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## A Quasi-Trefftz Method Based on a Flux Reconstruction Local Solver for the Time-Harmonic Maxwell System

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### Abstract

The simulation of time-harmonic electromagnetic waves in wide domains with classic methods induces large linear systems which are poorly adapted to iterative resolution. We introduce an iterative quasi-Trefftz method for the Maxwell's equations, where the basis functions are parameterised by a polynomial approximation of an impedance trace space, and computed thanks to a Flux Reconstruction method, allowing to overtake some limitations of the classic choice of plane waves.

**Keywords:** Trefftz method, Flux Reconstruction, Time-harmonic Maxwell's equations

### 1 The time-harmonic Maxwell's equations

We consider a linear isotropic material which occupies a domain  $\Omega \subset \mathbb{R}^3$ . The time-harmonic Maxwell's equations in their hyperbolic form, in absence of charges, stand in  $\Omega$  as

$$i\kappa \begin{pmatrix} \varepsilon_r \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mu_r \mathbf{I}_3 \end{pmatrix} \mathbb{E} + \sum_{j=1}^3 \frac{\partial \phi^j}{\partial x_j} = \mathbf{0}, \quad (1)$$

where  $\kappa > 0$  stands for the wavenumber,  $\varepsilon_r$  and  $\mu_r$  denote the relative permittivity and permeability respectively, and  $\mathbf{I}_3$  and  $\mathbf{0}_3$  are the identity and null matrices of  $\mathbb{R}^{3 \times 3}$ . Moreover,  $\mathbb{E} = (\mathbf{E}, \mathbf{H})^t$  and  $\phi^j := \mathbf{F}^j \mathbb{E}$  denote the electromagnetic field and the fluxes respectively, with the flux operators  $\mathbf{F}^j \in \mathbb{R}^{6 \times 6}$ , for  $1 \leq j \leq 3$ . We complete this equation with an impedance Boundary Condition (BC)

$$\gamma_t[\mathbf{n}_{\partial\Omega}] \mathbf{E} + Z_{\partial\Omega} \gamma_{\times}[\mathbf{n}_{\partial\Omega}] \mathbf{H} = \mathbf{g} \text{ on } \partial\Omega, \quad (2)$$

defined on the boundary of  $\Omega$ , denoted as  $\partial\Omega$ , for an impedance  $Z_{\partial\Omega} : \partial\Omega \rightarrow \mathbb{C}^3$  and a given tangential field  $\mathbf{g} \in \mathbf{L}_t^2(\partial\Omega)$ . Finally,  $\gamma_t[\mathbf{n}_{\partial\Omega}] = (\mathbf{n}_{\partial\Omega} \times \cdot) \times \mathbf{n}_{\partial\Omega}$  and  $\gamma_{\times}[\mathbf{n}_{\partial\Omega}] = \mathbf{n}_{\partial\Omega} \times \cdot$  stand for the tangential component and trace respectively, with  $\mathbf{n}_{\partial\Omega}$  the outgoing normal from  $\Omega$ .

### 2 The Trefftz method

The Trefftz method relies on a reciprocity formula which encompasses the energy conservation through the boundary fluxes. We introduce a mesh  $\mathcal{T}_h$  of the domain  $\Omega$ . For any cell  $T \in \mathcal{T}_h$ , we denote  $(\mathbf{E}^T, \mathbf{H}^T) := \mathbb{E}|_T$  in  $T$ ,  $\gamma_t \mathbf{E}^T = \gamma_t[\mathbf{n}_T] \mathbf{E}^T$  and  $\gamma_{\times}^T \mathbf{H}^T = \gamma_{\times}[\mathbf{n}_T] \mathbf{H}^T$  on  $\partial T$ , where  $\mathbf{n}_T$  is the outgoing normal of  $T$ . The method can be described as a DG approach for which the Galerkin space  $\mathbb{X} := \prod_{T \in \mathcal{T}_h} \mathbb{X}_T$  is made of local solutions (i.e. in each mesh cell  $T \in \mathcal{T}_h$ ) of the Maxwell's equations. It then writes [1]:

Find  $\mathbb{E} = (\mathbf{E}, \mathbf{H}) \in \mathbb{X}$  s.t.  $\forall \mathbb{E}' = (\mathbf{E}', \mathbf{H}') \in \mathbb{X}$ ,

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \widehat{\gamma_{\times}^T \mathbf{H}^T} \cdot \overline{\gamma_t \mathbf{E}'^T} + \widehat{\gamma_t \mathbf{E}^T} \cdot \overline{\gamma_{\times}^T \mathbf{H}'^T} = \ell(\mathbb{E}')$$

where  $\widehat{\cdot}$  denotes a numerical trace computation and  $\ell$  stands for a linear form which takes the BCs into account.

In particular, this approach limits the amount of degrees of freedom and is intrinsically adaptable to an iterative resolution through an inherent contraction property [1].

### 3 Quasi-Trefftz approach

The classic choice of plane waves as basis functions tends to cause numerical dependence phenomena which impact the conditioning properties of the matrices. To face such limitations, we consider a quasi-Trefftz approach [2]: the idea is then to parameterise the space of local solutions by a polynomial approximation of an impedance trace space. More specifically, let  $F$  be a face of a cell  $T \in \mathcal{T}_h$ . We introduce  $P$  polynomial face functions  $(f_j^F)_{j \in [1, P]} : F \rightarrow \mathbb{R}$  and the tangential vector fields

$$\mathbf{g}_{t,j}^F : \partial T \rightarrow \mathbb{C}^3 \text{ with } \mathbf{g}_{t,j}^F = \begin{cases} f_j^F \mathbf{e}_t^F & \text{on } F, \\ \mathbf{0} & \text{on } \partial T \setminus F, \end{cases}$$

where  $t \in \{1, 2\}$ ,  $\mathbf{e}_1^F$  and  $\mathbf{e}_2^F$  denote two unitary vectors tangential to  $F$ . Reporting these tangential fields in an (adapted) impedance BC on

$\partial T$ , the  $P$  local solutions  $\mathbb{E}_{t,j}^F$  of the associated Maxwell problem in  $T$  are then chosen to form a basis of  $\mathbb{X}_T$ . Yet, these are not analytically known, which implies the need of a local solver to compute approximate solutions  $\mathbb{E}_{t,j}^{F,\mathbf{h}}$ .

#### 4 Flux Reconstruction local solver

The local solver choice is then of special importance, and we chose to adapt the Flux Reconstruction method (FR) [3], which has already established itself for hyperbolic problems in the Computational Fluid Dynamics community, to this time-harmonic framework for an adapted mesh  $\mathcal{T}_{\mathbf{h}}^{FR}(T)$  of a cell  $T \in \mathcal{T}_{\mathbf{h}}$ . It relies on the strong form of the hyperbolic problem (1) by approximating the solution  $\mathbb{E}$  and the fluxes  $\phi^j$  by piecewise polynomial functions  $\mathbb{E}_{\mathbf{h}}$  and  $\tilde{\phi}_{\mathbf{h}}^j$ . For any mesh cell  $T^{FR} \in \mathcal{T}_{\mathbf{h}}^{FR}(T)$ , this leads to

$$i\kappa \begin{pmatrix} \varepsilon_r \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mu_r \mathbf{I}_3 \end{pmatrix} \mathbb{E}_{\mathbf{h}} + \sum_{j=1}^3 \frac{\partial \tilde{\phi}_{\mathbf{h}}^j}{\partial x_j} = \mathbf{0} \text{ in } T^{FR}.$$

To take BCs and flux continuity properties into account, the flux variables  $\tilde{\phi}_{\mathbf{h}}^j$  result from a polynomial correction of  $\phi_{\mathbf{h}} = \mathbf{F}^j \mathbb{E}_{\mathbf{h}}$  which ensures it fits with a given polynomial numerical trace on any face  $F$  orthogonal to  $\mathbf{e}_j$ .

This method allows to retrieve classic methods such as nodal Discontinuous Galerkin (DG) and Spectral Difference (SD) in usually less expensive ways, for specific flux correction polynomial functions [3], and specifically lends itself to Cartesian meshes.

#### 5 Quasi-Trefftz numerical experiments

This quasi-Trefftz framework is likely to bring pertinent adaptations to the Plane Wave-based Trefftz method. Indeed, the potential orthogonality characteristics of the tangential fields  $\mathbf{g}_{t,j}^F$  tend to improve the spectral properties of the matrices (see Figure 1). Moreover, this approach naturally allows to adapt the basis functions to the local mesh parameters and BC refinement, as they are independently computed for each mesh cell  $T \in \mathcal{T}_{\mathbf{h}}$ : physical heterogeneity and singularities may be locally taken into account. Yet, the introduction of an auxiliary local solver induces the use of basis functions which are no longer exact solutions of the Maxwell equations: its impact on the formulation properties has to be specified, in addition to the influence of the FR solver convergence (see Figure 2).

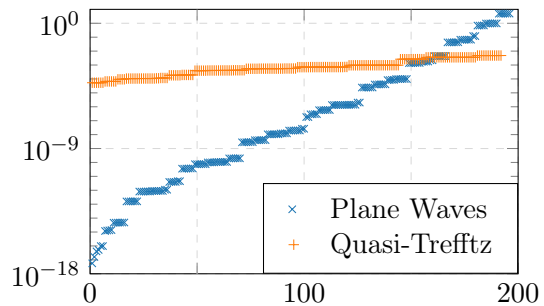


Figure 1: Eigenvalues of the scalar product matrix for local basis functions: Trefftz method based on 196 plane waves and quasi-Trefftz approach with  $P = 16$ .

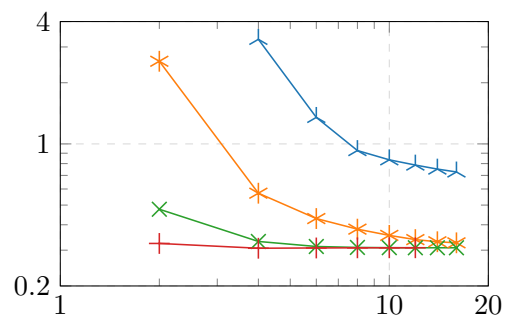


Figure 2: Relative volumic  $L^2$ -error for a quasi-Trefftz method with  $P = 9$  w.r.t. the number of mesh cells of  $\mathcal{T}_{\mathbf{h}}^{FR}$  in each direction, for different orders  $k^{FR}$  of the FR method ( $k^{FR} = 0$  ( $\blacktriangle$ ),  $k^{FR} = 1$  ( $\ast$ ),  $k^{FR} = 2$  ( $\times$ ),  $k^{FR} = 3$  ( $\oplus$ )).

In this presentation, we will focus on the different dependencies of the quasi-Trefftz approach (through the FR solver discretisation and the polynomial BC approximation): comparisons to a classic Trefftz method in terms of accuracy, iterative resolution and matrix conditioning will be held on challenging configurations.

#### References

- [1] S. Pernet et al., Ultra-weak variational formulation for heterogeneous Maxwell problem in the context of high performance computing, *ESAIM Proc.* **75** (2023), pp. 96-121.
- [2] H. Fure et al., A discontinuous Galerkin Trefftz type method for solving the two dimensional Maxwell equations, *SN PDEA* **1** (2020), pp. 1-25.
- [3] H. T. Huynh, A flux reconstruction approach to high-order schemes including discontinuous Galerkin methods, *18th AIAA CFD conference* (2007), p. 4079.