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Abstract- This paper presents new sufficient conditions for state and fault estimation of nonlinear systems. A new approach is proposed to deal with unmeasured premise variables and corrupted outputs. The adopted observer strategy consists of a scheduled gain that depends only on the estimated outputs provided by the observer. The design of such scheduled gain follows from a technical rewriting of the state error and uses the Lyapunov framework to derive sufficient conditions written as linear matrix inequalities. Such conditions are less conservative than existing approaches that deal with unmeasured premise variables. The effectiveness of the developed approach is assessed through a numerical single-link robotic arm and on a real car based experimental setup available at the laboratory.

Index Terms- Nonlinear observers, fault estimation, Takagi-Sugeno models, Linear Matrix Inequality (LMI).

I. INTRODUCTION

Observers are usually referred to estimate system states in presence of noise and of perturbations in both sensors and system [1]. Even though this task is made more difficult if the target system is nonlinear, different strategies are available in the literature to tackle such problem. For instance, the works of Mahony & al. [2], of Engel [3] or, more recently, of Bernard & Andrieu [4] adopt the so-called global approaches. Alternatively, as proposed in Shim & al [5], semi-global approaches can be also applied. Finally, local approaches exploiting quasi-linear parameter-varying (quasi-LPV) systems and Takagi-Sugeno (T-S) based models are discussed in Ichalal & Mammar [6], in Lendek & al [7] and in Bergsten & al [8].

As discussed in Perez & al [10], two fault-tolerant observer techniques are commonly referred in the literature. One relies on methods dedicated to deal with uncertainties and to ensure that the observer convergence is robust to the uncertainties caused by the mismatch between premise value and measurement. For example, Pourashgar & al [11] approximate the uncertainties (on premises and overall) to zonotopes and apply recursive approaches, under the assumption that the uncertainties are bounded. Also, Oliveira & Pereira [12] design discrete-time Unknown Input Observers (UIOs) to deal with uncertainties presenting a limited parameter variation.

The second approach directly tackles corrupted or unmeasured premises and relies on the input-to-state stability property. It ensures bounded error convergence, regardless of a perturbation term and of a Lipschitz condition in the LMI expression related to the perturbation. In Quadros & al [13], the authors design a discrete-time fault hiding observer for T-S systems under actuator and sensor faults with unmeasured premise variables, considering the LMI framework and the differential mean value theorem. H_∞ performance is guaranteed for both virtual actuators and sensors. In Xie & al [14], unknown premise problems are handled with dedicated observers, under the assumption of a relaxed form of the separation principle. The proposed functional observer-based design is shown to be stable and robust to external disturbance. In Gómez & al [15], sliding mode UIOs are proposed for T-S systems with bounded uncertainties on premise variables. Sliding mode is also used in Zhang & al [16], in which the problem is rewritten as a system with uncertainties and unknown input, and gain scheduling feedback is synthesized to cope with the estimation error. Slack matrices are introduced to reduce conservatism of the LMI formulation. Specific Fault Detection and Isolation (FDI) are also investigated in T-S systems with unknown premise variables, as in Hassani & al [17]. Due to the convenient UIO form, the isolation and even the estimation of additive faults are addressed directly. In this work, type-2 fuzzy sets are used to improve the estimation with (some) unmeasured premise variables.

Overall, the discussed research considers together a LMI constraints and Lyapunov frameworks under specific assumptions with respect to the premise uncertainties or faults. The results rely yet upon bounded uncertainties in the best case as in the well-known [16] or [24] for example. Also, the cited methods are not exhaustive: there exist other frameworks to tackle sensor faults in nonlinear systems under perturbation. For instance, Reppa & al [18] consider an adaptive approach and couple a neural network approximator with distributed sensor fault isolation. However, this research is focused in dealing specifically with unmeasured premises in T-S form. Adaptive neuro-fuzzy inference technique called MANFIS was used in [36] along with zonotopic observers for wind turbine robust fault detection under

uncertainties. Zonotopes are used as bound for uncertainties. An adaptive NL approach was also investigated in [24] for fault diagnosis and accommodation under sensor bias. Bounds are assumed on the state, on the nonlinearity and on the rate of sensor bias. Considering a different topic, related to missing feedback information, and the necessity to estimate this missing information, Ajwad et al. [35] proposed a solution for lead-follow control problem for multiagent systems using continuous-discrete observers [40,41]. Leader model is considered unknown. The exponential consensus of the agents under partially missing information was achieved based on time-scale theory and results of [33]. In a similar context, interval observer-based coordination control was tackled for multiagent systems over directed networks in [34], with uncertainties, measurement noise, and unknown initial states. However, bounding information and output information are considered known.

This paper investigates the design of fault-tolerant observers that rely on output estimates provided by the observer itself, both for premises and for the feedback injection. Theoretically and practically, the premise variables cannot be considered always measurable and moreover, even measured, they can include uncertainties and/or be corrupted, as in the case of sensors malfunction. Therefore, a dedicated fault-tolerant observer design is required. The feedback injection consists of a scheduled gain, which is obtained using the Lyapunov framework under specified conditions. The assumptions on the sensor faults (and subsequently on the unmeasured premises) are relaxed, provided that the faults are not considered strictly bounded, but expressed as a chain of integrators. It is an improvement over a previous work by the authors [19]. The approach is also different from recent works on the topic such as [29] or [30], where the scheduling variables are exogenous with assumed polynomial relationship with respect to the state. As in [30], the design doesn't require the preliminary design of a parameter-dependent observer and considers a broader range of uncertainties/perturbations. It is also necessary to emphasize the challenge of deriving observer design conditions for implicit premises over external scheduling parameters (see [10], [29], [30], [31]). Using the latter approaches implies to assume a bound (a measurement) of the uncertainty, contrary to the presented approach, where no such bound is assumed.
In this paper, general conditions are provided for the estimation and convergence of the state and the fault signals. These conditions are transformed into LMI constraints using the so-called quasi-LPV, also called Takagi-Sugeno, framework for practicality (see discussion in [32]). This framework is efficient to deal with nonlinearities that can be described by the sector nonlinearity concept, which depends on the precise knowledge of premise variables [9]. The methodology proposed in the paper allows to give a solution to this hard problem of state observer convergence conditions for systems with corrupted premise variables; as these variables impede known LPV design approaches, like gain scheduling.
The remaining of this paper is organized as follows: Section II gives some technical lemmas useful for the main result and formulates the problem presenting the considered class of nonlinear systems. Section III proposes a novel approach with LMI-based conditions to design a fault-tolerant nonlinear observer. Two illustrative examples, an academic one and one using real time experiments, are given in Section IV to illustrate the method. Section V concludes this paper and suggests possible future research topics.

Notations. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the real n-dimension Euclidean space and the set of $n\times m$ real matrices, respectively. X^T is the transpose of X. If $X\in\mathbb{R}^{n\times n}$, $\mathrm{He}\{X\}=X+X^T$ represents the Hermitian operator. If $X=X^T$, $X\succ 0$ ($X\prec 0$) implies that X is positive (negative) definite. S_+^n is the set of matrices $X\in\mathbb{R}^{n\times n}$ such that $X\succ 0$. The Kronecker product operator is represented by $\otimes$; $\mathrm{diag}(X,Y)=\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$. $\mathrm{co}(\cdot)$ stands for the convex hull, $\boldsymbol{I}_n$ is the identity matrix of size n. $\mathbf{0}_{n\times m}$ is a null matrix of dimension $n\times m$. Dimensions and function arguments will be omitted whenever their understanding is straightforward.

II. Preliminaries

II.A. System description

Consider a class of continuous-time affine-in-the-control quasi-LPV systems described by

$$\begin{cases} \dot{x}(t)=A(y(t))x(t)+B(y(t))u(t) \\ y(t)=Cx(t), \\ z(t)=y(t)+Ev(t), \end{cases} \tag{1}$$

where $x(t)\in\mathbb{R}^{n_x}$ is the state, $u(t)\in\mathbb{R}^{n_u}$ is the input vector, $y(t)\in\mathbb{R}^{n_y}$ is the output vector, $z(t)\in\mathbb{R}^{n_y}$ is the measured output vector corrupted by the fault $v(t)\in\mathbb{R}^{n_v}$, with $n_v\le n_y$, and $\Omega_x\subseteq\mathbb{R}^{n_x}$ is a given compact set. Systems (1) are supposed to satisfy to the following three assumptions.

Assumption 1. The scheduling variable of (1) is composed by continuously differentiable functions of $y(t)$.

Assumption 2. The pair $\left(A\left(y(t)\right),C\right)$ is observable.

Assumption 3. The fault $v(t)$ can be "captured" by a cascade of α integrators, such that $v^{(\alpha)}(t)=\dfrac{d^{\alpha}v(t)}{dt^{\alpha}}=0$, with $\alpha\in\mathbb{N}$.

Of course, the last assumption will be helpful to derive a so-called PI-observer [37-38]. Note that there is no fault/perturbation bounding assumption. Also, with Assumption 2 and as $rank(v)\le n_z$, this kind of observer can always be built.

II.B. Useful technical lemmas

The first lemma recalls the Differential Mean Value Theorem (DMVT) [20].

Lemma 1. (Differential Mean Value Theorem [20]). Let $a,b\in\mathbb{R}^n$ and $f(z):\mathbb{R}^n\to\mathbb{R}$ be a differentiable function on $[a,b]$. There exists $c\in\mathbb{R}^n$, with $c\in\text{co}(a,b)$, i.e., $a_i<c_i<b_i$ for all $i\in\{1\dots n\}$, such that

$$f(b)-f(a)=f'(c)(b-a) \tag{2}$$

with $f'(c)=\left.\dfrac{\partial f(z)}{\partial z}\right|_{z=c}\in\mathbb{R}^{1\times n}$.

A second technical lemma, useful for the main result, is also introduced. Consider a matrix $A(y)\in\mathbb{R}^{m\times n}$ whose entries depend linearly on the entries of a vector $y\in\mathbb{R}^p$, i.e., with $i\in\{1\dots p\}$ $A_i\in\mathbb{R}^{m\times n}$:

$$A(y)=A_0+\sum_{i=1}^{p}A_i y_i \tag{3}$$

Considering any vector $x\in\mathbb{R}^{n\times 1}$, the next lemma gives a rewriting of the vector $A(y)x\in\mathbb{R}^{m\times 1}$. First of all, notice that:

$$A(y)x=A_0x+\sum_{i=1}^{p}(A_ix)y_i \tag{4}$$

Therefore, if some of the matrices $A_i\in\mathbb{R}^{m\times n}$ are sparse-in-column, we may consider the matrices $\bar{A}_i\in\mathbb{R}^{m\times n_i}$, $n_i\le n$, for $i\in\{1\dots p\}$, that are extracted from $A_i\in\mathbb{R}^{m\times n}$ removing the $n-n_i$ null columns. Thus, defining the corresponding vector $\bar{x}^i\in\mathbb{R}^{n_i\times 1}$ removing its corresponding $n-n_i$ rows:

$$A(y)x=\bar{A}_0\bar{x}^0+\sum_{i=1}^{p}\left(\bar{A}_i\bar{x}^i\right)y_i \tag{5}$$

Lemma 2. Let $A(y)\in\mathbb{R}^{m\times n}$ be a matrix whose entries depend linearly on the entries of a vector $y\in\mathbb{R}^p$ (3) (respectively (5)), define $\boldsymbol{\mathcal{A}}=\begin{bmatrix}A_1 & \dots & A_p\end{bmatrix}\in\mathbb{R}^{m\times(n\cdot p)}$ according to (4), (respectively $\bar{\boldsymbol{\mathcal{A}}}=\begin{bmatrix}\bar{A}_1 & \dots & \bar{A}_p\end{bmatrix}\in\mathbb{R}^{m\times\sum_{i=1}^{p}n_i}$ according to (5)), then:

$$A(y)x=A_0x+\boldsymbol{\mathcal{A}}\times\left(I_p\otimes x\right)\times y \tag{6}$$

$$A(y)x=\bar{A}_0\bar{x}^0+\bar{\boldsymbol{\mathcal{A}}}\times diag\left(\begin{bmatrix}\bar{x}^1 & \dots & \bar{x}^p\end{bmatrix}\right)\times y \tag{7}$$

Proof of Lemma 2. (6) is direct considering that:

$$\boldsymbol{\mathcal{A}}\times\left(I_p\otimes x\right)\times y=\begin{bmatrix}A_1 & \dots & A_p\end{bmatrix}\times\begin{bmatrix}x & 0_{n\times1} & 0_{n\times1}\\ 0_{n\times1} & \ddots & 0_{n\times1}\\ 0_{n\times1} & 0_{n\times1} & x\end{bmatrix}\times y=\begin{bmatrix}A_1x & \dots & A_px\end{bmatrix}\times y=\sum_{i=1}^{p}(A_ix)y_i \tag{8}$$

which is exactly (4). The same way for (7):

$$\bar{\boldsymbol{\mathcal{A}}}\times diag\left(\begin{bmatrix}\bar{x}^1 & \dots & \bar{x}^p\end{bmatrix}\right)\times y=\begin{bmatrix}\bar{A}_1 & \dots & \bar{A}_p\end{bmatrix}\times\begin{bmatrix}\bar{x}^1 & 0_{n_1\times1} & \dots & 0_{n_1\times1}\\ 0_{n_2\times1} & \bar{x}^2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0_{n_{p-1}\times1}\\ 0_{n_p\times1} & \dots & 0_{n_p\times1} & \bar{x}^p\end{bmatrix}\times y=\begin{bmatrix}\bar{A}_1\bar{x}^1 & \dots & \bar{A}_p\bar{x}^p\end{bmatrix}\times y=\sum_{i=1}^{p}\left(\bar{A}_i\bar{x}^i\right)y_i \tag{9}$$

which is exactly (5). This concludes the proof. □

Example 1. Consider the vectors $x \in \mathbb{R}^3$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ and the matrix $A(y) = \begin{bmatrix} 3y_1 & 1-y_2 & 4 \\ 0 & 1+y_1 & 1 \\ y_1 & y_1+2y_2 & -2 \end{bmatrix} \in \mathbb{R}^{3\times 3}$. The matrix $A(y)$ can be rewritten in the form

$$A(y) = A_0 + A_1 y_1 + A_2 y_2 = \begin{bmatrix} 0 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} y_1 + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} y_2 . \tag{10}$$

Then, it follows that

$$\mathcal{A} = \begin{bmatrix} A_1 & A_2 \end{bmatrix} = \left[\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \right] \in \mathbb{R}^{3\times 6}.$$

Removing the null columns in $\mathcal{A}$ gives directly: $\bar{\mathcal{A}} = \left[\begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right] \in \mathbb{R}^{3\times(2+1)}$ associated with $\bar{x}^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ as the 3rd column of A_1 is removed and $\bar{x}^2 = \begin{bmatrix} x_2 \end{bmatrix}$ as the 1st and 3rd columns of A_2 are removed, from where:

$$A(y)x = \bar{A}_0 \bar{x}^0 + \bar{\mathcal{A}} \times diag\left(\begin{bmatrix} \bar{x}^1 & \bar{x}^2 \end{bmatrix} \right) \times y = \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \left[\begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right] \left[\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad 0 \right] \times y \tag{11}$$

II.C. Problem statement

Following the assumptions, a PI-observer can be built from (1) to reconstruct the fault $v(t)$. As usual, consider a cascade of integrators with $v^{(\alpha)}(t) = 0$, $\alpha \in \mathbb{N}$, that captures the fault dynamics. Let us define $V(t) \in \mathbb{R}^{\alpha n_v}$ an expanded state vector $V^T(t) = \begin{bmatrix} v^T(t) & v^{(1)^T}(t) & \dots & v^{(\alpha-1)^T}(t) \end{bmatrix}$. Then, using the extended vector $\begin{bmatrix} x(t) \\ V(t) \end{bmatrix}$, system (1) can be rewritten as

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{V}(t) \end{bmatrix} = \begin{bmatrix} A(y(t)) & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} x(t) \\ V(t) \end{bmatrix} + \begin{bmatrix} B(y(t)) \\ 0 \end{bmatrix} u(t) \\ y(t) = Cx(t),\ z(t) = \bar{C} \begin{bmatrix} x(t) \\ V(t) \end{bmatrix} \end{cases} \tag{12}$$

with $J = \begin{bmatrix} 0 & I_{(\alpha-1)n_v} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\alpha n_v \times \alpha n_v}$ and $\bar{C} = \begin{bmatrix} C & E & 0 \end{bmatrix} \in \mathbb{R}^{n_y \times (n_x + \alpha n_v)}$. From (12), it is direct to design a PI-observer as

$$\begin{cases} \begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{\hat{V}}(t) \end{bmatrix} = \begin{bmatrix} A(\hat{y}(t)) & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{V}(t) \end{bmatrix} + \begin{bmatrix} B(\hat{y}(t)) \\ 0 \end{bmatrix} u(t) + P^{-1} K(\cdot)\left(z(t) - \hat{z}(t) \right) \\ \hat{y}(t) = z(t) - E\hat{v}(t), \quad \hat{z}(t) = \bar{C} \begin{bmatrix} \hat{x}(t) \\ \hat{V}(t) \end{bmatrix} \end{cases} \tag{13}$$

where $P \in S_+^{n_x + \alpha n_v}$ and $K(\cdot) \in \mathbb{R}^{n_x + \alpha n_v \times n_y}$ are matrices to be determined whose arguments will be defined later on.

Remark 1. The form of (13) belongs to the family of the so-called non measurable premise variables, which appears to remain an open problem in the general nonlinear case. In comparison, classical approaches will define the observer using the measured variables, i.e., $A(z(t))$, $B(z(t))$ instead of $A(\hat{y}(t))$, $B(\hat{y}(t))$, with the inherent problem that it will propagate the fault into the state of the observer. The interest of the form (13) can be seen directly from the relationship $\hat{y}(t) = z(t) - E\hat{v}(t)$. On one side, it can resume to a no-fault case $\hat{y}(t) = z(t)$, and on the other, it recovers the true output variable $\hat{y}(t) \to y(t)$ if the estimated fault converges: $\hat{v}(t) \to v(t)$. In the following, conditions that enforce exponential convergence of the estimation are given in Theorem 1 and the related LMI formulation are in Theorem 2.

Remark 2. The gain $K(\cdot)$ and its argument are undetermined at this stage. Lyapunov framework and LMI constraints are the way to determine the gain. Therefore, the argument of $K(\cdot)$ can vary from parameter independent, i.e., constant K, to any variables that can depend on measured and/or internal observer variables only, i.e. $\hat{x}$, $\hat{y}$ and $\hat{V}$, as examples, either $K(\hat{x})$ or $K(\hat{y})$, or $K(\hat{x},\hat{y})$, or $K(\hat{x},\hat{y},\hat{V})$. Generally, the more variables used the better the result is, as we use a polytopic description of $K(\cdot)$ ($K(\cdot)=\sum_i h_i(\cdot)K_i$) and a gain K_i is derived for each of its vertex i. This gain scheduling and/or LPV design is known to allow better performance for controllers/observers than robust LTI design when scheduling parameters are available [37]. One of the contributions of this article is to extend the classical approaches for cases where a part of the scheduling parameters is unavailable or corrupted (see Remark 1).

Problem 1. Based solely on the corrupted measurements $z(t)$, find an observer of form (13) capable to asymptotically estimate together the state $x(t)$ and the fault vector $v(t)$ of system (1).

III. Main Results

In order to find a solution to Problem 1, consider the extended error $e(t)\in\mathbb{R}^{n_x+\alpha n_v}$ is defined as

$$e(t)=\begin{bmatrix}e_x(t)\\ e_V(t)\end{bmatrix}=\begin{bmatrix}x(t)-\hat{x}(t)\\ V(t)-\hat{V}(t)\end{bmatrix}. \tag{14}$$

The time derivative of $e(t)$ can be developed from (12) and (13) as follows:

$$\dot{e}(t)=\begin{bmatrix}A(y) & 0\\ 0 & J\end{bmatrix}\begin{bmatrix}x\\ V\end{bmatrix}-\begin{bmatrix}A(\hat{y}) & 0\\ 0 & J\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{V}\end{bmatrix}+\left(\begin{bmatrix}B(y)\\ 0\end{bmatrix}-\begin{bmatrix}B(\hat{y})\\ 0\end{bmatrix}\right)u-P^{-1}K(\cdot)\bar{C}e \tag{15}$$

Using $\hat{x}(t)=x(t)-e_x(t)$, (15) is equivalent to:

$$\dot{e}(t)=\left(\begin{bmatrix}A(\hat{y}) & 0\\ 0 & J\end{bmatrix}-P^{-1}K(\cdot)\bar{C}\right)e+\begin{bmatrix}A(y)-A(\hat{y})\\ 0\end{bmatrix}x+\begin{bmatrix}B(y)-B(\hat{y})\\ 0\end{bmatrix}u \tag{16}$$

In (16) appear the two quantities $(A(y)-A(\hat{y}))x$ and $(B(y)-B(\hat{y}))u$ on which the DMVT and the technical lemma can be applied. Using the DMVT, trivially, each of the entries of $A(y)-A(\hat{y})$ and $B(y)-B(\hat{y})$ will linearly depend on the vector $y-\hat{y}$. For example, with $a_{11}(y)=\dfrac{y_3}{1+y_2^2}$, $a_{11}(y)-a_{11}(\hat{y})=\left[\begin{matrix}0 & \dfrac{-2y_2y_3}{(1+y_2^2)^2} & \dfrac{1}{1+y_2^2}\end{matrix}\right]\Bigg|_{(y=c_A)}(y-\hat{y})$. Thus, it exists $c_A\in\mathrm{co}(y,\hat{y})$ and $c_B\in\mathrm{co}(y,\hat{y})$ (both understood component wise) such that:

$$\begin{aligned}A(y)-A(\hat{y})&=\sum_{i=1}^{n_v}A'_{(c_A)i}(y_i-\hat{y}_i)\\ B(y)-B(\hat{y})&=\sum_{i=1}^{n_v}B'_{(c_A)i}(y_i-\hat{y}_i)\end{aligned} \tag{17}$$

From (12) and (13), it follows that $y=z-Ev$, and $\hat{y}=z-E\hat{v}$, respectively. Therefore, using the previous notation (3) and introducing the faults error vector $e_v=E(v-\hat{v})$, it follows that

$$\begin{aligned}A(y)-A(\hat{y})&=-\sum_{i=1}^{n_v}A'_{(c_A)i}e_{vi}\\ B(y)-B(\hat{y})&=-\sum_{i=1}^{n_v}B'_{(c_B)i}e_{vi}\end{aligned} \tag{18}$$

Expression (18) exhibits two matrices $A(y)-A(\hat{y})$ and $B(y)-B(\hat{y})$ whose entries depend linearly on the vector e_{vi}. Therefore, Lemma 2 can be applied to both terms $\left(A(y)-A(\hat{y})\right)x$ and $\left(B(y)-B(\hat{y})\right)u$. Defining $\bar{\mathcal{A}}'(c_A)=\left[\bar{A}'_{(c_A)1} \quad \dots \quad \bar{A}'_{(c_A)p}\right]$ and $\bar{\mathcal{B}}'(c_B)=\left[\bar{B}'_{(c_B)1} \quad \dots \quad \bar{B}'_{(c_B)p}\right]$ in the same way as (5):

$$\begin{aligned}\left(A(y)-A(\hat{y})\right)x &= -\bar{\mathcal{A}}'(c_A)\times diag\left(\left[\bar{x}^1 \quad \dots \quad \bar{x}^p\right]\right)\times e_v = -\bar{\mathcal{A}}'(c_A)\times \mathcal{D}_x(\bar{x})\times e_v \\ \left(B(y)-B(\hat{y})\right)u &= -\bar{\mathcal{B}}'(c_B)\times diag\left(\left[\bar{u}^1 \quad \dots \quad \bar{u}^p\right]\right)\times e_v = -\bar{\mathcal{B}}'(c_B)\times \mathcal{D}_u(\bar{u})\times e_v\end{aligned} \tag{19}$$

Example 2. Consider a vector $y=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and a matrix $A(y)=\begin{bmatrix} 3f_1(y_1) & 1-f_3(y_1,y_2) & 4 \\ 0 & 1+2f_2(y_1) & 1 \\ f_2(y_1) & f_4(y_1,y_2) & -2 \end{bmatrix}\in\mathbb{R}^{3\times 3}$, with $f_i(\cdot)$, for $i\in\{1\dots 4\}$, continuously differentiable scalar functions. Expression (17) corresponds to

$$A(y)-A(\hat{y})=\begin{bmatrix} 3f'_1(c_{A1})(y_1-\hat{y}_1) & -\left[\frac{\partial f_3}{\partial y_1}(c_A) \quad \frac{\partial f_3}{\partial y_2}(c_A)\right]^T(y-\hat{y}) & 0 \\ 0 & 2f'_2(c_{A1})(y_1-\hat{y}_1) & 0 \\ f'_2(c_{A1})(y_1-\hat{y}_1) & \left[\frac{\partial f_4}{\partial y_1}(c_A) \quad \frac{\partial f_4}{\partial y_2}(c_A)\right]^T(y-\hat{y}) & 0 \end{bmatrix} \tag{20}$$

And (18) to:

$$A(y)-A(\hat{y})=-\begin{bmatrix} 3f'_1(c_{A1}) & -\frac{\partial f_3}{\partial y_1}(c_A) & 0 \\ 0 & 2f'_2(c_{A1}) & 0 \\ f'_2(c_{A1}) & \frac{\partial f_4}{\partial y_1}(c_A) & 0 \end{bmatrix}e_{v1}-\begin{bmatrix} 0 & -\frac{\partial f_3}{\partial y_2}(c_A) & 0 \\ 0 & 0 & 0 \\ 0 & \frac{\partial f_4}{\partial y_2}(c_A) & 0 \end{bmatrix}e_{v2} \tag{21}$$

Therefore, it is easy to build: $\bar{\mathcal{A}}'(c_A)=\left[\bar{A}'_{(c_A)1} \quad \bar{A}'_{(c_A)2}\right]=-\begin{bmatrix}\begin{bmatrix} 3f'_1(c_{A1}) & -\frac{\partial f_3}{\partial y_1}(c_A) \\ 0 & 2f'_2(c_{A1}) \\ f'_2(c_{A1}) & \frac{\partial f_4}{\partial y_1}(c_A)\end{bmatrix} & \begin{bmatrix} -\frac{\partial f_3}{\partial y_2}(c_A) \\ 0 \\ \frac{\partial f_4}{\partial y_2}(c_A)\end{bmatrix}\end{bmatrix}$ by removing the empty columns of (21), thus $\bar{x}^1=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\bar{x}^2=\left[x_2\right]$. Finally, expression (19) writes

$$\left(A(y)-A(\hat{y})\right)x=\bar{\mathcal{A}}'(c_A)\times diag\left(\left[\bar{x}^1 \quad \dots \quad \bar{x}^p\right]\right)\times e_v=-\begin{bmatrix} 3f'_1(c_{A1}) & \frac{\partial f_3}{\partial y_1}(c_A) & \frac{\partial f_3}{\partial y_2}(c_A) \\ 0 & 2f'_2(c_{A1}) & 0 \\ f'_2(c_{A1}) & \frac{\partial f_4}{\partial y_1}(c_A) & \frac{\partial f_4}{\partial y_2}(c_A) \end{bmatrix}\begin{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & 0 \\ 0 & \left[x_2\right]\end{bmatrix}\begin{bmatrix} e_{v1} \\ e_{v2} \end{bmatrix} \tag{22}$$

Theorem 1. With $J_v=\left[I_{n_v} \quad 0\right]\in\mathbb{R}^{n_v\times \alpha n_v}$, considering the matrices $\bar{\mathcal{A}}'(c_A)$, $\mathcal{D}_x(\bar{x})$, $\bar{\mathcal{B}}'(c_B)$ and $\mathcal{D}_u(\bar{u})$ defined in (19), given a scalar $\beta>0$, if there exist matrices $P\in S_+^{n_x+\alpha n_v}$ and $K(\cdot)\in\mathbb{R}^{(n_x+\alpha n_v)\times n_y}$ such that

$$He\left\{P\begin{bmatrix} A(\hat{y}) & -\left(\bar{\mathcal{A}}'(c_A)\times \mathcal{D}_x(\bar{x})+\bar{\mathcal{B}}'(c_B)\times \mathcal{D}_u(\bar{u})\right)J_v \\ 0 & J \end{bmatrix}-K(\cdot)\bar{C}\right\}+\beta P\prec 0. \tag{23}$$

Then, the observer (13) is able to estimate $x(t)$ and the fault vector $v(t)$ of system (1) with an exponential rate of convergence β.

Proof: From (19), the error dynamics (16) can be rewritten as

$$\dot{e}(t)=\left(\begin{bmatrix} A(\hat{y}) & 0 \\ 0 & J \end{bmatrix}-P^{-1}K(\cdot)\bar{C}\right)e-\begin{bmatrix} \bar{\mathcal{A}}'(c_A)\times \mathcal{D}_x(\bar{x})+\bar{\mathcal{B}}'(c_B)\times \mathcal{D}_u(\bar{u}) \\ 0 \end{bmatrix}e_v \tag{24}$$

with $\mathcal{D}_x(\bar{x}) = diag\left(\begin{bmatrix}\bar{x}^1 & \dots & \bar{x}^p\end{bmatrix}\right)$ and $\mathcal{D}_u(\bar{u}) = diag\left(\begin{bmatrix}\bar{u}^1 & \dots & \bar{u}^p\end{bmatrix}\right)$. Now, considering that $e_v = v - \hat{v} = \begin{bmatrix}0 & J_v\end{bmatrix}\begin{bmatrix}e_x \\ e_V\end{bmatrix}$ with $J_v = \begin{bmatrix}I_{n_v} & 0\end{bmatrix} \in \mathbb{R}^{n_v \times \alpha n_v}$, the error dynamics (24) becomes

$$\dot{e}(t) = \left(\begin{bmatrix}A(\hat{y}) & -\left(\bar{\mathcal{A}}'(c_A)\times\mathcal{D}_x(\bar{x}) + \bar{\mathcal{B}}'(c_B)\times\mathcal{D}_u(\bar{u})\right)J_v \\ 0 & J\end{bmatrix} - P^{-1}K(\cdot)\bar{C}\right)e \tag{25}$$

Consider the quadratic Lyapunov function

$$V(e(t)) = e^T(t)Pe(t),\ P = P^T \succ 0. \tag{26}$$

Taking the time-derivative of $V(e(t))$ along the trajectory of (25) while considering a decay rate $\beta > 0$, it follows that

$$\dot{V}(e(t)) + \beta V(e(t)) = 2e^T(t)P\dot{e}(t) + \beta e^T(t)Pe(t) \tag{27}$$

which corresponds to (23). This concludes the proof. □

In order to derive a feasible LMI problem to condition (23), we define expressions via a polytopic formulation using compact sets respectively of the output, state and control variables, $\hat{y} \in \Omega_y$, $\bar{x} \in \Omega_x$ and $\bar{u} \in \Omega_u$. First, consider that the gain of the observer can only be based on measured variables, thus, $\hat{y}$ and $\bar{u}$. Therefore, a polytopic representation of $K(\bar{u}, \hat{y}) \in \mathbb{R}^{(n_x+\alpha n_v)\times n_y}$ is given as

$$K(\bar{u}, \hat{y}) = \sum_{i_u=1}^{m_u}\sum_{i_y=1}^{m_y}\mu_{i_u}(\bar{u})\eta_{i_y}(\hat{y})K_{i_u i_y}. \tag{28}$$

where the matrices $K_{i_u i_y} \in \mathbb{R}^{(n_x+\alpha n_v)\times n_y}$, for $i_u \in \{1,\dots,m_u\}$ and $i_y \in \{1\dots m_y\}$, are the vertices to be determined, and

$$\eta_{i_y}(\hat{y}) \geq 0,\quad \sum_{i_y=1}^{m_y}\eta_{i_y}(\hat{y}) = 1,\quad \mu_{i_u}(\bar{u}) \geq 0,\quad \sum_{i_u=1}^{m_u}\mu_{i_u}(\bar{u}) = 1. \tag{29}$$

Writing $\bar{\mathcal{A}}'(c_A)\times\mathcal{D}_x(\bar{x}) + \bar{\mathcal{B}}'(c_B)\times\mathcal{D}_u(\bar{u})$ in a polytope can be done in many ways. Considering $c_A \in \text{co}(y, \hat{y})$ and $c_B \in \text{co}(y, \hat{y})$, a general writing is given by

$$\bar{\mathcal{A}}'(c_A)\times\mathcal{D}_x(\bar{x}) + \bar{\mathcal{B}}'(c_B)\times\mathcal{D}_u(\bar{u}) = \sum_{i_u=1}^{m_u}\sum_{i_p=1}^{m_p}\mu_{i_u}(\bar{u})\lambda_{i_p}(c_A, c_B, \bar{x})\mathcal{X}_{i_u i_p} \tag{30}$$

with $\sum_{i_p=1}^{m_p}\lambda_{i_p}(\cdot) = 1$, $\lambda_{i_p}(\cdot) \geq 0$, for $i_p \in \{1,\dots,m_p\}$, and the vertices $\mathcal{X}_{i_u i_p}$, for $i_u \in \{1,\dots,m_u\}$.

Theorem 2. Consider $J_v = \begin{bmatrix}I_{n_v} & 0\end{bmatrix} \in \mathbb{R}^{n_v \times \alpha n_v}$, the compact sets Ω_y, Ω_x, Ω_u such that $\hat{y} \in \Omega_y$, $\bar{x} \in \Omega_x$, $\bar{u} \in \Omega_u$ and considering the matrices $\mathcal{X}_{i_u i_p}$, with $i_u \in \{1,\dots,m_u\}$, for $i_p \in \{1,\dots,m_p\}$, defined in (30). Given a scalar $\beta > 0$, if there exist matrices $P \in S_+^{n_x+\alpha n_v}$ and $K_{i_u i_y} \in \mathbb{R}^{(n_x+\alpha n_v)\times n_y}$, for $i_y \in \{1,\dots,m_y\}$, such that the LMI constraints

$$He\left\{P\begin{bmatrix}A_{i_y} & -\mathcal{X}_{i_u i_p}J_v \\ 0 & J\end{bmatrix} - K_{i_u i_y}\bar{C}\right\} + \beta P \prec 0,\ \forall(i_u, i_y, i_p) \tag{31}$$

are satisfied. Then, the observer (13) is able to estimate $x(t)$ and the fault vector $v(t)$ of system (1) with an exponential rate of convergence β.

Proof: Using expressions (28) and (30) and considering that with the functions $\eta_{i_y}(\hat{y})$ defined for (28), $A(\hat{y}) = \sum_{i_y=1}^{m_y}\eta_{i_y}(\hat{y})A_{i_y}$, condition (23) can be written as

$$\sum_{i_y=1}^{m_y}\sum_{i_u}^{m_u}\sum_{i_p=1}^{m_p}\eta_{i_y}(\hat{y})\mu_{i_u}(\bar{u})\lambda_{i_p}(c_A, c_B, \bar{x})\left(He\left\{P\begin{bmatrix}A_{i_y} & -\mathcal{X}_{i_u i_p}J_v \\ 0 & J\end{bmatrix} - K_{i_u i_y}\bar{C}\right\} + \beta P\right) \prec 0,\ \forall(i_u, i_y, i_p) \tag{32}$$

which obviously holds if conditions (31) are satisfied. This concludes the proof. □

Remark 3. LMI constraints can be added to achieve enhanced performances. For example, limiting the observer gain can be relevant and obtained using the following constraints, given a scalar $\xi > 0$:

$$\begin{bmatrix} \xi P & K_{i_u i_y} \\ K_{i_u i_y}^T & I \end{bmatrix} \succ 0, \quad i_u \in \{1,\dots,m_u\}, i_y \in \{1,\dots,m_y\} \tag{33}$$

Remark 4. Note that the observer (13) does not use any of the variables $c_A \in \text{co}(y,\hat{y})$, $c_B \in \text{co}(y,\hat{y})$ and $\bar{x}$. These variables are only useful for deriving conditions that prove the convergence. In (23) they can be seen, and are treated like uncertainties, i.e., the conditions must hold in the full compact set, therefore in the domain defined by the vertices appearing in (31).

Remark 5. The numerical complexity of LMI-based optimizations can be evaluated by the number of scalar decision variables N_{var} and the number of rows N_{row} of all considered LMI conditions (number of LMI constraints x number of rows of each constraint). For Theorem 2, these numbers are given as follows:

$$\begin{aligned} N_{\text{var}} &= (n_x + \alpha n_v)\left(0.5(n_x + \alpha n_v + 1) + n_y m_u m_y\right) \\ N_{row} &= (n_x + \alpha n_v) m_p m_u m_y. \end{aligned} \tag{34}$$

These numbers characterizing the numerical complexity of LMI constraints are illustrated in Section IV for the proposed quasi-LPV observer design of two physical-motivated applications. Note that different LMI solvers may exhibit varying numerical complexities with respect to N_{var} and N_{row}. However, such a numerical complexity accounts only for the *offline* computations. Hence, in comparison to a non-fault tolerant PI observer, the practical use of the proposed quasi-LPV observer (13) just corresponds to an increase of computation due to the polytope based on (28).

IV. Illustrative Results

This section presents two physical-motivated examples to show the effectiveness of the proposed fault tolerant observer design. The LMI-based constraints in Theorem 2 are solved using the MOSEK [22] and YALMIP [23] packages alongside MATLAB 2021b.

IV.A. Numerical example (Single-link robotic arm)

The dynamics of a single-link robotic arm with a revolute elastic joint can be described by [24]:

$$\begin{cases} J_l \ddot{q}_1(t) + F_l \dot{q}_1(t) + k\left(q_1(t) - q_2(t)\right) + mgh\sin\left(q_1(t)\right) = 0, \\ J_m \ddot{q}_2(t) + F_m \dot{q}_2(t) - k\left(q_1(t) - q_2(t)\right) = k_\tau u(t), \end{cases} \tag{35}$$

in which $q_1(t)$ and $q_2(t)$ are respectively the angular positions of the link and of the motor of the robot, and $u(t)$ is the armature current flowing through the robot motor. The robot parameters are constant and set as [24]: link inertia $J_l = 4.5\,\text{kgm}^2$, motor rotor inertia $J_m = 1\,\text{kgm}^2$, elastic constant $k = 2\,\text{Nm}$, link mass $m = 4\,\text{kg}$, gravity constant $g = 9.8\,\text{m/s}^2$, distance to the center of mass $h = 0.5\,\text{m}$, torque coefficient $k_\tau = 1\,\text{Nm/A}$, viscous friction coefficients $F_l = 0.5\,\text{Nms}$ and $F_l = 1\,\text{Nms}$.

Let $x(t) = \left[\dot{q}_2(t) \;\; q_2(t) \;\; \dot{q}_1(t) \;\; q_1(t)\right]^T$ and $y(t) = \left[q_2(t) \;\; \dot{q}_1(t) \;\; q_1(t)\right]^T$. The validity region of (35) is defined as: $\Omega_x = \left\{|q_1(t)| < \pi, |\dot{q}_1(t)| < 10, |q_2(t)| < \pi, |\dot{q}_2(t)| < 10\right\}$. The nonlinearity is taken into account using $f\left(y(t)\right) = \frac{\sin q_1(t)}{q_1(t)} \in \left[\underline{f}, \overline{f}\right]$, with $\underline{f}$ and $\overline{f}$ being the minimum and maximum values of $f\left(y(t)\right)$ inside the compact set Ω_x. Thus, the model (35) can be represented in the state-space form (1), with

$$A(y) = A_0 + f\left(y(t)\right)A_1, \quad A_0 = \begin{bmatrix} -\frac{F_m}{J_m} & -\frac{k}{J_m} & 0 & \frac{k}{J_m} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{k}{J_l} & -\frac{F_l}{J_l} & \frac{k}{J_l} \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{mgh}{J_l} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{k_\tau}{J_m} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{36}$$

Then, it is direct that $\left(A(y) - A(\hat{y})\right)x = \left[0 \;\; 0 \;\; \frac{mgh}{J_l}\left(f(q_1) - f(\hat{q}_1)\right)x_4 \;\; 0\right]^T$. With $f'(q_1) = \dfrac{q_1\cos(q_1) - \sin(q_1)}{q_1^2}$ and considering notations (19), with $c_A \in \text{co}(q_1, \hat{q}_1)$, it corresponds to

$$\bar{\boldsymbol{A}}'(c_A) = \left[0 \;\; 0 \;\; -\frac{mgh}{J_l} f'(c_A) \;\; 0\right]^T \text{ and } diag\left(\left[\bar{x}^1 \;\; \dots \;\; \bar{x}^p\right]\right) = x_4 = q_1 \tag{37}$$

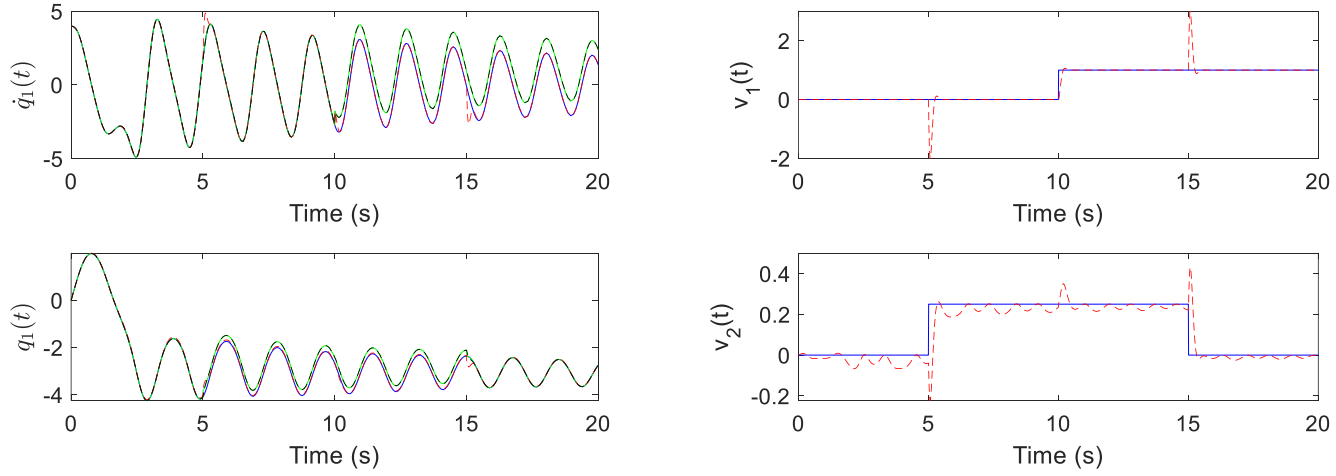
It turns out that $\left(A(y) - A(\hat{y})\right)x$ has only one nonlinearity $\alpha(c_A, q_1) = f'(c_A)q_1 \in \left[-\bar{\alpha}, \bar{\alpha}\right]$, with $\bar{\alpha} = 1.37$, the maximum value computed over the set $\left\{q_1(t), |q_1(t)| < \pi\right\}$. Therefore, expression (30) corresponds to

$$\bar{\mathcal{A}}'(c_A)\times\mathcal{D}_x(\bar{x})=\sum_{i_p=1}^{2}\lambda_{i_p}(c_A,q_1)\mathcal{K}_{i_p} \tag{38}$$

with $\lambda_1(c_A,q_1)=\dfrac{\alpha(c_A,q_1)+\bar{\alpha}}{2\bar{\alpha}}=1-\lambda_2(c_A,q_1)$, $\mathcal{K}_1=\begin{bmatrix}0 & 0 & \frac{mgh}{J_l}\bar{\alpha} & 0\end{bmatrix}^T$ and $\mathcal{K}_2=\begin{bmatrix}0 & 0 & -\frac{mgh}{J_l}\bar{\alpha} & 0\end{bmatrix}^T$. We consider the faults possibly impacting $x_3=\dot{q}_1$ and $x_4=q_1$, thus $E=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}^T$ and a cascade of two integrators is used, i.e., $\alpha=2$. Considering $\xi=10$ (Remark , gain limitation (33)) and a decay rate $\beta=10$, the gains of observer (13) are given by

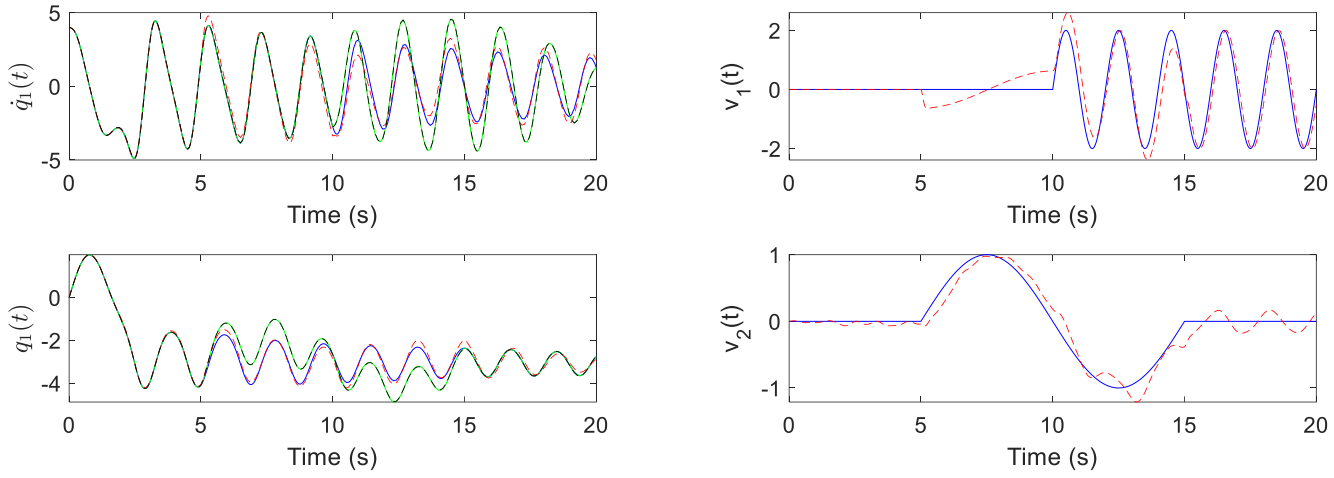
$$P^{-1}K_1=\begin{bmatrix}178.79 & 2.75 & 0.89\\ 20.75 & 0.20 & -0.49\\ 9.88 & 177.70 & 286.88\\ 330.19 & 7.07 & 48.38\\ -11.02 & -0.18 & -291.92\\ -329.67 & -6.07 & -25.49\\ 0.08 & 0.01 & -5.99\\ -3.69 & -0.07 & 0.16\end{bmatrix},\quad P^{-1}K_2=\begin{bmatrix}177.96 & -11.49 & -0.06\\ 20.71 & -0.83 & -0.57\\ 12.11 & 172.35 & 286.65\\ 328.19 & -24.68 & 46.47\\ -11.93 & 2.89 & -291.95\\ -327.55 & 25.52 & -23.57\\ 0.06 & 0.04 & -5.99\\ -3.67 & 0.28 & 0.19\end{bmatrix}. \tag{39}$$

The numbers characterizing the numerical complexity for this example are computed as $N_{\text{var}}=84$ and $N_{row}=32$, which represent a low level of computational burden for an LMI-based optimization problem. The robot model (36) is simulated with the application of the observer gains (39), as shown in Figure 1–Figure 5. Initial conditions are chosen as $x_0=\begin{bmatrix}5 & 0 & 4 & 0\end{bmatrix}^T$ and the input $u(t)=2\sin(t/2)$. Different types of fault $v(t)$ corrupting the output $y(t)$ are considered: steps (Figure 1b), sinuses (Figure 2b), trapezia (Figure 3b) and white noises (Figure 4b). For the sake of space, only the state $x(t)$ for the step fault $v(t)$ are depicted in Figure 5. Apart from the case of white-noise fault $v(t)$, notice that the estimated output $\hat{y}(t)$ and state $\hat{x}(t)$ converge respectively to $y(t)$ and to $x(t)$, even in the presence of fault $v(t)\in\mathbb{R}^2$. The fault $v(t)$ is also accurately estimated. On the other hand, Figure 4 illustrates a drawback of the observer design approach presented in this paper: as white-noise fault $v(t)$ cannot be captured by a cascade of integrators with α finite (ideally, $\alpha\to\infty$), the observer proposed in (13) is unable to properly estimate the fault $v(t)$. As a result, there is an inherent gap between the estimated output $\hat{y}(t)$ and the system output $y(t)$.



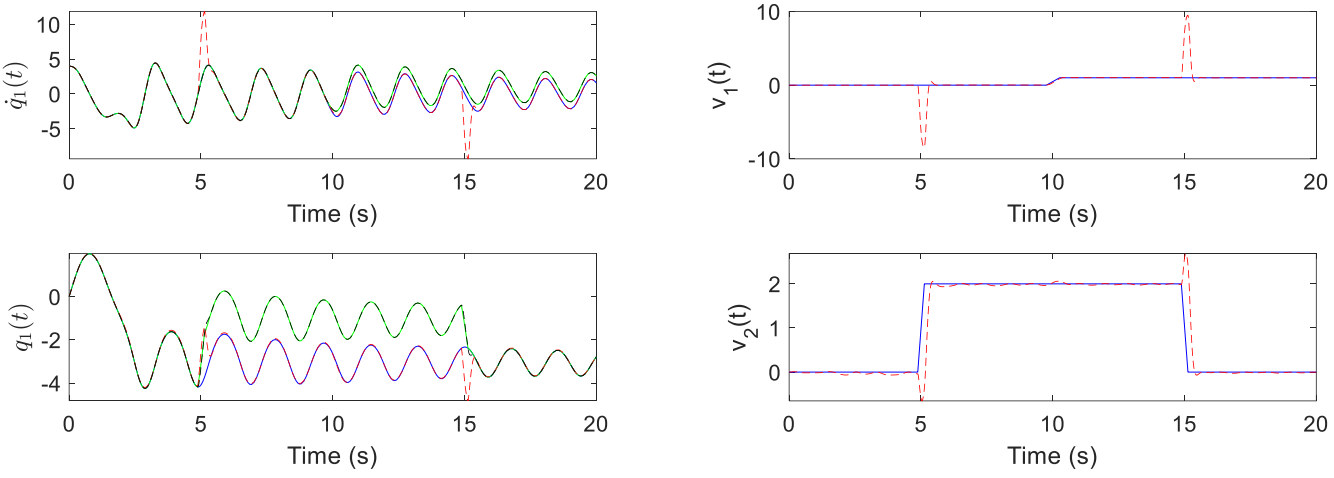
(a) (b)

Figure 1. (a) Output $y(t)=\begin{bmatrix}\dot{q}_1 & q_1\end{bmatrix}^T$ (blue solid line), measured output $z(t)$ (green solid line), estimated output $\hat{y}(t)$ (red dashed line), estimated measured output $\hat{z}(t)$ (black dashed line), (b) Step fault $v(t)$ (blue solid line) estimated fault $\hat{v}(t)$ (red dashed line).



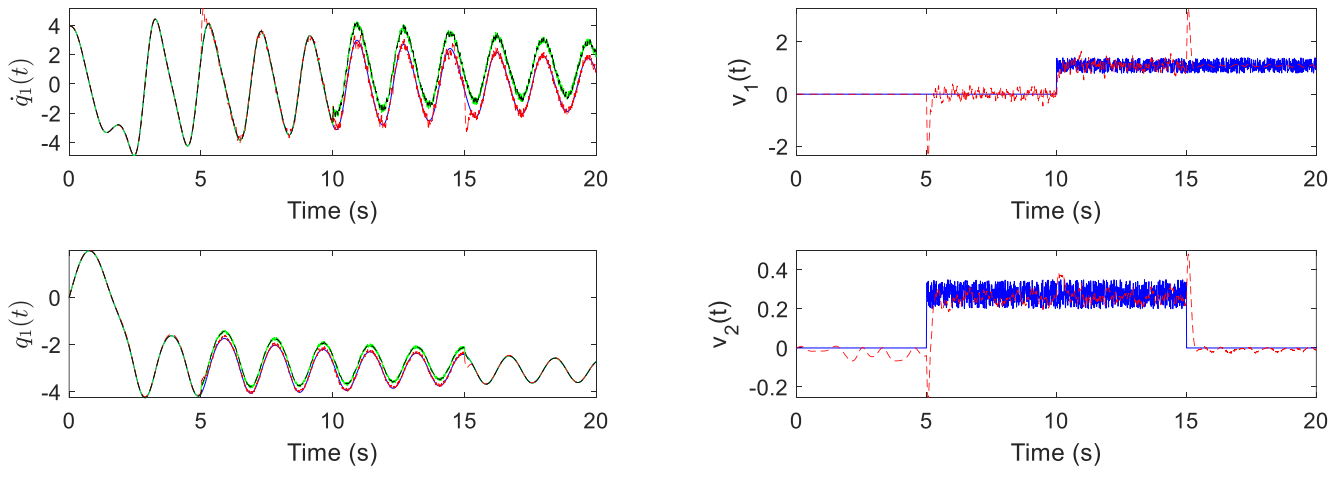
(a) (b)

Figure 2. Same as Figure 1, but considering sinusoidal fault $v(t)$ corrupting output $y(t)$ of system (36).



(a) (b)

Figure 3. Same as Figure 1, but considering trapezoidal fault $v(t)$ corrupting output $y(t)$ of system (36).



(a) (b)

Figure 4. Same as Figure 1, but considering white-noise fault $v(t)$ corrupting output $y(t)$ of system (36).

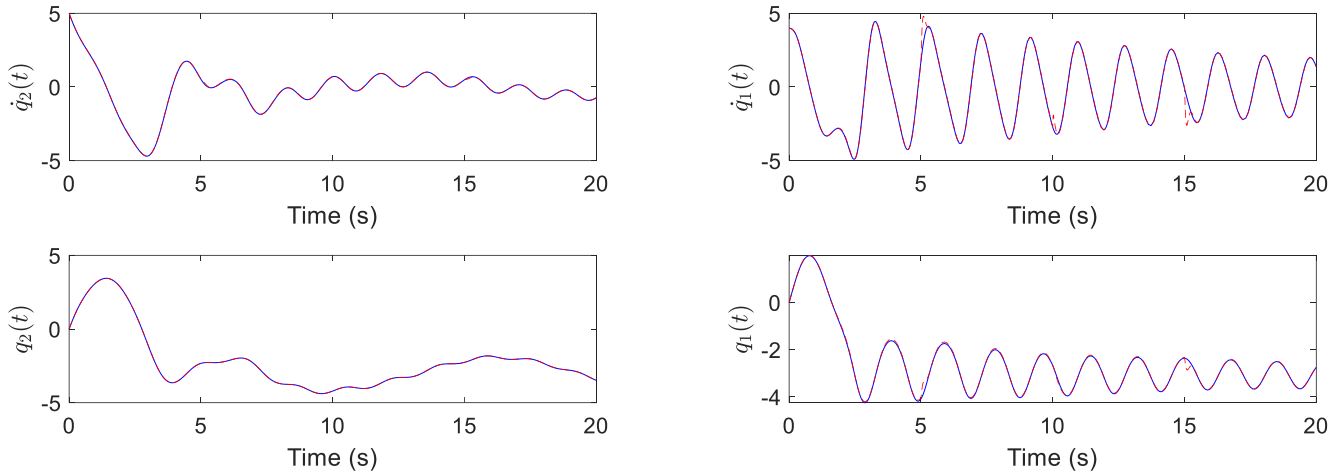


Figure 5. State $x(t)$ (blue solid line) and estimated state $\hat{x}(t)$ (red dashed line) responses applying the observer gains (39) for system (36), with step fault $v(t)$ corrupting its output $y(t)$.

IV.B. Real data example: vehicle driving

A real car-based experimental setup is proposed to validate the approach. The idea is to use real data coming from real experiments and show that with a "simple" bicycle model as core of the observer, it is possible to estimate together the state and the faults. It means that despite the inevitable errors due to the modeling, the approach is sufficiently robust to give good results. The experimental data is recorded from a car available at our Lab LAMIH UMR CNRS 8201 in Valenciennes (France) and shared with the engineering school INSA Hauts-de-France, Figure 6(a). The car is fully equipped and has two modes, manual and autonomous. A description of the plant, with links to videos demonstrating the maneuvering of such vehicle, is available in [28]. The yaw rate $y_r(t)$ and the longitudinal speed $v_x(t)$ are measured. The system is equipped with dSPACE MicroAutoBox where the control/observation algorithms are embedded. LMI constraints were computed on a desktop computer as in the previous example for convenience.

(a) (b)

Figure 6. (a) Vehicle for experimental tests (INSA HdF and LAMIH). (b) Schematic of the vehicle model in (40).

Figure 7 (blue lines) shows an example of experimental data. The first goal is to use a model to recover in normal situations the non-measured variable $v_y(t)$. We consider a bicycle model under regular driving conditions, with small angles assumption and no longitudinal slip that is represented Figure 6(b) and corresponds to the nonlinear model [25][26]:

$$\begin{cases} \dot{v}_y(t) = -\frac{2(C_f+C_r)v_y(t)}{m_v v_x(t)} + \frac{2(C_r l_r - C_f l_f)y_r(t)}{m_v v_x(t)} - y_r(t)v_x(t) + \frac{2C_f\delta(t)}{m_v} - \frac{C_{dy}\rho_a A_{fy} v_y^2(t)}{2m_v}, \\ \dot{y}_r(t) = \frac{2(C_r l_r - C_f l_f)v_y(t)}{I_z v_x(t)} - \frac{2(C_f l_f^2 + C_r l_r^2)y_r(t)}{I_z v_x(t)} + \frac{2C_f l_f \delta(t)}{I_z}, \\ \dot{v}_x(t) = y_r(t)v_y(t) - \frac{C_{dx}\rho_a A_{fx} v_x^2(t)}{2m_v} + \frac{T_w(t)}{m_v R_t}, \end{cases} \quad (40)$$

The parameters corresponding to the vehicle model (40) are given in Table 1, which are taken from [27]. Let us define $x(t) = \left[v_y(t) \quad y_r(t) \quad v_x(t)\right]^T$, $y(t) = \left[y_r(t) \quad v_x(t)\right]^T$ and $u(t) = \left[\delta(t) \quad T_w(t)\right]^T$. Following vehicle physical limitations under normal driving conditions [25], the validity region $x(t) \in \Omega_x$ for system (40) is defined as $\Omega_x = [-1.5, 1.5]^2 \times [1.5, 20]$. To derive a

polytopic representation of (40), we consider the 2 following functions $v_x^{-1}(t) \in \left[\frac{1}{20}, \frac{2}{3}\right]$ and $y_r(t) \in [-1.5, 1.5]$. As a result, the model (40) can be recast as

$$\begin{cases} \dot{x}(t) = A(y(t))x(t) + Bu(t) + g(x,y) \\ y(t) = Cx(t) \end{cases} \tag{41}$$

where $A(y) = A_1 v_x^{-1} + A_2 y_r$, and

$$A_1 = \begin{bmatrix} -\frac{2(C_f + C_r)}{m_v} & \frac{2(C_r l_r - C_f l_f)}{m_v} & 0 \\ \frac{2(C_r l_r - C_f l_f)}{I_z} & -\frac{2(C_f l_f^2 + C_r l_r^2)}{I_z} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{2C_f}{m_v} & 0 \\ \frac{2C_f l_f}{I_z} & 0 \\ 0 & \frac{1}{m_v R_t} \end{bmatrix}, \quad g(x,y) = -\begin{bmatrix} \frac{C_{dy}\rho_a A_{fy} x_1^2(t)}{2m_v} \\ 0 \\ \frac{C_{dx}\rho_a A_{fx} y_2^2(t)}{2m_v} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{42}$$

Table 1. Vehicle parameters for system (40).

Parameter	Description	Value
$v_y(t)$	Lateral speed	– [m/s]
$y_r(t)$	Yaw rate	– [rad/s]
$v_x(t)$	Longitudinal speed	– [m/s]
$\delta(t)$	Front wheel steering angle	– [rad]
$T_w(t)$	Longitudinal wheel torque force	– [Nm]
m_v	Vehicle mass	1077 [kg]
C_f	Front cornering stiffness	47135 [N/rad]
C_r	Rear cornering stiffness	56636 [N/rad]
l_f	Distance between front axle and gravity center	1.08 [m]
l_r	Distance between rear axle and gravity center	1.24 [m]
R_t	Tire radius	0.26 [m]
ρ_a	Air density	1.23 [kg/m^3]
C_{dy}	Lateral drag coefficient	0.35 [–]
C_{dx}	Longitudinal drag coefficient	0.32 [–]
A_{fy}	Lateral frontal area	2.01 [m^2]
A_{fx}	Longitudinal frontal area	1.97 [m^2]

Note that $g(x,y)$ is very small in comparison to other data and is therefore considered as an uncertainty and will not be included in the observer (13). Since $A(y) - A(\hat{y}) = A_1\left(v_x^{-1} - \hat{v}_x^{-1}\right) + A_2\left(y_r - \hat{y}_r\right) = -A_1 v_x^{-2}(c_1) e_{v_x} + A_2 e_{y_r}$, it follows that

$$\left(A(y) - A(\hat{y})\right)x = -A_1 v_x^{-2}(c_1)\, x e_{v_x} + A_2 x e_{y_r} \tag{43}$$

Moreover, using the notations from (19) and (43), it follows that

$$\left(A(y) - A(\hat{y})\right)x = -\bar{\mathcal{A}}'(c_A) \times \mathcal{B}_x(\bar{x}) \times e_v = -\begin{bmatrix} -\frac{2(C_f + C_r)}{m_v} & \frac{2(C_r l_r - C_f l_f)}{m_v} \\ \frac{2(C_r l_r - C_f l_f)}{I_z} & -\frac{2(C_f l_f^2 + C_r l_r^2)}{I_z} \\ 0 & 0 \end{bmatrix} v_x^{-2}(c_1) \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} v_y(t) \\ y_r(t) \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} v_y(t) \\ v_x(t) \end{bmatrix} \end{bmatrix} \begin{bmatrix} e_{v_x} \\ e_{y_r} \end{bmatrix} \tag{44}$$

The faults impact the lateral speed on $y_1 = v_y$, thus $E = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and a cascade of two integrators is used, i.e., $\alpha = 2$. Considering $\xi = 100$ (Remark , gain limitation (33)) and a decay rate $\beta = 5$, the gains of observer (13) are given by

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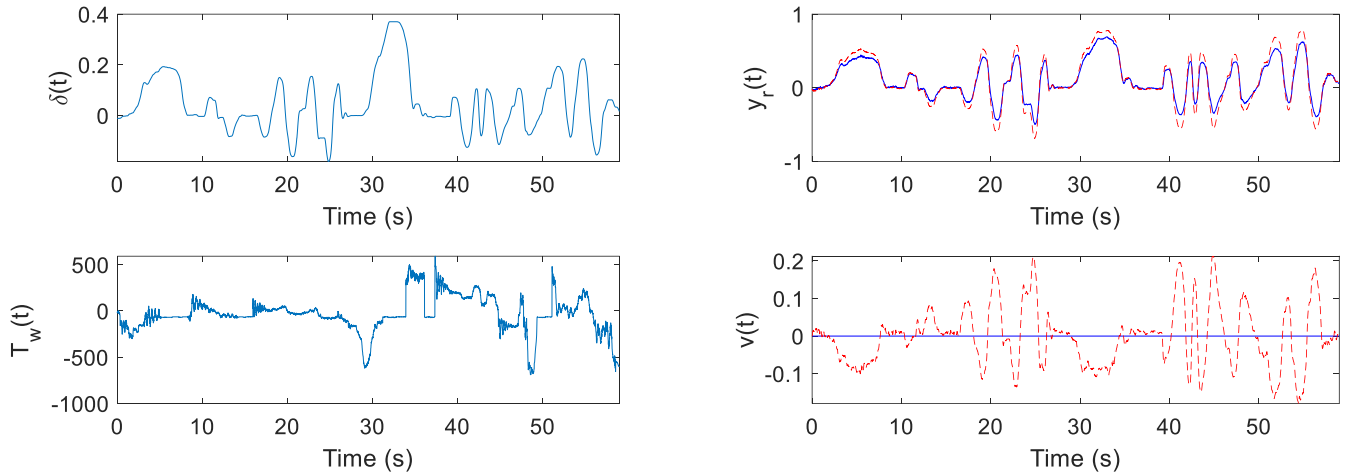
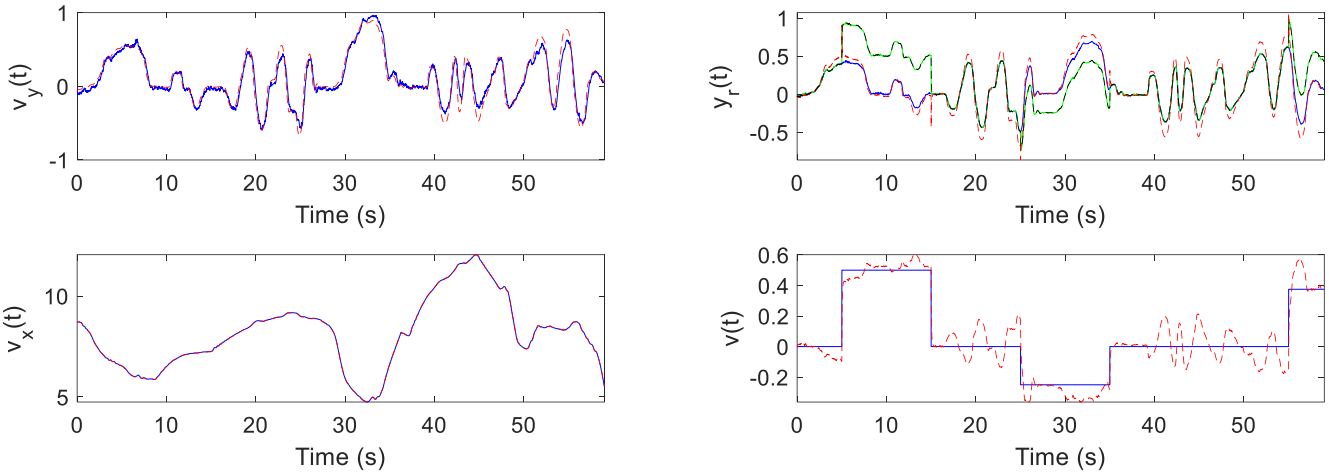
$$P^{-1}L_1 = \begin{bmatrix} 7.29 & -1.69 \\ 17.99 & 0.29 \\ -0.22 & 35.18 \\ 35.81 & -0.33 \\ 19.76 & -0.18 \end{bmatrix}, \quad P^{-1}L_2 = \begin{bmatrix} 7.29 & 1.69 \\ 17.99 & -0.29 \\ 0.22 & 35.18 \\ 35.81 & 0.33 \\ 19.76 & 0.18 \end{bmatrix}, \quad P^{-1}L_3 = \begin{bmatrix} 36.67 & -1.13 \\ -44.87 & -0.04 \\ -0.08 & 35.08 \\ 53.10 & -0.36 \\ 28.83 & -0.20 \end{bmatrix}, \quad P^{-1}L_4 = \begin{bmatrix} 36.67 & 1.13 \\ -44.87 & 0.04 \\ 0.08 & 35.08 \\ 53.10 & 0.36 \\ 28.83 & 0.20 \end{bmatrix}. \tag{45}$$

The numbers characterizing the numerical complexity for this example are computed as $N_{\text{var}} = 55$ and $N_{row} = 320$, which also represent a low level of computational burden for observer design of this real-world application. To show that the assumptions made are valid, i.e., a bicycle model and $g(x, y)$ excluded from the observer, the first trial Figure 7 is a scenario without fault, i.e., $v(t) = 0$. Therefore, Figure 7 shows the inputs left side (a), the outputs (real data and estimated ones) and fault signals on the right side (b). Despite a (voluntarily) shaken input torque $T_w(t)$ and the simplified observer, it shows a good capability of estimating the outputs. The estimated fault signal $\hat{v}(t)$ "absorbs" all the uncertainties with low values and a signal centered on 0. For the second trial, a fault $v(t)$ consisting of a series of step functions, is impacting the output $y_r(t)$. Figure 8 shows a comparison of the system output $y(t)$, fault $v(t)$, system state $x(t)$ responses and their corresponding estimations. Figure 8(b) bottom right shows the corrupted output $y_r(t)$. The figures illustrate that the observer (13) has good capabilities to estimate together the state $x(t)$ and the fault $v(t)$ using real data experiments. For the sake of comparison, Figure 9 shows an approach using measured premise variables, where the observer is designed without the presented novelties. It can be seen that the estimation is worse (Figure 9 (b)) for both the output (red dashed signal top curve) and the fault (bottom curve).



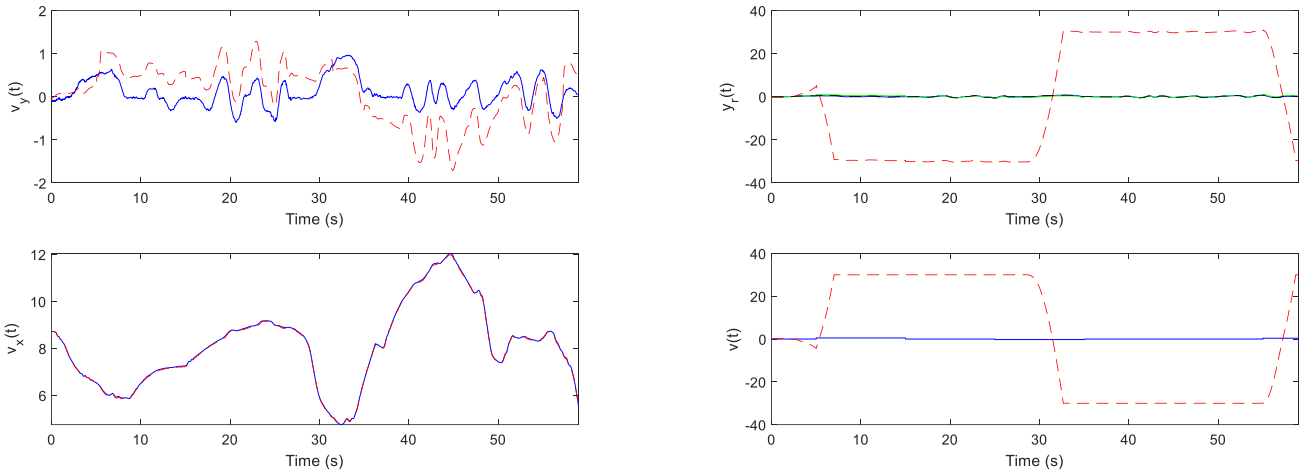
(a) (b)

Figure 7. (a) Inputs $\delta(t)$ and $T_w(t)$ (b) (Above) Output $y_r(t)$ (blue solid line), and estimated output $\hat{y}_r(t)$ (red dashed line), (Below) Fault $v(t)$ (blue solid line) and estimated fault $\hat{v}(t)$ (red dashed line). Scenario with fault $v(t) = 0$.



(a) (b)

Figure 8. States $x(t)=\left[v_y(t) \quad y_r(t) \quad v_x(t)\right]^T$ (blue solid line) and estimated state $\hat{x}(t)$ (red dashed line) (b) (Above) Output $y(t)$ (blue solid line), measured output $z(t)$ (green solid line), estimated output $\hat{y}(t)$ (red dashed line), and estimated measured output $\hat{z}(t)$ (black dashed line), (Below) Fault $v(t)$ (blue solid line) and estimated fault $\hat{v}(t)$ (red dashed line).



(a) (b)

Figure 9. States $x(t)=\left[v_y(t) \quad y_r(t) \quad v_x(t)\right]^T$ (blue solid line) and estimated state $\hat{x}(t)$ (red dashed line) (b) (Above) Output $y(t)$ (blue solid line), measured output $z(t)$ (green solid line), estimated output $\hat{y}(t)$ (red dashed line), and estimated measured output $\hat{z}(t)$ (black dashed line), (Below) Fault $v(t)$ (blue solid line) and estimated fault $\hat{v}(t)$ (red dashed line).

V. Conclusion

A novel approach has been proposed to the fault-tolerant observer design for continuous-time nonlinear systems based on a new formulation that allows to rewrite the extended state error without using classical Lipschitz assumptions. Moreover, this kind of problems belongs to the difficult class of problems with unmeasured premise variables, problem that remains, in its general form, open both for quasi-LPV and Takagi-Sugeno models. The fault is supposed to be captured by a cascade of integrators, which may concern nearly all kind of faults excepted high frequency ones. The nonlinear PI-observer based on this new writing gives rise to new conditions that can be written directly as LMI constraints that can be efficiently solved.

Two illustrative examples have been presented to assess the validity of the given approach. The first one is academic and presents the capabilities of the design and the second one shows that with data from real experiments for car control, even with uncertainties and a simplified model, the approach is able to deliver good results. Future works will focus on (i) coupling the observer together with a control design and deriving conditions and bounds for the convergence of the estimations and the stability and performances of the

closed loop architecture, (ii) designing nonlinear fault-tolerant PI-observers for nonlinear systems, in an attempt to reduce the complexity and the conservatism of the optimization problem.

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