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# Decoupled structure-preserving discretization of incompressible MHD equations with general boundary conditions

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## Abstract

In the framework of a mixed finite element method, a structure-preserving formulation for incompressible MHD equations with general boundary conditions is proposed. A leapfrog-type temporal scheme fully decouples the fluid part from the Maxwell part by means of staggered discrete time sequences and, in doing so, partially linearizes the system. Conservation and dissipation properties of the formulation before and after the decoupling are analyzed. We demonstrate optimal spatial and second-order temporal error convergence and conservation and dissipation properties of the proposed method using manufactured solutions, and apply it to the benchmark Orszag-Tang and lid-driven cavity test cases.

*Keywords:* MHD, mixed finite element method, structure-preserving discretization, general boundary conditions

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## 1. Introduction

Given a bounded, contractible domain  $\Omega \subset \mathbb{R}^d$ ,  $d \in \{2, 3\}$  with a Lipschitz boundary  $\partial\Omega$ , the incompressible, constant density magnetohydrodynamic (or simply incompressible MHD) equations [1] written as

$$\begin{aligned} (1a) \quad & \rho[\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}] - \tilde{\mu} \Delta \mathbf{u} - \mathbf{j} \times \mathbf{B} + \nabla p = \rho \mathbf{f}, \\ (1b) \quad & \nabla \cdot \mathbf{u} = 0, \\ (1c) \quad & \partial_t \mathbf{B} + \nabla \times \mathbf{E} = \mathbf{0}, \\ (1d) \quad & \mathbf{j} - \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) = \mathbf{0}, \\ (1e) \quad & \mathbf{j} - \nabla \times \mathbf{H} = \mathbf{0}, \\ (1f) \quad & \mathbf{B} = \mu \mathbf{H}, \end{aligned}$$

govern the dynamics of fluid velocity  $\mathbf{u}$ , electric current density  $\mathbf{j}$ , magnetic flux density  $\mathbf{B}$ , magnetic field strength  $\mathbf{H}$ , hydrodynamic pressure  $p$  and electric field strength  $\mathbf{E}$ , subject to body force field  $\mathbf{f}$ , a velocity initial condition  $\mathbf{u}^0 := \mathbf{u}(\mathbf{x}, 0)$  and a divergence-free magnetic initial condition  $\mathbf{B}^0 := \mathbf{B}(\mathbf{x}, 0)$ , and proper boundary conditions [2], in a space-time domain,  $\Omega \times (0, T]$ . The material parameters involved are the fluid density  $\rho$ , the dynamic viscosity  $\tilde{\mu}$ , the electric conductivity  $\sigma$ , and the magnetic permeability  $\mu$ .

The incompressible MHD equations, (1), are a coupled system of the incompressible Navier-Stokes equations and the Maxwell's equations. The two evolution equations, (1a) and (1c), stand for the conservation of momentum and the Faraday's law of induction, respectively. Ohm's law and Ampere's law are (1d) and (1e), respectively. (1f) sets up a constitutive relation between  $\mathbf{B}$  and  $\mathbf{H}$ . The divergence-free

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constraint on  $\mathbf{u}$ , (1b), represents mass conservation of the constant mass density flow. It is seen that, as  $\mathbf{B}^0 := \mathbf{B}(\mathbf{x}, 0)$  is divergence free, (1c) ensures

$$\nabla \cdot \mathbf{B} = 0 \quad \text{in } \Omega \times (0, T],$$

which is Gauss's law for magnetism. And, from (1e), we know  $\nabla \cdot \mathbf{j} = 0$  which stands for conservation of charge. The coupling between the flow field and the electromagnetic field is through the Lorentz force,  $\mathbf{j} \times \mathbf{B}$  in (1a), and the electromotive force,  $\mathbf{u} \times \mathbf{B}$  in (1d).

Let  $L$ ,  $U$ , and  $B$  denote characteristic quantities of length, velocity, and magnetic flux density, respectively. We scale the space-time domain and variables by

$$\begin{aligned} \mathbf{x} &\leftarrow \frac{\mathbf{x}}{L}, & t &\leftarrow \frac{Ut}{L}, \\ \mathbf{u} &\leftarrow \frac{\mathbf{u}}{U}, & p &\leftarrow \frac{p}{\rho U^2}, & \mathbf{f} &\leftarrow \frac{L\mathbf{f}}{U^2}, \\ \mathbf{B} &\leftarrow \frac{\mathbf{B}}{B}, & \mathbf{H} &\leftarrow \frac{\mu\mathbf{H}}{B}, & \mathbf{E} &\leftarrow \frac{\mathbf{E}}{UB}, & \mathbf{j} &\leftarrow \frac{\mu L\mathbf{j}}{B}. \end{aligned}$$

And the induced scaled operators are

$$\begin{aligned} D &\leftarrow LD, \quad D \in \{\nabla, \nabla \times, \nabla \cdot\}, \\ \Delta &\leftarrow L^2\Delta, \quad \partial_t \leftarrow \frac{L}{U}\partial_t, \quad \mathbf{u} \cdot \nabla \leftarrow \frac{L}{U}(\mathbf{u} \cdot \nabla). \end{aligned}$$

Note that we do not introduce new notations for the scaled objects and, after this point, they always refer to the scaled dimensionless ones.

Now (1) can be written in a dimensionless form, i.e.,

$$\begin{aligned} (2a) \quad \partial_t \mathbf{u} + \boldsymbol{\omega} \times \mathbf{u} + \text{R}_f^{-1} \nabla \times \boldsymbol{\omega} - c(\nabla \times \mathbf{H}) \times \mathbf{H} + \nabla P &= \mathbf{f}, & \text{in } \Omega \times (0, T], \\ (2b) \quad \boldsymbol{\omega} - \nabla \times \mathbf{u} &= \mathbf{0}, & \text{in } \Omega \times (0, T], \\ (2c) \quad \nabla \cdot \mathbf{u} &= 0, & \text{in } \Omega \times (0, T], \\ (2d) \quad \partial_t \mathbf{H} + \text{R}_m^{-1} \Delta \mathbf{H} - \nabla \times (\mathbf{u} \times \mathbf{H}) &= \mathbf{0}, & \text{in } \Omega \times (0, T], \end{aligned}$$

where, after introducing vorticity  $\boldsymbol{\omega} := \nabla \times \mathbf{u}$  and total pressure  $P := p + \frac{1}{2}\mathbf{u} \cdot \mathbf{u}$ , we have applied the relation  $(\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p \stackrel{(2c)}{=} \boldsymbol{\omega} \times \mathbf{u} + \nabla P$ . Meanwhile, by eliminating  $\mathbf{B}$ ,  $\mathbf{E}$  and  $\mathbf{j}$ , dimensionless forms of (1c)-(1f), i.e.,

$$\begin{aligned} (3a) \quad \partial_t \mathbf{B} + \nabla \times \mathbf{E} &= \mathbf{0}, \\ (3b) \quad \text{R}_m^{-1} \mathbf{j} - (\mathbf{E} + \mathbf{u} \times \mathbf{B}) &= \mathbf{0}, \\ (3c) \quad \mathbf{j} - \nabla \times \mathbf{H} &= \mathbf{0}, \\ (3d) \quad \mathbf{B} &= \mathbf{H}, \end{aligned}$$

have been written into (2d), which is the magnetic induction (or Helmholtz) equation.<sup>1</sup> The three dimensionless parameters are the fluid Reynolds number  $\text{R}_f = \frac{\rho UL}{\tilde{\mu}} = \frac{UL}{\nu}$  (with  $\nu = \frac{\tilde{\mu}}{\rho}$  being the kinematic viscosity), the magnetic Reynolds number  $\text{R}_m = \mu\sigma UL$ , and the coupling number  $c = A_l^{-2}$  ( $A_l = \frac{U\sqrt{\rho\tilde{\mu}}}{B} = \frac{U}{U_A}$  is the Alfvén number and  $U_A = \frac{B}{\sqrt{\rho\mu}}$  is the Alfvén speed). Sometimes, the coupling term in (2a),  $c(\nabla \times \mathbf{H}) \times \mathbf{H}$ , is written as  $\mathfrak{s}(\mathbf{E} + \mathbf{u} \times \mathbf{H}) \times \mathbf{H}$  where the coupling number becomes  $\mathfrak{s} = c\text{R}_m = \frac{\sigma LB^2}{\rho U}$ . For example, see [3, (2.2a)].

To close the problem, (2) is complemented with initial conditions  $\mathbf{u}^0, \mathbf{H}^0$  and boundary conditions. In this work, we consider the following general boundary condition configuration:

<sup>1</sup>Note that, after non-dimensionalization, we get  $\mathbf{B} = \mathbf{H}$ , i.e. (3d). This is the case in vector calculus. In differential forms, they appear as a pair of differently oriented dual forms connected by the Hodge  $\star$  operator, i.e.,  $\mathbf{B} = \star\mathbf{H}$ .

- For the fluid part, there are two pairs of boundary conditions,

$$(4) \quad \begin{cases} P = \widehat{P} & \text{on } \Gamma_{\widehat{P}} \times (0, T] \\ \mathbf{u} \cdot \mathbf{n} = \widehat{u} & \text{on } \Gamma_{\widehat{u}} \times (0, T] \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{u} \times \mathbf{n} = \widehat{\mathbf{u}} & \text{on } \Gamma_{\widehat{\mathbf{u}}} \times (0, T] \\ \mathbf{n} \times (\boldsymbol{\omega} \times \mathbf{n}) = \widehat{\boldsymbol{\omega}} & \text{on } \Gamma_{\widehat{\boldsymbol{\omega}}} \times (0, T] \end{cases},$$

where  $\mathbf{n}$  is the outward unit normal vector,  $\{\Gamma_{\widehat{P}}, \Gamma_{\widehat{u}}\}$  and  $\{\Gamma_{\widehat{\mathbf{u}}}, \Gamma_{\widehat{\boldsymbol{\omega}}}\}$  are two partitions of the boundary, such that  $\overline{\Gamma_{\widehat{P}}} \cup \overline{\Gamma_{\widehat{u}}} = \partial\Omega$ ,  $\Gamma_{\widehat{P}} \cap \Gamma_{\widehat{u}} = \emptyset$  and  $\overline{\Gamma_{\widehat{\mathbf{u}}}} \cup \overline{\Gamma_{\widehat{\boldsymbol{\omega}}}} = \partial\Omega$ ,  $\Gamma_{\widehat{\mathbf{u}}} \cap \Gamma_{\widehat{\boldsymbol{\omega}}} = \emptyset$ .

- For the Maxwell part, boundary conditions are given as

$$(5) \quad \begin{cases} \mathbf{E} \times \mathbf{n} = \widehat{\mathbf{E}} & \text{on } \Gamma_{\widehat{\mathbf{E}}} \times (0, T] \\ \mathbf{n} \times (\mathbf{H} \times \mathbf{n}) = \widehat{\mathbf{H}} & \text{on } \Gamma_{\widehat{\mathbf{H}}} \times (0, T] \end{cases},$$

where  $\{\Gamma_{\widehat{\mathbf{E}}}, \Gamma_{\widehat{\mathbf{H}}}\}$  is a partition of the boundary, satisfying  $\overline{\Gamma_{\widehat{\mathbf{E}}}} \cup \overline{\Gamma_{\widehat{\mathbf{H}}}} = \partial\Omega$ ,  $\Gamma_{\widehat{\mathbf{E}}} \cap \Gamma_{\widehat{\mathbf{H}}} = \emptyset$ . Note that the boundary condition  $\mathbf{E} \times \mathbf{n}$  implies the magnetic flux boundary condition  $\mathbf{H} \cdot \mathbf{n}$  (or  $\mathbf{B} \cdot \mathbf{n}$ ). This is because

$$(6) \quad \partial_t (\mathbf{H} \cdot \mathbf{n}) = \partial_t (\mathbf{B} \cdot \mathbf{n}) \stackrel{(3a)}{=} -(\nabla \times \mathbf{E}) \cdot \mathbf{n} \stackrel{\nabla \times \mathbf{n} = \mathbf{0}}{=} -(\nabla \cdot)_{\partial\Omega} (\mathbf{E} \times \mathbf{n})$$

on  $\partial\Omega \times (0, T]$ . Also see [4, Section 2.3].

For example, the boundary conditions considered in the work of Hu et al., see (7) of [5], are a particular case of this general configuration, i.e. the case when  $\Gamma_{\widehat{u}} = \Gamma_{\widehat{\boldsymbol{\omega}}} = \Gamma_{\widehat{\mathbf{H}}} = \emptyset$ ,  $\widehat{P} = 0$  and  $\widehat{\mathbf{u}} = \widehat{\mathbf{E}} = \mathbf{0}$ .

It is generally acknowledged that structure-preservation, especially the preservation (either strongly or weakly) of  $\nabla \cdot \mathbf{B} = 0$ , is an essential property of numerical methods for MHD [6]. Many of these structure-preserving methods can be classified into, for example, potential-based methods [7], divergence-cleaning methods [8, 9], constrained transport methods [10, 11] and Lagrange multiplier (or augmented) methods [12, 13, 14]. In the current paper neither of those techniques will be applied, but it will be shown that the absence of magnetic monopoles is weakly enforced through the decoupled variational formulation. Recently, methods that aim at preserving multiple structures in the mixed finite element setting become increasingly popular. See, for example, the work of Hu et al. for incompressible MHD that preserves cross- and magnetic-helicity, energy and Gauss law of magnetism [5], the work of Gawlik and Gay-Balmaz for incompressible MHD of a variable fluid density that preserves energy, cross-helicity (when the fluid density is constant) and magnetic-helicity, mass, total squared density, pointwise incompressibility, and Gauss's law for magnetism [15], the work of Laakmann et al. [16] and references therein. These methods preserve several physical quantities of interest, but usually are less computationally efficient because of their large (nonlinear) discrete systems to be solved. A compromise is to use a decoupled scheme which breaks the discrete system into several smaller ones, for example see [3, 12, 13, 17, 18]. These methods in the mixed finite element setting commonly deal with a particular configuration of boundary conditions.

In this paper, we introduce a structure-preserving mixed finite element discretization for the incompressible MHD under a general boundary condition configuration. The method is decoupled in the sense that the two evolution equations are temporally discretized at different time sequences such that they can be solved individually.

The outline of the rest of the paper is as follows. Relevant Hilbert spaces and the de Rham Hilbert complex are introduced in Section 2 followed by the proposed semi-discrete formulation and its conservation properties in Section 3. The temporal discretization that leads to the decoupled scheme is explained in Section 4. Numerical tests are presented in Section 5. Finally, in Section 6 conclusions are drawn.

## 2. Relevant Hilbert spaces and the de Rham complex

The function spaces used in this work are similar to those used in [19] where only the two-dimensional setting is discussed. For the sake of completeness, we give a brief introduction to the three-dimensional spaces as a complement to [19, Section 2.1]. We first introduce the relevant spaces and the de Rham complex in the infinite dimensional setting, and then introduce their finite dimensional counterparts. For an extensive introduction of function spaces, see [20] or any other textbook on functional analysis.

### 2.1. The infinite dimensional setting

Let  $L^2(\Omega)$  denote the space of square-integrable functions on  $\Omega$ ,

$$L^2(\Omega) := \left\{ \varphi \mid \langle \varphi, \varphi \rangle_\Omega := \int_\Omega \varphi \cdot \varphi \, d\Omega < \infty \right\}.$$

For  $\Omega \subset \mathbb{R}^3$ , we also consider the following Hilbert spaces,

$$\begin{aligned} H^1(\Omega) &:= \left\{ \psi \mid \psi \in L^2(\Omega), \nabla \psi \in [L^2(\Omega)]^3 \right\}, \\ H(\text{curl}; \Omega) &:= \left\{ \boldsymbol{\sigma} \mid \boldsymbol{\sigma} \in [L^2(\Omega)]^3, \nabla \times \boldsymbol{\sigma} \in [L^2(\Omega)]^3 \right\}, \\ H(\text{div}; \Omega) &:= \left\{ \boldsymbol{\phi} \mid \boldsymbol{\phi} \in [L^2(\Omega)]^3, \nabla \cdot \boldsymbol{\phi} \in L^2(\Omega) \right\}. \end{aligned}$$

They form the well known de Rham Hilbert complex [21, 22],

$$(7) \quad \mathbb{R} \hookrightarrow H^1(\Omega) \xrightarrow{\nabla} H(\text{curl}; \Omega) \xrightarrow{\nabla \times} H(\text{div}; \Omega) \xrightarrow{\nabla \cdot} L^2(\Omega) \rightarrow 0.$$

Note that, in particular, the  $\nabla \cdot$  maps surjectively from  $H(\text{div}; \Omega)$  onto  $L^2(\Omega)$ . Moreover, when  $\Omega$  is simply-connected, then the above complex is exact, i.e., the image of  $\nabla$  coincides with the kernel of  $\nabla \times$ , and the image of  $\nabla \times$  coincides with the kernel of  $\nabla \cdot$ .

The trace operator,  $\mathcal{T}$ , restricts an element of Hilbert spaces to a boundary section  $\Gamma \subseteq \partial\Omega$ . For  $\psi \in H^1(\Omega)$  and  $\boldsymbol{\phi} \in H(\text{div}; \Omega)$ , we define its action as

$$\mathcal{T}\psi = \psi|_\Gamma, \quad \mathcal{T}\boldsymbol{\phi} = \boldsymbol{\phi} \cdot \mathbf{n}|_\Gamma.$$

And we distinguish between  $\mathcal{T}$  and  $\mathcal{T}_\parallel$  for elements of  $H(\text{curl}; \Omega)$ , i.e., for  $\boldsymbol{\sigma}, \boldsymbol{\eta} \in H(\text{curl}; \Omega)$ ,

$$(8) \quad \mathcal{T}\boldsymbol{\sigma} = \boldsymbol{\sigma} \times \mathbf{n}|_\Gamma, \quad \mathcal{T}_\parallel \boldsymbol{\eta} = \mathbf{n} \times (\boldsymbol{\eta} \times \mathbf{n})|_\Gamma.$$

Using the above, we define the trace spaces on  $\Gamma$  as

$$\begin{aligned} H^{1/2}(\Omega, \Gamma) &:= \{ \mathcal{T}\psi \mid \psi \in H^1(\Omega) \}, \\ \mathcal{T}H(\text{curl}; \Omega, \Gamma) &:= \{ \mathcal{T}\boldsymbol{\sigma} \mid \boldsymbol{\sigma} \in H(\text{curl}; \Omega) \}, \\ \mathcal{T}_\parallel H(\text{curl}; \Omega, \Gamma) &:= \{ \mathcal{T}_\parallel \boldsymbol{\eta} \mid \boldsymbol{\eta} \in H(\text{curl}; \Omega) \}, \\ \mathcal{T}H(\text{div}; \Omega, \Gamma) &:= \{ \mathcal{T}\boldsymbol{\phi} \mid \boldsymbol{\phi} \in H(\text{div}; \Omega) \}. \end{aligned}$$

Note that, at the continuous level, although  $\mathcal{T}$  and  $\mathcal{T}_\parallel$  are different, see (8), trace spaces  $\mathcal{T}H(\text{curl}; \Omega, \Gamma)$  and  $\mathcal{T}_\parallel H(\text{curl}; \Omega, \Gamma)$  refer to the same tangential space.

### 2.2. The finite dimensional setting

Assume that we are given finite dimensional spaces  $G(\Omega) \subset H^1(\Omega)$ ,  $C(\Omega) \subset H(\text{curl}; \Omega)$ ,  $D(\Omega) \subset H(\text{div}; \Omega)$ ,  $S(\Omega) \subset L^2(\Omega)$ , which form a discretization of the continuous de Rham Hilbert complex,

$$(9) \quad \mathbb{R} \hookrightarrow G(\Omega) \xrightarrow{\nabla} C(\Omega) \xrightarrow{\nabla \times} D(\Omega) \xrightarrow{\nabla \cdot} S(\Omega) \rightarrow 0.$$

In particular, this means that if the complex in (7) is exact, then so is the one in (9). We will also consider the following subspaces with boundary conditions, where  $\Gamma \subset \partial\Omega$ ,

$$\begin{aligned} G_{\widehat{\psi}}(\Omega, \Gamma) &:= \left\{ \psi_h \mid \psi_h \in G(\Omega), \mathcal{T}\psi_h = \widehat{\psi} \text{ on } \Gamma \right\}, \\ C_{\widehat{\boldsymbol{\sigma}}}(\Omega, \Gamma) &:= \left\{ \boldsymbol{\sigma}_h \mid \boldsymbol{\sigma}_h \in C(\Omega), \mathcal{T}\boldsymbol{\sigma}_h = \widehat{\boldsymbol{\sigma}} \text{ on } \Gamma \right\}, \\ C_{\widehat{\boldsymbol{\eta}}}^\parallel(\Omega, \Gamma) &:= \left\{ \boldsymbol{\eta}_h \mid \boldsymbol{\eta}_h \in C(\Omega), \mathcal{T}_\parallel \boldsymbol{\eta}_h = \widehat{\boldsymbol{\eta}} \text{ on } \Gamma \right\}, \\ D_{\widehat{\boldsymbol{\phi}}}(\Omega, \Gamma) &:= \left\{ \boldsymbol{\phi}_h \mid \boldsymbol{\phi}_h \in D(\Omega), \mathcal{T}\boldsymbol{\phi}_h = \widehat{\boldsymbol{\phi}} \text{ on } \Gamma \right\}. \end{aligned}$$

For the corresponding homogeneous boundary conditions we have

$$\begin{aligned} G_0(\Omega, \Gamma) &:= \{\psi_h \mid \psi_h \in G(\Omega), \mathcal{T}\psi_h = 0 \text{ on } \Gamma\}, \\ C_0(\Omega, \Gamma) &:= \{\boldsymbol{\sigma}_h \mid \boldsymbol{\sigma}_h \in C(\Omega), \mathcal{T}\boldsymbol{\sigma}_h = \mathbf{0} \text{ on } \Gamma\}, \\ C_0^\parallel(\Omega, \Gamma) &:= \{\boldsymbol{\eta}_h \mid \boldsymbol{\eta}_h \in C(\Omega), \mathcal{T}_\parallel\boldsymbol{\eta}_h = \mathbf{0} \text{ on } \Gamma\}, \\ D_0(\Omega, \Gamma) &:= \{\boldsymbol{\phi}_h \mid \boldsymbol{\phi}_h \in D(\Omega), \mathcal{T}\boldsymbol{\phi}_h = 0 \text{ on } \Gamma\}. \end{aligned}$$

Moreover, the finite dimensional trace spaces will be denoted by  $\mathcal{T}G(\Omega, \Gamma)$ ,  $\mathcal{T}C(\Omega, \Gamma)$ ,  $\mathcal{T}_\parallel C(\Omega, \Gamma)$ ,  $\mathcal{T}D(\Omega, \Gamma)$ . Unlike their continuous counterparts, finite dimensional trace spaces  $\mathcal{T}C(\Omega, \Gamma)$  and  $\mathcal{T}_\parallel C(\Omega, \Gamma)$  usually are different.

We define a corresponding trilinear form,

$$(10) \quad a(\boldsymbol{\alpha}_h, \boldsymbol{\beta}_h, \boldsymbol{\gamma}_h) := \langle \boldsymbol{\alpha}_h \times \boldsymbol{\beta}_h, \boldsymbol{\gamma}_h \rangle_\Omega, \quad \boldsymbol{\alpha}_h, \boldsymbol{\beta}_h, \boldsymbol{\gamma}_h \in \{C(\Omega), D(\Omega)\},$$

which is skew-symmetric with respect to any two of the three entries. Note that, in the infinite dimensional setting, the  $L^2$ -boundedness of  $\langle \boldsymbol{\alpha} \times \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle_\Omega$  is not implied even if  $\boldsymbol{\alpha}$ ,  $\boldsymbol{\beta}$  and  $\boldsymbol{\gamma}$  are  $L^2$  vector fields. However, in the finite dimensional setting where the spaces  $C(\Omega)$  and  $D(\Omega)$  consist of piecewise-polynomial finite element vector fields, it is in fact  $L^2$ -bounded.

### 3. Spatially-discrete formulation and its conservation properties

In this section, we present a spatially-discrete weak formulation of (2) and analyze its conservation properties. As a remark, for neatness, we only discuss the three-dimensional formulation and its temporal discretization in this paper. Obtaining their two-dimensional versions is straightforward.

#### 3.1. Spatially-discrete formulation

To increase the notational clarity, we omit the part for the temporal domain. The proposed semi-discrete weak formulation of (2) in  $\mathbb{R}^3$  is written as: Given  $\mathbf{f} \in [L^2(\Omega)]^3$ , natural boundary conditions  $\widehat{P} \in H^{1/2}(\Omega; \Gamma_{\widehat{P}})$ ,  $\widehat{\mathbf{u}} \in \mathcal{T}H(\text{curl}; \Omega, \Gamma_{\widehat{\mathbf{u}}})$ ,  $\widehat{\mathbf{E}} \in \mathcal{T}H(\text{curl}; \Omega, \Gamma_{\widehat{\mathbf{E}}})$ , and initial conditions  $(\mathbf{u}_h^0, \mathbf{H}_h^0) \in D(\Omega) \times C(\Omega)$ , seek  $(\mathbf{u}_h, \boldsymbol{\omega}_h, P_h, \mathbf{H}_h) \in D_{\widehat{\mathbf{u}}}(\Omega, \Gamma_{\widehat{\mathbf{u}}}) \times C_0^\parallel(\Omega, \Gamma_{\widehat{\omega}}) \times S(\Omega) \times C_0^\parallel(\Omega, \Gamma_{\widehat{\mathbf{H}}})$ , such that  $\forall (\mathbf{v}_h, \mathbf{w}_h, q_h, \mathbf{b}_h) \in D_0(\Omega, \Gamma_{\widehat{\mathbf{u}}}) \times C_0^\parallel(\Omega, \Gamma_{\widehat{\omega}}) \times S(\Omega) \times C_0^\parallel(\Omega, \Gamma_{\widehat{\mathbf{H}}})$ ,

$$(11a) \quad \begin{aligned} \langle \partial_t \mathbf{u}_h, \mathbf{v}_h \rangle_\Omega + a(\boldsymbol{\omega}_h, \mathbf{u}_h, \mathbf{v}_h) + \text{R}_f^{-1} \langle \nabla \times \boldsymbol{\omega}_h, \mathbf{v}_h \rangle_\Omega \\ - c a(\nabla \times \mathbf{H}_h, \mathbf{H}_h, \mathbf{v}_h) - \langle P_h, \nabla \cdot \mathbf{v}_h \rangle_\Omega = \langle \mathbf{f}, \mathbf{v}_h \rangle_\Omega - \langle \widehat{P}, \mathcal{T}\mathbf{v}_h \rangle_{\Gamma_{\widehat{P}}}, \end{aligned}$$

$$(11b) \quad - \langle \mathbf{u}_h, \nabla \times \mathbf{w}_h \rangle_\Omega + \langle \boldsymbol{\omega}_h, \mathbf{w}_h \rangle_\Omega = - \langle \widehat{\mathbf{u}}, \mathcal{T}_\parallel \mathbf{w}_h \rangle_{\Gamma_{\widehat{\mathbf{u}}}},$$

$$(11c) \quad \langle \nabla \cdot \mathbf{u}_h, q_h \rangle_\Omega = 0,$$

$$(11d) \quad \langle \partial_t \mathbf{H}_h, \mathbf{b}_h \rangle_\Omega + \text{R}_m^{-1} \langle \nabla \times \mathbf{H}_h, \nabla \times \mathbf{b}_h \rangle_\Omega - a(\mathbf{u}_h, \mathbf{H}_h, \nabla \times \mathbf{b}_h) = \langle \widehat{\mathbf{E}}, \mathcal{T}_\parallel \mathbf{b}_h \rangle_{\Gamma_{\widehat{\mathbf{E}}}}.$$

Note that this assumes that the supplied boundary conditions are exactly representable as members of the chosen finite dimensional spaces; more generally, one can always project the boundary conditions into the appropriate finite dimensional trace spaces. See (4) and (5) for the general boundary setting.

#### 3.2. Conservation properties

In this section, we analyze some properties of the formulation (11). In particular, we show that this semi-discrete formulation preserves conservation of mass and conservation of charge strongly, preserves Gauss' law of magnetism weakly, and preserves the correct energy dissipation rate such that, when no body forces are applied and there is no net flux of energy through the boundary, it conserves the total energy in the ideal limit.

##### 3.2.1. Conservation of mass

For  $\mathbf{u}_h \in D(\Omega)$ , (11c) secures pointwise strong mass conservation, i.e.  $\nabla \cdot \mathbf{u}_h = 0$ , because of the fact that  $\nabla \cdot$  maps  $D(\Omega)$  surjectively onto  $S(\Omega)$ , see (9).

### 3.2.2. Conservation of charge

Since we have selected  $\mathbf{H}_h \in C(\Omega)$ , from (9), we know that we can always find  $\mathbf{j}_h = \nabla \times \mathbf{H}_h \in D(\Omega)$  which is pointwise divergence-free; conservation of charge is preserved by the formulation (11).

### 3.2.3. Weak conservation of Gauss's law for magnetism

Since (11d) is valid for all  $\mathbf{b}_h \in C_0^\parallel(\Omega, \Gamma_{\widehat{\mathbf{H}}})$ , let us choose  $\mathbf{b}_h = \nabla q_h$  for an arbitrary  $q_h \in G_0(\Omega, \Gamma_{\widehat{\mathbf{H}}})$ . Then, we have

$$(12) \quad \langle \partial_t \mathbf{H}_h, \nabla q_h \rangle_\Omega + \mathbf{R}_m^{-1} \langle \nabla \times \mathbf{H}_h, \nabla \times \nabla q_h \rangle_\Omega - a(\mathbf{u}_h, \mathbf{H}_h, \nabla \times \nabla q_h) = \left\langle \widehat{\mathbf{E}}, \mathcal{T}_\parallel \nabla q_h \right\rangle_{\Gamma_{\widehat{\mathbf{E}}}}.$$

The second and third terms on the left hand side are zero because we strongly satisfy  $\nabla \times \nabla q_h = 0$  since our discrete function spaces constitute a finite dimensional de Rham complex. We are left with the following equation,

$$(13) \quad \langle \partial_t \mathbf{H}_h, \nabla q_h \rangle_\Omega = \left\langle \widehat{\mathbf{E}}, \mathcal{T}_\parallel \nabla q_h \right\rangle_{\Gamma_{\widehat{\mathbf{E}}}}.$$

Using the fact that  $\nabla q_h \in C_0^\parallel(\Omega, \Gamma_{\widehat{\mathbf{H}}})$ , we have

$$(14) \quad \begin{aligned} \langle \partial_t \mathbf{H}_h, \nabla q_h \rangle_\Omega &= \left\langle \widehat{\mathbf{E}}, \mathcal{T}_\parallel \nabla q_h \right\rangle_{\Gamma_{\widehat{\mathbf{E}}}} = \left\langle \widehat{\mathbf{E}}, \mathcal{T}_\parallel \nabla q_h \right\rangle_{\partial\Omega} \\ &= - \left\langle (\nabla \cdot)_{\partial\Omega} \widehat{\mathbf{E}}, \mathcal{T} q_h \right\rangle_{\partial\Omega} = - \left\langle (\nabla \cdot)_{\Gamma_{\widehat{\mathbf{E}}}} \widehat{\mathbf{E}}, \mathcal{T} q_h \right\rangle_{\Gamma_{\widehat{\mathbf{E}}}}. \end{aligned}$$

Finally, using the relation between  $\mathbf{H} \cdot \mathbf{n}$  and  $\widehat{\mathbf{E}}$  from (6), we can rewrite the rightmost term to obtain the following equation,

$$(15) \quad \frac{d}{dt} \left[ \langle \mathbf{H}_h, \nabla q_h \rangle_\Omega - \langle \mathbf{H} \cdot \mathbf{n}, \mathcal{T} q_h \rangle_{\Gamma_{\widehat{\mathbf{E}}}} \right] = 0.$$

The expression inside the square brackets is precisely the weak divergence-free constraint on  $\mathbf{H}_h$ . The above equation thus states that the time-derivative of this constraint on  $\mathbf{H}_h$  is zero, and implies that  $\mathbf{H}_h$  will remain weakly divergence-free for all  $t$  if the initial condition  $\mathbf{H}_h^0$  is weakly divergence-free.

### 3.2.4. Energy Conservation

The (semi-)discrete (total) energy is defined as

$$(16) \quad \mathcal{E}_h := \mathcal{K}_h + \mathcal{M}_h$$

whose  $\mathcal{K}_h := \frac{1}{2} \langle \mathbf{u}_h, \mathbf{u}_h \rangle_\Omega$  is the kinetic energy of the fluid and  $\mathcal{M}_h := \frac{c}{2} \langle \mathbf{H}_h, \mathbf{H}_h \rangle_\Omega$  is the magnetic energy. Suppose

$$(17) \quad \widehat{u} = 0, \quad \Gamma_{\widehat{\mathbf{u}}} = \Gamma_{\widehat{\mathbf{E}}} = \partial\Omega, \quad \Gamma_{\widehat{\omega}} = \Gamma_{\widehat{\mathbf{H}}} = \emptyset.$$

By replacing  $\mathbf{v}_h$  in (11a) with  $\mathbf{u}_h$ , we can obtain

$$(18) \quad \partial_t \mathcal{K}_h + \mathbf{R}_f^{-1} \langle \nabla \times \boldsymbol{\omega}_h, \mathbf{u}_h \rangle_\Omega - c a(\mathbf{j}_h, \mathbf{H}_h, \mathbf{u}_h) = \langle \mathbf{f}, \mathbf{u}_h \rangle_\Omega - \left\langle \widehat{P}, \mathcal{T} \mathbf{u}_h \right\rangle_{\Gamma_{\widehat{P}}},$$

where  $\mathbf{j}_h = \nabla \times \mathbf{H}_h \in D(\Omega)$ , and the fluid advection term  $a(\boldsymbol{\omega}_h, \mathbf{u}_h, \mathbf{u}_h)$  and the total pressure term  $\langle P_h, \nabla \cdot \mathbf{u}_h \rangle_\Omega$  have vanished because of the skew-symmetry of the trilinear (10) and  $\nabla \cdot \mathbf{u}_h = 0$ , respectively. Meanwhile, because (11b) holds  $\forall \mathbf{w}_h \in C(\Omega)$  and  $\boldsymbol{\omega}_h \in C(\Omega)$ , we know that

$$- \langle \mathbf{u}_h, \nabla \times \boldsymbol{\omega}_h \rangle_\Omega + \langle \boldsymbol{\omega}_h, \boldsymbol{\omega}_h \rangle_\Omega = - \langle \widehat{\mathbf{u}}, \mathcal{T}_\parallel \boldsymbol{\omega}_h \rangle_{\partial\Omega}.$$

As a results, (18) can be expressed as

$$(19) \quad \partial_t \mathcal{K}_h + \mathbf{R}_f^{-1} \langle \boldsymbol{\omega}_h, \boldsymbol{\omega}_h \rangle_\Omega - c a(\mathbf{j}_h, \mathbf{H}_h, \mathbf{u}_h) = \langle \mathbf{f}, \mathbf{u}_h \rangle_\Omega - \left\langle \widehat{P}, \mathcal{T} \mathbf{u}_h \right\rangle_{\Gamma_{\widehat{P}}} + \langle \widehat{\mathbf{u}}, \mathcal{T}_\parallel \boldsymbol{\omega}_h \rangle_{\partial\Omega}.$$

Similarly, replacing  $\mathbf{b}_h$  in (11d) with  $\mathbf{H}_h$  leads to

$$(20) \quad \frac{1}{c} \partial_t \mathcal{M}_h + \mathbf{R}_m^{-1} \langle \mathbf{j}_h, \mathbf{j}_h \rangle_\Omega - a(\mathbf{u}_h, \mathbf{H}_h, \mathbf{j}_h) = \langle \widehat{\mathbf{E}}, \mathcal{T}_\parallel \mathbf{H}_h \rangle_{\partial\Omega}.$$

With (19) and (20), we can finally derive the rate of change of the discrete energy,

$$(21) \quad \begin{aligned} \partial_t \mathcal{E}_h &= \partial_t \mathcal{K}_h + \partial_t \mathcal{M}_h \\ &= \underbrace{\langle \mathbf{f}, \mathbf{u}_h \rangle_\Omega}_{(i)} - \underbrace{\mathbf{R}_f^{-1} \mathcal{S}_h}_{(ii)} - \underbrace{c \mathbf{R}_m^{-1} \mathcal{J}_h}_{(iii)} - \underbrace{\langle \widehat{P}, \mathcal{T} \mathbf{u}_h \rangle_{\Gamma_{\widehat{P}}}}_{(iv)} + \underbrace{\langle \widehat{\mathbf{u}}, \mathcal{T}_\parallel \boldsymbol{\omega}_h \rangle_{\partial\Omega}}_{(v)} + c \underbrace{\langle \widehat{\mathbf{E}}, \mathcal{T}_\parallel \mathbf{H}_h \rangle_{\partial\Omega}}_{(vi)}, \end{aligned}$$

where  $\mathcal{S}_h := \langle \boldsymbol{\omega}_h, \boldsymbol{\omega}_h \rangle_\Omega$ ,  $\mathcal{J}_h := \langle \mathbf{j}_h, \mathbf{j}_h \rangle_\Omega$ . The two nonlinear terms have canceled each other due to the skew-symmetry of the trilinear form (10); the exchange of energy between  $\mathcal{K}_h$  and  $\mathcal{M}_h$  is exactly captured.

In (21), (i) represents amount of mechanical work done on the system per unit time, (ii) and (iii) are dissipation rates due to viscosity and electric resistance, respectively, and terms (iv) - (vi) are the net flux of energy through the domain boundary.

Clearly, if  $\mathbf{f} = \mathbf{0}$ , when the MHD flow is ideal (i.e.  $\mathbf{R}_f = \mathbf{R}_m = \infty$ ) and there is no net flux of energy through the domain boundary, formulation (11) preserves energy, i.e.,

$$\partial_t \mathcal{E}_h = \partial_t \mathcal{K}_h + \partial_t \mathcal{M}_h = 0.$$

#### 4. Fully-discrete weak formulation

If a conservative temporal discrete scheme, for example, the Crank-Nicolson scheme, is applied to the semi-discrete formulation (11) directly, we can obtain a fully-discrete formulation of same conservation and dissipation properties (except that the time derivative of energy is now expressed in a temporally discrete format). The proof is straightforward; it mimics the analyses in Section 3.2. Therefore, the rest of this paper is devoted to a decoupled formulation with computational efficiency in mind.

##### 4.1. Decoupled temporal discretization

Consider a set of time instants,

$$\left\{ t^0, t^{\frac{1}{2}}, t^1, t^{1+\frac{1}{2}}, t^2, \dots \right\},$$

where  $t^0 = 0$ ,  $\Delta t = t^k - t^{k-1} > 0$ ,  $k \in \{1, 2, 3, \dots\}$ , is a constant time interval, and  $t^{k-\frac{1}{2}} = \frac{t^{k-1} + t^k}{2}$ . We use a superscript to denote the evaluation of a variable at a particular time instant. For example,  $\mathbf{u}_h^k := \mathbf{u}_h(\mathbf{x}, t^k)$ .

A fully discrete, temporally decoupled formulation of (11) (hereinafter referred to as the decoupled formulation) is expressed as follows. Given  $\mathbf{f} \in [L^2(\Omega)]^3$ , natural boundary conditions  $\widehat{P} \in H^{1/2}(\Omega; \Gamma_{\widehat{P}})$ ,  $\widehat{\mathbf{u}} \in \mathcal{TH}(\text{curl}; \Omega, \Gamma_{\widehat{\mathbf{u}}})$ ,  $\widehat{\mathbf{E}} \in \mathcal{TH}(\text{curl}; \Omega, \Gamma_{\widehat{\mathbf{E}}})$ , and initial conditions  $(\mathbf{u}_h^0, \boldsymbol{\omega}_h^0, \mathbf{H}_h^{\frac{1}{2}}) \in D(\Omega) \times C(\Omega) \times C(\Omega)$ , for time step index  $k = 1, 2, 3, \dots$  successively,

**(step 1)** seek  $(\mathbf{u}_h^k, \boldsymbol{\omega}_h^k, P_h^{k-\frac{1}{2}}) \in D_{\widehat{\mathbf{u}}}(\Omega, \Gamma_{\widehat{\mathbf{u}}}) \times C_{\widehat{\mathbf{E}}}^\parallel(\Omega, \Gamma_{\widehat{\mathbf{E}}}) \times S(\Omega)$ , such that,  $\forall (\mathbf{v}_h, \mathbf{w}_h, q_h) \in D_0(\Omega, \Gamma_{\widehat{\mathbf{u}}}) \times C_0^\parallel(\Omega, \Gamma_{\widehat{\mathbf{E}}}) \times S(\Omega)$ ,

$$(22a) \quad \left\langle \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t}, \mathbf{v}_h \right\rangle_\Omega + a \left( \frac{\boldsymbol{\omega}_h^{k-1} + \boldsymbol{\omega}_h^k}{2}, \frac{\mathbf{u}_h^{k-1} + \mathbf{u}_h^k}{2}, \mathbf{v}_h \right) + \mathbf{R}_f^{-1} \left\langle \nabla \times \frac{\boldsymbol{\omega}_h^{k-1} + \boldsymbol{\omega}_h^k}{2}, \mathbf{v}_h \right\rangle_\Omega - c a \left( \nabla \times \mathbf{H}_h^{k-\frac{1}{2}}, \mathbf{H}_h^{k-\frac{1}{2}}, \mathbf{v}_h \right) - \left\langle P_h^{k-\frac{1}{2}}, \nabla \cdot \mathbf{v}_h \right\rangle_\Omega = \left\langle \mathbf{f}^{k-\frac{1}{2}}, \mathbf{v}_h \right\rangle_\Omega - \left\langle \widehat{P}^{k-\frac{1}{2}}, \mathcal{T} \mathbf{v}_h \right\rangle_{\Gamma_{\widehat{P}}},$$

$$(22b) \quad - \langle \mathbf{u}_h^k, \nabla \times \mathbf{w}_h \rangle_\Omega + \langle \boldsymbol{\omega}_h^k, \mathbf{w}_h \rangle_\Omega = - \langle \widehat{\mathbf{u}}^k, \mathcal{T}_\parallel \mathbf{w}_h \rangle_{\Gamma_{\widehat{\mathbf{u}}}},$$

$$(22c) \quad \langle \nabla \cdot \mathbf{u}_h^k, q_h \rangle_\Omega = 0.$$

(step 2) seek  $\mathbf{H}_h^{k+\frac{1}{2}} \in C_{\widehat{\mathbf{H}}}^{\parallel}(\Omega, \Gamma_{\widehat{\mathbf{H}}})$ , such that,  $\forall \mathbf{b}_h \in C_0^{\parallel}(\Omega, \Gamma_{\widehat{\mathbf{H}}})$ ,

$$(23) \quad \left\langle \frac{\mathbf{H}_h^{k+\frac{1}{2}} - \mathbf{H}_h^{k-\frac{1}{2}}}{\Delta t}, \mathbf{b}_h \right\rangle_{\Omega} + \mathbf{R}_m^{-1} \left\langle \nabla \times \frac{\mathbf{H}_h^{k-\frac{1}{2}} + \mathbf{H}_h^{k+\frac{1}{2}}}{2}, \nabla \times \mathbf{b}_h \right\rangle_{\Omega} - a \left( \mathbf{u}_h^k, \frac{\mathbf{H}_h^{k-\frac{1}{2}} + \mathbf{H}_h^{k+\frac{1}{2}}}{2}, \nabla \times \mathbf{b}_h \right) = \left\langle \widehat{\mathbf{E}}^k, \mathcal{T}_{\parallel} \mathbf{b}_h \right\rangle_{\Gamma_{\widehat{\mathbf{E}}}}.$$

Note that the decoupled formulation takes  $\mathbf{H}_h^{\frac{1}{2}}$ , instead of  $\mathbf{H}_h^0$ , as an initial condition. Thus, to initiate the iterations, we need to compute  $\mathbf{H}_h^{\frac{1}{2}}$  by applying, for example, an explicit Euler scheme or the Crank-Nicolson scheme as discussed in Section 4 to (11) at the half time step from  $t^0$  to  $t^{\frac{1}{2}}$ . An illustration of the overall temporal scheme for the decoupled formulation is show in Fig. 1.

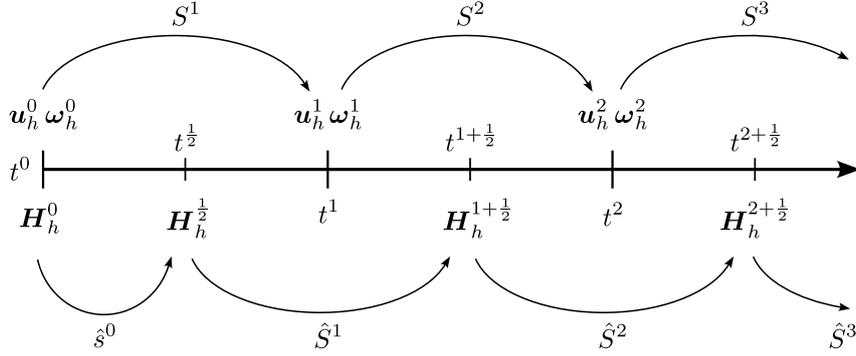


Figure 1: An illustration of the leapfrog temporal scheme for the decoupled formulation. The time step sequence is  $\hat{s}^0 \rightarrow S^1 \rightarrow \hat{S}^1 \rightarrow S^2 \rightarrow \hat{S}^2 \rightarrow \dots$ . Steps  $S^1, S^2, S^3, \dots$  refer to the step 1, see (22), of the decoupled formulation, and steps  $\hat{S}^1, \hat{S}^2, \hat{S}^3, \dots$  refer to the step 2, see (23), of the decoupled formulation. The pre-step  $\hat{s}^0$  computes  $\mathbf{H}_h^{\frac{1}{2}}$ . A similar scheme has been used in a dual-field discretization of incompressible Navier-Stokes equations, see [23, Fig. 1].

This leapfrog-type temporal scheme is analogous to the symplectic Störmer–Verlet leapfrog integrator [24] in Hamiltonian mechanics. Seeing that the decoupled formulation has linearized the Maxwell part, see (23), it does not fully linearize the Navier-Stokes part. More specifically, the second term of (22a), namely, the fluid convection term, is yet left nonlinear. Thus, a nonlinear system solver, for example, the Picard method or the Newton-Raphson method, still needs to be employed.

#### 4.2. Conservation properties

Clearly, the formulation (22) strongly preserves conservation of mass, i.e.,  $\nabla \cdot \mathbf{u}_h^k = 0$ , and conservation of charge, i.e.,  $\nabla \cdot \mathbf{j}_h^{k+\frac{1}{2}} = 0$ , pointwise, and weakly preserves Gauss’s law for magnetism, for same reasons as in Section 3.2.1, Section 3.2.2 and Section 3.2.3, respectively; the temporal discretization scheme maintains these conservation properties at the fully discrete level.

For energy conservation, it has been shown that, for the semi-discrete formulation, a key to derive it is the canceling of the two nonlinear terms, see (19) - (21). We now study whether a similar canceling happens for the decoupled formulation. And, to simplify the notations, we use a periodic domain, i.e., whose  $\partial\Omega = \emptyset$  or vanishing boundary conditions, i.e.  $\widehat{P} = 0$  and  $\widehat{\mathbf{u}} = \widehat{\mathbf{E}} = \mathbf{0}$  on  $\partial\Omega \times (0, T]$ . And throughout the paper, we use the midpoint rule, i.e., for example,

$$\mathbf{u}_h^{k-\frac{1}{2}} := \frac{\mathbf{u}_h^{k-1} + \mathbf{u}_h^k}{2}.$$

If we replace  $\mathbf{v}_h$  in (22a) with  $\mathbf{u}_h^{k-\frac{1}{2}}$ , by performing similar analyses as those in Section 3.2.4, we can find that

$$\left\langle \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t}, \frac{\mathbf{u}_h^{k-1} + \mathbf{u}_h^k}{2} \right\rangle_{\Omega} + \mathbf{R}_f^{-1} \left\langle \boldsymbol{\omega}_h^{k-\frac{1}{2}}, \boldsymbol{\omega}_h^{k-\frac{1}{2}} \right\rangle_{\Omega} - c a \left( \mathbf{j}_h^{k-\frac{1}{2}}, \mathbf{H}_h^{k-\frac{1}{2}}, \mathbf{u}_h^{k-\frac{1}{2}} \right) = \left\langle \mathbf{f}^{k-\frac{1}{2}}, \mathbf{u}_h^{k-\frac{1}{2}} \right\rangle_{\Omega},$$

where  $\mathbf{j}_h^{k-\frac{1}{2}} = \nabla \times \mathbf{H}_h^{k-\frac{1}{2}}$ . This leads to, if  $\mathcal{K}^k := \frac{1}{2} \langle \mathbf{u}^k, \mathbf{u}_h^k \rangle_\Omega$  denotes the discrete kinetic energy,

$$(24) \quad \frac{\mathcal{K}_h^k - \mathcal{K}_h^{k-1}}{\Delta t} = \mathcal{F}^{k-\frac{1}{2}} - \mathbf{R}_f^{-1} \mathcal{S}_h^{k-\frac{1}{2}} + c \mathcal{A}_h^{k-\frac{1}{2}},$$

where  $\mathcal{F}^{k-\frac{1}{2}} := \langle \mathbf{f}^{k-\frac{1}{2}}, \mathbf{u}_h^{k-\frac{1}{2}} \rangle_\Omega$ ,  $\mathcal{S}_h^{k-\frac{1}{2}} := \langle \boldsymbol{\omega}_h^{k-\frac{1}{2}}, \boldsymbol{\omega}_h^{k-\frac{1}{2}} \rangle_\Omega$  and  $\mathcal{A}_h^{k-\frac{1}{2}} := a \left( \mathbf{j}_h^{k-\frac{1}{2}}, \mathbf{H}_h^{k-\frac{1}{2}}, \mathbf{u}_h^{k-\frac{1}{2}} \right)$ . Similarly, by replacing  $\mathbf{b}_h$  in (23) with  $\mathbf{H}_h^k$ , we can obtain

$$(25) \quad \frac{\mathcal{M}_h^{k+\frac{1}{2}} - \mathcal{M}_h^{k-\frac{1}{2}}}{\Delta t} = -c \mathbf{R}_m^{-1} \mathcal{J}_h^k - c \mathcal{A}_h^k,$$

where  $\mathcal{M}_h^{k+\frac{1}{2}} = \frac{c}{2} \langle \mathbf{H}^{k+\frac{1}{2}}, \mathbf{H}^{k+\frac{1}{2}} \rangle_\Omega$  denotes the discrete magnetic energy, and  $\mathcal{J}_h^k := \langle \mathbf{j}_h^k, \mathbf{j}_h^k \rangle_\Omega$ . Repeating this analysis for the time step indexed  $(k-1)$  gives

$$(26) \quad \frac{\mathcal{M}_h^{k-\frac{1}{2}} - \mathcal{M}_h^{k-\frac{3}{2}}}{\Delta t} = -c \mathbf{R}_m^{-1} \mathcal{J}_h^{k-1} - c \mathcal{A}_h^{k-1}.$$

(25) and (26) together shows

$$(27) \quad \frac{\tilde{\mathcal{M}}_h^k - \tilde{\mathcal{M}}_h^{k-1}}{\Delta t} = -c \mathbf{R}_m^{-1} \tilde{\mathcal{J}}_h^{k-\frac{1}{2}} - c \tilde{\mathcal{A}}_h^{k-\frac{1}{2}},$$

where we have used the tilde sign to denote that it is an average of two integral quantities, i.e., for example,

$$(28) \quad \tilde{\mathcal{M}}_h^k := \frac{\mathcal{M}_h^{k-\frac{1}{2}} + \mathcal{M}_h^{k+\frac{1}{2}}}{2}$$

which differs from  $\mathcal{M}_h^k = \frac{c}{2} \langle \mathbf{H}^k, \mathbf{H}^k \rangle_\Omega$ . With (24) and (27), we can conclude that the decoupled formulation dissipates a discrete energy defined as

$$(29) \quad \tilde{\mathcal{E}}_h^k := \mathcal{K}_h^k + \tilde{\mathcal{M}}_h^k$$

with a discrete rate

$$(30) \quad \frac{\tilde{\mathcal{E}}_h^k - \tilde{\mathcal{E}}_h^{k-1}}{\Delta t} = \mathcal{F}^{k-\frac{1}{2}} - \mathbf{R}_f^{-1} \mathcal{S}_h^{k-\frac{1}{2}} - c \mathbf{R}_m^{-1} \tilde{\mathcal{J}}_h^{k-\frac{1}{2}} + c \left( \mathcal{A}_h^{k-\frac{1}{2}} - \tilde{\mathcal{A}}_h^{k-\frac{1}{2}} \right).$$

Furthermore, when  $\mathbf{f} = \mathbf{0}$  and the flow is ideal, we have

$$\frac{\tilde{\mathcal{E}}_h^k - \tilde{\mathcal{E}}_h^{k-1}}{\Delta t} = c \left( \mathcal{A}_h^{k-\frac{1}{2}} - \tilde{\mathcal{A}}_h^{k-\frac{1}{2}} \right).$$

As generally  $\mathcal{A}_h^{k-\frac{1}{2}} - \tilde{\mathcal{A}}_h^{k-\frac{1}{2}} \neq 0$ , the decoupled formulation does not conserve the discrete energy (29). Exact preservation of the discrete energy would require a fully implicit scheme coupling the fluid and Maxwell part. This can be achieved for instance using the midpoint rule as the Hamiltonian is quadratic in the state variable and the non-linearity is confined in the Poisson bracket. This choice would make the scheme computationally more demanding and is therefore non pursued in this paper.

In short, the decoupled formulation breaks the nonlinear MHD formulation into two evolution equations representing the Navier-Stokes part and the Maxwell part, respectively, partially linearizes the former and fully linearizes the latter. It keeps strong conservation of mass, strong conservation of charge and weak conservation of Gauss's law for magnetism while sacrificing conservation of energy.

## 5. Numerical tests

In this section, we present some results of numerical tests using the proposed method. The test cases are manufactured solution tests, the Orszag-Tang vortex flow, magnetic lid-driven cavity flow. We use the mimetic spectral elements [25, 26] for all tests. The degree of the basis functions is denoted by  $N$ . Any other set of finite elements that also satisfies the discrete de Rham Hilbert complex (9) and the  $L^2$ -regularity for the triple products, see Section 2.2 works for the proposed method. A combination of Lagrange (continuous Galerkin) elements of degree  $N$ , the first kind Nédélec  $H(\text{curl})$ -conforming elements of degree  $N$  [27], Raviart-Thomas elements of degree  $N$  [28], and discontinuous Galerkin elements of degree  $(N-1)$  [29] is a classic candidate [22, 30].

### 5.1. Manufactured solution tests

Two sets of manufactured solutions are employed for convergence tests and conservation and dissipation tests, respectively.

#### 5.1.1. Convergence tests

Suppose three-dimensional manufactured solutions

$$\begin{aligned}\mathbf{u} &= [\cos(x) \sin(y) \sin(z)e^t \quad \sin(x) \cos(y) \sin(z)e^t \quad -2 \sin(x) \sin(y) \cos(z)e^t]^\top, \\ \mathbf{E} &= [\cos(x) \sin(y) \sin(z/2)e^t \quad \sin(x/2) \cos(y) \sin(z)e^t \quad -\sin(x) \sin(y) \cos(z)e^t]^\top, \\ P &= \cos(x) \cos(y) \cos(z)e^{-t} \quad \text{and} \quad \mathbf{H}^0 = \mathbf{B}^0 = \mathbf{0},\end{aligned}$$

solve (2). Manufactured solutions of remaining variables then can be obtained from  $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ ,  $\mathbf{H} = \mathbf{B} = \int_0^t (-\nabla \times \mathbf{E}) dt$ ,  $\mathbf{j} = \nabla \times \mathbf{H}$ . Given  $\mathbf{R}_f$  and  $\mathbf{c}$ , we can get the analytical expression of  $\mathbf{f}$  through (2a). And if  $\mathbf{R}_m$  is known, we can find an extra source, denoted by  $\mathbf{e}$ , needed to balance relation

$$\mathbf{e} = \mathbf{R}_m^{-1} \mathbf{j} - (\mathbf{E} + \mathbf{u} \times \mathbf{B}).$$

This will lead to an extra term,  $\langle \mathbf{e}, \nabla \times \mathbf{b}_h \rangle_\Omega$ , in the right hand sides of, for example, (11d) and (23).

For the convergence tests, parameters are set to  $\mathbf{R}_f = 1$ ,  $\mathbf{R}_m = 1$ ,  $\mathbf{c} = 1$ . And the domain is selected to be  $\Omega = (x, y, z) \in [0, 2\pi]^3$  whose boundary is partitioned, see (4) and (5), such that

$$\begin{aligned}\Gamma_{\hat{p}} &= \Gamma_x^- \cup \Gamma_y^+ \cup \Gamma_z^+, \\ \Gamma_{\hat{u}} &= \Gamma_x^+ \cup \Gamma_y^- \cup \Gamma_z^+, \\ \Gamma_{\hat{E}} &= \Gamma_x^+ \cup \Gamma_y^+ \cup \Gamma_z^-, \end{aligned}$$

where, for example,  $\Gamma_x^-$  represents the face  $(x, y, z) \in 0 \times (0, 2\pi) \times (0, 2\pi)$  and  $\Gamma_x^+$  represents the face  $(x, y, z) \in 2\pi \times (0, 2\pi) \times (0, 2\pi)$ . Let  $K$  and  $\kappa$  be two positive integers. We generate a mesh of  $K^3$  uniform cubic elements in  $\Omega$ , and  $h = \frac{2\pi}{K}$  is the size, i.e. the edge length, of elements. On this mesh, given initial conditions, boundary conditions,  $\mathbf{f}$  and  $\mathbf{e}$  according to the manufactured solutions, the decoupled formulation, (22) and (23), is solved with a constant time step interval  $\Delta t = \frac{1}{\kappa}$ . Errors between simulation and manufactured solutions are measured at  $t = 1$ , i.e., when time step index  $k = \kappa$ . Results in Fig. 2 show that the decoupled temporal discretization has a second-order accuracy for both evolution equations. And in Fig. 3 optimal spatial convergence rates are observed for all variables. Results support that the formulation weakly preserves Gauss' law for magnetism are presented in Fig. 4.

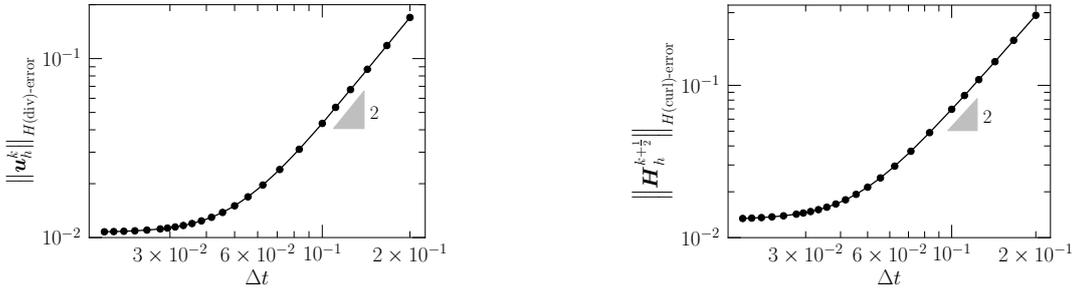


Figure 2: Results of  $\|\mathbf{u}_h^k\|_{H(\text{div})\text{-error}}$  and  $\|\mathbf{H}_h^{k+\frac{1}{2}}\|_{H(\text{curl})\text{-error}}$  for the temporal convergence tests at  $N = 3$ ,  $K = 16$ ,  $\Delta t \in \{\frac{1}{5}, \frac{1}{6}, \dots, \frac{1}{10}, \frac{1}{12}, \dots, \frac{1}{36}, \frac{1}{40}, \dots, \frac{1}{56}\}$ .

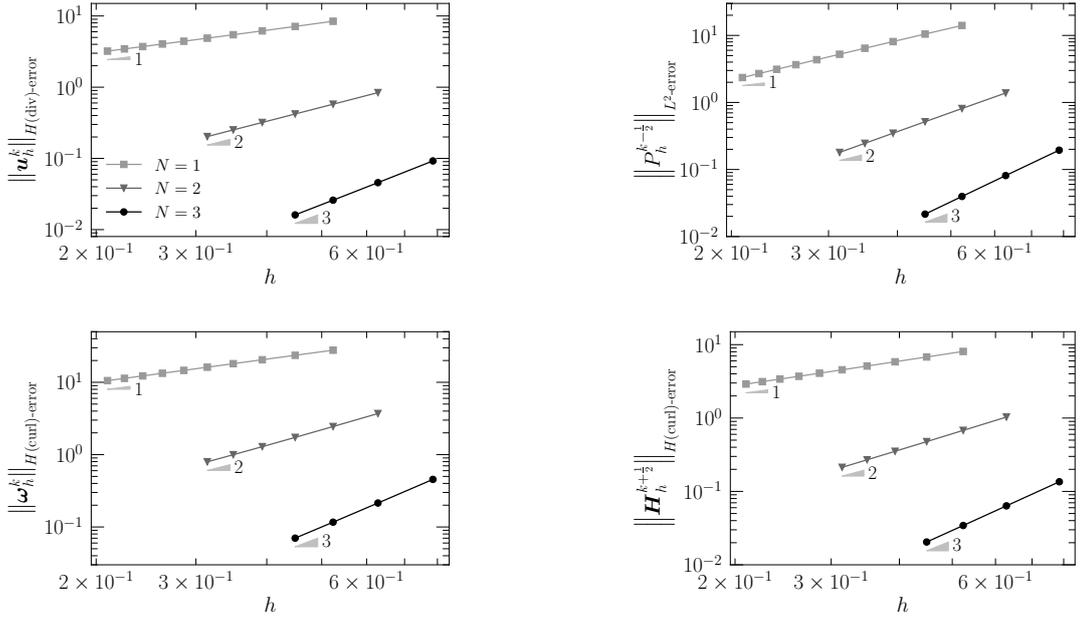


Figure 3: Results of  $\|\mathbf{u}_h^k\|_{H(\text{div})\text{-error}}$ ,  $\|P_h^{k-\frac{1}{2}}\|_{L^2\text{-error}}$ ,  $\|\boldsymbol{\omega}_h^k\|_{H(\text{curl})\text{-error}}$  and  $\|\mathbf{H}_h^{k+\frac{1}{2}}\|_{H(\text{curl})\text{-error}}$  for the spatial convergence tests. We use  $K \in \{12, 14, \dots, 32\}$  for  $N = 1$ ,  $K \in \{10, 12, \dots, 20\}$  for  $N = 2$  and  $K \in \{8, 10, \dots, 14\}$  for  $N = 3$ . And  $\Delta t = \frac{1}{100}$  is employed to avoid pollution of the temporal discretization error, see Fig. 2.

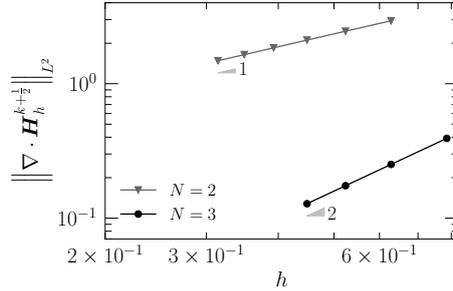


Figure 4: Results of  $\|\nabla \cdot \mathbf{H}_h^{k+\frac{1}{2}}\|_{L^2}$  for the spatial convergence tests. We use  $K \in \{10, 12, \dots, 20\}$  for  $N = 2$ ,  $K \in \{8, 10, \dots, 14\}$  for  $N = 3$ , and  $\Delta t = \frac{1}{100}$ . In this work, the employed finite dimensional space for  $\mathbf{H}_h \in C(\Omega)$  is the mimetic spectral space which is a polynomial space  $\mathcal{P}^{N-1, N, N} \times \mathcal{P}^{N, N-1, N} \times \mathcal{P}^{N, N, N-1}$  on orthogonal meshes. So we can compute exact divergence of  $\mathbf{H}_h$  element-wise, and, when  $N = 1$ ,  $\nabla \cdot \mathbf{H}_h \equiv 0$ . For the present results,  $\|\nabla \cdot \mathbf{H}_h^{k+\frac{1}{2}}\|_{L^2} := \sqrt{\sum_i H_i^2}$  where  $H_i$  is the  $L^2$ -norm of  $\nabla \cdot \mathbf{H}_h^{k+\frac{1}{2}}$  in the  $i$ th element.

### 5.1.2. Conservation and dissipation tests

We test conservation and dissipation properties of the decoupled formulation in a spatial domain  $\Omega = (x, y, z) \in [0, 1]^3$  using manufactured solutions. The initial conditions, taken from [5, Section 4.2], are

$$\mathbf{u}^0 = \begin{bmatrix} -\sin\left(\pi\left(x - \frac{1}{2}\right)\right) \cos\left(\pi\left(y - \frac{1}{2}\right)\right) z(z-1) \\ \cos\left(\pi\left(x - \frac{1}{2}\right)\right) \sin\left(\pi\left(y - \frac{1}{2}\right)\right) z(z-1) \\ 0 \end{bmatrix}$$

and

$$\mathbf{H}^0 = [-\sin(\pi x) \cos(\pi y) \quad \cos(\pi x) \sin(\pi y) \quad 0]^\top$$

which possess the initial energy of  $\mathcal{E}^0 \approx 0.25833$  if  $c = 1$ . The boundary conditions are

$$\hat{\mathbf{u}} = \hat{\mathbf{E}} = \mathbf{0} \quad \text{and} \quad \hat{P} = 0 \quad \text{on} \quad \partial\Omega \times (0, T];$$

we use natural boundary conditions on the whole boundary and set them to zero for these tests. And the external body force is zero;  $\mathbf{f} = \mathbf{0}$ . A mesh of uniform cubic elements and a constant time step interval is employed. The element edge length is denoted by  $h$ . The decoupled formulation, (22) and (23), is solved with different  $R_f$  and  $R_m$ . Results presented in Fig. 5 show that mass conservation is always satisfied strongly. As for the energy, a dissipation rate of a machine precision error is obtained.

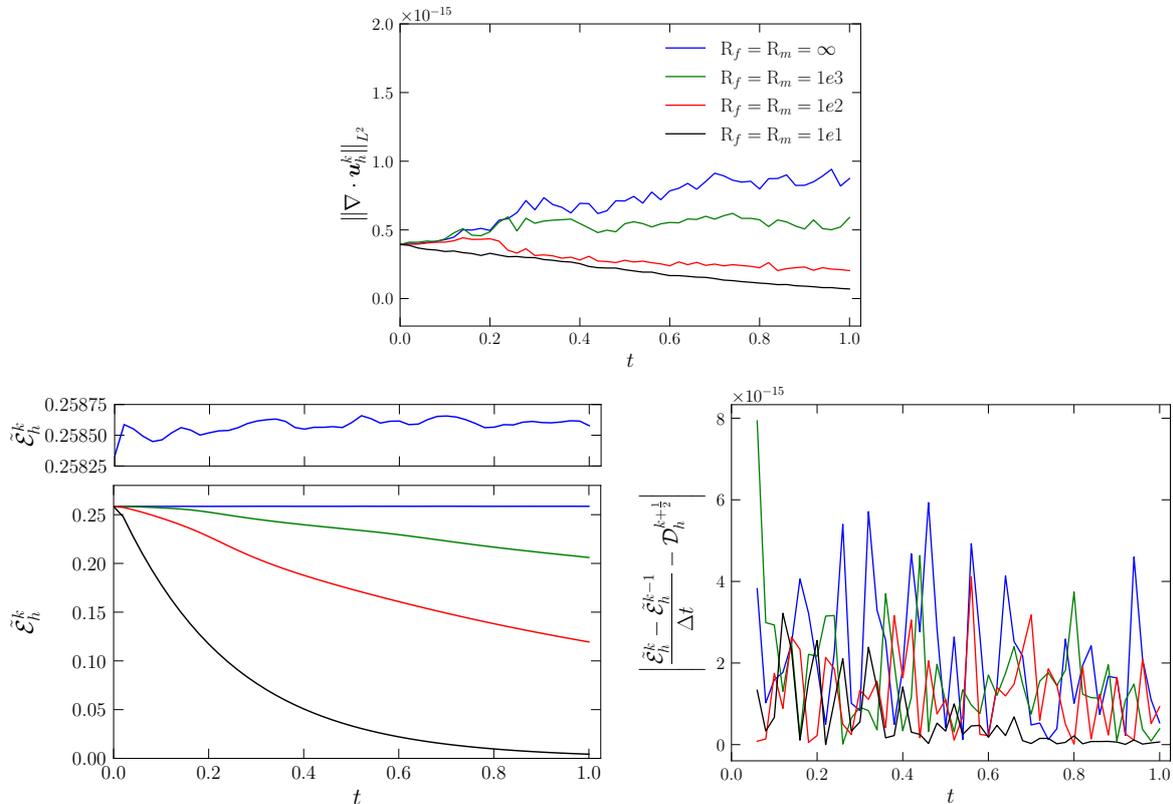


Figure 5: Some results of  $\|\nabla \cdot \mathbf{u}_h^k\|_{L^2}$ ,  $\tilde{\mathcal{E}}_h^k$ ,  $\left| \frac{\tilde{\mathcal{E}}_h^k - \tilde{\mathcal{E}}_h^{k-1}}{\Delta t} - \mathcal{D}_h^{k+\frac{1}{2}} \right|$  over time for the conservation and dissipation tests at  $c = 1$ ,  $N = 2$ ,  $h = \frac{1}{8}$  and  $\Delta t = \frac{1}{50}$ , where  $\mathcal{D}_h^{k-\frac{1}{2}} := -R_f^{-1} \mathcal{S}_h^{k-\frac{1}{2}} - c R_m^{-1} \tilde{\mathcal{J}}_h^{k-\frac{1}{2}} + c \left( \mathcal{A}_h^{k-\frac{1}{2}} - \tilde{\mathcal{A}}_h^{k-\frac{1}{2}} \right)$ , cf. (30).

As can be seen from Figure 5 the total energy in the absence of dissipative terms, viscosity and resistivity, is not constant although it does not deviate too far from its initial value. This is due to the fact that  $\mathcal{A}_h^{k-\frac{1}{2}} - \tilde{\mathcal{A}}_h^{k-\frac{1}{2}} \neq 0$  as was shown in Section 4. The variations in total energy, however, remain bounded as is expected from a symplectic integrator.

## 5.2. Orszag-Tang vortex

The Orszag-Tang vortex initially studied in [31] is a well-known two-dimensional incompressible MHD test case. In a periodic square, the Orszag-Tang vortex gradually develops narrow corridors (that will eventually become singularities) of extreme current density where the magnetic field changes its sign suddenly [32], which makes this test case a challenging one.

Giving a stream function  $\psi$  and a magnetic potential  $A$ , also see [32, Section 5.3],

$$\psi = 2 \sin(y) - 2 \cos(x), \quad A = \cos(2y) - 2 \cos(x),$$

the initial conditions are  $\mathbf{u}^0 = \nabla \times \psi$ , and  $\mathbf{H}^0 = \nabla \times A$ . The fully, orthogonally periodic square is  $\Omega = (x, y) \in [0, 2\pi]^2$  and  $\mathbf{f} = \mathbf{0}$ . On a mesh of  $K^2$  uniform cubic elements, we solve the two-dimensional version of the decoupled formulation with a time step interval  $\Delta t = \frac{1}{200}$  and parameters are  $R_f = 100$ ,  $R_m = 100$  and  $c = 1$ . Some snapshots of  $\mathbf{j}_h^{k+\frac{1}{2}} = \nabla \times \mathbf{H}_h^{k+\frac{1}{2}}$  are shown in Fig. 6, and a good match

to reference results in [32] is observed. Element-wise  $\log_{10} \left( \left| \nabla \cdot \mathbf{H}_h^{k+\frac{1}{2}} \right| \right)$  at  $t^k = t = 1$  for different combinations of  $N$  and  $K$  are shown in Fig. 7 where we can see that the convergence of weak conservation of Gauss's law for magnetism under  $ph$ -refinement.

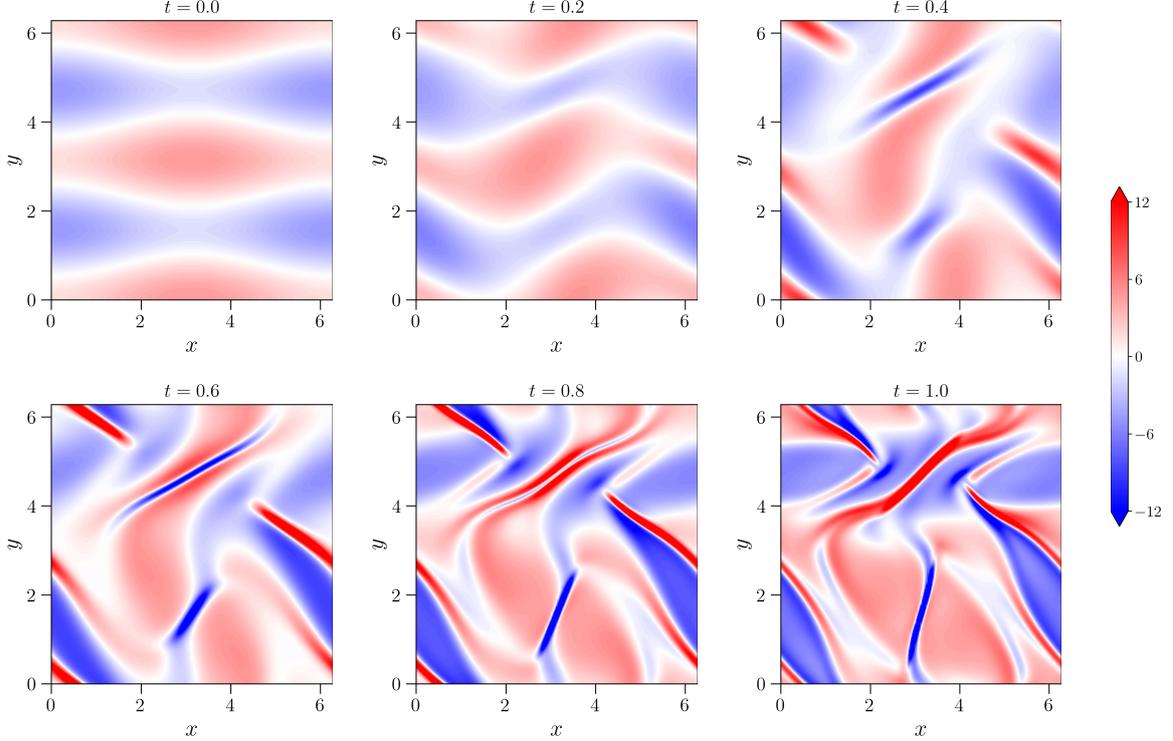


Figure 6:  $J_h^{k+\frac{1}{2}}$  at  $t^k = t \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$  for the Orszag-Tang vortex test using  $N = 4, K = 48, \Delta t = \frac{1}{200}$ .

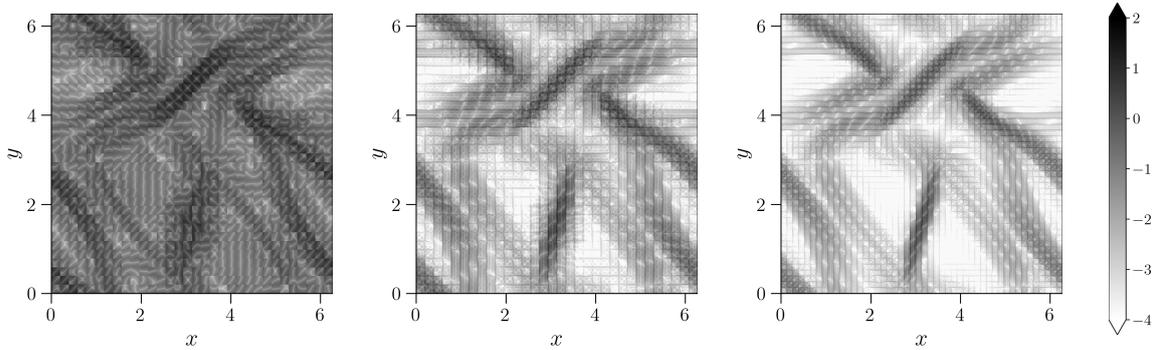


Figure 7: Element-wise  $\log_{10} \left( \left| \nabla \cdot \mathbf{H}_h^{k+\frac{1}{2}} \right| \right)$  at  $t^k = t = 1$  for the Orszag-Tang vortex test using  $\Delta t = \frac{1}{200}$ . Left:  $N = 2, K = 32$ . Middle:  $N = 4, K = 32$ . Right:  $N = 4, K = 48$ .

### 5.3. Lid-driven cavity

The two-dimensional lid-driven cavity is a domain  $\Omega = (x, y) \in [0, 1]^2$  with four infinite no-slip solid walls, and there is no external body force;  $\mathbf{f} = \mathbf{0}$ . Initially, the fluid is at rest, i.e.,  $\mathbf{u}^0 = \mathbf{0}$ , and is immersed in a magnetic field  $\mathbf{H}^0 = [0 \ 1]^\top$ . At  $t > 0$ , the top wall, namely the lid, moves right with a constant speed 1, i.e.  $\mathbf{u}_{\text{lid}} = [1 \ 0]^\top$ , and drives the fluid due to viscous entrainment. Over the whole boundary, the tangential electric field is zero all the time. In summary, we have boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = \hat{u} = 0, \quad \mathbf{E} \times \mathbf{n} = \hat{\mathbf{E}} = 0 \quad \text{on } \partial\Omega \times (0, T],$$

and

$$\mathbf{u} \times \mathbf{n} = \widehat{u}_{\parallel} = \begin{cases} 1 & \text{on } \Gamma_y^+ \times (0, T] \\ 0 & \text{else} \end{cases},$$

where  $\Gamma_y^+$  is the face  $(x, y) \in (0, 1) \times 1$ .

To simulate the lid-driven cavity, an orthogonal spatial mesh of  $32^2$  elements that are locally refined near the boundary, see Fig. 8a, is used. The time step interval is  $\Delta t = \frac{1}{1000}$ , Reynolds numbers are  $R_f = R_m = 400$ , the coupling number is  $c = R_m^{-1}$  (such that  $s = cR_m = 1$ ) and the polynomial degree is  $N = 3$ . The two-dimensional version of the decoupled formulation is solved until steady state is reached. The criterion of the steady state is

$$\frac{1}{\Delta t} \max \left( \|\mathbf{u}_h^k - \mathbf{u}_h^{k-1}\|_{L^2}, \|\mathbf{H}_h^{k+\frac{1}{2}} - \mathbf{H}_h^{k-\frac{1}{2}}\|_{L^2} \right) < 10^{-5}.$$

The flow reaches its steady state at  $t = 82.049$ . Some results are presented in Fig. 8. They have a good agreement to the reference results taken from [3]. In Table 1 and Table 2, solutions along two centerlines are provided as quantitative benchmarks.

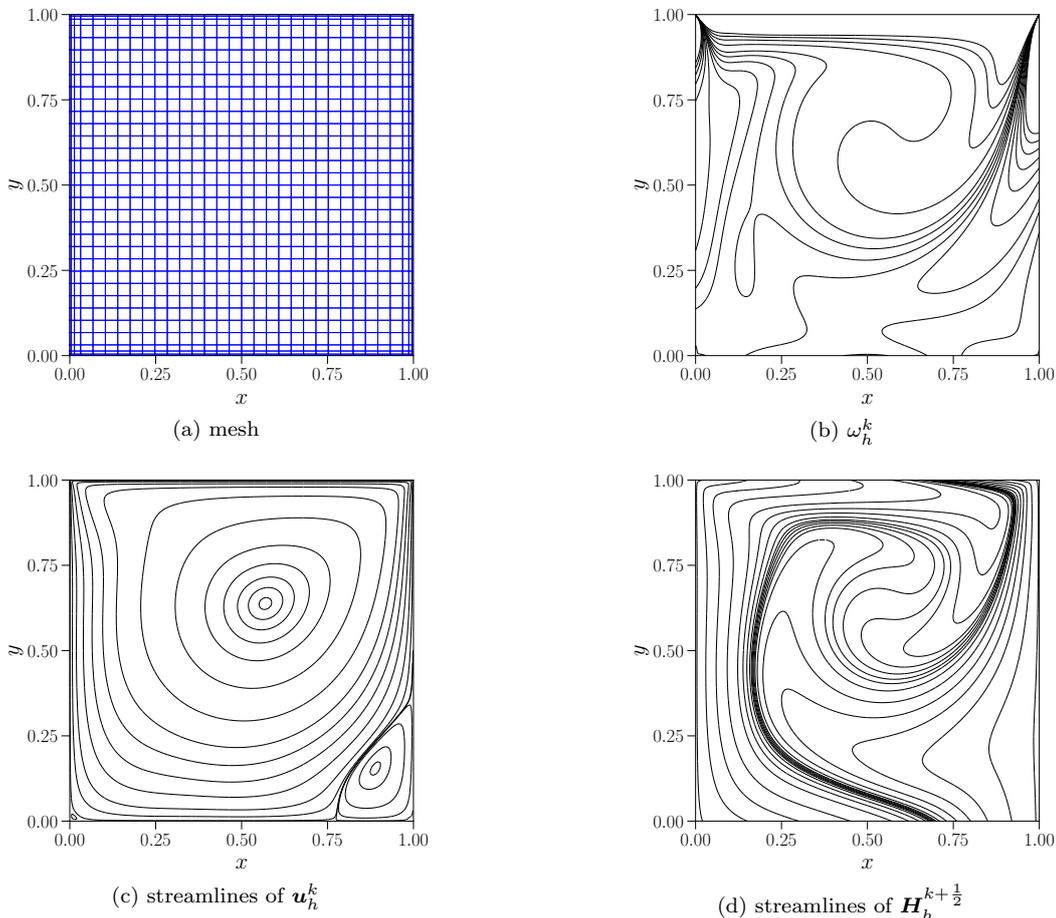


Figure 8: The mesh and some results at the steady state for the lid-driven cavity test. (a): The mesh. (b): Vorticity  $\omega_h^k$  for contour lines in  $\{-5, -4, -3, -2, -1, -0.5, 0, 0.5, 1, 2, 3, 4, 5, 6, 7\}$ . (c): Streamlines of  $\mathbf{u}_h^k$ . (d): Streamlines of  $\mathbf{H}_h^{k+\frac{1}{2}}$ . Note that, for diagram (d), a direct numerical integral over a grid of discrete values of  $\mathbf{H}_h^{k+\frac{1}{2}}$  is applied. Since  $\mathbf{H}_h^{k+\frac{1}{2}}$  is not strongly divergence free, there does not exist a stream function  $\psi$  such that  $\mathbf{H}_h^{k+\frac{1}{2}} = \nabla \times \psi$ . However, as we have shown that  $\mathbf{H}_h^{k+\frac{1}{2}}$  is weakly divergence free for the proposed method, see Fig. 4 and Fig. 7, the error in the stream function by this approach is limited. Stream functions of contour lines in (c) and (d) are given in See Appendix A . A good match to reference results in [3, Fig. 6, Fig. 7] is obtained.

Table 1: Solution along the  $x$ -direction centerline, i.e.,  $(x, y) = [0, 1] \times 0.5$ , at the steady state for the lid-driven cavity test.  $\mathbf{u}_h^k = [u_h^k \ v_h^k]^\top$  and  $\mathbf{H}_h^{k+\frac{1}{2}} = [H_{x;h}^{k+\frac{1}{2}} \ H_{y;h}^{k+\frac{1}{2}}]^\top$ .

| $x$  | $u_h^k$  | $v_h^k$  | $\omega_h^k$ | $p_h^{k-\frac{1}{2}}$ | $H_{x;h}^{k+\frac{1}{2}}$ | $H_{y;h}^{k+\frac{1}{2}}$ |
|------|----------|----------|--------------|-----------------------|---------------------------|---------------------------|
| 0    | 0        | 0        | 2.88484      | -0.01992              | 0                         | 4.98952                   |
| 0.05 | -0.00152 | 0.11308  | 1.63098      | -0.01879              | 0.09510                   | 4.90996                   |
| 0.1  | -0.01043 | 0.17267  | 0.63099      | -0.01028              | 0.15636                   | 4.15657                   |
| 0.15 | -0.02841 | 0.19891  | 0.09245      | 0.00211               | 0.14927                   | 2.49169                   |
| 0.25 | -0.07561 | 0.22147  | -0.48799     | -0.00019              | 0.13626                   | -0.07002                  |
| 0.5  | -0.14277 | 0.05402  | -2.19175     | -0.03870              | 0.05387                   | -0.07419                  |
| 0.75 | -0.20138 | -0.25127 | -2.33726     | -0.02413              | 0.41737                   | 0.35175                   |
| 0.85 | -0.16772 | -0.33732 | 0.53077      | -0.00351              | 0.73888                   | 1.37109                   |
| 0.9  | -0.09707 | -0.26638 | 2.68539      | 0.00303               | 0.56165                   | 1.55770                   |
| 0.95 | -0.02697 | -0.12420 | 3.14441      | 0.00396               | 0.27997                   | 1.49902                   |
| 1    | 0        | 0        | 1.72384      | 0.00160               | 0                         | 1.46331                   |

Table 2: Solution along the  $y$ -direction centerline, i.e.,  $(x, y) = 0.5 \times [0, 1]$ , at the steady state for the lid-driven cavity test.  $\mathbf{u}_h^k = [u_h^k \ v_h^k]^\top$  and  $\mathbf{H}_h^{k+\frac{1}{2}} = [H_{x;h}^{k+\frac{1}{2}} \ H_{y;h}^{k+\frac{1}{2}}]^\top$ .

| $y$  | $u_h^k$  | $v_h^k$  | $\omega_h^k$ | $p_h^{k-\frac{1}{2}}$ | $H_{x;h}^{k+\frac{1}{2}}$ | $H_{y;h}^{k+\frac{1}{2}}$ |
|------|----------|----------|--------------|-----------------------|---------------------------|---------------------------|
| 0    | 0        | 0        | 1.06180      | 0.01293               | -2.74879                  | 1                         |
| 0.05 | -0.03939 | -0.00056 | 0.60008      | 0.01526               | -2.40069                  | 0.80054                   |
| 0.1  | -0.07135 | -0.00329 | 0.58025      | 0.01911               | -1.65057                  | 0.60078                   |
| 0.15 | -0.11225 | -0.00676 | 0.70413      | 0.02128               | -0.73480                  | 0.39444                   |
| 0.25 | -0.21581 | -0.00329 | 0.35426      | 0.01513               | 0.52230                   | 0.06335                   |
| 0.5  | -0.14277 | 0.05402  | -2.19175     | -0.03870              | 0.05387                   | -0.07419                  |
| 0.75 | 0.12722  | 0.07170  | -1.81455     | -0.04325              | -0.11813                  | -0.04774                  |
| 0.85 | 0.22351  | 0.05999  | -1.41003     | -0.03233              | 0.33644                   | -0.08662                  |
| 0.9  | 0.28772  | 0.04373  | -2.18569     | -0.03120              | 1.71570                   | -0.08702                  |
| 0.95 | 0.47275  | 0.01801  | -6.41153     | -0.03569              | 3.05614                   | 0.20349                   |
| 1    | 1        | 0        | -15.51828    | -0.05632              | -5.29567                  | 1                         |

## 6. Conclusion

In this work, we present a decoupled structure-preserving discretization for the incompressible MHD equations with general boundary conditions. A spatially discrete formulation in a mixed finite element setting is firstly proposed. It preserves conservation of mass and conservation of charge strongly, preserves Gauss's law for magnetism weakly and preserves the correct energy dissipation rate. A leapfrog-type temporal integrator then is applied to the fluid part and the Maxwell part at two staggered time sequences, respectively, such that they are decoupled at the fully discrete level. Other invariants, for example, cross- and magnetic-helicity, are preserved by the incompressible MHD equations. Possible extensions of this work include the preservation of these quantities.

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## Appendix A. Stream functions

In Fig. 8c, the streamlines are for stream functions in

$$\{-0.093, -0.092, -0.09, -0.086, -0.078, -0.06, -0.035, -0.018, -0.01, -5\text{E}-3, -2\text{E}-3, -5\text{E}-4, -8\text{E}-5, -1\text{E}-6, 5\text{E}-8, 6\text{E}-6, 8\text{E}-5, 3\text{E}-4, 4.1\text{E}-4\}.$$

In Fig. 8d, the streamlines are for stream functions in

$$\{-0.99, -0.92, -0.85, -0.80, -0.76, -0.73, -0.71, -0.7, -0.695, -0.69, \\ -0.685, -0.68, -0.67, -0.65, -0.62, -0.57, -0.45, -0.3, -0.15, -0.02\}.$$

For both cases, the reference stream function is set to 0 at the bottom-left corner.

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