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# Nonconcave Robust Utility Maximization under Projective Determinacy

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We study a robust utility maximization problem in a general discrete-time frictionless market. The investor is assumed to have a random, nonconcave and nondecreasing utility function, which may or may not be finite on the whole real-line. She also faces model ambiguity on her beliefs about the market, which is modeled through a set of priors. We prove, using only primal methods, the existence of an optimal investment strategy when the utility function is also upper-semicontinuous. For that, we introduce the new notion of projectively measurable functions. We show basic properties of these functions as stability under sums, differences, products, suprema, infima and compositions but also assuming the set-theoretical axiom of Projective Determinacy (PD) stability under integration and existence of  $\epsilon$ -optimal selectors. We consider projectively measurable random utility function and price process and assume that the graphs of the sets of local priors are projective sets. Our other assumptions are stated on a prior-by-prior basis and correspond to generally accepted assumptions in the literature on markets without ambiguity.

*Key words:* optimal investment; nondominated Knightian uncertainty; (PD) axiom ; Projective sets  
*MSC2020 subject classification:* Primary: 91B16, 91G80, 54H05, 28B20, 60A10, 93E20

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**1. Introduction** We are interested in the existence of an optimal investment strategy in a discrete-time frictionless market where there is uncertainty on the true probability of the events. Take the example of an urn with red and black balls from which a ball is drawn. There is only risk and no uncertainty if the composition is exactly known, e.g. 50 red and 50 black balls. There is uncertainty if the composition is known only up to some bounds, e.g. at most 90 reds, at least 40, the rest being black. In the first case, the probability of drawing a red ball is 0.5 and this is called the “known unknowns”. In the second case, it is between 0.4 and 0.9 and this is called the “unknown unknowns”. This notion of uncertainty, coming initially from Knight (see [28]), is named Knightian uncertainty. Elsberg (see [16]) measures that people strictly prefer the first urn to the second one, showing uncertainty aversion. In line with this observation, Gilboa and Schmeidler [21] propose axioms so that the individuals’ preferences have the following numerical representation:  $X \mapsto \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q U(X)$  where  $U$  is a utility function and  $\mathcal{Q}$  is the set of probability measures (called priors) that model all the individual’s beliefs about the future. Thus, the existence of an optimal investment strategy is equivalent to the existence of a solution to a maxmin expected utility problem. When  $\mathcal{Q}$  is reduced to a singleton, there is no uncertainty and our problem amounts to solving a classical utility maximization problem. We refer to [17] and the references therein for a detailed overview of these results. The first attempts to solve the maxmin utility maximization problem, when  $\mathcal{Q}$  is not a singleton, were made under the assumption that  $\mathcal{Q}$  is dominated by a given probability measure. We refer to [18] for a comprehensive survey of the dominated case. However, this setting excludes models with uncertainty about volatility and Bouchard and Nutz propose the notion of quasi-sure uncertainty in [10]. First, random sets of “local” priors, representing the

investor's beliefs between times  $t$  and  $t+1$  are defined. The Fubini products of these "local" priors then form the set of inter-temporal priors  $\mathcal{Q}$ . The framework of [10] assumes that the graphs of these random sets are analytic sets, see [5] for a comprehensive presentation of these sets. Apart from this,  $\mathcal{Q}$  is neither assumed to be compact nor dominated by any particular measure. Since beliefs are uncertain, utility functions are usually assumed to be random. In the discrete time quasi-sure setting, [37] solves the maxmin utility maximization problem for concave and bounded from above utility functions, when only positive wealth is admissible i.e. the utility functions are defined on the positive axis. The same result is later proved in [8] for concave and unbounded utility functions. We now focus on the case of potentially negative wealth. The existence of a solution to the maxmin utility maximization problem is proved in [6] for a unbounded and concave utility functions but in a one-period market. This result is then extended in [12] to a general multiperiod market. For nonconcave and bounded from above utility functions, [36] obtains existence under the rather strong assumption that the investor's asset positions belong to a discrete set. All these results are proved using primal methods. Using a dual approach, [2] solves the maxmin problem for an exponential utility function assuming a strong local no-arbitrage condition. Some of these results are extended in [3] to concave and bounded utility functions under the assumption that medial limits exist. The existence of medial limits is a set-theoretic assumption, since it cannot be proved in the usual Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC). Finally, [38] derived existence results for concave and unbounded utility functions in a completely different framework, where uncertainty is represented by a set of stochastic processes.

To the best of our knowledge, there are no general results on the existence of an optimal investment strategy in the discrete-time nondominated quasi-sure setting of [10], when the utility function is nonconcave and the wealth of the investor can be negative. Our main contribution to financial mathematics is to remove the concavity and the continuity assumptions on  $U(\omega, \cdot)$ : we only assume that it is nonincreasing. Nonconcave utility functions are more realistic to represent individual's beliefs. Indeed, in [25], it has been shown experimentally that there should be a wealth threshold under which an individual is risk seeking (her utility function is convex below that threshold) and above which she is risk averse (her utility function is concave above that threshold). This is the so-called S-shape utility functions. It is also natural that an individual may have jumps on her preferences when certain levels of wealth are reached.

From a mathematical point of view, one may wonder which kind of measurability is required. In the case of bounded utility functions, [37] and [36] assume that  $U$  is lower-semianalytic (lsa). In the unbounded case, [8] and [12] suppose that  $U(\cdot, x)$  is Borel measurable, which implies that  $U$  is jointly Borel measurable (and thus lsa) as  $U(\omega, \cdot)$  is also assumed to be upper-semicontinuous and nondecreasing. In this literature, the price process is assumed to be also Borel measurable and the graph of the set of "local" priors to be analytic. Here we propose a new setup based on projective sets. The projective sets are known generalization of analytic sets, which construction is made recursively. The class of complements of analytic sets, called coanalytic sets, does not coincide with the class of analytic sets and is not stable by projection. An analytic set of order 2 is then the projection of some coanalytic set. Again the class of coanalytic sets of order 2 does not coincide with the class of analytic sets of order 2 and is not stable by projection and we call an analytic set of order 3 the projection of some coanalytic set of order 2. We define recursively the classes of analytic and coanalytic sets of order  $n$  and called  $\Delta_n^1$  their intersection. A set is called projective if it belongs to  $\Delta_n^1$  for some  $n$ . Borel sets and analytic sets are projective sets and also universally measurable sets. To get that all projective sets are universally measurable sets, we can not stay in ZFC: we need a set-theoretic axiom called Projective Determinacy (PD). This axiom postulates that certain set-theoretic two-player games, with perfect information and played on a projective set, are determined in the sense that one of the two players always has a winning strategy. The seemingly only other use of the (PD) axiom in mathematical finance is in [11] which, in a different

uncertainty setting called model-independent, provides a pointwise fundamental theorem of asset pricing as well as superhedging duality results assuming that the set of scenarios is analytic. A non-trivial result from set-theory shows that the (PD) axiom is implied, for example, by the existence of infinitely many Woodin cardinals, see [33]. Essentially, under the (PD) axiom, all properties that are true for analytic sets, are also true for the projective sets and in particular projective sets are universally measurable. Moreover, projectively measurable selection can be made on a projective set, see [27]. To generalize lower and upper-semianalytic functions (lsa or usa), we introduce the new notion of projectively measurable function. It is a function  $f$  for which there exists some  $n$  such that  $f$  is a  $\Delta_n^1$ -measurable. Our main mathematical results are about the properties of the projectively measurable functions. All the properties are new and some of the proofs are completely innovative (for example the ones of Lemmata 9 and 11 and Propositions 11 and 12). We show that sums, differences, products, suprema, infima and compositions of projectively measurable functions remain projectively measurable. We also get that Borel measurable as well as lsa and usa functions are projectively measurable. Under the (PD) axiom, we show that the integrals of projectively measurable functions remain projectively measurable and that a projectively measurable  $\epsilon$ -optimal selector exists for projectively measurable functions. All this preparation allows us to work in an extended setting of [10] for uncertainty, where we require that the graph of “local” priors to be projective sets rather than analytic sets. We also assume that the price process and the utility function are projectively measurable instead of Borel measurable.

Then, we show that under the (PD) axiom and well-accepted conditions on the market and on the nonconcave utility function, an optimal investment strategy exists. Our two main financial results are Theorems 1 and 2. Theorem 1 gives the existence of an investment strategy such that, if admissible, a bound on the optimality error can be derived that depends mainly on the jumps of the utility function. In particular, if the utility function is upper-semicontinuous (usc), then this investment strategy is an optimal investment strategy. In Theorem 2, we show that for a given type of nonconcave and usc random utility called of type (A) (introduced in [12]) that includes  $S$ -shaped functions, the optimal solution of Theorem 1 is automatically admissible and, thus optimal. This is done under additional integrability assumptions on the market, that are weaker than the ones of [12]. We now comment on the assumptions of Theorem 1, starting with three of them that can be tested directly when the utility and the market are specified. The first one is that  $U(\omega, \cdot)$  is nondecreasing. Note again that we do not assume that  $U(\omega, \cdot)$  is concave or continuous. The second one is the classical Asymptotic Elasticity constraints introduced in [29] and [41]. The third one requires that  $U$  is “negative enough” and is automatically satisfied for deterministic utility functions that are unbounded from below. If one of this two last assumptions is not verified, then an optimal solution may not exist. We also assume that some set  $\mathcal{H}^T$  of priors  $P$ , for which the  $P$  no-arbitrage condition holds true in a quasi-sure sense, is nonempty. The existence of such priors has been proved in [8] under the quasi-sure no-arbitrage condition of [10] and in the context of uncertainty of [10]. We leave the proof of the equivalence in our projective setup between  $\mathcal{H}^T \neq \emptyset$  and the quasi-sure no-arbitrage condition for further study. Our last two conditions are not directly verifiable. The first one asserts that the  $P$  prior value function  $U_0^P$  at time 0 is finite for each prior  $P \in \mathcal{H}$ . This is the well-accepted assumption when there is no uncertainty of the seminal work [39] (extended by [13] to non concave functions). This assumption provides a control from above for the value functions. The last assumption asserts that the expectations of the  $P$ -prior value functions  $U_t^P$  are well-defined for all time  $t$  and for each prior  $P \in \mathcal{H}$ . These assumptions are made on a prior-by-prior basis and not for the supremum over all priors. Thus, they can be verified as in a setting without ambiguity.

The proof of Theorem 1 is done by dynamic programming. First, we solve a one-period maximization problem. In this part, our expression of the value function does not involve an “obvious” maxmin, since it is now defined using an additional upper closure. The reason for this choice is

that our utility function is not assumed to be regular in any way and is in particular nonconcave. Note that the “obvious” maxmin one-period problem of [12] may not have a solution. Most of the assumptions that we postulate in the one-period case are taken from [13] and stated for each prior in  $\mathcal{H}$ . The method for finding a one-period optimal strategy is the same as in [12] and uses arguments of regularity and coercivity obtained through the definition of the value function and the assumptions of quasi-sure no arbitrage and asymptotic elasticity. We then return to the general multiperiod problem and specify the (multiperiod) value functions. Our dynamic programming procedure is much simpler than those in [7] and [12]. In fact, our value functions do not involve a countable supremum or closure. Such a choice is made possible by the (PD) axiom which ensures that these value functions exist and are projectively measurable. The (PD) axiom also provides a measurable selection theorem that we apply to multiperiod versions of our one-period problem in order to find one-step optimal strategies. The resulting “glued” strategy is the desired investment strategy of Theorem 1 and its admissibility as well as the properties of the optimal one-step strategies provide the desired bound.

The rest of this article is organized as follows. In Section 2, we introduce the axiom of Projective Determinacy (PD). We then describe the financial model, the assumptions as well as the main results. In Section 3, we solve a utility maximization problem in a one-period market, while in Section 4, we prepare the dynamic programming procedure. Section 5 provides the proof of Theorem 1, while Section 6 the one of Theorem 2. Section 7 presents projective sets and projectively measurable functions, and the implications of the (PD) axiom for them. The appendix contains further proofs and results, in particular on the measurability of the graph of the affine hull of the conditional support of the price process.

## 2. Setting and main result

**2.1. Projective Determinacy** We introduce the axiom of Projective Determinacy (PD) which will be assumed for our main financial results Theorems 1 and 2. For that, we need to define a game and a projective set. We start with the notion of game. Let  $A \subset \mathbb{N}^{\mathbb{N}}$  be a non-empty set. Imagine a two-player infinite game played as follows :

$$\begin{array}{llll} \text{I} & a_0 & a_1 & \cdots \\ \text{II} & b_0 & b_1 & \cdots \end{array} \tag{1}$$

Player I plays  $a_0 \in \mathbb{N}$ , then Player II plays  $b_0 \in \mathbb{N}$ , then Player I plays  $a_1 \in \mathbb{N}$ , etc. A play is thus a sequence  $(a_0, b_0, a_1, b_1, \dots) \in \mathbb{N}^{\mathbb{N}}$ . We say that Player I wins the game if  $(a_0, b_0, a_1, b_1, \dots) \in A$ . Else, if  $(a_0, b_0, a_1, b_1, \dots) \in \mathbb{N}^{\mathbb{N}} \setminus A$ , Player II wins. A strategy for Player I is a function  $\sigma$  taking values in  $\mathbb{N}$  and defined on the set of finite sequences of integer having an even length. For the sequence  $(a_0, b_0, a_1, b_1, \dots)$ , the strategy  $\sigma$  of Player I is defined as follows:  $a_0 = \sigma(\emptyset)$ ,  $a_1 = \sigma((a_0, b_0))$ ,  $a_2 = \sigma((a_0, b_0, a_1, b_1))$ , and so on. Note that if Player I follows the strategy  $\sigma$ , then the complete play is described by  $\sigma$  and  $b = (b_0, b_1, \dots)$ , the actions of Player II, and denoted by  $\sigma * b$ . A winning strategy  $\sigma$  for Player I is a strategy under which Player I always wins, i.e.  $\sigma * b \in A$  for all  $b \in \mathbb{N}^{\mathbb{N}}$ . Similarly, a strategy for Player II is a function  $\tau$  taking values in  $\mathbb{N}$  and defined on the set of finite sequences of odd length. We denote by  $a * \tau$ , the play in which Player I plays  $a$  and Player II plays according to strategy  $\tau$ . A winning strategy for Player II is a strategy  $\tau$  under which Player II always wins, i.e.  $a * \tau \in \mathbb{N}^{\mathbb{N}} \setminus A$  for all  $a \in \mathbb{N}^{\mathbb{N}}$ . Now, we say that  $A$  is *determined* if there exists a winning strategy for one of the two players.

One may wonder what kind of sets are determined and if there exists a set that is not determined. It turns out that any answer to these questions depends on the set paradigm we use. In the usual Zermelo-Fraenkel theory with the Axiom of Choice (ZFC), any closed set  $A$  is determined (see [20,

Corollary 7]) and more generally any Borel set is determined (see [31, Theorem, p371]). However, a more general result can not be obtained in the ZFC theory, see [27, 36.E, p307]. For the second question, in ZFC there exists a set that is not determined, see [24, Lemma 33.1, p628].

REMARK 1. Imagine a finite version of the two-player game defined above with  $2N + 2$  steps (instead of infinity). Let  $A \subset \mathbb{N}^{2N+2}$ . Then, “Player I has a winning strategy” if and only if

$$\exists a_0 \in \mathbb{N}, \forall b_0 \in \mathbb{N}, \exists a_1 \in \mathbb{N}, \forall b_1 \in \mathbb{N}, \dots, \exists a_N \in \mathbb{N}, \forall b_N \in \mathbb{N}, (a_0, b_0, a_1, b_1, \dots, a_N, b_N) \in A. \quad (2)$$

Taking the contraposition of Assertion (2), Morgan’s law shows that “Player I has no winning strategy” if and only if

$$\forall a_0 \in \mathbb{N}, \exists b_0 \in \mathbb{N}, \forall a_1 \in \mathbb{N}, \exists b_1 \in \mathbb{N}, \dots, \forall a_N \in \mathbb{N}, \exists b_N \in \mathbb{N}, (a_0, b_0, a_1, b_1, \dots, a_N, b_N) \in \mathbb{N}^{2N+2} \setminus A. \quad (3)$$

But, Assertion (3) is exactly “Player II has a winning strategy”. Thus, the set  $A$  is always determined: if Player I has a winning strategy,  $A$  is determined, and if Player I has no winning strategy, Player II has a winning strategy and  $A$  is again determined. Thus, all finite sets  $A$  are determined in a finite sense. In the infinite case,  $A$  is determined amongs to say that Morgan’s law applies (formally) to infinite sequences of  $\forall$  and  $\exists$ .

Now, we turn to projective sets. Projective sets are generalization of analytic sets. Let  $X$  be a Polish space. An analytic set of  $X$  is the projection into  $X$  of the Borel subsets of  $X \times \mathbb{N}^{\mathbb{N}}$ , see [5, Proposition 7.41, p166]. We denote by  $\Sigma_1^1(X)$ , the class of analytic sets of  $X$ . Then, the projection into  $X$  of any set of  $\Sigma_1^1(X \times \mathbb{N}^{\mathbb{N}})$  is in  $\Sigma_1^1(X)$ , see [5, Proposition 7.39, p165]. However,  $\Sigma_1^1(X)$  is not stable by complement, see [5, Proposition B.6, p292]. The complement of an analytic set is called a coanalytic set. We denote by  $\Pi_1^1(X)$ , the class of coanalytic sets of  $X$  and by  $\Delta_1^1(X)$ , the intersection of  $\Sigma_1^1(X)$  and  $\Pi_1^1(X)$ . Now, the projection into  $X$  of any set of  $\Pi_1^1(X \times \mathbb{N}^{\mathbb{N}})$  has no reason to be coanalytic or analytic and we denote by  $\Sigma_2^1(X)$  the class of these sets. The elements of  $\Sigma_2^1(X)$  can be seen as “analytic sets of level 2”. Repeating inductively this scheme and taking the (increasing) union of the  $\Delta_n^1(X)$  gives the class  $\mathbf{P}(X)$  of projective sets, see also Definition 8. We provide more details on projective sets and their properties in Section 7. In particular, we show in Proposition 8 that the projective sets behave “almost” exactly as analytic sets. The issue is with the “almost”. There is two (bothersome) differences. First, while an analytic set is universally measurable (see [5, p171]), one cannot prove and seemingly cannot refute in the ZFC theory, that this is still true for projective sets (see Remark 9). Moreover, while measurable selection can be performed on analytic sets using the Jankov-von Neumann theorem (see [5, Proposition 7.49 p182]), there seems to be no such general theorem for projective sets within the ZFC theory. These two main differences vanishes if we postulate the (PD) axiom that we are now in position to state.

AXIOM 1. Let  $A \subset \mathbb{N}^{\mathbb{N}}$ . Assume that  $A \neq \emptyset$  and that  $A$  is a projective set. Then,  $A$  is determined.

We will not use the (PD) axiom directly, but either the two consequences mention above: under the (PD) axiom, a projective set is always universally measurable (see Theorem 3) and measurable selection can be performed on projective sets (see Theorem 4). We develop further the importance of these results and their link with axiom (PD) in Remark 9.

One may wonder if it is really legitimate to add the (PD) axiom to the other axioms of the usual ZFC theory. As a first observation, the (PD) axiom cannot be proved in the ZFC theory. Indeed, otherwise a projective set will be universally measurable, which is not true in the ZFC theory (see Remark 9). The negation of the (PD) axiom in the ZFC theory is still an open problem. The (PD) axiom is a fruitful axiom and extends properties of analytic sets in the ZFC theory



to projective sets. Indeed as mentioned above, all projective sets are universally measurable and admit a uniformization. Moreover, every uncountable projective set contains a Cantor set (and so satisfy the Continuum hypothesis) and every projective set has the Baire property (see [27, Theorem 38.17, p326]). One may wonder if the reciprocal holds true, i.e. if we assume that every projective set is Lebesgue measurable, has the Baire property and that projective uniformization holds true, does the (PD) axiom holds true? This question, asked by Woodin in 1981 and known as the 12th Defino problem, was solved in 1997 and the answer is no. Nevertheless, what make the (PD) axiom “plausible” is that it is implied by numerous set-theoretical statements, as for example the Proper Forcing Axiom, and many of them come from areas of set theory with apparently no connection with projective sets. Examples of such statements can be found in [42]. A remarkable discovery was the close link between the (PD) axiom and the notion of large cardinal. A large cardinal is a cardinal number (see [24]) that is so large that its existence cannot be proved in the ZFC theory, see [26] and [15] for an overview on the large cardinal theory. An example of large cardinals are the inaccessible cardinals and the Woodin cardinals, see [24, p58] and [26, p16]. Now, it has been shown in [33] that the (PD) axiom is provable in the ZFC theory assuming the existence of infinitely many Woodin cardinals which are themselves inaccessible large cardinals. The following approximated reciprocal has been proved in [42, p597] : postulating the (PD) axiom is equivalent to postulating the existence of Woodin cardinals. An enlightening thought experiment, suggesting the existence of a inaccessible cardinal, can be found in [22, p279].

REMARK 2. The (PD) axiom will be assumed for our main financial results except in Section 3. Nevertheless, we will always write “under the (PD) axiom” to stress where this axiom is indeed used. As already mentioned, under the (PD) axiom, any projective set  $A$  is universally measurable. This will be used to define  $p(A)$  for any probability measure  $p$  and more generally to use classical measure theory results in the projective context. First, any projectively measurable function  $f$  is universally measurable (see Lemma 5 (iii)) so that  $\int f dp$  is well-defined (in the sense of (107)). Moreover, Fubini’s theorem (see [5, Proposition 7.45 p175]) holds true for projectively measurable functions and projectively measurable stochastic kernels. From now, we will use Fubini’s theorem without further reference. An important consequence is that the sets  $(Q^t)_{0 \leq t \leq T}$  (see (4)) are indeed well-defined. In Section 7 where projective sets are study the (PD) axiom is of course not postulated everywhere.

**2.2. Financial Setting** We fix a time horizon  $T$  and introduce a family of Polish spaces  $(\Omega_t)_{1 \leq t \leq T}$ . For some  $0 \leq t \leq T$ , let  $\Omega^t := \Omega_1 \times \dots \times \Omega_t$  with the convention that  $\Omega^0$  is a singleton. For all Polish space  $X$ , we denote by  $\mathfrak{P}(X)$  the set of probability measures defined on the measurable space  $(X, \mathcal{B}(X))$ , where  $\mathcal{B}(X)$  is the Borel sigma-algebra on  $X$ . We denote by  $\mathcal{B}_c(X)$  the completion of  $(X, \mathcal{B}(X))$  with respect to all  $P \in \mathfrak{P}(X)$ .

Let  $S := (S_t)_{0 \leq t \leq T}$  be a  $\mathbb{R}^d$ -valued process representing the discounted price of  $d$  risky assets over time. For all  $0 \leq t \leq T$ , we assume that  $S_t$  is projectively measurable. This requires that there exists some  $n \geq 1$  such that  $S_t^{-1}(B) \in \Delta_n^1(\Omega^t)$  for all  $B \in \mathcal{B}(\mathbb{R}^d)$ , see Definition 9. Note that in the setting of Bouchard and Nutz,  $S_t$  is usually assumed to be Borel measurable. As Borel measurable functions are projectively measurable without the (PD) axiom (see Lemma 5), our assumption is thus weaker.

ASSUMPTION 1. *For all  $0 \leq t \leq T$ ,  $S_t$  is projectively measurable.*

We consider a random utility function defined on the whole real line which models the investor’s preference on the market in the case of possible negative wealth.

DEFINITION 1. A random utility  $U : \Omega^T \times \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  is a function such that  $U$  is projectively measurable and  $U(\omega^T, \cdot)$  is nondecreasing for all  $\omega^T \in \Omega^T$ .

The utility function is chosen random because the law of nature is uncertain. Note that  $U(\cdot, x)$  is usually assumed to be Borel measurable and  $U(\omega^T, \cdot)$  to be nondecreasing and upper-semicontinuous (usc), see [7], [8], [9], [12] and [37]. This implies that  $U$  is  $\mathcal{B}(\Omega^T) \otimes \mathcal{B}(\mathbb{R})$ -measurable (see [9, Lemmata 5.10 and 5.13]). Thus,  $U$  is projectively measurable (see Lemma 5 without the (PD) axiom). So, our assumption is again weaker. Moreover, we do not assume that  $U(\omega^T, \cdot)$  is continuous or upper-semicontinuous. However, as  $U(\omega^T, \cdot)$  is nondecreasing,  $U(\omega^T, \cdot)$  must be continuous except on a countable set of points. And of course, we do not assume that  $U(\omega^T, \cdot)$  is concave.

We now construct the set  $\mathcal{Q}^T$  of all priors prevailing on the market. The set  $\mathcal{Q}^T$  captures all the investor's beliefs about the law of nature. For all  $0 \leq t \leq T-1$ , let<sup>1</sup>  $\mathcal{Q}_{t+1} : \Omega^t \rightarrow \mathfrak{P}(\Omega_{t+1})$  where  $\mathcal{Q}_{t+1}(\omega^t)$  can be seen as the set of all possible priors for the  $t+1$ -th period given the state  $\omega^t$  at time  $t$ . The following assumption allows us to perform measurable selection, see Proposition 9.

ASSUMPTION 2. *The set  $\mathcal{Q}_1$  is nonempty and convex. For all  $1 \leq t \leq T-1$ ,  $\mathcal{Q}_{t+1}$  is a nonempty and convex-valued random set such that  $\text{Graph}(\mathcal{Q}_{t+1}) := \{(\omega^t, p) \in \Omega^t \times \mathfrak{P}(\Omega_{t+1}), p \in \mathcal{Q}_{t+1}(\omega^t)\}$  is a projective set.*

Projective sets have been introduced in Section 2.1, see also Definition 8. Note that in the Bouchard and Nutz setting (see among others [10], [37], [7], [2], [8], [36] and [12]),  $\text{Graph}(\mathcal{Q}_{t+1})$  is assumed to be a convex analytic set. We will show in (102) in Proposition 8 that an analytic set is a projective set again without the (PD) axiom. Thus in ZFC theory, our setting includes the classical quasi-sure financial one without postulating continuity and concavity assumptions on  $U(\omega^T, \cdot)$ . Explicit examples of nondominated financial markets satisfying Assumption 2 can be adapted from [8] and [2]. Among them is a robust discrete time Black-Scholes model and a robust binomial model where the uncertainty affects the probability of jumps and their size.

For all  $0 \leq t \leq T-1$ , let  $SK_{t+1}$  be the set of projectively measurable stochastic kernels on  $\Omega_{t+1}$  given  $\Omega^t$ . For  $q_{t+1}(\cdot|\cdot) : \mathcal{B}(\Omega_{t+1}) \times \Omega^t \rightarrow \mathbb{R}$ , we say that  $q_{t+1} \in SK_{t+1}$  if for all  $\omega^t \in \Omega^t$ ,  $q_{t+1}(\cdot|\omega^t) \in \mathfrak{P}(\Omega_{t+1})$  and  $\omega^t \mapsto q_{t+1}(\cdot|\omega^t)$  is projectively measurable (see Definition 9). Under the (PD) axiom, for all  $q_{t+1} \in SK_{t+1}$ , we have that  $\omega^t \mapsto q_{t+1}(\cdot|\omega^t)$  is universally measurable (see Lemma 5) and  $q_{t+1}$  is an universally measurable stochastic kernel, see [5, Definition 7.12, p134]. Then, still under the (PD) axiom, Assumption 2 and Proposition 9 show that there exists  $q_{t+1} \in SK_{t+1}$  such that for all  $\omega^t \in \Omega^t$  (recall that  $\mathcal{Q}_{t+1} \neq \emptyset$ ),  $q_{t+1}(\cdot|\omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$ . Now, for all  $1 \leq t \leq T$ , we can use Fubini's theorem, see Remark 2, and define the product measure  $q_1 \otimes \dots \otimes q_t$  which belongs to  $\mathfrak{P}(\Omega^t)$ . Let  $\mathcal{Q}^t \subset \mathfrak{P}(\Omega^t)$  be defined by

$$\mathcal{Q}^t := \{q_1 \otimes q_2 \otimes \dots \otimes q_t, q_1 \in \mathcal{Q}_1, q_{s+1} \in SK_{s+1}, q_{s+1}(\cdot|\omega^s) \in \mathcal{Q}_{s+1}(\omega^s), \forall \omega_s \in \Omega^s, \forall 1 \leq s \leq t-1\} \quad (4)$$

We also set  $\mathcal{Q}^0 := \{\delta_{\omega_0}\}$ , where  $\delta_{\omega_0}$  is the Dirac measure on the single element  $\omega_0$  of  $\Omega^0$ . If  $P := q_1 \otimes q_2 \otimes \dots \otimes q_T \in \mathcal{Q}^T$ , we write for any  $1 \leq t \leq T$ ,  $P^t := q_1 \otimes q_2 \otimes \dots \otimes q_t$  and  $P^t \in \mathcal{Q}^t$ . In this paper, most of the time, we work directly on the disintegration of  $P$  rather than  $P$ . Thus, from now, we will precise the fixed disintegration for which the required result holds true.

<sup>1</sup> The notation  $\rightarrow$  stands for set-valued mapping.



Trading strategies are represented by  $d$ -dimensional processes  $\phi := \{\phi_t, 1 \leq t \leq T\}$  representing the investor's holdings in each of the  $d$  risky assets over time. We assume that  $\phi_t : \Omega^{t-1} \rightarrow \mathbb{R}^d$  is projectively measurable for all  $1 \leq t \leq T$ . The set of all such trading strategies is denoted by  $\Phi$ . Under the (PD) axiom, if  $\phi \in \Phi$ , then  $\phi$  is universally measurable (see Lemma 5) which is the usual assumption in the quasi-sure literature. So, again our assumption is weaker but this times assuming the (PD) axiom. Trading is assumed to be self-financing and the value at time  $t$  of a portfolio  $\phi \in \Phi$  starting from initial capital  $x \in \mathbb{R}$  is thus given by

$$V_t^{x,\phi} = x + \sum_{s=1}^t \phi_s \Delta S_s.$$

Note that if  $x, y \in \mathbb{R}^d$  then the concatenation  $xy$  stands for their scalar product. The symbol  $|\cdot|$  refers to the Euclidean norm on  $\mathbb{R}^d$  (or on  $\mathbb{R}$ ) and  $|\cdot|_1$  is the norm on  $\mathbb{R}^d$  defined by  $|x|_1 := \sum_{i=1}^d |x_i|$  for all  $x \in \mathbb{R}^d$ .

We now introduce a condition which will play the role of the no-arbitrage condition. For that, we need the conditional supports of the price increments. Let  $0 \leq t \leq T-1$  and  $P \in \mathfrak{P}(\Omega^T)$  with the fixed disintegration  $P := q_1^P \otimes \cdots \otimes q_T^P$ , the multiple-priors conditional support  $D^{t+1} : \Omega^t \rightarrow \mathbb{R}^d$  and the conditional support relatively to  $P$ ,  $D_P^{t+1} : \Omega^t \rightarrow \mathbb{R}^d$  are defined by

$$D^{t+1}(\omega^t) := \bigcap \{A \subset \mathbb{R}^d, \text{ closed, } p(\Delta S_{t+1}(\omega^t, \cdot) \in A) = 1, \forall p \in \mathcal{Q}_{t+1}(\omega^t)\} \quad (5)$$

$$D_P^{t+1}(\omega^t) := \bigcap \{A \subset \mathbb{R}^d, \text{ closed, } q_{t+1}^P(\Delta S_{t+1}(\omega^t, \cdot) \in A | \omega^t) = 1\}. \quad (6)$$

Additionally, for some  $R \subset \mathbb{R}^d$ , let

$$\text{Aff}(R) := \bigcap \{A \subset \mathbb{R}^d, \text{ affine, } R \subset A\} \quad \text{Conv}(R) := \bigcap \{C \subset \mathbb{R}^d, \text{ convex, } R \subset C\} \quad (7)$$

and if  $R$  is convex,  $\text{ri}(R)$  is the interior of  $R$  relatively to  $\text{Aff}(R)$ .

Recall that a set  $A \subset \Omega^T$  is a  $\mathcal{Q}^T$ -polar set if there exists  $N \in \mathcal{B}(\Omega^T)$  such that  $A \subset N$  and  $P(N) = 0$  for all  $P \in \mathcal{Q}^T$ . A property holds true  $\mathcal{Q}^T$ -quasi-surely (q.s.) if it holds true outside of a  $\mathcal{Q}^T$ -polar set. The complement of a  $\mathcal{Q}^T$ -polar set is called a  $\mathcal{Q}^T$ -full-measure set. Of course, any  $\mathcal{Q}^T$ -full-measure set is a  $P$ -full-measure set for all  $P \in \mathcal{Q}^T$ . Note that under the (PD) axiom, if  $A \subset \Omega^T$  is a projective set, then  $A \in \mathcal{B}_c(\Omega^T)$ , see Theorem 3. Thus,  $A$  is a  $\mathcal{Q}^T$ -full-measure set if  $P(A) = 1$  for all  $P \in \mathcal{Q}^T$ .

We can now define the set  $\mathcal{H}^T$  :

$$\mathcal{H}^T := \{P \in \mathcal{Q}^T, 0 \in \text{ri}(\text{conv}(D_P^{s+1}))(\cdot) \mathcal{Q}^s\text{-q.s., } \text{Aff}(D_P^{s+1})(\cdot) = \text{Aff}(D^{s+1})(\cdot) \mathcal{Q}^s\text{-q.s., } \forall 0 \leq s \leq T-1\} \quad (8)$$

ASSUMPTION 3. We have that  $\mathcal{H}^T \neq \emptyset$ .

We first comment on Assumption 3 and its link with the  $NA(\mathcal{Q}^T)$ -condition of Bouchard and Nutz [10, Definition 1.1]. We also recall the classical uni-prior no-arbitrage condition.

DEFINITION 2. The  $NA(\mathcal{Q}^T)$  condition holds true if  $V_T^{0,\phi} \geq 0$   $\mathcal{Q}^T$ -q.s. for some  $\phi \in \Phi$  implies that  $V_T^{0,\phi} = 0$   $\mathcal{Q}^T$ -q.s.

Let  $P \in \mathfrak{P}(\Omega^T)$ . The  $NA(P)$  condition holds true if  $V_T^{0,\phi} \geq 0$   $P$ -a.s. for some  $\phi \in \Phi$  implies that  $V_T^{0,\phi} = 0$   $P$ -a.s.

LEMMA 1. Assume the (PD) axiom. Assume that Assumptions 1, 2 and 3 hold true. We have that (i)  $\mathcal{H}^T \subset \mathcal{Q}^T$ , (ii)  $\mathcal{H}^T$  and  $\mathcal{Q}^T$  have the same polar sets, (iii)  $NA(P)$  holds true for all  $P \in \mathcal{H}^T$ , (iv) the  $NA(\mathcal{Q}^T)$ -condition holds true and (v) for all  $P^* := q_1^{P^*} \otimes \cdots \otimes q_T^{P^*} \in \mathcal{H}^T$ ,

$$\mathcal{P}^T := \{(\lambda_1 q_1^{P^*} + (1 - \lambda_1) q_1^Q) \otimes \cdots \otimes (\lambda_T q_T^{P^*} + (1 - \lambda_T) q_T^Q), 0 < \lambda_i \leq 1, Q \in \mathcal{Q}^T\} \subset \mathcal{H}^T. \quad (9)$$

*Proof.* See Appendix 8.1.  $\square$

REMARK 3. It should be possible to show that Assumption 3 is in fact equivalent to the  $NA(\mathcal{Q}^T)$ -condition in our projective setup as this is done in [10] for the Bouchard and Nutz analytic one. This is kept for further research.

As in [12], we use an adhoc integral  $\int_-$ . This integral corresponds to the usual integral when finite. Else it is computed using the convention :

$$-\infty + \infty = +\infty - \infty = -\infty. \quad (10)$$

This is the usual convention for maximization problems. Note however that [5] adopts the opposite convention. Otherwise, we adopt the usual arithmetic rules in calculations involving  $(+\infty)$  and  $(-\infty)$  described in [5, Section 7.44, p26-27] :

$$+(-\infty) = -\infty, \quad -(+\infty) = -\infty \quad 0 \times (\pm\infty) = (\pm\infty) \times 0 = 0. \quad (11)$$

We will show in Section 7.5 some measurability properties for such an integral and in particular that under the (PD) axiom, the integral of a projectively measurable function (which exists, see Remark 2) is projectively measurable, see Proposition 12. From now, we will simply write  $\int$  (or  $\mathbb{E}$ ) instead of  $\int_-$  except when we need to clarify the difference between both integrals.

We can introduce the set of admissible strategies for the utility maximization problem.

DEFINITION 3. Let  $U$  be a random utility function as in Definition 1. Let  $P \in \mathfrak{P}(\Omega^T)$  and  $x \in \mathbb{R}$ .

$$\begin{aligned} \Phi(x, U, P) &:= \{\phi \in \Phi, \mathbb{E}_P U^-(\cdot, V_T^{x, \phi}(\cdot)) < +\infty\} \\ \Phi(x, U, \mathcal{Q}^T) &:= \bigcap_{P \in \mathcal{Q}^T} \Phi(x, U, P). \end{aligned}$$

We are now in position to state our utility maximization problem when the uncertainty about the true probability of future events is modeled by  $\mathcal{Q}^T$  :

$$u(x) := \sup_{\phi \in \Phi(x, U, \mathcal{Q}^T)} \inf_{P \in \mathcal{Q}^T} \mathbb{E}_P U(\cdot, V_T^{x, \phi}(\cdot)). \quad (12)$$

Note that  $u(x) = \sup_{\phi \in \Phi} \inf_{P \in \mathcal{Q}^T} \mathbb{E}_P U(\cdot, V_T^{x, \phi}(\cdot))$ . Indeed, if  $\phi \in \Phi \setminus \Phi(x, U, \mathcal{Q}^T)$ , then  $\inf_{P \in \mathcal{Q}^T} \mathbb{E}_P U(\cdot, V_T^{x, \phi}(\cdot)) = -\infty$  thanks to convention (10).

We now introduce the different assumptions needed for the existence of an optimal strategy in (12).

**2.3. Direct Assumptions on  $U$**  We start with the assumptions that can be directly (and easily) checked. The first one is on the asymptotic behavior of the random utility  $U$ . This kind of condition already appeared in [13, Proposition 4] for a finite nondecreasing continuous and non necessarily concave function in the uni-prior setting. See also [39, Proposition 5.1], [7, Proposition 3.24] and [12, Assumption 3].

ASSUMPTION 4. *There exist  $0 < \underline{\gamma} < \bar{\gamma}$  and a projectively measurable random variable  $C : \Omega^T \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  such that  $\sup_{P \in \mathcal{Q}^T} \mathbb{E}_P C < +\infty$  and such that for all  $\omega^T \in \Omega^T$  satisfying  $C(\omega^T) < +\infty$ , for all  $\lambda \geq 1$  and  $x \in \mathbb{R}$ ,*

$$U(\omega^T, \lambda x) \leq \lambda^{\bar{\gamma}} (U(\omega^T, x) + C(\omega^T)) \quad (13)$$

$$U(\omega^T, \lambda x) \leq \lambda^{\underline{\gamma}} (U(\omega^T, x) + C(\omega^T)). \quad (14)$$

Recall from Remark 2 that  $\mathbb{E}_P C$  exists for all  $P \in \mathcal{Q}^T$ . A consequence of Assumption 4 is that a one-step strategy must be bounded in order to be optimal for problem (30) (see Proposition 2) and this compactness result will allow us to prove existence of an optimal strategy for (12). Note that [12, Assumption 3] is a weaker version of Assumption 4. Indeed, when  $U(\omega^T, \cdot)$  is assumed to be concave, only one inequality in Assumption 4 suffices (see [12, Lemma 7]). Moreover, Assumption 4 is related to the notion of Reasonable Asymptotic Elasticity (RAE) introduced in [29] and [41]. We refer to [12] and [13] for a detailed discussion on the link between RAE and Assumption 4. Assumption 4 asserts that  $\bar{\gamma}$ , the Asymptotic Elasticity (AE) in  $-\infty$  must be greater than  $\underline{\gamma}$ , the AE in  $+\infty$ .

The next assumption ensures that  $U$  takes negative values and is similar to [12, Assumption 4].

ASSUMPTION 5. *There exists a projectively measurable random variable  $\underline{X} : \Omega^T \rightarrow \mathbb{R}$  such that for all  $\omega^T \in \Omega^T$ ,  $\underline{X}(\omega^T) < 0$  and  $U(\cdot, \underline{X}(\cdot)) < -C(\cdot)$   $\mathcal{Q}^T$  q.s, where  $C(\cdot) \geq 0$  has been introduced in Assumption 4.*

Assumption 5 is of course satisfied if  $U$  is deterministic, nondecreasing and such that  $\lim_{x \rightarrow -\infty} U(x) = -\infty$ . Moreover, if Assumption 5 is not satisfied, then (12) may have no solution, see [12, Remark 5] in a one-period concave and usc setting.

**2.4. Value functions and Assumption on  $U_0^P$**  We first introduce the dynamic programming procedure and the associated value functions. We do it for the multiple-priors utility maximization problem (12) with the value functions  $U_t$  and also for the utility problem related to a given prior  $P$  with the value functions  $U_t^P$ . Fix  $P \in \mathcal{Q}^T$  with fixed disintegration  $P := q_1^P \otimes \dots \otimes q_T^P$ . For all  $0 \leq t \leq T-1$ , we define

$$\begin{cases} U_T^P(\omega^T, x) & := U(\omega^T, x) \\ u_t^P(\omega^t, x, h) & := \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)} U_{t+1}^P(\omega^t, \cdot, x + h\Delta S_{t+1}(\omega^t, \cdot)) \\ U_t^P(\omega^t, x) & := \sup_{h \in \mathbb{R}^d} u_t^P(\omega^t, x, h) \end{cases} \quad (15)$$

$$\begin{cases} U_T(\omega^T, x) & := U(\omega^T, x) \\ u_t(\omega^t, x, h) & := \inf_{p \in \mathcal{Q}_{t+1}(\omega^t)} \mathbb{E}_p U_{t+1}(\omega^t, \cdot, x + h\Delta S_{t+1}(\omega^t, \cdot)) \\ U_t(\omega^t, x) & := \sup_{h \in \mathbb{R}^d} u_t(\omega^t, x, h). \end{cases} \quad (16)$$

The existence and the measurability of  $U_t$  and  $U_t^P$  are not trivial especially as we take uncountable suprema. These results will be proved in Section 4 under the (PD) axiom. Here, no regularity is assumed for the random utility function and the (PD) axiom provides a measurability framework where uncountable supremum or infimum of measurable functions remain measurable and measurable selection can be performed (see Section 7). Our definition of the value functions differs

from the usual one as we take suprema over all  $h$  in  $\mathbb{R}^d$  and not in  $\mathbb{Q}^d$  and as we do not take the closure of these suprema, see [7], [12], [36] and [37]. Taking a countable supremum and the closure ensure that  $U_t$  is lower-semianalytic, i.e. that the measurability of  $U_t$  stays at the lowest level of the projective hierarchy. Low enough to be able to use measurable selection methods in the ZFC theory only. One may wonder if we can simplify proofs and obtain a better result in Theorem 1 if we take a countable infimum in (15) and (16) and if we take the closure as it is usually done. The answer is no, see Remarks 4 and 6.

We now introduce an assumption that provides a control from above on each  $U_t^P$  for  $P \in \mathcal{H}^T$ .

ASSUMPTION 6. *For all  $P \in \mathcal{H}^T$ ,  $U_0^P(1) < +\infty$ .*

Assumption 6 is similar to [39, Assumption 2.3 (1)] and thus well-accepted in the uni-prior setting. What is nice here is that the assumption is postulated prior by prior and not uniformly on all of them. This assumption is obviously satisfied if  $U$  is bounded from above. Else, Assumption 6 is not easy to verify. We will propose a still general context where it holds automatically true (see Theorem 2). Assumption 6 was also postulated in [12, Assumption 6] or [13, Assumption 2 (11)]. We refer to [12, Lemma 2] and the discussion after for a detailed comparison of Assumption 6 to the similar assumptions in the literature. Note that  $U_0(1) < +\infty$  is not enough to get existence of an optimal strategy, see the counter-example in [12, Remark 2].

The last assumption states that the  $P$ -prior problems are well-defined in the following sense.

ASSUMPTION 7. *For all  $1 \leq t \leq T$ ,  $P \in \mathcal{H}^T$  and  $\theta \in \{-1, 1\}^d \cup \{0\}$ ,  $\mathbb{E}_P U_t^P(\cdot, 1 + \theta \Delta S_t(\cdot))$  is well defined in the generalized sense i.e. we have that*

$$\mathbb{E}_P(U_t^P)^+(\cdot, 1 + \theta \Delta S_t(\cdot)) < +\infty \quad \text{or} \quad \mathbb{E}_P(U_t^P)^-(\cdot, 1 + \theta \Delta S_t(\cdot)) < +\infty.$$

This assumption is satisfied if either  $U$  is bounded from above or from below. We compare Assumption 7 to [12, Assumption 5], i.e. that  $\sup_{P \in \mathcal{Q}^T} \mathbb{E}_P U^-(\cdot, x + h \Delta S_t(\cdot)) < +\infty$  for all  $1 \leq t \leq T$ ,  $(x, h) \in \mathbb{Q} \times \mathbb{Q}^d$ . One trivial observation is that contrary to [12, Assumption 5], Assumption 7 allows that  $U = -\infty$  on a non-polar set. Moreover, using [12, (52) and Proposition 6 (iv)], we can show that Assumption 7 holds true under Assumptions 1, 2, 4 and [12, Assumption 5] when  $U$  is as in [12, Definition 1].

**2.5. Main result** As  $U(\omega^T, \cdot)$  is not continuous, our main result is not the existence of an optimal solution for (12). Such a result will be obtained if  $U(\omega^T, \cdot)$  is usc or if the candidate for the optimal wealth does not belong to the set of discontinuity of  $U(\omega^T, \cdot)$ , see (22). What we obtain in general is a strategy  $\phi^{*,x}$  for which a bound on the optimality error, i.e. on  $u(x) - \inf_{P \in \mathcal{Q}^T} \mathbb{E}_P U(\cdot, V_T^{x, \phi^{*,x}}(\cdot))$  can be derived. This bound will depend on the jump size of  $U(\omega^T, \cdot)$ , see (20). To do that, we first need to introduce the closure operator for a function  $F : \Omega^t \times \mathbb{R}^p \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  with  $p \geq 1$ . Fix  $\omega^t \in \Omega^t$ . Then,  $\mathbb{R}^p \ni x \mapsto F_{\omega^t}(x) := F(\omega^t, x)$  is an extended real-valued function and its closure, denoted by  $\text{Cl}(F_{\omega^t})$ , is the smallest usc function  $w : \mathbb{R}^p \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  such that  $F_{\omega^t} \leq w$ . Now  $\text{Cl}(F) : \Omega^t \times \mathbb{R}^p \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  is defined by  $\text{Cl}(F)(\omega^t, x) := \text{Cl}(F_{\omega^t})(x)$ . We will use the closure operator for functions defined on  $\Omega^t \times \mathbb{R}$  or on  $\Omega^t \times \mathbb{R} \times \mathbb{R}^d$ . When  $F$  is defined on  $\Omega^t \times \mathbb{R}$  and  $F(\omega^t, \cdot)$  is nondecreasing, using [40, 1(7)] applied to  $-F(\omega^t, \cdot)$ , we get that

$$\text{Cl}(F)(\omega^t, x) = \lim_{\epsilon \rightarrow 0^+} F(\omega^t, x + \epsilon) = F(\omega^t, x) + \Delta_+ F(\omega^t, x), \quad (17)$$

where  $\Delta_+ F(\omega^t, x) := \lim_{\epsilon \rightarrow 0^+} F(\omega^t, x + \epsilon) - F(\omega^t, x)$  is the jump of  $F(\omega^t, \cdot)$  at  $x$ .

We will solve the following value functions associated to  $\text{Cl}(u_t)$  for all  $0 \leq t \leq T$ .

$$u_t^{\text{cl}}(\omega^t, x) := \sup_{h \in \mathbb{R}^d} \text{Cl}(u_t)(\omega^t, x, h). \quad (18)$$

As  $u_t$  has no reason to be usc, we do not have that  $U_t = u_t^{\text{cl}}$  and we do not solve (16) directly. This is discussed again in Section 3 (see Problem (30)).

We are now in position to state our first main theorem.

**THEOREM 1.** *Assume the (PD) axiom. Let  $U$  be a random utility function (see Definition 1). Assume that Assumptions 1, 2, 3, 4, 5, 6 and 7 hold true. Then, there exist some  $(\phi^{*,x})_{x \in \mathbb{R}} \subset \Phi$  and for all  $0 \leq t \leq T-1$ , a  $\mathcal{Q}^t$ -full measure set  $\hat{\Omega}^t \in \mathcal{B}(\Omega^t)$  such that for all  $x \in \mathbb{R}$  and  $\omega^t \in \hat{\Omega}^t$ , we have that  $\phi_{t+1}^{*,x}(\omega^t) \in \text{Aff}(D_{t+1})(\omega^t)$  and that*

$$\begin{aligned} U_t(\omega^t, V_t^{x, \phi^{*,x}}(\omega^t)) &\leq \text{Cl}(U_t)(\omega^t, V_t^{x, \phi^{*,x}}(\omega^t)) \leq u_t^{\text{cl}}(\omega^t, V_t^{x, \phi^{*,x}}(\omega^t)) = \sup_{h \in \mathbb{R}^d} \text{Cl}(u_t)(\omega^t, V_t^{x, \phi^{*,x}}(\omega^t), h) \\ &= \text{Cl}(u_t)(\omega^t, V_t^{x, \phi^{*,x}}(\omega^t), \phi_{t+1}^{*,x}(\omega^t)). \end{aligned} \quad (19)$$

We have the following bounds for the optimality error

$$0 \leq u(x) - \inf_{P \in \mathcal{Q}^T} \mathbb{E}_P U(\cdot, V_T^{x, \phi^{*,x}}(\cdot)) \leq \sup_{P \in \mathcal{Q}^T} \mathbb{E}_P \Delta_+ U(\cdot, V_T^{x, \phi^{*,x}}(\cdot)), \quad (20)$$

If  $\phi^{*,x} \in \Phi(x, U, \mathcal{Q}^T)$ , then

$$u(x) \leq U_0(x) \leq \text{Cl}(U_0)(x) \leq u_0^{\text{cl}}(x) = \inf_{P \in \mathcal{Q}^T} \mathbb{E}_P \text{Cl}(U)(\cdot, V_T^{x, \phi^{*,x}}(\cdot)). \quad (21)$$

Moreover, if  $U(\omega^T, \cdot)$  is usc for all  $\omega^T$  in a  $\mathcal{Q}^T$ -full measure set or  $V_T^{x, \phi^{*,x}}(\cdot) \notin \mathcal{D}(\cdot)$   $\mathcal{Q}^T$ -q.s., where  $\mathcal{D}(\omega^T)$  is the (countable) set of discontinuity of  $U(\omega^T, \cdot)$ , then  $U_0$  is usc in  $x$  and

$$u(x) = U_0(x) = \inf_{P \in \mathcal{Q}^T} \mathbb{E}_P U(\cdot, V_T^{x, \phi^{*,x}}(\cdot)). \quad (22)$$

*Proof.* The last part of Theorem 1 is an easy consequence of the first one. If  $U(\omega^T, \cdot)$  is usc for all  $\omega^T$  in a  $\mathcal{Q}^T$ -full measure set, we have that

$$U(\cdot, V_T^{x, \phi^{*,x}}(\cdot)) = \text{Cl}(U)(\cdot, V_T^{x, \phi^{*,x}}(\cdot)) \quad \mathcal{Q}^T\text{-q.s.} \quad (23)$$

So, (21) shows the desired result. Now, if  $V_T^{x, \phi^{*,x}}(\cdot) \notin \mathcal{D}(\cdot)$   $\mathcal{Q}^T$ -q.s., (23) still holds true using (17). The rest of the proof of Theorem 1 is quite involved and is delayed to Section 5. The general idea is still the one of dynamic programming, i.e. to glue together all the one-step optimal strategies constructed in Section 3 for Problem (18).  $\square$

**REMARK 4.** One may wonder if we can prove that  $\phi^{*,x}$  is an optimal strategy for Problem (12). The answer is no without further assumptions. To prove that  $\phi^{*,x}$  is optimal for (12), we should first prove that  $\phi_{t+1}^{*,x}$  is an optimal one-step strategy between  $t$  and  $t+1$  starting from an initial wealth equal to  $x + \sum_{s=1}^t \phi_s^{*,x} \Delta S_s$ , i.e. if one have already followed the strategies  $(\phi_1^{*,x}, \dots, \phi_t^{*,x})$  until time  $t$ . This means that

$$\begin{aligned} U_t(\omega^t, V_t^{x, \phi^{*,x}}(\omega^t)) &= \sup_{h \in \mathbb{R}^d} \inf_{p \in \mathcal{Q}_{t+1}(\omega^t)} \mathbb{E}_p U_{t+1} \left( \omega^t, \cdot, V_t^{x, \phi^{*,x}}(\omega^t) + h \Delta S_{t+1}(\omega^t, \cdot) \right) \\ &= \inf_{p \in \mathcal{Q}_{t+1}(\omega^t)} \mathbb{E}_p U_{t+1} \left( \omega^t, \cdot, V_t^{x, \phi^{*,x}}(\omega^t) + \phi_{t+1}^{*,x}(\omega^t) \Delta S_{t+1}(\omega^t, \cdot) \right). \end{aligned} \quad (24)$$

We show below in Example 1 that the last equality in (24) can be false. We also prove that we can have  $u(x) > \inf_{P \in \mathcal{Q}^T} \mathbb{E}_P U(\cdot, V_T^{x, \phi^{*,x}}(\cdot))$ .

Now, if  $u_t(\omega^t, \cdot, \cdot)$  is usc (recall (16)), then (19) implies that (24) holds true. However,  $u_t(\omega^t, \cdot, \cdot)$  has no reason to be usc. One might think that if  $U_{t+1}(\omega^{t+1}, \cdot)$  is usc, then  $u_t(\omega^t, \cdot, \cdot)$  would also be usc. This is not necessarily true with our one-period assumptions even if  $U_{t+1}(\omega^{t+1}, \cdot)$  is assumed to be concave and finite (and in particular usc), see Example 2. So, even if  $U_{t+1}$  was defined in a closure “way”, i.e. as the closure of  $x \mapsto \sup_{h \in \mathbb{R}^d} u_{t+1}(\omega^{t+1}, x, h)$  for all  $\omega^{t+1} \in \Omega^{t+1}$  as in [7] and [12], it would not guarantee that  $(x, h) \mapsto u_t(\omega^t, x, h)$  would be usc for  $\omega^t \in \Omega^t$  in some full-measure set, and so would be pointless. In any case, the closures in (19) should still be needed.

**EXAMPLE 1.** The existence of an optimal strategy may fail when the utility function is not usc and concave. Let  $T = 1$ ,  $d = 1$ ,  $\Omega^1 = \mathbb{R}$  and for all  $x \in \mathbb{R}$ ,  $U(x) := x$  if  $x \leq 0$  and  $U(x) := 1$  otherwise. Then,  $U$  is nonconcave, lower-semicontinuous (lsc) but not usc at  $x = 0$ . Let  $\Delta S_1$  and  $P_0 \in \mathfrak{P}(\Omega^1)$  be such that  $\Delta S_1$  is a  $P_0$ -integrable Borel measurable function and  $D_{P_0}^1 = \mathbb{R}$ . Note that in particular,  $P_0(\Delta S_1 < 0) > 0$  and  $P_0(\Delta S_1 > 0) > 0$ . Set  $\mathcal{Q}^1 := \{P_0\}$ . The function  $U$  is a (nonrandom) utility in the sense of Definition 1, and Assumptions 1 and 2 hold true (see Lemma 5). Here,  $D^1 = D_{P_0}^1 = \mathbb{R}$  and so,  $\text{Aff}(D^1) = \text{Aff}(D_{P_0}^1) = \mathbb{R}$ ,  $0 \in \text{ri}(\text{Conv}(D_{P_0}^1))$  and Assumption 3 holds true. Assumption 4 holds true for  $\bar{\gamma} := 1$ ,  $\underline{\gamma} := 0.5$  and  $C := 1$ . Indeed, let  $\lambda \geq 1$ . If  $x > 0$ , then  $U(\lambda x) = 1$  and  $U(x) = 1$ . So,  $\lambda^{0.5}(U(x) + C) = 2\lambda^{0.5} \geq 1 = U(\lambda x)$  and  $\lambda(U(x) + C) = 2\lambda \geq 1 = U(\lambda x)$ . If  $x \leq 0$ , then  $U(\lambda x) = \lambda x$  and  $U(x) = x$ . So,  $U(\lambda x) = \lambda U(x) \leq \lambda(U(x) + C)$  and  $U(\lambda x) = \lambda U(x) \leq \lambda^{0.5} U(x) \leq \lambda^{0.5}(U(x) + C)$  as  $U(x) \leq 0$ . Assumption 5 holds true for  $\underline{X} = -2$  as  $U(\underline{X}) = U(-2) = -2 < -1 = -C$ . Now, as  $U$  is bounded from above (by 1), so is  $U_0$  and Assumptions 6 and 7 hold true. We claim that an optimal strategy fails to exist when  $x = 0$ . Let  $\phi : \mathbb{R} \ni h \mapsto \mathbb{E}_{P_0} U(h \Delta S_1)$ . For all  $h > 0$ ,  $\phi(h) = P_0(\Delta S_1 > 0) - h \mathbb{E}_{P_0}(\Delta S_1)^-$ . For all  $h < 0$ ,  $\phi(h) = P_0(\Delta S_1 < 0) + h \mathbb{E}_{P_0}(\Delta S_1)^+$ . Thus,  $\phi$  is nondecreasing on  $(-\infty, 0)$  and nonincreasing  $(0, +\infty)$  and the only possible maximizer for  $\phi$  is  $h = 0$ . But, as  $\phi(0) = 0$ ,  $\phi(0^+) = P_0(\Delta S_1 > 0) > 0$  and  $\phi(0^-) = P_0(\Delta S_1 < 0) > 0$ , we see that  $h = 0$  cannot be a maximizer and there is no optimal strategy for (12).

We now claim that there is a strict inequality at  $x = 0$  in (19) or (21) and also in (24). From the previous computations,  $u(0) = \max(P_0(\Delta S_1 > 0), P_0(\Delta S_1 < 0)) \in (0, 1)$ . We have that  $\text{Cl}(U)(x) = x$  if  $x < 0$  and  $\text{Cl}(U)(x) = 1$  otherwise. As  $\text{Cl}(U)(x) \leq 1 = \text{Cl}(U)(0)$  for all  $x \in \mathbb{R}$ , we have that  $\mathbb{E}_{P_0} \text{Cl}(U)(h \Delta S_1) \leq 1 = \mathbb{E}_{P_0} \text{Cl}(U)(0 \Delta S_1)$  for all  $h \in \mathbb{R}$  and  $h^* = 0$  is a maximizer of  $h \mapsto \mathbb{E}_{P_0} \text{Cl}(U)(h \Delta S_1)$ . But,

$$\mathbb{E}_{P_0} U(h^* \Delta S_1) = 0 < u(0) = U_0(0) < 1 = \mathbb{E}_{P_0} \text{Cl}(U)(h^* \Delta S_1) = \sup_{h \in \mathbb{R}} \mathbb{E}_{P_0} \text{Cl}(U)(h \Delta S_1).$$

**2.6. Application** Note that the strategy  $\phi^{*,x}$  belongs to  $\Phi$  but may fail to be admissible. This is not specific to our quasi-sure setting. Already in [39, Theorem 2.7] or [13, Theorem 1], one has to assume that  $\phi^{*,x}$  is admissible in order to be optimal (see [13, Remark 15]). Interestingly, we see that if  $\phi^{*,x}$  is admissible and if  $V_T^{x, \phi^{*,x}}(\omega^T)$  does not belong to the set of discontinuity of  $U(\omega^T, \cdot)$ , then  $\phi^{*,x}$  is a solution of (12). The condition that  $\phi^{*,x} \in \Phi(x, U, \mathcal{Q}^T)$  is obviously not easy to verify and we would like to check that Theorem 1 applies for a concrete, broad class of market models and random utility functions. For that, we first define some sets of random variables which are integrable enough.

**DEFINITION 4.** Fix  $0 \leq t \leq T$  and  $P \in \mathfrak{P}(\Omega^t)$ .

$$\begin{aligned} \mathcal{M}^t(P) &:= \{X : \Omega^t \rightarrow \mathbb{R} \cup \{-\infty, +\infty\} \text{ projectively measurable such that } \mathbb{E}_P |X|^r < +\infty, \forall r \geq 1\} \\ \mathcal{M}^t &:= \bigcap_{P \in \mathcal{Q}^t} \mathcal{M}^t(P). \end{aligned}$$



Note that  $\mathcal{M}^0 = \mathcal{M}^0(P) = \mathbb{R}$ . It is also clear that  $\mathcal{M}^t \subset \mathcal{M}^t(P)$  for all  $P \in \mathcal{Q}^t$ . The set  $\mathcal{M}^T$  is a robust extension in the projective setting of the set  $\mathcal{M}$  defined in [39] (see also [13]).

DEFINITION 5. A random utility as in Definition 1 is of type (A) if  $U(\omega^T, \cdot)$  is usc for all  $\omega^t \in \Omega^T$ ,  $U^+(\cdot, 1) \in \mathcal{M}^T$ , Assumption 4 holds true for some  $C \in \mathcal{M}^T$ , Assumption 5 holds true for some  $\underline{X} \in \mathcal{M}^T$  such that  $1/|U(\cdot, \underline{X}(\cdot)) + C(\cdot)| \in \mathcal{M}^T$ , and if there exist  $p \geq 1$  and a non-negative and projectively measurable random variable  $C_1 \in \mathcal{M}^T$  such that for all  $\omega^T \in \Omega^T$  and  $x \in \mathbb{R}$

$$U(\omega^T, x) \geq -C_1(\omega^T)(1 + |x|^p). \quad (25)$$

This notion of random utility function of type (A) was already defined in [12, Definition 5] but for concave utility functions and assuming that  $U(\cdot, x)$  is Borel measurable and  $C_1 \in \mathcal{W}^T$ , where  $\mathcal{W}^T$  is the set of  $X$  such that  $\sup_{P \in \mathcal{Q}^T} \mathbb{E}_P |X|^r < +\infty$  for all  $r \geq 1$  (see [12, Definition 4]). It is indeed better to work with  $\mathcal{M}^T$  as the integrability conditions of  $\mathcal{M}^T$  can be checked on a prior-by-prior basis.

An example of utility functions of type (A) are utility functions with random benchmark, see [12, Definition 6 and Proposition 13], where the utility function is no more concave and the benchmark is only assumed to be projectively measurable and in  $\mathcal{M}^T$ .

We propose the following theorem that shows the existence of an optimal strategy for random utility of type (A) under some integrability conditions on the market especially on the process  $\alpha^P$  introduced below and related to the ‘‘quantitative’’ no-arbitrage condition, see [8, Definition 3.19]. This lemma generalizes [8, Proposition 3.35] to our projective setup

LEMMA 2. Assume the (PD) axiom. Assume that Assumptions 1, 2 and 3 hold true. Fix  $P := q_1^P \otimes \dots \otimes q_T^P \in \mathcal{H}^T$ . For all  $0 \leq t \leq T-1$ , there exists some projectively measurable function  $\alpha_t^P(\cdot) \in (0, 1]$  such that  $\Omega_{q_{NA}}^{t,P}$  is a  $\mathcal{Q}^t$ -full-measure set, where

$$\Omega_{q_{NA}}^{t,P} := \left\{ \omega^t \in \Omega^t, \forall h \in \text{Aff}(D^{t+1})(\omega^t), h \neq 0, q_{t+1}^P (h \Delta S_{t+1}(\omega^t, \cdot) < -\alpha_t^P(\omega^t) |h| |\omega^t|) \geq \alpha_t^P(\omega^t) \right\}. \quad (26)$$

*Proof.* See Appendix 8.1.  $\square$

THEOREM 2. Assume the (PD) axiom. Let  $U$  be a random utility of type (A). Assume that Assumptions 1, 2 and 3 hold true. Moreover, suppose that for all  $P \in \mathcal{H}^T$  and for all  $0 \leq t \leq T-1$ ,  $1/\alpha_t^P \in \mathcal{M}^t$  and  $|\Delta S_{t+1}| \in \mathcal{M}^{t+1}$ . Then, for all  $x \in \mathbb{R}$ , there exists  $\phi^{*,x} \in \Phi(x, U, \mathcal{Q}^T)$  such that

$$u(x) = \sup_{\phi \in \Phi(x, U, \mathcal{Q}^T)} \inf_{P \in \mathcal{Q}^T} \mathbb{E}_P U(\cdot, V_T^{x, \phi}(\cdot)) = \inf_{P \in \mathcal{Q}^T} \mathbb{E}_P U(\cdot, V_T^{x, \phi^{*,x}}(\cdot)).$$

The integrability conditions of Theorem 2 are quite classical in the uni-prior literature on unbounded utility functions, see for example [39, Proposition 7.1], [9, Theorem 4.16] and [13, Proposition 7], and also close to the ones of the multiple-priors literature, see for example [7, Theorem 3.6], [8, Corollary 3.16], [12, Theorem 2] and [38, Theorem 3.11]. Theorem 2 extends [12, Theorem 2] to utility functions that are no longer necessarily concave and to integrability conditions stated through  $\mathcal{M}^t$  and not  $\mathcal{W}^t$ .

**3. One period case** As mentioned after Theorem 1, we construct the candidate for optimal solutions of (12) by gluing together one-step optimal strategies for (18). This is the reason why we start with a one-period model. Let  $\bar{\Omega}$  be a Polish space,  $\mathfrak{P}(\bar{\Omega})$  be the set of all probability measures on  $(\bar{\Omega}, \mathcal{B}(\bar{\Omega}))$  and  $\mathcal{Q}$  be a nonempty convex subset of  $\mathfrak{P}(\bar{\Omega})$ . In the one-period model, we will not assume the (PD) axiom because we don't do measurable selection. We will also not be using projective sets or projective measurability. Thus, here the  $\mathbb{R}^d$ -vector  $Y(\cdot) := (Y_1(\cdot), \dots, Y_d(\cdot))$ , which could represent the change of values of the price process during the period, is assumed to be  $\mathcal{B}_c(\bar{\Omega})$ -measurable and not projectively measurable. Like previously,  $D \subset \mathbb{R}^d$  is the support of the distribution of  $Y(\cdot)$  under  $\mathcal{Q}$  and  $D_p \subset \mathbb{R}^d$  is the one of  $Y(\cdot)$  under  $p \in \mathcal{Q}$ . We first assume the one-period counterpart of Assumption 3.

ASSUMPTION 8. *There exists  $p^* \in \mathcal{Q}$  such that  $0 \in \text{ri}(\text{conv}(D_{p^*}))$  and  $\text{Aff}(D) = \text{Aff}(D_{p^*})$ .*

In the one-period case, Assumption 8 is equivalent to the NA( $\mathcal{Q}$ ) condition, see [4, Lemma 2.7] and also [8, Theorem 3.29] for  $T = 1$ . In the rest of this Section, we fix some  $p^*$  as in Assumption 8. As NA( $p^*$ ) holds true, the ‘‘quantitative’’ no-arbitrage condition is satisfied and [39, Proposition 3.3] shows that there exists some  $0 < \alpha^* \leq 1$  such that for all  $h \in \text{Aff}(D_{p^*}) = \text{Aff}(D)$  (see Assumption 8),  $h \neq 0$ ,

$$p^*(hY < -\alpha^*|h|) \geq \alpha^*. \quad (27)$$

ASSUMPTION 9. *A random utility  $V : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  is a  $\mathcal{B}_c(\bar{\Omega} \times \mathbb{R})$ -measurable function such that  $V(\omega, \cdot)$  is nondecreasing for all  $\omega \in \bar{\Omega}$ .*

The measurability of  $V$  implies that the expectations in (28) or (34) below are well-defined. Indeed, let  $(x, h) \in \mathbb{R} \times \mathbb{R}^d$ . Noting that  $\omega \mapsto (\omega, x + hY(\omega))$  is  $\mathcal{B}_c(\bar{\Omega})$ -measurable (recall that  $Y$  is  $\mathcal{B}_c(\bar{\Omega})$ -measurable) and using [5, Proposition 7.44, p172], we get that  $V(\cdot, x + hY(\cdot))$  is also  $\mathcal{B}_c(\bar{\Omega})$ -measurable.

REMARK 5. The value functions  $U_{t+1}^P(\omega^t, \cdot, \cdot)$  and  $U_{t+1}(\omega^t, \cdot, \cdot)$  (see (15) and (16)) are the multiperiod counterparts of  $V(\cdot, \cdot)$  and they are projectively measurable (see Proposition 5 (i)) and thus under the (PD) axiom,  $\mathcal{B}_c(\Omega^t \times \mathbb{R})$ -measurable (see Lemma 5 (iii)). One may wonder if the one-period analysis can be done with an arbitrary measurable space  $(\bar{\Omega}, \mathcal{G})$  and under the assumption that  $V(\cdot, x)$  is  $\mathcal{G}$ -measurable and  $V(\omega, \cdot)$  is non-decreasing as in [7, Section 3] and [12, Section 3]. The answer is no. As we do not assume here that  $V(\omega, \cdot)$  is usc,  $V$  has no reason to be jointly measurable and (28) and (34) may not exist.

The pendant of (12) is :

$$v(x) := \sup_{h \in \mathbb{R}^d} \inf_{p \in \mathcal{Q}} \mathbb{E}_p V(\cdot, x + hY(\cdot)). \quad (28)$$

Let  $\Psi : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  be the worst case expected utility function defined for all  $(x, h) \in \mathbb{R} \times \mathbb{R}^d$  by

$$\Psi(x, h) := \inf_{p \in \mathcal{Q}} \mathbb{E}_p V(\cdot, x + hY(\cdot)). \quad (29)$$

Then,  $v(x) = \sup_{h \in \mathbb{R}^d} \Psi(x, h)$ . The function  $\Psi$  is the pendant of  $u_t$ , while  $v$  is the pendant of  $U_t$ , see (16). Problem (28) have already been solved in [6] and [12] in the case of concave and usc utility functions. The fact that  $\Psi(x, \cdot)$  was usc played a significant role to prove the existence of an optimizer. In [6, Lemma 3.5.12], a strong integrability assumption on  $V^+$  and the continuity and the concavity of  $V(\omega, \cdot)$  allow to show that  $\Psi(x, \cdot)$  is usc using Fatou's lemma. In [12, Proposition

1], it is a strong integrability assumption on  $V^-$  (and a weaker integrability assumption on  $V^+$ ), as well as the concavity of  $V(\omega, \cdot)$ , which imply that  $\Psi(x, \cdot)$  is finite and concave, and thus usc.

Here, even if  $V(\omega, \cdot)$  is assumed to be usc and concave, there is no reason for  $\Psi(x, \cdot)$  to be usc and also to have a maximizer. This is primarily because we do not assume enough integrability conditions on  $V^-$  and  $V^+$ . Example 2 proposes an example of this phenomenon. Without continuity and concavity assumption, we do not aim to solve (28) but instead, we focus on the related problem

$$v^{cl}(x) := \sup_{h \in \mathbb{R}^d} \text{Cl}(\Psi)(x, h). \quad (30)$$

Except for the concavity and the continuity of  $V(\omega, \cdot)$  and the measurability of  $V$ , our other assumptions are very similar to the ones of [12, Section 3]. We will comment as we go along on the differences.

ASSUMPTION 10. *For all  $\theta \in \{-1, 1\}^d$ ,  $\mathbb{E}_{p^*} V^+(\cdot, 1 + \theta Y(\cdot)) < +\infty$ .*

Assumption 10 is similar to [12, Assumption 10], [6, Assumption 3.5.6 (3.19)] and [7, Assumption 3.16] and provides some upper bound for the value function. As [12, Assumption 10], Assumption 10 is stated only for  $p^*$  in contrast to [6, Assumption 3.5.6 (3.19)] which is postulated for all  $p \in \mathcal{Q}$ . However, Assumption 10 is stronger than [12, Assumption 10] which only requires that  $\mathbb{E}_{p^*} V^+(\cdot, 1) < +\infty$ . This counterbalances [12, Assumption 9], which is replaced by Assumption 13, as here we don't need a control from below on  $v$  in order to transfer concavity results to the value function. We now give the assumption related to RAE in discrete time.

ASSUMPTION 11. *There exist some constants  $0 < \underline{\gamma} < \bar{\gamma}$  and a  $\mathcal{B}_c(\bar{\Omega})$ -measurable random variable  $C : \bar{\Omega} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  such that  $c^* := \mathbb{E}_{p^*}(C) < +\infty$  and such that for all  $\omega \in \bar{\Omega}$  satisfying  $C(\omega) < +\infty$ , for all  $\lambda \geq 1$ ,  $x \in \mathbb{R}$ ,*

$$V(\omega, \lambda x) \leq \lambda^{\underline{\gamma}}(V(\omega, x) + C(\omega)) \quad (31)$$

$$V(\omega, \lambda x) \leq \lambda^{\bar{\gamma}}(V(\omega, x) + C(\omega)). \quad (32)$$

From now, we choose some  $0 < \eta < 1$  such that  $\underline{\gamma} < \eta \bar{\gamma}$ .

Equation (31) provides a control on  $V^-$ , while (32) gives a control on  $V^+$ . Indeed, we see easily that for all  $\omega \in \bar{\Omega}$  such that  $C(\omega) < +\infty$ ,  $x \in \mathbb{R}$  and  $\lambda \geq 1$ ,

$$\begin{aligned} V^-(\omega, \lambda x) &\geq \lambda^{\bar{\gamma}}(V^-(\omega, x) - C(\omega)). \\ V^+(\omega, \lambda x) &\leq \lambda^{\underline{\gamma}}(V^+(\omega, x) + C(\omega)). \end{aligned} \quad (33)$$

The coefficient  $\eta$  will play an important role to establish bounds for the value function in (30) and the optimal strategy : it is crucial that the control  $\bar{\gamma}$  on  $V^-$  is strictly larger than the control  $\underline{\gamma}$  on  $V^+$ . As  $U$  is not concave we don't have the condition that  $\underline{\gamma} \leq 1 \leq \bar{\gamma}$  which is required in [12, Assumption 11].

ASSUMPTION 12. *There exists some  $n_0^* \in \mathbb{N} \setminus \{0\}$  such that,*

$$p^* \left( V(\cdot, -n_0^*) \leq - \left( 1 + 2 \frac{c^*}{\alpha^*} \right) \right) \geq 1 - \frac{\alpha^*}{2},$$

where  $\alpha^*$  is defined in (27) and  $c^*$  in Assumption 11.

Assumption 12 (see also [13, Assumption 8] and [12, Assumption 12]) is the one-period counterpart of Assumption 5 and ensures that the functions  $v$  and  $v^{cl}$  (see (28) and (30)) can take arbitrary negative values. Note that if  $\lim_{x \rightarrow -\infty} V(\cdot, x) = -\infty$   $p^*$ -almost surely, then Assumption 12 is verified. If Assumption 12 fails, then there may be no solution to the one-step utility maximization

problem even in the concave uni-prior case, see [12, Remark 5].

Let  $p \in \mathcal{Q}$ . We introduce the function  $\Psi_p : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  which is defined for all  $(x, h) \in \mathbb{R} \times \mathbb{R}^d$  by

$$\Psi_p(x, h) := \mathbb{E}_p V(\cdot, x + hY(\cdot)). \quad (34)$$

Then,  $\Psi(x, h) = \inf_{p \in \mathcal{Q}} \Psi_p(x, h)$ , recall (29). Recall that the expectation in (29) and (34) are defined in the generalized sense using that  $V(\cdot, x + hY(\cdot))$  is  $\mathcal{B}_c(\bar{\Omega})$ -measurable and the convention  $+\infty - \infty = -\infty + \infty = -\infty$ . Thus,  $\Psi_p$  and  $\Psi$  are well-defined although being potentially infinite. The function  $\Psi_p$  is introduced as a tool to prove properties on  $\Psi$  and  $v$  but also to pass Assumption 6 through the dynamic programming procedure. The results on  $\Psi_p$  can be find in [12] (see also [39] and [6]) but under a different set of assumptions.

EXAMPLE 2. Even if  $V(\omega, \cdot)$  is assumed to be finite and concave (and thus continuous),  $\Psi$  may fail to be usc and Problem (28) may have no solution.

Let  $N_1, N_2$  and  $N_3$  be Borel measurable functions defined on  $\mathbb{R}$ . Let  $p$  be a probability measure on  $\mathbb{R}$  such that  $(N_1, N_2, N_3)$  are independent standard Gaussians under  $p$ . Set  $\bar{Y} := \exp(\exp(N_3))$ . We choose  $\bar{\Omega} = \mathbb{R} \times \{0, 1\}$ ,  $d = 2$  and define  $Y(\cdot) = (Y_1(\cdot), Y_2(\cdot))$  as follows for all  $\omega = (x, i) \in \bar{\Omega}$ :

$$Y_1(\omega) := N_1(x)1_{\{0\}}(i) + \bar{Y}(x)1_{\{1\}}(i) \quad Y_2(\omega) := N_2(x)1_{\{0\}}(i) - \bar{Y}(x)1_{\{1\}}(i).$$

We also define  $p^*$  and  $p_0$  on  $\bar{\Omega}$  as follows:  $p^* := p \otimes \delta_{\{0\}}$  and  $p_0 := p \otimes \delta_{\{1\}}$ . Set  $\mathcal{Q} := \text{Conv}(p_0, p^*)$ . We see easily that  $(Y_1, Y_2) = (N_1, N_2)$   $p^*$ -a.s. Thus, under  $p^*$ ,  $Y_1$  and  $Y_2$  are independent and follow some standard Gaussian laws. We also have that  $(Y_1, Y_2) = (\bar{Y}, -\bar{Y})$   $p_0$ -a.s.

Now, Assumption 8 holds true as  $D_{p^*} = \mathbb{R}^2 = \text{Aff}(D_{p^*}) \subset \text{Aff}(D) \subset \mathbb{R}^2$  and  $0 \in \text{ri}(\text{Conv}(D_{p^*})) = \mathbb{R}^2$ . We now define  $V$ . Let  $Z(\omega) := \exp(N_3(x))1_{\{1\}}(i)$ . Then,  $Z = 0$   $p^*$ -a.s. and  $Z = \exp(N_3)$   $p_0$ -a.s. For all  $x \in \mathbb{R}$ , let  $\tilde{V}(x) := 1 - \exp(-x)$  if  $x \leq 0$  and  $\tilde{V}(x) := x$  otherwise and set  $V(\omega, x) := \tilde{V}(x - Z(\omega))$ . Then,  $V(\omega, \cdot)$  is finite and strictly concave for all  $\omega \in \bar{\Omega}$ . Assumption 9 trivially holds true as  $Z$  and  $V$  are Borel measurable. As  $V^+(\omega, x) = (x - Z(\omega))^+$  for all  $(\omega, x) \in \bar{\Omega} \times \mathbb{R}$  and  $Z = 0$   $p^*$ -a.s., for all  $h = (h_1, h_2) \in \mathbb{R}^2$ , we have that

$$\mathbb{E}_{p^*} V^+(\cdot, x + hY(\cdot)) \leq |x| + \mathbb{E}_p |h_1 N_1 + h_2 N_2| < +\infty, \quad (35)$$

as  $h_1 N_1 + h_2 N_2$  follows a Gaussian law under  $p$  and Assumption 10 follows. We show now that Assumption 11 holds true. Note first that

$$\text{AE}_{-\infty}(V(\omega, \cdot)) := \liminf_{x \rightarrow -\infty} \frac{xV'(\omega, \cdot)}{V(\omega, \cdot)} = +\infty.$$

Thus, applying [12, Lemma 8 and Proposition 10] to  $V(\omega, \cdot)$  shows that for all  $\lambda \geq 1$  and  $x \in \mathbb{R}$ ,

$$V(\omega, \lambda x) \leq \lambda(V(\omega, x) + C(\omega)) \quad \text{and} \quad V(\omega, \lambda x) \leq \lambda^2(V(\omega, x) + C(\omega)),$$

where  $C(\omega) := \tilde{V}^+(-Z(\omega)) + \tilde{V}^-(-Z(\omega)) + \tilde{V}^-(-3)$ . As  $Z = 0$   $p^*$ -a.s., we have that  $c^* = \mathbb{E}_{p^*} C < +\infty$  and Assumption 11 follows. We turn to Assumption 12. As  $\tilde{V}$  is unbounded from below, there exists some  $n_0^* \geq 1$  such that  $\tilde{V}(-n_0^*) \leq -(1 + 2c^*/\alpha^*)$ . Recalling again that  $Z = 0$   $p^*$ -a.s., we have that  $V(\cdot, -n_0^*) = \tilde{V}(-n_0^*)$   $p^*$ -a.s. and Assumption 12 holds true.

We claim that  $\Psi$  is not usc. We first compute  $\Psi_{p_0}(0, h)$ . Let  $h = (h_1, h_2) \in \mathbb{R}^2$ ,  $\bar{h} := h_1 - h_2$  and  $f(x) := \bar{h} \exp(x) - x$ . Then,

$$\Psi_{p_0}(0, h) = \mathbb{E}_{p_0} V(\cdot, hY(\cdot)) = \mathbb{E}_{p_0} \tilde{V}(hY(\cdot) - Z(\cdot)) = \mathbb{E}_p \tilde{V}(f(\exp(N_3(\cdot)))) ,$$

as  $hY - Z = h_1\bar{Y} - h_2\bar{Y} - Z = \bar{h}\exp(\exp(N_3)) - \exp(N_3) = f(\exp(N_3))$   $p_0$ -a.s. If  $\bar{h} \leq 0$ ,  $hY - Z \leq 0$  and

$$\Psi_{p_0}(0, h) = 1 - \mathbb{E}_p \exp(-\bar{h}\exp(\exp(N_3(\cdot))) + \exp(N_3(\cdot))) \leq 1 - \mathbb{E}_p \exp(\exp(N_3(\cdot))) = -\infty.$$

Assume now that  $\bar{h} > 0$ . It is easy to see that  $f$  is strictly nonincreasing on  $(-\infty, -\ln(\bar{h}))$  and that  $f$  is strictly nondecreasing on  $(-\ln(\bar{h}), +\infty)$ , that  $\lim_{x \rightarrow -\infty} f(x) = +\infty$  and  $\lim_{x \rightarrow +\infty} f(x) = +\infty$  and so that  $\min_{x \in \mathbb{R}} f(x) = f(-\ln(\bar{h})) = 1 + \ln(\bar{h})$ . Thus if  $\bar{h} \geq \exp(-1)$ ,  $f(\exp(N_3)) = hY - Z \geq 0$   $p_0$ -a.s and  $\Psi_{p_0}(0, h) = \mathbb{E}_{p_0}(hY(\cdot) - Z(\cdot)) = \mathbb{E}_{p_0}(\bar{h}\bar{Y}(\cdot) - Z(\cdot)) = +\infty$  as  $\mathbb{E}_{p_0}Z(\cdot) = \mathbb{E}_p \exp(N_3(\cdot)) = \exp(1/2) < +\infty$  and  $\mathbb{E}_{p_0}(\bar{h}\bar{Y}(\cdot)) = \bar{h}\mathbb{E}_p \exp(\exp(N_3(\cdot))) = +\infty$ .

Assume now that  $\bar{h} \in (0, \exp(-1))$ . As  $\min_{x \in \mathbb{R}} f(x) = f(-\ln(\bar{h})) < 0$  and recalling the variations of  $f$ , there exist  $x_1, x_2$  such that  $f(x) \leq 0$  if and only if  $x \in [x_1, x_2]$ . Thus,  $p_0$ -a.s,  $Z = \exp(N_3) \in [x_1, x_2]$  is equivalent to  $hY - Z = f(\exp(N_3)) \leq 0$ . Now, as  $V^-(\omega, x) = \tilde{V}^-(x - Z(\omega)) = (\exp(-x)\exp(Z(\omega)) - 1)1_{x \leq Z(\omega)}$ , we have that

$$\begin{aligned} \mathbb{E}_{p_0} V^-(\cdot, hY(\cdot)) &= \mathbb{E}_{p_0}((\exp(-\bar{h}\bar{Y}(\cdot))\exp(Z(\cdot)) - 1)1_{hY(\cdot) \leq Z(\cdot)}) \\ &\leq \mathbb{E}_p((\exp(\exp(N_3(\cdot))) - 1)1_{\exp(N_3(\cdot)) \in [x_1, x_2]}) \leq \exp(x_2) - 1 < +\infty, \end{aligned}$$

where the first inequality come from  $\bar{h} > 0$  and  $\bar{Y} > 0$ . Moreover,

$$\mathbb{E}_{p_0} V^+(\cdot, hY(\cdot)) = \mathbb{E}_{p_0}(\bar{h}\bar{Y}(\cdot) - Z(\cdot))^+ \geq \mathbb{E}_{p_0}(\bar{h}\bar{Y}(\cdot) - Z(\cdot)) = \bar{h}\mathbb{E}_p \exp(\exp(N_3(\cdot))) - \mathbb{E}_p \exp(N_3(\cdot)) = +\infty.$$

As a result,  $\Psi_{p_0}(0, h) = -\infty$  if  $h_1 \leq h_2$  and  $\Psi_{p_0}(0, h) = +\infty$  otherwise. Thus, for all  $h \in \mathbb{R}^2$ ,

$$\Psi(0, h) = \inf_{p \in \text{Conv}(p_0, p^*)} \Psi_p(0, h) = \min(\Psi_{p_0}(0, h), \Psi_{p^*}(0, h)) = -\infty 1_{h_1 \leq h_2} + \Psi_{p^*}(0, h) 1_{h_1 > h_2}$$

as  $\Psi_{p^*}$  is finite, see (35). Recalling that  $Z = 0$   $p^*$ -a.s, for all  $(x, h) \in \mathbb{R} \times \mathbb{R}^2$ ,

$$\Psi_{p^*}(x, h) = \mathbb{E}_{p^*} \tilde{V}(x + hY(\cdot) - Z(\cdot)) = \mathbb{E}_{p^*} \tilde{V}(x + hY(\cdot)) = \mathbb{E}_{p^*} \tilde{V}(x + h_1 N_1(\cdot) + h_2 N_2(\cdot)). \quad (36)$$

The fact that  $\tilde{V}$  is strictly concave implies that  $\Psi_{p^*}$  is strictly concave and (35) that  $\Psi_{p^*}$  is finite. Thus,  $\Psi_{p^*}$  is continuous. But,  $\Psi$  is not usc at  $(0, 0)$  as

$$\limsup_{n \rightarrow +\infty} \Psi(0, (1/n, 0)) = \limsup_{n \rightarrow +\infty} \Psi_{p^*}(0, (1/n, 0)) = \Psi_{p^*}(0, (0, 0)) = 0 > \Psi(0, (0, 0)) = -\infty.$$

We claim now that there is no maximizer for  $\Psi(0, \cdot)$ . Using Jensen's inequality in (36), we get that for all  $h \in \mathbb{R}^2$ ,

$$\Psi_{p^*}(0, h) = \mathbb{E}_{p^*} \tilde{V}(hY(\cdot)) \leq \tilde{V}(h\mathbb{E}_{p^*} Y(\cdot)) = \tilde{V}(0) = \Psi_{p^*}(0, (0, 0))$$

and  $(0, 0)$  is a maximizer for  $\Psi_{p^*}(0, \cdot)$ . Now, as  $\Psi_{p^*}(0, \cdot)$  is strictly concave,  $(0, 0)$  is the only global and local maximizer of  $\Psi_{p^*}(0, \cdot)$ . Assume now that a maximizer  $h^* = (h_1^*, h_2^*)$  exists for  $\Psi(0, \cdot)$ . Then, as  $\Psi_{p^*}$  is finite,  $\Psi(0, h^*) = \Psi_{p^*}(0, h^*)$  and  $h_1^* > h_2^*$ . This implies that  $h^* \neq (0, 0)$ , a contradiction. Thus, there is no maximizer for  $\Psi(0, \cdot)$  and Problem (28) doesn't have a solution for  $x = 0$ .

**PROPOSITION 1.** *Assume that Assumptions 8, 9, 10 and 11 hold true. Then,  $\Psi_{p^*} < +\infty$  and for all  $(x, h) \in \mathbb{R} \times \mathbb{R}^d$ ,*

$$\Psi_{p^*}(x, h) \leq \mathbb{E}_{p^*} V^+(\cdot, x + hY(\cdot)) \leq (|h| \vee x^+ \vee 1)^\gamma (l^* + c^*), \quad (37)$$

where  $c^* = \mathbb{E}_{p^*} C < +\infty$ ,  $l^* := \sum_{\theta \in \{-1, 1\}^d} \mathbb{E}_{p^*} V^+(\cdot, 1 + \theta Y(\cdot)) < +\infty$  and  $a \vee b = \max(a, b)$ . Moreover, the function  $\Psi_{p^*}^{cl} : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  defined by

$$\Psi_{p^*}^{cl}(x, h) := \mathbb{E}_{p^*} \text{Cl}(V)(\cdot, x + hY(\cdot)) \quad (38)$$

is usc.

*Proof.* For  $i \in \{1, \dots, d\}$ , let  $\widehat{\theta}_i(\cdot) := \text{sgn}(Y_i(\cdot))$  where for all  $y \in \mathbb{R}$ ,  $\text{sgn}(y) = 1$  if  $y \geq 0$  and  $\text{sgn}(y) = -1$  otherwise. Then,  $\widehat{\theta}$  is a  $\{-1, 1\}^d$ -valued process and

$$|Y(\cdot)| \leq |Y(\cdot)|_1 = \sum_{i=1}^d \text{sgn}(Y_i(\cdot)) Y_i(\cdot) = \widehat{\theta}(\cdot) Y(\cdot). \quad (39)$$

Let  $\omega \in \overline{\Omega}$  such that  $C(\omega) < +\infty$  and  $(x, h) \in \mathbb{R} \times \mathbb{R}^d$ . As  $V^+(\omega, \cdot)$  is nondecreasing, using Cauchy-Schwarz inequality, (33) and (39), we obtain that

$$\begin{aligned} V^+(\omega, x + hY(\omega)) &\leq V^+(\omega, (|h| \vee x^+ \vee 1)(1 + |Y(\omega)|)) \\ &\leq (|h| \vee x^+ \vee 1)^{\mathfrak{L}} (V^+(\omega, 1 + |Y(\omega)|) + C(\omega)) \\ &\leq (|h| \vee x^+ \vee 1)^{\mathfrak{L}} \left( \sum_{\theta \in \{-1, 1\}^d} V^+(\omega, 1 + \theta Y(\omega)) + C(\omega) \right). \end{aligned} \quad (40)$$

As  $c^* < +\infty$  by Assumption 11,  $p^*(C < +\infty) = 1$  and taking the expectation under  $p^*$  in (40) shows (37). As  $l^* < +\infty$  by Assumption 10, we conclude that  $\Psi_{p^*} < +\infty$ .

Let  $(x, h) \in \mathbb{R} \times \mathbb{R}^d$  and  $(x_n, h_n) \in \mathbb{R} \times \mathbb{R}^d$  such that  $\lim_{n \rightarrow +\infty} (x_n, h_n) = (x, h)$ . Let  $M := \sup_{n \geq 0} |x_n| + \sup_{n \geq 0} |h_n|$ . Then,  $M < +\infty$ . Using (17) ( $V(\omega, \cdot)$  is nondecreasing), we get that for all  $\omega \in \overline{\Omega}$  and  $x \in \mathbb{R}$

$$\text{Cl}(V)(\omega, x) = \inf_{n \geq 1} V\left(\omega, x + \frac{1}{n}\right) \leq V(\omega, x + 1). \quad (41)$$

Thus, as  $V$  is  $\mathcal{B}_c(\overline{\Omega} \times \mathbb{R})$ -measurable,  $\text{Cl}(V)$  is also  $\mathcal{B}_c(\overline{\Omega} \times \mathbb{R})$ -measurable. Using now (40), we find that for all  $\omega \in \overline{\Omega}$  such that  $C(\omega) < +\infty$ ,

$$(\text{Cl}(V))^+(\omega, x_n + h_n Y(\omega)) \leq (M + 1)^{\mathfrak{L}} \left( \sum_{\theta \in \{-1, 1\}^d} V^+(\omega, 1 + \theta Y(\omega)) + C(\omega) \right).$$

As  $c^* < +\infty$  and  $l^* < +\infty$ , we deduce that  $(\text{Cl}(V))^+(\cdot, x_n + h_n Y(\cdot))$  is dominated by a  $p^*$ -integrable random variable. So,  $(\limsup)$  Fatou's Lemma and the fact that  $\text{Cl}(V)(\omega, \cdot)$  is usc for all  $\omega \in \overline{\Omega}$  show that

$$\limsup_{n \rightarrow +\infty} \Psi_{p^*}^{cl}(x_n, h_n) \leq \mathbb{E}_{p^*} \text{Cl}(V)(\cdot, x + hY(\cdot)) = \Psi_{p^*}^{cl}(x, h).$$

Thus,  $\Psi_{p^*}^{cl}$  is usc.  $\square$

We give one additional assumption under which a polynomial control on  $\text{Cl}(\Psi)$  as well as a bound on the optimal strategies for  $v^{cl}(x)$  in (30) can be derived.

**ASSUMPTION 13.** For all  $k \in \mathbb{N} \setminus \{0\}$ ,  $\inf_{p \in \mathcal{Q}} \mathbb{E}_p V(\cdot, -k) > -\infty$ .

Assumption 13 is trivially weaker than [12, Assumption 9], which requires that  $\sup_{p \in \mathcal{Q}} \mathbb{E}_p V^-(\cdot, x + hY(\cdot)) < +\infty$  for all  $(x, h) \in \mathbb{Q} \times \mathbb{Q}^d$ . Note that Assumption 13 is not required to get the existence of an optimal strategy in (30).

**PROPOSITION 2.** Suppose that Assumptions 8, 9, 10, 11 and 12 hold true. Then, for all  $x \in \mathbb{R}$ , we have that

$$v(x) \leq v^{cl}(x) = \sup_{h \in \mathbb{R}^d} \text{Cl}(\Psi)(x, h) = \sup_{h \in \text{Aff}(D)} \text{Cl}(\Psi)(x, h). \quad (42)$$



For all  $x \in \mathbb{R}$ , we define  $K_0$  and  $K_1$  as follows

$$\begin{aligned} K_0(x) &:= \max \left( 1, x^+, \frac{x^+ + n_0^*}{\alpha^*}, \left( \frac{x^+ + n_0^*}{\alpha^*} \right)^{\frac{1}{1-\eta}} \right) \\ K_1(x) &:= \max \left( K_0(x), \left( \frac{6l^*}{\alpha^*} \right)^{\frac{1}{\eta\bar{\gamma}-\underline{\gamma}}}, \left( \frac{6c^*}{\alpha^*} \right)^{\frac{1}{\eta\bar{\gamma}-\underline{\gamma}}}, \left( \frac{6}{\alpha^*} \Psi^-(x, 0) \right)^{\frac{1}{\eta\bar{\gamma}}} \right), \end{aligned}$$

where  $\alpha^*$  is defined in (27),  $c^*$ ,  $\eta$ ,  $\bar{\gamma}$  and  $\underline{\gamma}$  in Assumption 11,  $l^*$  in Proposition 1 and  $n_0^*$  in Assumption 12. Then,  $K_0 < +\infty$  and for all  $h \in \text{Aff}(D)$  and  $x \in \mathbb{R}$ , we get that

$$|h| > K_0(x) \implies \text{Cl}(\Psi)(x, h) \leq |h|^{\underline{\gamma}}(l^* + c^*) - |h|^{\eta\bar{\gamma}} \frac{\alpha^*}{2}. \quad (43)$$

Assume now that Assumption 13 also holds true. Then,  $K_1 < +\infty$  and we have that for all  $h \in \text{Aff}(D)$  and  $x \in \mathbb{R}$ ,

$$|h| > K_1(x) \implies \text{Cl}(\Psi)(x, h) < \text{Cl}(\Psi)(x, 0). \quad (44)$$

In this case, we also obtain that for all  $x \in \mathbb{R}$ ,

$$v(x) \leq \sup_{h \in \mathbb{R}^d} \text{Cl}(\Psi)(x, h) = \sup_{h \in \text{Aff}(D)} \text{Cl}(\Psi)(x, h) = \sup_{\substack{|h| \leq K_1(x) \\ h \in \text{Aff}(D)}} \text{Cl}(\Psi)(x, h). \quad (45)$$

*Proof.* We first show (42). As  $v(x) = \sup_{h \in \mathbb{R}^d} \Psi(x, h)$  and  $\Psi \leq \text{Cl}(\Psi)$ , the first inequality is immediate. Let  $(x, h) \in \mathbb{R} \times \mathbb{R}^d$ . Let  $h^\perp$  be the orthogonal projection of  $h$  on  $\text{Aff}(D)$ . Using [7, Remark 3.10], we get that  $hY = h^\perp Y$   $\mathcal{Q}$ -q.s. We remark now that

$$\text{Cl}(\Psi)(x, h) = \text{Cl}(\Psi)(x, h^\perp). \quad (46)$$

Indeed, using [40, 1(7)] (with  $-\Psi$ ), we get that

$$\begin{aligned} \text{Cl}(\Psi)(x, h) &= \inf_{\substack{\epsilon_1 > 0 \\ \epsilon_2 > 0}} \sup_{\substack{|x' - x| < \epsilon_1 \\ |h' - h| < \epsilon_2}} \Psi(x', h') = \inf_{\substack{\epsilon_1 > 0 \\ \epsilon_2 > 0}} \sup_{\substack{|x' - x| < \epsilon_1 \\ |h' - h| < \epsilon_2}} \Psi(x', h' + (h^\perp - h)) \\ &= \inf_{\substack{\epsilon_1 > 0 \\ \epsilon_2 > 0}} \sup_{\substack{|x' - x| < \epsilon_1 \\ |h' - h^\perp| < \epsilon_2}} \Psi(x', h') = \text{Cl}(\Psi)(x, h^\perp). \end{aligned} \quad (47)$$

Thus, the second equality in (42) follows.

The proof that  $K_0 < +\infty$  and that for all  $x \in \mathbb{R}$  and  $h \in \text{Aff}(D)$ ,

$$|h| \geq K_0(x) \implies \Psi(x, h) \leq |h|^{\underline{\gamma}}(l^* + c^*) - |h|^{\eta\bar{\gamma}} \frac{\alpha^*}{2} \quad (48)$$

is exactly the same as the proof of [12, Proposition 2 (29)] and is thus omitted. We now prove (43). Let  $(x, h) \in \mathbb{R} \times \mathbb{R}^d$ . We set  $w(x, h) := |h^\perp|^{\underline{\gamma}}(l^* + c^*) - |h^\perp|^{\eta\bar{\gamma}} \frac{\alpha^*}{2}$  if  $|h^\perp| > K_0(x)$  and  $w(x, h) := +\infty$  otherwise. Then, as  $hY = h^\perp Y$   $\mathcal{Q}$ -q.s., (48) shows that  $\Psi(x, h) = \Psi(x, h^\perp) \leq w(x, h)$ . Assume for a moment that  $w$  is usc. Then, by definition of the closure,  $\text{Cl}(\Psi) \leq w$ . Thus, for all  $x \in \mathbb{R}$  and  $h \in \text{Aff}(D)$  with  $|h| > K_0(x)$ , as  $h^\perp = h$ ,  $\text{Cl}(\Psi)(x, h) \leq w(x, h) = |h|^{\underline{\gamma}}(l^* + c^*) - |h|^{\eta\bar{\gamma}} \frac{\alpha^*}{2}$  and (43) is proved.

We show now that  $w$  is usc. Let  $(x, h) \in \mathbb{R} \times \mathbb{R}^d$  and  $(x_n, h_n) \in \mathbb{R} \times \mathbb{R}^d$  such that  $\lim_{n \rightarrow +\infty} (x_n, h_n) = (x, h)$ . If  $|h^\perp| \leq K_0(x)$ , we have that  $\limsup_{n \rightarrow +\infty} w(x_n, h_n) \leq +\infty = w(x, h)$ .

Assume now that  $|h^\perp| > K_0(x)$ . As  $h \mapsto h^\perp$  and  $x \mapsto K_0(x)$  are continuous,  $\{(x, h) \in \mathbb{R} \times \mathbb{R}^d, |h^\perp| > K_0(x)\}$  is an open set and there exists  $k \geq 0$  such that for all  $n \geq k$ ,  $|h_n^\perp| > K_0(x_n)$ . Thus,

$$\limsup_{n \rightarrow +\infty} w(x_n, h_n) = \lim_{n \rightarrow +\infty} |h_n^\perp|^\gamma (l^* + c^*) - |h_n^\perp|^{\eta\bar{\gamma}} \frac{\alpha^*}{2} = w(x, h).$$

So,  $w$  is indeed usc.

We now also postulate Assumption 13. Let  $x \in \mathbb{R}$ . As  $V(\omega, \cdot)$  is nondecreasing for all  $\omega \in \bar{\Omega}$  and as Assumption 13 holds true,  $\Psi(x, 0) > \Psi(-\lceil x \rceil, 0) > -\infty$  where  $\lceil x \rceil$  is the smallest natural number greater than  $x$ , and thus,  $\Psi^-(x, 0) < +\infty$ . Recall that  $K_0(x) < +\infty$ . Moreover,  $\alpha^* > 0$  (see (27)),  $l^* < +\infty$  (see Assumption 10),  $\eta\bar{\gamma} - \gamma > 0$  and  $c^* < +\infty$  (see Assumption 11). Thus,  $K_1(x) < +\infty$ . Assume now that  $h \in \text{Aff}(D)$  and  $|h| > K_1(x)$ . As  $\underline{\gamma} < \eta\bar{\gamma}$ , we get that  $|h| > K_0(x)$  and

$$|h|^\gamma l^* < |h|^{\eta\bar{\gamma}} \frac{\alpha^*}{6} \quad |h|^\gamma c^* < |h|^{\eta\bar{\gamma}} \frac{\alpha^*}{6} \quad \Psi^-(x, 0) < |h|^{\eta\bar{\gamma}} \frac{\alpha^*}{6}.$$

Then, using (43) and the definition of the closure, we obtain that

$$\text{Cl}(\Psi)(x, h) < -|h|^{\eta\bar{\gamma}} \frac{\alpha^*}{6} < -\Psi^-(x, 0) \leq \Psi(x, 0) \leq \text{Cl}(\Psi)(x, 0) \quad (49)$$

and (44) is proved. Assumption 8 shows that  $0 \in \text{ri}(\text{conv}(D_{p^*})) \subset \text{Aff}(D_{p^*}) = \text{Aff}(D)$  and  $\text{Aff}(D)$  is a vector space. Thus, (42), (49) and the fact that  $0 \in \text{Aff}(D)$  show (45).  $\square$

We now show that under Assumptions 8 to 12, an optimal strategy exists for (30).

**PROPOSITION 3.** *Assume that Assumptions 8, 9, 10, 11 and 12 hold true. Let  $x \in \mathbb{R}$ . Then, there exists  $\hat{h}_x \in \text{Aff}(D)$  such that*

$$v(x) \leq \text{Cl}(v)(x) \leq v^{cl}(x) = \sup_{h \in \text{Aff}(D)} \text{Cl}(\Psi)(x, h) = \sup_{h \in \mathbb{R}^d} \text{Cl}(\Psi)(x, h) = \text{Cl}(\Psi)(x, \hat{h}_x). \quad (50)$$

If Assumption 13 also holds true, then  $|\hat{h}_x| \leq K_1(x)$  where  $K_1$  is defined in Proposition 2.

*Proof.* We start with the easy assertions in (50). The first inequality holds true by definition of the closure and the first and second equalities are proved in (42).

Assume for a moment that  $v^{cl}$  is usc. Recalling again (42),  $v \leq v^{cl}$  and by definition of the closure,  $\text{Cl}(v) \leq v^{cl}$ . Thus, the second inequality in (50) is proved. We now show that  $v^{cl}$  is usc. Remark first that as  $V(\omega, \cdot)$  is nondecreasing, we have successively that  $\Psi(\cdot, h)$  and  $\text{Cl}(\Psi)(\cdot, h)$  are nondecreasing for all  $h \in \text{Aff}(D)$  (see (47)) and thus  $v^{cl}$  is nondecreasing. We prove now that  $v^{cl}$  is right-continuous, which will imply that  $v^{cl}$  is usc.

*$v^{cl}$  is right-continuous.*

Let  $x \in \mathbb{R}$  and  $(x_n)_{n \geq 0} \subset \mathbb{R}$  such that  $x < x_n$  for all  $n \geq 0$  and  $\lim_{n \rightarrow +\infty} x_n = x$ .

*Case 1 : There exists some  $p \geq 0$  such that  $v^{cl}(x_n) < +\infty$  for all  $n \geq p$ .*

We can extract a subsequence that we still denote by  $(x_n)_{n \geq 0}$  such that  $v^{cl}(x_n) < +\infty$  for all  $n \geq 0$ . As  $v^{cl}$  is nondecreasing and by definition of the sup in  $v^{cl}(x_n)$ , there exists  $(h_n)_{n \geq 0} \subset \text{Aff}(D)$  such that for all  $n \geq 0$ ,

$$v^{cl}(x) \leq v^{cl}(x_n) \leq \text{Cl}(\Psi)(x_n, h_n) + \frac{1}{n+1}. \quad (51)$$

Assume that  $(|h_n|)_{n \geq 0}$  is not bounded. Then, as  $\lim_{n \rightarrow +\infty} K_0(x_n) = K_0(x)$ , we can extract a subsequence that we also denote by  $(x_n, h_n)_{n \geq 0}$  such that  $|h_n|$  goes to infinity and  $|h_n| \geq K_0(x) + 1 > K_0(x_n)$ . Thus, (43) in Proposition 2 implies that for all  $n \geq 0$ ,

$$v^{cl}(x) \leq v^{cl}(x_n) \leq \text{Cl}(\Psi)(x_n, h_n) + \frac{1}{n+1} \leq |h_n|^\gamma (l^* + c^*) - |h_n|^{\eta\bar{\gamma}} \frac{\alpha^*}{2} + \frac{1}{n+1}.$$

As  $\underline{\gamma} < \eta\bar{\gamma}$ , taking the limit, we get that  $v^{cl}(x) \leq \lim_{n \rightarrow +\infty} v^{cl}(x_n) \leq -\infty$  and  $v^{cl}$  is right-continuous in  $x$ .

Assume now that  $(|h_n|)_{n \geq 0}$  is bounded. Then, one can extract a subsequence, that we still denote by  $(x_n, h_n)_{n \geq 0}$ , such that  $\lim_{n \rightarrow +\infty} h_n = \bar{h}$  for some  $\bar{h} \in \mathbb{R}^d$ . Moreover,  $\bar{h} \in \text{Aff}(D)$ . As  $\text{Cl}(\Psi)$  is usc by definition of the closure, taking the limit in (51), we get that

$$v^{cl}(x) \leq \lim_{n \rightarrow +\infty} v^{cl}(x_n) \leq \limsup_{n \rightarrow +\infty} \text{Cl}(\Psi)(x_n, h_n) \leq \text{Cl}(\Psi)(x, \bar{h}) \leq v^{cl}(x).$$

Thus,  $v^{cl}$  is again right-continuous in  $x$ .

*Case 2 : One can extract a subsequence (still denoted by  $(x_n)_{n \geq 0}$ ) such that  $v^{cl}(x_n) = +\infty$  for all  $n \geq 0$ .*

Then, by definition of the sup in  $v^{cl}(x_n)$ , there exists  $(h_n)_{n \geq 0} \subset \text{Aff}(D)$  such that for all  $n \geq 0$ ,

$$\text{Cl}(\Psi)(x_n, h_n) \geq n. \quad (52)$$

We show by contradiction that  $(h_n)_{n \geq 0}$  is bounded. Assume that  $(h_n)_{n \geq 0}$  is unbounded. As done earlier in the proof, one can extract a subsequence that we also denote by  $(x_n, h_n)_{n \geq 0}$  such that  $|h_n|$  goes to infinity and  $|h_n| > K_0(x_n)$ . Using (52) and (43), we obtain that for all  $n \geq 0$ ,

$$n \leq \text{Cl}(\Psi)(x_n, h_n) \leq |h_n|^{\underline{\gamma}(l^* + c^*)} - |h_n|^{\eta\bar{\gamma}} \frac{\alpha^*}{2}.$$

As  $\underline{\gamma} < \eta\bar{\gamma}$ , taking the limit, we get that  $+\infty \leq -\infty$ , a contradiction. Thus,  $(h_n)_{n \geq 0}$  is bounded and one can extract a subsequence, that we still denote by  $(x_n, h_n)_{n \geq 0}$ , such that  $\lim_{n \rightarrow +\infty} h_n = \bar{h}$  for some  $\bar{h} \in \text{Aff}(D)$ . As  $\text{Cl}(\Psi)$  is usc, taking the limit in (52), we get that

$$+\infty \leq \lim_{n \rightarrow +\infty} \text{Cl}(\Psi)(x_n, h_n) = \limsup_{n \rightarrow +\infty} \text{Cl}(\Psi)(x_n, h_n) \leq \text{Cl}(\Psi)(x, \bar{h}) \leq v^{cl}(x).$$

Thus, as  $\text{Cl}(\Psi)(x_n, h_n) \leq v^{cl}(x_n)$ , we obtain that  $\lim_{n \rightarrow +\infty} v^{cl}(x_n) = +\infty = v^{cl}(x)$  and  $v^{cl}$  is right-continuous in  $x$ .

*Existence of a maximizer for  $\text{Cl}(\Psi)(x, \cdot)$ .*

Fix  $x \in \mathbb{R}$ . Let  $w$  be defined by  $w(h) := \text{Cl}(\Psi)(x, h)$  when  $h \in \text{Aff}(D)$  and  $w(h) := -\infty$  otherwise. Assume first that  $-w$  is not proper in the sense of [40, p5] i.e. that for all  $h \in \mathbb{R}^d$ ,  $w(h) = -\infty$  or that there exists some  $h^* \in \mathbb{R}^d$  such that  $w(h^*) = +\infty$ . In the first case,  $\text{Cl}(\Psi)(x, h) = -\infty$  for all  $h \in \mathbb{R}^d$  using the second equality in (42) and any  $h \in \mathbb{R}^d$  is a maximizer. In the second case, we have that  $w(h^*) = \text{Cl}(\Psi)(x, h^*) = +\infty$  and  $h^*$  is then a maximizer. Assume now that  $-w$  is proper. Remark that  $w$  is usc as  $\text{Aff}(D)$  is a closed set and  $\text{Cl}(\Psi)(x, \cdot)$  is usc. Now, using (43) in Proposition 2 and recalling that  $\underline{\gamma} < \eta\bar{\gamma}$ , we have that  $\lim_{|h| \rightarrow +\infty} -w(h) = +\infty$ . Thus,  $-w$  is level bounded (see [40, Definition 1.8, p11] and the short text below this definition) and [40, Theorem 1.9, p11] shows that there exists  $\hat{h}_x \in \mathbb{R}^d$  such that  $v^{cl}(x) = \sup_{h \in \mathbb{R}^d} w(h) = w(\hat{h}_x) > -\infty$ . As a result,  $\hat{h}_x \in \text{Aff}(D)$ ,  $w(\hat{h}_x) = \text{Cl}(\Psi)(x, \hat{h}_x)$  and the last equality in (50) is proved.

Assume now that Assumption 13 also holds true. Then, then (44) immediately shows that  $|\hat{h}_x| \leq K_1(x)$ .  $\square$

**4. Multiple-period case** In this section, we prepare the proofs of Theorems 1 and 2. For that we will apply the one-period results in two contexts. The first one, called the robust context, assumes that  $\mathcal{Q} := \mathcal{Q}_{t+1}(\omega^t)$  and  $V := U_{t+1}(\omega^t, \cdot, \cdot)$  and is used to prove Theorem 1. For  $P := q_1^P \otimes \dots \otimes q_T^P \in \mathcal{H}^T$ , the second one, called the  $P$ -prior context, suppose that  $\mathcal{Q} := \{q_{t+1}^P(\cdot | \omega^t)\}$  and  $V := U_{t+1}^P(\omega^t, \cdot, \cdot)$  and is used to prove Theorem 2. Note that in the  $P$ -prior context,  $\text{Graph}(\mathcal{Q}) = \text{Graph}(q_{t+1}^P)$  may not be a projective set. This will not be an issue as in the one-period case, we

did not assume that  $\text{Graph}(\mathcal{Q})$  is projective. We will construct a  $\mathcal{Q}^t$ -full-measure set  $\tilde{\Omega}^t$  (resp. a  $P^t$ -full-measure set  $\tilde{\Omega}^{t,P}$ ) where Assumptions 8 to 12 hold true in the robust (resp.  $P$ -prior) context (see Lemma 4 and Proposition 6). To do that, we first introduce and prove properties for the dynamic version of  $C$  that appears in Assumption 4 (see Proposition 4). Then, Proposition 5 gives fundamental properties of the value functions  $U_t$  and  $U_t^P$ . To prove measurability results, we will also use Lemma 12 and Proposition 12 state and prove in Section 7.5. Recall that our approach differs from the ones of [12] or [7]. It is indeed simplest and more direct when the results on projective sets and projectively measurable functions are proved. For example our definitions of  $U_t$  and  $U_t^P$  don't use closure or sup on  $\mathbb{Q}^d$  and the proof of Proposition 5 is much easier. Moreover, we don't need to introduce functions to control from below the value functions. We will comment on the other differences as we go along.

For the rest of this part, we fix some random utility function  $U$  in the sense of Definition 1 and some random variable  $C$  as in Assumption 4.

For  $0 \leq t \leq T$ , we define by induction the function  $C_t : \Omega^t \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  as follows :

$$\begin{cases} C_T(\omega^T) := C(\omega^T) \\ C_t(\omega^t) := \sup_{p \in \mathcal{Q}_{t+1}(\omega^t)} \mathbb{E}_p C_{t+1}(\omega^t, \cdot). \end{cases} \quad (53)$$

The function  $C_t$  will appear in (13) and (14) in Assumption 4 stated for  $U_t$ . It has already been introduced in [12, (49)].

**PROPOSITION 4.** *Assume the (PD) axiom. Assume that Assumptions 2 and 4 hold true. Then, for all  $0 \leq t \leq T$ ,  $C_t$  is non-negative, projectively measurable and satisfies that*

$$C_t(\cdot) < +\infty \quad \mathcal{Q}^t - q.s. \quad (54)$$

*Proof.* We show by backward induction the following claim :  $C_t$  is non-negative, projectively measurable and satisfies that

$$\sup_{P \in \mathcal{Q}^t} \mathbb{E}_P C_t < +\infty. \quad (55)$$

The claim at time  $T$  follows from Assumption 4 as  $C_T := C$ . Let  $0 \leq t \leq T-1$  and assume that the claim holds true at time  $t+1$ . Let  $c_t : \Omega^t \times \mathfrak{P}(\Omega_{t+1}) \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  be defined for all  $(\omega^t, p) \in \Omega^t \times \mathfrak{P}(\Omega_{t+1})$  by  $c_t(\omega^t, p) := \mathbb{E}_p C_{t+1}(\omega^t, \cdot)$ . Then (53) shows that  $C_t(\omega^t) = \sup_{p \in \mathcal{Q}_{t+1}(\omega^t)} c_t(\omega^t, p)$ . The non-negativity of  $c_t$  and  $C_t$  follows from the one of  $C_{t+1}$ . Considering the Borel measurable and thus projectively measurable (see Lemma 5) stochastic kernel  $q$  defined by  $q(\cdot | \omega^t, p) := p(\cdot)$  and as  $C_{t+1}$  is projectively measurable, Proposition 12 (ii) shows that  $c_t$  is projectively measurable. Then, Assumption 2 and Proposition 10 show that  $C_t$  is projectively measurable and that given any  $\epsilon > 0$ , there exists  $q^\epsilon : \Omega^t \rightarrow \mathfrak{P}(\Omega_{t+1})$ , which is projectively measurable, such that for all  $\omega^t \in \Omega^t$  (recall that  $\mathcal{Q}_{t+1} \neq \emptyset$ ),  $q^\epsilon(\cdot | \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$  and

$$c_t(\omega^t, q^\epsilon(\cdot | \omega^t)) \geq \begin{cases} \frac{1}{\epsilon} & \text{if } C_t(\omega^t) = +\infty, \\ C_t(\omega^t) - \epsilon & \text{otherwise.} \end{cases} \quad (56)$$

For all  $\bar{P} \in \mathcal{Q}^t$ , taking the expectation under  $\bar{P}$  and using Fubini's theorem (see Remark 2) as  $C_{t+1} \geq 0$  we get

$$\mathbb{E}_{\bar{P} \otimes q^\epsilon} C_{t+1} \geq \frac{1}{\epsilon} \bar{P}(C_t = +\infty) + \mathbb{E}_{\bar{P}} ((C_t - \epsilon) 1_{\{C_t < +\infty\}}). \quad (57)$$

As  $\bar{P} \otimes q^\epsilon \in \mathcal{Q}^{t+1}$  and  $C_t \geq 0$ , we get that

$$\sup_{P \in \mathcal{Q}^{t+1}} \mathbb{E}_P C_{t+1} \geq \frac{1}{\epsilon} \bar{P}(C_t = +\infty) - \epsilon. \quad (58)$$

If  $\bar{P}(C_t = +\infty) > 0$ , taking the limit when  $\epsilon$  goes to 0 in (58), we find that  $\sup_{P \in \mathcal{Q}^{t+1}} \mathbb{E}_P C_{t+1} = +\infty$ , which contradicts (55) at time  $t+1$ . Thus,  $\bar{P}(C_t = +\infty) = 0$  and (57) implies that

$$\sup_{P \in \mathcal{Q}^{t+1}} \mathbb{E}_P C_{t+1} \geq \mathbb{E}_{\bar{P}} C_t - \epsilon.$$

So, letting  $\epsilon$  go to 0, taking the supremum over all  $\bar{P} \in \mathcal{Q}^t$  and using (55) for  $t+1$ , we get that  $\sup_{P \in \mathcal{Q}^t} \mathbb{E}_P C_t < +\infty$ . This concludes the induction.

We now prove (54). Let  $0 \leq t \leq T-1$ . Assume that there exists some  $P \in \mathcal{Q}^t$  such that  $P(C_t = +\infty) > 0$ . Then, as  $C_t \geq 0$ ,  $\mathbb{E}_P C_t = +\infty$  and also  $\sup_{P \in \mathcal{Q}^t} \mathbb{E}_P C_t = +\infty$ , a contradiction to (55). Thus,  $P(C_t < +\infty) = 1$  for all  $P \in \mathcal{Q}^t$ , which shows (54).  $\square$

We now provide some fundamental properties of the value functions  $U_t$  and  $U_t^P$ .

**PROPOSITION 5.** *Assume the (PD) axiom. Assume that Assumptions 1 and 2 hold true. Let  $P \in \mathcal{Q}^T$  and  $0 \leq t \leq T$ . We have that (i)  $U_t$  and  $U_t^P$  are projectively measurable, (ii) for all  $\omega^t \in \Omega^t$ ,  $U_t(\omega^t, \cdot), U_t^P(\omega^t, \cdot) : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  are nondecreasing, (iii)  $U_t \leq U_t^P$ , (iv)  $\text{Cl}(U_t)$  and  $\text{Cl}(U_t^P)$  are projectively measurable, (v) for all  $\omega^t \in \Omega^t$  and  $x \in \mathbb{R}$ ,*

$$\text{Cl}(U_t)(\omega^t, x) \leq U_t(\omega^t, x+1) \text{ and } \text{Cl}(U_t^P)(\omega^t, x) \leq U_t^P(\omega^t, x+1).$$

*Assume furthermore that Assumption 4 holds true. Then, for all  $\omega^t \in \Omega^t$  such that  $C_t(\omega^t) < +\infty$ ,  $\lambda \geq 1$ ,  $x \in \mathbb{R}$ , we get that*

$$U_t(\omega^t, \lambda x) \leq \lambda^{\bar{\gamma}}(U_t(\omega^t, x) + C_t(\omega^t)) \text{ and } U_t^P(\omega^t, \lambda x) \leq \lambda^{\bar{\gamma}}(U_t^P(\omega^t, x) + C_t(\omega^t)). \quad (59)$$

$$U_t(\omega^t, \lambda x) \leq \lambda^{\underline{\gamma}}(U_t(\omega^t, x) + C_t(\omega^t)) \text{ and } U_t^P(\omega^t, \lambda x) \leq \lambda^{\underline{\gamma}}(U_t^P(\omega^t, x) + C_t(\omega^t)). \quad (60)$$

*Proof. Proof of (i) to (iii) under Assumptions 1 and 2.*

Let  $P := q_1^P \otimes \dots \otimes q_T^P \in \mathcal{Q}^T$ . We show by backward induction that (i) to (iii) hold true.

*Initialization step.*

As  $U_T = U_T^P = U$  and  $U$  is projectively measurable (see Definition 1), (i) and (iii) hold true. Definition 1 shows that for all  $\omega^T \in \Omega^T$ ,  $U_T(\omega^T, \cdot) = U_T^P(\omega^T, \cdot) = U(\omega^T, \cdot)$  is nondecreasing and (ii) holds true.

*Heredity step.*

Fix  $0 \leq t \leq T-1$  and assume that the induction hypothesis holds at time  $t+1$ . Using (i) at time  $t+1$  and Lemma 12 for  $f = U_{t+1}$  and  $f = U_{t+1}^P$ , we find that  $U_t$  and  $U_t^P$  are projectively measurable and thus, (i) at  $t$  is proved. Now, (ii) and (iii) at  $t$  follow from (ii) and (iii) at  $t+1$  and from (15) and (16). This concludes the induction.

*Proof of (iv) and (v) under Assumptions 1 and 2.*

Let  $0 \leq t \leq T$ . As  $U_t$  is nondecreasing, using (17) we get that for all  $(\omega^t, x) \in \Omega^t \times \mathbb{R}$ ,

$$\text{Cl}(U_t)(\omega^t, x) = \inf_{n \geq 1} U_t \left( \omega^t, x + \frac{1}{n} \right) \leq U_t(\omega^t, x+1).$$

Moreover, as  $U_t$  is projectively measurable, Lemma 8 shows that  $\text{Cl}(U_t)$  is also projectively measurable. The same holds true for  $U_t^P$  and (iv) and (v) at  $t$  hold true.

*Proof of (59) and (60) under Assumptions 1, 2 and 4.*

We proceed again by backward induction. We only show the left-hand side of (59) as the proofs

of its right-hand side and of (60) are very similar and thus omitted. Assumption 4 ensures that the left-hand side of (59) holds true at time  $T$  as  $C_T = C$ . Fix  $0 \leq t \leq T - 1$  and assume that the left-hand side of (59) holds true at  $t + 1$ . Let  $\omega^t \in \Omega^t$  such that  $C_t(\omega^t) < +\infty$ ,  $(x, h) \in \mathbb{R} \times \mathbb{R}^d$  and  $\lambda \in \mathbb{R}$ . Take any  $p \in \mathcal{Q}_{t+1}(\omega^t)$ . Then,  $\{\omega_{t+1} \in \Omega_{t+1}, C_{t+1}(\omega^t, \omega_{t+1}) < +\infty\}$  is a  $p$ -full measure set. Otherwise, we get a contradiction with (54). So, the left-hand side of (59) at time  $t + 1$  implies

$$\mathbb{E}_p U_{t+1}(\omega^t, \cdot, \lambda x + \lambda h \Delta S_{t+1}(\omega^t, \cdot)) \leq \lambda^{\bar{\gamma}} (\mathbb{E}_p U_{t+1}(\omega^t, \cdot, x + h \Delta S_{t+1}(\omega^t, \cdot)) + \mathbb{E}_p C_{t+1}(\omega^t, \cdot)).$$

Thus, taking the infimum over all  $p \in \mathcal{Q}_{t+1}(\omega^t)$  and using (53)

$$\begin{aligned} \inf_{p \in \mathcal{Q}_{t+1}(\omega^t)} \mathbb{E}_p U_{t+1}(\omega^t, \cdot, \lambda x + \lambda h \Delta S_{t+1}(\omega^t, \cdot)) &\leq \lambda^{\bar{\gamma}} \inf_{p \in \mathcal{Q}_{t+1}(\omega^t)} (\mathbb{E}_p U_{t+1}(\omega^t, \cdot, x + h \Delta S_{t+1}(\omega^t, \cdot)) + \mathbb{E}_p C_{t+1}(\omega^t, \cdot)) \\ &\leq \lambda^{\bar{\gamma}} \inf_{p \in \mathcal{Q}_{t+1}(\omega^t)} \mathbb{E}_p U_{t+1}(\omega^t, \cdot, x + h \Delta S_{t+1}(\omega^t, \cdot)) + \lambda^{\bar{\gamma}} C_t(\omega^t). \end{aligned}$$

Now, we get the left-hand side of (59) at  $t$  by taking the supremum over every  $h \in \mathbb{R}^d$ .  $\square$

REMARK 6. Proposition 5 is the pendent of [12, Proposition 6]. The main difference is that  $U_t$  (resp.  $U_t^P$ ) is now only projectively measurable and no more lower-semianalytic (resp. universally measurable). Recall that  $U_t$  in [12, (8)] is defined using countable supremum and closure, under the assumption that  $U$  is concave, so that it remains lower-semianalytic through dynamic programming. Nevertheless, [12, Proposition 8] shows that it coincides with the definition that we take in this paper on a full-measure set. Such a result seems to be possible only for concave random utility function. Else, there is no guarantee that a countable supremum would coincide with an uncountable one.

Assume that Assumptions 1, 2 and 3 hold true. Then,  $\mathcal{H}^T \neq \emptyset$  and we fix for the rest of the paper some  $P^* \in \mathcal{H}^T$  with the following given disintegration

$$P^* := q_1^{P^*} \otimes \cdots \otimes q_T^{P^*}. \quad (61)$$

Lemma 2 shows the existence of the functions  $\alpha_t^{P^*} : \Omega^t \rightarrow (0, 1]$  and also that  $\Omega_{q_{NA}^{t, P^*}}$  defined in (26) is a  $\mathcal{Q}^t$ -full-measure set. The stochastic kernels  $(q_t^{P^*})_{1 \leq t \leq T}$  will be of special interest for the statements of the one-period Assumptions 8, 10 and 11 in the multiple-period contexts. On the other hand,  $(\alpha_t^{P^*})_{0 \leq t \leq T-1}$  will serve for the one of Assumption 12.

We now formally present the two different contexts, already introduced in [12, Definition 7] (with a different choice of  $c_t^P$ ), where we will apply the one-period results. The robust context will be used to prove Theorem 1 while the  $P$ -prior one will be used to prove Theorem 2.

DEFINITION 6. Let  $0 \leq t \leq T - 1$  and  $\omega^t \in \Omega^t$ . We call context  $(t + 1)$ , the following one-period market :  $\bar{\Omega} := \Omega_{t+1}$ ,  $Y(\cdot) := \Delta S_{t+1}(\omega^t, \cdot)$ ,  $C(\cdot) := C_{t+1}(\omega^t, \cdot)$  and  $\underline{\gamma}$  and  $\bar{\gamma}$  are introduced in Assumption 4.

Then, we are in the robust  $(t + 1)$  context if in addition  $\mathcal{Q} := \mathcal{Q}_{t+1}(\omega^t)$ ,  $p^* := q_{t+1}^{P^*}(\cdot | \omega^t)$ ,  $\alpha^* := \alpha_t^{P^*}(\omega^t)$  and  $V(\cdot, \cdot) := U_{t+1}(\omega^t, \cdot, \cdot)$ . As a consequence,  $\Psi(x, h) = u_t(\omega^t, x, h)$ ,  $v(x) = U_t(\omega^t, x)$ , see (16), (28) and (29).

Now, let  $P := q_1^P \otimes \cdots \otimes q_T^P \in \mathcal{H}^T$ . We are in the  $P$ -prior  $(t + 1)$  context if  $\mathcal{Q} := \{q_{t+1}^P(\cdot | \omega^t)\}$ ,  $p^* := q_{t+1}^P(\cdot | \omega^t)$ ,  $\alpha^* := \alpha_t^P(\omega^t)$  and  $V(\cdot, \cdot) := U_{t+1}^P(\omega^t, \cdot, \cdot)$ . Then,  $\Psi(x, h) = u_t^p(\omega^t, x, h)$ ,  $v(x) = U_t^P(\omega^t, x)$ , see (15), (28) and (29).



Let  $0 \leq t \leq T-1$  and  $P := q_1^P \otimes \cdots \otimes q_T^P \in \mathcal{H}^T$ . For all  $\omega^t \in \Omega^t$ , we define

$$c_t^P(\omega^t) := \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)} C_{t+1}(\omega^t, \cdot) \quad (62)$$

$$i_t^P(\omega^t) := 1 + 2 \frac{c_t^P(\omega^t)}{\alpha_t^P(\omega^t)}. \quad (63)$$

Fix  $\omega^t \in \Omega^t$ . The multiple-period counterpart of  $c^*$ ,  $l^*$ ,  $n_0^*$  (see Assumption 11, Proposition 1, Assumption 12) in the robust  $(t+1)$  context are  $c_t^{P*}(\omega^t)$ ,

$$\begin{aligned} l_t^*(\omega^t) &:= \sum_{\theta \in \{-1,1\}^d} \mathbb{E}_{q_{t+1}^{P*}(\cdot|\omega^t)} U_{t+1}^+(\omega^t, \cdot, 1 + \theta \Delta S_{t+1}(\omega^t, \cdot)), \\ N_t^*(\omega^t) &:= \inf \left\{ k \geq 1, q_{t+1}^{P*} \left( U_{t+1}(\omega^t, \cdot, -k) \leq -i_t^{P*}(\omega^t) | \omega^t \right) \geq 1 - \frac{\alpha_t^{P*}(\omega^t)}{2} \right\}, \end{aligned} \quad (64)$$

with the convention (which will be used until the end of the paper) that  $\inf \emptyset = +\infty$ . Now, the counterpart of  $c^*$ ,  $l^*$ ,  $n_0^*$  in the  $P$ -prior  $(t+1)$  context are  $c_t^P(\omega^t)$ ,

$$l_t^P(\omega^t) := \sum_{\theta \in \{-1,1\}^d} \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)} (U_{t+1}^P)^+(\omega^t, \cdot, 1 + \theta \Delta S_{t+1}(\omega^t, \cdot)), \quad (65)$$

$$N_t^P(\omega^t) := \inf \left\{ k \geq 1, q_{t+1}^P (U_{t+1}^P(\omega^t, \cdot, -k) \leq -i_t^P(\omega^t) | \omega^t) \geq 1 - \frac{\alpha_t^P(\omega^t)}{2} \right\}. \quad (66)$$

Note that for all  $\omega^t \in \Omega^t$ , using (62), (53) and Proposition 4, we get that

$$0 \leq c_t^P(\omega^t) \leq \sup_{p \in \mathcal{Q}_{t+1}(\omega^t)} \mathbb{E}_p C_{t+1}(\omega^t, \cdot) = C_t(\omega^t). \quad (67)$$

We first show that all the previous random variables are projectively measurable.

**LEMMA 3.** *Assume the (PD) axiom. Assume that Assumptions 1, 2, 3 and 4 hold true. Let  $P \in \mathcal{H}^T$  and  $0 \leq t \leq T-1$ . Then,  $l_t^*$ ,  $N_t^*$ ,  $i_t^P$ ,  $c_t^P$ ,  $l_t^P$  and  $N_t^P$  are projectively measurable.*

*Proof.* Let  $P := q_1^P \otimes \cdots \otimes q_T^P \in \mathcal{H}^T$  and  $0 \leq t \leq T-1$ . As  $q_{t+1}^P$  is a projectively measurable stochastic kernel and  $C_{t+1}$  is projectively measurable (see Proposition 4), Proposition 12 (ii) shows that  $c_t^P$  is projectively measurable. Recalling that  $\alpha_t^P$  is projectively measurable (see Lemma 2), we then deduce that  $i_t^P$  is also projectively measurable. Proposition 5 shows that  $U_{t+1}^+$  and  $(U_{t+1}^P)^+$  are projectively measurable. Now, using Lemma 12 for  $f = U_{t+1}^+$  (resp.  $f = (U_{t+1}^P)^+$ ) and Lemmata 8 and 10, we get that  $l_t^*$  (resp.  $l_t^P$ ) is projectively measurable.

We now show that  $N_t^P$  is projectively measurable. The proof for  $N_t^*$  is completely similar and thus omitted. Let  $n \geq 1$ . By definition of  $N_t^P$  in (66),

$$\{N_t^P \leq n\} = \bigcup_{k=1}^n \left\{ \omega^t \in \Omega^t, \int_{\Omega^t} 1_{A(k)}(\omega^t, \omega_{t+1}) q_{t+1}^P(d\omega_{t+1} | \omega^t) - 1 + \frac{\alpha_t^P(\omega^t)}{2} \geq 0 \right\},$$

where  $A(k) := \{(\omega^t, \omega_{t+1}) \in \Omega^t \times \Omega_{t+1}, U_{t+1}^P(\omega^t, \omega_{t+1}, -k) + i_t^P(\omega^t) \leq 0\}$ . As  $i_t^P$ ,  $\alpha_t^P$  and  $U_{t+1}^P$  are projectively measurable, we have that  $U_{t+1}^P$  is  $\Delta_p^1(\Omega^t \times \Omega_{t+1} \times \mathbb{R})$ -measurable,  $i_t^P$  is  $\Delta_q^1(\Omega^t)$ -measurable and  $\alpha_t^P$  is  $\Delta_r^1(\Omega^t)$ -measurable for some  $p, q, r \geq 1$ . We may assume that  $r \leq q \leq p$ . So, using (iv) in Proposition 8,  $i_t^P$  and  $\alpha_t^P$  are  $\Delta_{p+1}^1(\Omega^t)$ -measurable. Fix some  $k \geq 1$ . We get that  $U_{t+1}^P(\cdot, \cdot, -k)$  is  $\Delta_{p+1}^1(\Omega^t \times \Omega_{t+1})$ -measurable using Lemma 10. Thus,  $A(k) \in \Delta_{p+1}^1(\Omega^t \times \Omega_{t+1})$  (see Lemma 8). Now, recalling that  $q_{t+1}^P$  is a projectively measurable stochastic kernel, there exists some  $i \geq 1$  such that  $\omega^t \mapsto q_{t+1}^P(\cdot | \omega^t)$  is  $\Delta_i^1(\Omega^t)$ -measurable. Then, Proposition 12 (i) shows that  $\omega^t \mapsto$

$\int_{\Omega^t} 1_{A(k)}(\omega^t, \omega_{t+1}) q_{t+1}^P(d\omega_{t+1}|\omega^t)$  is  $\Delta_{p+i+3}^1(\Omega^t)$ -measurable. As  $\alpha_t^P$  is  $\Delta_{p+1}^1(\Omega^t)$ -measurable and also  $\Delta_{p+i+3}^1(\Omega^t)$ -measurable, we find that

$$\left\{ \omega^t \in \Omega^t, \int_{\Omega^t} 1_{A(k)}(\omega^t, \omega_{t+1}) q_{t+1}^P(d\omega_{t+1}|\omega^t) - 1 + \frac{\alpha_t^P(\omega^t)}{2} \geq 0 \right\} \in \Delta_{p+i+3}^1(\Omega^t).$$

Finally, as  $\Delta_{p+i+3}^1(\Omega^t)$  is a  $\sigma$ -algebra (see (iii) in Proposition 8) and  $p+i+3$  is independent of  $k$ , we obtain that  $\{N_t^P \leq n\} \in \Delta_{p+i+3}^1(\Omega^t)$  for all  $n \geq 1$  and so that  $N_t^P$  is projectively measurable.  $\square$

The following sets describe the paths  $\omega^t \in \Omega^t$  for which the one-period Assumptions 8 to 12 are satisfied in the robust  $(t+1)$  context and/or in the  $P$ -prior  $(t+1)$  context for a prior  $P := q_1^P \otimes \dots \otimes q_T^P \in \mathcal{H}^T$ . Note that Assumption 13 will only be used in the proof of Theorem 2.

DEFINITION 7. Let  $0 \leq t \leq T-1$ . For  $i \in \{8, 9, 10, 11\}$ , let

$$\begin{aligned} \Omega_i^t &:= \{\omega^t \in \Omega^t, \text{Assumption } i \text{ holds true in the robust } (t+1) \text{ context}\} \\ \Omega_i^{t,P} &:= \{\omega^t \in \Omega^t, \text{Assumption } i \text{ holds true in the } P\text{-prior } (t+1) \text{ context}\} \\ \Omega_{12}^t &:= \Omega_{q_{NA}}^{t,P^*} \cap \{N_t^* < +\infty\} \\ \Omega_{12}^{t,P} &:= \Omega_{q_{NA}}^{t,P} \cap \{N_t^P < +\infty\}, \end{aligned}$$

recall (26) for the definitions of  $\Omega_{q_{NA}}^{t,P^*}$  and  $\Omega_{q_{NA}}^{t,P}$ . Moreover, we set

$$\tilde{\Omega}^t := \bigcap_{i=8}^{12} \Omega_i^t \quad \text{and} \quad \tilde{\Omega}^{t,P} := \bigcap_{i=8}^{12} \Omega_i^{t,P}. \quad (68)$$

The next lemma shows that if we choose  $\omega^t$  in  $\tilde{\Omega}^t$  or  $\tilde{\Omega}^{t,P}$ , the one-period assumptions are true in the associated  $(t+1)$  context.

LEMMA 4. Assume the (PD) axiom. Assume that Assumptions 1, 2, 3 and 4 hold true. Let  $P \in \mathcal{H}^T$  and  $0 \leq t \leq T-1$ . If  $\omega^t \in \tilde{\Omega}^t$  (resp.  $\omega^t \in \tilde{\Omega}^{t,P}$ ), then Assumptions 8, 9, 10, 11 and 12 hold true in the robust  $(t+1)$  context (resp.  $P$ -prior  $(t+1)$  context).

*Proof.* The proof is trivial and thus omitted.  $\square$

We now prove that the  $\Omega_i^t$  are  $\mathcal{Q}^t$ -full-measure sets while the  $\Omega_i^{t,P}$  are  $P^t$ -full-measure sets. The proof needs the technical Lemmata 13 and 14 which are relegated to Appendix 8.2.

PROPOSITION 6. Assume the (PD) axiom. Assume that Assumptions 1, 2, 3 and 4 hold true. Let  $P \in \mathcal{H}^T$  and  $0 \leq t \leq T-1$ . Then, for all  $i \in \{8, 9, 11\}$ ,  $\Omega_i^t$  is a  $\mathcal{Q}^t$ -full-measure set and  $\Omega_i^{t,P}$  is a  $P^t$ -full-measure set. Assume furthermore that Assumptions 5, 6 and 7 hold true. Then,  $\Omega_{10}^t$  and  $\Omega_{12}^t$  are  $\mathcal{Q}^t$ -full-measure sets while  $\Omega_{10}^{t,P}$  and  $\Omega_{12}^{t,P}$  are  $P^t$ -full-measure sets. Thus, there exists a  $\mathcal{Q}^t$ -full-measure set  $\hat{\Omega}^t \in \mathcal{B}(\Omega^t)$  that satisfies  $\hat{\Omega}^t \subset \tilde{\Omega}^t$ .

*Proof.* Fix  $P := q_1^P \otimes \dots \otimes q_T^P \in \mathcal{H}^T$ .

Let  $0 \leq t \leq T-1$  and  $i \in \{8, 9, 11\}$ . The sets  $\Omega_i^t$  and  $\Omega_i^{t,P}$  are of  $\mathcal{Q}^t$ -full-measure and also of  $P^t$ -full-measure under Assumptions 1, 2, 3 and 4.

Recall  $P^*$  from (61). By Definitions 6 and 7, we see that for  $Q \in q_1^Q \otimes \dots \otimes q_T^Q \in \mathcal{H}^T$ ,

$$\begin{aligned} \Omega_8^{t,Q} &:= \{\omega^t \in \Omega^t, 0 \in \text{ri}(\text{conv}(D_Q^{t+1})(\omega^t))\} \\ \Omega_8^t &:= \{\omega^t \in \Omega^t, 0 \in \text{ri}(\text{conv}(D_{P^*}^{t+1})(\omega^t)), \text{Aff}(D_{P^*}^{t+1})(\omega^t) = \text{Aff}(D^{t+1})(\omega^t)\}, \end{aligned}$$

and  $\Omega_8^t \subset \Omega_8^{t,P^*}$ . As  $P^*$  and  $Q$  belong to  $\mathcal{H}^T$ , (8) shows that  $\Omega_8^t$ ,  $\Omega_8^{t,P^*}$  and  $\Omega_8^{t,Q}$  are  $\mathcal{Q}^t$ -full-measure sets.

Proposition 5 at  $t+1$  is now in force. Under the (PD) axiom, Assertion (i) and Lemma 5 (iii) show that  $U_{t+1}$  and  $U_{t+1}^P$  are  $\mathcal{B}_c(\Omega^{t+1} \times \mathbb{R})$ -measurable and [5, Lemma 7.29, p174] gives that  $U_{t+1}(\omega^t, \cdot, \cdot)$  and  $U_{t+1}^P(\omega^t, \cdot, \cdot)$  are  $\mathcal{B}_c(\Omega_{t+1} \times \mathbb{R})$ -measurable for all  $\omega^t \in \Omega^t$ . Thus, assertion (ii) shows that  $\Omega_9^t = \Omega_9^{t,P} = \Omega^t$ , which is of course of  $\mathcal{Q}^t$ -full-measure.

Now,  $C_{t+1}(\omega^t, \cdot)$  is  $\mathcal{B}_c(\Omega_{t+1})$ -measurable, see Proposition 4 and again Lemma 5 (iii) and [5, Lemma 7.29, p174]. We also have that  $C_{t+1}(\omega^t, \cdot) \geq 0$ . Fix  $\omega^t \in \Omega^t$  such that  $C_t(\omega^t) < +\infty$ . Then,  $c_t^{P*}(\omega^t) = \mathbb{E}_{q_{t+1}^{P*}(\cdot|\omega^t)} C_{t+1}(\omega^t, \cdot) \leq C_t(\omega^t) < +\infty$ , see (67). Moreover, the left-hand sides of (59) and (60) in Proposition 5 show that the inequalities in (31) and (32) are indeed satisfied for  $\omega_{t+1} \in \{C_{t+1}(\omega^t, \cdot) < +\infty\}$  in the robust  $(t+1)$  context. Thus,  $\{\omega^t \in \Omega^t, C_t(\omega^t) < +\infty\} \subset \Omega_{11}^t$ . So, (54) in Proposition 4 shows that  $\Omega_{11}^t$  is a  $\mathcal{Q}^t$ -full-measure set. The same arguments apply for  $U_{t+1}^P$  (using the right-hand sides of (59) and (60) instead of the left-hand sides and  $c_t^P(\omega^t)$  instead of  $c_t^{t,P}(\omega^t)$ ) and  $\Omega_{11}^{t,P}$  is also a  $\mathcal{Q}^t$ -full-measure set.

Note that, as  $\mathcal{H}^T \subset \mathcal{Q}^T$  (see Lemma 1),  $\Omega_i^t$  and  $\Omega_i^{t,P}$  are also  $P^t$ -full-measure sets for all  $i \in \{8, 9, 11\}$ .

Let  $0 \leq t \leq T-1$  and  $i \in \{10, 12\}$ . The set  $\Omega_i^t$  is of  $\mathcal{Q}^t$ -full-measure and  $\Omega_i^{t,P}$  is of  $P^t$ -full-measure if we also assume Assumptions 5, 6 and 7.

The assertions for  $\Omega_{10}^t$  and  $\Omega_{10}^{t,P}$  are proved in Lemma 13 in the Appendix. Recall  $\Omega_{12}^t$  and  $\Omega_{12}^{t,P}$  from Definition 7. We prove by backward induction that  $\Omega_{12}^t$  is a  $\mathcal{Q}^t$ -full-measure set and that  $\Omega_{12}^{t,P}$  is a  $P^t$ -full-measure set for  $P \in \mathcal{H}^T$ . The initialization step at  $T-1$  is a direct consequence of (117) in Lemma 14 and of the fact that  $\Omega_{qNA}^{T-1, P*}$  and  $\Omega_{qNA}^{T-1, P}$  are  $\mathcal{Q}^{T-1}$ -full-measure sets (see Lemma 2). Assume now that the induction hypothesis holds true for some  $1 \leq t \leq T-1$ . We have already proved for all  $i \in \{8, 9, 10, 11\}$  that the set  $\Omega_i^{t,P}$  is of  $P^t$ -full-measure for any  $P \in \mathcal{H}^T$ . So, the induction hypothesis implies that  $\tilde{\Omega}^{t,P}$  (see (68)) is also a  $P^t$ -full-measure set for all  $P \in \mathcal{H}^T$  and Lemma 14 can be applied for  $t$ . Thus, (118) and the fact that  $\Omega_{qNA}^{t-1, P*}$  and  $\Omega_{qNA}^{t-1, P}$  are  $\mathcal{Q}^{t-1}$ -full-measure sets show the heredity step. This concludes the backward induction.

Finally, under all the assumptions,  $\tilde{\Omega}^t$  is a  $\mathcal{Q}^t$ -full-measure set and we choose  $\hat{\Omega}^t \in \mathcal{B}(\Omega^t)$  such that  $\hat{\Omega}^t$  is a  $\mathcal{Q}^t$ -full-measure set and  $\hat{\Omega}^t \subset \tilde{\Omega}^t$ .  $\square$

**5. Proof of Theorem 1** We now turn to the proof of Theorem 1. First, we show in Proposition 7 that for all  $0 \leq t \leq T-1$ , there exists a projectively measurable optimal investment strategy at time  $t$  for problem (18) when starting with a cash position  $x$  (see (69)). Proposition 7 is not provable in the usual ZFC theory as  $U_t$  is only shown to be projectively measurable. Our proof is simpler than the one of [12, Proposition 8]. Indeed, as measurable selection is now performed on projectively measurable function, there is no need to show that some mathematical objects are normal integrands (see [40, Definition 14.27]). Nevertheless, we have to extend the result on  $D^{t+1}$  of [8, Lemma 2.6] to projective sets (see Proposition 13).

**PROPOSITION 7.** *Assume the (PD) axiom. Assume that Assumptions 1, 2, 3, 4, 5, 6 and 7 hold true. Let  $0 \leq t \leq T-1$ . There exists a projectively measurable function  $H_{t+1}^* : \Omega^t \times \mathbb{R} \rightarrow \mathbb{R}^d$  such that  $H_{t+1}(\omega^t, \cdot) \in \text{Aff}(D^{t+1})(\omega^t)$  for all  $\omega^t \in \hat{\Omega}^t$  (where  $\hat{\Omega}^t$  has been defined in Proposition 6). Moreover, for all  $\omega^t \in \hat{\Omega}^t$  and  $x \in \mathbb{R}$ ,*

$$U_t(\omega^t, x) \leq \text{Cl}(U_t)(\omega^t, x) \leq u_t^{cl}(\omega^t, x) = \sup_{h \in \mathbb{R}^d} \text{Cl}(u_t)(\omega^t, x, h) = \text{Cl}(u_t)(\omega^t, x, H_{t+1}^*(\omega^t, x)), \quad (69)$$

where  $u_t$  is defined in (16) and  $u_t^{cl}$  in (18).

*Proof.* We first remark that  $u_t$  is projectively measurable (see Lemma 12 and (16)). We now show that  $\text{Cl}(u_t)$  is projectively measurable. For all  $(\omega^t, x, h) \in \Omega^t \times \mathbb{R} \times \mathbb{R}^d$ , we have that

$$\text{Cl}(u_t)(\omega^t, x, h) = \inf_{\substack{p \geq 1 \\ q \geq 1}} \sup_{|x'| < \frac{1}{p}, |h'| < \frac{1}{q}} u_t(\omega^t, x + x', h + h'). \quad (70)$$

Note that  $(\omega^t, x, h, x', h') \mapsto u_t(\omega^t, x + x', h + h')$  is projectively measurable (see Lemma 10). Let  $D_{pq} := \Omega^t \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{B}(0, 1/p) \times \mathbb{B}(0, 1/q)$  where for all  $r > 0$ ,  $\mathbb{B}(0, r)$  is an open ball (in  $\mathbb{R}$  or  $\mathbb{R}^d$ ) centered at 0 and of radius  $r$ . Then,  $D_{pq}$  is a Borel set and also a projective set (see Proposition 8 (vi)). Thus, Proposition 10 implies that for all  $p, q \geq 1$ ,  $(\omega^t, x, h) \mapsto \sup_{|x'| < 1/p, |h'| < 1/q} u_t(\omega^t, x + x', h + h')$  is projectively measurable. So, using (70) and Lemma 8 we get that  $\text{Cl}(u_t)$  is also projectively measurable. Now, choosing  $D := \Omega^t \times \mathbb{R} \times \mathbb{R}^d$  (which is a projective set) in Proposition 10 proves that  $\sup_{l \in \mathbb{R}^d} \text{Cl}(u_t)(\cdot, \cdot, l)$  is projectively measurable. So, Lemma 8 implies that  $(\omega^t, x, h) \mapsto \sup_{l \in \mathbb{R}^d} \text{Cl}(u_t)(\omega^t, x, l) - \text{Cl}(u_t)(\omega^t, x, h)$  is projectively measurable. Let

$$\bar{A} := \left\{ (\omega^t, x, h) \in \Omega^t \times \mathbb{R} \times \mathbb{R}^d, \sup_{l \in \mathbb{R}^d} \text{Cl}(u_t)(\omega^t, x, l) = \text{Cl}(u_t)(\omega^t, x, h) \right\}.$$

We have proved that  $\bar{A} \in \mathbf{P}(\Omega^t \times \mathbb{R} \times \mathbb{R}^d)$ . Let  $A := T^{-1}(\text{Graph}(\text{Aff}(D^{t+1})) \cap \bar{A})$  where  $T(\omega^t, x, h) := (\omega^t, h)$ . As  $T$  is Borel measurable, using Propositions 13 and 8 (vi), we find that  $A \in \mathbf{P}(\Omega^t \times \mathbb{R} \times \mathbb{R}^d)$ . Proposition 9 applied to  $A$  shows the existence of a projectively measurable function  $H_{t+1} : \text{proj}_{\Omega^t \times \mathbb{R}}(A) \rightarrow \mathbb{R}^d$  such that for all  $(\omega^t, x) \in \text{proj}_{\Omega^t \times \mathbb{R}}(A)$ ,  $H_{t+1}(\omega^t, x) \in \text{Aff}(D^{t+1})(\omega^t)$  and

$$\sup_{h \in \mathbb{R}^d} \text{Cl}(u_t)(\omega^t, x, h) = \text{Cl}(u_t)(\omega^t, x, H_{t+1}(\omega^t, x)). \quad (71)$$

Let  $(\omega^t, x) \in \widehat{\Omega}^t \times \mathbb{R}$ . Note that the set  $\widehat{\Omega}^t \subset \widetilde{\Omega}^t$  introduced in Proposition 6 does not depend from  $x$ . As  $\omega^t \in \widehat{\Omega}^t$ , Lemma 4 shows that Assumptions 8 to 12 hold true in the robust  $(t+1)$  context and (50) in Proposition 3 shows immediately that  $(\omega^t, x) \in \text{proj}_{\Omega^t \times \mathbb{R}}(A)$ . Note that (50) in Proposition 3 also shows the inequalities in (69) for such an  $\omega^t$ . As  $\widehat{\Omega}^t \times \mathbb{R} \subset \text{proj}_{\Omega^t \times \mathbb{R}}(A)$ , we can extend  $H_{t+1}$  on  $\Omega^t \times \mathbb{R}$  as follows. Let  $H_{t+1}^*$  be defined by  $H_{t+1}^*(\omega^t, x) := H_{t+1}(\omega^t, x) 1_{\widehat{\Omega}^t}(\omega^t)$  for all  $(\omega^t, x) \in \Omega^t \times \mathbb{R}$ . As  $\widehat{\Omega}^t \in \mathcal{B}(\Omega^t)$ ,  $H_{t+1}^*$  remains projectively measurable, see Lemmata 5 (i) and 8. Moreover, for all  $(\omega^t, x) \in \widehat{\Omega}^t \times \mathbb{R}$ ,  $H_{t+1}^*(\omega^t, x) = H_{t+1}(\omega^t, x)$  and (69) remains true.  $\square$

We are now in position to prove Theorem 1.

*Proof of Theorem 1* Using Assumption 6, we show first that an admissible strategy in a  $P$ -prior context leads to a finite value function. Then, we show that  $U_0(x) \geq u(x)$  and that the admissibility of a given subset of strategies is stable in time. After that, we prove that there exists an appropriate strategy satisfying (19). Finally, we show that when this strategy is admissible, (21) holds true. We now define another set of admissible strategies on which we will do some computations in the proof. Let for all  $0 \leq t \leq T$  and  $x \in \mathbb{R}$ ,

$$\widehat{\Phi}_{|t}(x, U, \mathcal{Q}^T) := \left\{ \phi \in \Phi, \inf_{P \in \mathcal{Q}^t} \mathbb{E}_P U_t(\cdot, V_t^{x, \phi}(\cdot)) > -\infty \right\}.$$

We have that  $\widehat{\Phi}_{|t}(x, U_t, \mathcal{Q}^t) \subset \Phi_{|t}(x, U_t, \mathcal{Q}^t)$  where  $\Phi_{|t}(x, U_t, \mathcal{Q}^t)$  is the set of admissible strategies for the random utility  $U_t$  with time horizon  $t$ , see Definition 3. Indeed, let  $\phi \in \widehat{\Phi}_{|t}(x, U_t, \mathcal{Q}^t)$  and assume that  $\phi \notin \Phi_{|t}(x, U_t, \mathcal{Q}^t)$ . Then, there exists some  $P \in \mathcal{Q}^t$  such that  $\mathbb{E}_P U_t^-(\cdot, V_t^{x, \phi}(\cdot)) = +\infty$ . Recalling convention (10),  $\mathbb{E}_P U_t(\cdot, V_t^{x, \phi}(\cdot)) = -\infty$  and also  $\inf_{P \in \mathcal{Q}^t} \mathbb{E}_P U_t(\cdot, V_t^{x, \phi}(\cdot)) = -\infty$ , a contradiction. Moreover, we still trivially have that

$$u(x) = \sup_{\phi \in \widehat{\Phi}_{|T}(x, U, \mathcal{Q}^T)} \inf_{P \in \mathcal{Q}^T} \mathbb{E}_P U(\cdot, V_T^{x, \phi}(\cdot)).$$

*Finiteness of  $\mathbb{E}_{P_t} U_t(\cdot, V_t^{x, \phi}(\cdot))$  for  $0 \leq t \leq T$ ,  $P \in \mathcal{H}^T$ ,  $x \in \mathbb{R}$  and  $\phi \in \Phi(x, U, \mathcal{Q}^T)$ .*

For all  $0 \leq t \leq T$ ,  $P \in \mathcal{H}^T$ ,  $x \in \mathbb{R}$ ,  $\phi \in \Phi(x, U, \mathcal{Q}^T)$  and  $\omega^t \in \Omega^t$ , we have using (15) that,

$$U_t^P(\omega^t, V_t^{x, \phi}(\omega^t)) \geq \mathbb{E}_{q_{t+1}^P(\cdot | \omega^t)} U_{t+1}^P(\omega^t, \cdot, V_{t+1}^{x, \phi}(\omega^t, \cdot)) \geq -\mathbb{E}_{q_{t+1}^P(\cdot | \omega^t)} (U_{t+1}^P)^-(\omega^t, \cdot, V_{t+1}^{x, \phi}(\omega^t, \cdot)). \quad (72)$$

We show that for all  $0 \leq t \leq T$ ,  $x \in \mathbb{R}$ ,  $P \in \mathcal{H}^T$  and  $\phi \in \Phi(x, U, \mathcal{Q}^T)$ , we have that,

$$\mathbb{E}_{P^t}(U_t^P)^-(\cdot, V_t^{x,\phi}(\cdot)) < +\infty \text{ and } \mathbb{E}_{P^t}(U_t^P)^+(\cdot, V_t^{x,\phi}(\cdot)) < +\infty. \quad (73)$$

First, we show by backward induction that,  $\mathbb{E}_{P^t}(U_t^P)^-(\cdot, V_t^{x,\phi}(\cdot)) < +\infty$ . The initialization step follows from the definition of  $\Phi(x, U, \mathcal{Q}^T)$  and  $U_T = U$ . Let  $0 \leq t \leq T - 1$  and assume that the induction hypothesis holds true at time  $t + 1$ . Then, (72) implies that  $(U_t^P)^-(\omega^t, V_t^{x,\phi}(\omega^t)) \leq \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)}(U_{t+1}^P)^-(\omega^t, \cdot, V_{t+1}^{x,\phi}(\omega^t, \cdot))$  and Fubini's theorem together with the induction hypothesis show that

$$\mathbb{E}_{P^t}(U_t^P)^-(\cdot, V_t^{x,\phi}(\cdot)) \leq \mathbb{E}_{P^{t+1}}(U_{t+1}^P)^-(\cdot, V_{t+1}^{x,\phi}(\cdot)) < +\infty.$$

This concludes the proof of the left-hand side of (73). We now prove the right-hand side by forward induction. For the initialization step, we show that  $U_0^P(x) < +\infty$  for all  $x \in \mathbb{R}$ . As  $U_0^P(1) < +\infty$  (by Assumption 6) and  $U_0^P$  is nondecreasing (see Proposition 5), we deduce that for all  $x \leq 1$ , that  $U_0^P(x) < +\infty$ . Now, using the right-hand side of (59), we find that for all  $x > 1$ ,  $U_0^P(x) \leq x^{\bar{\gamma}}(U_0^P(1) + C_0) < +\infty$  as  $C_0 < +\infty$  by Proposition 4. This shows the initialization step as  $(U_0^P)^+ = U_0^P + (U_0^P)^-$  and  $(U_0^P)^- < +\infty$  from the first induction. Let  $0 \leq t \leq T - 1$  and assume that the induction hypothesis holds true at time  $t$ . Thus, using (72), we get that

$$\begin{aligned} \mathbb{E}_{P^{t+1}}U_{t+1}^P(\cdot, V_{t+1}^{x,\phi}(\cdot)) &= \mathbb{E}_{P^t} \left( \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)}U_{t+1}^P(\omega^t, \cdot, V_{t+1}^{x,\phi}(\omega^t, \cdot)) \right) \leq \mathbb{E}_{P^t}U_t^P(\cdot, V_t^{x,\phi}(\cdot)) \\ &\leq \mathbb{E}_{P^t}(U_t^P)^+(\cdot, V_t^{x,\phi}(\cdot)) < +\infty, \end{aligned}$$

where we have used for the first equality Fubini's theorem which can be applied because of the left-hand side of (73) at  $t + 1$ . This concludes the second induction and shows (73) at  $t + 1$ .

*Upper bound for  $u(x)$  and stability of  $\widehat{\Phi}_{|T}$ .*

We show that for any  $x \in \mathbb{R}$ ,

$$U_0(x) \geq \sup_{\phi \in \widehat{\Phi}_{|T}(x, U, \mathcal{Q}^T)} \inf_{P \in \mathcal{Q}^T} \mathbb{E}_P U(\cdot, V_T^{x,\phi}(\cdot)) = u(x), \quad (74)$$

and that if  $\phi \in \widehat{\Phi}_{|T}(x, U, \mathcal{Q}^T)$ , then  $\phi \in \widehat{\Phi}_{|t}(x, U_t, \mathcal{Q}^t)$  for all  $0 \leq t \leq T$ . Fix  $x \in \mathbb{R}$  and  $\phi \in \widehat{\Phi}_{|T}(x, U, \mathcal{Q}^T)$ . We proceed by backward induction with the following induction hypothesis :

$$\phi \in \widehat{\Phi}_{|t}(x, U_t, \mathcal{Q}^t) \text{ and } \inf_{P \in \mathcal{Q}^t} \mathbb{E}_P U(\cdot, V_T^{x,\phi}(\cdot)) \leq \inf_{P \in \mathcal{Q}^t} \mathbb{E}_P U_t(\cdot, V_t^{x,\phi}(\cdot)). \quad (75)$$

The initialization step is trivial as  $U_T = U$ . Let  $0 \leq t \leq T - 1$  and assume that the induction hypothesis holds true at time  $t + 1$ . Proposition 5 shows that  $U_{t+1}$  is projectively measurable and Lemma 12 for  $f = U_{t+1}$  that

$$(\omega^t, x, h, p) \mapsto \bar{u}_t(\omega^t, x, h, p) := \mathbb{E}_p U_{t+1}(\omega^t, \cdot, x + h\Delta S_{t+1}(\omega^t, \cdot)) \quad (76)$$

is projectively measurable. Note that  $\bar{u}_t(\omega^t, \cdot, \cdot, p)$  is equal to  $\Psi_p$  (see (34)) in the robust ( $t + 1$ ) context. Let  $\epsilon > 0$ . Proposition 10 applied to the projective set  $D = \{(\omega^t, x, h, p), (\omega^t, p) \in \text{Graph}(\mathcal{Q}_{t+1})\}$  (see Assumption 2 and (102) in Proposition 8) show that there exists a projectively measurable function  $q_{t+1}^\epsilon : \Omega^t \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathfrak{P}(\Omega_{t+1})$  such that  $\forall (\omega^t, x, h) \in \Omega^t \times \mathbb{R} \times \mathbb{R}^d$  (recall that  $\mathcal{Q}_{t+1} \neq \emptyset$  and  $\text{proj}_{\Omega^t \times \mathbb{R} \times \mathbb{R}^d}(D) = \Omega^t \times \mathbb{R} \times \mathbb{R}^d$ ),  $q_{t+1}^\epsilon(\cdot|\omega^t, x, h) \in \mathcal{Q}_{t+1}(\omega^t)$  and

$$\bar{u}_t(\omega^t, x, h, q_{t+1}^\epsilon(\cdot|\omega^t, x, h)) \leq -\frac{1}{\epsilon} \mathbf{1}_{\{u_t(\omega^t, x, h) = -\infty\}} + (u_t(\omega^t, x, h) + \epsilon) \mathbf{1}_{\{u_t(\omega^t, x, h) > -\infty\}}, \quad (77)$$

as  $u_t(\omega^t, x, h) = \inf_{p \in \mathcal{Q}_{t+1}(\omega^t)} \bar{u}_t(\omega^t, x, h, p)$ , see (16). Fix  $\omega^t \in \Omega^t$ . As  $U_t(\omega^t, x) \geq u_t(\omega^t, x, h)$  for all  $(x, h) \in \mathbb{R} \times \mathbb{R}^d$ , using (77) with  $x = V_t^{x, \phi}(\omega^t)$  and  $h = \phi_{t+1}(\omega^t)$  and setting  $\bar{q}_{t+1}^\epsilon(\cdot | \omega^t) := q_{t+1}^\epsilon(\cdot | \omega^t, V_t^{x, \phi}(\omega^t), \phi_{t+1}(\omega^t))$ , we have that for all  $\omega^t \in \Omega^t$ ,

$$\bar{u}_t(\omega^t, V_t^{x, \phi}(\omega^t), \phi_{t+1}(\omega^t), \bar{q}_{t+1}^\epsilon(\cdot | \omega^t)) \leq -\frac{1}{\epsilon} \mathbf{1}_{\{u_t(\omega^t, V_t^{x, \phi}(\omega^t), \phi_{t+1}(\omega^t)) = -\infty\}} + (U_t(\omega^t, V_t^{x, \phi}(\omega^t)) + \epsilon) \mathbf{1}_{\{u_t(\omega^t, V_t^{x, \phi}(\omega^t), \phi_{t+1}(\omega^t)) > -\infty\}}. \quad (78)$$

We prove now that  $\bar{q}_{t+1}^\epsilon$  is a projectively measurable stochastic kernel and that  $\bar{q}_{t+1}^\epsilon(\cdot | \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$  for all  $\omega^t \in \Omega^t$ . First,  $V_t^{x, \phi} = x + \sum_{s=1}^t \phi_s \Delta S_s$  is projectively measurable. Indeed, by Assumption 1 and Lemma 7 (ii),  $\Delta S_t$  is projectively measurable. As  $\phi \in \Phi$ ,  $\phi_t$  is projectively measurable and so is  $\phi_t \Delta S_t$  and  $V_t^{x, \phi}$  using Lemma 7 (ii) again. As  $(\omega^t, x, h) \mapsto q_{t+1}^\epsilon(\cdot | \omega^t, x, h)$  is projectively measurable, we obtain using Lemmata 6 and 10 (iii) that  $\omega^t \mapsto \bar{q}_{t+1}^\epsilon(\cdot | \omega^t)$  is projectively measurable and so that  $\bar{q}_{t+1}^\epsilon$  is a projectively measurable stochastic kernel. Now, we get that for  $\omega^t \in \Omega^t$ ,  $\bar{q}_{t+1}^\epsilon(\cdot | \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$  as  $q_{t+1}^\epsilon(\cdot | \omega^t, x, h) \in \mathcal{Q}_{t+1}(\omega^t)$  for all  $(\omega^t, x, h) \in \Omega^t \times \mathbb{R} \times \mathbb{R}^d$ .

Let  $P := q_1^P \otimes \dots \otimes q_T^P \in \mathcal{Q}^T$  and  $P^* := q_1^{P^*} \otimes \dots \otimes q_T^{P^*} \in \mathcal{H}^T$  defined in (61). We set

$$\bar{P} := \frac{q_1^{P^*} + q_1^P}{2} \otimes \dots \otimes \frac{q_t^{P^*} + q_t^P}{2} \otimes q_{t+1}^{P^*} \otimes \dots \otimes q_T^{P^*}.$$

Then  $\bar{P} \in \mathcal{H}^T$  (see (9)). Taking the expectation in (78) under  $\bar{P}^t$  and using Fubini's theorem as  $\phi \in \hat{\phi}_{|t+1}(x, U_{t+1}, \mathcal{Q}^{t+1}) \subset \phi_{|t+1}(x, U_{t+1}, \mathcal{Q}^{t+1})$ , we get that

$$\mathbb{E}_{\bar{P}^t \otimes \bar{q}_{t+1}^\epsilon} U_{t+1}(\cdot, V_{t+1}^{x, \phi}(\cdot)) \leq -\frac{1}{\epsilon} \bar{P}^t(u_t(\cdot, V_t^{x, \phi}(\cdot), \phi_{t+1}(\cdot)) = -\infty) + \mathbb{E}_{\bar{P}^t} U_t^+(\cdot, V_t^{x, \phi}(\cdot)) + \epsilon. \quad (79)$$

As  $\bar{P}^t \otimes q^\epsilon \in \mathcal{Q}^{t+1}$ , we obtain that

$$\inf_{P \in \mathcal{Q}^{t+1}} \mathbb{E}_P U_{t+1}(\cdot, V_{t+1}^{x, \phi}(\cdot)) \leq -\frac{1}{\epsilon} \bar{P}^t(u_t(\cdot, V_t^{x, \phi}(\cdot), \phi_{t+1}(\cdot)) = -\infty) + \mathbb{E}_{\bar{P}^t} U_t^+(\cdot, V_t^{x, \phi}(\cdot)) + \epsilon. \quad (80)$$

Recall from (73) and from  $U_t \leq U_t^P$  (see Proposition 5) that  $\mathbb{E}_{\bar{P}^t} U_t^+(\cdot, V_t^{x, \phi}(\cdot)) < +\infty$ . So, if  $\bar{P}^t(u_t(\cdot, V_t^{x, \phi}(\cdot), \phi_{t+1}(\cdot)) = -\infty) > 0$ , taking the limit when  $\epsilon$  goes to 0 in (80) gives that  $\inf_{P \in \mathcal{Q}^{t+1}} \mathbb{E}_P U_{t+1}(\cdot, V_{t+1}^{x, \phi}(\cdot)) = -\infty$ , a contradiction with  $\phi \in \hat{\Phi}_{|t+1}(x, U_{t+1}, \mathcal{Q}^{t+1})$  of the induction hypothesis. Thus,  $\bar{P}^t(u_t(\cdot, V_t^{x, \phi}(\cdot), \phi_{t+1}(\cdot)) = -\infty) = 0$ . Using now [12, Proposition 12], we have that  $P^t \ll \bar{P}^t$  and  $P^t(u_t(\cdot, V_t^{x, \phi}(\cdot), \phi_{t+1}(\cdot)) = -\infty) = 0$ . As  $P$  is arbitrary, we have that

$$u_t(\cdot, V_t^{x, \phi}(\cdot), \phi_{t+1}(\cdot)) > -\infty \quad \mathcal{Q}^t\text{-q.s.}$$

Thus, taking again the expectation in (78) but under  $P^t$  and using Fubini's theorem, we get that for all  $P^t \in \mathcal{Q}^t$

$$\inf_{P \in \mathcal{Q}^{t+1}} \mathbb{E}_P U_{t+1}(\cdot, V_{t+1}^{x, \phi}(\cdot)) \leq \mathbb{E}_{P^t} U_t(\cdot, V_t^{x, \phi}(\cdot)) + \epsilon.$$

Taking the infimum over all  $P^t \in \mathcal{Q}^t$  on the right-hand side, letting  $\epsilon$  go to 0 and using the second part of the induction hypothesis (75), we get that

$$\inf_{P \in \mathcal{Q}^T} \mathbb{E}_P U(\cdot, V_T^{x, \phi}(\cdot)) \leq \inf_{P \in \mathcal{Q}^{t+1}} \mathbb{E}_P U_{t+1}(\cdot, V_{t+1}^{x, \phi}(\cdot)) \leq \inf_{P \in \mathcal{Q}^t} \mathbb{E}_P U_t(\cdot, V_t^{x, \phi}(\cdot)).$$

In particular, as  $\phi \in \hat{\Phi}_{|t+1}(x, U_{t+1}, \mathcal{Q}^{t+1})$ ,  $-\infty < \inf_{P \in \mathcal{Q}^{t+1}} \mathbb{E}_P U_{t+1}(\cdot, V_{t+1}^{x, \phi}(\cdot)) \leq \inf_{P \in \mathcal{Q}^t} \mathbb{E}_P U_t(\cdot, V_t^{x, \phi}(\cdot))$ , and so that  $\phi \in \hat{\phi}_{|t}(x, U_t, \mathcal{Q}^t)$ . This concludes the induction.



Now, using (75) at  $t = 0$  shows that  $\inf_{P \in \mathcal{Q}^T} \mathbb{E}_P U(\cdot, V_T^{x, \phi}(\cdot)) \leq U_0(x)$  for  $\phi \in \widehat{\Phi}|_T(x, U, \mathcal{Q}^T)$ . Thus, (74) follows by taking the supremum over all such  $\phi$ .

*Existence of a one-step optimal strategy.*

Let  $x \in \mathbb{R}$ . We define recursively the strategy  $\phi^{*,x}$  as follows. Let  $\phi_1^{*,x} := x$  and for all  $1 \leq t \leq T-1$  and  $\omega^t \in \Omega^t$ ,  $\phi_{t+1}^{*,x}(\omega^t) := H_{t+1}^*(\omega^t, x + \sum_{s=1}^t \phi_s^{*,x}(\omega^s) \Delta S_s(\omega^t))$  where  $H_{t+1}^* : \Omega^t \times \mathbb{R} \rightarrow \mathbb{R}^d$  is defined in Proposition 7. Let  $1 \leq t \leq T-1$ . Proposition 7 shows that for all  $(\omega^t, x) \in \widehat{\Omega}^t \times \mathbb{R}$ ,  $\phi_{t+1}^{*,x}(\omega^t) \in \text{Aff}(D^{t+1})(\omega^t)$  and that (69) holds true for  $x = V_t^{x, \phi^{*,x}}(\omega^t)$  (recall that  $\widehat{\Omega}^t$  does not depend from  $x$ ). Thus, (19) holds also true.

Now, we show by induction that  $\phi_{t+1}^{*,x}$  is projectively measurable for all  $0 \leq t \leq T-1$ . At  $t = 0$ , this is trivial (recall that  $\Omega^0$  is a singleton). Suppose that this holds true for all  $0 \leq k \leq t-1$ . Then, recalling Assumption 1,  $V_t^{x, \phi^{*,x}} = x + \sum_{k=1}^t \phi_k^{*,x}(S_k - S_{k-1})$  is projectively measurable (see Lemma 7 (ii)). Thus, as  $H_{t+1}^*$  is projectively measurable (see Proposition 7), Lemmata 6 and 10 (iii) show that  $\phi_{t+1}^{*,x} = H_{t+1}^*(\cdot, V_t^{x, \phi^{*,x}}(\cdot))$  is projectively measurable. This concludes the induction and  $\phi^{*,x} \in \Phi$ .

*Optimality of an admissible one-step optimal strategy.*

Assume now that  $\phi^{*,x} \in \Phi(x, U, \mathcal{Q}^T)$ . Let  $0 \leq t \leq T-1$  and  $P^* := q_1^{P^*} \otimes \cdots \otimes q_T^{P^*} \in \mathcal{H}^T$  as in (61). Let  $U_t^{cl} : \Omega^t \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  be defined by

$$U_t^{cl}(\omega^t, x, h) := \mathbb{E}_{q_{t+1}^{P^*}(\cdot|\omega^t)} \text{Cl}(U_{t+1})(\omega^t, \cdot, x + h \Delta S_{t+1}(\omega^t, \cdot)).$$

Note that  $u_t \leq u_t^{P^*} \leq U_t^{cl}$  by definition of the closure of  $U_{t+1}$  (see (15) and (16)) and that  $U_t^{cl}$  is projectively measurable. Indeed,  $\text{Cl}(U_{t+1})$  is projectively measurable (see Proposition 5 (iv)) and applying Lemma 12 to  $f = \text{Cl}(U_{t+1})$  gives the desired result. We have proved in the preceding step that (19) holds true. Thus, we have for all  $\omega^t \in \widehat{\Omega}^t$  that

$$\text{Cl}(U_t)(\omega^t, V_t^{x, \phi^{*,x}}(\omega^t)) \leq \text{Cl}(u_t)(\omega^t, V_t^{x, \phi^{*,x}}(\omega^t), \phi_{t+1}^{*,x}(\omega^t)) \leq \text{Cl}(U_t^{cl})(\omega^t, V_t^{x, \phi^{*,x}}(\omega^t), \phi_{t+1}^{*,x}(\omega^t)). \quad (81)$$

Now, we show that for all  $\omega^t \in \widehat{\Omega}^t$ ,  $U_t^{cl}(\omega^t, \cdot, \cdot)$  is usc and so that  $\text{Cl}(U_t^{cl})(\omega^t, \cdot, \cdot) = U_t^{cl}(\omega^t, \cdot, \cdot)$ . Remark that  $U_t^{cl}(\omega^t, \cdot, \cdot) = \Psi_{P^*}^{cl}(\cdot, \cdot)$  in the robust  $(t+1)$  context where  $\Psi_{P^*}^{cl}$  is defined in (38). As Assumptions 8, 9, 10 and 11 are satisfied in the robust  $(t+1)$  context for  $\omega^t \in \widehat{\Omega}^t$  (see Lemma 4), Proposition 1 shows that  $U_t^{cl}(\omega^t, \cdot, \cdot)$  is usc. Thus, (81) implies that

$$\text{Cl}(U_t)(\omega^t, V_t^{x, \phi^{*,x}}(\omega^t)) \leq U_t^{cl}(\omega^t, V_t^{x, \phi^{*,x}}(\omega^t), \phi_{t+1}^{*,x}(\omega^t)) = \mathbb{E}_{q_{t+1}^{P^*}(\cdot|\omega^t)} \text{Cl}(U_{t+1})(\omega^t, \cdot, V_{t+1}^{x, \phi^{*,x}}(\omega^t, \cdot)) \quad (82)$$

Applying recursively (82) from  $t = 0$  to  $t = T-1$ , we obtain that

$$\text{Cl}(U_0)(x) \leq \int_{\Omega_1} \cdots \int_{\Omega_T} \text{Cl}(U)(\omega^T, V_T^{x, \phi^{*,x}}(\omega^T)) q_T^{P^*}(d\omega^T | \omega_{T-1}) \cdots q_1^{P^*}(d\omega_1). \quad (83)$$

Now  $(\text{Cl}(U))^- \leq U^-$  and  $\phi^{*,x} \in \Phi(x, U, \mathcal{Q}^T)$ , we get that for all  $Q \in \mathcal{Q}^T$ ,

$$\mathbb{E}_Q(\text{Cl}(U))^- (\cdot, V_T^{x, \phi^{*,x}}(\cdot)) \leq \mathbb{E}_Q U^- (\cdot, V_T^{x, \phi^{*,x}}(\cdot)) < +\infty, \quad (84)$$

see (73) as  $U_T^P = U$ . Thus, we can apply Fubini's theorem in (83) and we obtain that  $\text{Cl}(U_0)(x) \leq \mathbb{E}_{P^*} \text{Cl}(U)(\cdot, V_T^{x, \phi^{*,x}}(\cdot))$ . As  $P^*$  is arbitrary in  $\mathcal{H}^T$ , we have that

$$\text{Cl}(U_0)(x) \leq \inf_{P \in \mathcal{H}^T} \mathbb{E}_P \text{Cl}(U)(\cdot, V_T^{x, \phi^{*,x}}(\cdot)). \quad (85)$$

Now, using Proposition 5 (v), we have that for all  $x \in \mathbb{R}$  and for all  $P \in \mathcal{H}^T$ ,

$$\mathbb{E}_P(\text{Cl}(U))^+ (\cdot, V_T^{x, \phi^{*,x}}(\cdot)) \leq \mathbb{E}_P U^+ (\cdot, V_T^{x+1, \phi^{*,x}}(\cdot)) < +\infty, \quad (86)$$

see again (73). By definition of the closure of  $U_0$  and recalling (85), we get that

$$U_0(x) \leq \text{Cl}(U_0)(x) \leq \inf_{P \in \mathcal{H}^T} \mathbb{E}_P \text{Cl}(U)(\cdot, V_T^{x, \phi^{*,x}}(\cdot)) = \inf_{Q \in \mathcal{Q}^T} \mathbb{E}_Q \text{Cl}(U)(\cdot, V_T^{x, \phi^{*,x}}(\cdot)),$$

where the last equality follows from Lemma 15 applied to  $X = \text{Cl}(U)(\cdot, V_T^{x, \phi^{*,x}}(\cdot))$  as (84) and (86) hold true. Finally, let  $x \in \mathbb{R}$ . Then,

$$\inf_{P \in \mathcal{Q}^T} \mathbb{E}_P U(\cdot, V_T^{x, \phi^{*,x}}(\cdot)) \leq u(x) \leq \inf_{P \in \mathcal{Q}^T} \mathbb{E}_P U(\cdot, V_T^{x, \phi^{*,x}}(\cdot)) + \sup_{P \in \mathcal{Q}^T} \mathbb{E}_P \Delta_+ U(\cdot, V_T^{x, \phi^{*,x}}(\cdot)),$$

where the first inequality follows from the definition of  $u(x)$  as  $\phi^{*,x} \in \Phi(x, U, \mathcal{Q}^T)$  and the second one from (21) and (17). Thus, (20) holds true which concludes the proof.  $\square$

**6. Proof of Theorem 2** In this part, we prove Theorem 2. We want to apply Theorem 1 and verify the different conditions needed for that. More precisely, we prove that if Assumptions 1, 2 and 3 hold true and if  $U$  is a random utility of type (A), Assumptions 6 and 7 hold true. Note that Assumptions 4 and 5 hold true by definition of a random utility of type (A). For that, we use the results of Section 4 for  $\Omega^{t,P}$  and  $U_t^P$ . The proof follows the same path than the proof of [12, Theorem 2]. Nevertheless, there are three main differences. First, the expression of  $c_t^P$  is different of the one of [12, (61)]. Second, we have for the moment introduced no control from below on the value function  $U_t$ . Third, Assumption 13 is now needed to get a bound on the optimal strategy in Proposition 2. So, we introduce for all  $0 \leq t \leq T-1$  and  $P := q_1^P \otimes \dots \otimes q_T^P \in \mathcal{H}^T$ , the set of paths  $\omega^t$  for which Assumption 13 holds true (recall (15), (16) and Definition 7) :

$$\Omega_{13}^t := \{\omega^t \in \Omega^t, u_t(\omega^t, -k, 0) = \inf_{p \in \mathcal{Q}_{t+1}(\omega^t)} \mathbb{E}_p U_{t+1}(\omega^t, \cdot, -k) > -\infty, \forall k \in \mathbb{N} \setminus \{0\}\} \quad (87)$$

$$\Omega_{13}^{t,P} := \{\omega^t \in \Omega^t, u_t^P(\omega^t, -k, 0) = \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)} U_{t+1}^P(\omega^t, \cdot, -k) > -\infty, \forall k \in \mathbb{N} \setminus \{0\}\}. \quad (88)$$

*Proof of Theorem 2.*

*Assumption 6 holds true.*

Let  $P := q_1^P \otimes \dots \otimes q_T^P \in \mathcal{H}^T$ . Recall that  $u_t^P$ ,  $u_t$ ,  $c_t^P$ ,  $i_t^P$ ,  $l_t^P$ ,  $N_t^P$  and  $\tilde{\Omega}^{t,P}$  are defined respectively in (15), (16), (62), (63), (65), (66) and (68) for all  $0 \leq t \leq T-1$ . Recall also that the sets  $\mathcal{M}^t(P)$ ,  $\mathcal{M}^t$  are defined in Definition 4 for all  $0 \leq t \leq T$ . For the need of the induction, we need to define some of them also for  $t = -1$  or  $t = T$  and we set  $u_T^P := 0$ ,  $u_T := 0$ ,  $N_{-1}^P := 0$ ,  $l_T^P := 0$ ,  $\tilde{\Omega}^{T,P} := \Omega^T$  and  $\mathcal{M}^{-1}(P) := \{0\}$ . We also set  $\Omega_{13}^T := \Omega^T$ ,  $\Omega_{13}^{T,P} := \Omega^T$ .

For all  $0 \leq t \leq T$ , we prove by backward induction the following induction hypothesis :  $(U_t^P)^+(\cdot, 1)$  and  $l_t^P$  belong to  $\mathcal{M}^t(P)$ ,  $C_t$ ,  $u_t^-(\cdot, x, 0)$ ,  $(u_t^P)^-(\cdot, x, 0)$  belong to  $\mathcal{M}^t$  for all  $x \in \mathbb{R}$ ,  $N_{t-1}^P \in \mathcal{M}^{t-1}(P)$ ,  $\tilde{\Omega}^{t,P} \cap \Omega_{13}^{t,P}$  is a  $P^t$ -full-measure set and there exists  $C_{1,t} \in \mathcal{M}^t$  such that  $C_{1,t} \geq 0$  and for all  $\omega^T \in \Omega^T$  and for all  $x \in \mathbb{R}$

$$U_t(\omega^t, x) \geq -C_{1,t}(\omega^t)(1 + |x|^p). \quad (89)$$

Then, we will obtain from the induction hypothesis at  $t = 0$  that Assumption 6 holds true. Indeed,  $\mathcal{M}^0(P) = \mathbb{R}$  and  $U_0^P(1) \leq (U_0^P)^+(1) < +\infty$ .

*Initialization step.*

We trivially have that  $l_T^P = 0 \in \mathcal{M}^T(P)$ ,  $u_T^- = (u_T^P)^- = 0 \in \mathcal{M}^T$  and that  $\tilde{\Omega}^{T,P} \cap \Omega_{13}^{T,P} = \Omega^T$  is a  $P$ -full-measure set. Additionally, using Definition 5,  $C_T = C \in \mathcal{M}^T$  (see (53)),  $(U_T^P)^+(\cdot, 1) = U^+(\cdot, 1) \in \mathcal{M}^T \subset \mathcal{M}^T(P)$  and (89) holds true with  $C_{1,T} := C_1$ , see (25). By assumption of Theorem 2, we have that  $1/\alpha_{T-1}^P \in \mathcal{M}^{T-1} \subset \mathcal{M}^{T-1}(P)$ . Again, as  $U$  is of type (A), we know that  $\underline{X}$  and  $1/|U(\cdot, \underline{X}(\cdot)) + C(\cdot)|$  belong to  $\mathcal{M}^T$  and also to  $\mathcal{M}^T(P)$ . Thus, we can use assertion (A1) in Lemma

14 and we get that  $N_{T-1}^P \in \mathcal{M}^{T-1}(P)$ .

Assume that the induction hypothesis holds true at time  $t+1$  for some  $0 \leq t \leq T-1$ .

*Heredity step 1* :  $C_t \in \mathcal{M}^t$ , there exists  $C_{1,t} \in \mathcal{M}^t$  such that  $C_{1,t} \geq 0$  and (89) holds true at  $t$  and  $u_t^-(\cdot, x, 0), (u_t^P)^-(\cdot, x, 0) \in \mathcal{M}^t$  for all  $x \in \mathbb{R}$ .

Using (54), we have that  $C_t < +\infty$   $\mathcal{Q}^t$ -q.s. Thus, (56) with  $\epsilon = 1$  provides some  $q_{t+1} \in SK_{t+1}$  such that for all  $\omega^t \in \Omega^t$ ,  $q_{t+1}(\cdot|\omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$  and for all  $\omega^t$  in the  $\mathcal{Q}^t$ -full-measure set where  $C_t < +\infty$ ,

$$\mathbb{E}_{q_{t+1}(\cdot|\omega^t)} C_{t+1}(\omega^t, \cdot) \geq C_t(\omega^t) - 1. \quad (90)$$

By the induction hypothesis  $C_{t+1} \in \mathcal{M}^{t+1}$  and Lemma 18 shows that  $\omega^t \mapsto \mathbb{E}_{q_{t+1}(\cdot|\omega^t)} C_{t+1}(\omega^t, \cdot)$  belongs to  $\mathcal{M}^t$ . Proposition 4 shows that  $C_t(\cdot)$  is non-negative and projectively measurable. Thus, (90) and Lemma 17 ensure that  $C_t \in \mathcal{M}^t$ .

Now, let  $\omega^t \in \Omega^t$  and  $x \in \mathbb{R}$ , using (16) and (89) at  $t+1$ , we get that

$$U_t(\omega^t, x) \geq u_t(\omega^t, x, 0) = \inf_{p \in \mathcal{Q}_{t+1}(\omega^t)} \mathbb{E}_p U_{t+1}(\omega^t, \cdot, x) \geq -C_{1,t}(\omega^t)(1 + |x|^p), \quad (91)$$

where  $C_{1,t}(\omega^t) := \sup_{p \in \mathcal{Q}_{t+1}(\omega^t)} \mathbb{E}_p C_{1,t+1}(\omega^t, \cdot)$ . As in the proof of Proposition 4, we find that  $C_{1,t}$  is non-negative and projectively measurable. As  $C_{1,t+1} \in \mathcal{M}^{t+1}$ , a very similar reasoning to the one that shows that  $C_t \in \mathcal{M}^t$  proves that  $C_{1,t} \in \mathcal{M}^t$ . So, (89) holds true at  $t$ .

Let  $x \in \mathbb{R}$ . Using (91), we have that  $u_t^-(\cdot, x, 0) \leq C_{1,t}(\cdot)(1 + |x|^p)$ . Thus, as  $u_t^-(\cdot, x, 0)$  is projectively measurable (see Lemmata 12, 8 and 10 (iv)) and  $C_{1,t} \in \mathcal{M}^t$ , Lemma 17 shows that  $u_t^-(\cdot, x, 0) \in \mathcal{M}^t$ . Now, as  $U_{t+1} \leq U_{t+1}^P$  (see Proposition 5), we have that  $(u_t^P)^-(\cdot, x, 0) \leq u_t^-(\cdot, x, 0)$ . As  $(u_t^P)^-(\cdot, x, 0)$  is projectively measurable, we have that  $(u_t^P)^-(\cdot, x, 0) \in \mathcal{M}^t$ , see again Lemmata 12, 8, 10 (iv) and 17.

*Heredity step 2* :  $l_t^P \in \mathcal{M}^t(P)$  and  $\tilde{\Omega}^{t,P} \cap \Omega_{13}^{t,P}$  is a  $P^t$ -full-measure set.

We first show that  $l_t^P \in \mathcal{M}^t(P)$ . Let  $\theta \in \{-1, 1\}^d$ . Using (54),  $C_{t+1} < +\infty$   $\mathcal{Q}^{t+1}$ -q.s. Thus, Cauchy-Schwarz inequality, (ii) and the right-hand side of (59) at time  $t+1$  in Proposition 5 show that  $\mathcal{Q}^{t+1}$ -q.s.

$$\begin{aligned} (U_{t+1}^P)^+(\cdot, 1 + \theta \Delta S_{t+1}(\cdot)) &\leq (U_{t+1}^P)^+(\cdot, 1 + |\theta| |\Delta S_{t+1}(\cdot)|) \\ &\leq \left(1 + \sqrt{d} |\Delta S_{t+1}(\cdot)|\right)^{\overline{7}} ((U_{t+1}^P)^+(\cdot, 1) + C_{t+1}(\cdot)). \end{aligned} \quad (92)$$

As  $U_{t+1}^P$  is projectively measurable (see Proposition 5) and Assumption 1 holds true,  $(U_{t+1}^P)^+(\cdot, 1 + \theta \Delta S_{t+1}(\cdot))$  is also projectively measurable using Lemmata 8 and 10. Now, recalling that  $(U_{t+1}^P)^+(\cdot, 1) \in \mathcal{M}^{t+1}(P)$  and  $C_{t+1} \in \mathcal{M}^{t+1}$  by the induction hypothesis and that  $|\Delta S_{t+1}| \in \mathcal{M}^{t+1}$  by assumption of Theorem 2, we deduce from (92) and Lemma 17 that

$$(U_{t+1}^P)^+(\cdot, 1 + \theta \Delta S_{t+1}(\cdot)) \in \mathcal{M}^{t+1}(P). \quad (93)$$

Thus,  $\omega^t \mapsto \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)} (U_{t+1}^P)^+(\omega^t, \cdot, 1 + \theta \Delta S_{t+1}(\omega^t, \cdot))$  belongs to  $\mathcal{M}^t(P)$  for all  $\theta \in \{-1, 1\}^d$  thanks to Lemma 18. So, (65) and Lemma 17 shows that  $l_t^P \in \mathcal{M}^t(P)$ .

We now prove that  $\tilde{\Omega}^{t,P} \cap \Omega_{13}^{t,P}$  is a  $P^t$ -full-measure set. For that, we first show that  $\Omega_{12}^{t,P}$  is a  $P^t$ -full-measure set. Indeed, by the induction hypothesis,  $N_t^P \in \mathcal{M}^t(P)$ , which implies that  $N_t^P < +\infty$   $P^t$  - a.s. Lemma 2 shows that  $\Omega_{qNA}^{t,P}$  is a  $\mathcal{Q}^t$ -full-measure set and also a  $P^t$ -full-measure set as  $\mathcal{H}^T \subset \mathcal{Q}^T$ . Thus,  $\Omega_{12}^{t,P}$  is a  $P^t$ -full-measure set. Moreover, using the first part of Proposition 6, which do not require Assumptions 6 and 7, we also have that  $\Omega_i^{t,P}$  is a  $P^t$ -full-measure set for all  $i \in \{8, 9, 11\}$ . As  $l_t^P \in \mathcal{M}^t(P)$ , we have that  $l_t^P < +\infty$   $P^t$  - a.s. and so that  $\Omega_{10}^{t,P}$  and also  $\tilde{\Omega}^{t,P}$  are  $P^t$ -full-measure sets. As  $(u_t^P)^-(\cdot, -k, 0) \in \mathcal{M}^t$  (see Heredity step 1),  $(u_t^P)^-(\cdot, -k, 0) < +\infty$   $P^t$ -a.s.

So,  $\Omega_{13}^{t,P}$  is a  $P^t$ -full-measure set (see (88)). Thus, we can find a  $P^t$ -full-measure set  $\widehat{\Omega}^{t,P} \in \mathcal{B}(\Omega^t)$  such that  $\widehat{\Omega}^{t,P} \subset \widetilde{\Omega}^{t,P} \cap \Omega_{13}^{t,P}$ .

*Heredity step 3* :  $(U_t^P)^+(\cdot, 1) \in \mathcal{M}^t(P)$ .

We first define  $K_t^P(\omega^t)$  as follows. If  $\omega^t \in \widehat{\Omega}^{t,P}$ , let  $K_t^P(\omega^t) := K_1(1)$  where  $K_1$  is defined in Proposition 2 applied in the  $P$ -prior  $(t+1)$  context (see Definition 6). This is possible as  $\omega^t \in \widehat{\Omega}^{t,P}$  and Assumptions 8 to 12 are satisfied in this context. Recall that in the  $P$ -prior  $(t+1)$  context,  $\alpha^* = \alpha_t^P(\omega^t)$ ,  $n_0^* = N_t^P(\omega^t)$ ,  $c^* = c_t^P(\omega^t)$ ,  $l^* = l_t^P(\omega^t)$  and  $\Psi^-(x, 0) = (u_t^P)^-(\omega^t, x, 0)$ . When  $\omega^t \notin \widehat{\Omega}^{t,P}$ , we set  $K_t^P(\omega^t) := 1$ . We prove that  $K_t^P \in \mathcal{M}^t(P)$ . As  $C_t \in \mathcal{M}^t(P)$  (see Heredity step 1), we have using (67), Lemmata 3 and 17 that  $c_t^P \in \mathcal{M}^t(P)$ . Recall then that  $N_t^P \in \mathcal{M}^t(P)$  by the induction hypothesis and that  $1/\alpha_t^P \in \mathcal{M}^t(P)$  by assumption of Theorem 2. We have proved in Heredity step 1 that  $(u_t^P)^-(\cdot, x, 0) \in \mathcal{M}^t$  for all  $x \in \mathbb{R}$  and in Heredity step 2 that  $l_t^P \in \mathcal{M}^t(P)$ . Thus, we deduce from Lemma 17 that  $K_t^P$  restricted to  $\widehat{\Omega}^{t,P}$  belongs to  $\mathcal{M}^t(P)$ . So,  $K_t^P \in \mathcal{M}^t(P)$  as  $\widehat{\Omega}^{t,P} \in \mathcal{B}(\Omega^t)$ , see Lemmata 5 (i) and 8. We now prove that  $(U_t^P)^+(\cdot, 1) \in \mathcal{M}^t(P)$ . For all  $\omega^t \in \widehat{\Omega}^{t,P}$ , Assumptions 8 to 13 are satisfied in the  $P$ -prior  $(t+1)$  context. Thus, we can apply (45) in Proposition 2 in this context and

$$U_t^P(\omega^t, 1) \leq \sup_{\substack{|h| \leq K_t^P(\omega^t) \\ h \in \text{Aff}(D_{t+1}^P(\omega^t))}} \text{Cl}(u_t^P)(\omega^t, 1, h) \leq \sup_{|h| \leq K_t^P(\omega^t)} \text{Cl}(u_t^P)(\omega^t, 1, h). \quad (94)$$

Now, let  $U_t^{P,cl} : \Omega^t \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  be defined by

$$U_t^{P,cl}(\omega^t, x, h) := \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)} \text{Cl}(U_{t+1}^P)(\omega^t, \cdot, x + h\Delta S_{t+1}(\omega^t, \cdot)).$$

Then,  $u_t^P \leq U_t^{P,cl}$  by definition of the closure (see (15)) and  $U_t^{P,cl}$  is projectively measurable. Indeed,  $\text{Cl}(U_{t+1}^P)$  is projectively measurable (see Proposition 5 (iv)) and applying Lemma 12 to  $f = \text{Cl}(U_{t+1}^P)$  gives the desired result. Now, we show that for all  $\omega^t \in \widehat{\Omega}^{t,P}$ ,  $U_t^{P,cl}(\omega^t, \cdot, \cdot)$  is usc. Remark that  $U_t^{P,cl}(\omega^t, \cdot, \cdot) = \Psi_p^{cl}(\cdot, \cdot)$  in the  $P$ -prior  $(t+1)$  context where  $\Psi_p^{cl}$  is defined in (38). As Assumptions 8, 9, 10 and 11 are satisfied in the  $P$ -prior  $(t+1)$  context for  $\omega^t \in \widehat{\Omega}^{t,P}$  (see Lemma 4), Proposition 1 shows that  $U_t^{P,cl}(\omega^t, \cdot, \cdot)$  is usc. Moreover, using Proposition 5 (v), we get that for all  $\omega^{t+1} \in \Omega^{t+1}$  and  $h \in \mathbb{R}^d$ ,

$$\text{Cl}(U_{t+1}^P)(\omega^{t+1}, 1 + h\Delta S_{t+1}(\omega^{t+1})) \leq U_{t+1}^P(\omega^{t+1}, 2 + h\Delta S_{t+1}(\omega^{t+1})). \quad (95)$$

Now, using (94) and (95) and recalling that  $U_t^{P,cl}(\omega^t, \cdot, \cdot)$  is usc and that  $u_t^P \leq U_t^{P,cl}$ , we obtain that for  $\omega^t \in \widehat{\Omega}^{t,P}$  that

$$\begin{aligned} U_t^P(\omega^t, 1) &\leq \sup_{|h| \leq K_t^P(\omega^t)} \text{Cl}(u_t^P)(\omega^t, 1, h) \leq \sup_{|h| \leq K_t^P(\omega^t)} \text{Cl}(U_t^{P,cl})(\omega^t, 1, h) = \sup_{|h| \leq K_t^P(\omega^t)} U_t^{P,cl}(\omega^t, 1, h) \\ &= \sup_{|h| \leq K_t^P(\omega^t)} \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)} \text{Cl}(U_{t+1}^P)(\omega^t, \cdot, 1 + h\Delta S_{t+1}(\omega^t, \cdot)) \\ &\leq \sup_{|h| \leq K_t^P(\omega^t)} \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)} U_{t+1}^P(\omega^t, \cdot, 2 + h\Delta S_{t+1}(\omega^t, \cdot)) \\ &\leq \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)} (U_{t+1}^P)^+(\omega^t, \cdot, 2 + K_t^P(\omega^t)|\Delta S_{t+1}(\omega^t, \cdot)|), \end{aligned} \quad (96)$$

where the last inequality follows because  $U_{t+1}^P(\omega^t, \cdot)$  is nondecreasing. Using the right-hand side of (59), we get that,

$$(U_{t+1}^P)^+(\cdot, 2 + K_t^P(\cdot)|\Delta S_{t+1}(\cdot)|) \leq (2 + K_t^P(\cdot)|\Delta S_{t+1}(\cdot)|)^{\bar{\gamma}} ((U_{t+1}^P)^+(\cdot, 1) + C_{t+1}(\cdot)).$$

Recalling that we have proved that  $K_t^P \in \mathcal{M}^t(P)$ , that  $|\Delta S_{t+1}(\cdot)| \in \mathcal{M}^{t+1}$  by assumption of Theorem 2 and that  $C_{t+1} \in \mathcal{M}^{t+1}$  and  $(U_{t+1}^P)^+(\cdot, 1) \in \mathcal{M}^{t+1}(P)$  from the induction hypothesis, we get

that  $(U_{t+1}^P)^+(\cdot, 2 + K_t^P(\cdot)|\Delta S_{t+1}(\cdot)|) \in \mathcal{M}^{t+1}(P)$ . Indeed, as  $U_{t+1}^P$ ,  $|\Delta S_{t+1}(\cdot)|$  and  $K_t^P$  are projectively measurable, Lemmata 8 and 10 show that  $(U_{t+1}^P)^+(\cdot, 2 + K_t^P(\cdot)|\Delta S_{t+1}(\cdot)|)$  is projectively measurable and we can apply Lemma 17. Now, Lemma 18 shows that  $\omega^t \mapsto \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)}(U_{t+1}^P)^+(\omega^t, \cdot, 2 + K_t^P(\omega^t)|\Delta S_{t+1}(\omega^t, \cdot)|)$  belongs to  $\mathcal{M}^t(P)$ . Proposition 5 (i) and Lemma 10 (iv) show that  $(U_t^P)^+(\cdot, 1)$  is projectively measurable. Thus, (96), Lemma 17 and the fact that  $\widehat{\Omega}^{t,P}$  is a  $P^t$ -full-measure set imply that  $(U_t^P)^+(\cdot, 1) \in \mathcal{M}^t(P)$ .

*Heredity step 4* :  $N_{t-1}^P \in \mathcal{M}^{t-1}(P)$ .

If  $t = 0$ , we trivially have that  $N_{-1}^P = 0 \in \mathcal{M}^{-1}(P) = \{0\}$ . So, assume that  $t \geq 1$ . Recall from Heredity steps 1 and 2 that  $l_t^P \in \mathcal{M}^t(P)$ ,  $C_t \in \mathcal{M}^t \subset \mathcal{M}^t(P)$ , from the induction hypothesis that  $N_t^P \in \mathcal{M}^t(P)$  and from the assumptions of Theorem 2 that  $1/\alpha_{t-1}^P \in \mathcal{M}^{t-1} \subset \mathcal{M}^{t-1}(P)$  and  $1/\alpha_t^P \in \mathcal{M}^t \subset \mathcal{M}^t(P)$ . Thus, assertion (A2) in Lemma 14 for  $1 \leq t \leq T-1$  (recall that  $\widetilde{\Omega}^{t,P}$  is a  $P^t$ -full-measure set for all  $P \in \mathcal{H}^T$  from Heredity step 2) shows that  $N_{t-1}^P \in \mathcal{M}^{t-1}(P)$ . This concludes the heredity step.

*Assumption 7 holds true.*

Assumption 7 follows from (93) at  $t$  for the case  $\theta \in \{-1, 1\}^d$  and from the fact that  $(U_t^P)^+(\cdot, 1) \in \mathcal{M}^t(P)$  for all  $P \in \mathcal{H}^T$  for the case  $\theta = 0$ .

*Application of Theorem 1.*

Let  $x \in \mathbb{R}$ . As Assumptions 1, 2, 3, 4, 5, 6 and 7 hold true, we can apply Theorem 1 and there exists  $\phi^{*,x} \in \Phi$  satisfying (19). As  $U$  is usc ( $U$  is of type (A)), if we prove that  $\phi^{*,x} \in \Phi(x, U, \mathcal{Q}^T)$ , (22) in the second part of Theorem 1 shows that  $\phi^{*,x}$  is an optimal strategy for (12), which will conclude the proof.

*We have that  $\phi^{*,x} \in \Phi(x, U, \mathcal{Q}^T)$ .*

We prove now that for all  $P \in \mathcal{Q}^T$ ,  $\mathbb{E}_P U^-(\cdot, V_T^{x, \phi^{*,x}}(\cdot)) < +\infty$ . As (25) holds true and  $C_1 \in \mathcal{M}^T$ , we only need to check that  $V_T^{x, \phi^{*,x}} \in \mathcal{M}^T(P)$  for all  $P \in \mathcal{Q}^T$ , see Lemma 17. Fix  $0 \leq t \leq T-1$  and recall  $\widehat{\Omega}^t$  from Proposition 6. We know from the induction that for all  $k \geq 1$ ,  $u_t^-(\cdot, -k, 0) \in \mathcal{M}^t$  and so that  $u_t^-(\cdot, -k, 0) < +\infty$   $\mathcal{Q}^t$ -q.s. Thus,  $\Omega_{13}^t$  is a  $\mathcal{Q}^t$ -full-measure set and so is  $\widehat{\Omega}^t \cap \Omega_{13}^t$ . We can find a  $\mathcal{Q}^t$ -full-measure set  $\overline{\Omega}^t \in \mathcal{B}(\Omega^t)$  such that  $\overline{\Omega}^t \subset \widehat{\Omega}^t \cap \Omega_{13}^t$ . For all  $\omega^t \in \overline{\Omega}^t$ , Assumptions 8 to 13 are satisfied in the robust  $(t+1)$  context (see Definition 7) and recalling the definition of  $\phi^{*,x}$ , Proposition 3 in the robust  $(t+1)$  context shows that for all  $\omega^t \in \overline{\Omega}^t$ ,  $|\phi_{t+1}^{*,x}(\omega^t)| \leq K_t(\omega^t)$  where for  $\omega^t \in \overline{\Omega}^t$ ,  $K_t(\omega^t) := K_1(V_t^{x, \phi^{*,x}})$  with  $K_1$  defined in Proposition 2 in the robust  $(t+1)$  context and  $K_t(\omega^t) := 1$  when  $\omega^t \notin \overline{\Omega}^t$ . We set  $K_{-1} := 0$  for the need of the next induction.

Let  $P \in \mathcal{Q}^T$ . We show by induction on  $0 \leq t \leq T$  that  $V_t^{x, \phi^{*,x}} \in \mathcal{M}^t(P)$  and  $K_{t-1} \in \mathcal{M}^{t-1}(P)$ . The initialization step is trivial as  $V_0^{x, \phi^{*,x}} = x$  and  $K_{-1} = 0$ . Assume now that the induction hypothesis holds at  $t$ . As  $V_{t+1}^{x, \phi^{*,x}} = V_t^{x, \phi^{*,x}} + \phi_{t+1}^{*,x} \Delta S_{t+1}$  and  $V_t^{x, \phi^{*,x}}$ ,  $\phi_{t+1}^{*,x}$  and  $\Delta S_{t+1}$  are projectively measurable, we get that  $V_{t+1}^{x, \phi^{*,x}}$  is also projectively measurable, see Lemma 7. As  $\overline{\Omega}^t$  is a  $\mathcal{Q}^t$ -full-measure set, we have that

$$|V_{t+1}^{x, \phi^{*,x}}| \leq |V_t^{x, \phi^{*,x}}| + K_t |\Delta S_{t+1}| \quad \mathcal{Q}^t\text{-q.s.} \quad (97)$$

Thus, as  $V_t^{x, \phi^{*,x}} \in \mathcal{M}^t(P)$  by the induction hypothesis and  $|\Delta S_{t+1}| \in \mathcal{M}^t$  by assumption of Theorem 2, if we prove that  $K_t \in \mathcal{M}^t(P)$ , Lemma 17 shows that  $V_{t+1}^{x, \phi^{*,x}} \in \mathcal{M}^{t+1}(P)$  and the induction hypothesis holds true at  $t+1$ . To prove that  $K_t \in \mathcal{M}^t(P)$ , as  $P$  belongs to  $\mathcal{Q}^T$  but not necessarily to  $\mathcal{H}^T$ , we use  $\widehat{P}_{t+1} \in \mathcal{H}^T$  defined in (116) in the Appendix. First, recall that  $\mathcal{M}^t(\widehat{P}_{t+1}) \subset \mathcal{M}^t(P)$ , see Lemma 16. Recall also that in the robust  $(t+1)$  context,  $\alpha^* = \alpha_t^{P^*}(\omega^t)$ ,  $n_0^* = N_t^*(\omega^t)$ ,  $c^* = c_t^{P^*}(\omega^t)$ ,  $l^* = l_t^*(\omega^t)$  and  $\Psi(x, 0) = u_t(\omega^t, x, 0)$ . By assumption of Theorem 2,  $1/\alpha_t^{P^*} \in \mathcal{M}^t$ . So,  $1/\alpha_t^{P^*}$  belong both to  $\mathcal{M}^t(P)$  and  $\mathcal{M}^t(\widehat{P}_{t+1})$ . We first prove that  $l_t^* \in \mathcal{M}^t(\widehat{P}_{t+1}) \subset \mathcal{M}^t(P)$ . Using that  $U_t \leq U_t^{\widehat{P}_{t+1}}$  (see Proposition 5), we have that  $l_t^* \leq l_t^{\widehat{P}_{t+1}}$ . So, as  $l_t^{\widehat{P}_{t+1}} \in \mathcal{M}^t(\widehat{P}_{t+1})$  (see Heredity step 2) and  $l_t^*$  is projectively measurable (see Lemma 3), we get that  $l_t^* \in \mathcal{M}^t(\widehat{P}_{t+1})$  by Lemma 17. We now prove that  $N_t^* \in \mathcal{M}^t(P)$ . Recall assertions (B1) and (B2) from Lemma 14. Assertion (B1) applies and shows

that  $N_{T-1}^* \in \mathcal{M}^{T-1}(P)$  as  $1/\alpha_{T-1}^{P^*} \in \mathcal{M}^{T-1} \subset \mathcal{M}^{T-1}(\widehat{P}_T)$  and  $\underline{X}$ ,  $1/|U(\cdot, \underline{X}(\cdot)) + C(\cdot)|$ ,  $C$  belong to  $\mathcal{M}^t \subset \mathcal{M}^T(\widehat{P}_T)$ . Now, Assertion (B2) also applies and shows that  $N_t^* \in \mathcal{M}^t(P)$  as  $\widehat{\Omega}^{t+1, P}$  is a  $P^{t+1}$ -full-measure set for all  $P \in \mathcal{H}^T$ ,  $1/\alpha_t^{P^*} \in \mathcal{M}^t \subset \mathcal{M}^t(\widehat{P}_{t+1})$ ,  $1/\alpha_{t+1}^{\widehat{P}_{t+1}}$ ,  $C_{t+1} \in \mathcal{M}^{t+1} \subset \mathcal{M}^{t+1}(\widehat{P}_{t+1})$  and  $l_{t+1}^{\widehat{P}_{t+1}}$ ,  $N_{t+1}^{\widehat{P}_{t+1}} \in \mathcal{M}^{t+1}(\widehat{P}_{t+1})$ , see Heredity steps 1, 2 and 4 for  $P = \widehat{P}_{t+1} \in \mathcal{H}^T$ . Now, recalling (91) and that  $C_{1,t} \in \mathcal{M}^t$  and  $V_t^{x, \phi^*, x} \in \mathcal{M}^t(P)$ , we have that  $u_t^-(\cdot, V_t^{x, \phi^*, x}(\cdot), 0) \in \mathcal{M}^t(P)$  (see Lemma 17 as  $u_t^-(\cdot, V_t^{x, \phi^*, x}(\cdot), 0)$  is projectively measurable, see Heredity step 1 and Lemma 10). Finally, as  $C_t \in \mathcal{M}^t$ ,  $c_t^{P^*} \in \mathcal{M}^t$  using (67) and Lemma 17. Thus, recalling again that  $V_t^{x, \phi^*, x} \in \mathcal{M}^t(P)$  and  $1/\alpha_t^{P^*} \in \mathcal{M}^t$ , we deduce from Lemma 17 that  $K_t \in \mathcal{M}^t(P)$ . This concludes the induction and the proof.  $\square$

**7. Projective sets and projectively measurable functions** In this part, we present the projective sets, the projectively measurable functions and the consequences of the (PD) axiom. In Section 7.1, we define the projective sets (see Definition 8). First, we prove that these sets have almost the same properties as the analytic sets (see Proposition 8). The proof of these results will be provided in Appendix, see Section 8.4. Then, in Section 7.2, we present two crucial results that are implied by the (PD) axiom (see Theorems 3 and 4). Section 7.3 introduces the projectively measurable functions (see Definition 9) and some of their basic properties. In Section 7.4, we show that like for universally measurable functions the composition of projectively measurable functions remains projectively measurable (see Lemma 10). We also show that under the (PD) axiom, a projectively measurable selection on a projective set is always possible (see Proposition 9). Then, we prove that projectively measurable  $\epsilon$ -optimal selectors always exist for projectively measurable functions and that like for lower (resp. upper) semicontinuous functions, the class of projectively measurable functions is stable by uncountable infimum (resp. supremum). (see Proposition 10). Finally, in Section 7.5, we prove some results related to the  $(-\infty)$  integrals and its link with the projective hierarchy. We show that the integral of a projectively measurable function remains projectively measurable under the  $(-\infty)$  integration (see Lemma 12 and Proposition 12).

**7.1. Definition of projective sets and first properties** We now introduce projective sets. The reader can find an elaborate construction of them in [27, p313].

**DEFINITION 8.** Let  $X$  be a Polish space. For each  $n \geq 1$ , we define recursively the classes  $\Sigma_n^1(X)$  and  $\Pi_n^1(X)$  of  $X$ . First,  $\Sigma_1^1(X)$  is the class of analytic sets of  $X$  while  $\Pi_1^1(X)$  is the class of coanalytic sets of  $X$ . Then, for all  $n \geq 1$ , we set

$$\Sigma_{n+1}^1(X) := \{\text{proj}_X(C), C \in \Pi_n^1(X \times \mathcal{N})\} \quad (98)$$

$$\Pi_{n+1}^1(X) := \{X \setminus C, C \in \Sigma_{n+1}^1(X)\}, \quad (99)$$

where  $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$  is the Baire space. Then,

$$\Delta_n^1(X) := \Sigma_n^1(X) \cap \Pi_n^1(X). \quad (100)$$

The class  $\mathbf{P}(X)$  of projective sets of  $X$  is defined by

$$\mathbf{P}(X) := \bigcup_{n \geq 1} \Delta_n^1(X).$$

The following proposition justifies the term ‘‘hierarchy’’ and gives crucial properties for  $\Sigma_n^1(X)$ ,  $\Pi_n^1(X)$ ,  $\Delta_n^1(X)$  and  $\mathbf{P}(X)$  that are similar to the ones of the analytic sets.



PROPOSITION 8. Let  $n \geq 1$  and let  $X$  and  $Y$  be Polish spaces. Let  $f : X \rightarrow Y$  be Borel measurable<sup>2</sup>.

(i) The class  $\Sigma_n^1(X)$  is closed under countable intersections and unions. Moreover, we have that  $f^{-1}(B) \in \Sigma_n^1(X)$  for all  $B \in \Sigma_n^1(Y)$  and that  $f(A) \in \Sigma_n^1(Y)$  for all  $A \in \Sigma_n^1(X)$ .

(ii) The class  $\Pi_n^1(X)$  is closed under countable intersections and unions. Moreover, we have that  $f^{-1}(B) \in \Pi_n^1(X)$  for all  $B \in \Pi_n^1(Y)$ .

(iii) The class  $\Delta_n^1(X)$  is a  $\sigma$ -algebra of  $X$ ,  $\Delta_1^1(X) = \mathcal{B}(X)$  and  $f^{-1}(B) \in \Delta_n^1(X)$  for all  $B \in \Delta_n^1(Y)$ .

(iv) The sequences  $(\Sigma_n^1(X))_{n \geq 1}$ ,  $(\Pi_n^1(X))_{n \geq 1}$  and  $(\Delta_n^1(X))_{n \geq 1}$  are nondecreasing.

(v) We have that  $\Sigma_n^1(X) \times \Sigma_n^1(Y) \subset \Sigma_n^1(X \times Y)$ ,  $\Pi_n^1(X) \times \Pi_n^1(Y) \subset \Pi_n^1(X \times Y)$ ,  $\Delta_n^1(X) \times \Delta_n^1(Y) \subset \Delta_n^1(X \times Y)$  and that

$$\Sigma_n^1(X) \cup \Pi_n^1(X) \subset \Delta_{n+1}^1(X) \text{ and } \bigcup_{n \geq 1} \Sigma_n^1(X) = \bigcup_{n \geq 1} \Pi_n^1(X) = \bigcup_{n \geq 1} \Delta_n^1(X). \quad (101)$$

(vi) The class  $\mathbf{P}(X)$  is closed under finite unions and intersections. It is also closed under complements. Moreover, we have that  $f^{-1}(B) \in \mathbf{P}(X)$  for all  $B \in \mathbf{P}(Y)$  and that  $f(A) \in \mathbf{P}(Y)$  for all  $A \in \mathbf{P}(X)$ . Finally,

$$\mathcal{B}(X) \subset \mathbf{P}(X), \Sigma_1^1(X) \subset \mathbf{P}(X), \mathbf{P}(X) \times \mathbf{P}(Y) \subset \mathbf{P}(X \times Y). \quad (102)$$

*Proof.* See Appendix 8.4.  $\square$

REMARK 7. Proposition 8 does not follow directly from [27, Proposition 37.1, p314]. In fact, their classes of sets  $\Sigma_n^1$  and  $\Pi_n^1$  differ from ours in that they are defined by

$$\Sigma_n^1 := \bigcup_{X \text{ Polish space}} \Sigma_n^1(X) \text{ and } \Pi_n^1 := \bigcup_{X \text{ Polish space}} \Pi_n^1(X).$$

REMARK 8. The classes  $\Pi_n^1(X)$  and  $\Delta_n^1(X)$  are not stable per direct image. Indeed, (ii) and (iii) follow from (i) as  $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$  but  $f(X \setminus C)$  and  $Y \setminus f(C)$  have no reason to be included in each other. Moreover, while the complement of a projective set is still a projective set (in opposition to the case of analytic sets), a countable union of projective sets may not be a projective set. Let  $X$  be an uncountable Polish space and  $(A_n)_{n \geq 2}$  be such that  $A_n \in \Sigma_n^1(X)$  and  $A_n \notin \Sigma_{n-1}^1(X)$  for all  $n \geq 2$ . Such a sequence exists (see [27, Theorem 37.7, p316] and (100)). Let  $B_n := \{(n, x), x \in A_n\}$  for all  $n \geq 2$ . We prove below that  $B_n \in \mathbf{P}(\mathbb{N} \times X)$  for all  $n \geq 2$ , but that  $B := \bigcup_{n \geq 2} B_n \notin \mathbf{P}(\mathbb{N} \times X)$ . Thus,  $\mathbf{P}(\mathbb{N} \times X)$  is not a  $\sigma$ -algebra of  $\mathbb{N} \times X$ .

Note first that for all  $n \geq 2$ ,  $B_n = l_n(A_n)$  where  $l_n(x) := (n, x)$  for all  $x \in X$ . Using then (i) in Proposition 8, as  $A_n \in \Sigma_n^1(X)$  and  $l_n$  is Borel measurable, we get that  $B_n \in \Sigma_n^1(\mathbb{N} \times X)$ . So, (101) in Proposition 8 implies that  $B_n \in \Sigma_n^1(\mathbb{N} \times X) \subset \Delta_{n+1}^1(\mathbb{N} \times X) \subset \mathbf{P}(\mathbb{N} \times X)$  for all  $n \geq 2$ . Assume now that  $B \in \mathbf{P}(\mathbb{N} \times X)$ . Then, there exists  $p \geq 1$  such that  $B \in \Delta_p^1(\mathbb{N} \times X) \subset \Sigma_p^1(\mathbb{N} \times X)$  (see (100)). Remark that

$$l_{p+1}^{-1}(B) = \{x \in X, (p+1, x) \in B\} = \bigcup_{n \geq 2} \{x \in X, (p+1, x) \in B_n\} = \{x \in X, (p+1, x) \in B_{p+1}\} = A_{p+1}.$$

As  $B \in \Sigma_p^1(\mathbb{N} \times X)$  and  $l_{p+1}$  is Borel measurable, using again (i) in Proposition 8,  $A_{p+1} \in \Sigma_p^1(X)$ , a contradiction. So,  $B \notin \mathbf{P}(\mathbb{N} \times X)$ .

<sup>2</sup> In particular, Proposition 8 will be used when  $f$  is a projection.

**7.2. Measurability and uniformization under the (PD) axiom** We now present two crucial consequences of the (PD) axiom (see Axiom 1). The first one asserts that projective sets are universally measurable and the second one that the class of projective sets satisfies the uniformization property.

**THEOREM 3.** *Assume the (PD) axiom. Let  $X$  be a Polish space. Then,  $\mathbf{P}(X) \subset \mathcal{B}_c(X)$ .*

*Proof.* See [27, Theorem 38.17, p326] (see also [35, Remark 2, p71]).  $\square$

This implies that  $\Sigma_n^1(X)$ ,  $\Pi_n^1(X)$  and  $\Delta_n^1(X)$  are included in  $\mathcal{B}_c(X)$  for all  $n \geq 1$ , see (101) in Proposition 8.

**THEOREM 4.** *Assume the (PD) axiom. Let  $X$  and  $Y$  be Polish spaces,  $n \geq 0$  and  $A \in \Pi_{2n+1}^1(X \times Y)$ . Then, there exists  $A^* \in \Pi_{2n+1}^1(X \times Y)$  such that  $A^* \subset A$  and for all  $x \in X$ ,*

$$\exists y \in Y, (x, y) \in A \iff \exists! y \in Y, (x, y) \in A^*. \quad (103)$$

The set  $A^*$  is called a  $\Pi_{2n+1}^1(X \times Y)$ -uniformization of  $A$ .

*Proof.* See [27, Corollary 39.9, p339] and [27, p120] for the definition of a uniformization (see also [34, Theorem 1]).  $\square$

**REMARK 9.** An accessible proof of Theorem 3 for  $\Sigma_2^1(X)$  sets is available in [27, p308]. Interestingly, [30] shows that the universal measurability of all value functions associated to a Borel gambling problem is in fact equivalent to the fact that the sets of  $\Sigma_2^1(X)$  are universally measurable. Theorem 3 for  $\Sigma_2^1(X)$  sets is false under other axioms, for example in the ZFC theory assuming the Axiom of Constructibility, see [24, Corollary 25.28, p495]. Note also that [32] shows that in the ZFC theory assuming the Martin's Axiom and the negation of the Continuum hypothesis, all sets  $A \in \Sigma_2^1(X)$  are Lebesgue-measurable.

Theorem 4 provides a uniformization of any set  $A \in \Pi_{2n+1}^1(X \times Y)$ . We will see in Proposition 9 that a uniformization of  $A$  is the graph of a projectively measurable selection (on  $A$ ). The proof of Theorem 4 is much more complex than the one of Theorem 3. It relies on the notion of scale. A scale on a given set  $A$  is a sequence of ordinal functions on the set  $A$  that satisfy sufficient continuity conditions to be able to select an element out of  $A$ . The reader can check [27, 36(B), p299] for a complete description of scales. Note that when  $n = 0$  i.e. for the class  $\Pi_1^1(X \times Y)$  of coanalytic set, Theorem 4 can be shown inside the ZFC theory (see [27, Theorem 36.14, p306]). Nevertheless, any resulting selection function has no reason to be universally measurable under the ZFC theory only.

**7.3. Projectively measurable functions** We now define the key concept of projectively measurable functions which will allow us to perform measurable selection.

**DEFINITION 9.** Let  $X$  and  $Y$  be Polish spaces and  $D \subset X$ . A function  $f : D \rightarrow Y$  is projectively measurable if  $D \in \mathbf{P}(X)$  and if there exists some  $n \geq 1$  such that  $f$  is  $\Delta_n^1(X)$ -measurable in the sense that for all  $B \in \mathcal{B}(Y)$ ,  $f^{-1}(B) = \{x \in D, f(x) \in B\} \in \Delta_n^1(X)$ .

**REMARK 10.** It is important in Definition 9 that the  $n$  is the same for all  $B \in \mathcal{B}(Y)$ .

**REMARK 11.** As  $\Delta_n^1(X)$  is a  $\sigma$ -algebra (see (iii) in Proposition 8), to prove that  $f : X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  is  $\Delta_n^1(X)$ -measurable (resp. projectively measurable), it is enough to prove that  $\{f < c\}$  or  $\{f \leq c\}$  belong to  $\Delta_n^1(X)$  for all  $c \in \mathbb{R}$  (resp. for some given  $n \geq 1$ ). This will be used in the rest of the paper without further mention.

The following lemmata are used several times in the paper. The first lemma shows that Borel measurable functions as well as lower-semianalytic functions are projectively measurable, and also that projectively measurable functions are universally measurable under the (PD) axiom.

LEMMA 5. *Let  $X$  and  $Y$  be Polish spaces and let  $f : X \rightarrow Y$ .*

- (i) *If  $f$  is  $\mathcal{B}(X)$ -measurable, then  $f$  is projectively measurable.*
- (ii) *If  $Y = \mathbb{R} \cup \{-\infty, +\infty\}$  and  $f$  is a lower-semianalytic function (see [5, Definition 7.21, p177]), then  $f$  is projectively measurable.*
- (iii) *Assume the (PD) axiom. If  $f$  is projectively measurable, then  $f$  is universally measurable (see [5, Definition 7.20, p171]).*

*Proof.* (i) The first assertion follows from  $\mathcal{B}(X) \subset \mathbf{P}(X)$ , see (102). (ii) Let  $c \in \mathbb{R}$ , as  $f$  is lower-semianalytic,  $\{f < c\} \in \Sigma_1^1(X)$  and we conclude as  $\Sigma_1^1(X) \subset \Delta_2^1(X)$  (see (101)). (iii) The third assertion follows from Theorem 3 as for all  $n \geq 1$ ,  $\Delta_n^1(X) \subset \mathbf{P}(X) \subset \mathcal{B}_c(X)$ .  $\square$

The second lemma shows that a vector of projectively measurable functions is projectively measurable.

LEMMA 6. *Let  $X, Y$  and  $Z$  be Polish spaces and let  $f : X \rightarrow Y$  and  $g : X \rightarrow Z$ . Let  $h : X \rightarrow Y \times Z$  be defined by  $h(x) := (f(x), g(x))$  for all  $x \in X$ .*

- (i) *If  $f$  and  $g$  are  $\Delta_p^1(X)$ -measurable for some  $p \geq 1$ , then  $h$  is  $\Delta_p^1(X)$ -measurable.*
- (ii) *If  $f$  and  $g$  are projectively measurable, then  $h$  is projectively measurable.*

*Proof.* (i) Let  $A \in \mathcal{B}(Y)$  and  $B \in \mathcal{B}(Z)$ . We have that  $h^{-1}(A \times B) = f^{-1}(A) \cap g^{-1}(B)$  and that  $f^{-1}(A)$  and  $g^{-1}(B)$  belong to  $\Delta_p^1(X)$ . Thus, Proposition 8 (iii) ensures that  $h^{-1}(A \times B) \in \Delta_p^1(X)$  and  $h$  is  $\Delta_p^1(X)$ -measurable.

(ii) As  $f$  and  $g$  are projectively measurable, there exist  $n, p \geq 1$  such that  $f$  is  $\Delta_n^1(X)$ -measurable and  $g$  is  $\Delta_p^1(X)$ -measurable. We may assume that  $n \leq p$ . Then, Proposition 8 (iv) shows that  $f$  is  $\Delta_p^1(X)$ -measurable and (ii) follows from (i).  $\square$

The third lemma shows that the set of finite vector-valued projectively measurable functions is closed under usual operations.

LEMMA 7. *Let  $X$  be a Polish space and let  $f : X \rightarrow \mathbb{R}^n$  and  $g : X \rightarrow \mathbb{R}^n$  for some  $n \geq 1$ .*

- (i) *If  $f$  and  $g$  are  $\Delta_p^1(X)$ -measurable for some  $p \geq 1$ , then  $f + g$ ,  $fg$  and  $-f$  are  $\Delta_p^1(X)$ -measurable.*
- (ii) *If  $f$  and  $g$  are projectively measurable, then  $f + g$ ,  $fg$  and  $-f$  are projectively measurable.*

*Proof.* (i) Let  $B_1 \in \mathcal{B}(\mathbb{R}^d)$ ,  $B_2 \in \mathcal{B}(\mathbb{R})$  and  $B_3 \in \mathcal{B}(\mathbb{R}^d)$ . We have that

$$\begin{aligned} (f + g)^{-1}(B_1) &= \{x \in X, f(x) + g(x) \in B_1\} = \{x \in X, (f(x), g(x)) \in \sigma_1^{-1}(B_1)\} \\ (fg)^{-1}(B_2) &= \{x \in X, f(x)g(x) \in B_2\} = \{x \in X, (f(x), g(x)) \in \sigma_2^{-1}(B_2)\} \\ (-f)^{-1}(B_3) &= \{x \in X, -f(x) \in B_3\} = \{x \in X, f(x) \in \sigma_3^{-1}(B_3)\}, \end{aligned}$$

where  $\sigma_1(x, y) := x + y$ ,  $\sigma_2(x, y) := xy$ ,  $\sigma_3(x) := -x$  for all  $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ . As  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  are Borel measurable, Lemma 6 and Proposition 8 (iii) show (i).

(ii) This is the same proof as in Lemma 6 (ii).  $\square$

The last lemma extends the results of Lemma 7 to real, possibly infinite, valued functions under conventions (10) and (11).

LEMMA 8. *Let  $X$  be a Polish space and for all  $n \geq 0$ , let  $f, f_n, g : X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ .*

*Let  $p \geq 1$ . Assume that  $f, f_n$  and  $g$  are  $\Delta_p^1(X)$ -measurable for all  $n \geq 0$ . Then,  $fg$  (under convention (11)),  $f + g$  (under convention (10)),  $-f$ ,  $\min(f, g)$ ,  $\max(f, g)$ ,  $\inf_{n \geq 0} f_n$ ,  $\sup_{n \geq 0} f_n$  are  $\Delta_p^1(X)$ -measurable. If  $f \geq 0$ ,  $f^a$  is also  $\Delta_p^1(X)$ -measurable for all  $a > 0$ .*

*Now, if  $f$  and  $g$  are projectively measurable, then the previous functions are also projectively measurable.*

*Proof.* The first part of the proof follows from the fact that  $\Delta_p^1(X)$  is a  $\sigma$ -algebra and from conventions (10) and (11) and is given for sake of completeness. We first prove the sup and inf results. As for all  $c \in \mathbb{R}$ ,  $\{\min(f, g) < c\} = \{f < c\} \cup \{g < c\} \in \Delta_p^1(X)$  and  $\{\max(f, g) < c\} = \{f < c\} \cap \{g < c\} \in \Delta_p^1(X)$ ,  $\min(f, g)$  and  $\max(f, g)$  are  $\Delta_p^1(X)$ -measurable. Similarly,  $\{\sup_{n \geq 0} f_n \leq c\} = \bigcap_{n \geq 0} \{f_n \leq c\} \in \Delta_p^1(X)$  and  $\{\inf_{n \geq 0} f_n < c\} = \bigcup_{n \geq 0} \{f_n < c\} \in \Delta_p^1(X)$  and  $\sup_{n \geq 0} f_n$  and  $\inf_{n \geq 0} f_n$  are also  $\Delta_p^1(X)$ -measurable. We now prove that  $-f$  is  $\Delta_p^1(X)$ -measurable. As  $\{-f < c\} = f^{-1}((-c, +\infty]) \in \Delta_p^1(X)$ , we have that  $-f$  is  $\Delta_p^1(X)$ -measurable. We now show that  $f + g$  is  $\Delta_p^1(X)$ -measurable. Let  $N := \{f = +\infty, g = -\infty\} \cup \{f = -\infty, g = +\infty\}$ . Then, as  $\{f = +\infty\} = \bigcap_{n \geq 0} \{f \geq n\}$  and  $\{f = -\infty\} = \bigcap_{n \geq 0} \{f \leq -n\}$ ,  $N \in \Delta_p^1(X)$ . Now, for all  $c \in \mathbb{R}$ , recalling convention (10),

$$\{x \in X, f(x) + g(x) < c\} = N \cup \left( (X \setminus N) \cap \bigcup_{r \in \mathbb{Q}} \{x \in X, f(x) < r\} \cap \{x \in X, g(x) < c - r\} \right) \in \Delta_p^1(X).$$

Thus,  $f + g$  is  $\Delta_p^1(X)$ -measurable. Now, we prove that  $fg$  is  $\Delta_p^1(X)$ -measurable. We start with the case where  $f \geq 0$  and  $g \geq 0$ . We have that  $\{fg < c\} = \emptyset \in \Delta_p^1(X)$  for all  $c \leq 0$ . Due to convention (11), we have that for all  $c > 0$ ,

$$\{x \in X, f(x)g(x) < c\} = \{x \in X, g(x) = 0\} \cup \bigcup_{r \in \mathbb{Q}_+^*} \{x \in X, f(x) < c/r\} \cap \{x \in X, g(x) < r\} \in \Delta_p^1(X).$$

So,  $fg$  is  $\Delta_p^1(X)$ -measurable when  $f \geq 0$  and  $g \geq 0$ . The general case follows from the equality  $fg = (f^+g^+ + f^-g^-) - (f^+g^- + f^-g^+)$  and the fact that the sum (and the difference) of  $\Delta_p^1(X)$ -measurable functions are  $\Delta_p^1(X)$ -measurable. Assume now that  $f \geq 0$  and let  $a > 0$ . Then,  $\{f^a < c\} = \emptyset \in \Delta_p^1(X)$  for all  $c \leq 0$ . Moreover,  $\{f^a < c\} = \{f < c^{1/a}\} \in \Delta_p^1(X)$  for all  $c > 0$  and  $f^a$  is  $\Delta_p^1(X)$ -measurable.

Now if  $f$  and  $g$  are projectively measurable, the same proof as in Lemma 6 (ii).  $\square$

**7.4. Consequences of the (PD) axioms** The next proposition is an important consequence of Theorems 3 and 4 and shows that the (PD) axiom allows to perform a projectively measurable selection on any projective set. This proposition can be seen as an extension of [5, Proposition 7.49, p182].

**PROPOSITION 9.** *Assume the (PD) axiom. Let  $X$  and  $Y$  be Polish spaces and  $A \in \mathbf{P}(X \times Y)$ . Then, there exists a projectively measurable function  $\phi: \text{proj}_X(A) \rightarrow Y$  such that  $\text{Graph}(\phi) \subset A$ .*

*Proof.* As  $A \in \mathbf{P}(X \times Y)$ , there exists some  $n \geq 0$  such that  $A \in \Delta_n^1(X \times Y) \subset \Pi_n^1(X \times Y) \subset \Pi_{2n+1}^1(X \times Y)$  using (100) and (iv) in Proposition 8. Applying now Theorem 4 to  $A$ , there exists  $A^* \in \Pi_{2n+1}^1(X \times Y)$  such that  $A^* \subset A$  and (103) holds true. So, we get that

$$x \in \text{proj}_X(A) \iff \exists y_0 \in Y \text{ such that } (x, y_0) \in A \iff \exists! y \in Y \text{ such that } (x, y) \in A^*.$$

For all  $x \in \text{proj}_X(A)$ , we set  $\phi(x) := y$ . Then,  $\phi$  is a function because of the unicity obtained in (103). By definition of  $\phi$ ,  $\text{Graph}(\phi) := \{(x, y) \in X \times Y, x \in \text{proj}_X(A), y = \phi(x)\} \subset A^*$ . We prove now that  $\phi$  is projectively measurable. Let  $B \in \mathcal{B}(Y)$ .

$$\begin{aligned} \phi^{-1}(B) &= \{x \in \text{proj}_X(A), \phi(x) \in B\} = \{x \in \text{proj}_X(A), \exists y \in B, (x, y) \in A^*\} \\ &= \text{proj}_X((X \times B) \cap A^*) \cap \text{proj}_X(A) = \text{proj}_X((X \times B) \cap A^*), \end{aligned}$$

where the last equality follows from  $A^* \subset A$ . Note that  $X \times B \in \mathcal{B}(X \times Y) = \Delta_1^1(X \times Y) \subset \Delta_{2n+2}^1(X \times Y)$  using again (iii) and (iv) in Proposition 8. Recall also from (101) that  $A^* \in \Pi_{2n+1}^1(X \times Y) \subset \Delta_{2n+2}^1(X \times Y)$ . So,  $(X \times B) \cap A^* \subset \Delta_{2n+2}^1(X \times Y) \subset \Sigma_{2n+2}^1(X \times Y)$  and  $\text{proj}_X((X \times B) \cap A^*) \in \Sigma_{2n+2}^1(X)$  using (i) in Proposition 8. Finally, (101) shows that  $\phi^{-1}(B) \in \Sigma_{2n+2}^1(X) \subset \Delta_{2n+3}^1(X) \subset \mathbf{P}(X)$  and  $\phi$  is projectively measurable.  $\square$

We now extend (iii) in Proposition 8 to  $\Delta_p^1(X)$ -measurable functions. This result will be useful to show that the composition of projectively measurable functions remains projectively measurable.

LEMMA 9. *Let  $X$  and  $Y$  be Polish spaces. Let  $p \geq 1$  and assume that  $f : X \rightarrow Y$  is  $\Delta_p^1(X)$ -measurable. Then, for all  $n \geq 1$  and for all  $B \in \Delta_n^1(Y)$ ,  $f^{-1}(B) \in \Delta_{p+n}^1(X)$ .*

*Proof.* First, we show by induction on  $n$  that for all  $n \geq 1$ ,  $p \geq 1$ , for all Polish spaces  $X$  and  $Y$ , for all  $\Delta_p^1(X)$ -measurable  $f : X \rightarrow Y$  and for all  $B \in \Sigma_n^1(Y)$ , we have that  $f^{-1}(B) \in \Sigma_{n+p-1}^1(X)$ . We begin with the heredity step. Assume that the induction hypothesis holds true for  $n \geq 1$ . Let  $B \in \Sigma_{n+1}^1(Y)$ . There exists some  $C \in \Pi_n^1(Y \times \mathcal{N})$  such that  $B = \text{proj}_Y(C)$ . Let  $p \geq 1$ . Let  $f : X \rightarrow Y$  be  $\Delta_p^1(X)$ -measurable. For all  $(x, u) \in X \times \mathcal{N}$ , let  $\Psi(x, u) := (f(x), u) \in Y \times \mathcal{N}$ . Then,

$$\begin{aligned} f^{-1}(B) &= \{x \in X, f(x) \in B\} = \{x \in X, \exists u \in \mathcal{N}, (f(x), u) \in C\} \\ &= \{x \in X, \exists u \in \mathcal{N}, (x, u) \in \Psi^{-1}(C)\} = \text{proj}_X(\Psi^{-1}(C)). \end{aligned} \quad (104)$$

Note that  $(x, u) \mapsto f(x)$  and  $(x, u) \mapsto u$  are  $\Delta_p^1(X \times \mathcal{N})$ -measurable using (iii) to (v) in Proposition 8. So, Lemma 6 shows that  $\Psi$  is a  $\Delta_p^1(X \times \mathcal{N})$ -measurable. As  $C \in \Pi_n^1(Y \times \mathcal{N})$ , (99) implies that  $(Y \times \mathcal{N}) \setminus C \in \Sigma_n^1(X \times \mathcal{N})$  and using the induction hypothesis for  $X \times \mathcal{N}$  and  $Y \times \mathcal{N}$ , we get that  $\Psi^{-1}((Y \times \mathcal{N}) \setminus C) \in \Sigma_{n+p-1}^1(X \times \mathcal{N})$ . As  $\Psi^{-1}((Y \times \mathcal{N}) \setminus C) = (X \times \mathcal{N}) \setminus \Psi^{-1}(C)$ , we obtain that  $\Psi^{-1}(C) \in \Pi_{n+p-1}^1(X \times \mathcal{N})$ . So, (104) and (98) show that  $f^{-1}(B) \in \Sigma_{n+p}^1(X)$ . We now turn to the initialization step. Let  $B \in \Sigma_1^1(Y)$ . Then, [5, Proposition 7.39, p165] shows that there exists  $C \in \mathcal{B}(Y \times \mathcal{N})$  such that  $B = \text{proj}_Y(C)$ . Let  $p \geq 1$ ,  $f : X \rightarrow Y$  be  $\Delta_p^1(X)$ -measurable and  $\Psi$  be defined as above. Then,  $\Psi$  is  $\Delta_p^1(X \times \mathcal{N})$ -measurable and  $\Psi^{-1}(C) \in \Delta_p^1(X \times \mathcal{N}) \subset \Sigma_p^1(X \times \mathcal{N})$  (see (100)). So, (104) and (i) in Proposition 8 show that  $f^{-1}(B) \in \Sigma_p^1(X)$ . This concludes the induction. Let  $n \geq 1$  and  $B \in \Delta_n^1(Y) \subset \Sigma_n^1(Y)$ . Let  $p \geq 1$  and  $f : X \rightarrow Y$  be  $\Delta_p^1(X)$ -measurable. Then,  $f^{-1}(B) \in \Sigma_{n+p-1}^1(X) \subset \Delta_{n+p}^1(X)$  (see (101)) which is the desired result.  $\square$

We now show that the composition of projectively measurable functions is still projectively measurable. Note that the universally measurable functions also have this property (see [5, Proposition 7.44, p172]). This is a very important result for the dynamic programming procedure.

LEMMA 10. *Let  $X, Y$  and  $Z$  be Polish spaces. Let  $h : X \times Y \rightarrow Z$ ,  $g : D \rightarrow Y$  and  $f : E \rightarrow Z$  where  $D \subset X$  and  $g(D) \subset E \subset Y$ .*

- (i) *Assume that  $f$  is  $\Delta_p^1(Y)$ -measurable and that  $g$  is  $\Delta_q^1(X)$ -measurable for some  $p, q \geq 1$ . Then,  $f \circ g$  is  $\Delta_{p+q}^1(X)$ -measurable.*
- (ii) *Assume that  $h$  is  $\Delta_p^1(X \times Y)$ -measurable for some  $p \geq 1$ . Then  $h(x, \cdot) : y \mapsto h(x, y)$  is  $\Delta_{p+1}^1(Y)$ -measurable for all  $x \in X$  and  $h(\cdot, y) : x \mapsto h(x, y)$  is  $\Delta_{p+1}^1(X)$ -measurable for all  $y \in Y$ .*
- (iii) *Assume that  $f$  and  $g$  are projectively measurable. Then,  $f \circ g$  is projectively measurable.*
- (iv) *Assume that  $h$  is projectively measurable. Then  $h(x, \cdot) : y \mapsto h(x, y)$  is projectively measurable for all  $x \in X$  and  $h(\cdot, y) : x \mapsto h(x, y)$  is projectively measurable for all  $y \in Y$ .*

*Proof.* We show (i). Let  $B \in \mathcal{B}(Z)$ . We have that  $(f \circ g)^{-1}(B) = g^{-1}(f^{-1}(B))$ . As  $f$  is  $\Delta_p^1(Y)$ -measurable,  $f^{-1}(B) \in \Delta_p^1(Y)$  and Lemma 9 shows that  $g^{-1}(f^{-1}(B)) \in \Delta_{p+q}^1(X)$  and (i) is proved. If  $f$  and  $g$  are projectively measurable, then there exist some  $p \geq 1$  and  $q \geq 1$  such that  $f$  is  $\Delta_p^1(X)$ -measurable and  $g$  is  $\Delta_q^1(X)$ -measurable and (i) shows that  $f \circ g$  is  $\Delta_{p+q}^1(X)$ -measurable and thus projectively measurable : (iii) is proved. We show (ii). Let  $x \in X$  and  $\sigma : y \mapsto (x, y)$ . Then,  $\sigma$  is Borel measurable and also  $\Delta_1^1(Y)$ -measurable by Proposition 8 (iii). As  $h(x, \cdot) = h \circ \sigma$  and  $h$  is  $\Delta_p^1(X \times Y)$ -measurable, we deduce from (i) that  $h(x, \cdot)$  is  $\Delta_{p+1}^1(Y)$ -measurable. The proof that  $h(\cdot, y)$  is  $\Delta_{p+1}^1(X)$ -measurable for all  $y \in Y$  is similar and omitted. Finally, if  $h$  is projectively measurable, there exists some  $p \geq 1$  such that  $h$  is  $\Delta_p^1(X \times Y)$  and (ii) shows that  $h(x, \cdot)$  and  $h(\cdot, y)$  are respectively  $\Delta_{p+1}^1(Y)$  and  $\Delta_{p+1}^1(X)$ -measurable for all  $x \in X$  and  $y \in Y$ , and thus projectively measurable.  $\square$



We now show that uncountable supremum and infimum of projectively measurable functions may remain projectively measurable. We also prove that assuming the (PD) axiom,  $\epsilon$ -optimal selectors exist for projectively measurable functions. This proposition can be seen as an extension of [5, Proposition 7.47, p179] and [5, Proposition 7.50, p184] which hold true for lower-semianalytic functions.

**PROPOSITION 10.** *Let  $X$  and  $Y$  be Polish spaces. Let  $D \in \mathbf{P}(X \times Y)$  and  $f : X \times Y \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  be a projectively measurable function. Let  $D_x := \{y \in Y, (x, y) \in D\}$  for all  $x \in X$ . Then, the functions  $f_*, f^* : \text{proj}_X(D) \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  defined by*

$$f_*(x) := \inf_{y \in D_x} f(x, y) \text{ and } f^*(x) := \sup_{y \in D_x} f(x, y)$$

*are projectively measurable.*

*Assume the (PD) axiom. Let  $\epsilon_* > 0$  and  $\epsilon^* > 0$ . Then, there exist projectively measurable functions  $\phi_*, \phi^* : \text{proj}_X(D) \rightarrow Y$  such that  $\text{Graph}(\phi_*) \subset D$ ,  $\text{Graph}(\phi^*) \subset D$  and for all  $x \in \text{proj}_X(D)$ ,*

$$f(x, \phi_*(x)) < \begin{cases} f_*(x) + \epsilon_* & \text{if } f_*(x) > -\infty, \\ -\frac{1}{\epsilon_*} & \text{if } f_*(x) = -\infty. \end{cases} \quad (105)$$

$$f(x, \phi^*(x)) > \begin{cases} f^*(x) - \epsilon^* & \text{if } f^*(x) < +\infty, \\ \frac{1}{\epsilon^*} & \text{if } f^*(x) = +\infty. \end{cases} \quad (106)$$

*Proof. (i)  $f_*$  and  $f^*$  are projectively measurable.*

Note first that there exists some  $p \geq 1$  such that  $D \in \Delta_p^1(X \times Y) \in \Sigma_p^1(X \times Y)$  (see (100)). An application of (i) in Proposition 8 shows that  $\text{proj}_X(D) \in \Sigma_p^1(X) \subset \Delta_{p+1}^1(X) \subset \mathbf{P}(X)$  (see (101)). We now show that  $f_*$  is projectively measurable. Let  $c \in \mathbb{R}$ . We have that

$$f_*^{-1}([-\infty, c]) = \{x \in X, \exists y \in Y, (x, y) \in D \text{ and } f(x, y) < c\} = \text{proj}_X(D \cap f^{-1}([-\infty, c])).$$

As  $f$  is projectively measurable, there exists some  $q \geq 1$  such that  $f$  is  $\Delta_q^1(X \times Y)$ -measurable and  $f^{-1}([-\infty, c]) \in \Delta_q^1(X \times Y) \subset \Delta_{p+q}^1(X \times Y) \subset \Sigma_{p+q}^1(X \times Y)$  (see (iv) in Proposition 8 and (100)). We also have that  $D \in \Delta_p^1(X \times Y) \subset \Sigma_{p+q}^1(X \times Y)$ . Thus, using again (i) in Proposition 8 and (101), we get that  $f_*^{-1}([-\infty, c]) \in \Sigma_{p+q}^1(X) \subset \Delta_{p+q+1}^1(X)$  and as  $p$  and  $q$  do not depend from  $c$ , Remark 11 shows that  $f_*$  is projectively measurable. Lemma 8 shows that  $-f$  is projectively measurable and so,  $(-f)_*$  is also projectively measurable. Thus, as  $f^* = -(-f)_*$ ,  $f^*$  is projectively measurable.

*(ii) Measurable selection.*

Assume the (PD) axiom. We show (105). Let  $\epsilon_* > 0$  and  $E := (A_1 \cap B_1) \cup (A_2 \cap B_2) \subset X \times Y$  where,

$$\begin{aligned} A_1 &:= \{(x, y) \in D, f(x, y) < f_*(x) + \epsilon_*\} \\ A_2 &:= \{(x, y) \in D, f(x, y) < -1/\epsilon_*\} \\ B_1 &:= \{x \in \text{proj}_X(D), f_*(x) > -\infty\} \times Y \\ B_2 &:= \{x \in \text{proj}_X(D), f_*(x) = -\infty\} \times Y. \end{aligned}$$

We show that  $\text{proj}_X(E) = \text{proj}_X(D)$ . The “ $\subset$ ” inclusion is trivial as  $E \subset D$ . Let  $x \in \text{proj}_X(D)$ . Then,  $D_x \neq \emptyset$ . If  $f_*(x) > -\infty$ , by definition of the infimum, there exists  $y \in D_x$  such that  $f(x, y) < f_*(x) + \epsilon_*$  and  $(x, y) \in A_1 \cap B_1 \subset E$ . Similarly, if  $f_*(x) = -\infty$ , there exists  $y \in D_x$  such that  $(x, y) \in A_2 \cap B_2 \subset E$ . Thus,  $x \in \text{proj}_X(E)$  and the reverse inclusion is proved. Assume for a moment that  $E \in \mathbf{P}(X \times Y)$ . Let  $\phi_* : \text{proj}_X(E) \rightarrow Y$  be the function given by Proposition 9 i.e.  $\text{Graph}(\phi_*) \subset E$ . Then, (105) holds true for  $\phi_*$  for all  $x \in \text{proj}_X(E) = \text{proj}_X(D)$  and  $\phi_*$  is the desired function as  $E \subset D$ . It remains to show that  $E \in \mathbf{P}(X \times Y)$ . As  $f$  is projectively measurable and  $D \in \mathbf{P}(X \times Y)$ ,  $A_2 = f^{-1}([-\infty, -1/\epsilon_*]) \cap D \in \mathbf{P}(X \times Y)$ , see Proposition 8 (vi). As  $f_*$  is also projectively measurable by (i),  $A_1 \in \mathbf{P}(X \times Y)$ , see Lemma 8. Now, Proposition 8 (vi) again shows that  $\text{proj}_X(D) \in \mathbf{P}(X)$ , that  $\{x \in \text{proj}_X(D), f_*(x) > -\infty\}$  and  $\{x \in \text{proj}_X(D), f_*(x) = -\infty\}$  belong to  $\mathbf{P}(X)$ ,  $B_1$  and  $B_2$  belong to  $\mathbf{P}(X \times Y)$  and thus that  $E \in \mathbf{P}(X \times Y)$ . Finally, recalling that  $f^* = -(-f)_*$ , (106) follows from (105) applied to  $-f$ .  $\square$



**7.5. Integration and measurability** Let  $X$  be a Polish space. Let  $f : X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  be a universally measurable function and let  $p \in \mathfrak{P}(X)$ . We define the  $(-\infty)$  integral denoted by  $\int_- f dp$  and the  $(+\infty)$  integral denoted by  $\int^- f dp$  as follows. When  $\int f^+ dp < +\infty$  or  $\int f^- dp < +\infty$ , both integrals are equal and are defined as the extended integral of  $f$  i.e.

$$\int_- f dp = \int^- f dp := \int f^+ dp - \int f^- dp. \quad (107)$$

Otherwise,  $\int_- f dp := -\infty$  and  $\int^- f dp := +\infty$ . We adopt the usual arithmetic rules in calculations involving  $+\infty$  and  $-\infty$  (see (11)) and that  $+\infty - \infty = -\infty + \infty = -\infty$  (see (10)). The main result of this section is an extension of [5, Proposition 7.43, p169] (where  $+\infty - \infty = +\infty$ ) to the projective hierarchy (see Proposition 11). This result is crucial to show that integrals of projectively measurable functions remain projectively measurable (see Proposition 12). Recall that these integrals exist (see Remark 2). In order to show that, we first establish a lemma which roughly states that the sets in  $\Sigma_{n+1}^1(X)$ , and also the projective sets (see (101)), are close to analytic sets in terms of measure. We need to assume the (PD) axiom as Theorem 3 and Proposition 9 are used in the proof.

**LEMMA 11.** *Assume the (PD) axiom. Let  $X$  be a Polish space,  $\mu \in \mathfrak{P}(X)$ ,  $n \geq 1$  and  $A := \text{proj}_X(C)$  for some  $C \in \Pi_n^1(X \times \mathcal{N})$ . Then, there exists  $C' \in \mathcal{B}(X \times \mathcal{N})$  such that  $C' \subset C$  and  $A \setminus \text{proj}_X(C')$  is a  $\mu$ -null-set.*

*Proof.* As  $A \in \Sigma_{n+1}^1(X) \subset \Delta_{n+2}^1(X) \subset \mathbf{P}(X) \subset \mathcal{B}_c(X)$  (see (98), (101) and Theorem 3), there exists some  $A' \in \mathcal{B}(X)$  such that  $A' \subset A$  and  $A \setminus A'$  is a  $\mu$ -null-set. As  $C \in \Pi_n^1(X \times \mathcal{N}) \subset \Delta_{n+1}^1(X \times \mathcal{N}) \subset \mathbf{P}(X \times \mathcal{N})$  (see (101)), Proposition 9 provides a projectively measurable (and thus universally measurable by Lemma 5 (iii)) function  $\phi : A = \text{proj}_X(C) \rightarrow \mathcal{N}$  such that  $\text{Graph}(\phi) \subset C$ . Let  $\tilde{\phi}(x) := \phi(x)$  if  $x \in A'$  and  $\tilde{\phi}(x) := u_0$  if  $x \in X \setminus A'$  where  $u_0$  is an arbitrary element of  $\mathcal{N}$ . Then,  $\phi$  is also universally measurable. Thus, [5, Lemma 7.27, p173] provides a  $\mathcal{B}(X)$ -measurable function  $\tilde{\psi} : X \rightarrow \mathcal{N}$  such that  $\tilde{\phi} = \tilde{\psi}$  except on a  $\mu$ -null-set. Let  $B' \in \mathcal{B}(X)$  such that  $X \setminus B'$  is a  $\mu$ -null-set and  $\tilde{\phi}(x) = \tilde{\psi}(x)$  for all  $x \in B'$ . Let  $C' := \{(x, u) \in X \times \mathcal{N}, u = \tilde{\psi}(x) \text{ and } x \in A' \cap B'\}$ . Then,  $C' \in \mathcal{B}(X \times \mathcal{N})$  (see [5, Corollary 7.14.1, p121]). Moreover,  $C' \subset C$ . Indeed, let  $(x, u) \in C'$ . Then,  $u = \tilde{\psi}(x) = \tilde{\phi}(x) = \phi(x)$  as  $x \in A' \cap B'$ . As  $A' \subset A$ ,  $(x, u) \in \text{Graph}(\phi) \subset C$  and  $C' \subset C$ . Let  $x \in A \setminus \text{proj}_X(C')$ . If  $x \in A' \cap B'$ , then  $\phi(x) = \tilde{\phi}(x) = \tilde{\psi}(x)$  and  $x \in \text{proj}_X(C')$  which is a contradiction. Thus,  $A \setminus \text{proj}_X(C') \subset (A \setminus A') \cup (X \setminus B')$  and  $A \setminus \text{proj}_X(C')$  is indeed a  $\mu$ -null-set.  $\square$

The next proposition extends [5, Proposition 7.43, p169], which holds true for  $\Sigma_1^1$  i.e. analytic sets, to  $\Sigma_n^1$  and thus projective sets. It is based on Lemma 11.

**PROPOSITION 11.** *Assume the (PD) axiom. Let  $X$  be a Polish space,  $n \geq 1$  and  $A \in \Sigma_n^1(X)$ . Then, for all  $r \in \mathbb{R}$ ,*

$$W_r := \{\mu \in \mathfrak{P}(X), \mu(A) \geq r\} \in \Sigma_n^1(\mathfrak{P}(X)). \quad (108)$$

First recall from Remark 2 that  $\mu(A)$  is well-defined.

*Proof.* When  $n = 1$ , the result is shown in [5, Proposition 7.43, p169]. Let  $n \geq 1$ . We prove (108) at  $n + 1$ . Let  $A \in \Sigma_{n+1}^1(X)$ . There exists  $C \in \Pi_n^1(X \times \mathcal{N})$  such that  $A = \text{proj}_X(C)$ . We have that

$$W_r = \{\mu \in \mathfrak{P}(X), \exists C' \in \mathcal{B}(X \times \mathcal{N}), C' \subset C \text{ and } \mu(\text{proj}_X(C')) \geq r\}. \quad (109)$$

Indeed, the “ $\subset$ ” inclusion is a direct consequence of Lemma 11 and the “ $\supset$ ” one is immediate from the monotony of  $\mu$ .

We introduce now a parametrization of  $\mathcal{B}(X \times \mathcal{N}) = \Delta_1^1(X \times \mathcal{N})$  (see (iii) in Proposition 8). Let  $\mathcal{C} := \{0, 1\}^{\mathbb{N}}$  (see [27, p13]). Using [27, Theorem 35.5, p283], there exist  $\mathcal{D} \in \Pi_1^1(\mathcal{C})$  and  $S \in \Sigma_1^1(\mathcal{C} \times X \times \mathcal{N})$  such that  $\Delta_1^1(X \times \mathcal{N}) = \{S_d, d \in \mathcal{D}\}$  where  $S_d := \{(x, u) \in X \times \mathcal{N}, (d, x, u) \in S\}$  for all  $d \in \mathcal{D}$ . Thus,

$$W_r = \{\mu \in \mathfrak{P}(X), \exists d \in \mathcal{D}, S_d \subset C \text{ and } \mu(\text{proj}_X(S_d)) \geq r\} = \text{proj}_{\mathfrak{P}(X)}(\mathcal{Z}_r^1 \cap \mathcal{Z}_r^2 \cap \mathcal{Z}_r^3),$$

where

$$\begin{aligned} \mathcal{Z}_r^1 &:= \{(\mu, d) \in \mathfrak{P}(X) \times \mathcal{C}, d \in \mathcal{D}\} \\ \mathcal{Z}_r^2 &:= \{(\mu, d) \in \mathfrak{P}(X) \times \mathcal{C}, S_d \subset C\} \\ \mathcal{Z}_r^3 &:= \{(\mu, d) \in \mathfrak{P}(X) \times \mathcal{C}, \mu(\text{proj}_X(S_d)) \geq r\}. \end{aligned}$$

Assume for a moment that  $\mathcal{Z}_r^1, \mathcal{Z}_r^2$  and  $\mathcal{Z}_r^3$  belong to  $\Sigma_{n+1}^1(\mathfrak{P}(X) \times \mathcal{C})$ . Then (i) in Proposition 8 ensures that  $W_r \in \Sigma_{n+1}^1(\mathfrak{P}(X))$  which is the desired result.

(i)  $\mathcal{Z}_r^1 \in \Sigma_{n+1}^1(\mathfrak{P}(X) \times \mathcal{C})$ .

We have that  $\mathcal{Z}_r^1 = \mathfrak{P}(X) \times \mathcal{D} \in \Pi_1^1(\mathfrak{P}(X)) \times \Pi_1^1(\mathcal{C}) \subset \Pi_1^1(\mathfrak{P}(X) \times \mathcal{C})$  (see (v) in Proposition 8) and we conclude as  $\Pi_1^1 \subset \Pi_n^1 \subset \Delta_{n+1}^1 \subset \Sigma_{n+1}^1$ , see (iv) and (101) in Proposition 8.

(ii)  $\mathcal{Z}_r^2 \in \Sigma_{n+1}^1(\mathfrak{P}(X) \times \mathcal{C})$ .

Taking the complement, we obtain that

$$\begin{aligned} (\mathfrak{P}(X) \times \mathcal{C}) \setminus \mathcal{Z}_r^2 &= \{(\mu, d) \in \mathfrak{P}(X) \times \mathcal{C}, \exists (x, u) \in X \times \mathcal{N}, (d, x, u) \in S \text{ and } (x, u) \in (X \times \mathcal{N}) \setminus C\} \\ &= \text{proj}_{\mathfrak{P}(X) \times \mathcal{C}}((\mathfrak{P}(X) \times S) \cap (\mathfrak{P}(X) \times \mathcal{C} \times ((X \times \mathcal{N}) \setminus C))). \end{aligned}$$

As  $S \in \Sigma_1^1(\mathcal{C} \times X \times \mathcal{N}) \subset \Sigma_n^1(\mathcal{C} \times X \times \mathcal{N})$  (see (iv) in Proposition 8),  $\mathfrak{P}(X) \times S \in \Sigma_n^1(\mathfrak{P}(X) \times \mathcal{C} \times X \times \mathcal{N})$  (see (v) in Proposition 8). Now, as  $C \in \Pi_n^1(X \times \mathcal{N})$ , (99) shows that  $(X \times \mathcal{N}) \setminus C \in \Sigma_n^1(X \times \mathcal{N})$ . So, we deduce from (i) and (v) in Proposition 8 that  $\mathfrak{P}(X) \times \mathcal{C} \times ((X \times \mathcal{N}) \setminus C) \in \Sigma_n^1(\mathfrak{P}(X) \times \mathcal{C} \times X \times \mathcal{N})$  and that  $(\mathfrak{P}(X) \times \mathcal{C}) \setminus \mathcal{Z}_r^2 \in \Sigma_n^1(\mathfrak{P}(X) \times \mathcal{C})$ . Thus, using (99) and (101),  $\mathcal{Z}_r^2 \in \Pi_n^1(\mathfrak{P}(X) \times \mathcal{C}) \subset \Sigma_{n+1}^1(\mathfrak{P}(X) \times \mathcal{C})$ .

(iii)  $\mathcal{Z}_r^3 \in \Sigma_{n+1}^1(\mathfrak{P}(X) \times \mathcal{C})$ .

Remark first that for all  $d \in \mathcal{C}$ ,  $\text{proj}_X(S_d) = (\text{proj}_{\mathcal{C} \times X}(S))_d$ . As  $S \in \Sigma_1^1(\mathcal{C} \times X \times \mathcal{N})$ , (i) in Proposition 8 shows that  $\text{proj}_{\mathcal{C} \times X}(S) \in \Sigma_1^1(\mathcal{C} \times X)$ . Note now that for all  $(\mu, d) \in \mathfrak{P}(X) \times \mathcal{C}$ ,  $\mu(\text{proj}_X(S_d)) = \mu((\text{proj}_{\mathcal{C} \times X}(S))_d) = (\delta_d \otimes \mu)(\text{proj}_{\mathcal{C} \times X}(S))$ , using Fubini's theorem. So,

$$\mathcal{Z}_r^3 = \zeta^{-1}(\{\nu \in \mathfrak{P}(\mathcal{C} \times X), \nu(\text{proj}_{\mathcal{C} \times X}(S)) \geq r\}),$$

where  $\zeta : \mathfrak{P}(X) \times \mathcal{C} \rightarrow \mathfrak{P}(\mathcal{C} \times X)$  is defined by  $\zeta(\mu, d) := \delta_d \otimes \mu$  for all  $(\mu, d) \in \mathfrak{P}(X) \times \mathcal{C}$ . Then,  $\zeta$  is continuous (see [5, Lemma 7.12, p144] and [5, Corollary 7.21.1, p130]). Recalling that  $\text{proj}_{\mathcal{C} \times X}(S) \in \Sigma_1^1(\mathcal{C} \times X)$  and using (108) for  $n = 1$ , we obtain that  $\{\nu \in \mathfrak{P}(\mathcal{C} \times X), \nu(\text{proj}_{\mathcal{C} \times X}(S)) \geq r\} \in \Sigma_1^1(\mathfrak{P}(\mathcal{C} \times X))$ . Thus,  $\mathcal{Z}_r^3 \in \Sigma_1^1(\mathfrak{P}(X) \times \mathcal{C}) \subset \Sigma_{n+1}^1(\mathfrak{P}(X) \times \mathcal{C})$  using (i) and (iv) in Proposition 8.  $\square$

We are now in position to give an extension of [5, Proposition 7.48, p180]. This is a key result to show that the dynamic programming procedure of Section 4 is well-defined. Proposition 11 will be in force.

**PROPOSITION 12.** *Assume the (PD) axiom. Let  $X$  and  $Y$  be Polish spaces. Let  $f : X \times Y \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  and let  $q$  be a stochastic kernel on  $Y$  given  $X$ . Let  $\lambda : X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  be defined by*

$$\lambda(x) := \int_{-} f(x, y)q(dy|x).$$

(i) *Assume that  $x \mapsto q(\cdot|x)$  is  $\Delta_r^1(X)$ -measurable for some  $r \geq 1$  and that  $f$  is  $\Delta_p(X \times Y)$ -measurable for some  $p \geq 1$ . Then,  $\lambda$  is  $\Delta_{p+r+2}^1(X)$ -measurable.*

(ii) *Assume that  $x \mapsto q(\cdot|x)$  is projectively measurable and that  $f$  is projectively measurable. Then,  $\lambda$  is projectively measurable.*

*Proof.* The proof of (i) is close to the one of [5, Proposition 7.48, p180] but recall that [5] uses the reverse convention to (10). Assume first that  $f \geq 0$ . In this case  $\int_- f(x, y)q(dy|x) = \int f(x, y)q(dy|x)$ . Let  $\theta_f : \mathfrak{P}(X \times Y) \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  be defined by  $\theta_f(\nu) := \int_{X \times Y} f d\nu$  and  $\sigma : X \rightarrow \mathfrak{P}(X \times Y)$  be defined by  $\sigma(x) := \delta_x \otimes q(\cdot|x)$ . Then,  $\lambda(x) = \theta_f(\sigma(x))$ . First, we show that  $\sigma$  is  $\Delta_{r+1}^1(X)$ -measurable. As  $x \mapsto q(\cdot|x)$  is  $\Delta_r^1(X)$ -measurable and  $x \mapsto \delta_x$  is continuous (see [5, Corollary 7.21.1, p130]) and thus  $\Delta_r^1(X)$ -measurable (see (iii) and (iv) in Proposition 8), we get using Lemma 6 (i) that  $x \mapsto (q(\cdot|x), \delta_x)$  is  $\Delta_r^1(X)$ -measurable. Recalling that  $(\nu, \mu) \mapsto \nu \otimes \mu$  is continuous and thus  $\mathcal{B}(\mathfrak{P}(X) \times \mathfrak{P}(Y)) = \Delta_1^1(\mathfrak{P}(X) \times \mathfrak{P}(Y))$ -measurable (see (iii) in Proposition 8), Lemma 10 (i) proves that  $\sigma$  is  $\Delta_{r+1}^1(X)$ -measurable. Assume for a moment that  $\theta_f$  is  $\Delta_{p+1}^1(\mathfrak{P}(X \times Y))$ -measurable. Then,  $\lambda = \theta_f \circ \sigma$  is  $\Delta_{p+r+2}^1(X)$ -measurable using again Lemma 10. We now prove that  $\theta_f$  is  $\Delta_{p+1}^1(\mathfrak{P}(X \times Y))$ -measurable. For all  $n \geq 0$ , let  $f_n(x, y) := \min(n, f(x, y))$  and  $E_n := \{(x, y, b) \in X \times Y \times \mathbb{R}, f_n(x, y) \leq b \leq n\}$ . Then,  $f_n \uparrow f$ . Let  $\mu$  be the Lebesgue measure on  $\mathbb{R}$  and let  $\nu \in \mathfrak{P}(X \times Y)$ . By Fubini's theorem, we get that

$$(\nu \otimes \mu)(E_n) = \int_{X \times Y} \int_{\mathbb{R}} 1_{E_n} d\mu d\nu = \int_{X \times Y} [n - f_n] d\nu = n - \int_{X \times Y} f_n d\nu.$$

Let  $c \in \mathbb{R}$ . As the sequence  $(f_n)_{n \geq 0}$  is nondecreasing, the monotone convergence theorem shows that

$$\begin{aligned} \left\{ \nu \in \mathfrak{P}(X \times Y), \int_{X \times Y} f d\nu \leq c \right\} &= \bigcap_{\substack{n=1 \\ +\infty}}^{+\infty} \left\{ \nu \in \mathfrak{P}(X \times Y), \int_{X \times Y} f_n d\nu \leq c \right\} \\ &= \bigcap_{\substack{n=1 \\ +\infty}}^{+\infty} \{ \nu \in \mathfrak{P}(X \times Y), (\nu \otimes \mu)(E_n) \geq n - c \} \\ &= \bigcap_{n=1}^{+\infty} \Psi^{-1}(\{ \nu \in \mathfrak{P}(X \times Y \times \mathbb{R}), \nu(E_n) \geq n - c \}), \end{aligned}$$

where  $\Psi : \mathfrak{P}(X \times Y) \rightarrow \mathfrak{P}(X \times Y \times \mathbb{R})$  is such that  $\Psi(\nu) := \nu \otimes \mu$  for all  $\nu \in \mathfrak{P}(X \times Y)$ . Fix  $n \geq 0$ . Assume that  $E_n \in \Delta_p^1(X \times Y \times \mathbb{R}) \subset \Sigma_p^1(X \times Y \times \mathbb{R})$ . Proposition 11 shows that  $\{ \nu \in \mathfrak{P}(X \times Y \times \mathbb{R}), \nu(E_n) \geq n - c \} \in \Sigma_p^1(\mathfrak{P}(X \times Y \times \mathbb{R}))$ . As  $\Psi$  is continuous (see [5, Lemma 7.12, p144]), (i) and (101) in Proposition 8 show that  $\{ \nu \in \mathfrak{P}(X \times Y), \int_{X \times Y} f d\nu \leq c \} \in \Sigma_p^1(\mathfrak{P}(X \times Y)) \subset \Delta_{p+1}^1(\mathfrak{P}(X \times Y))$  and  $\theta_f$  is  $\Delta_{p+1}^1(\mathfrak{P}(X \times Y))$ -measurable (see Remark 11).

We still have to prove that  $E_n \in \Delta_p^1(X \times Y \times \mathbb{R})$ . Remark first that  $E_n = \{(x, y, b) \in X \times Y \times \mathbb{R}, f_n(x, y) \leq b\} \cap (X \times Y \times (-\infty, n])$  and that  $X \times Y \times (-\infty, n] \in \mathcal{B}(X \times Y \times \mathbb{R}) = \Delta_1^1(X \times Y \times \mathbb{R}) \subset \Delta_p^1(X \times Y \times \mathbb{R})$  (see (iii) and (iv) in Proposition 8). Then,  $(x, y, b) \mapsto f_n(x, y)$  is  $\Delta_p^1(X \times Y \times \mathbb{R})$ -measurable (see Lemma 8 and (v) in Proposition 8). As  $(x, y, b) \mapsto b$  is  $\mathcal{B}(X \times Y \times \mathbb{R})$ -measurable, Lemma 8 shows that  $(x, y, b) \mapsto f_n(x, y) - b$  is  $\Delta_p^1(X \times Y \times \mathbb{R})$ -measurable and so that  $E_n \in \Delta_p^1(X \times Y \times \mathbb{R})$  which concludes the proof of (i) when  $f \geq 0$ . In the general case, if  $f : X \times Y \mapsto \mathbb{R} \cup \{-\infty, +\infty\}$ , (107) implies that

$$\int_- f(x, y)q(dy|x) = \int f^+(x, y)q(dy|x) - \int f^-(x, y)q(dy|x),$$

using the convention  $+\infty - \infty = -\infty + \infty = -\infty$ . Lemma 8 shows that  $f^+ = \max(f, 0)$  and  $f^- = \max(-f, 0)$  are  $\Delta_p^1(X \times Y)$ -measurable. So,  $x \mapsto \int f^\pm(x, y)q(dy|x)$  are  $\Delta_{p+r+2}^1(X)$ -measurable and finally  $\lambda$  is  $\Delta_{p+r+2}^1(X)$ -measurable (see Lemma 8 again). This shows (i).

If  $f$  is a projectively measurable function and  $q$  is a projectively measurable stochastic kernel, then there exist some  $p \geq 1$  and  $r \geq 1$  such that  $f$  is  $\Delta_p^1(X)$ -measurable and  $x \mapsto q(\cdot|x)$  is  $\Delta_r^1(X)$ -measurable and (i) shows (ii).  $\square$

This lemma is an application of Propositions 10 and 11 and allows to solve measurability issues in Section 4.

LEMMA 12. *Assume that the (PD) axiom and Assumptions 1 and 2 hold true. Let  $0 \leq t \leq T-1$ . Let  $f : \Omega^{t+1} \times \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  be projectively measurable. We define  $\lambda : \Omega^t \times \mathbb{R} \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}) \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ ,  $\lambda_{\inf} : \Omega^t \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  and  $\lambda_{\sup} : \Omega^t \times \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  as follows*

$$\begin{aligned} \lambda(\omega^t, x, h, p) &:= \int_{-} f(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))p(d\omega_{t+1}) \\ \lambda_{\inf}(\omega^t, x, h) &:= \inf_{p \in \mathcal{Q}_{t+1}(\omega^t)} \lambda(\omega^t, x, h, p) \quad \text{and} \quad \lambda_{\sup}(\omega^t, x) := \sup_{h \in \mathbb{R}^d} \lambda_{\inf}(\omega^t, x, h). \end{aligned}$$

Let  $q \in SK_{t+1}$ . We also define  $\lambda^q : \Omega^t \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  and  $\lambda_{\sup}^q : \Omega^t \times \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  as follows

$$\begin{aligned} \lambda^q(\omega^t, x, h) &:= \int_{-} f(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))q(d\omega_{t+1}|\omega^t) \\ \lambda_{\sup}^q(\omega^t, x) &:= \sup_{h \in \mathbb{R}^d} \lambda^q(\omega^t, x, h). \end{aligned}$$

Then  $\lambda$ ,  $\lambda_{\inf}$ ,  $\lambda_{\sup}$ ,  $\lambda^q$  and  $\lambda_{\sup}^q$  are projectively measurable.

*Proof.* Let  $g : (\omega^t, x, h, p, \omega_{t+1}) \mapsto f(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))$ . As Assumption 1 holds true, Lemmata 6, 7, 5 and 10 show that  $g$  is projectively measurable. Consider the projectively measurable stochastic kernel  $q$  defined by  $q(\cdot|\omega^t, x, h, p) := p(\cdot)$ . Indeed,  $(\omega^t, x, h, p) \mapsto p(\cdot)$  is Borel and thus projectively measurable, see Lemma 5. Now, we need the (PD) axiom. Proposition 12 (ii) shows that  $\lambda$  is projectively measurable. Let  $q \in SK_{t+1}$  and consider the projectively measurable stochastic kernel  $\hat{q}$  defined by  $\hat{q}(\cdot|\omega^t, x, h, p) := q(\cdot|\omega^t)$  (recall that  $q \in SK_{t+1}$  and Proposition 8). So, Proposition 12 (ii) again proves that  $\lambda^q$  is projectively measurable. Assumption 2 shows that  $\text{proj}_{\Omega^t}(\text{Graph}(\mathcal{Q}_{t+1})) = \Omega^t$  as  $\mathcal{Q}_{t+1}$  is non empty. Let  $D := \{(\omega^t, x, h, p) \in \Omega^t \times \mathbb{R} \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}), p \in \mathcal{Q}_{t+1}(\omega^t)\}$ . Using Assumption 2 again and (102) in Proposition 8,  $D \in \mathbf{P}(\Omega^t \times \mathbb{R} \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}))$ . So, Proposition 10 proves that  $\lambda_{\inf}$  is projectively measurable. Finally, Proposition 10 with  $D = \Omega^t \times \mathbb{R} \times \mathbb{R}^d$  shows that  $\lambda_{\sup}$  and  $\lambda_{\sup}^q$  are also projectively measurable.  $\square$

**8. Appendix** The first part collects the missing proofs of Section 2 (proofs of Lemmata 1 and 2) and also Proposition 13 which shows that the graph of the affine hull of the conditional support of  $\Delta S_{t+1}$  is a projective set. This generalizes [10, Lemma 4.3] and [8, Lemma 2.6]. The second part presents the missing results of Sections 4 and 5. Lemma 13 ensures that Assumption 10 is preserved through dynamic programming while Lemma 14 shows that  $N_t^*$  (see (64)) and  $N_t^P$  (see (66)) are almost-surely finite and may be integrable. Lemma 15 is used directly in the proof of Theorem 1 and shows that an infimum on  $\mathcal{Q}^T$  can coincide with an infimum on  $\mathcal{H}^T$ . The third part provides some properties of the sets  $\mathcal{M}^t$  and  $\mathcal{M}^t(P)$  (see Lemmata 16, 17 and 18) which are used in the proof of Theorem 2. Finally, the fourth part provides the proof of Proposition 8.

**8.1. Proofs of Section 2** We provide the proofs of Lemmata 1 and 2 of Section 2. We also state and prove Proposition 13 which is used in the proof of Lemma 2 and also in the proof of Proposition 7. We first give the proof of Lemma 1 which is an extension of [8, Theorem 3.6, Remark 3.31] to the projective setup.

**Proof of Lemma 1** Assertion (i) is trivially true.

*Proof of (v).*

Fix  $P^* := q_1^{P^*} \otimes \cdots \otimes q_T^{P^*} \in \mathcal{H}^T$ . Let  $P \in \mathcal{P}^T$ . Then, there exist  $Q \in \mathcal{Q}^T$  and  $(\lambda_i)_{1 \leq i \leq T} \subset (0, 1]$  such that  $P = (\lambda_1 q_1^{P^*} + (1 - \lambda_1) q_1^Q) \otimes \cdots \otimes (\lambda_T q_T^{P^*} + (1 - \lambda_T) q_T^Q)$ . As  $\mathcal{Q}_{t+1}$  is convex-valued for all  $0 \leq t \leq T - 1$  (see Assumption 2), we have that  $P \in \mathcal{Q}^T$ . Fix  $0 \leq t \leq T - 1$ . Let  $\omega^t \in \Omega^t$  and  $h \in D_{P^*}^{t+1}(\omega^t)$ . Then for all  $\epsilon > 0$ ,  $q_{t+1}^{P^*}(\Delta S_{t+1}(\omega^t, \cdot) \in \mathbb{B}(h, \epsilon) | \omega^t) > 0$  where  $\mathbb{B}(h, \epsilon)$  is an open ball centered at  $h$  with radius  $\epsilon$ . So,  $\lambda_{t+1} q_{t+1}^{P^*}(\Delta S_{t+1}(\omega^t, \cdot) \in \mathbb{B}(h, \epsilon) | \omega^t) + (1 - \lambda_{t+1}) q_{t+1}^Q(\Delta S_{t+1}(\omega^t, \cdot) \in \mathbb{B}(h, \epsilon) | \omega^t) > 0$  and  $h \in D_P^{t+1}(\omega^t)$ . Thus,  $D_{P^*}^{t+1}(\omega^t) \subset D_P^{t+1}(\omega^t)$ . Moreover, (5) and (6) imply that  $D_P^{t+1}(\omega^t) \subset D^{t+1}(\omega^t)$ . As a result, we find that  $\text{Aff}(D_{P^*}^{t+1}) \subset \text{Aff}(D_P^{t+1}) \subset \text{Aff}(D^{t+1})$  and  $\text{ri}(\text{conv}(D_{P^*}^{t+1})) \subset \text{ri}(\text{conv}(D_P^{t+1}))$ . As  $P^* \in \mathcal{H}^T$ ,  $0 \in \text{ri}(\text{conv}(D_{P^*}^{t+1}))(\cdot)$   $\mathcal{Q}^t$ -q.s. and  $\text{Aff}(D_{P^*}^{t+1})(\cdot) = \text{Aff}(D^{t+1})(\cdot)$   $\mathcal{Q}^t$ -q.s. for all  $0 \leq t \leq T - 1$ . Thus, we conclude that  $P \in \mathcal{H}^T$ .

*Proof of (ii).*

As  $\mathcal{H}^T \subset \mathcal{Q}^T$ , any  $\mathcal{Q}^T$  polar set is a  $\mathcal{H}^T$  polar set. Let  $A$  be a  $\mathcal{H}^T$ -polar set. Then, there exists  $N \in \mathcal{B}(\Omega^T)$  such that  $A \subset N$  and  $P(N) = 0$ , for all  $P \in \mathcal{H}^T$ . Let  $Q := q_1^Q \otimes \cdots \otimes q_T^Q \in \mathcal{Q}^T$ ,  $P := q_1^P \otimes \cdots \otimes q_T^P \in \mathcal{H}^T$  and  $R := \frac{q_1^Q + q_1^P}{2} \otimes \cdots \otimes \frac{q_T^Q + q_T^P}{2}$ . Then, (v) proves that  $R \in \mathcal{P}^T \subset \mathcal{H}^T$  and so that  $R(N) = 0$ . Now, [12, Proposition 12] shows that  $Q \ll R$  and so that  $Q(N) = 0$ . As  $Q$  is arbitrary in  $\mathcal{Q}^T$ ,  $Q(N) = 0$  for all  $Q \in \mathcal{Q}^T$  and  $A$  is thus a  $\mathcal{Q}^T$ -polar set.

*Proof of (iii).*

Let  $P := q_1^P \otimes \cdots \otimes q_T^P \in \mathcal{H}^T$ . By definition of  $\mathcal{H}^T$ , for all  $0 \leq t \leq T - 1$ ,  $\omega^t \mapsto q_{t+1}(\cdot | \omega^t)$  is projectively measurable and Lemma 5 shows that it is also universally measurable under the (PD) axiom. So,  $q_{t+1}$  is also a universally measurable stochastic kernel. Similarly, Assumption 1 and Lemma 5 imply that  $S_t$  is universally measurable for all  $0 \leq t \leq T$ . Thus, using [5, Lemma 7.27, p173], one can find for all  $0 \leq t \leq T$ , a  $\mathcal{B}(\Omega^t)$ -measurable function  $\hat{S}_t$  such that  $\hat{S}_t = S_t$   $P^t$ -a.s. In particular,  $\Delta \hat{S}_{t+1} = \Delta S_{t+1}$   $P^{t+1}$ -a.s. As a result, using Fubini's theorem, we have that for all  $0 \leq t \leq T - 1$ ,  $E^t := \{\omega^t \in \Omega^t, q_{t+1}^P(\Delta \hat{S}_{t+1}(\omega^t, \cdot) = \Delta S_{t+1}(\omega^t, \cdot) | \omega^t) = 1\}$  is a  $P^t$ -full-measure set. For all  $0 \leq t \leq T - 1$ , let  $\hat{D}_P^{t+1}$  be the conditional support of  $\Delta \hat{S}_{t+1}$  relatively to  $P$ , see (6). Let  $0 \leq t \leq T - 1$  and  $\omega^t \in E^t$ . Then, for all closed subset  $A$  of  $\mathbb{R}^d$ ,  $q_{t+1}^P(\Delta \hat{S}_{t+1}(\omega^t, \cdot) \in A | \omega^t) = q_{t+1}^P(\Delta S_{t+1}(\omega^t, \cdot) \in A | \omega^t)$ . So,  $D_P^{t+1}(\omega^t) = \hat{D}_P^{t+1}(\omega^t)$  and also  $\text{ri}(\text{conv}(D_P^{t+1}))(\omega^t) = \text{ri}(\text{conv}(\hat{D}_P^{t+1}))(\omega^t)$ . Now, recalling that  $E^t$  is a  $P^t$ -full measure set, we have that  $\text{ri}(\text{conv}(D_P^{t+1}))(\cdot) = \text{ri}(\text{conv}(\hat{D}_P^{t+1}))(\cdot)$   $P^t$ -a.s. Then, by definition of  $\mathcal{H}^T$ , we deduce that  $0 \in \text{ri}(\text{conv}(D_P^{t+1}))(\cdot) = \text{ri}(\text{conv}(\hat{D}_P^{t+1}))(\cdot)$   $P^t$ -a.s. As  $\hat{S}$  is Borel measurable, we can apply [23, Theorem 3] and we have that  $NA(P)$  holds true for the price process  $\hat{S} = (\hat{S}_t)_{0 \leq t \leq T}$ . Now, let  $\phi \in \Phi$ . Note that  $V_T^{0, \phi} = \sum_{s=1}^T \phi_s \Delta S_s = \sum_{s=1}^T \phi_s \Delta \hat{S}_s =: \hat{V}_T^{0, \phi}$   $P$ -a.s. Assume that  $V_T^{0, \phi} \geq 0$   $P$ -a.s. Then,  $\hat{V}_T^{0, \phi} \geq 0$   $P$ -a.s. and  $\hat{V}_T^{0, \phi} = 0$   $P$ -a.s. as  $NA(P)$  holds true for  $\hat{S}$ . It follows that  $V_T^{0, \phi} = 0$   $P$ -a.s. and  $NA(P)$  also holds true for  $S$ .

*Proof of (iv).*

Let  $\phi \in \Phi$  such that  $V_T^{0, \phi} \geq 0$   $\mathcal{Q}^T$ -q.s. Then, using (i),  $V_T^{0, \phi} \geq 0$   $P$ -a.s for all  $P \in \mathcal{H}^T$ . Using (iii), we have that  $V_T^{0, \phi} = 0$   $P$ -a.s for all  $P \in \mathcal{H}^T$ . Thus,  $V_T^{0, \phi} = 0$   $\mathcal{H}^T$ -q.s. and (ii) implies that  $V_T^{0, \phi} = 0$   $\mathcal{Q}^T$ -q.s. Thus,  $NA(\mathcal{Q}^T)$  holds true.  $\square$

Now, we provide the proof of Lemma 2 which extends [8, Proposition 3.35] to the projective setup.

### Proof of Lemma 2

Let  $P := q_1^P \otimes \cdots \otimes q_T^P \in \mathcal{H}^T$ . Let  $0 \leq t \leq T - 1$  and define

$$E^t := \{(\omega^t, \alpha) \in \Omega^t \times \mathbb{R}, \forall h \in \text{Aff}(D^{t+1})(\omega^t), h \neq 0, \alpha \in (0, 1], q_{t+1}^P(h \Delta S_{t+1}(\omega^t, \cdot) < -\alpha |h| \omega^t) \geq \alpha\}.$$

*Existence of projectively measurable  $\alpha_t^P(\cdot)$*

Assume for a moment that  $E^t \in \mathbf{P}(\Omega^t \times \mathbb{R})$ . Proposition 9 shows that there exists some projectively measurable  $\bar{\alpha}_t^P : \text{proj}_{\Omega^t}(E^t) \rightarrow \mathbb{R}$  such that  $\text{Graph}(\bar{\alpha}_t^P) \subset E^t$ . As  $E^t$  is a projective set, (vi) in Proposition 8 shows that  $\text{proj}_{\Omega^t}(E^t)$  (and its complement) is also a projective set. Let  $\alpha_t^P : \Omega^t \rightarrow$

$(0, 1]$  be defined by  $\alpha_t^P(\omega^t) = \bar{\alpha}_t^P(\omega^t)$  if  $\omega^t \in \text{proj}_{\Omega^t}(E^t)$  and  $\alpha_t^P(\omega^t) = 1$  otherwise. Then,  $\alpha_t^P$  is projectively measurable, see Lemmata 8 and 10. Let  $\omega^t \in \text{proj}_{\Omega^t}(E^t)$ . As  $\text{Graph}(\bar{\alpha}_t^P) \subset E^t$ , for all  $h \in \text{Aff}(D^{t+1})(\omega^t)$  such that  $h \neq 0$ ,  $q_{t+1}^P(h\Delta S_{t+1}(\omega^t, \cdot) < -\alpha_t^P(\omega^t)|h||\omega^t) \geq \alpha_t^P(\omega^t)$ . Thus,  $\text{proj}_{\Omega^t}(E^t) \subset \Omega_{qNA}^{t,P}$ .

$\Omega_{qNA}^{t,P}$  is a  $\mathcal{Q}^t$ -full-measure set. Let

$$A^t := \{\omega^t \in \Omega^t, 0 \in \text{ri}(\text{Conv}(D_P^{t+1}))(\omega^t), \text{Aff}(D_P^{t+1})(\omega^t) = \text{Aff}(D^{t+1})(\omega^t)\}.$$

As  $P \in \mathcal{H}^T$ ,  $A^t$  is  $\mathcal{Q}^t$ -full-measure set. Let  $\omega^t \in A^t$ . Then, using for example [39, Proposition 3.3], there exist some constants  $\beta, \kappa \in (0, 1]$  such that for all  $h \in \text{Aff}(D_P^{t+1})(\omega^t)$ ,  $h \neq 0$ ,

$$q_{t+1}^P(h\Delta S_{t+1}(\omega^t, \cdot) < -\beta|h||\omega^t) \geq \kappa.$$

As  $\omega^t \in A^t$ ,  $\text{Aff}(D_P^{t+1})(\omega^t) = \text{Aff}(D^{t+1})(\omega^t)$  and setting  $\alpha := \min(\beta, \kappa)$ , we find that  $\omega^t \in \text{proj}_{\Omega^t}(E^t)$ . Thus  $A^t \subset \text{proj}_{\Omega^t}(E^t)$  and as  $\text{proj}_{\Omega^t}(E^t) \subset \Omega_{qNA}^{t,P}$  using the first part of the proof, it follows that  $\Omega_{qNA}^{t,P}$  is a  $\mathcal{Q}^t$ -full-measure set.

$$E^t \in \mathbf{P}(\Omega^t \times \mathbb{R})$$

For all  $(\omega^t, \alpha, h) \in \Omega^t \times \mathbb{R} \times \mathbb{R}^d$ , let

$$\lambda(\omega^t, \alpha, h) := q_{t+1}^P(h\Delta S_{t+1}(\omega^t, \cdot) < -\alpha|h||\omega^t) - \alpha = \int_{\Omega_{t+1}} f(\omega^t, \omega_{t+1}, \alpha, h) q_{t+1}^P(d\omega_{t+1}|\omega^t),$$

where for all  $(\omega^t, \omega_{t+1}, \alpha, h) \in \Omega^t \times \Omega_{t+1} \times \mathbb{R} \times \mathbb{R}^d$ ,  $f(\omega^t, \omega_{t+1}, \alpha, h) := \mathbf{1}_{\{h\Delta S_{t+1}(\omega^t, \omega_{t+1}) + \alpha|h| < 0\}} - \alpha$ . As  $f$  is projectively measurable (see Assumption 1 and Lemmata 5, 7, 8 and 10), Proposition 12 (ii) shows that  $\lambda$  is projectively measurable. Now, we have that

$$\begin{aligned} (\Omega^t \times \mathbb{R}) \setminus E^t &= \{(\omega^t, \alpha) \in \Omega^t \times \mathbb{R}, \exists h \in \text{Aff}(D^{t+1})(\omega^t), h \neq 0, \alpha \in (0, 1], \lambda(\omega^t, \alpha, h) < 0\} \\ &= \text{proj}_{\Omega^t \times \mathbb{R}}(\mathcal{Z}_1 \cap \mathcal{Z}_2 \cap \mathcal{Z}_3), \end{aligned}$$

where  $\mathcal{Z}_1 := \sigma^{-1}(\text{Graph}(\text{Aff}(D^{t+1})) \times (0, 1])$  with  $\sigma(\omega^t, \alpha, h) := (\omega^t, h, \alpha)$ ,  $\mathcal{Z}_2 := \Omega^t \times \mathbb{R} \times (\mathbb{R}^d \setminus \{0\})$  and  $\mathcal{Z}_3 := \lambda^{-1}((-\infty, 0))$ . Proposition 13 below and (vi) in Proposition 8 show that  $\mathcal{Z}_1 \in \mathbf{P}(\Omega^t \times \mathbb{R} \times \mathbb{R}^d)$ . We have that  $\mathcal{Z}_2 \in \mathcal{B}(\Omega^t \times \mathbb{R} \times \mathbb{R}^d) \subset \mathbf{P}(\Omega^t \times \mathbb{R} \times \mathbb{R}^d)$  using again (vi) in Proposition 8. Finally,  $\mathcal{Z}_3 \in \mathbf{P}(\Omega^t \times \mathbb{R} \times \mathbb{R}^d)$  as  $\lambda$  is projectively measurable. Thus, using again (vi) in Proposition 8,  $\mathcal{Z}_1 \cap \mathcal{Z}_2 \cap \mathcal{Z}_3 \in \mathbf{P}(\Omega^t \times \mathbb{R} \times \mathbb{R}^d)$ ,  $(\Omega^t \times \mathbb{R}) \setminus E^t \in \mathbf{P}(\Omega^t \times \mathbb{R})$  and  $E^t \in \mathbf{P}(\Omega^t \times \mathbb{R})$ .  $\square$

The next proposition, which is used in the proof of Lemma 2 and Proposition 7, allows to find a measurable optimal strategy for Problem 18 that stays in the affine hull of the quasi-sure support. It generalizes [10, Lemma 4.3] and [8, Lemma 2.6].

**PROPOSITION 13.** *Assume the (PD) axiom. Assume that Assumptions 1 and 2 hold true. Let  $0 \leq t \leq T - 1$ . Then, there exists some  $q \geq 1$  such that the set valued mapping  $D^{t+1}$  is  $\Delta_q^1(\Omega^t)$ -measurable (in the sense of [40, Definition 14.1, p643]). Moreover,*

$$\text{Graph}(\text{Aff}(D^{t+1})) \in \mathbf{P}(\Omega^t \times \mathbb{R}^d). \quad (110)$$

*Proof.* Let  $O$  be an open set of  $\mathbb{R}^d$ . The set-valued mapping  $D^{t+1}$  is  $\Delta_q^1(\Omega^t)$ -measurable for some  $q \geq 1$  if  $D_{-1}^{t+1}(O) \in \Delta_q^1(\Omega^t)$  where

$$D_{-1}^{t+1}(O) := \{\omega^t \in \Omega^t, D^{t+1}(\omega^t) \cap O \neq \emptyset\}.$$



Note that for all  $\omega^t \in \Omega^t$ ,  $h \in D^{t+1}(\omega^t)$  if and only if for all  $n \geq 1$ , there exists  $p \in \mathcal{Q}_{t+1}(\omega^t)$  such that  $p(\Delta S_{t+1}(\omega^t, \cdot) \in \mathbb{B}(h, 1/n)) > 0$  where  $\mathbb{B}(h, 1/n)$  is the open ball centered at  $h$  with radius  $1/n$ . So,

$$D_{-1}^{t+1}(O) = \{\omega^t \in \Omega^t, \exists h \in O, \forall n \geq 1, \exists p \in \mathcal{Q}_{t+1}(\omega^t), p(\Delta S_{t+1}(\omega^t, \cdot) \in \mathbb{B}(h, 1/n)) > 0\}.$$

Let

$$\begin{aligned} F^n &:= \{(\omega^t, h, \omega_{t+1}) \in \Omega^t \times \mathbb{R}^d \times \Omega_{t+1}, \Delta S_{t+1}(\omega^t, \omega_{t+1}) \in \mathbb{B}(h, 1/n)\} \\ F_{\omega^t, h}^n &:= \{\omega_{t+1} \in \Omega_{t+1}, (\omega^t, h, \omega_{t+1}) \in F^n\} \text{ for all } (\omega^t, h) \in \Omega^t \times \mathbb{R}^d \\ E^n &:= \{(\omega^t, h, p) \in \Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}), p(F_{\omega^t, h}^n) > 0\}. \end{aligned}$$

Then,

$$\begin{aligned} D_{-1}^{t+1}(O) &= \{\omega^t \in \Omega^t, \exists h \in O, \forall n \geq 1, \exists p \in \mathcal{Q}_{t+1}(\omega^t), p(F_{\omega^t, h}^n) > 0\} \\ &= \text{proj}_{\Omega^t} \left( (\Omega^t \times O) \cap \bigcap_{n \geq 1} \{(\omega^t, h) \in \Omega^t \times \mathbb{R}^d, \exists p \in \mathcal{Q}_{t+1}(\omega^t), p(F_{\omega^t, h}^n) > 0\} \right) \\ &= \text{proj}_{\Omega^t} \left( (\Omega^t \times O) \cap \bigcap_{n \geq 1} \text{proj}_{\Omega^t \times \mathbb{R}^d} (\sigma^{-1}(\text{Graph}(\mathcal{Q}_{t+1}) \times \mathbb{R}^d) \cap E^n) \right), \end{aligned} \quad (111)$$

where  $\sigma(\omega^t, h, p) = (\omega^t, p, h)$  for all  $(\omega^t, h, p) \in \Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1})$ . Recalling Assumptions 1 and 2, there exist some  $p, r \geq 1$  such that  $\text{Graph}(\mathcal{Q}_{t+1}) \in \Delta_p^1(\Omega^t \times \mathfrak{P}(\Omega_{t+1}))$  and  $\Delta S_{t+1}$  is  $\Delta_r^1(\Omega^{t+1})$ -measurable (see Lemma 7). We may assume that  $r \leq p$ . Then,  $\Delta S_{t+1}$  is  $\Delta_p^1(\Omega^{t+1})$ -measurable, see Proposition 8 (iv). Assume for a moment that  $E^n \in \Sigma_p^1(\Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}))$ . We have that  $\Omega^t \times O \in \mathcal{B}(\Omega^t \times \mathbb{R}^d) = \Delta_1^1(\Omega^t \times \mathbb{R}^d) \subset \Sigma_1^1(\Omega^t \times \mathbb{R}^d) \subset \Sigma_p^1(\Omega^t \times \mathbb{R}^d)$  (see (100) and (iii) and (i) in Proposition 8). As  $\text{Graph}(\mathcal{Q}_{t+1}) \times \mathbb{R}^d \subset \Sigma_p^1(\Omega^t \times \mathfrak{P}(\Omega_{t+1})) \times \Sigma_p^1(\mathbb{R}^d) \subset \Sigma_p^1(\Omega^t \times \mathfrak{P}(\Omega_{t+1}) \times \mathbb{R}^d)$  (see (v) in Proposition 8) and  $\sigma$  is Borel measurable, we find using (i) in Proposition 8 and (111) that  $D_{-1}^{t+1}(O) \in \Sigma_p^1(\Omega^t) \subset \Delta_{p+1}^1(\Omega^t)$  (see (v) in Proposition 8). Thus, the set valued mapping  $D^{t+1}$  is  $\Delta_{p+1}^1(\Omega^t)$ -measurable. Applying successively [40, Exercise 14.12, p652] and [40, Theorem 14.8, p648] we get that  $\text{Graph}(\text{Aff}(D^{t+1})) \in \Delta_{p+1}^1(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$ . Now, using (iii) to (v) in Proposition 8,

$$\Delta_{p+1}^1(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d) \subset \Delta_{p+1}^1(\Omega^t) \times \Delta_{p+1}^1(\mathbb{R}^d) \subset \Delta_{p+1}^1(\Omega^t \times \mathbb{R}^d) \subset \mathbf{P}(\Omega^t \times \mathbb{R}^d),$$

and (110) is proved. It remains to show that  $E^n \in \Sigma_p^1(\Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}))$ . Let for all  $q \geq 1$ ,

$$E_q^n := \{(\omega^t, h, p) \in \Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}), p(F_{\omega^t, h}^n) \geq 1/q\}.$$

Then,  $E^n = \bigcup_{q \geq 1} E_q^n$ . If we show that for all  $q \geq 1$ ,  $E_q^n \in \Sigma_p^1(\Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}))$ , Proposition 8 (i) proves the claim. First, remark that the function  $(\omega^t, h, \omega_{t+1}) \mapsto \Delta S_{t+1}(\omega^t, \omega_{t+1})$  is  $\Delta_p^1(\Omega^t \times \mathbb{R}^d \times \Omega_{t+1})$ -measurable, see Lemmata 5 and 10. So,  $F^n \in \Delta_p^1(\Omega^t \times \mathbb{R}^d \times \Omega_{t+1}) \subset \Sigma_p^1(\Omega^t \times \mathbb{R}^d \times \Omega_{t+1})$  (see Proposition 8 (iii) and (100)). We have using Fubini's theorem that  $p(F_{\omega^t, h}^n) = (\delta_{\omega^t, h} \otimes p)(F^n)$  for all  $(\omega^t, h, p) \in \Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1})$ . Let  $q \geq 1$ . Then,

$$E_q^n = \kappa^{-1}(\{\nu \in \mathfrak{P}(\Omega^t \times \mathbb{R}^d \times \Omega_{t+1}), \nu(F^n) \geq 1/q\}),$$

where  $\kappa : \Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}) \rightarrow \mathfrak{P}(\Omega^t \times \mathbb{R}^d \times \Omega_{t+1})$  is such that  $\kappa(\omega^t, h, p) = \delta_{\omega^t, h} \otimes p$  for all  $(\omega^t, h, p) \in \Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1})$ . Then,  $\kappa$  is continuous (see [5, Lemma 7.12, p144] and [5, Corollary 7.21.1, p130]). Recalling that  $F^n \in \Sigma_p^1(\Omega^t \times \mathbb{R}^d \times \Omega_{t+1})$  and using (108), we obtain that  $\{\nu \in \mathfrak{P}(\Omega^t \times \mathbb{R}^d \times \Omega_{t+1}), \nu(F^n) \geq 1/q\} \in \Sigma_p^1(\mathfrak{P}(\Omega^t \times \mathbb{R}^d \times \Omega_{t+1}))$ . Thus, (i) in Proposition 8 implies that  $E_q^n \in \Sigma_p^1(\Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}))$ .  $\square$

**8.2. Proofs of Sections 4 and 5** We now provide the missing proofs and results of Section 4 and 5. The first lemma shows that  $\Omega_{10}^t$  is a  $\mathcal{Q}^t$ -full-measure set and  $\Omega_{10}^{t,P}$  is a  $P^t$ -full-measure set. It is used in the proof of Proposition 6.

LEMMA 13. *Assume the (PD) axiom. Assume that Assumptions 1, 2 and 3 hold true. Let  $P := q_1^P \otimes \cdots \otimes q_T^P \in \mathcal{H}^T$  and  $0 \leq t \leq T - 1$ . Then,  $\Omega_{10}^t$  and  $\Omega_{10}^{t,P}$  are projective sets and also universally measurable sets. Assume furthermore that Assumption 7 holds true and that  $U_0^P(1) < +\infty$ , then  $P^t(\Omega_{10}^{t,P}) = 1$ .*

Finally, if Assumptions 1, 2, 3, 6 and 7 hold true,  $\Omega_{10}^t$  is a  $\mathcal{Q}^t$ -full-measure set.

*Proof.* Assume that Assumptions 1, 2 and 3 holds true. For all  $\theta \in \{-1, 1\}^d \cup \{0\}$ , let

$$\begin{aligned} \Omega_{10}^{t,P,\theta} &:= \left\{ \omega^t \in \Omega^t, \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)}(U_{t+1}^P)^+(\omega^t, \cdot, 1 + \theta \Delta S_{t+1}(\omega^t, \cdot)) < +\infty \right\} \\ \Omega_{10}^{t,\theta} &:= \left\{ \omega^t \in \Omega^t, \mathbb{E}_{q_{t+1}^{P^*}(\cdot|\omega^t)} U_{t+1}^+(\omega^t, \cdot, 1 + \theta \Delta S_{t+1}(\omega^t, \cdot)) < +\infty \right\} \\ \Omega_{10,-}^{t,P,\theta} &:= \left\{ \omega^t \in \Omega^t, \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)}(U_{t+1}^P)^-(\omega^t, \cdot, 1 + \theta \Delta S_{t+1}(\omega^t, \cdot)) < +\infty \right\}. \end{aligned} \quad (112)$$

Then, Definitions 6 and 7 imply that  $\Omega_{10}^{t,P} = \bigcap_{\theta \in \{-1, 1\}^d} \Omega_{10}^{t,P,\theta}$  and  $\Omega_{10}^t = \bigcap_{\theta \in \{-1, 1\}^d} \Omega_{10}^{t,\theta}$ . Let  $\theta \in \{-1, 1\}^d \cup \{0\}$ . As  $(U_{t+1}^P)^+$  is projectively measurable (see Proposition 5 (i) and Lemma 8), Lemma 12 applied to  $f = (U_{t+1}^P)^+$  together with Lemma 10 show that  $\omega^t \mapsto \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)}(U_{t+1}^P)^+(\omega^t, \cdot, 1 + \theta \Delta S_{t+1}(\omega^t, \cdot))$  is projectively measurable. This implies that  $\Omega_{10}^{t,P,\theta}$  is a projective set. Similarly,  $\Omega_{10,-}^{t,P,\theta}$  and  $\Omega_{10}^{t,\theta}$  are also projective sets. Thus, using Proposition 8 (vi),  $\Omega_{10}^{t,P}$  and  $\Omega_{10}^t$  are projective sets. Now, Theorem 3 shows that all these sets are universally measurable.

Assume that Assumptions 1, 2, 3 and 7 holds true. If  $U_0^P(1) < +\infty$ , then  $P^t(\Omega_{10}^{t,P,\theta}) = 1$  for all  $\theta \in \{-1, 1\}^d \cup \{0\}$  (and so,  $P^t(\Omega_{10}^{t,P}) = 1$ ).

Assume that  $U_0^P(1) < +\infty$ . We proceed by contraposition and assume that there exists some  $\theta \in \{-1, 1\}^d \cup \{0\}$  such that  $P^t(\Omega_{10}^{t,P,\theta}) < 1$ . Then, for all  $\omega^t \notin \Omega_{10}^{t,P,\theta}$ ,

$$\mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)}(U_{t+1}^P)^+(\omega^t, \cdot, 1 + \theta \Delta S_{t+1}(\omega^t, \cdot)) = +\infty. \quad (113)$$

For all  $0 \leq k \leq t$ , we show by backward induction the following property : there exists some  $\tilde{\Omega}_+^k \in \mathbf{P}(\Omega^k)$  such that  $P^k(\tilde{\Omega}_+^k) > 0$  and  $U_k^P(\omega^k, 1) = +\infty$  for all  $\omega^k \in \tilde{\Omega}_+^k$ . The property at  $k = 0$  will show that  $U_0^P(1) = +\infty$  : a contradiction that proves the claim.

We start with  $k = t$ . Using (113) and Fubini's theorem, we obtain that  $\mathbb{E}_{P_{t+1}}(U_{t+1}^P)^+(\cdot, 1 + \theta \Delta S_{t+1}(\cdot)) = +\infty$ . Thus, Assumption 7 implies that  $\mathbb{E}_{P_{t+1}}(U_{t+1}^P)^-(\cdot, 1 + \theta \Delta S_{t+1}(\cdot)) < +\infty$  and using again Fubini's theorem,

$$\int_{\Omega^t} \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)}(U_{t+1}^P)^-(\omega^t, \cdot, 1 + \theta \Delta S_{t+1}(\omega^t, \cdot)) P^t(d\omega^t) < +\infty$$

and we deduce that  $P^t(\Omega_{10,-}^{t,P,\theta}) = 1$ . Let  $\tilde{\Omega}_+^t := (\Omega^t \setminus \Omega_{10}^{t,P,\theta}) \cap \Omega_{10,-}^{t,P,\theta}$ . Then,  $\tilde{\Omega}_+^t \in \mathbf{P}(\Omega^t)$  (see Proposition 8 (vi)) and  $P^t(\tilde{\Omega}_+^t) = P^t(\Omega^t \setminus \Omega_{10}^{t,P,\theta}) > 0$ . For all  $\omega^t \in \tilde{\Omega}_+^t$ , (15) implies that

$$U_t^P(\omega^t, 1) \geq \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)}(U_{t+1}^P)^+(\omega^t, \cdot, 1 + \theta \Delta S_{t+1}(\omega^t, \cdot)) - \mathbb{E}_{q_{t+1}^P(\cdot|\omega^t)}(U_{t+1}^P)^-(\omega^t, \cdot, 1 + \theta \Delta S_{t+1}(\omega^t, \cdot)). \quad (114)$$

So, using (112), (113) and (114),  $U_t^P(\omega^t, 1) = +\infty$  for all  $\omega^t \in \tilde{\Omega}_+^t$  and the property is proved for  $k = t$ .

Now, we prove the induction step. Assume that the property holds true for some  $1 \leq k + 1 \leq t$ . Define  $\tilde{\Omega}_+^k := \{\omega^k \in \Omega^k, q_{k+1}^P(\tilde{\Omega}_+^{k+1} | \omega^k) > 0\}$ , where

$$\begin{aligned} \tilde{\Omega}_+^{k+1} &= \{\omega_{k+1} \in \Omega_{k+1}, (\omega^k, \omega_{k+1}) \in \tilde{\Omega}_+^{k+1}\} \subset \{\omega_{k+1} \in \Omega_{k+1}, U_{k+1}^P(\omega^k, \omega_{k+1}, 1) = +\infty\} \\ &= \{\omega_{k+1} \in \Omega_{k+1}, (U_{k+1}^P)^+(\omega^k, \omega_{k+1}, 1) = +\infty\}. \end{aligned}$$

As  $\tilde{\Omega}_+^{k+1} \in \mathbf{P}(\Omega^{k+1})$ , Proposition 8 (vi) shows that  $\tilde{\Omega}_{+, \omega^k}^{k+1} \in \mathbf{P}(\Omega_{k+1})$ . Thus,  $\hat{\Omega}_+^k \in \mathbf{P}(\Omega^k)$  using Proposition 12 (ii). Moreover, we have that

$$\begin{aligned} P^{k+1}(\tilde{\Omega}_+^{k+1}) &= P^k \otimes q_{k+1}^P(\tilde{\Omega}_+^{k+1}) = \int_{\hat{\Omega}_+^k} q_{k+1}^P(\tilde{\Omega}_{+, \omega^k}^{k+1} | \omega^k) P^k(d\omega^k) + \int_{\Omega^k \setminus \hat{\Omega}_+^k} q_{k+1}^P(\tilde{\Omega}_{+, \omega^k}^{k+1} | \omega^k) P^k(d\omega^k) \\ &= \int_{\hat{\Omega}_+^k} q_{k+1}^P(\tilde{\Omega}_{+, \omega^k}^{k+1} | \omega^k) P^k(d\omega^k). \end{aligned}$$

As  $P^{k+1}(\tilde{\Omega}_+^{k+1}) > 0$ , we get that  $P^k(\hat{\Omega}_+^k) > 0$ . Now, for all  $\omega^k \in \hat{\Omega}_+^k$ ,

$$\mathbb{E}_{q_{k+1}^P(\cdot | \omega^k)}(U_{k+1}^P)^+(\omega^k, \cdot, 1) \geq \mathbb{E}_{q_{k+1}^P(\cdot | \omega^k)}\left((U_{k+1}^P)^+(\omega^k, \cdot, 1) 1_{\tilde{\Omega}_{+, \omega^k}^{k+1}}(\cdot)\right) = (+\infty) q_{k+1}^P(\tilde{\Omega}_{+, \omega^k}^{k+1} | \omega^k) = +\infty \quad (115)$$

using that  $q_{k+1}^P(\tilde{\Omega}_{+, \omega^k}^{k+1} | \omega^k) > 0$  as  $\omega^k \in \hat{\Omega}_+^k$ . Now, Fubini's theorem and  $P^k(\hat{\Omega}_+^k) > 0$  show that  $\mathbb{E}_{P^{k+1}}(U_{k+1}^P)^+(\cdot, 1) = +\infty$ . Assumption 7 implies then that  $\mathbb{E}_{P^{k+1}}(U_{k+1}^P)^-(\cdot, 1) < +\infty$ . Using again Fubini's theorem,

$$\int_{\Omega^k} \mathbb{E}_{q_{k+1}^P(\cdot | \omega^k)}(U_{k+1}^P)^-(\omega^k, \cdot, 1) P^k(d\omega^k) < +\infty$$

and we deduce that  $P^k(\Omega_{10, -}^{k, P, 0}) = 1$ . Let  $\tilde{\Omega}_+^k := \hat{\Omega}_+^k \cap \Omega_{10, -}^{k, P, 0}$ . Then,  $\tilde{\Omega}_+^k \in \mathbf{P}(\Omega^k)$  (using Proposition 8 (vi)) and  $P^k(\tilde{\Omega}_+^k) = P^k(\hat{\Omega}_+^k) > 0$ . Let  $\omega^k \in \tilde{\Omega}_+^k$ . Using (15) and (115), we see that

$$\begin{aligned} U_k^P(\omega^k, 1) &\geq \mathbb{E}_{q_{k+1}^P(\cdot | \omega^k)} U_{k+1}^P(\omega^k, \cdot, 1) \\ &\geq \mathbb{E}_{q_{k+1}^P(\cdot | \omega^k)} \left( (+\infty) 1_{(\omega^k, \cdot) \in \tilde{\Omega}_{+, \omega^k}^{k+1}} - (U_{k+1}^P)^-(\omega^k, \cdot, 1) \right) \\ &\geq (+\infty) q_{k+1}^P(\tilde{\Omega}_{+, \omega^k}^{k+1} | \omega^k) - \mathbb{E}_{q_{k+1}^P(\cdot | \omega^k)} \left( (U_{k+1}^P)^-(\omega^k, \cdot, 1) \right), \end{aligned}$$

using for the last inequality [5, Lemma 7.11 (a), p139] adapted to convention (10). Recalling (112) and that  $q_{k+1}^P(\tilde{\Omega}_{+, \omega^k}^{k+1} | \omega^k) > 0$  for all  $\omega^k \in \hat{\Omega}_+^k$ , we find that  $U_k^P(\omega^k, 1) = +\infty$  for all  $\omega^k \in \tilde{\Omega}_+^k$ .

If Assumptions 1, 2, 3, 6 and 7 hold true,  $\Omega_{10}^t$  is a  $\mathcal{Q}^t$ -full-measure set.

Let  $\tilde{P} := q_1^{\tilde{P}} \otimes \dots \otimes q_T^{\tilde{P}} \in \mathcal{Q}^T$  and  $\hat{P} := \frac{q_1^{P^*} + q_1^{\tilde{P}}}{2} \otimes \dots \otimes \frac{q_T^{P^*} + q_T^{\tilde{P}}}{2}$ , where  $P^*$  is defined in (61). As  $\hat{P} \in \mathcal{H}^T$  (see (9)), we have that  $U_0^{\hat{P}}(1) < +\infty$  by Assumption 6. So, the preceding step shows that  $\Omega_{10}^{t, \hat{P}}$  is a  $\hat{P}^t$ -full-measure set. Using now (iii) in Proposition 5 at  $t+1$ ,  $U_{t+1} \leq U_{t+1}^{\hat{P}}$  and we get that  $\Omega_{10}^{t, \hat{P}} \subset \Omega_{10}^t$ . Thus,  $\Omega_{10}^t$  is also a  $\hat{P}^t$ -full-measure set. Now, [12, Proposition 12] shows that  $\tilde{P} \ll \hat{P}$  and  $\Omega_{10}^t$  is a  $\tilde{P}^t$ -full-measure set. As  $\tilde{P}$  is arbitrary  $\Omega_{10}^t$  is a  $\mathcal{Q}^t$ -full-measure set.  $\square$

The next lemma ensures that Assumption 12 is preserved through dynamic programming and is used in Proposition 6. It also provides some properties on  $N_t^P$  and  $N_t^*$  which are used in the proof of Theorem 2.

LEMMA 14. *Assume the (PD) axiom. Assume that Assumptions 1, 2, 3, 4 and 5 hold true. For all  $P := q_1^P \otimes \dots \otimes q_T^P \in \mathcal{Q}^T$ , let*

$$\hat{P}_t := \frac{q_1^{P^*} + q_1^P}{2} \otimes \dots \otimes q_t^{P^*} \otimes \dots \otimes \frac{q_T^{P^*} + q_T^P}{2}. \quad (116)$$

Then,  $\hat{P}_t \in \mathcal{H}^T$  and we have that

$$N_{T-1}^P < +\infty \quad P^{T-1} - a.s. \text{ and } N_{T-1}^* < +\infty \quad \mathcal{Q}^{T-1} - q.s. \quad (117)$$

Moreover, for all  $P \in \mathcal{H}^T$ , Assertions (A1) and (B1) below hold true.

(A1) : If  $1/\alpha_{T-1}^P \in \mathcal{M}^{T-1}(P)$  and  $\underline{X}$ ,  $1/|U(\cdot, \underline{X}(\cdot)) + C(\cdot)|$ ,  $C \in \mathcal{M}^T(P)$ , then  $N_{T-1}^P \in \mathcal{M}^{T-1}(P)$ .

(B1) : If  $1/\alpha_{T-1}^{P^*} \in \mathcal{M}^{T-1}(\widehat{P}_T)$  and  $\underline{X}$ ,  $1/|U(\cdot, \underline{X}(\cdot)) + C(\cdot)|$ ,  $C \in \mathcal{M}^T(\widehat{P}_T)$ , then  $N_{T-1}^* \in \mathcal{M}^{T-1}(P)$ . Assume now that there exists some  $1 \leq t \leq T-1$  such that  $\widetilde{\Omega}^{t,P}$  (see (68)) is a  $P^t$ -full-measure set for all  $P \in \mathcal{H}^T$ . Then,

$$N_{t-1}^P < +\infty \quad P^{t-1} - \text{a.s.} \quad \text{and} \quad N_{t-1}^* < +\infty \quad \mathcal{Q}^{t-1} - \text{q.s.} \quad (118)$$

Moreover, for all  $P \in \mathcal{H}^T$ , Assertions (A2) and (B2) below hold true.

(A2) : If  $1/\alpha_{t-1}^P \in \mathcal{M}^{t-1}(P)$  and  $1/\alpha_t^P$ ,  $N_t^P$ ,  $l_t^P$ ,  $C_t \in \mathcal{M}^t(P)$ , then  $N_{t-1}^P \in \mathcal{M}^{t-1}(P)$ .

(B2) : If  $1/\alpha_{t-1}^{P^*} \in \mathcal{M}^{t-1}(\widehat{P}_t)$  and  $1/\alpha_t^{\widehat{P}_t}$ ,  $N_t^{\widehat{P}_t}$ ,  $l_t^{\widehat{P}_t}$ ,  $C_t \in \mathcal{M}^t(\widehat{P}_t)$ , then  $N_{t-1}^* \in \mathcal{M}^{t-1}(P)$ .

*Proof.* The proof is completely similar to [12, Lemma 14] and thus omitted. One only needs to be careful about the definition of  $c_t^P$  which differs here.  $\square$

The following lemma is used in the proof of Theorem 1, see [8, Proposition 3.12] for a related result.

LEMMA 15. Assume the (PD) axiom. Let  $X : \Omega^T \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  be projectively measurable. Assume that  $\mathcal{H}^T \neq \emptyset$  and that  $\mathbb{E}_P X^+ < +\infty$  and  $\mathbb{E}_Q X^- < +\infty$  for all  $P \in \mathcal{H}^T$  and  $Q \in \mathcal{Q}^T$ . Then,  $\inf_{P \in \mathcal{Q}^T} \mathbb{E}_P X = \inf_{P \in \mathcal{H}^T} \mathbb{E}_P X$ .

*Proof.* As  $\mathcal{H}^T \neq \emptyset$ , fix  $P^* := q_1^{P^*} \otimes \dots \otimes q_T^{P^*} \in \mathcal{H}^T$ . Let  $P := q_1^P \otimes \dots \otimes q_T^P \in \mathcal{Q}^T$ . Define for all  $n \geq 1$  and  $1 \leq t \leq T$ ,  $P_n^t = (\frac{1}{n}q_1^{P^*} + (1 - \frac{1}{n})q_1^P) \otimes \dots \otimes (\frac{1}{n}q_t^{P^*} + (1 - \frac{1}{n})q_t^P)$ . Fix  $n \geq 1$ . We show by induction on  $t$  that there exist some  $(R_k^t)_{0 \leq k \leq t-1} \subset \text{Conv}(\mathcal{Q}^t)$  which are independent of  $n$  and satisfy

$$P_n^t = \left(1 - \frac{1}{n}\right)^t P^t + \frac{1}{n^t} \sum_{k=0}^{t-1} \binom{t}{k} (n-1)^k R_k^t, \quad (119)$$

where

$$\text{Conv}(\mathcal{Q}^t) = \left\{ \sum_{i=0}^p \lambda_i Q_i, p \geq 0, \lambda_i \geq 0, \sum_{i=0}^p \lambda_i = 1, (Q_i)_{0 \leq i \leq p} \subset \mathcal{Q}^t \right\},$$

see [40, Theorem 2.27, p53]. We have that  $P_n^1 = \frac{1}{n}q_1^{P^*} + (1 - \frac{1}{n})q_1^P = (1 - \frac{1}{n})P^1 + \frac{1}{n}q_1^{P^*}$ . Let  $R_0^1 := q_1^{P^*}$ . Then,  $R_0^1 \in \mathcal{Q}^1 \subset \text{Conv}(\mathcal{Q}^1)$  and the initialization step holds true. Assume now that the induction hypothesis holds true at  $n$ . Then,

$$\begin{aligned} P_n^{t+1} &= P_n^t \otimes \left( \frac{1}{n}q_{t+1}^{P^*} + \left(1 - \frac{1}{n}\right)q_{t+1}^P \right) = \left(1 - \frac{1}{n}\right) P_n^t \otimes q_{t+1}^P + \frac{1}{n} P_n^t \otimes q_{t+1}^{P^*} \\ &= \left(1 - \frac{1}{n}\right)^{t+1} P^{t+1} + \left(1 - \frac{1}{n}\right) \frac{1}{n^t} \sum_{k=0}^{t-1} \binom{t}{k} (n-1)^k R_k^t \otimes q_{t+1}^P + \\ &\quad \frac{1}{n} \left(1 - \frac{1}{n}\right)^t P^t \otimes q_{t+1}^{P^*} + \frac{1}{n^{t+1}} \sum_{k=0}^{t-1} \binom{t}{k} (n-1)^k R_k^t \otimes q_{t+1}^{P^*}, \end{aligned}$$

using the induction hypothesis at  $n$ . Now,

$$\begin{aligned} \left(1 - \frac{1}{n}\right) \frac{1}{n^t} \sum_{k=0}^{t-1} \binom{t}{k} (n-1)^k R_k^t \otimes q_{t+1}^P &= \frac{1}{n^{t+1}} \sum_{k=0}^{t-1} \binom{t}{k} (n-1)^{k+1} R_k^t \otimes q_{t+1}^P \\ &= \frac{(n-1)^t}{n^{t+1}} t R_{t-1}^t \otimes q_{t+1}^P + \frac{1}{n^{t+1}} \sum_{k=1}^{t-1} \binom{t}{k-1} (n-1)^k R_{k-1}^t \otimes q_{t+1}^P \\ \frac{1}{n^{t+1}} \sum_{k=0}^{t-1} \binom{t}{k} (n-1)^k R_k^t \otimes q_{t+1}^{P^*} &= \frac{1}{n^{t+1}} R_0^t \otimes q_{t+1}^{P^*} + \frac{1}{n^{t+1}} \sum_{k=1}^{t-1} \binom{t}{k} (n-1)^k R_k^t \otimes q_{t+1}^{P^*} \\ \frac{1}{n} \left(1 - \frac{1}{n}\right)^t P^t \otimes q_{t+1}^{P^*} &= \frac{1}{n^{t+1}} (n-1)^t P^t \otimes q_{t+1}^{P^*}. \end{aligned}$$

As a result,

$$\begin{aligned}
P_n^{t+1} &= \left(1 - \frac{1}{n}\right)^{t+1} P^{t+1} + \frac{1}{n^{t+1}}(n-1)^t \left(P^t \otimes q_{t+1}^{P^*} + tR_{t-1}^t \otimes q_{t+1}^P\right) + \frac{1}{n^{t+1}}R_0^t \otimes q_{t+1}^{P^*} \\
&\quad + \frac{1}{n^{t+1}} \sum_{k=1}^{t-1} (n-1)^k \left( \binom{t}{k-1} R_{k-1}^t \otimes q_{t+1}^P + \binom{t}{k} R_k^t \otimes q_{t+1}^{P^*} \right) \\
&= \left(1 - \frac{1}{n}\right)^{t+1} P^{t+1} + \frac{1}{n^{t+1}} \binom{t+1}{t} (n-1)^t \frac{P^t \otimes q_{t+1}^{P^*} + tR_{t-1}^t \otimes q_{t+1}^P}{\binom{t+1}{t}} + \frac{1}{n^{t+1}} R_0^t \otimes q_{t+1}^{P^*} \\
&\quad + \frac{1}{n^{t+1}} \sum_{k=1}^{t-1} \binom{t+1}{k} (n-1)^k \frac{\binom{t}{k-1} R_{k-1}^t \otimes q_{t+1}^P + \binom{t}{k} R_k^t \otimes q_{t+1}^{P^*}}{\binom{t+1}{k}} \\
&= \left(1 - \frac{1}{n}\right)^{t+1} P^{t+1} + \frac{1}{n^{t+1}} \binom{t+1}{t} (n-1)^t R_t^{t+1} + \frac{1}{n^{t+1}} R_0^{t+1} + \frac{1}{n^{t+1}} \sum_{k=1}^{t-1} \binom{t+1}{k} (n-1)^k R_k^{t+1} \\
&= \left(1 - \frac{1}{n}\right)^{t+1} P^{t+1} + \frac{1}{n^{t+1}} \sum_{k=0}^t \binom{t+1}{k} (n-1)^k R_k^{t+1},
\end{aligned}$$

where we have set  $R_0^{t+1} := R_0^t \otimes q_{t+1}^{P^*}$  and for all  $1 \leq k \leq t-1$ ,

$$R_k^{t+1} := \frac{\binom{t}{k-1} R_{k-1}^t \otimes q_{t+1}^P + \binom{t}{k} R_k^t \otimes q_{t+1}^{P^*}}{\binom{t}{k-1} + \binom{t}{k}} = \frac{\binom{t}{k-1} R_{k-1}^t \otimes q_{t+1}^P + \binom{t}{k} R_k^t \otimes q_{t+1}^{P^*}}{\binom{t+1}{k}} \quad (120)$$

$$R_t^{t+1} := \frac{P^t \otimes q_{t+1}^{P^*} + tR_{t-1}^t \otimes q_{t+1}^P}{t+1} = \frac{P^t \otimes q_{t+1}^{P^*} + tR_{t-1}^t \otimes q_{t+1}^P}{\binom{t+1}{t}}. \quad (121)$$

Thus, (119) at  $t+1$  holds true for  $(R_k^{t+1})_{0 \leq k \leq t}$ . It remains to show that  $(R_k^{t+1})_{0 \leq k \leq t} \subset \text{Conv}(\mathcal{Q}^{t+1})$ . We show first that for all  $Q \in \text{Conv}(\mathcal{Q}^t)$  and  $q_{t+1} \in \{q_{t+1}^{P^*}, q_{t+1}^P\}$ ,  $Q \otimes q_{t+1} \in \text{Conv}(\mathcal{Q}^{t+1})$ . Let  $Q \in \text{Conv}(\mathcal{Q}^t)$  and  $q_{t+1} \in \{q_{t+1}^{P^*}, q_{t+1}^P\}$ . Then, there exist  $p \geq 0$ ,  $(Q_i)_{0 \leq i \leq p} \subset \mathcal{Q}^t$  and  $(\lambda_i)_{0 \leq i \leq p} \subset [0, +\infty)^{p+1}$  such that  $\sum_{i=0}^p \lambda_i = 1$  and  $Q = \sum_{i=0}^p \lambda_i Q_i$ . Thus,  $Q \otimes q_{t+1} = \sum_{i=0}^p \lambda_i Q_i \otimes q_{t+1}$ . As  $Q_i \otimes q_{t+1} \in \mathcal{Q}^{t+1}$ , we get that  $Q \otimes q_{t+1} \in \text{Conv}(\mathcal{Q}^{t+1})$ . By the induction hypothesis,  $(R_k^t)_{0 \leq k \leq t-1} \subset \text{Conv}(\mathcal{Q}^t)$  and so,  $R_0^t \otimes q_{t+1}^{P^*}$ ,  $P^t \otimes q_{t+1}^{P^*}$ ,  $R_{t-1}^t \otimes q_{t+1}^P$  and for all  $1 \leq k \leq t-1$ ,  $R_{k-1}^t \otimes q_{t+1}^P$ ,  $R_k^t \otimes q_{t+1}^{P^*}$  belong to  $\text{Conv}(\mathcal{Q}^{t+1})$ . Thus, recalling (120) and (121) and as  $\text{Conv}(\mathcal{Q}^{t+1})$  is trivially a convex set, we have that  $(R_k^{t+1})_{0 \leq k \leq t} \subset \text{Conv}(\mathcal{Q}^{t+1})$ . This concludes the induction.

Now,  $P_n^T \in \mathcal{H}^T$  (see (9)) and using (119),

$$\mathbb{E}_{P_n^T} X = \left(1 - \frac{1}{n}\right)^T \mathbb{E}_P X + \frac{1}{n^T} \sum_{k=0}^{T-1} \binom{T}{k} (n-1)^k \mathbb{E}_{R_k^T} X. \quad (122)$$

We show that  $\mathbb{E}_P X$  and  $\mathbb{E}_{R_k^T} X$  are finite for all  $0 \leq k \leq T-1$ . First, by assumption of the lemma,  $\mathbb{E}_Q X^- < +\infty$  for all  $Q \in \mathcal{Q}^T$  and thus  $\mathbb{E}_R X^- < +\infty$  for all  $R \in \text{Conv}(\mathcal{Q}^T)$ . So,  $\mathbb{E}_P X^-$  and  $\mathbb{E}_{R_k^T} X^-$  are finite. Now, if  $\mathbb{E}_P X^+ = +\infty$  or  $\mathbb{E}_{R_k^T} X^+ = +\infty$  for some  $0 \leq k \leq T-1$ , then  $\mathbb{E}_{P_n^T} X^+ = +\infty$  using (122). But this contradicts the assumption of the lemma as  $P_n^T \in \mathcal{H}^T$ . So,  $\mathbb{E}_P X^+ < +\infty$  and  $\mathbb{E}_{R_k^T} X^+ < +\infty$  for all  $0 \leq k \leq T-1$  and the claim is proved. Now, using again (122), we have that

$$\inf_{P \in \mathcal{H}^T} \mathbb{E}_P X \leq \mathbb{E}_{P_n^T} X = \left(1 - \frac{1}{n}\right)^T \mathbb{E}_P X + \frac{1}{n^T} \sum_{k=0}^{T-1} \binom{T}{k} (n-1)^k \mathbb{E}_{R_k^T} X.$$

Taking the limit in  $n$  in the last expression (recall that the  $(R_k^T)_{0 \leq k \leq T-1}$  do not depend of  $n$ ), we obtain that  $\inf_{P \in \mathcal{H}^T} \mathbb{E}_P X \leq \mathbb{E}_P X$ . As  $P$  is arbitrary in  $\mathcal{Q}^T$ , we deduce that  $\inf_{P \in \mathcal{H}^T} \mathbb{E}_P X \leq \inf_{P \in \mathcal{Q}^T} \mathbb{E}_P X$ . The reverse inequality is trivial as  $\mathcal{H}^T \subset \mathcal{Q}^T$ .  $\square$

**8.3. The sets  $\mathcal{M}^t(P)$  and  $\mathcal{M}^t$**  We give properties of these sets defined in Definition 4. All the results state in this part were already proved in [12] for universally measurable functions. Note that contrary to [12] which construct  $\mathcal{Q}^T$  using universally measurable stochastic kernels, we construct  $\mathcal{Q}^T$  with projectively measurable stochastic kernels (which are in particular universally measurable stochastic kernels under the (PD) axiom, see Lemma 5).

LEMMA 16. *Assume the (PD) axiom. Let  $1 \leq l \leq T$ . Let  $P := q_1 \otimes \cdots \otimes q_l \in \mathcal{Q}^l$ ,  $\tilde{P} := \tilde{q}_1 \otimes \cdots \otimes \tilde{q}_l \in \mathcal{Q}^l$  and*

$$\widehat{P} := (\lambda_1 q_1 + (1 - \lambda_1) \tilde{q}_1) \otimes \cdots \otimes (\lambda_l q_l + (1 - \lambda_l) \tilde{q}_l), \quad (123)$$

where  $\lambda_i \in (0, 1]$  for all  $1 \leq i \leq l$ . Then,  $\mathcal{M}^t(\widehat{P}) \subset \mathcal{M}^t(P)$  for all  $0 \leq t \leq l$ .

*Proof.* The proof is completely similar to the proof of [12, Lemma 10] and thus omitted.  $\square$

LEMMA 17. *Assume the (PD) axiom. Fix  $0 \leq t \leq T$ ,  $P \in \mathcal{Q}^T$  and  $a \geq 0$ . If  $X, Y \in \mathcal{M}^t$  (resp.  $\mathcal{M}^t(P)$ ), then  $X + Y$ ,  $XY$ ,  $\min(X, Y)$ ,  $\max(X, Y)$  belong to  $\mathcal{M}^t$  (resp.  $\mathcal{M}^t(P)$ ) and  $X^a$  also belongs to  $\mathcal{M}^t$  (resp.  $\mathcal{M}^t(P)$ ) if  $X \geq 0$ . Let  $Z : \Omega^t \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  be projectively measurable. If  $0 \leq Z \leq Y$   $\mathcal{Q}^t$ -q.s. with  $Y \in \mathcal{M}^t$ , then  $Z \in \mathcal{M}^t$ . If  $0 \leq Z \leq Y$   $P^t$ -a.s. with  $Y \in \mathcal{M}^t(P)$ , then  $Z \in \mathcal{M}^t(P)$ .*

*Proof.* The fact that  $X + Y$ ,  $XY$ ,  $\min(X, Y)$ ,  $\max(X, Y)$  and  $X^a$  (when  $X \geq 0$ ) are projectively measurable follows from Lemma 8. The rest of the proof is trivial and thus omitted.  $\square$

LEMMA 18. *Assume the (PD) axiom. Let  $0 \leq t \leq T - 1$ . Let  $X : \Omega^{t+1} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  be projectively measurable and choose  $q_{t+1} \in SK_{t+1}$  such that  $q_{t+1}(\cdot | \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$  for all  $\omega^t \in \Omega^t$ . Let  $\lambda_X : \Omega^t \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  be defined by  $\lambda_X(\omega^t) := \mathbb{E}_{q_{t+1}(\cdot | \omega^t)} X(\omega^t, \cdot)$ . Let  $P \in \mathcal{Q}^t$ . If  $X \in \mathcal{M}^{t+1}(P \otimes q_{t+1})$ , then  $\lambda_X \in \mathcal{M}^t(P)$ . If  $X \in \mathcal{M}^{t+1}$ , then  $\lambda_X \in \mathcal{M}^t$ .*

*Proof.* As  $q_{t+1}$  is a projectively measurable stochastic kernel, Proposition 12 shows that  $\lambda_X$  is a projectively measurable function. The rest of the proof is completely similar to the proof of [12, Lemma 12] and thus omitted.  $\square$

**8.4. Proof of Proposition 8** First, the equality  $\Delta_1^1(X) = \mathcal{B}(X)$  in (iii) is proved in [27, Theorem 14.11, p88].

*Proof of (iv).*

We show by induction that for all  $n \geq 1$  and for all Polish spaces  $X$ ,  $\Sigma_n^1(X) \subset \Sigma_{n+1}^1(X)$ . Let  $X$  be a Polish space and  $A \in \Sigma_1^1(X)$  be an analytic set of  $X$ . Using [5, Proposition 7.39, p165], there exists some  $C \in \mathcal{B}(X \times \mathcal{N})$  such that  $A = \text{proj}_X(C)$ . As  $X \times \mathcal{N}$  is a Polish space,  $\mathcal{B}(X \times \mathcal{N}) = \Delta_1^1(X \times \mathcal{N}) \subset \Pi_1^1(X \times \mathcal{N})$ , see (100), and  $C \in \Pi_1^1(X \times \mathcal{N})$ . So,  $A \in \Sigma_2^1(X)$ . This shows the initialization step. Assume now that the induction hypothesis holds true at  $n \geq 1$ . Let  $X$  be a Polish space and  $A \in \Sigma_{n+1}^1(X)$ . Then, there exists some  $C \in \Pi_n^1(X \times \mathcal{N})$  such that  $A = \text{proj}_X(C)$ . Using the induction hypothesis for  $X \times \mathcal{N}$ , we get that  $\Sigma_n^1(X \times \mathcal{N}) \subset \Sigma_{n+1}^1(X \times \mathcal{N})$  and (99) implies that  $\Pi_n^1(X \times \mathcal{N}) \subset \Pi_{n+1}^1(X \times \mathcal{N})$ . Thus,  $C \in \Pi_{n+1}^1(X \times \mathcal{N})$  and  $A \in \Sigma_{n+2}^1(X)$ . This concludes the induction and the sequence  $(\Sigma_n^1(X))_{n \geq 1}$  is nondecreasing. Then, we trivially have, using again (99) and (100), that the sequences  $(\Pi_n^1(X))_{n \geq 1}$  and  $(\Delta_n^1(X))_{n \geq 1}$  are nondecreasing.

*Proof of (i), (ii) and (iii).*

We show by induction that for all  $n \geq 1$  and for all Polish spaces  $X$ , (i) (ii) and (iii) hold true. Let  $X$  be a Polish space. Assertion (i) at  $n = 1$  follows from [5, Corollary 7.35.2, p160] and [5, Proposition 7.40, p165]. Moreover, (ii) and (iii) at  $n = 1$  follow easily from (i) at  $n = 1$ , (99), (100) and from the fact that  $\mathcal{B}(X) = \Delta_1^1(X)$ . This shows the initialization step. Assume now that for all Polish spaces  $X$ , (i), (ii) and (iii) hold true at  $n \geq 1$ . Assume for a moment that (i) holds true



at  $n+1$ . Then, the stability by countable unions and intersections and by preimage in (ii) and (iii) is a direct consequence, as in the initialization step. To prove that  $\Delta_{n+1}^1(X)$  is a  $\sigma$ -algebra, it remains to prove the stability by complement. Let  $A \in \Delta_{n+1}^1(X) = \Sigma_{n+1}^1(X) \cap \Pi_{n+1}^1(X)$ . Using (99),  $A \in \Sigma_{n+1}^1(X)$  implies that  $X \setminus A \in \Pi_{n+1}^1(X)$  and  $A \in \Pi_{n+1}^1(X)$  implies that  $X \setminus A \in \Sigma_{n+1}^1(X)$ . Thus,  $X \setminus A \in \Delta_{n+1}^1(X)$ . It remains to prove (i).

*Proof of (i) at  $n+1$  : Countable unions.*

Let  $X$  be a Polish space. Let  $(A_i)_{i \geq 0} \subset \Sigma_{n+1}^1(X)$  and  $(C_i)_{i \geq 0} \subset \Pi_n^1(X \times \mathcal{N})$  be such that  $A_i = \text{proj}_X(C_i)$  for all  $i \geq 0$ . We have that

$$\begin{aligned} x \in \bigcup_{i \geq 0} A_i &\iff \exists i \geq 0, x \in A_i \iff \exists i \geq 0, \exists u \in \mathcal{N}, (x, u) \in C_i \\ &\iff \exists u \in \mathcal{N}, (x, u) \in \bigcup_{i \geq 0} C_i \iff x \in \text{proj}_X \left( \bigcup_{i \geq 0} C_i \right). \end{aligned}$$

So,  $\bigcup_{i \geq 0} A_i = \text{proj}_X(\bigcup_{i \geq 0} C_i)$ . Now, (ii) at  $n$  and the fact that  $X \times \mathcal{N}$  is a Polish space show that  $\bigcup_{i \geq 0} C_i \in \Pi_n^1(X \times \mathcal{N})$ . Thus,  $\bigcup_{i \geq 0} A_i \in \Sigma_{n+1}^1(X)$ .

*Proof of (i) at  $n+1$  : Borel preimages.*

Let  $X$  and  $Y$  be Polish spaces. Let  $f : X \rightarrow Y$  be Borel measurable. Let  $B \in \Sigma_{n+1}^1(Y)$  and  $C \in \Pi_n^1(Y \times \mathcal{N})$  such that  $B = \text{proj}_Y(C)$ . Then,

$$f^{-1}(B) = \{x \in X, \exists u \in \mathcal{N}, (f(x), u) \in C\} = \{x \in X, \exists u \in \mathcal{N}, (x, u) \in \Psi^{-1}(C)\} = \text{proj}_X(\Psi^{-1}(C)),$$

where for all  $(x, u) \in X \times \mathcal{N}$ ,  $\Psi(x, u) := (f(x), u) \in Y \times \mathcal{N}$  is Borel measurable. Using (ii) at  $n$ , we have that  $\Psi^{-1}(C) \in \Pi_n^1(X \times \mathcal{N})$  and thus that  $f^{-1}(B) \in \Sigma_{n+1}^1(X)$ .

*Proof of (i) at  $n+1$  : Borel images.*

Let  $X$  and  $Y$  be Polish spaces. Let  $f : X \rightarrow Y$  be Borel measurable. Let  $A \in \Sigma_{n+1}^1(X)$  and  $C \in \Pi_n^1(X \times \mathcal{N})$  be such that  $A = \text{proj}_X(C)$ . As  $X \times \mathcal{N}$  is a non-empty Polish space, [27, Theorem 7.9, p38] shows that there exists a continuous surjection  $\Phi$  from  $\mathcal{N}$  to  $X \times \mathcal{N}$ . Then,

$$\begin{aligned} f(A) &= \{y \in Y, \exists x \in A, y = f(x)\} = \{y \in Y, \exists (x, u) \in X \times \mathcal{N}, y = f(x) \text{ and } (x, u) \in C\} \\ &= \{y \in Y, \exists w \in \mathcal{N}, y = g(w) \text{ and } \Phi(w) \in C\} = \text{proj}_Y(\xi^{-1}(\text{Graph}(g)) \cap (Y \times \Phi^{-1}(C))), \end{aligned}$$

where for all  $w \in \mathcal{N}$ ,  $g(w) := f(\text{proj}_X(\Phi(w)))$  and for all  $(y, w) \in Y \times \mathcal{N}$ ,  $\xi(y, w) := (w, y)$ . Assume for a moment that both  $Y \times \Phi^{-1}(C)$  and  $\xi^{-1}(\text{Graph}(g))$  belong to  $\Pi_n^1(Y \times \mathcal{N})$ , then (ii) at  $n$  shows that their intersection remains in  $\Pi_n^1(Y \times \mathcal{N})$  and thus that  $f(A) \in \Sigma_{n+1}^1(Y)$ . Note first that  $\Phi^{-1}(C) \in \Pi_n^1(\mathcal{N})$  using (ii) at  $n$  as  $C \in \Pi_n^1(X \times \mathcal{N})$  and  $\Phi$  is continuous and thus Borel measurable. Now, as  $Y \times \mathcal{N} \ni (a, b) \mapsto b$  is Borel measurable, (ii) at  $n$  implies that  $Y \times \Phi^{-1}(C) \in \Pi_n^1(Y \times \mathcal{N})$ . As  $g$  is Borel measurable, [5, Corollary 7.14.1, p121] shows that  $\text{Graph}(g) \in \mathcal{B}(\mathcal{N} \times Y) = \Delta_1^1(\mathcal{N} \times Y) \subset \Pi_1^1(\mathcal{N} \times Y)$ . But (iv) shows that  $\Pi_1^1(\mathcal{N} \times Y) \subset \Pi_n^1(\mathcal{N} \times Y)$  and thus that  $\text{Graph}(g) \in \Pi_n^1(\mathcal{N} \times Y)$ . Using again (ii) at  $n$  as  $\xi$  is Borel measurable, we deduce that  $\xi^{-1}(\text{Graph}(g)) \in \Pi_n^1(Y \times \mathcal{N})$ .

*Proof of (i) at  $n+1$  : Countable intersections.*

Let  $X$  be a Polish space. Let  $(A_i)_{i \geq 0} \subset \Sigma_{n+1}^1(X)$  and  $(C_i)_{i \geq 0} \subset \Pi_n^1(X \times \mathcal{N})$  be such that  $A_i = \text{proj}_X(C_i)$  for all  $i \geq 0$ . Then,

$$\begin{aligned} \bigcap_{i \geq 0} A_i &= \{x \in X, \forall i \geq 0, \exists u_i \in \mathcal{N}, (x, u_i) \in C_i\} \\ &= \{x \in X, \exists u = (u_i)_{i \geq 0} \in \mathcal{N}^{\mathbb{N}}, \forall i \geq 0, (x, u_i) \in C_i\} \\ &= \{x \in X, \exists z \in \mathcal{N}, \forall i \geq 0, (x, \Phi(z)_i) \in C_i\} = \text{proj}_X \left( \bigcap_{i \geq 0} D_i \right), \end{aligned}$$

where  $\Phi$  is a continuous surjection from  $\mathcal{N}$  to the Polish space  $\mathcal{N}^{\mathbb{N}}$  (see [5, Proposition 7.4, p108]) given by [27, Theorem 7.9, p38] and for all  $i \geq 0$ ,  $D_i := \{(x, z) \in X \times \mathcal{N}, (x, \Phi(z)_i) \in C_i\}$ . As for all  $i \geq 0$ ,  $(x, z) \mapsto (x, \Phi(z)_i)$  is Borel measurable and  $C_i \in \Pi_n^1(X \times \mathcal{N})$ , (ii) at  $n$  shows that  $D_i \in \Pi_n^1(X \times \mathcal{N})$ . So, again (ii) at  $n$  shows that  $\bigcap_{i \geq 0} D_i \in \Pi_n^1(X \times \mathcal{N})$ . Thus,  $\bigcap_{i \geq 0} A_i \in \Sigma_{n+1}^1(X)$ . This concludes the induction for (i).

*Proof of (v).*

Let  $n \geq 1$  and  $X$  be a Polish space. We show the first inclusion in (101). For that, we prove that  $\Pi_n^1(X) \subset \Sigma_{n+1}^1(X)$ . Let  $A \in \Pi_n^1(X)$ . Then  $A = \text{proj}_X(A \times \mathcal{N})$ . As  $X \times \mathcal{N} \ni (a, b) \mapsto a$  is Borel measurable, (ii) at  $n$  implies that  $A \times \mathcal{N} \in \Pi_n^1(X \times \mathcal{N})$ . Thus,  $A \in \Sigma_{n+1}^1(X)$ , see (98). Now, let  $B \in \Sigma_n^1(X)$ . Then, using (99),  $X \setminus B \in \Pi_n^1(X) \subset \Sigma_{n+1}^1(X)$  and so,  $B \in \Pi_{n+1}^1(X)$ . Thus,  $\Sigma_n^1(X) \subset \Pi_{n+1}^1(X)$ . Recalling then (iv),  $\Sigma_n^1(X) \subset \Sigma_{n+1}^1(X)$ . So,  $\Sigma_n^1(X) \subset \Delta_{n+1}^1(X)$  using (100). As  $\Delta_{n+1}^1(X)$  is a  $\sigma$ -algebra, (99) proves that  $\Pi_n^1(X) \subset \Delta_{n+1}^1(X)$ . This shows that  $\Sigma_n^1(X) \cup \Pi_n^1(X) \subset \Delta_{n+1}^1(X)$ . Using this last inclusion and (100), we have the following inclusions for all  $n \geq 1$ :

$$\Pi_n^1(X) \subset \Delta_{n+1}^1(X) \subset \Sigma_{n+1}^1(X) \subset \Delta_{n+2}^1(X) \subset \Pi_{n+2}^1(X).$$

Thus, taking the union over  $n \geq 1$  and using (iv), we get that the unions of  $(\Sigma_n^1(X))_{n \geq 1}$ ,  $(\Pi_n^1(X))_{n \geq 1}$  and  $(\Delta_n^1(X))_{n \geq 1}$  are equal, which achieves the proof of (101). We now prove that  $\Sigma_n^1(X) \times \Sigma_n^1(Y) \subset \Sigma_n^1(X \times Y)$ . Let  $A \in \Sigma_n^1(X)$  and  $B \in \Sigma_n^1(Y)$ . As the mappings  $X \times Y \ni (a, b) \mapsto a$  and  $X \times Y \ni (a, b) \mapsto b$  are Borel measurable, (i) shows that  $A \times Y$  and  $X \times B$  belong to  $\Sigma_n^1(X \times Y)$  and also that  $A \times B = (A \times Y) \cap (X \times B) \in \Sigma_n^1(X \times Y)$ . The same method, using (ii) and (iii) instead of (i), shows that  $\Pi_n^1(X) \times \Pi_n^1(Y) \subset \Pi_n^1(X \times Y)$  and  $\Delta_n^1(X) \times \Delta_n^1(Y) \subset \Delta_n^1(X \times Y)$ .

*Proof of (vi).*

We first show that  $\mathbf{P}(X)$  is closed under complement. Let  $A \in \mathbf{P}(X)$ . Then, there exists some  $n \geq 1$  such that  $A \in \Delta_n^1(X)$ . As  $\Delta_n^1(X)$  is a  $\sigma$ -algebra,  $X \setminus A \in \Delta_n^1(X) \subset \mathbf{P}(X)$ . We now prove that  $\mathbf{P}(X)$  is stable by finite unions and intersections. Let  $A \in \mathbf{P}(X)$  and  $B \in \mathbf{P}(X)$ , there exist  $n, p \geq 1$  such that  $A \in \Delta_n^1(X)$  and  $B \in \Delta_p^1(X)$ . We may assume that  $n \leq p$ . Then, (iv) and (iii) show that  $A \in \Delta_p^1(X)$  and that  $A \cup B$  and  $A \cap B$  belong to  $\Delta_p^1(X) \subset \mathbf{P}(X)$ .

Now, let  $f : X \rightarrow Y$  be Borel measurable. Let  $B \in \mathbf{P}(Y)$ . Then, there exists some  $q \geq 1$  such that  $B \in \Delta_q^1(Y)$  and it follows from (iii) that  $f^{-1}(B) \in \Delta_q^1(X) \subset \mathbf{P}(X)$ . Let  $A \in \mathbf{P}(X)$ . Then, there exists some  $q \geq 1$  such that  $A \in \Delta_q^1(X) \subset \Sigma_q^1(X)$  (see (100)) and it follows from (i) that  $f(A) \in \Sigma_q^1(Y) \subset \Delta_{q+1}^1(Y) \subset \mathbf{P}(Y)$  (see (101)).

We now prove (102). As  $\Delta_1^1(X) = \mathcal{B}(X)$ ,  $\mathcal{B}(X) \subset \mathbf{P}(X)$ . Now, (101) shows that  $\Sigma_1^1(X) \subset \Delta_2^1(X) \subset \mathbf{P}(X)$ . Let  $A \in \mathbf{P}(X)$  and  $B \in \mathbf{P}(Y)$ . There exist  $n, p \geq 1$  such that  $A \in \Delta_n^1(X)$  and  $B \in \Delta_p^1(Y)$ . Again, we may assume that  $n \leq p$  and we have that  $A \in \Delta_p^1(X)$ . Using (v), we get that  $A \times B \in \Delta_p^1(X \times Y) \subset \mathbf{P}(X \times Y)$ .  $\square$

## References

- [1] ALIPRANTIS, C. D. AND BORDER, K. C. (2006). Infinite Dimensional Analysis : A Hitchhiker's Guide 3rd ed. Springer, Berlin. MR2378491.
- [2] BARTL, D. (2019). Exponential utility maximization under model uncertainty for unbounded endowments. *Annals of Applied Probability*, 29(1) 577-612.
- [3] BARTL, D AND P, CHERIDITO AND KUPPER, M (2019). Robust expected utility maximization with medial limit. *Journal of Mathematical Analysis and Applications*, 471(1-2) 752-775.
- [4] BAYRAKTAR, E. AND ZHOU, Z.(2004). On arbitrage and duality under model uncertainty and portfolio constraints. *Mathematical Finance*, 27(4) 988-1012.
- [5] BERTSEKAS, D. P. AND SHREVE, S.E. (1978). Stochastic optimal control: the discrete time case. *New York: Academic Press*.
- [6] BLANCHARD, R. (2017). Stochastic control applied in the theory of decision in a discrete time non-dominated multiple-priors. Phd Thesis, *Université de Reims Champagne-Ardenne*.

- 
- [7] BLANCHARD R. AND CARASSUS L. (2018). Multiple-priors optimal investment in discrete time for unbounded utility function. *Annals of Applied Probability*, 28(3) 1856-1892.
- [8] BLANCHARD R. AND CARASSUS L. (2019). No-arbitrage with multiple-priors in discrete time. *Stochastic Processes and their Applications*, 130(11) 6657-6688.
- [9] BLANCHARD, R. AND CARASSUS, L. AND RASONYI, M. (2018). No-arbitrage and optimal investment with possibly non-concave utilities: a measure theoretical approach. *Mathematical Methods of Operations Research*, 88.
- [10] BOUCHARD, B. AND NUTZ, M. (2015). Arbitrage and Duality in Nondominated Discrete-Time Models. *Annals of Applied Probability*, 25 823-859.
- [11] BURZONI, M., FRITTELI, M., HOU, Z. AND OBLOJ, J. (2019). Pointwise Arbitrage Pricing Theory in Discrete Time. *Mathematics of Operations Research*, 44(3) 1034-1057.
- [12] CARASSUS, L. AND FERHOUNE, M. (2023). Discrete time optimal investment under model uncertainty. *arXiv:2307.11919*.
- [13] CARASSUS, L AND RASONYI M. (2016). Maximization of Non-Concave Utility Functions in Discrete-Time Financial Market Models. *Mathematics of Operations Research*, 41(1).
- [14] DONOGHUE, WILLIAM F. (1969). Distributions and Fourier transforms. *New York: Academic Press*.
- [15] DRAKE, F. R. (1974). Set Theory : An Introduction to Large Cardinals. *Amsterdam New York: North-Holland Pub. Co. American Elsevier Pub*.
- [16] ELSBERG, D. (1961). Risk, Ambiguity, and the Savage Axioms. *Quarterly Journal of Economics*, 75(4) 643-669.
- [17] FÖLLMER, H. AND SCHIED, A. (2002). Stochastic Finance: An Introduction in Discrete Time *Walter de Gruyter & Co, Berlin*.
- [18] FÖLLMER, H AND SCHIED, A AND WEBER, S. (2007). Robust Preferences and Robust Portfolio Choice. *Handbook of Numerical Analysis*, 15.
- [19] FRITTELI, M. AND ROSAZZA GIANIN, E.(2004). Equivalent formulations of reasonable asymptotic elasticity. *Technical Report no. 12, Department of Matematica per le Decisioni, University of Florence*.
- [20] GALE, D AND STEWART, F. M. (1953). Infinite games with perfect information. *Ann. Math. Studies*, 28 245-266.
- [21] GILBOA, I. AND SCHMEIDLER, D. (1989). Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*,18(2) 141-153.
- [22] HRBACEK, K AND JECH, T. (1999). Introduction to Set Theory, Third Edition, Revised and Expanded *CRC Press*
- [23] JACOD, J. AND SHIRYAEV, A. (1998). Local martingales and the fundamental asset pricing theorems in the discrete-time case. *Finance Stochast*,(2) 259-273.
- [24] JECH, T (2003). Set Theory: The Third Millennium Edition, Revised and Expanded. *Springer*
- [25] KAHNEMAN, D AND TVERSKY, A. (1979). Prospect Theory: An Analysis of Decision under Risk. *Econometrica, Econometric Society*, 47(2) 263-291.
- [26] KANAMORI, A. (2005). The higher infinite : large cardinals in set theory from their beginnings. *Springer Monographs in Mathematics*.
- [27] KECHRIS, S. A. (1995). Classical Descriptive Set Theory. *Graduate Texts in Mathematics*.
- [28] KNIGHT, F. (1921). Risk, Uncertainty, and Profit. *Houghton Mifflin Co*.
- [29] KRAMKOV, D. AND SCHACHERMAYER, W. (1999). The Asymptotic Elasticity of Utility Functions and Optimal Investment in Incomplete Markets. *Annals of Applied Probability*, 9(3), 904-950
- [30] MAITRA, A. AND PURVES, R. AND SUDDERTH, W. (1990). Leavable Gambling Problems with Unbounded Utilities. *Transactions of the American Mathematical Society*, 320(2) 543-567
- [31] MARTIN, D. A. (1975). Borel Determinacy. *Annals of Mathematics*, 102(2) 363-371.
- [32] MARTIN, D. A. AND SOLOVAY, R. M. (1970). Internal Cohen extensions. *Annals of Mathematical Logic*, 2(2) 143-178.

- 
- [33] MARTIN, D. A. AND STEEL, R. J. (1989). A Proof of Projective Determinacy. *Journal of the American Mathematical Society*, 2(1) 71-125.
- [34] MOSCHOVAKIS, Y. N. (1971). Uniformization in a playful universe. *Bull. Amer. Math. Soc*, 77(5) 731-736.
- [35] MYCIELSKI, J. AND ŚWIERCZKOWSKI, S. (1964). On the Lebesgue measurability and the axiom of determinateness. *Fundamenta Mathematicae*, 54(1) 67-71.
- [36] NEUFELD, A AND SIKIC, M. (2019). Nonconcave robust optimization with discrete strategies under Knightian uncertainty. *Mathematical Methods of Operations Research*, 90(2) 229-253.
- [37] NUTZ, M. (2013). Utility Maximization under Model Uncertainty in Discrete Time. *Mathematical Finance*, 26.
- [38] RASONYI, M. AND MEIRELES-RODRIGUES, A. (2020). On utility maximization under model uncertainty in discrete time markets. *Mathematical Finance*, 31 149 - 175.
- [39] RASONYI, M. AND STETTNER, L. (2005). On utility maximization in discrete-time financial market models. *Annals of Applied Probability*, 15(2) 1367-1395.
- [40] ROCKAFELLAR, R. T. AND WETS, R. (2010). Variational Analysis. Heidelberg: Springer.
- [41] SCHACHERMAYER, W. (2001). Optimal investment in incomplete markets when wealth may become negative. *Annals of Applied Probability*, 11 694-734
- [42] WOODIN, W. H. (2001). The Continuum Hypothesis, Part I. *Notices of the American Mathematical Society* 48