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► **To cite this version:**

Emilio Molina, Alain Rapaport. Optimal synthesis for a class of Linfty optimal control problems in the plane with L1 constraint on the input. Optimization, In press. hal-04842043

**HAL Id: hal-04842043**

**<https://hal.science/hal-04842043v1>**

Submitted on 19 Dec 2024

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# Optimal synthesis for a class of $L^\infty$ optimal control problems in the plane with $L^1$ constraint on the input

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November 29, 2024

## Abstract

For a particular class of planar dynamics that are linear with respect to the control variable, we show that the feedback strategy "null-singular-null" is minimizing the maximum of a coordinate over infinite horizon, under a  $L^1$  budget constraint on the control. Moreover, we characterize the optimal cost as a function of the budget. The proof is based on an unusual use of the clock form. This result generalizes the one obtained formerly for the SIR epidemiological model to more general Kolmogorov dynamics, that we illustrate on other biological models.

**Key-words.** Optimal control, maximum cost, infinite horizon, feedback strategy, Green's Theorem, singular arc.

**Mathematics Subject Classification (2000).** 49J35, 49K10, 49K35, 49N35, 26B20.

## 1 Introduction

The synthesis of optimal solutions of control problems with maximum cost has received relatively few attention in the literature, apart characterizations of the value function in terms of Hamilton-Jacobi-Bellman variational inequality [2, 6]. For concrete problems, it is often very difficult or merely impossible to find analytical solutions, but these characterizations have led to dedicated numerical schemes [3, 5]. On the other hand, necessary optimality conditions cannot be provided by a direct application of the Pontryagin's Maximum Principle, because the maximum cost is not a criterion in Mayer or Bolza form. However, several equivalent formulations in Mayer form have been proposed by

augmenting the state dynamics and adding a state constraint (see [4, 12]) or as a minimax problem [14]. In practice, dealing with the state constraint remains an issue to derive analytical optimal strategies. The authors of the present work have recently investigated in [11] the optimal control problem which consists in minimizing the peak of an epidemic for the well-known epidemiological SIR model (a model that has been formerly proposed in [9]), and provided the optimal solution when controlling the transmission rate under a budget or  $L^1$  constraint on the control. This problem, whose trajectories lie in the plane, presents a singular arc and has been solved analytically by applying a clock form. The clock form technique is well-known for planar optimal control problem with integral or minimal time criterion [8, 10] and fixed terminal state, as a tool to compare a candidate optimal solution with any other admissible solutions. Therefore, it cannot be applied in this way for comparing maximum costs. However, for this particular epidemiological problem, the authors of [11] (who are, moreover, the authors of the present work) have used the clock form in reasoning by the absurd, showing that a possible better solution would require a larger budget. While the proof has been derived for the particular dynamics of the SIR model, the aim of the present work is to extend this technique to more generic problems of minimizing the peak of one coordinate of planar dynamics under a  $L^1$  constraint on the control. We characterize a class of problems for which the optimal solution presents the same structure of the control strategy, which consists in three phases: 1. go as fast as possible to the optimal peak value 2. apply a control to maintain the peak value constant until the budget is exhausted (singular arc), and 3. release the control when entering a domain of the state space for which the peak cannot increase applying the null control. Moreover, we give conditions on particular Kolmogorov dynamics in plane, for which this result generalizes the one obtained previously for the SIR epidemiological model, which can be then regarded as a simple application of our result.

More precisely, we consider a controlled dynamics on an positively invariant<sup>1</sup> domain  $\mathcal{D}$  of  $\mathbb{R}^2$

$$\begin{cases} \dot{x} &= f_1(x, y) + g_1(x, y)u, \\ \dot{y} &= f_2(x, y) + g_2(x, y)u, \end{cases} \quad (1)$$

where the maps  $f_1, f_2, g_1, g_2$  are assumed to be smooth (at least  $C^1$ ). Given a positive number  $K$  and an initial condition  $(x_0, y_0) \in \mathcal{D}$ , we consider the optimal control problem over infinite horizon

$$\text{minimize } \sup_{t \geq 0} y(t) \text{ over } u(\cdot) \in \mathcal{U}, \quad (2)$$

where  $\mathcal{U}$  denotes the set of time measurable functions  $u(\cdot)$  that takes values in  $U := [0, 1]$ , subject to the  $L^1$  constraint

$$\int_0^{+\infty} u(t) dt \leq K. \quad (3)$$

---

<sup>1</sup>We recall that a domain is positively invariant for a controlled system if for any initial condition in this domain, the solutions of the system remain in the domain for any future time and any control.

The problem consists then in minimizing the "peak" of the variable  $y$  under a "budget" constraint on the control. We shall say that a control  $u(\cdot)$  is *admissible* if it belongs to  $\mathcal{U}$  and satisfy the constraint (3), and that a pair of absolutely continuous functions  $(x(\cdot), y(\cdot))$  with values in  $\Omega$  is an *admissible* solution of (1) if there exists an admissible control  $u(\cdot)$  such that one has equations (1) are satisfied for almost any  $t$  in  $[0, +\infty)$ . Note that the uniqueness of solution of (1) for given initial condition in  $\Omega$  and admissible control is guaranteed as the maps  $f_1, f_2, g_1, g_2$  are at least  $C^1$ .

The paper is structured as follows. In the next section, we give assumptions and some preliminary results, which ensure that the problem (2) under infinite horizon and the control strategy that we consider later are well defined. Section 3 defines the control strategy that what we propose to name "NSN", and gives a characterization of it. Section 4 proves our main result about the optimality of the NSN strategy, under a flux-like condition. Finally, we consider in Section 5 a class of controlled Kolmogorov dynamics for which the former assumptions are satisfied. As examples, we show how our result applies straightforwardly to the SIR model and to more sophisticated biological models.

## 2 Assumptions and preliminaries

We shall consider the following assumptions to deal with infinite horizon, where  $\text{clo}$  and  $\text{proj}_2$  denote respectively the closure and the second projection in the  $(x, y)$  coordinates.

**Assumptions 1.** *One has*

- i. For any initial condition in  $\mathcal{D}$ , the solution of the uncontrolled dynamics (that is with  $u = 0$ ) is bounded, and any other admissible solution with a cost  $\sup_t y(t)$  lower than that for the uncontrolled dynamics is also bounded.*

- ii. The strict sub and super level sets of the function  $f_2$ :*

$$\mathcal{D}_- := \{(x, y) \in \mathcal{D} ; f_2(x, y) < 0\}, \quad \mathcal{D}_+ := \{(x, y) \in \mathcal{D} ; f_2(x, y) > 0\},$$

*are non empty.*

- iii. The 0-level set of  $f_2$  is the graph of a function  $\tilde{x} : \text{proj}_2(\mathcal{D}) \mapsto \mathbb{R}$ :*

$$\mathcal{D}_0 := \{(x, y) \in \mathcal{D} ; f_2(x, y) = 0\} = \{(\tilde{x}(y), y), y \in \text{proj}_2(\mathcal{D})\}. \quad (4)$$

Note that the optimal control problem (2) over infinite horizon is well defined (and finite) under Assumption 1.i.

We shall require a certain behavior of the vector fields  $f$  and  $g$  on the sets  $\mathcal{D}_+$  and  $\mathcal{D}_0$ .

**Assumptions 2.** *In  $\mathcal{D}_+$ , one has the properties*

- i.  $f_1$  is negative and strictly decreasing w.r.t.  $x$  and  $y$ ,*
- ii.  $g_1$  is strictly increasing w.r.t.  $x$  and  $y$  and  $f_1 + g_1$  is non positive,*

iii.  $\partial_x f_2$  is positive,

iv.  $g_2$  is strictly decreasing w.r.t.  $x$  and  $y$  and  $f_2 + g_2$  is negative,

and moreover on  $\mathcal{D}_0$

v.  $f_1$  is negative,

vi.  $\partial_x f_2$  is positive and  $\partial_y f_2$  is non-positive.

Then, the following Lemma gives properties of the trajectories in  $\mathcal{D}_+$  and its complement in  $\mathcal{D}$ , related to the infinite horizon.

**Lemma 2.1.** *Under Assumptions 1 and 2,*

1. *With the control  $u = 0$ , the domain  $\mathcal{D} \setminus \mathcal{D}_+$  is positively invariant, and from any initial condition  $(x_0, y_0)$  in  $\mathcal{D}_+$ , the solution of (1) reaches  $\mathcal{D}_0$  in finite time.*
2. *For any initial condition  $(x_0, y_0)$  in  $\mathcal{D}_+$  and optimal control  $u(\cdot)$ , the solution of (1) reaches (possibly in infinite time) the domain  $\mathcal{D}_0$ .*

*Proof.* 1. At a point  $(x, y) \in \mathcal{D}_0$ , the Lie derivative of the function  $f_2$  with respect to the vector field of the dynamics with  $u = 0$  is given by the expression

$$\partial_x f_2(x, y) f_1(x, y),$$

which is negative from Assumptions 2.v and 2.vi. We deduce that the sub level set  $\{(x, y) \in \mathcal{D} ; f_2(x, y) \leq 0\} = \mathcal{D} \setminus \mathcal{D}_+$  is positively invariant.

Let  $(x(\cdot), y(\cdot))$  be a solution for an initial condition in  $\mathcal{D}_+$  and the control  $u = 0$ . As long as  $(x(t), y(t))$  remains in  $\mathcal{D}_+$ , one has  $\dot{y}(t) = f_2(x(t), y(t)) > 0$ . Therefore, one has  $y(t) \geq y_0$ . Moreover, as  $f_2$  is strictly increasing w.r.t.  $x$  on  $\mathcal{D}_+ \cup \mathcal{D}_0$  (Assumptions 2.iii and 2.vi), one gets

$$f_2(x(t), y(t)) > 0 = f_2(\tilde{x}(y(t)), y(t)) \Rightarrow x(t) > \tilde{x}(y(t)).$$

Then,  $f_1$  being strictly decreasing w.r.t.  $x$  and  $y$  on  $\mathcal{D}_+$  (Assumption 2.i), one has also

$$\dot{x}(t) = f_1(x(t), y(t)) \leq f_1(\tilde{x}(y(t)), y_0).$$

On the other hand, on  $\mathcal{D}_0$  one has from the implicit function theorem

$$f_2(\tilde{x}(y), y) = 0 \Rightarrow \tilde{x}'(y) = -\frac{\partial_y f_2(x, y)}{\partial_x f_2(x, y)} \geq 0,$$

under Assumption 2.vi. The map  $\tilde{x}$  is thus non decreasing and one has then  $\tilde{x}(y(t)) \geq \tilde{x}(y_0)$ , from which one obtains

$$\dot{x}(t) \leq f_1(\tilde{x}(y_0), y_0),$$

as long as  $(x(t), y(t))$  remains in  $\mathcal{D}_+$ . From Assumption 2.v, one has also  $f_1(\tilde{x}(y_0), y_0) < 0$ . If  $(x(t), y(t))$  belongs to  $\mathcal{D}_+$  for any  $t \geq 0$ , then  $x(\cdot)$  is unbounded, which contradicts Assumption 1.1. We deduce that  $(x(\cdot), y(\cdot))$  has

to escape  $\mathcal{D}_+$  in finite time, and consequently  $y(\cdot)$  reaches its maximum  $y_{max}^0$  at finite time.

2. Let  $(x(\cdot), y(\cdot))$  be an optimal solution, and  $\bar{y} := \sup_t y(t) \leq y_{max}^0 < +\infty$ . Note that  $x(\cdot)$  is non increasing in the domain  $\mathcal{D}_+$ , with Assumption 2.ii. Then, with Assumption 2.iv, one has

$$\dot{y}(t) \geq f_2(x(t), y(t)) + g_2(x_0, \bar{y})u(t),$$

as long as  $(x(t), y(t))$  remains in  $\mathcal{D}_+$ . Therefore, one has

$$\bar{y} \geq y(t) \geq y_0 + \int_0^t f_2(x(t), y(t))dt + g_2(x_0, \bar{y}) \int_0^t u(t)dt,$$

and as  $g_2 < -f_2 < 0$  is negative, one obtains

$$\int_0^t f_2(x(t), y(t))dt < \bar{y} - y_0 - g_2(x_0, \bar{y})K < +\infty.$$

If  $(x(t), y(t))$  remains in  $\mathcal{D}_+$  for any  $t \geq 0$ , this last integral converges when  $t$  tends to  $+\infty$ , the function  $f_2$  being positive in  $\mathcal{D}_+$ . The solution  $(x(\cdot), y(\cdot))$  being bounded from Assumption 1.i, the function  $t \mapsto f_2(x(t), y(t))$  is uniformly continuous on  $[0, +\infty)$  and thus tends to 0 when  $t$  tends to  $+\infty$  (from Barbalat's Lemma [1]). We conclude that any solution reaches the domain  $\mathcal{D}_0$  possibly in infinite time.  $\square$

**Remark 2.1.** *For initial conditions  $(x(0), y(0))$  in  $\mathcal{D} \setminus \mathcal{D}_+$ , the control  $u$  identically null generates a trajectory that stays in  $\mathcal{D} \setminus \mathcal{D}_+$ , according to Lemma 2.1, with  $\dot{y} = f_2(x, y) \leq 0$ . Therefore, this control is optimal with a peak of  $y$  reached at initial time. In the following, we shall consider the nontrivial case of initial condition in  $\mathcal{D}_+$  only.*

### 3 The NSN strategy

In this section, we shall assume that Assumptions 1 and 2 are fulfilled. Let us fix an initial condition  $(x_0, y_0)$  in  $\mathcal{D}_+$ , and consider the uncontrolled dynamics, i.e. with  $u = 0$ . We denote  $(x^0(\cdot), y^0(\cdot))$  its solution. As long as  $(x^0(t), y^0(t))$  belongs to  $\mathcal{D}_+$ ,  $y^0(\cdot)$  is strictly increasing. From Lemma 2.1, we know that  $y^0(\cdot)$  reaches in finite time the domain  $\mathcal{D}_-$ , where it is strictly decreasing, and finally remains in  $\mathcal{D}_-$ . Therefore  $y^0(\cdot)$  reaches its maximum  $y_{max}^0 < +\infty$  in finite time, and for any  $\bar{y} \in [y_0, y_{max}^0]$ , we can define

$$\bar{x}(\bar{y}) := x^0(\bar{t}_{\bar{y}}) \text{ where } \bar{t}_{\bar{y}} := \inf\{t \geq 0; y^0(t) = \bar{y}\} < +\infty. \quad (5)$$

Note that  $x^0(\cdot)$  is strictly decreasing by Assumption 2.i, and therefore the map  $\bar{y} \mapsto \bar{x}(\bar{y})$  is smooth with  $\bar{x}' < 0$ .

We define the NSN (for "Null-Singular-Null") strategy as follows.

**Definition 1.** *Under Assumptions 1 and 2, consider for  $\bar{y} \in [y_0, y_{max}^0]$  the feedback control*

$$\psi_{\bar{y}}(x, y) := \begin{cases} k(x) := -\frac{f_2(x, \bar{y})}{g_2(x, \bar{y})}, & \text{if } y = \bar{y} \text{ and } (x, \bar{y}) \in \mathcal{D}_+, \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

This strategy consists in applying a null control until time  $\bar{t}_{\bar{y}}$ . Then, the singular arc consists in keeping  $y$  constant equal to  $\bar{y}$  with the feedback control  $k(x)$  until the solution exits the domain  $\mathcal{D}_+$ . Finally, the control is kept equal to 0. Let us show that the solution with the discontinuous feedback  $\psi_{\bar{y}}$  is well defined.

**Lemma 3.1.** *Under Assumptions 1 and 2, for any initial condition  $(x_0, y_0)$  in  $\mathcal{D}_+$  and number  $\bar{y} \in [y_0, y_{max}^0]$ , there exists a unique absolutely continuous solution which verifies*

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} f_1(x(t), y(t)) \\ f_2(x(t), y(t)) \end{bmatrix} + \begin{bmatrix} g_1(x(t), y(t)) \\ g_2(x(t), y(t)) \end{bmatrix} \psi_{\bar{y}}(x(t), y(t)), \text{ a.e. } t \geq 0$$

where  $\psi_{\bar{y}}(x(t), y(t))$  is well defined and takes values in  $[0, 1]$  for any  $t \geq 0$ .

*Proof.* With control  $u = 0$ , the solution is well defined, absolutely continuous and unique as solution of an o.d.e. with Lipschitz right side, up to time  $\bar{t}_{\bar{y}}$ . In the domain  $\mathcal{D}_+$ , note that one has  $g_2(x, \bar{y}) < -f_2(x, \bar{y}) < 0$  with Assumption A2.iv. Then,  $k(x)$  is well defined,  $C^1$  and takes values in  $[0, 1]$  for any  $x$  such that  $(x, \bar{y})$  belongs to  $\mathcal{D}_+$ . From time  $\bar{t}_{\bar{y}}$ , the solution with the control  $u = k(x)$  is thus well defined, absolutely continuous and unique as solution of an o.d.e. with Lipschitz right side, as long as it stays in  $\mathcal{D}_+$ . With this control, one can straightforwardly check that one has  $\dot{y}(t) = 0$  i.e. the solution verifies  $y(t) = \bar{y}$  for  $t \geq \bar{t}_{\bar{y}}$ . Therefore, the solution reaches a possible point of discontinuity of the feedback  $\psi_{\bar{y}}$  exactly when it exits the domain  $\mathcal{D}_+$ , say at a time  $t_+$ . From that time, the solution with control  $u = 0$  stays in  $\mathcal{D} \setminus \mathcal{D}_+$ , according to Lemma 2.1, and thus never reaches again a locus of discontinuity of the feedback  $\psi_{\bar{y}}$ . The solution is thus unique and well defined for any future time. Finally, we obtain by concatenation an unique absolutely continuous solution which satisfies the right side of the closed loop system excepted a times  $\bar{t}_{\bar{y}}, t_+$ .  $\square$

We shall denote the  $L^1$  norm associated with the NSN control by

$$\mathcal{L}(\bar{y}) := \int_0^{+\infty} u^{\psi_{\bar{y}}}(t) dt, \quad \bar{y} \in [y_0, y_{max}^0],$$

where  $u^{\psi_{\bar{y}}}(\cdot)$  is the open-loop control generated by the feedback (6).

Let us now define the function

$$\Delta(x, y) := f_2(x, y)g_1(x, y) - f_1(x, y)g_2(x, y), \quad (x, y) \in \mathcal{D}.$$

Note that  $\Delta$  is necessarily negative on  $\mathcal{D}_+$  under Assumption 2. We consider the additional assumption.

**Assumption 3.** *One has  $\frac{\partial}{\partial \bar{y}}(\frac{f_2}{\Delta}) \geq 0$  on  $\mathcal{D}_+$ .*

Then, the function  $\mathcal{L}$  can be characterized as follows.

**Proposition 3.1.** *Under Assumptions 1, 2 and 3, one has*

$$\mathcal{L}(\bar{y}) = \int_{\bar{x}(\bar{y})}^{\tilde{x}(\bar{y})} \frac{-f_2(x, \bar{y})}{\Delta(x, \bar{y})} dx, \quad \bar{y} \in [y_0, y_{max}^0], \quad (7)$$

where  $\tilde{x}(\bar{y})$  and  $\bar{x}(\bar{y})$  are defined in (4) and (5) respectively. Moreover, the map  $\bar{y} \mapsto \mathcal{L}(\bar{y})$  is strictly decreasing.

*Proof.* Let  $(x(\cdot), y(\cdot))$  be the solution generated by the feedback control (6), and  $u(\cdot)$  the corresponding open loop control. We consider the time function

$$\gamma_2(t) := f_2(x(t), \bar{y}), \quad t \geq \bar{t}_{\bar{y}},$$

where  $\bar{t}_{\bar{y}} = \inf\{t \geq 0; y(t) = \bar{y}\} < +\infty$ . As long as  $(x(t), \bar{y}) \in \mathcal{D}_+$ , one has  $\dot{y}(t) = 0$  with the control  $u(t) = k(x(t))$  and

$$\dot{x} = h(x) := f_1(x, \bar{y}) + g_1(x, \bar{y})k(x) = -\frac{\Delta(x, \bar{y})}{g_2(x, \bar{y})},$$

which is negative by Assumption 2, and then

$$\dot{\gamma}_2 = \frac{\partial f_2(x(t), \bar{y})}{\partial x} h(x) < 0,$$

from Assumption 2.iii. The function  $\gamma_2$  is thus strictly decreasing and therefore  $\mathcal{D}_0$  is reached at a time  $T \geq \bar{t}_{\bar{y}}$  (possibly equal to  $+\infty$ ). One has then

$$x(T) = \max\{x \leq \bar{x}(\bar{y}); f_2(x, \bar{y}) = 0\} = \tilde{x}(\bar{y}).$$

If  $T < +\infty$ , then  $u(t) = 0$  for any  $t > T$  because the state cannot reach again  $\mathcal{D}_+$  by Lemma 2.1. Therefore,  $T$  being finite or not, one has

$$\mathcal{L}(\bar{y}) = \int_{\bar{t}_{\bar{y}}}^T k(x(t)) dt.$$

Note that the map  $[\bar{t}_{\bar{y}}, T] \mapsto [\tilde{x}(\bar{y}), \bar{x}(\bar{y})]$  is onto, and one can then write

$$\mathcal{L}(\bar{y}) = - \int_{\tilde{x}(\bar{y})}^{\bar{x}(\bar{y})} \frac{k(x)}{h(x)} dx = \int_{\tilde{x}(\bar{y})}^{\bar{x}(\bar{y})} \frac{-f_2(x, \bar{y})}{\Delta(x, \bar{y})} dx.$$

The function  $\bar{y} \mapsto \tilde{x}(\bar{y})$  is not necessarily differentiable. However, the integrand in the above expression of  $\mathcal{L}$  is null at  $\tilde{x}(\bar{y})$  for any  $\bar{y}$ . Therefore,  $\mathcal{L}$  is differentiable with

$$\mathcal{L}'(\bar{y}) = \left( -\frac{f_2(\tilde{x}(\bar{y}), \bar{y})}{\Delta(\tilde{x}(\bar{y}), \bar{y})} \right) \tilde{x}' - \int_{\tilde{x}}^{\bar{x}} \frac{\partial}{\partial y} \left( \frac{f_2(x, \bar{y})}{\Delta(x, \bar{y})} \right) dx.$$

By Assumptions 2 and 3, one has  $-\frac{f_2}{\Delta} < 0$  and  $\frac{\partial}{\partial y} \left( \frac{f_2}{\Delta} \right) \geq 0$  on  $\mathcal{D}_+$ , and as  $\tilde{x}'$  is negative, we deduce that one has  $\mathcal{L}'(\bar{y}) < 0$ .  $\square$

**Remark 3.1.** *When applying the feedback (6), it generates only one discontinuity point of the open loop control  $u(\cdot)$ , when the solution  $y(\cdot)$  reaches  $\bar{y}$  in  $\mathcal{D}_+$ , but not when  $y(\cdot)$  leaves the singular arc  $y = \bar{y}$  as the control is null when reaching  $\mathcal{D}_0$ . Consequently, the trajectory tangentially leaves the singular arc.*

## 4 An optimal synthesis

In this section, we give our main result about the optimality of the NSN strategy, which is expressed in terms of positivity of a certain flux on the domain  $\mathcal{D}_+$ . The proof is using the clock form but in an unusual way (compared for instance to [10, 8]), which requires some assumption about the dynamics on the boundary of the domain  $\mathcal{D}_+$ , given below.



**Assumption 4.**  $\mathcal{D}_0$  is connected and has  $g_2 < 0$ ,  $\nabla f_2 \cdot (f + g) \geq 0$  on  $\mathcal{D}_0$ .

**Proposition 4.1.** Under Assumptions 1, 2, 3 and 4, let  $(x_0, y_0)$  be an initial condition in  $\mathcal{D}_+$  such that  $\mathcal{L}(y_0) \geq K$ . If one has

$$\frac{\partial}{\partial y} \left( \frac{f_2(x, y)}{\Delta(x, y)} \right) + \frac{\partial}{\partial x} \left( \frac{f_1(x, y)}{\Delta(x, y)} \right) > 0, \quad (x, y) \in \mathcal{D}_+, \quad y \leq y_{max}^0, \quad (8)$$

then the feedback  $\psi_{\bar{y}^*}$ , with  $\bar{y}^* \in [y_0, y_{max}^0]$  such that  $\mathcal{L}(\bar{y}^*) = K$ , is optimal.

*Proof.* Note first that one has  $\mathcal{L}(y_{max}^0) = 0$  as the NSN control is identically null for  $\bar{y} = y_{max}^0$ . As the map  $\mathcal{L}$  is strictly decreasing by Proposition 3.1, we deduce that there exists a unique  $\bar{y}^* \in [y_0, y_{max}^0]$  such that  $\mathcal{L}(\bar{y}^*) = K$  when  $\mathcal{L}(y_0) \geq K$ .

For a fixed initial condition  $(x_0, y_0)$  in  $\mathcal{D}_+$ , we denote by  $(x^*(\cdot), y^*(\cdot))$  the solution generated by the NSN strategy with  $\bar{y} = \bar{y}^*$  (which is unique according to Lemma 3.1), and  $u^*(\cdot)$  its open loop control. Consider the curve  $\mathcal{C}^*$  in the plane

$$\mathcal{C}^* := \{(x^*(t), y^*(t)) ; t \in [0, t_h^*]\},$$

where  $t_h^* > 0$  is such that  $x^*(t_h^*) = \bar{x}(\bar{y}^*)$ .  $\mathcal{C}^*$  is the part of the orbit for which  $y^*(\cdot)$  is non decreasing, and its extremity belongs to  $\mathcal{D}_0$ .

Let  $\bar{t}^* \in [0, t_h^*]$  be such that  $x^*(\bar{t}^*) = \bar{x}(\bar{y}^*) \leq x_0$ . For any  $t \in [0, \bar{t}^*]$ , the control  $u^*(t)$  is null. Then, at any  $(x, y) \in \mathcal{C}^*$  with  $x > \bar{x}(\bar{y}^*)$ , the curve  $\mathcal{C}^*$  admits an upward normal in the  $(x, y)$  plane given by

$$\vec{n}(x, y) = \begin{bmatrix} f_2(x, y) \\ -f_1(x, y) \end{bmatrix}.$$

Let  $(x, y) \mapsto \vec{v}(x, y, u)$  be the vector field in the plane for the control  $u$ . For any  $(x, y) \in \mathcal{C}^*$  with  $x > \bar{x}(\bar{y}^*)$ , one has

$$\vec{n}(x, y) \cdot \vec{v}(x, y, u) = \Delta(x, y)u \leq 0.$$

Therefore, the forward orbit with any other control  $u(\cdot)$  lies below the curve  $\mathcal{C}^*$  in the  $(x, y)$  plane for  $x \in [\bar{x}(\bar{y}^*), x_0]$ .

Assume that there exists another solution  $(x(\cdot), y(\cdot))$  with  $(x(0), y(0)) = (x_0, y_0) \in \mathcal{D}_+$ , generated by an optimal control  $u(\cdot)$  such that  $\sup_t y(t) < \bar{y}^*$ . From Lemma 2.1, we know that  $(x(\cdot), y(\cdot))$  reaches the level set  $\mathcal{D}_0$  at a time  $t^0$  (possibly infinite). The trajectory being bounded, by Assumption 1.i, the point  $(x(t^0), y(t^0)) \in \mathcal{D}_0$  is finite, with  $y(t^0) < \bar{y}^*$ . Let us now consider the curve in the plane

$$\mathcal{C} := \{(x(t), y(t)) ; t \in [0, t^0]\},$$

that has to be below the curve  $\mathcal{C}^*$ , according to the above.

From the point  $(x^*(t_h^*), y^*(t_h^*)) \in \mathcal{D}_0$ , there exists an admissible trajectory that stays in the level set  $\mathcal{D}_0$  if for any  $(x, y) \in \mathcal{D}_0$  there is a control  $u$  in  $[0, 1]$  such that

$$\nabla f_2(x, y) \cdot (f(x, y) + g(x, y)u) = 0.$$

On the set  $\mathcal{D}_0$ , one has  $\nabla f_2 \cdot f = \partial_x f_2 f_1$  which is negative by Assumptions 2.v-vi. Then the function

$$\psi^\dagger(x, y) := -\frac{\nabla f_2(x, y) \cdot f(x, y)}{\nabla f_2(x, y) \cdot g(x, y)} > 0, \quad (x, y) \in \mathcal{D}_0,$$

is well defined and belongs to  $[0, 1]$  by Assumption 4. Let  $(x^\dagger(\cdot), y^\dagger(\cdot))$  be the solution of (1) with  $(x^\dagger(t_h^*), y^\dagger(t_h^*)) = (x^*(t_h^*), y^*(t_h^*)) \in \mathcal{D}_0$  and the feedback control  $\psi^\dagger$ . We denote  $u^\dagger(\cdot)$  the corresponding open loop control. The trajectory remains in  $\mathcal{D}_0$ , and as  $g_2$  is negative on  $\mathcal{D}_0$  (Assumption 4) and  $u^\dagger$  is positive, one has  $\dot{y}^\dagger = g_2(x, y)\psi^\dagger(x, y) < 0$ . Then, when  $t$  tends to  $+\infty$ , either  $y^\dagger(t)$  tends to  $-\infty$  or  $y^\dagger(t)$  converges to some finite limit  $y_\infty^\dagger$  with  $\lim_{t \rightarrow +\infty} \dot{y}^\dagger(t) = 0$ . In this last case, one has  $g_2(x^\dagger(y_\infty^\dagger), y_\infty^\dagger)\psi^\dagger(x^\dagger(y_\infty^\dagger), y_\infty^\dagger) = 0$  ( $\tilde{x}$  being continuous) i.e.  $(x^\dagger(y_\infty^\dagger), y_\infty^\dagger) \notin \mathcal{D}_0$ , and as  $\mathcal{D}_0$  is connected one gets  $y_\infty^\dagger = \inf \text{proj}_2(\mathcal{D})$ . Therefore, there exists  $t^\dagger < +\infty$  such that  $y^\dagger(t^\dagger) = y(t^0)$ , with  $x^\dagger(t^\dagger) = \tilde{x}(y(t^0)) = x(t^0)$ . Let us finally consider the curve in the plane

$$\mathcal{C}^\dagger := \{(x^\dagger(t), y^\dagger(t)) ; t \in [t_h^*, t^\dagger]\}.$$

We consider now the concatenation of the three curves  $\mathcal{C}^*$ ,  $\mathcal{C}^\dagger$  and  $\mathcal{C}$  (see Figure 1), which defines a simple closed curve  $\Gamma = \{(\tilde{x}(\tau), \tilde{y}(\tau)) ; \tau \in [0, t^\dagger + t^0]\}$  with

$$(\tilde{x}(\tau), \tilde{y}(\tau)) = \begin{cases} (x^*(\tau), y^*(\tau)), & \tau \in [0, t_h^*), \\ (x^\dagger(\tau), y^\dagger(\tau)), & \tau \in [t_h^*, t^\dagger), \\ (x(t^0 - t^\dagger - \tau), y(t^0 - t^\dagger - \tau)), & \tau \in [t^\dagger, t^\dagger + t^0), \end{cases}$$

that is anti-clockwise oriented in the  $(x, y)$  plane by  $\tau \in [0, t^\dagger + t^0]$  (see Figure 1). Let  $\mathcal{E}$  be the region bounded by  $\Gamma$ , which belongs to  $\mathcal{D}_+$ . By Assumption 2,  $\Delta$  is non null on  $\mathcal{D}_+$  and one can then write from equations (1) the 1-form in  $\mathcal{E}$

$$u(t)dt = \frac{f_2(x, y)}{\Delta(x, y)} dx - \frac{f_1(x, y)}{\Delta(x, y)} dy.$$

Applying Green's Theorem, one obtains

$$\oint_{\Gamma} u(t)dt = \iint_{\mathcal{E}} \frac{\partial}{\partial x} \left( -\frac{f_1(x, y)}{\Delta(x, y)} \right) - \frac{\partial}{\partial y} \left( \frac{f_2(x, y)}{\Delta(x, y)} \right) dx dy,$$

which is negative by condition (8). Consequently, one has

$$\oint_{\Gamma} u(t)dt = \int_0^{t_h^*} u^*(t)dt + \int_{t_h^*}^{t^\dagger} u^\dagger(t)dt - \int_0^{t^0} u(t)dt < 0,$$

that is

$$\int_0^{t^0} u(t)dt > K + \int_{t_h^*}^{t^\dagger} u^\dagger(t)dt > K,$$

which contradicts the optimality of the control  $u(\cdot)$  under the constraint (3).  $\square$

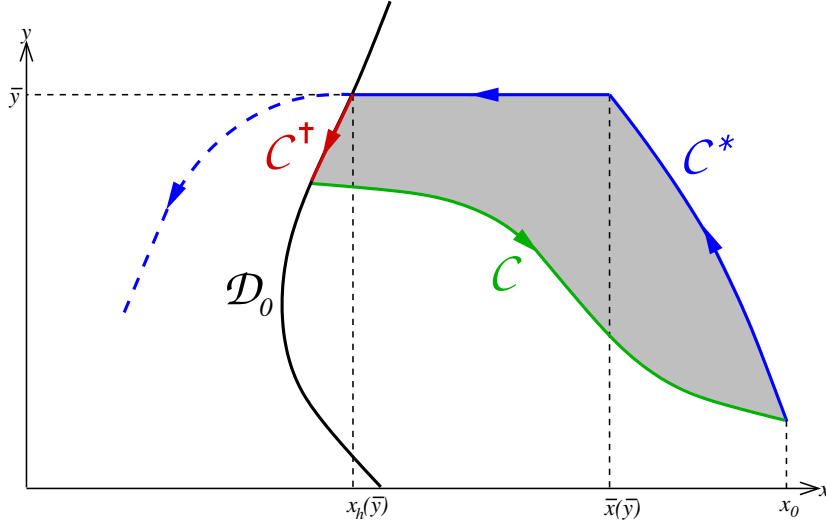


Figure 1: Application of the Green's Theorem on the closed domain (in gray) delimited by the curves  $C^*$  (in blue),  $C^\dagger$  (in red) and  $C$  (in green). The curve (in black) represents the level-set  $\mathcal{D}_0$  which delimits the domain  $\mathcal{D}_-$  on the right.

**Remark 4.1.** When  $K > \mathcal{L}(y_0)$ , the budget  $K$  is large enough to ensure  $y(t) \leq y_0$  for any  $t \geq 0$ . One can apply for instance the feedback strategy  $\psi_{y_0}$ , which is optimal with a  $L^1$  norm of the control less than  $K$ , equal to  $\mathcal{L}(y_0)$ .

Let us illustrate our results on an example for which the optimal control can be determined analytically.

**Example 1.**

$$\begin{cases} \dot{x} &= -(x+1)^2 y + (x+1)^2 y u, \\ \dot{y} &= x y - (x+1) y u, \end{cases} \quad u \in [0, 1].$$

Whatever is the control  $u$ , one has  $\dot{x} = 0$  at  $x = -1$  and  $\dot{y} = 0$  at  $y = 0$ . Therefore the domain

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2; x > -1, y > 0\},$$

is positively invariant. Note that any solution  $(x(\cdot), y(\cdot))$  in  $\mathcal{D}$  verifies  $\dot{x} \leq 0$ . Therefore,  $x(\cdot)$  is always bounded. Consider now the control  $u = 0$  and the function  $V(x, y) = x + 2y$ . One has  $\frac{d}{dt}V = -x^2 y - y^2 < 0$ . The function  $V$  is thus strictly decreasing along the solutions in  $\mathcal{D}$ , from which one deduces the inequalities  $-1 \leq x(t) \leq x(0) + 2y(0)$  and  $0 \leq y(t) \leq (x(0) + y(0) - 1)/2$ . The solutions in  $\mathcal{D}$  with  $u = 0$  are thus bounded.

The sub and super sets of the function  $f_2$  are  $\mathcal{D}_- = \{(x, y) \in \mathcal{D}; x < 0\}$ ,  $\mathcal{D}_+ = \{(x, y) \in \mathcal{D}; x > 0\} = \{(x, y) \in \mathbb{R}^2; x > 0, y > 0\}$  and the function  $\tilde{x}$  is simply the null function. Assumption 1 is satisfied.

In  $\mathcal{D}_+$ , the function  $f_1(x, y) = -(x+1)^2 y$  is negative and strictly decreasing with respect to  $x$  and  $y$ , while the function  $g_1(x, y) = -f_1(x, y)$  is strictly increasing with respect to  $x$  and  $y$  with  $f_1 + g_1 = 0 \leq 0$ . The function  $f_2(x, y) = xy$

verifies  $\partial_x f_2 = y > 0$ , while the function  $g_2(x, y) = -(x+1)y$  is strictly decreasing with respect to  $x$  and  $y$  with  $f_2 + g_2 = -y < 0$ . In  $\mathcal{D}_0$ , one has  $f_1 = -y < 0$  and  $\partial_x f_2 = y > 0$ ,  $\partial_y f_2 = 0 \leq 0$ . Assumption 2 is thus fulfilled. One has

$$\frac{f_2}{\Delta} = -\frac{x}{(x+1)^2 y} \Rightarrow \frac{\partial}{\partial y} \left( \frac{f_2}{\Delta} \right) = \frac{x}{(x+1)^2 y^2}$$

which is positive on  $\mathcal{D}_+$ . Assumption 3 is satisfied.

In  $\mathcal{D}_0$ ,  $g_2 = -y$  is negative and  $\nabla f_2 \cdot (f + g) = 0 \geq 0$ . Assumption 4 is satisfied.

Finally, one has

$$\partial_y \left( \frac{f_2}{\Delta} \right) + \partial_x \left( \frac{f_1}{\Delta} \right) = \frac{x}{(x+1)^2 y},$$

that is positive on  $\mathcal{D}_+$ .

Now, from Proposition 3.1, one can determine the function  $\mathcal{L}$  as follows. Firstly, the solution of the system with  $u = 0$  for an initial condition  $(x_0, y_0)$  in  $\mathcal{D}_+$  can be parameterized by  $x$  as the map  $t \mapsto x(t)$  is strictly decreasing, that is

$$y(t) = y_0 - \int_{x_0}^{x(t)} \frac{x}{(x+1)^2} dx,$$

which gives

$$\int_{x_0}^{\bar{x}(\bar{y})} \frac{x}{(x+1)^2} dx = y_0 - \bar{y}, \quad (9)$$

and

$$\begin{aligned} y_{max}^0 &= y_0 - \int_{x_0}^0 \frac{x}{(x+1)^2} dx = y_0 - \left[ \frac{1}{x+1} + \log(x+1) \right]_0^{x_0} \\ &= \frac{1}{x_0+1} + \log(x_0+1) - 1 + y_0. \end{aligned} \quad (10)$$

Secondly, one has

$$\begin{aligned} \mathcal{L}(\bar{y}) &= \int_0^{\bar{x}(\bar{y})} \frac{-f_2(x, \bar{y})}{\Delta(x, \bar{y})} dx = \frac{1}{\bar{y}} \int_0^{\bar{x}(\bar{y})} \frac{x}{(x+1)^2} dx \\ &= \frac{1}{\bar{y}} \left( \int_0^{x_0} \frac{x}{(x+1)^2} dx + \int_{x_0}^{\bar{x}(\bar{y})} \frac{x}{(x+1)^2} dx \right), \end{aligned}$$

which gives with (9) and (10) the expression

$$\mathcal{L}(\bar{y}) = \frac{y_{max}^0 - y_0 + y_0 - \bar{y}}{\bar{y}} = \frac{y_{max}^0}{\bar{y}} - 1,$$

that is defined for  $\bar{y} \in [y_0, y_{max}^0]$ . Finally, from Proposition 4.1, we obtain that for a budget  $K \leq \frac{y_{max}^0}{y_0} - 1$ , the NSN strategy (6) with

$$\bar{y} = \bar{y}^* := \frac{y_{max}^0}{K+1}$$

is optimal. Therefore, the feedback

$$\psi^*(x, y) := \begin{cases} \frac{x}{x+1}, & \text{if } y = \bar{y}^* \text{ and } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

is optimal. An example of optimal solution is drawn on Figure 2, where one can see that the optimal trajectory leaves tangentially the singular arc and the optimal control is continuous at that point, as underlined in Remark 3.1.

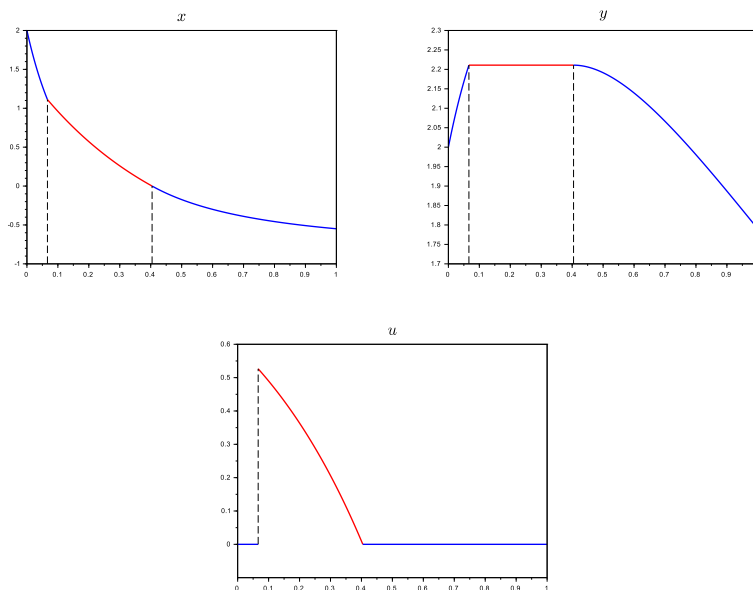


Figure 2: Optimal solution of Example 1 for initial condition  $(x_0, y_0) = (2, 2)$  and budget  $K = 0.1$ . Singular arc is depicted in red.

## 5 The case of Kolmogorov dynamics

In this Section we particularize the results of Proposition 4.1 to a class of Kolmogorov dynamics [13] in  $\mathbb{R}_+^2$ , for which it is easier to verify the required assumptions.

$$\begin{cases} \dot{x} &= -(\phi_1(x, y) - \phi_2(x, y)u)x, \\ \dot{y} &= (\phi_3(x, y) - \phi_4(x, y)u)y, \end{cases} \quad u \in [0, 1], \quad (11)$$

where  $\phi_i$  are smooth maps. The positive orthant  $\mathcal{D} = \{(x, y) \in \mathbb{R}^2; x > 0, y > 0\}$  is clearly positively invariant by (11).

**Hypotheses 5.** On  $\mathcal{D}$ , one has

- i.  $\phi_1$  and  $\phi_2$  are positive, with  $\phi_2 - \phi_1 \leq M < +\infty$ ,
- ii.  $\partial_x \phi_3$  is positive and  $\partial_y \phi_3$  is non positive, with  $\phi_3(0, y) < 0 < \lim_{x \rightarrow +\infty} \phi_3(x, y)$  for any  $y > 0$ ,

- iii.  $\phi_1 \geq \phi_2$  with  $\partial_y \phi_1 \geq \partial_y \phi_2 > 0$  when  $\phi_3 \geq 0$ , and  $\phi_1 = \phi_2$  when  $\phi_3 = 0$ .  
The maps  $x \mapsto \phi_1(x, y)x$ ,  $x \mapsto \phi_2(x, y)x$  are strictly increasing for any  $y$ ,
- iv. When  $\phi_3 > 0$ ,  $\phi_4$  is strictly increasing with respect to  $x$  with  $\phi_4 > \phi_3$  and  $[\phi_3, \phi_4]_y := \phi_3 \partial_y \phi_4 - \phi_4 \partial_y \phi_3 \geq 0$ . The map  $y \mapsto \phi_4(x, y)y$  is strictly increasing for any  $x$ .

**Lemma 5.1.** *Under Hypotheses 5, Assumptions 1, 2, 3, 4 are fulfilled.*

*Proof.* From Hypothesis 5.ii, there exists a unique function  $y \mapsto \tilde{x}(y) > 0$  such that  $\phi_3(\tilde{x}(y), y) = 0$  for any  $y > 0$ , which is moreover non decreasing with respect to  $y$  (by the implicit function theorem). The sub and super level sets  $\mathcal{D}_-$ ,  $\mathcal{D}_+$ , are thus non empty and defined as

$$\mathcal{D}_- = \{(x, y) \in \mathcal{D}; x < \tilde{x}(y)\}, \quad \mathcal{D}_+ = \{(x, y) \in \mathcal{D}; x > \tilde{x}(y)\},$$

and the level set  $\mathcal{D}_0$  is  $\{(x, y) \in \mathcal{D}; x = \tilde{x}(y)\}$ .

From Hypothesis 5.i, one has  $\dot{x} \leq (\phi_1(x, y)(u - 1) + Mu)x$  from which one can write for any admissible solution the inequality

$$x(t) \leq x_0 e^{MK} \exp\left(\int_0^t \phi_1(x(\tau), y(\tau))(u(\tau) - 1) d\tau\right) \leq x_0 e^{MK} < +\infty,$$

for any  $t \geq 0$ . For the uncontrolled dynamics, one has  $\dot{x} = -\phi_1(x, y)x < 0$  i.e.  $x(\cdot)$  is strictly decreasing. Let us show that  $\mathcal{D} \setminus \mathcal{D}_+ = \mathcal{D}_0 \cup \mathcal{D}_-$  is reached in finite time. If not, one has  $x(t) \geq \tilde{x}(y(t))$  for any  $t \geq 0$  and  $y(\cdot)$  is strictly increasing. Then, one should have  $\dot{x}(t) \leq -\kappa x(t)$  for any  $t \geq 0$ , where  $\kappa = \min_{\xi \in [\tilde{x}(y_0), x(0)]} \phi_1(\xi, y(0)) > 0$ . Therefore,  $x(\cdot)$  converges to 0, while  $\tilde{x}(y(t)) \geq \tilde{x}(y(0)) > 0$  for any  $t \geq 0$ , and thus a contradiction. On  $\mathcal{D}_0$ , one has

$$\frac{d}{dt} \phi_3(x, y) = -\partial_x \phi_3(x, y) \phi_1(x, y) x < 0,$$

with Hypotheses 5i. and ii. The domain  $\mathcal{D} \setminus \mathcal{D}_+$  is thus positively invariant. Moreover one has  $\dot{y} \leq 0$  in  $\mathcal{D} \setminus \mathcal{D}_+$ . We conclude that the solutions for the uncontrolled dynamics are such that  $y(\cdot)$  is either non decreasing, or strictly increasing up to a finite time and then non decreasing (and thus bounded). Moreover, any other controlled solution with a lower peak value of  $y$  is also bounded, as  $x(\cdot)$  is always bounded. Assumption 1 is verified.

Clearly, the map  $f_1 = -\phi_1 x$  is negative in  $\mathcal{D}_+ \cup \mathcal{D}_0$  and strictly decreasing with respect to  $x$  and  $y$  from Hypotheses 5.i. and iii. The map  $g_1 = \phi_2 x$  is strictly increasing with respect to  $x$  and  $y$  also from Hypothesis 5.iii, and  $f_1 + g_1 = (\phi_2 - \phi_1)x \leq 0$  in  $\mathcal{D}_+$ . The map  $f_2 = \phi_3 y$  verifies  $\partial_x f_2 = \partial_x \phi_3 y$  which is positive on  $\mathcal{D}$  from Hypothesis 5.ii, and  $\partial_y f_2 = \phi_3 + \partial_y \phi_3 y$  is non positive on  $\mathcal{D}_0$ . The map  $g_2 = -\phi_4 y$  is strictly decreasing with respect to  $x$  and  $y$  from Hypothesis 5.iv, and  $f_2 + g_2 = (\phi_3 - \phi_4)y < 0$  in  $\mathcal{D}_+$ . Assumption 2 is verified.

One has  $\Delta = (\phi_3 \phi_2 - \phi_1 \phi_4)xy$  and with inequality  $\phi_1 \geq \phi_2$ , one obtains  $\Delta \leq (\phi_3 - \phi_4)\phi_1 xy$  on  $\mathcal{D}_+$ , which is negative by Hypothesis 5.iv. One gets

$$\frac{f_2}{\Delta} = \frac{\phi_3}{(\phi_3 \phi_2 - \phi_1 \phi_4)x} \Rightarrow \frac{\partial}{\partial y} \left( \frac{f_2}{\Delta} \right) = \frac{\phi_3(\phi_4 \partial_y \phi_1 - \phi_3 \partial_y \phi_2) + \phi_1[\phi_3, \phi_4]_y}{(\phi_3 \phi_2 - \phi_1 \phi_4)^2 x}.$$

With Hypotheses 5.iii. and iv., one has  $[\phi_3, \phi_4]_y \geq 0$  and  $\phi_4 \partial_y \phi_1 - \phi_3 \partial_y \phi_2 > 0$  on  $\mathcal{D}_+$ , which implies that  $\partial_y(f_2/\Delta)$  is positive on  $\mathcal{D}_+$ . Assumption 3 is thus verified.

From Hypothesis 5.iv, one has  $\phi_4 > 0$  on  $\mathcal{D}_0$ . Then, the map  $g_2 = -\phi_4 y$  is negative on  $\mathcal{D}_0$ . Moreover, one has

$$\nabla f_2 = \begin{bmatrix} \partial_x \phi_3 \\ \partial_y \phi_3 \end{bmatrix} y, \quad (x, y) \in \mathcal{D}_0,$$

which gives

$$\nabla f_2 \cdot (f + g) = \partial_x \phi_3 (\phi_2 - \phi_1) x y + \partial_y \phi_3 (\phi_3 - \phi_4) y^2,$$

that is non negative on  $\mathcal{D}_0$  with Hypotheses 5.ii, iii and iv. Assumption 4 is fulfilled.  $\square$

Let us posit

$$\delta(x, y) = \phi_3(x, y) \phi_2(x, y) - \phi_1(x, y) \phi_4(x, y), \quad (x, y) \in \mathcal{D}.$$

The application of Propositions 3.1 and 4.1 gives the following result.

**Proposition 5.1.** *Under Hypotheses 5, one has*

$$\mathcal{L}(\bar{y}) = \int_{\bar{x}(\bar{y})}^{\bar{x}(\bar{y})} \frac{-\phi_3(x, \bar{y})}{\delta(x, \bar{y}) x} dx.$$

For initial conditions  $(x_0, y_0)$  in  $\mathcal{D}_+$  such that  $\mathcal{L}(y_0) \geq K$  and

$$\begin{aligned} & (\phi_3(\phi_4 \partial_y \phi_1 - \phi_3 \partial_y \phi_2) + \phi_1[\phi_3, \phi_4]_y) y + \\ & (\phi_3[\phi_1, \phi_2]_x + \phi_1[\phi_2 \partial_x \phi_3 - \phi_1 \partial_x \phi_4]) x > 0, \quad (x, y) \in \mathcal{D}_+, \quad y \leq y_{max}^0 \end{aligned} \quad (12)$$

(where  $[\phi_1, \phi_2]_x := \phi_1 \partial_x \phi_2 - \phi_2 \partial_x \phi_1$ ), then there exists  $y^* \in [y_0, y_{max}^0]$  such that  $\mathcal{L}(y^*) = K$  and the feedback

$$\psi_{y^*}(x) = \begin{cases} \frac{\phi_3(x, y^*)}{\phi_4(x, y^*)}, & \text{if } y = y^* \text{ and } x > \tilde{x}(y^*), \\ 0, & \text{otherwise,} \end{cases} \quad (13)$$

is optimal.

*Proof.* One has

$$\frac{f_1}{\Delta} = \frac{-\phi_1}{(\phi_3 \phi_2 - \phi_1 \phi_4) y} \Rightarrow \frac{\partial}{\partial x} \left( \frac{f_1(x, y)}{\Delta(x, y)} \right) = \frac{\phi_3[\phi_1, \phi_2]_x + \phi_1[\phi_2 \partial_x \phi_3 - \phi_1 \partial_x \phi_4]}{(\phi_3 \phi_2 - \phi_1 \phi_4)^2 y},$$

and then condition (8) amounts exactly to have (12).  $\square$

Let us underline that the first term in (12) is necessarily positive, under Hypotheses 5.i, iii and iv. A simple way to guarantee condition (12) to be fulfilled is to have the second term non-negative, which can be obtained for instance as follows.

**Corollary 5.1.** *Under Hypotheses 5 with  $\phi_1 = \phi_2$  and  $\phi_4 = \phi_3 + \alpha$  ( $\alpha > 0$ ) in  $\mathcal{D}$ , the feedback (6) is optimal for any initial condition in  $\mathcal{D}_+$  with  $\mathcal{L}(y_0) \geq K$ .*

We present below some concrete examples within the biological field, which satisfy the conditions of Corollary 5.1 and allow to conclude directly about the optimality of the NSN strategy.

**Example 2.** *We consider the SIR model [9] which is very popular in epidemiology. With non-pharmaceutic interventions which consist in reducing the contact between susceptible and infected populations (by means of reducing social distance for human disease for instance), the model writes:*

$$\begin{cases} \dot{x} &= -\beta(1-u)xy, \\ \dot{y} &= \beta(1-u)xy - \alpha y, \end{cases} \quad u \in [0, 1],$$

where  $x$  and  $y$  stand for the density of susceptible and infected populations, respectively, and  $u$  is the control variable (naturally subject to a budget constraint). Parameter  $\beta$  is the infection rate (without intervention), and  $\alpha$  is the recovery rate. Without control (i.e.  $u = 0$ ), it is well known that the condition for an epidemics outbreak is given by the reproduction number

$$\mathcal{R}_0 := \frac{\beta}{\alpha},$$

that has to be larger than one. Then, the size of the infected population  $y(\cdot)$  increases up to a peak value that could be very high. The objective of the control is to reduce this peak value. Here the domain  $\mathcal{D}$  is

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2; x > 0, y > 0, x + y \leq 1\},$$

and one has the following expressions of the functions  $\phi_i$  ( $i = 1 \dots 4$ )

$$\phi_1(x, y) = \phi_2(x, y) = \beta y, \quad \phi_3(x, y) = \beta x - \alpha, \quad \phi_4(x, y) = \beta x.$$

The separatrix  $\mathcal{D}_0$  between  $\mathcal{D}_-$  and  $\mathcal{D}_+$  is a vertical segment

$$\mathcal{D}_0 = \left\{ (x, y) \in \mathcal{D}; x = \frac{1}{\mathcal{R}_0} \right\}.$$

One can straightforwardly check that Hypotheses 5 and conditions of Corollary 5.1 are fulfilled when  $\mathcal{R}_0 > 1$ . We can then conclude that the NSN strategy is optimal under a  $L^1$  budget control on the control  $u(\cdot)$ , as in [11]. Let us underline that when the initial density  $y_0$  of the infected population is very low (which is often the case in face to a new epidemics), the time to reach the minimum peak can be very large, justifying the consideration of an unbounded time horizon. In [11], it is shown that the optimal control can be determined analytically for the limiting case of of an arbitrary small  $y_0$  with an initial density of the susceptible population equal to  $1 - y_0$ .

**Example 3.** *We consider the classical resource-consumer (or "batch" bio-process) model, which is very popular in microbiology (see e.g. [7])*

$$\begin{cases} \dot{x} &= -\frac{1}{Y}\mu(x)y(1-u), \\ \dot{y} &= \mu(x)y(1-u) - my, \end{cases} \quad u \in [0, 1],$$



where  $x$  and  $y$  are the concentrations of the resource and the consumer, respectively. The function  $\mu$  is the specific growth rate, that is assumed to follow the well-known Monod's expression

$$\mu(x) := \frac{x}{K + x}.$$

The parameter  $Y$  is the yield coefficient of the transformation of the resource into consumer growth, while the parameter  $m > 0$  is the mortality rate of the consumer (supposed to be relatively low compared to the growth term). Here the control  $u$  is an isolation factor (by biological or physical means) which limits the access to the resource for the consumer. When the consumer is a living species that proliferates on the resource in an undesirable way (e.g. bacteria presenting some health risks), an objective is to reduce its peak value for a given budget on the control. For this model, the domain  $\mathcal{D}$  is

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2; x > 0, y > 0\},$$

with the functions

$$\phi_1(x, y) = \phi_2(x, y) = \frac{1}{Y} \frac{y}{1 + x}, \quad \phi_3(x, y) = \frac{x}{1 + x} - m, \quad \phi_4(x, y) = \frac{x}{1 + x},$$

for which can easily check that Hypotheses 5 and conditions of Corollary 5.1 are fulfilled for a mortality rate  $m < 1$ . Here also, the level set  $\mathcal{D}_0$  which splits the domain  $\mathcal{D}$  into between  $\mathcal{D}_-$  and  $\mathcal{D}_+$  is a vertical line

$$\mathcal{D}_0 = \left\{ (x, y) \in \mathcal{D}; x = \frac{m}{1 - m} \right\}.$$

Then, we can conclude that the NSN strategy is also optimal for this problem.

**Example 4.** We consider here the same resource-consumer model as in Example 3 but with a ratio-dependent growth rate (see e.g. [7])

$$\begin{cases} \dot{x} &= -\frac{1}{Y} \mu(x, y) y (1 - u), \\ \dot{y} &= \mu(x, y) y (1 - u) - m y, \end{cases} \quad u \in [0, 1],$$

where  $\mu$  is the Contois function

$$\mu(x, y) = \frac{x}{x + y}.$$

This model aims to take into consideration a crowding effect when the population of consumers is high, or equivalently that the growth is driven by the ratio "resource by consumer"  $x/y$  rather than simply the level of the resource  $x$ . Here also, one can easily check that the corresponding functions

$$\phi_1(x, y) = \phi_2(x, y) = \frac{1}{Y} \frac{y}{x + y}, \quad \phi_3(x, y) = \frac{x}{x + y} - m, \quad \phi_4(x, y) = \frac{x}{x + y},$$

satisfy Hypotheses 5 and conditions of Corollary 5.1 for  $m < 1$ . Let us underline that the function  $\phi_3$  depends on both variables, differently to Examples 2 and 3, and consequently the function  $\tilde{x}(\cdot)$  is not constant here. The NSN strategy is again optimal for  $m < 1$  and the level set  $\mathcal{D}_0$ , which gives to the end of the singular arc, is no longer a vertical line:

$$\mathcal{D}_0 = \left\{ (x, y) \in \mathcal{D}; x = \frac{m}{1 - m} y \right\}.$$

## 6 Conclusion

In this work, we have provided a method for solving a class of optimal control problems with  $L^\infty$  cost under  $L^1$  budget for planar nonlinear dynamics with control appearing linearly. It uses Green's Theorem as a comparison tool, generalizing a method formerly developed for problems with a prescribed terminal state and integral cost. Differently to such problems, two admissible solutions do not have necessarily the same terminal condition, which necessitated a significant modification of the method in order to obtain a closed curve on which Green's Theorem is informative.

For the class of problems under investigation, we showed the structure of the optimal solutions as "bang-singular-bang", with a geometrical characterization of the locus of the singular arc. The price to pay to obtain such a nice structure required a set of hypotheses on the data of the problem. Future directions of improvement and extension of the results could be to relax these hypotheses. In particular, our conditions are related to the existence of a singular arc whose control is not saturated (i.e. which does not take extreme values in the controls set). The possibility of having saturated singular arcs would significantly change the structure of the optimal solutions, but we believe that the comparison method in the plane could be some how extended. Considering dynamics of higher dimensions is another challenge. Particular "cascade" structures could be investigated in view of using comparison argumentation in two-dimensional subspace.

## Acknowledgment

This work has been partially supported by MIAI@Grenoble Alpes (ANR19-P3IA-0003) and the NOCIME project (ANR-23-CE48-0004-03).

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