_k b_j a_i$, we again get six different listings. For example,

$$\begin{aligned} 1. & \{a_i c_k b_j : i = 1, 2; j = 1, 2, 3; k = 1, 2\} \quad (a_i b_j c_k \text{ changed to } a_i c_k b_j \text{ fixing } (i, j, k)) \\ &= \left\{ a_1 c_1 b_1, a_1 c_1 b_2, a_1 c_1 b_3, a_1 c_2 b_1, a_1 c_2 b_2, a_1 c_2 b_3, a_2 c_1 b_1, a_2 c_1 b_2, a_2 c_1 b_3, a_2 c_2 b_1, a_2 c_2 b_2, a_2 c_2 b_3 \right\} \\ 2. & \{b_j c_k a_i : i = 1, 2; j = 1, 2, 3; k = 1, 2\} \quad (a_i b_j c_k \text{ changed to } b_j c_k a_i \text{ fixing } (i, j, k)) \\ &= \left\{ b_1 c_1 a_1, b_1 c_1 a_2, b_1 c_2 a_1, b_1 c_2 a_2, b_2 c_1 a_1, b_2 c_1 a_2, b_2 c_2 a_1, b_2 c_2 a_2, b_3 c_1 a_1, b_3 c_1 a_2, b_3 c_2 a_1, b_3 c_2 a_2 \right\}. \end{aligned}$$

Thus, we observe that by varying the element form $a_i b_j c_k$ and the ordering of the loops (i, j, k) , we can obtain a total of 36 different listings. In some cases, if there are commutative relations between a_i, b_j , and c_k , so there may be some repetition among the 36 different listings.

4.2 Group ring $\mathbb{F}_q(C_l \times C_m)$

To build intuition before addressing the general case of arbitrary l and m , we start by discussing specific values, setting $l = 3$ and $m = 5$. Consider the direct product $C_3 \times C_5$, where $C_3 = \{1, x, x^2\}$ and $C_5 = \{1, y, y^2, y^3, y^4\}$ are cyclic groups of orders 3 and 5, respectively. To list the elements of $C_3 \times C_5$, we consider the two types of multiplications discussed above:

$$\begin{aligned} \text{Form 1.} & \left\{ \{x^i y^j : i = 0, 1, 2\}, j = 0, 1, 2, 3, 4 \right\} \\ &= \left\{ \{y^j, x y^j, x^2 y^j\}, j = 0, 1, 2, 3 \right\} \\ &= \left\{ 1, x, x^2, y, x y, x^2 y, y^2, x y^2, x^2 y^2, y^3, x y^3, x^2 y^3, y^4, x y^4, x^2 y^4 \right\}, \end{aligned}$$

and

$$\begin{aligned}
\text{Form 2. } & \left\{ \{x^i y^j : j = 0, 1, 2, 3, 4\}, i = 0, 1, 2 \right\} \\
& = \left\{ \{x^i, x^i y, x^i y^2, x^i y^3, x^i y^4\}, i = 0, 1, 2 \right\} \\
& = \left\{ 1, y, y^2, y^3, y^4, x, xy, xy^2, xy^3, xy^4, x^2, x^2 y, x^2 y^2, x^2 y^3, x^2 y^4 \right\}.
\end{aligned}$$

We again emphasize that the elements in the two listings are identical but appear in different index orders. Our discussion primarily focuses on these two types of listings.

If we consider the Form 1, then any element of the group ring $\mathbb{F}_q(C_3 \times C_5)$ is of the form

$$\begin{aligned}
f_1 = & \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 y + \alpha_5 xy + \alpha_6 x^2 y + \alpha_7 y^2 \\
& + \alpha_8 xy^2 + \alpha_9 x^2 y^2 + \alpha_{10} y^3 + \alpha_{11} xy^3 + \alpha_{12} x^2 y^3 + \alpha_{13} y^4 + \alpha_{14} xy^4 + \alpha_{15} x^2 y^4.
\end{aligned}$$

Multiplying $g_i f_1$ and considering the coefficients we get the i -th row of the generator matrix, where g_i belongs to the element listing of Form 1. So the first row of the generator matrix correspond to the coefficients of $g_1 f = f$, i.e. is

$$(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \alpha_7 \quad \alpha_8 \quad \alpha_9 \quad \alpha_{10} \quad \alpha_{11} \quad \alpha_{12} \quad \alpha_{13} \quad \alpha_{14} \quad \alpha_{15}).$$

Multiplying $g_2 = x$ with f_1 we get

$$\begin{aligned}
g_2 f_1 = & \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 xy + \alpha_5 x^2 y + \alpha_6 x^3 y + \alpha_7 xy^2 + \alpha_8 x^2 y^2 \\
& + \alpha_9 x^3 y^2 + \alpha_{10} xy^3 + \alpha_{11} x^2 y^3 + \alpha_{12} x^3 y^3 + \alpha_{13} xy^4 + \alpha_{14} x^2 y^4 + \alpha_{15} x^3 y^4.
\end{aligned}$$

Using the fact $x^3 = 1$, and considering coefficients of $g_2 f_1 = x f_1$ gives the second row:

$$(\alpha_3 \quad \alpha_1 \quad \alpha_2 \quad \alpha_6 \quad \alpha_4 \quad \alpha_5 \quad \alpha_9 \quad \alpha_7 \quad \alpha_8 \quad \alpha_{12} \quad \alpha_{10} \quad \alpha_{11} \quad \alpha_{15} \quad \alpha_{13} \quad \alpha_{14}).$$

Similarly, from $g_3 f = x^2 f_1$ we get

$$(\alpha_2 \quad \alpha_3 \quad \alpha_1 \quad \alpha_5 \quad \alpha_6 \quad \alpha_4 \quad \alpha_8 \quad \alpha_9 \quad \alpha_7 \quad \alpha_{11} \quad \alpha_{12} \quad \alpha_{10} \quad \alpha_{14} \quad \alpha_{15} \quad \alpha_{13}).$$

Considering the coefficients of $g_i f_1$ for $i = 1, 2, \dots, 15$ we get the following matrix, where for better readability we replace α_i by i in the matrix:

$$\mathcal{G}_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 3 & 1 & 2 & 6 & 4 & 5 & 9 & 7 & 8 & 12 & 10 & 11 & 15 & 13 & 14 \\ 2 & 3 & 1 & 5 & 6 & 4 & 8 & 9 & 7 & 11 & 12 & 10 & 14 & 15 & 13 \\ 13 & 14 & 15 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 15 & 13 & 14 & 3 & 1 & 2 & 6 & 4 & 5 & 9 & 7 & 8 & 12 & 10 & 11 \\ 14 & 15 & 13 & 2 & 3 & 1 & 5 & 6 & 4 & 8 & 9 & 7 & 11 & 12 & 10 \\ 10 & 11 & 12 & 13 & 14 & 15 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 12 & 10 & 11 & 15 & 13 & 14 & 3 & 1 & 2 & 6 & 4 & 5 & 9 & 7 & 8 \\ 11 & 12 & 10 & 14 & 15 & 13 & 2 & 3 & 1 & 5 & 6 & 4 & 8 & 9 & 7 \\ 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 1 & 2 & 3 & 4 & 5 & 6 \\ 9 & 7 & 8 & 12 & 10 & 11 & 15 & 13 & 14 & 3 & 1 & 2 & 6 & 4 & 5 \\ 8 & 9 & 7 & 11 & 12 & 10 & 14 & 15 & 13 & 2 & 3 & 1 & 5 & 6 & 4 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 1 & 2 & 3 \\ 6 & 4 & 5 & 9 & 7 & 8 & 12 & 10 & 11 & 15 & 13 & 14 & 3 & 1 & 2 \\ 5 & 6 & 4 & 8 & 9 & 7 & 11 & 12 & 10 & 14 & 15 & 13 & 2 & 3 & 1 \end{pmatrix}.$$

If we look into the matrix, \mathcal{G}_1 we observe that, it is actually a block circulant matrix, and can be written as

$$\mathcal{G}_1 = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ A_5 & A_1 & A_2 & A_3 & A_4 \\ A_4 & A_5 & A_1 & A_2 & A_3 \\ A_3 & A_4 & A_5 & A_1 & A_2 \\ A_2 & A_3 & A_4 & A_5 & A_1 \end{pmatrix} = \text{Circ}(A_1, A_2, A_3, A_4, A_5),$$

where A_i for $i = 1, 2, 3, 4, 5$ are denoted as follows

$$A_1 = \text{Circ}(1, 2, 3), A_2 = \text{Circ}(4, 5, 6), A_3 = \text{Circ}(7, 8, 9), A_4 = \text{Circ}(10, 11, 12), \text{ and } A_5 = \text{Circ}(13, 14, 15).$$

Using similar arguments in case of the Form 2 type multiplication, we get the following

$$\mathcal{G}_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 5 & 1 & 2 & 3 & 4 & 10 & 6 & 7 & 8 & 9 & 15 & 11 & 12 & 13 & 14 \\ 4 & 5 & 1 & 2 & 3 & 9 & 10 & 6 & 7 & 8 & 14 & 15 & 11 & 12 & 13 \\ 3 & 4 & 5 & 1 & 2 & 8 & 9 & 10 & 6 & 7 & 13 & 14 & 15 & 11 & 12 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 9 & 10 & 6 & 12 & 13 & 14 & 15 & 11 \\ 11 & 12 & 13 & 14 & 15 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 15 & 11 & 12 & 13 & 14 & 5 & 1 & 2 & 3 & 4 & 10 & 6 & 7 & 8 & 9 \\ 14 & 15 & 11 & 12 & 13 & 4 & 5 & 1 & 2 & 3 & 9 & 10 & 6 & 7 & 8 \\ 13 & 14 & 15 & 11 & 12 & 3 & 4 & 5 & 1 & 2 & 8 & 9 & 10 & 6 & 7 \\ 12 & 13 & 14 & 15 & 11 & 2 & 3 & 4 & 5 & 1 & 7 & 8 & 9 & 10 & 6 \\ 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 1 & 2 & 3 & 4 & 5 \\ 10 & 6 & 7 & 8 & 9 & 15 & 11 & 12 & 13 & 14 & 5 & 1 & 2 & 3 & 4 \\ 9 & 10 & 6 & 7 & 8 & 14 & 15 & 11 & 12 & 13 & 4 & 5 & 1 & 2 & 3 \\ 8 & 9 & 10 & 6 & 7 & 13 & 14 & 15 & 11 & 12 & 3 & 4 & 5 & 1 & 2 \\ 7 & 8 & 9 & 10 & 6 & 12 & 13 & 14 & 15 & 11 & 2 & 3 & 4 & 5 & 1 \end{pmatrix}.$$

Then

$$\mathcal{G}_2 = \begin{pmatrix} A_1 & A_2 & A_3 \\ A_3 & A_1 & A_2 \\ A_2 & A_3 & A_1 \end{pmatrix} = \text{Circ}(A_1, A_2, A_3),$$

where $A_1 = \text{Circ}(1, 2, 3, 4, 5)$, $A_2 = \text{Circ}(6, 7, 8, 9, 10)$, and $A_3 = \text{Circ}(11, 12, 13, 14, 15)$.

With this understanding, we can now present the results for general values of l and m . Let C_l and C_m be two finite cyclic groups consisting of l and m elements, respectively. We denote their direct product as $C_l \times C_m$, where

$$C_l \times C_m = \langle x, y \mid x^l = y^m = 1, xy = yx \rangle.$$

As discussed above, the elements of the product groups $C_l \times C_m$, denoted by $\{x^i y^j : i = 0, 1, \dots, l; j = 0, 1, \dots, m\}$, can be represented in the two different following forms

$$\text{Form 1. } \left\{ \{x^i y^j : i = 0, 1, \dots, l\} : j = 0, 1, \dots, m \right\},$$

and

$$\text{Form 2. } \left\{ \{x^i y^j : j = 0, 1, \dots, m\}, i = 0, 1, \dots, l \right\}.$$

4.2.1 Form 1:

Let $C_l \times C_m = \{g_1, g_2, \dots, g_n\}$, where $n = lm$ and g_r for $r = 1, 2, \dots, n$ is defined as follows:

$$g_{1+i+l_j} = x^i y^j, \quad 0 \leq i \leq l-1; 0 \leq j \leq m-1.$$

We consider an element $f \in \mathbb{F}_q(C_l \times C_m)$ such that

$$f_1 = \sum_{r=1}^n \alpha_r g_r.$$

By multiplying g_r with f_1 and considering the resulting coefficients, we obtain the r -th row of the generator matrix, defined as follows:

$$\mathcal{G}_1 = \text{Circ}(A_1, A_2, \dots, A_m),$$

where A_s are $l \times l$ circulant matrices over \mathbb{F}_q defined as

$$A_s = \text{Circ}(\alpha_{1+l(s-1)}, \alpha_{2+l(s-1)}, \dots, \alpha_{l+l(s-1)}), \quad s = 1, 2, \dots, m.$$

4.2.2 Form 2:

Let $C_l \times C_m = \{g_1, g_2, \dots, g_n\}$, where $n = lm$ and g_r for $r = 1, 2, \dots, n$ is defined as follows:

$$g_{1+j+mi} = x^i y^j, \quad 0 \leq j \leq m-1; 0 \leq i \leq l-1.$$

We consider an element $f_2 \in \mathbb{F}_q(C_l \times C_m)$ such that

$$f_2 = \sum_{r=1}^n \alpha_r g_r.$$

By multiplying g_r with f_2 and considering the resulting coefficients, we obtain the r -th row of the generator matrix, defined as follows:

$$\mathcal{G}_2 = \text{Circ}(A_1, A_2, \dots, A_l),$$

where A_t are $m \times m$ circulant matrices over \mathbb{F}_q defined as

$$A_t = \text{Circ}(\alpha_{1+m(t-1)}, \alpha_{2+m(t-1)}, \dots, \alpha_{m+m(t-1)}), \quad t = 1, 2, \dots, l.$$

Remark 4.4. We can extend the above discussion to the finite direct product of cyclic groups. For example, if we consider the direct product of three cyclic groups, $C_k \times C_l \times C_m = \{g_1, g_2, \dots, g_n\}$, where $n = klm$ and g_r for $r = 1, 2, \dots, n$, the listing of elements can be given in the following ways:

$$\text{Form 1: } g_{1+a+kb+klc} = x^a y^b z^c, \quad 0 \leq a \leq k-1; 0 \leq b \leq l-1; 0 \leq c \leq m-1,$$

$$\text{Form 2: } g_{1+a+kc+klb} = x^a y^b z^c, \quad 0 \leq a \leq k-1; 0 \leq c \leq m-1; 0 \leq b \leq l-1,$$

$$\text{Form 3: } g_{1+b+ka+klc} = x^a y^b z^c, \quad 0 \leq b \leq l-1; 0 \leq a \leq k-1; 0 \leq c \leq m-1,$$

$$\text{Form 4: } g_{1+b+kc+kla} = x^a y^b z^c, \quad 0 \leq b \leq l-1; 0 \leq c \leq m-1; 0 \leq a \leq k-1,$$

$$\text{Form 5: } g_{1+c+ka+klb} = x^a y^b z^c, \quad 0 \leq c \leq m-1; 0 \leq a \leq k-1; 0 \leq b \leq l-1,$$

$$\text{Form 6: } g_{1+c+kb+kla} = x^a y^b z^c, \quad 0 \leq c \leq m-1; 0 \leq b \leq l-1; 0 \leq a \leq k-1.$$

Similarly, we can extend this concept to any finite product of cyclic groups.

Remark 4.5. Considering the semidirect product of two cyclic groups C_l and C_m , we can express it as

$$C_l \rtimes C_m = \langle x, y \mid x^l = 1, y^m = 1, yxy^{-1} = x^k \rangle,$$

where $\gcd(k, l) = 1$. By fixing l and m , we can also construct generator matrices for the semidirect product of these two cyclic groups.

4.3 Group ring $\mathbb{F}_q(C_l \times D_m)$

The number of generators in a group and the relations between them influence the number of distinct ways in which elements can be listed. In the case of the group ring $\mathbb{F}_q(C_l \times D_m)$, there are three generators and three relations among them, leading to multiple possible listings. Before discussing the general case for arbitrary l and m , we examine specific values of l and m to help illustrate the general structure.

If we let $l = 3$ and $m = 3$, we have $C_3 = \{1, x, x^2\}$, which is the cyclic group of order 3 generated by x , where 1 is the identity element. We also define $D_3 = \langle y, z \mid y^3 = z^2 = 1, yzy^{-1} = z^{-1} \rangle$, the dihedral group of order 6. According to our discussion on the group ring $\mathbb{F}_q D_m$, we can list the elements of D_3 in two distinct ways $D_3 = \{1, y, y^2, z, zy, zy^2\}$, and $D_3 = \{1, y, y^2, z, yz, y^2z\}$.

For the direct product $C_3 \times D_3$, we can list the elements in two different ways based on how we order the elements in D_3 . The two possible listings are: $\left\{ \{1, x, x^2\} \times \{1, y, y^2, z, zy, zy^2\} \right\}$, and $\left\{ \{1, x, x^2\} \times \{1, y, y^2, z, yz, y^2z\} \right\}$.

From $\left\{ \{1, x, x^2\} \times \{1, y, y^2, z, zy, zy^2\} \right\}$, we consider the following two types of listing

$$\begin{aligned}
\text{Form 1 : } & \left\{ \{cd : c \in C_3\}, d \in D_3 \right\} \\
& = \{1, x, x^2, y, xy, x^2y, y^2, xy^2, x^2y^2, z, xz, x^2z, zy, xzy, x^2zy, zy^2, xzy^2, x^2zy^2\} \\
& = \left\{ \{ \{x^i z^j y^k : i = 0, 1, 2\}, k = 0, 1, 2\}, j = 0, 1 \right\} \\
& = \{x^i z^j y^k; \quad i = 0, 1, 2; \quad k = 0, 1, 2; \quad j = 0, 1\}.
\end{aligned} \tag{1}$$

$$\begin{aligned}
\text{Form 2 : } & \left\{ \{cd : d \in D_3\}, c \in C_3 \right\} \\
& = \{1, y, y^2, z, zy, zy^2, x, yx, y^2x, zx, zyx, zy^2x, x^2, yx^2, y^2x^2, zx^2, zyx^2, zy^2x^2\} \\
& = \left\{ \{ \{z^j y^k x^i : k = 0, 1, 2\}, j = 0, 1\}, i = 0, 1, 2 \right\} \\
& = \{z^j y^k x^i; \quad k = 0, 1, 2; \quad j = 0, 1; \quad i = 0, 1, 2\}.
\end{aligned} \tag{2}$$

From $\{1, x, x^2\} \times \{1, y, y^2, z, yz, y^2z\}$, we get

$$\begin{aligned}
\text{Form 3 : } & \left\{ \{cd : c \in C_3\}, d \in D_3 \right\} \\
& = \{1, x, x^2, y, xy, x^2y, y^2, xy^2, x^2y^2, z, xz, x^2z, yz, xyz, x^2yz, y^2z, xy^2z, x^2y^2z\} \\
& = \left\{ \{ \{x^i y^k z^j : i = 0, 1, 2\}, k = 0, 1, 2\}, j = 0, 1 \right\} \\
& = \{x^i y^k z^j; \quad i = 0, 1, 2; \quad k = 0, 1, 2; \quad j = 0, 1\}.
\end{aligned} \tag{3}$$

$$\begin{aligned}
\text{Form 4 : } & \left\{ \{cd : d \in D_3\}, c \in C_3 \right\} \\
& = \{1, y, y^2, z, yz, y^2z, x, yx, y^2x, zx, yzx, y^2zx, x^2, yx^2, y^2x^2, zx^2, yzx^2, y^2zx^2\} \\
& = \left\{ \{ \{y^k z^j x^i : k = 0, 1, 2\}, j = 0, 1\}, i = 0, 1, 2 \right\} \\
& = \{y^k z^j x^i; \quad k = 0, 1, 2; \quad j = 0, 1; \quad i = 0, 1, 2\}.
\end{aligned} \tag{4}$$

Remark 4.6. As discussed earlier, for a fixed form $x^i y^j z^k$, depending on the choice of the inner, middle, and outer loops, there are six possible orderings of the indices. Similarly, each of $x^i z^j y^k$, $y^i x^j z^k$, $y^i z^j x^k$, $z^i x^j y^k$, and $z^i y^j x^k$ can also be written in six different ways. This results in a total of 36 possible orderings. While some of these orderings may be identical, we primarily focus on the four Forms of listings discussed above. \square

We define the direct product of the cyclic group C_l of order l and the dihedral group D_m of order $2m$ as follows:

$$C_l \times D_m = \langle x, y, z \mid x^l = y^m = z^2 = 1, xy = yx, xz = zx, yzy^{-1} = y^{-1} \rangle.$$

This group has $2ml$ elements. We denote $n = 2ml$.

4.3.1 Form 1:

Let $C_l \times D_m = \{g_1, g_2, \dots, g_n\}$, where each g_r for $r = 1, 2, \dots, n$ is defined as follows:

$$g_{1+i+km+jml} = x^i z^j y^k, \quad 0 \leq i \leq l-1; \quad 0 \leq k \leq m-1; \quad 0 \leq j \leq 1.$$

We consider an element $f_1 \in \mathbb{F}_q(C_l \times D_m)$ such that

$$f_1 = \sum_{r=1}^n \alpha_r g_r.$$

By multiplying g_r with f_1 and considering the resulting coefficients, we obtain the r -th row of the generator matrix, defined as follows:

$$\mathcal{G}_1 = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix} = \text{Circ}(M_1, M_2),$$

where $M_1 = \text{Circ}(A_1, A_2, \dots, A_m)$ and $M_2 = \text{RevCirc}(A_{m+1}, A_{m+2}, \dots, A_{2m})$, with A_s denoting $l \times l$ circulant matrices defined as

$$A_s = \text{Circ}(\alpha_{(s-1)l+1}, \alpha_{(s-1)l+2}, \dots, \alpha_{(s-1)l+l}), \quad s = 1, 2, \dots, 2m.$$

Example 4.7. Let $l = 3$ and $m = 3$. Then, according to the structure given in Form 1, the elements are listed as:

$$g_{1+i+km+jml} = x^i z^j y^k, \quad 0 \leq i \leq l-1, 0 \leq k \leq m-1, 0 \leq j \leq 1.$$

Following the procedure outlined above, we obtain the generator matrix as follows:

$$\mathcal{G}_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 3 & 1 & 2 & 6 & 4 & 5 & 9 & 7 & 8 & 12 & 10 & 11 & 15 & 13 & 14 & 18 & 16 & 17 \\ 2 & 3 & 1 & 5 & 6 & 4 & 8 & 9 & 7 & 11 & 12 & 10 & 14 & 15 & 13 & 17 & 18 & 16 \\ 7 & 8 & 9 & 1 & 2 & 3 & 4 & 5 & 6 & 13 & 14 & 15 & 16 & 17 & 18 & 10 & 11 & 12 \\ 9 & 7 & 8 & 3 & 1 & 2 & 6 & 4 & 5 & 15 & 13 & 14 & 18 & 16 & 17 & 12 & 10 & 11 \\ 8 & 9 & 7 & 2 & 3 & 1 & 5 & 6 & 4 & 14 & 15 & 13 & 17 & 18 & 16 & 11 & 12 & 10 \\ 4 & 5 & 6 & 7 & 8 & 9 & 1 & 2 & 3 & 16 & 17 & 18 & 10 & 11 & 12 & 13 & 14 & 15 \\ 6 & 4 & 5 & 9 & 7 & 8 & 3 & 1 & 2 & 18 & 16 & 17 & 12 & 10 & 11 & 15 & 13 & 14 \\ 5 & 6 & 4 & 8 & 9 & 7 & 2 & 3 & 1 & 17 & 18 & 16 & 11 & 12 & 10 & 14 & 15 & 13 \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 12 & 10 & 11 & 15 & 13 & 14 & 18 & 16 & 17 & 3 & 1 & 2 & 6 & 4 & 5 & 9 & 7 & 8 \\ 11 & 12 & 10 & 14 & 15 & 13 & 17 & 18 & 16 & 2 & 3 & 1 & 5 & 6 & 4 & 8 & 9 & 7 \\ 13 & 14 & 15 & 16 & 17 & 18 & 10 & 11 & 12 & 7 & 8 & 9 & 1 & 2 & 3 & 4 & 5 & 6 \\ 15 & 13 & 14 & 18 & 16 & 17 & 12 & 10 & 11 & 9 & 7 & 8 & 3 & 1 & 2 & 6 & 4 & 5 \\ 14 & 15 & 13 & 17 & 18 & 16 & 11 & 12 & 10 & 8 & 9 & 7 & 2 & 3 & 1 & 5 & 6 & 4 \\ 16 & 17 & 18 & 10 & 11 & 12 & 13 & 14 & 15 & 4 & 5 & 6 & 7 & 8 & 9 & 1 & 2 & 3 \\ 18 & 16 & 17 & 12 & 10 & 11 & 15 & 13 & 14 & 6 & 4 & 5 & 9 & 7 & 8 & 3 & 1 & 2 \\ 17 & 18 & 16 & 11 & 12 & 10 & 14 & 15 & 13 & 5 & 6 & 4 & 8 & 9 & 7 & 2 & 3 & 1 \end{pmatrix}.$$

We can also write $\mathcal{G}_1 = \text{Circ}(M_1, M_2)$, where $M_1 = \text{Circ}(A_1, A_2, A_3)$, and $M_2 = \text{RevCirc}(A_4, A_5, A_6)$ with A_i the 3×3 circulant matrices defined as

$$A_1 = \text{Circ}(1, 2, 3), \quad A_2 = \text{Circ}(4, 5, 6), \quad A_3 = \text{Circ}(7, 8, 9),$$

$$A_4 = \text{Circ}(10, 11, 12), \quad A_5 = \text{Circ}(13, 14, 15), \quad \text{and} \quad A_6 = \text{Circ}(16, 17, 18).$$

□

4.3.2 Form 2:

Let $C_l \times D_m = \{g_1, g_2, \dots, g_n\}$, where each g_r for $r = 1, 2, \dots, n$ is defined as follows:

$$g_{1+k+jm+iml} = z^j y^k x^i, \quad 0 \leq k \leq m-1; 0 \leq j \leq 1; 0 \leq i \leq l-1.$$

We consider an element $f_2 \in \mathbb{F}_q(C_l \times D_m)$ such that

$$f_2 = \sum_{r=1}^n \alpha_r g_r.$$

Then the generator matrix corresponding to this element listing is defined as follows:

$$\mathcal{G}_2 = \text{Circ}(M_1, M_2, \dots, M_l),$$

where each M_s for $s = 1, 2, \dots, l$ is defined as

$$M_s = \text{Circ}(A_{2s-1}, A_{2s}),$$

and A_t is an $m \times m$, defined as follows:

- If t is odd, then

$$A_t = \text{Circ}(\alpha_{(t-1)m+1}, \alpha_{(t-1)m+2}, \dots, \alpha_{(t-1)m+m}).$$

- If t is even, then

$$A_t = \text{RevCirc}(\alpha_{(t-1)m+1}, \alpha_{(t-1)m+2}, \dots, \alpha_{(t-1)m+m}),$$

where $t = 1, 2, \dots, 2t$.

Example 4.8. We consider $l = 2, m = 5$ with element listing following Form 2. The generator matrix corresponding to this listing is

$$\mathcal{G}_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ 5 & 1 & 2 & 3 & 4 & 7 & 8 & 9 & 10 & 6 & 15 & 11 & 12 & 13 & 14 & 17 & 18 & 19 & 20 & 16 & 17 \\ 4 & 5 & 1 & 2 & 3 & 8 & 9 & 10 & 6 & 7 & 14 & 15 & 11 & 12 & 13 & 18 & 19 & 20 & 16 & 17 & 18 \\ 3 & 4 & 5 & 1 & 2 & 9 & 10 & 6 & 7 & 8 & 13 & 14 & 15 & 11 & 12 & 19 & 20 & 16 & 17 & 18 & 18 \\ 2 & 3 & 4 & 5 & 1 & 10 & 6 & 7 & 8 & 9 & 12 & 13 & 14 & 15 & 11 & 20 & 16 & 17 & 18 & 19 & 19 \\ 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 & 16 & 17 & 18 & 19 & 20 & 11 & 12 & 13 & 14 & 15 & 15 \\ 7 & 8 & 9 & 10 & 6 & 5 & 1 & 2 & 3 & 4 & 17 & 18 & 19 & 20 & 16 & 15 & 11 & 12 & 13 & 14 & 14 \\ 8 & 9 & 10 & 6 & 7 & 4 & 5 & 1 & 2 & 3 & 18 & 19 & 20 & 16 & 17 & 14 & 15 & 11 & 12 & 13 & 13 \\ 9 & 10 & 6 & 7 & 8 & 3 & 4 & 5 & 1 & 2 & 19 & 20 & 16 & 17 & 18 & 13 & 14 & 15 & 11 & 12 & 12 \\ 10 & 6 & 7 & 8 & 9 & 2 & 3 & 4 & 5 & 1 & 20 & 16 & 17 & 18 & 19 & 12 & 13 & 14 & 15 & 11 & 11 \\ 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 10 \\ 15 & 11 & 12 & 13 & 14 & 17 & 18 & 19 & 20 & 16 & 5 & 1 & 2 & 3 & 4 & 7 & 8 & 9 & 10 & 6 & 6 \\ 14 & 15 & 11 & 12 & 13 & 18 & 19 & 20 & 16 & 17 & 4 & 5 & 1 & 2 & 3 & 8 & 9 & 10 & 6 & 7 & 7 \\ 13 & 14 & 15 & 11 & 12 & 19 & 20 & 16 & 17 & 18 & 3 & 4 & 5 & 1 & 2 & 9 & 10 & 6 & 7 & 8 & 8 \\ 12 & 13 & 14 & 15 & 11 & 20 & 16 & 17 & 18 & 19 & 2 & 3 & 4 & 5 & 1 & 10 & 6 & 7 & 8 & 9 & 9 \\ 16 & 17 & 18 & 19 & 20 & 11 & 12 & 13 & 14 & 15 & 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 & 5 \\ 17 & 18 & 19 & 20 & 16 & 15 & 11 & 12 & 13 & 14 & 7 & 8 & 9 & 10 & 6 & 5 & 1 & 2 & 3 & 4 & 4 \\ 18 & 19 & 20 & 16 & 17 & 14 & 15 & 11 & 12 & 13 & 8 & 9 & 10 & 6 & 7 & 4 & 5 & 1 & 2 & 3 & 3 \\ 19 & 20 & 16 & 17 & 18 & 13 & 14 & 15 & 11 & 12 & 9 & 10 & 6 & 7 & 8 & 3 & 4 & 5 & 1 & 2 & 2 \\ 20 & 16 & 17 & 18 & 19 & 12 & 13 & 14 & 15 & 11 & 10 & 6 & 7 & 8 & 9 & 2 & 3 & 4 & 5 & 1 & 1 \end{pmatrix}.$$

We can express \mathcal{G}_2 as

$$\mathcal{G}_2 = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix} = \text{Circ}(M_1, M_2),$$

where $M_1 = \text{Circ}(A_1, A_2)$ and $M_2 = \text{Circ}(A_3, A_4)$. Here, A_1 and A_3 are 5×5 circulant matrices defined by

$$A_1 = \text{Circ}(1, 2, 3, 4, 5), \quad A_3 = \text{Circ}(11, 12, 13, 14, 15),$$

and A_2 and A_4 are 5×5 reverse circulant matrices defined by

$$A_2 = \text{RevCirc}(6, 7, 8, 9, 10), \quad A_4 = \text{RevCirc}(16, 17, 18, 19, 20).$$

□

4.3.3 Form 3:

Let $C_l \times D_m = \{g_1, g_2, \dots, g_n\}$, where each g_r for $r = 1, 2, \dots, n$ is defined as follows:

$$g_{1+i+km+jml} = x^i y^k z^j, \quad 0 \leq i \leq l-1; \quad 0 \leq k \leq m-1; \quad 0 \leq j \leq 1.$$

We consider an element $f_3 \in \mathbb{F}_q(C_l \times D_m)$ such that

$$f_3 = \sum_{r=1}^n \alpha_r g_r.$$

Then the generator matrix corresponding to this element listing is defined as follows:

$$\mathcal{G}_3 = \begin{pmatrix} M_1 & M_2 \\ M_2^T & M_1^T \end{pmatrix},$$

where $M_1 = \text{Circ}(A_1, A_2, \dots, A_m)$ and $M_2 = \text{Circ}(A_{m+1}, A_{m+2}, \dots, A_{2m})$ and A_s are $l \times l$ circulant matrices defined as

$$A_s = \text{Circ}(\alpha_{(s-1)l+1}, \alpha_{(s-1)l+2}, \dots, \alpha_{(s-1)l+l}), \quad s = 1, 2, \dots, 2m.$$

Remark 4.9. If M is a block circulant matrix defined as $M = \text{Circ}(B_1, B_2, B_3)$, then M^T is defined as

$$M^T = \begin{pmatrix} B_1 & B_2 & B_3 \\ B_3 & B_1 & B_2 \\ B_2 & B_3 & B_1 \end{pmatrix}^T = \begin{pmatrix} B_1 & B_3 & B_2 \\ B_2 & B_1 & B_3 \\ B_3 & B_2 & B_1 \end{pmatrix}.$$

Example 4.10. We consider $l = 5, m = 3$ and the element listing following Form 3. The generator matrix \mathcal{G}_3 corresponding to this listing is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 \\ 5 & 1 & 2 & 3 & 4 & 10 & 6 & 7 & 8 & 9 & 15 & 11 & 12 & 13 & 14 & 20 & 16 & 17 & 18 & 19 & 25 & 21 & 22 & 23 & 24 & 30 & 26 & 27 & 28 & 29 \\ 4 & 5 & 1 & 2 & 3 & 9 & 10 & 6 & 7 & 8 & 14 & 15 & 11 & 12 & 13 & 19 & 20 & 16 & 17 & 18 & 24 & 25 & 21 & 22 & 23 & 29 & 30 & 26 & 27 & 28 \\ 3 & 4 & 5 & 1 & 2 & 8 & 9 & 10 & 6 & 7 & 13 & 14 & 15 & 11 & 12 & 18 & 19 & 20 & 16 & 17 & 23 & 24 & 25 & 21 & 22 & 28 & 29 & 30 & 26 & 27 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 9 & 10 & 6 & 12 & 13 & 14 & 15 & 11 & 17 & 18 & 19 & 20 & 16 & 22 & 23 & 24 & 25 & 21 & 27 & 28 & 29 & 30 & 26 \\ 11 & 12 & 13 & 14 & 15 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 26 & 27 & 28 & 29 & 30 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 \\ 15 & 11 & 12 & 13 & 14 & 5 & 1 & 2 & 3 & 4 & 10 & 6 & 7 & 8 & 9 & 30 & 26 & 27 & 28 & 29 & 20 & 16 & 17 & 18 & 19 & 25 & 21 & 22 & 23 & 24 \\ 14 & 15 & 11 & 12 & 13 & 4 & 5 & 1 & 2 & 3 & 9 & 10 & 6 & 7 & 8 & 29 & 30 & 26 & 27 & 28 & 19 & 20 & 16 & 17 & 18 & 24 & 25 & 21 & 22 & 23 \\ 13 & 14 & 15 & 11 & 12 & 3 & 4 & 5 & 1 & 2 & 8 & 9 & 10 & 6 & 7 & 28 & 29 & 30 & 26 & 27 & 18 & 19 & 20 & 16 & 17 & 23 & 24 & 25 & 21 & 22 \\ 12 & 13 & 14 & 15 & 11 & 2 & 3 & 4 & 5 & 1 & 7 & 8 & 9 & 10 & 6 & 27 & 28 & 29 & 30 & 26 & 17 & 18 & 19 & 20 & 16 & 22 & 23 & 24 & 25 & 21 \\ 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 1 & 2 & 3 & 4 & 5 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 16 & 17 & 18 & 19 & 20 \\ 10 & 6 & 7 & 8 & 9 & 15 & 11 & 12 & 13 & 14 & 5 & 1 & 2 & 3 & 4 & 25 & 21 & 22 & 23 & 24 & 30 & 26 & 27 & 28 & 29 & 20 & 16 & 17 & 18 & 19 \\ 9 & 10 & 6 & 7 & 8 & 14 & 15 & 11 & 12 & 13 & 4 & 5 & 1 & 2 & 3 & 24 & 25 & 21 & 22 & 23 & 29 & 30 & 26 & 27 & 28 & 19 & 20 & 16 & 17 & 18 \\ 8 & 9 & 10 & 6 & 7 & 13 & 14 & 15 & 11 & 12 & 3 & 4 & 5 & 1 & 2 & 23 & 24 & 25 & 21 & 22 & 28 & 29 & 30 & 26 & 27 & 18 & 19 & 20 & 16 & 17 \\ 7 & 8 & 9 & 10 & 6 & 12 & 13 & 14 & 15 & 11 & 2 & 3 & 4 & 5 & 1 & 22 & 23 & 24 & 25 & 21 & 27 & 28 & 29 & 30 & 26 & 17 & 18 & 19 & 20 & 16 \\ 16 & 17 & 18 & 19 & 20 & 26 & 27 & 28 & 29 & 30 & 21 & 22 & 23 & 24 & 25 & 1 & 2 & 3 & 4 & 5 & 11 & 12 & 13 & 14 & 15 & 6 & 7 & 8 & 9 & 10 \\ 20 & 16 & 17 & 18 & 19 & 30 & 26 & 27 & 28 & 29 & 25 & 21 & 22 & 23 & 24 & 5 & 1 & 2 & 3 & 4 & 15 & 11 & 12 & 13 & 14 & 10 & 6 & 7 & 8 & 9 \\ 19 & 20 & 16 & 17 & 18 & 29 & 30 & 26 & 27 & 28 & 24 & 25 & 21 & 22 & 23 & 4 & 5 & 1 & 2 & 3 & 14 & 15 & 11 & 12 & 13 & 9 & 10 & 6 & 7 & 8 \\ 18 & 19 & 20 & 16 & 17 & 28 & 29 & 30 & 26 & 27 & 23 & 24 & 25 & 21 & 22 & 3 & 4 & 5 & 1 & 2 & 13 & 14 & 15 & 11 & 12 & 8 & 9 & 10 & 6 & 7 \\ 17 & 18 & 19 & 20 & 16 & 27 & 28 & 29 & 30 & 26 & 22 & 23 & 24 & 25 & 21 & 2 & 3 & 4 & 5 & 1 & 12 & 13 & 14 & 15 & 11 & 7 & 8 & 9 & 10 & 6 \\ 21 & 22 & 23 & 24 & 25 & 16 & 17 & 18 & 19 & 20 & 26 & 27 & 28 & 29 & 30 & 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 & 11 & 12 & 13 & 14 & 15 \\ 25 & 21 & 22 & 23 & 24 & 20 & 16 & 17 & 18 & 19 & 30 & 26 & 27 & 28 & 29 & 10 & 6 & 7 & 8 & 9 & 5 & 1 & 2 & 3 & 4 & 15 & 11 & 12 & 13 & 14 \\ 24 & 25 & 21 & 22 & 23 & 19 & 20 & 16 & 17 & 18 & 29 & 30 & 26 & 27 & 28 & 9 & 10 & 6 & 7 & 8 & 4 & 5 & 1 & 2 & 3 & 14 & 15 & 11 & 12 & 13 \\ 23 & 24 & 25 & 21 & 22 & 18 & 19 & 20 & 16 & 17 & 28 & 29 & 30 & 26 & 27 & 8 & 9 & 10 & 6 & 7 & 3 & 4 & 5 & 1 & 2 & 13 & 14 & 15 & 11 & 12 \\ 22 & 23 & 24 & 25 & 21 & 17 & 18 & 19 & 20 & 16 & 27 & 28 & 29 & 30 & 26 & 7 & 8 & 9 & 10 & 6 & 2 & 3 & 4 & 5 & 1 & 12 & 13 & 14 & 15 & 11 \\ 26 & 27 & 28 & 29 & 30 & 21 & 22 & 23 & 24 & 25 & 16 & 17 & 18 & 19 & 20 & 11 & 12 & 13 & 14 & 15 & 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 \\ 30 & 26 & 27 & 28 & 29 & 25 & 21 & 22 & 23 & 24 & 20 & 16 & 17 & 18 & 19 & 15 & 11 & 12 & 13 & 14 & 10 & 6 & 7 & 8 & 9 & 5 & 1 & 2 & 3 & 4 \\ 29 & 30 & 26 & 27 & 28 & 24 & 25 & 21 & 22 & 23 & 19 & 20 & 16 & 17 & 18 & 14 & 15 & 11 & 12 & 13 & 9 & 10 & 6 & 7 & 8 & 4 & 5 & 1 & 2 & 3 \\ 28 & 29 & 30 & 26 & 27 & 23 & 24 & 25 & 21 & 22 & 18 & 19 & 20 & 16 & 17 & 13 & 14 & 15 & 11 & 12 & 8 & 9 & 10 & 6 & 7 & 3 & 4 & 5 & 1 & 2 \\ 27 & 28 & 29 & 30 & 26 & 22 & 23 & 24 & 25 & 21 & 17 & 18 & 19 & 20 & 16 & 12 & 13 & 14 & 15 & 11 & 7 & 8 & 9 & 10 & 6 & 2 & 3 & 4 & 5 & 1 \end{pmatrix}.$$

Here \mathcal{G}_3 is of the form

$$\mathcal{G}_3 = \begin{pmatrix} M_1 & M_2 \\ M_2^T & M_1^T \end{pmatrix},$$

where $M_1 = \text{Circ}(A_1, A_2, A_3)$ and $M_2 = \text{Circ}(A_4, A_5, A_6)$ and A_s are 5×5 circulant matrices defined as

$$A_1 = \text{Circ}(1, 2, 3, 4, 5), \quad A_2 = \text{Circ}(6, 7, 8, 9, 10), \quad A_3 = \text{Circ}(11, 12, 13, 14, 15),$$

$$A_4 = \text{Circ}(16, 17, 18, 19, 20), \quad A_5 = \text{Circ}(21, 22, 23, 24, 25), \quad \text{and} \quad A_6 = \text{Circ}(26, 27, 28, 29, 30).$$

□

4.3.4 Form 4:

Let $C_l \times D_m = \{g_1, g_2, \dots, g_n\}$, where each g_r for $r = 1, 2, \dots, n$ is defined as follows:

$$g_{1+k+jm+iml} = y^k z^j x^i, \quad 0 \leq k \leq m-1; \quad 0 \leq j \leq 1; \quad 0 \leq i \leq l-1.$$

We consider an element $f_4 \in \mathbb{F}_q(C_l \times D_m)$ such that

$$f_4 = \sum_{r=1}^n \alpha_r g_r.$$

Then the generator matrix corresponding to this element listing is defined as follows:

$$\mathcal{G}_4 = \text{Circ}(M_1, M_2, \dots, M_l),$$

where each M_s is defined as

$$M_s = \begin{pmatrix} A_{2s-1} & A_{2s} \\ A_{2s}^T & A_{2s-1}^T \end{pmatrix},$$

for $s = 1, 2, \dots, m$, and A_t are $m \times m$ circulant matrices defined as

$$A_t = \text{Circ}(\alpha_{(t-1)m+1}, \alpha_{(t-1)m+2}, \dots, \alpha_{(t-1)m+m}); \quad t = 1, 2, \dots, 2l.$$

Example 4.11. We consider $l = 2, m = 5$, and the element listing following Form 4. The generator matrix corresponding to this listing is

$$\mathcal{G}_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ 5 & 1 & 2 & 3 & 4 & 10 & 6 & 7 & 8 & 9 & 15 & 11 & 12 & 13 & 14 & 20 & 16 & 17 & 18 & 19 \\ 4 & 5 & 1 & 2 & 3 & 9 & 10 & 6 & 7 & 8 & 14 & 15 & 11 & 12 & 13 & 19 & 20 & 16 & 17 & 18 \\ 3 & 4 & 5 & 1 & 2 & 8 & 9 & 10 & 6 & 7 & 13 & 14 & 15 & 11 & 12 & 18 & 19 & 20 & 16 & 17 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 9 & 10 & 6 & 12 & 13 & 14 & 15 & 11 & 17 & 18 & 19 & 20 & 16 \\ 6 & 10 & 9 & 8 & 7 & 1 & 5 & 4 & 3 & 2 & 16 & 20 & 19 & 18 & 17 & 11 & 15 & 14 & 13 & 12 \\ 7 & 6 & 10 & 9 & 8 & 2 & 1 & 5 & 4 & 3 & 17 & 16 & 20 & 19 & 18 & 12 & 11 & 15 & 14 & 13 \\ 8 & 7 & 6 & 10 & 9 & 3 & 2 & 1 & 5 & 4 & 18 & 17 & 16 & 20 & 19 & 13 & 12 & 11 & 15 & 14 \\ 9 & 8 & 7 & 6 & 10 & 4 & 3 & 2 & 1 & 5 & 19 & 18 & 17 & 16 & 20 & 14 & 13 & 12 & 11 & 15 \\ 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 \\ 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 15 & 11 & 12 & 13 & 14 & 20 & 16 & 17 & 18 & 19 & 5 & 1 & 2 & 3 & 4 & 10 & 6 & 7 & 8 & 9 \\ 14 & 15 & 11 & 12 & 13 & 19 & 20 & 16 & 17 & 18 & 4 & 5 & 1 & 2 & 3 & 9 & 10 & 6 & 7 & 8 \\ 13 & 14 & 15 & 11 & 12 & 18 & 19 & 20 & 16 & 17 & 3 & 4 & 5 & 1 & 2 & 8 & 9 & 10 & 6 & 7 \\ 12 & 13 & 14 & 15 & 11 & 17 & 18 & 19 & 20 & 16 & 2 & 3 & 4 & 5 & 1 & 7 & 8 & 9 & 10 & 6 \\ 16 & 20 & 19 & 18 & 17 & 11 & 15 & 14 & 13 & 12 & 6 & 10 & 9 & 8 & 7 & 1 & 5 & 4 & 3 & 2 \\ 17 & 16 & 20 & 19 & 18 & 12 & 11 & 15 & 14 & 13 & 7 & 6 & 10 & 9 & 8 & 2 & 1 & 5 & 4 & 3 \\ 18 & 17 & 16 & 20 & 19 & 13 & 12 & 11 & 15 & 14 & 8 & 7 & 6 & 10 & 9 & 3 & 2 & 1 & 5 & 4 \\ 19 & 18 & 17 & 16 & 20 & 14 & 13 & 12 & 11 & 15 & 9 & 8 & 7 & 6 & 10 & 4 & 3 & 2 & 1 & 5 \\ 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

Here, \mathcal{G}_4 is defined as

$$\mathcal{G}_4 = \text{Circ}(M_1, M_2),$$

where

$$M_1 = \begin{pmatrix} A_1 & A_2 \\ A_2^T & A_1^T \end{pmatrix}, \quad M_2 = \begin{pmatrix} A_3 & A_4 \\ A_4^T & A_3^T \end{pmatrix}.$$

The matrices A_1, A_2, A_3 , and A_4 are defined as follows:

$$\begin{aligned} A_1 &= \text{Circ}(1, 2, 3, 4, 5), & A_2 &= \text{Circ}(6, 7, 8, 9, 10), \\ A_3 &= \text{Circ}(11, 12, 13, 14, 15), & A_4 &= \text{Circ}(16, 17, 18, 19, 20). \end{aligned}$$

□

Remark 4.12. Using a similar line of argument as in the above cases, we can also find generator matrices for the group ring $\mathbb{F}_q(C_l \times Q_{4n})$, where Q_{4n} is the generalized quaternion group of order $4n$. □

5 QECCs from self orthogonal codes

Let G be a finite group of order n , and let $a = \sum_{i=1}^n \alpha_i g_i \in \mathbb{F}_q G$, where $g_i \in G$ for $i = 1, 2, \dots, n$. We define $a^T = \sum_{i=1}^n \alpha_i g_i^{-1} \in \mathbb{F}_q G$. By [10], $\sigma(a^T) = \sigma(a)^T$, here $\sigma(a)^T$ denotes the transpose matrix of $\sigma(a)$.

5.1 QECCs from Euclidean self orthogonal group ring codes

Proposition 5.1. *Let M be a generator matrix for a linear code C . Then, C is an Euclidean self-orthogonal code if and only if the matrix MM^T is the zero matrix.*

Proof. Suppose C is an Euclidean self-orthogonal code, meaning $C \subseteq C^{\perp_e}$. This implies that

$$\begin{aligned} C \subseteq C^{\perp_e} &\iff \forall x \in C, \quad x \in C^{\perp_e}, \\ &\iff \forall x \in C, \quad Mx^T = 0, \\ &\iff \text{for all rows } r \text{ of } M, \quad Mr^T = 0, \\ &\iff MM^T = O, \end{aligned}$$

where O is the zero matrix. □

Using this property, we derive a necessary and sufficient condition for a G -code to be Euclidean self-orthogonal.

Proposition 5.2. *Let C be a linear G -code of length n over \mathbb{F}_q , with a generator matrix $M = \sigma(a)$ for some $a \in \mathbb{F}_q G$, where G is a finite group. Then C is Euclidean self-orthogonal if and only if $aa^T = 0$.*

Proof. Suppose C is a linear code of length n over \mathbb{F}_q with generator matrix $M = \sigma(a)$. Since $M^T = \sigma(a)^T$, by Proposition 5.1, we have

$$\begin{aligned} MM^T &= \sigma(a)\sigma(a)^T = O, \\ &\iff \sigma(aa^T) = O, \\ &\iff aa^T = 0. \end{aligned}$$

□

Corollary 5.3. [10] *If $a = a^T$ and $a^2 = 0$, then C is Euclidean self orthogonal.*

A QECC with length n , dimension k and minimum distance d over \mathbb{F}_q is denoted by $[[n, k, d]]_q$. Here, we present the result from [2], which enables the construction of quantum codes from any self-orthogonal linear code.

Theorem 5.4. ([2]) *Let C be a $[n, k, d]$ linear code over \mathbb{F}_q . If $C \subseteq C^{\perp_e}$, then there exists a QECC with parameters $[[n, n - 2k, \geq d_H]]_q$, where d_H is the minimum Hamming weight of $C^{\perp_e} \setminus C$.*

Using Theorem 5.4 and Proposition 5.2, we derive the main result for constructing quantum codes from group ring codes.

Theorem 5.5. *Let C be an $[n, k, d]$ linear G -code over \mathbb{F}_q , and let $a \in \mathbb{F}_q G$, where G is a finite group. If $aa^T = 0$, then there exists a quantum error-correcting code (QECC) with parameters $[[n, n - 2k, \geq d_H]]_q$, where d_H is the minimum Hamming weight of $C^{\perp_e} \setminus C$.*

Proof. Let C be a linear G -code with parameters $[n, k, d]$ over \mathbb{F}_q , with a generator matrix $M = \sigma(a)$ for some $a \in \mathbb{F}_q G$, where G is a finite group. If $aa^T = 0$, then by Proposition 5.2, C is Euclidean self-orthogonal. The result then follows directly from Theorem 5.4. □

Example 5.6. We consider the group ring $\mathbb{F}_q(C_l \times C_m)$ with $q = 2$, $l = 5$, and $m = 3$. Let the cyclic group C_5 be generated by x and C_3 be generated by y , so that $C_5 = \{e_l, x, x^2, x^3, x^4\}$ and $C_3 = \{e_m, y, y^2\}$. The elements of the direct product group $C_l \times C_m$ are given by:

$$C_l \times C_m = \{e_{lm}, y, y^2, x, xy, xy^2, x^2, x^2y, x^2y^2, x^3, x^3y, x^3y^2, x^4, x^4y, x^4y^2\}.$$

Consider $a \in \mathbb{F}_2(C_l \times C_m)$ defined as:

$$a = e_{lm} + y^2 + x^2 + x^2y^2 + x^3y + x^3y^2 + x^4y + x^4y^2.$$

Then, its transpose is:

$$a^T = e_{lm} + y + xy + xy^2 + x^2y + x^2y^2 + x^3 + x^3y.$$

Let $\mathcal{G} = \sigma(a)$ and C be the G -code generated by \mathcal{G} . The code C is a linear code over \mathbb{F}_2 with parameters $[[15, 4, 8]]$. By direct calculation we get $aa^T = 0$, it follows from Proposition 5.2 that C is Euclidean self-orthogonal. Therefore, using Theorem 5.5, we construct a QECC with parameters $[[15, 7, 3]]_2$. These parameters correspond to the best-known QECC parameters of length 15, as noted in [12]. \square

5.2 QECCs from Hermitian self orthogonal group ring codes

Suppose $q = p^2$, where p is a prime number. Let $\mathbf{a} = (a_1, a_2, \dots, a_n), \mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbb{F}_q^n$. Then, in \mathbb{F}_q^n , the Hermitian inner product is defined as

$$\langle \mathbf{a}, \mathbf{b} \rangle_h = a_1b_1^p + \dots + a_nb_n^p.$$

The dual code of C with respect to the Hermitian inner product is defined as

$$C^{\perp_h} = \{\mathbf{c} \in \mathbb{F}_{p^2}^n \mid \langle \mathbf{c}, \mathbf{d} \rangle_h = 0, \text{ for all } \mathbf{d} \in C\},$$

and C is a Hermitian self-orthogonal code if $C \subseteq C^{\perp_h}$.

Let $a = \sum_{i=1}^n \alpha_i g_i \in \mathbb{F}_q G$, where $g_i \in G$ for $i = 1, 2, \dots, n$. We define a^p by

$$a^p = \sum_{i=1}^n \alpha_i^p g_i \in \mathbb{F}_q G.$$

If $M = (m_{ij})$, then we define $\overline{M} = (m_{ij}^p)$ for $i, j = 1, 2, \dots, n$.

Proposition 5.7. *Suppose M is a generator matrix of a linear code C of length n over \mathbb{F}_{q^2} . Then, C is a Hermitian self-orthogonal code if and only if the matrix $M\overline{M}^T$ is the zero matrix.*

Proof. Suppose C is a Hermitian self-orthogonal code, meaning $C \subseteq C^{\perp_h}$. This implies that for any $x, y \in C$, the Hermitian inner product $\langle x, y \rangle_H = 0$. We can express each codeword $x \in C$ as $x = uM$ for some $u \in \mathbb{F}_q^k$, where M is the $k \times n$ generator matrix of C .

Then, for any two codewords $x = uM$ and $y = vM$ in C , the Hermitian inner product of x and y is

$$\langle x, y \rangle_H = (uM)(\overline{vM})^T = uM\overline{M}^T \overline{v}^T.$$

We note that, $\langle x, y \rangle_H = 0$ for all u, v , if and only if $M\overline{M}^T = 0$. Thus, C is Hermitian self-orthogonal if and only if $M\overline{M}^T = 0$. \square

Proposition 5.8. *Suppose C is a linear G -code of length n over \mathbb{F}_q , with a generator matrix $M = \sigma(a)$ for some $a \in \mathbb{F}_q G$, where G is a finite group. Then, C is a Hermitian self-orthogonal code if and only if $a(a^p)^T = 0$.*

Proof. Suppose C is a linear code of length n over \mathbb{F}_q with generator matrix $M = \sigma(a)$, then $\overline{M}^T = \sigma(a^p)^T$. By Proposition 5.7, we have

$$\begin{aligned} M\overline{M}^T &= \sigma(a)\sigma(a^p)^T = O, \\ &\iff \sigma(a(a^p)^T) = O, \\ &\iff a(a^p)^T = 0. \end{aligned}$$

\square

We recall the following result from [2], which provides a necessary and sufficient condition for a Hermitian self-orthogonal code to construct a quantum code.

Theorem 5.9. ([2]) *Let C be a $[n, k, d]$ linear code over \mathbb{F}_{p^2} . If $C \subseteq C^{\perp_h}$, then there exists a QECC with parameters $[[n, n - 2k, \geq d_H]]_p$, where d_H is the minimum Hamming weight of $C^{\perp_h} \setminus C$.*

We now present the following result for constructing a quantum code from a Hermitian self-orthogonal G -code over \mathbb{F}_q .

Theorem 5.10. *Let C be a $[n, k, d]$ linear G -code over \mathbb{F}_q , and $a \in \mathbb{F}_q G$, where G is a finite group. If $a(a^p)^T = 0$, then there exists a QECC with parameters $[[n, n - 2k, \geq d_H]]_p$, where d_H is the minimum Hamming weight of $C^{\perp_h} \setminus C$.*

Proof. This proof follows from Proposition 5.8 and Theorem 5.9. □

Example 5.11. We consider the group ring $\mathbb{F}_q D_m$ with $q = 3^2$, and $m = 5$. Then element listing of the dihedral group D_5 is as follows

$$D_5 = \{1, b, b^2, b^3, b^4, a, ab, ab^2, ab^3, ab^4\}.$$

We consider $a \in \mathbb{F}_9 D_5$ such that

$$a = 1 + w^5 b + 2b^2 + w^6 b^3 + w^3 b^4 + w^2 a + w^6 ab + w^7 ab^2 + w^7 ab^3 + w^7 ab^4,$$

where w is a generator of the multiplicative cyclic group \mathbb{F}_9^* . Then a^3 and the transpose of a^3 are as follows,

$$\begin{aligned} a^3 &= 1 + w^7 b + 2b^2 + w^2 b^3 + w b^4 + w^6 a + w^2 ab + w^5 ab^2 + w^5 ab^3 + w^5 ab^4, \\ (a^3)^T &= 1 + w b + w^2 b^2 + 2b^3 + w^7 b^4 + w^6 a + w^2 ab + w^5 ab^2 + w^5 ab^3 + w^5 ab^4. \end{aligned}$$

Let $\mathcal{G} = \sigma(a)$ and C be a G -code generated by \mathcal{G} . The code C is a linear code over \mathbb{F}_9 with parameters $[10, 4, 6]$. By direct calculation we get $(a^3)^T = 0$, then from Proposition 5.8 that C is Hermitian self-orthogonal code. Therefore, using Theorem 5.10, we construct a QECC with parameters $[[10, 2, 4]]_3$. These parameters correspond to the best-known QECC parameters of length 10 ([12]). □

5.3 QECCs from symplectic self orthogonal group ring codes

In this subsection, we discuss two methods for implementing a necessary and sufficient condition for a G -code to be symplectic self-orthogonal. Before going into these methods, we recall the following result, which provides a necessary condition for constructing a QECC from a $2n$ -length symplectic self-orthogonal code. We note that the length $2n$ is connected to the definition of the symplectic inner product, as outlined in Section 2.

Theorem 5.12. ([1, 2]) *Let C be a $[2n, k, d]$ linear code over \mathbb{F}_q . If $C \subseteq C^{\perp_s}$, then there exists a QECC with parameters $[[n, n - k, \geq d_S]]_q$, where d_S is the minimum symplectic weight of $C^{\perp_s} \setminus C$.*

First, in Theorem 5.14, we present a result demonstrating how a $2n$ -length group ring code can be symplectic self-orthogonal, and then in Theorem 5.17, we show an alternative approach that does not require a $2n$ -length code; instead, two n -length group ring codes can be used and horizontally joined. The key difference is that for the $2n$ -length case, a group ring of order $2n$ is needed, whereas for the two n -length codes, a group ring of order n suffices.

We recall the following result, which provides a necessary and sufficient condition for a $2n$ length linear code to be symplectic self-orthogonal.

Theorem 5.13. [22] *Let C be a linear code of length $2n$ over \mathbb{F}_q with generator matrix M . Then C is a symplectic self-orthogonal code if and only if $M\Omega M^T$ is the zero matrix, we recall that $\Omega = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}$, O is the zero matrix and I is the identity matrices of order n .*

Proof. Let us assume C is a symplectic dual-containing code, which means $C^{\perp_s} \subseteq C$. This gives us:

$$\begin{aligned} C \subseteq C^{\perp_s} &\iff \forall x \in C, \quad x \in C^{\perp_s}, \\ &\iff \forall x \in C, \quad M\Omega x^T = 0, \\ &\iff \text{for all rows } r \text{ of } M, \quad M\Omega r^T = 0, \\ &\iff M\Omega M^T = O. \end{aligned}$$

□

Using the above result, we present a necessary and sufficient condition for the symplectic self-orthogonality of a $2n$ -length G -code.

Theorem 5.14. *Suppose C is a linear G -code of length $2n$ over \mathbb{F}_q with generator matrix $M = \sigma(a)$ for some $a \in \mathbb{F}_q G$, where G is a group of order $2n$. Then C is a symplectic self-orthogonal code if and only if $M\Omega M^T$ is the zero matrix.*

We have a following framework to construct a QECC from a $2n$ -length symplectic self-orthogonality G -code.

Theorem 5.15. *Let C be a $[2n, k, d]$ linear code over \mathbb{F}_q with generator matrix $M = \sigma(a)$, where $a \in \mathbb{F}_q G$ and G is of order $2n$. If $M\Omega M^T$ is the zero matrix, then there exists a QECC with parameters $[[n, n - k, \geq d_S]]_q$, where d_S is the minimum symplectic weight of $C^{\perp_s} \setminus C$.*

Proof. This proof follows from the proof of Theorem 5.14 and Theorem 5.12. □

Example 5.16. We consider the group ring $\mathbb{F}_q D_m$ with $q = 3$, and $m = 11$. The element listing of the dihedral group D_{11} is as follows

$$D_{11} = \{1, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, a, ab, ab^2, ab^3, ab^4, ab^5, ab^6, ab^7, ab^8, ab^9, ab^{10}\}.$$

We consider $a \in \mathbb{F}_3 D_{11}$ such that

$$\begin{aligned} a &= b^2 + 2b^4 + b^5 + 2b^6 + 2b^7 + 2b^8 + 2b^9 + 2b^{10} \\ &\quad + 2a + 2ab + 2ab^2 + 2ab^3 + 2ab^4 + 2ab^5 + 2ab^6 + 2ab^7 + 2ab^8 + 2ab^9 + 2ab^{10}. \end{aligned}$$

The transpose of a is as follows

$$\begin{aligned} a^T &= 2b + 2b^2 + 2b^3 + 2b^4 + 2b^5 + b^6 + 2b^7 + b^9 \\ &\quad + 2a + 2ab + 2ab^2 + 2ab^3 + 2ab^4 + 2ab^5 + 2ab^6 + 2ab^7 + 2ab^8 + 2ab^9 + 2ab^{10}. \end{aligned}$$

Let $\mathcal{G} = \sigma(a)$ and C be a G -code generated by \mathcal{G} . The code C is a linear code over \mathbb{F}_3 with parameters $[22, 11, 6]$. By calculation we get $\mathcal{G}\Omega\mathcal{G}^T$ is the zero matrix, then from Proposition 5.14, C is symplectic self-orthogonal code. Therefore, using Theorem 5.15, we construct a QECC with parameters $[[11, 0, 5]]_3$. These parameters correspond to the best-known QECC parameters of length 11 ([12]). □

Now, we present the above results from the perspective of an alternative form for symplectic self-orthogonal G -codes. To fully use the definition of the symplectic inner product, we can state the necessary and sufficient condition as follows:

Suppose $a, b \in \mathbb{F}_q G$, where G has order n , and consider two generator matrices $\mathcal{G}_1 = \sigma(a)$ and $\mathcal{G}_2 = \sigma(b)$. The horizontal join of these two matrices, denoted as $\mathcal{G} = (\mathcal{G}_1 \mid \mathcal{G}_2)$, generates a $2n$ -length G -code, where \mid denotes the horizontal join. We note that in this context, G is assumed to have order n . This approach provides a more intuitive and structured characterization of symplectic self-orthogonal codes in terms of elements of the group ring.

Theorem 5.17. *Let C be a linear $2n$ -length code over \mathbb{F}_q , with generator matrix $\mathcal{G} = (\mathcal{G}_1 \mid \mathcal{G}_2)$, where $\mathcal{G}_1 = \sigma(a)$ and $\mathcal{G}_2 = \sigma(b)$, with $a, b \in \mathbb{F}_q G$ and G of order n . Then $C \subseteq C^{\perp_s}$ if and only if $ab^T = ba^T$.*

Proof. Suppose $\mathcal{G} = (\mathcal{G}_1 \mid \mathcal{G}_2)$ is the generator matrix of C . Then, we consider

$$\mathcal{G}\Omega\mathcal{G}^T = (\mathcal{G}_1 \quad \mathcal{G}_2) \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix} (\mathcal{G}_1 \quad \mathcal{G}_2)^T = (-\mathcal{G}_2 \quad \mathcal{G}_1) \begin{pmatrix} \mathcal{G}_1^T \\ \mathcal{G}_2^T \end{pmatrix} = -\mathcal{G}_2\mathcal{G}_1^T + \mathcal{G}_1\mathcal{G}_2^T.$$

Therefore,

$$\begin{aligned} \mathcal{G}\Omega\mathcal{G}^T &= O \\ \mathcal{G}_1\mathcal{G}_2^T - \mathcal{G}_2\mathcal{G}_1^T &= O \\ \sigma(a)\sigma(b)^T - \sigma(b)\sigma(a)^T &= O \\ \sigma(a)\sigma(b^T) - \sigma(b)\sigma(a^T) &= O \\ \sigma(ab^T - ba^T) &= O \\ ab^T - ba^T &= 0. \end{aligned}$$

□

Here, we present our main result for constructing a QECC using this alternative form.

Theorem 5.18. *Let C be a $[2n, k, d]$ linear code over \mathbb{F}_q with a generator matrix of the form $\mathcal{G} = (\mathcal{G}_1 \mid \mathcal{G}_2)$, where $\mathcal{G}_1 = \sigma(a)$ and $\mathcal{G}_2 = \sigma(b)$, with $a, b \in \mathbb{F}_q G$ and G of order n . If $ab^T - ba^T = 0$, then there exists a QECC with parameters $[[n, n-k, \geq d_S]]_q$, where d_S is the minimum symplectic weight of $C^{\perp_s} \setminus C$.*

Proof. This proof follows from Theorem 5.12 and Theorem 5.17. □

Example 5.19. We consider the group ring $\mathbb{F}_q D_m$ with $q = 2$, and $m = 5$. The element listing of the dihedral group D_5 is as follows

$$D_5 = \{1, b, b^2, b^3, b^4, a, ab, ab^2, ab^3, ab^4\}.$$

We consider $a, b \in \mathbb{F}_3 D_{11}$ such that

$$a = 1 + b + b^2 + b^3 + ab + ab^2, \quad b = 1 + a + ab + ab^3.$$

Then we get

$$a^T = 1 + b^2 + b^3 + b^4 + ab + ab^2, \quad b^T = 1 + a + ab + ab^3.$$

We note that $ab^T - ba^T = 0$. Thus, the linear code C with parameters $[20, 9, 6]$ over \mathbb{F}_2 , and a generator matrix of the form $\mathcal{G} = (\mathcal{G}_1 \mid \mathcal{G}_2)$, where $\mathcal{G}_1 = \sigma(a)$ and $\mathcal{G}_2 = \sigma(b)$, is symplectic self-orthogonal. By Theorem 5.18, we obtain a QECC with parameters $[[10, 1, 4]]_3$. These are also the parameters of the best-known QECC of length 10, see [12]. □

Remark 5.20. In Proposition 3.10, we present the self-orthogonality conditions for two-dimensional codes. We note that this proposition corresponds to the standard Euclidean inner product. Using Theorem 5.4, we can construct quantum codes from two-dimensional self-orthogonal cyclic codes over \mathbb{F}_q . Furthermore, using Theorem 5.14, we can construct quantum codes from two-dimensional cyclic codes in a symplectic structure. To achieve this, we require a generator matrix of a two-dimensional code, which can be constructed from the basis vectors provided in Theorem 3.2. All computations are done using MAGMA [6]. □

6 Conclusion

Group rings provide a powerful framework for exploring coding possibilities, allowing us to extend our intuition and methods to work with various finite groups. This approach can serve as an alternative to studying the corresponding polynomial rings, which involve additional variables (generators) and complex relations between them. This study presents code formulations over certain finite group rings $\mathbb{F}_q G$, where $G = D_n, Q_{4n}, C_l \times D_m$, and $C_l \times C_m$. We also establish necessary and sufficient conditions for these group ring codes to be self-orthogonal under standard inner products, including Euclidean, Hermitian, and symplectic inner products. Additionally, we propose a method for constructing quantum codes from two-dimensional cyclic codes and group ring codes. These codes come with multiple generator matrices for a fixed-length code. In this paper, we construct codes using only a subset of these generator matrices. Exploring a larger selection of generator matrices could be interesting for identifying codes with improved parameters. We believe that further studies in this area may reveal additional structural properties and lead to codes with improved parameters.

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