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# Constrained Euler Buckling: the von Karman approximation

Jiayu Wang<sup>a,\*</sup>, Stéphanie Deboeuf<sup>a</sup>, Arnaud Antkowiak<sup>a</sup>, Sébastien Neukirch<sup>a</sup>

<sup>a</sup> Sorbonne Université, CNRS, Institut Jean Le Rond d'Alembert, F-75005 Paris, France

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## Abstract

We consider the classical problem of the buckling of a planar elastica inside a rectangular cavity. We compute the equilibrium solutions analytically in the (von Karman) small deflection approximation. We list the different equilibrium states and their domain of validity in terms of the imposed horizontal  $\Delta$  and vertical  $H$  displacements. We compute the horizontal  $P$  and the vertical  $F$  applied forces and show how they increase and scale when the compaction ratio  $\sqrt{\Delta}/H$  is increased. Finally, we introduce an approximate response state, where the system adopts a periodic configuration with a noninteger number of repeated folds. This solution represents an average response of the system and brings information on its global behaviour.

*Keywords:* elastic rods, bifurcation diagram, contact, packing

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## 1. Introduction

The buckling of elastic beams is a classical subject, especially in the planar case (Euler, 1744), (Thomson and Tait, 1883, section 611, p. 148), (Goss, 2009), (Levien, 2009, Chap. 5). Analytical solutions are known for simple (Love, 1944; Bigoni, 2012) and less simple (Djordjorov et al., 2011) cases. The global buckling behavior is nevertheless often studied numerically, and the goal is to achieve a thorough understanding of the phase diagram of the problem (Domokos, 1994) which in some cases may be cluttered (Holmes et al., 1999, Fig. 7), (Domokos and Healey, 2005, Fig. 3), (Henderson and Neukirch, 2004, Fig. 16), (Coleman and Swigon, 2004, Fig. 4). The equilibrium solutions may be computed either by direct resolution of the equilibrium equations (boundary value problem – strong form approach), finite element method (weak form approach), or minimization of the elastic energy (Charronière et al., 2024).

The writhing and coiling of elastic rods in cavities has numerous applications. Examples comprise the helical buckling of tubings in the drilling industry (Lubinski and Althouse, 1962; Miller et al., 2015), the interaction of endoscopes and arteries in vascular surgery, the stuff box crimping in the textile industry (Hearle,

2014), DNA viral capsids (Vetter et al., 2014), spider threads in liquid drops (Elettro et al., 2016); see other examples in Judah and Givli (2024). However, the buckling of elastic beams in cavities brings the complication of inequality constraints (Villaggio, 1979) (arising from the contact condition) and their non-classical features in the bifurcation diagram (for example, corner points (Schulz and Pellegrino, 2000)). Their numerical treatment involves more sophisticated approaches, among which are linear and nonlinear complementary problems (Daviet et al., 2011), interior point methods (Wachter and Biegler, 2006), or barrier potentials (Manning and Bulman, 2005).

The present study considers the simpler case of a planar beam buckled in a rectangular cavity. To our knowledge, this problem was first addressed by Chateau and Nguyen (1991), then Adan et al. (1994) considered the case where the constraining surface has imperfections, Domokos et al. (1997) studied pinned-pinned boundary conditions, and Pocheau and Roman (2004) unveiled the presence of multiple solutions and their self-similar aspect, as well as tilted solutions (Roman and Pocheau, 1999, 2002). The effect of the thickness of the beam was studied by Chai (1998), while Tzokova (2020) combined Abaqus FEM and experiments on beams and plates, and Katz and Givli (2015) considered the case of springy walls, see also (Judah and Givli, 2024). Finally, the influence of shear deformations in the section of the beam has been studied by Bosten et al. (2023) where this contact problem is used as a benchmark scenario to

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\*Corresponding author

Email address: jiayu.wang@dalembert.upmc.fr (Jiayu Wang)

test finite element simulations.

Here we use what is generally called the von Karman approximation (Woinowsky-Krieger, 1950; Eringen, 1952; Thomas, 2025). In this approximation, the moment balance is linearized, and a von Kármán-type axial strain measure, first introduced for the buckling of elastic plates (von Kármán, 1907; Easley, 1964), is used. This model has been widely used in the literature and has shown its efficiency in computing approximate solutions for the equilibrium (Bazant and Cedolin, 2010, Section 1.9), (Neukirch et al., 2021) and dynamics (Lacarbonara and Yabuno, 2006; Pandey et al., 2014; Thomas et al., 2016) of elastic rods in the weakly nonlinear regime. It turns out that Kirchhoff himself introduced this approximated model in his book, see Kirchhoff (1876, Eq. (16), p. 441). We show in this paper that constrained Euler buckling can be studied and partially understood with this von-Karman kinematics approximation; see also (Chai, 1998; Judah and Givli, 2024).

### Paper contributions

In the case of small vertical ( $H$ ) and horizontal ( $\Delta$ ) displacements:

- We present an analytical study of the Euler buckling problem with contact constraints.
- We give closed-form formulas for the vertical  $F$  and horizontal  $P$  forces and the number of folds as functions of  $H$  and  $\Delta$ , and identify the scaling  $F H \sim P \Delta$ .
- We show that a reduced kinematic loading parameter  $\sqrt{\Delta}/H$  can be used to rationalize the bifurcation curves.
- We introduce a cellular model, where the number of folds is a noninteger, that provides an all-in-one view of the response of the system.

## 2. Problem setup

### 2.1. The planar Elastica

We consider an inextensible, unshearable, elastic beam buckled inside a planar, rectangular cavity. The beam has total length  $L$ , bending rigidity  $YI$ , and is horizontally clamped at both ends. The shape of the beam is studied parametrically as  $(x(s), y(s))$ , where  $s$  is the arc length along the beam. We introduce the angle  $\theta(s)$  between the tangent to the beam and the horizontal axis  $\mathbf{e}_x$ , see Figure 1. As the beam is considered inextensible and unshearable, we have

$$x'(s) = \cos \theta(s), \quad y'(s) = \sin \theta(s) \quad (1)$$

where  $(\cdot)' = d(\cdot)/ds$ . We consider clamped boundary conditions

$$x(0) = 0 \quad y(0) = 0 \quad \theta(0) = 0 \quad (2a)$$

$$x(L) = L(1 - \Delta) \quad y(L) = 0 \quad \theta(L) = 0 \quad (2b)$$

where  $\Delta$  is the (dimensionless) end-shortening, ranging from  $\Delta = 0$  at buckling to  $\Delta = 1$  when the two ends meet and the elastica adopts a Lemniscate-like shape (Goss, 2009). Additionally, as the beam is constrained to stay in a cavity of height  $H$ , we have the inequality constraint

$$\forall s, 0 \leq y(s) \leq H \quad (3)$$

The elastic deformation energy only involves the curvature  $\kappa(s)$ :

$$E_\kappa = \int_0^L \frac{1}{2} YI \kappa^2(s) ds \quad \text{where } \kappa(s) = \theta'(s) \quad (4)$$

This quadratic deformation energy is associated with the linear constitutive relation relating the internal moment  $m(s)$  and the curvature

$$m(s) = YI \kappa(s) \quad (5)$$

We look for equilibrium solutions of the system, that is stationary points of this energy under constraints (2) and (3). The equilibrium condition on the internal moment reads

$$m'(s) = n_x(s) \sin \theta(s) - n_y(s) \cos \theta(s) \quad (6)$$

As we study the frictionless case, the horizontal component  $n_x(s)$  of the internal force is uniform across the system and we note  $P = -n_x(s)$ . The vertical component  $n_y(s)$  is uniform in each free rod section and jumps every time the rod contacts the lower or the upper wall. Please also note that the moment  $m(s)$  and hence the curvature  $\kappa(s)$  do not experience any jump at contact points and hence are continuous along the entire rod (Bigoni, 2012). We will note  $F$  the total (vertical) force applied by the upper wall down to the rod, see Figure 1. In this frictionless-contact case, the Hamiltonian invariant (Dichmann et al., 1996; Kehrbbaum and Maddocks, 1997)

$$I_{\text{nv}} = \frac{1}{2} YI \kappa^2(s) - P \cos \theta(s) + n_y(s) \sin \theta(s) \quad (7)$$

takes the same value in the entire system.

### 2.2. The von Karman approximation

In the limit where the rod is only slightly bent, the deflection  $\theta(s)$  stays small and we use the first terms in

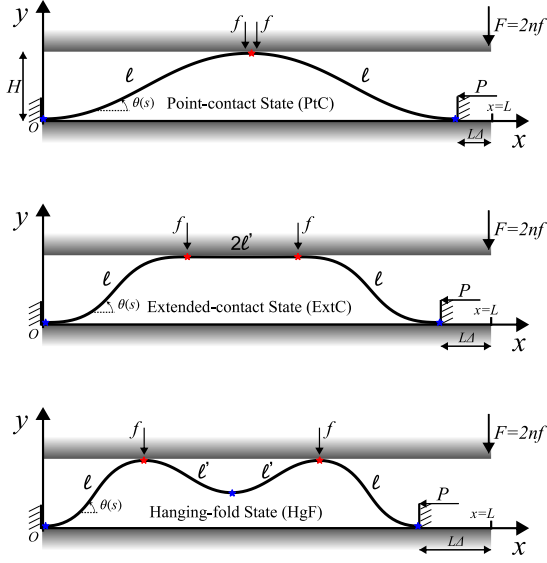


Figure 1: Planar elastica bent in a cavity. The left end is horizontally clamped at the origin, while the right end is horizontally clamped on a block constrained to slide along the horizontal axis. The elastica has a total length  $L$ , and the angle  $\theta$  is defined between the tangent to the beam and the horizontal axis. An external force is applied to the system at  $s = L$ , and we note its horizontal component  $-P$ .

the Taylor expansion of the sin and cos functions. The equilibrium equations (1), (5) and (6) simplify to

$$x'(s) = 1 - (1/2)\theta^2(s) \quad (8a)$$

$$y'(s) = \theta(s) \quad (8b)$$

$$YI\theta''(s) = -P\theta(s) - n_y(s) \quad (8c)$$

Note that the only nonlinear remaining term is in (8a), see Thomas (2025) for more details.

### 2.3. Non-dimensionalization

In this equilibrium problem, we introduce non-dimensionalized quantities

$$\hat{s} = s/L, \quad \hat{x} = x/L, \quad \hat{y} = y/L, \quad \hat{H} = H/L, \quad \hat{\theta} = \theta \quad (9a)$$

$$\hat{P} = \frac{PL^2}{YI}, \quad \hat{F} = \frac{FL^2}{YI}, \quad \hat{m} = \frac{mL}{YI}, \quad \hat{\kappa} = \kappa L \quad (9b)$$

Note that it boils down to choosing  $L$  as length unit and  $YI/L^2$  as unit force, as can be done in most equilibrium problems of elastic rods. Please also note the non-dimensionalized version of the equations can be readily obtained by formally setting  $L = 1$  and  $YI = 1$  in the equations of Sections 2.1 and 2.2.

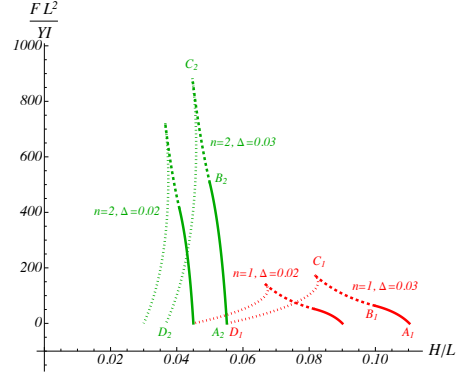


Figure 2: The total vertical force  $F$  applied on the rod from the upper wall as a function of the height of the enclosing cavity. For  $\Delta = 0.02$  and  $0.03$  and  $n = 1$  and  $n = 2$ . Legend: Point-Contact state (continuous line), Extended-Contact state (dashed line, only  $k = 1$ ), and Hanging-Fold state (dotted line).

Anticipating the results of section 4, we plot in Figure 2 a typical bifurcation diagram where the vertical force  $\hat{F} = \frac{FL^2}{YI}$  is plotted as a function of the height  $\hat{H} = H/L$  of the cavity. We see that even if non-dimensionalized quantities have been used to draw the diagram, the different curves still depend on both the Atlas number  $n$  (to be defined in Section 4) and the end shortening  $\Delta$ . See also Figure 3 for the dual experiment where the end-shortening  $\Delta$  is varied while keeping the height  $\hat{H}$  fixed. In this experiment, we find a dependence of the curves on both  $n$  and  $\hat{H}$ . One of the goal of the present paper is to introduce rescaled quantities on the axes that induce partial collapses of the curves of Figures 2 and 3. Moreover, we will show that the important loading measure is  $\sqrt{\Delta}/\hat{H}$ .

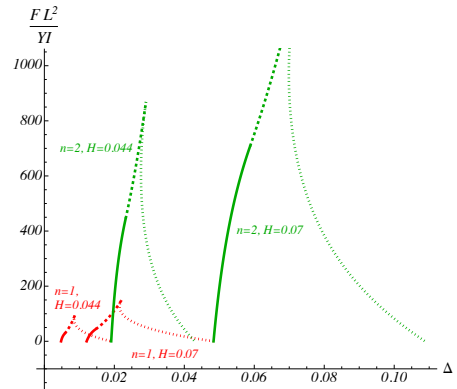


Figure 3: The total vertical force  $F$  applied on the rod from the upper wall as a function of the end-shortening  $\Delta$ . For  $H = 0.044$  and  $0.07$  and  $n = 1$  and  $n = 2$ . Legend: Point-Contact state (continuous line), Extended-Contact state (dashed line, only  $k = 1$ ), and Hanging-Fold state (dotted line).

117 To keep notations simple, we will drop the hats for  
 118 the non-dimensionalized quantities in the remainder of  
 119 the paper.

### 120 3. The Arch solution

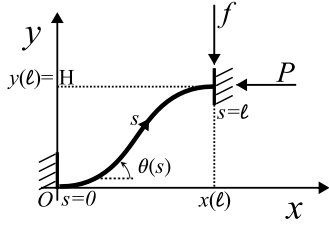


Figure 4: The arch solution is the building block with which we compute the Point-Contact, Extended-Contact, and Hanging-Fold states presented in Section 4, as well as the cellular model presented in Section 5. This solution is in contact with both the lower and upper walls, has length  $\ell$ , and inner force  $n_x = -P$  and  $n_y = -f$ .

We will describe the different solution types in Section 4. They are all based on the following fundamental equilibrium solution, which we denote as the 'Arch' solution, see Figure 4 and (Chai, 1998; Judah and Givli, 2024). This solution has boundary conditions (2a) at  $s = 0$ . Its length is  $\ell$  and the boundary conditions at  $s = \ell$  are

$$y(\ell) = H \text{ and } \theta(\ell) = 0 \quad (10)$$

Hence the boundary value problem to solve is system (8) with  $n_y(s) \equiv -f$  together with (2a) and (10). The solution is

$$m(s) = \frac{f}{\sqrt{P}} \left( \sin \sqrt{P}s - \tan \frac{\sqrt{P}\ell}{2} \cos \sqrt{P}s \right) \quad (11a)$$

$$\theta(s) = \frac{f}{P} \left( 1 - \cos \sqrt{P}s - \tan \frac{\sqrt{P}\ell}{2} \sin \sqrt{P}s \right) \quad (11b)$$

$$y(s) = \frac{f}{P\sqrt{P}} \left( \sqrt{P}s - \sin \sqrt{P}s + \tan \frac{\sqrt{P}\ell}{2} \left[ \cos \sqrt{P}s - 1 \right] \right) \quad (11c)$$

where  $\pi \leq \sqrt{P}\ell \leq 2\pi$ . In the following, we will need the two geometrical quantities  $H$  and  $x(\ell)$ , as well as the

energy  $E_k$ , which are here computed as

$$H = y(\ell) = \frac{f}{P\sqrt{P}} \left( \sqrt{P}\ell - 2 \tan \frac{\sqrt{P}\ell}{2} \right) \quad (12a)$$

$$x(\ell) = \int_0^\ell 1 - (1/2)\theta^2(s) ds = \ell - f^2 \frac{\sqrt{P}\ell \left[ 2 + \cos \sqrt{P}\ell \right] - 3 \sin \sqrt{P}\ell}{2P^2 \sqrt{P} (\cos \sqrt{P}\ell + 1)} \quad (12b)$$

$$E_\kappa = \frac{1}{2} \int_0^\ell m^2(s) ds = \frac{f^2}{2P\sqrt{P}} \frac{\sqrt{P}\ell - \sin \sqrt{P}\ell}{1 + \cos \sqrt{P}\ell} \quad (12c)$$

Please note that, by symmetry, the moment at  $s = 0$  and  $s = \ell$  have the same magnitude and opposite sign

$$m(\ell) = -m(0) = \frac{f}{\sqrt{P}} \tan \frac{\sqrt{P}\ell}{2} \quad (13)$$

### 121 4. The different equilibrium states

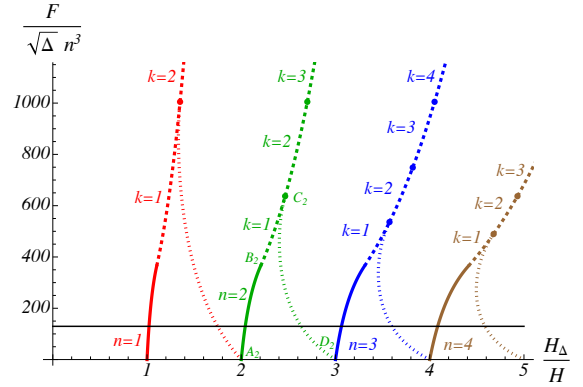


Figure 5: Vertical force  $F$  plotted as a function of the compaction ratio  $H_\Delta/H$ . This plot does not depend on the value of  $\Delta$ . The threshold  $H_\Delta$  is defined in Eq. (14). The black line corresponds to the continuous  $n$  model presented in Section 5. Legend: Point-Contact state (continuous line), Extended-Contact state (dashed line), and Hanging-Fold state (dotted line). We show Extended-Contact states where the flat region is fragmented into  $k$  pieces. For  $n = 2$ , we show the limit points between the different states.

In this section, we describe three different fundamental states the system visits as either  $H$  is lowered or  $\Delta$  is increased (Domokos et al., 1997; Roman and Pocheau, 2002). For large values of  $H$  or small values of  $\Delta$ , the elastica buckles when the horizontal force  $P$  reaches the threshold  $P = 4\pi^2$  (Euler, 1744). Contact with the upper wall does not happen in this first regime where, in the present von Karman approximation,  $P$  stays constant

$P = 4\pi^2$  while  $H$  is lowered or  $\Delta$  increased. Eventually, contact first happens when  $H$  and  $\Delta$  are such that

$$H_\Delta = \frac{2}{\pi} \sqrt{\Delta} \quad (14)$$

We define the *Atlas* number  $n$  as half the number of rod segments spreading out from  $y = 0$  to  $y = H$  (i.e. touching both walls), such as the Arch of Section 3. In the following section, we first describe the solution for  $n = 1$  and then generalize formulas for any  $n$  value. This Atlas number has been called the number of folds in previous works.

#### 4.1. Point-Contact states

We first describe configurations in which the elastica contacts the upper wall only at isolated point(s), see Figure 1. We call these solutions *Point-Contact states*.

When  $n = 1$ , we have  $\ell = 1/2$ , the total force from the upper wall  $F$  takes the value  $F = 2f$  and the end shortening is  $\Delta = 1 - 2x(\ell)$ . From the Arch solution, Eqs. (12) and (13), we have

$$H = \frac{F}{4P\sqrt{P}} \left( \sqrt{P} - 4 \tan \frac{\sqrt{P}}{4} \right) \quad (15a)$$

$$\Delta = F^2 \frac{\sqrt{P} \left[ 2 + \cos(\sqrt{P}/2) \right] - 6 \sin(\sqrt{P}/2)}{8P^2 \sqrt{P} [\cos(\sqrt{P}/2) + 1]} \quad (15b)$$

$$m(\ell) = -m(0) = \frac{F}{2\sqrt{P}} \tan \frac{\sqrt{P}}{4} \quad (15c)$$

This branch of solutions starts at point  $A_1$  (see Figure 2) when  $P = 4\pi^2$ ,  $F = 0$ ,  $H = H_\Delta$  and ends with point  $B_1$  when  $P = 16\pi^2$ ,  $F = 3^{-1/2} 64\pi^2 \sqrt{\Delta}$ , and  $H = 3^{-1/2} \sqrt{\Delta}$ .

These formulas generalize directly to the case with  $n \geq 1$  by setting  $\ell = 1/(2n)$ ,  $F = 2nf$ , and  $\Delta = 1 - 2nx(\ell)$ . Using Eqs. (12) and (13) from the Arch solution, we find

$$H_u = \frac{F_u}{4P_u \sqrt{P_u}} \left( \sqrt{P_u} - 4 \tan \frac{\sqrt{P_u}}{4} \right) \quad (16a)$$

$$\Delta = F_u^2 \frac{\sqrt{P_u} \left[ 2 + \cos(\sqrt{P_u}/2) \right] - 6 \sin(\sqrt{P_u}/2)}{8P_u^2 \sqrt{P_u} [\cos(\sqrt{P_u}/2) + 1]} \quad (16b)$$

$$\frac{m(\ell)}{n} = -\frac{m(0)}{n} = \frac{F_u}{2\sqrt{P_u}} \tan \frac{\sqrt{P_u}}{4} \quad (16c)$$

with  $H_u = nH$ ,  $P_u = P/n^2$ , and  $F_u = F/n^3$ . These branches of solutions each start at point  $A_n$  when  $P_u =$

$4\pi^2$ ,  $F_u = 0$ ,  $H_u = H_\Delta$  and end at point  $B_n$  when  $P_u = 16\pi^2$ ,  $F_u = 3^{-1/2} 64\pi^2 \sqrt{\Delta}$ , and  $H_u = 3^{-1/2} \sqrt{\Delta}$ . Using (16a) et (16b), one can write a single equation for  $P_u$  having the form  $\Delta/H_u^2 = \phi(P_u)$ . For  $4\pi^2 \leq P_u \leq 16\pi^2$ , the function  $\phi(\cdot)$  is monotonously increasing, taking values from  $(\pi/2)^2$  to 3, which means that, for all  $n$ , there is always a unique Point-Contact solution.

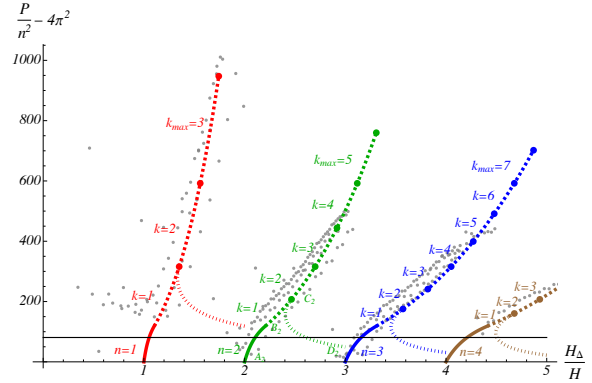


Figure 6: Horizontal force  $P$  plotted as a function of the compaction ratio  $H_\Delta/H$ . This plot does not depend on the value of  $\Delta$ . The contact threshold  $H_\Delta$  is defined in Eq. (14). The black line corresponds to the continuous  $n$  model presented in Section 5. Legend: Point-Contact state (continuous line), Extended-Contact state (dashed line), and Hanging-Fold state (dotted line). We show Extended-Contact states where the flat region is fragmented into  $k$  pieces. For  $n = 2$ , we show the limit points between the different states. Gray dots correspond to experimental data from Figure 5 of Deboeuf et al. (2024).

For this point-contact state, the vertical force  $F$  and the horizontal force  $P$  are plotted in plain line against the height  $H$  of the cavity for different values of the end shortening  $\Delta$  in Figures 5 and 6. We remark that, contrary to Figure 2, the rescaling with  $H_\Delta$  induces a collapse of the bifurcation curves, which now do not depend on the value of  $\Delta$ . Furthermore, we show in Figures 7 and 8 (continuous line) that, using the Atlas number  $n$ , the Point-Contact branches all collapse on a single curve, for any  $\Delta$  and any  $n$ . We also plot in Figures 6 and 8 the experimental data from Figure 5 of Deboeuf et al. (2024), where  $\Delta < 0.16$ ,  $0.042 \leq H \leq 0.085$ , and  $1 \leq n \leq 4$ . In these data, friction tends to increase the measured horizontal force  $P$  which otherwise is in good agreement with the present theory.

As the first contact occurs, when  $H = H_\Delta$ , the vertical force  $F$  increases from  $F = 0$ . This is repeated for each  $n$  as we have  $F = 0$  each time  $nH = H_u = H_\Delta$ . At this point, we compute the slope of the function  $F = F(H)$  by performing a Taylor expansion of Eqs. (16a) and (16b) for  $H_u$  near  $H_\Delta$  and find

$$F_u = \frac{16\pi^4}{10 - \pi^2} [H_\Delta - H_u] + \dots \quad (17)$$



161 The large value of the prefactor (nearly  $1.2 \cdot 10^4$ ) shows  
 162 that the vertical force steeply increases as one com-  
 163 presses the system vertically.

At  $P_u = 16\pi^2$ , when the Point-Contact branch ends,  
 the deflection angle at the inflexion point  $s = \ell/2$  takes  
 the value

$$\theta_{\max} = \frac{F_u}{P_u} = \frac{4}{\sqrt{3}} \sqrt{\Delta} \quad (18)$$

164 This maximum value yields an upper bound for  $\Delta$  if we  
 165 require  $\theta_{\max}$  to stay small, e.g.  $\theta_{\max} < 0.5$  implies  $\Delta <$   
 166  $0.06$ .

#### 167 4.2. Extended-Contact states

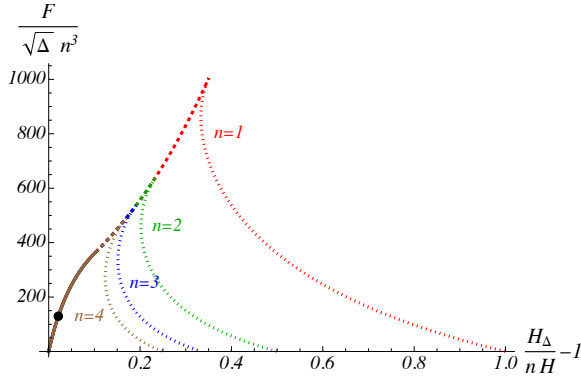


Figure 7: Vertical force  $F$  plotted as a function of the rescaled quantity  $H_\Delta/(nH) - 1$ . For the Point-Contact and Extended-Contact states the plot does not depend on the Atlas number  $n$ . The entire plot does not depend on the value of  $\Delta$ . The single black point at  $(0.02, 130)$  corresponds to the response of the cellular model. Legend: Point-Contact state (continuous line), Extended-Contact state (dashed line, only  $k = 1$ ), and Hanging-Fold state (dotted line).

168 In Figure 2, as the vertical force  $F$  is increased above  
 169 point  $B$ , the system exhibits a region where the contact  
 170 with the upper (or lower) wall is flat; We call these so-  
 171 lutions *Extended-Contact states*, see Figure 1. For these  
 172 solutions, both the deflection angle  $\theta(s)$  and the moment  
 173  $m(s)$  at the start of the flat region vanishes, and conse-  
 174 quently the Hamiltonian invariant  $I_{nv} = -P$ , see Eq. (7).  
 175 It has been explained that the total length  $2\ell'$  of flat con-  
 176 tact may be distributed within several sections of the  
 177 system with no change of its energy  $E_\kappa$  (Pocheau and  
 178 Roman, 2004).

When  $n = 1$ , we have  $2\ell + 2\ell' = 1$ . The value of  
 $\ell$  is fixed through the condition  $m(\ell) = 0$  which yields  
 $\ell = 2\pi/\sqrt{P}$ , see (13). Using the Arch solution (12) with

$F = 2f$ , we find

$$H = \frac{\pi F}{P \sqrt{P}} \quad (19a)$$

$$\Delta = 1 - [2x(\ell) + 2\ell'] = \frac{3\pi}{4} \frac{F^2}{P^2 \sqrt{P}} \quad (19b)$$

$$m(\ell) = m(0) = 0 \quad (19c)$$

179 This branch of solutions starts at point  $B_1$  (see Figure  
 180 2) when  $\ell' = 0$ ,  $P = 16\pi^2$ ,  $F = 3^{-1/2} 64 \pi^2 \sqrt{\Delta}$ , and  
 181  $H = 3^{-1/2} \sqrt{\Delta}$  and ends when the flat region buck-  
 182 les, that is when  $P\ell'^2 = \pi^2$  (buckling threshold of a  
 183 clamped-clamped beam of length  $2\ell'$ ). As we do not  
 184 know whether the flat region stands in one piece or is  
 185 fragmented in  $k$  pieces, we can only give lower and up-  
 186 per bounds on the end of the branch, as follows.

187 If the flat region stands in one piece,  $k = 1$ , the branch  
 188 ends at its lower bound, point  $C_1$ , when  $\ell' = 1/6$ ,  $P =$   
 189  $36\pi^2$ ,  $F = 72 \sqrt{2} \pi^2 \sqrt{\Delta}$ , and  $H = (\sqrt{2}/3) \sqrt{\Delta}$ .

190 In this  $n = 1$  case, the flat region can be, at most, frag-  
 191 mented into three pieces. This yields the upper bound  
 192 for the end of the branch with  $\ell' = 3/10$ ,  $P = 100\pi^2$ ,  
 193  $F = 100 \sqrt{40/3} \pi^2 \sqrt{\Delta}$ , and  $H = \sqrt{2/15} \sqrt{\Delta}$ .

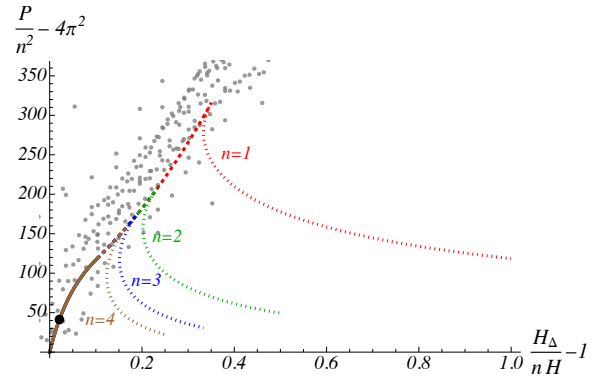


Figure 8: Horizontal force  $P$  plotted as a function of the rescaled compaction ratio  $H_\Delta/H$ . For the Point-Contact and Extended-Contact states the plot does not depend on the Atlas number  $n$ . The entire plot does not depend on the value of  $\Delta$ . The single black point at  $(0.02, 41)$  corresponds to the response of the cellular model. Legend: Point-Contact state (continuous line), Extended-Contact state (dashed line, only  $k = 1$ ), and Hanging-Fold state (dotted line). Gray dots correspond to experimental data from Figure 5 of Deboeuf et al. (2024).

When  $n \geq 1$ , we have  $2n\ell + 2\ell' = 1$ ,  $F = 2nf$ , and  
 the condition  $m(\ell) = 0$  still yields  $\ell = 2\pi/\sqrt{P}$ . As in

Section 4.1, formulas (19) are directly generalized to

$$H_u = \frac{\pi F_u}{P_u \sqrt{P_u}} \quad (20a)$$

$$\Delta = 1 - [2nx(\ell) + 2\ell'] = \frac{3\pi}{4} \frac{F_u^2}{P_u^2 \sqrt{P_u}} \quad (20b)$$

$$m(\ell) = m(0) = 0 \quad (20c)$$

This rescaling of the Extended-Contact solution for any  $n$  and any  $\Delta$  is illustrated in Figures 7 and 8 (dashed line) where we see that all the ( $n \geq 1$ ) Extended-Contact solution branches collapse. This branch of solutions starts at point  $B_n$  when  $\ell' = 0$ ,  $P_u = 16\pi^2$ ,  $F_u = 3^{-1/2} 64\pi^2 \sqrt{\Delta}$ , and  $H_u = 3^{-1/2} \sqrt{\Delta}$  and ends when the flat region buckles. As the flat region could be fragmented into  $k$  pieces (with  $k = 1$  to  $k = 2n + 1$ ), the branch of solutions ends when

$$\ell' = k/(4n + 2k) \text{ and } \ell = 1/(2n + k) \quad (21a)$$

$$P_u = 4\pi^2 \left( \frac{2n + k}{n} \right)^2 \quad (21b)$$

$$F_u = 8\pi^2 \sqrt{\Delta} \sqrt{2/3} \left( \frac{2n + k}{n} \right)^{5/2} \quad (21c)$$

$$H_u = \sqrt{\Delta} \sqrt{2/3} \sqrt{\frac{n}{2n + k}} \quad (21d)$$

194 which means that for large  $n$  and with a flat region  
195 fragmented into a maximum number of pieces ( $k =$   
196  $2n + 1$ ) the vertical force can take values up to  $F \simeq$   
197  $2000 n^3 \sqrt{\Delta} \simeq 140 \Delta^2 / H^3$ , and the horizontal force val-  
198 ues up to  $P \simeq 600 n^2 \simeq 100 \Delta / H^2$ .

### 199 4.3. Hanging-fold states

200 Along the Extended-Contact solution branch, when  
201 the flat region is in one piece,  $k = 1$ , a new branch bifur-  
202 cates as the compression force  $P$  exceeds the buckling  
203 threshold given in (21b). A hanging fold is created, re-  
204 placing the flat region, see Figure 1. We call these con-  
205 figurations *Hanging-Fold states*. Please note that a con-  
206 figuration having a fold and a flat region is not possible,  
207 as it would imply two different values of the Hamilto-  
208 nian invariant  $I_{nv}$ : the invariant value for solutions with  
209 a flat region is  $I_{nv} = -P$  and the invariant value for solu-  
210 tions with a fold is  $I_{nv} = -P + (1/2) \kappa^2(\ell) > -P$ .

When  $n = 1$ , we have  $2\ell + 2\ell' = 1$  where the value of  $\ell'$  is fixed by Eq. (A.1) in Appendix A. We have  $F = 2f$  and  $\Delta = 1 - 2[x(\ell) + x(\ell')]$  where  $x(\ell)$  is given by Eq. (12b) and  $x(\ell')$  by Eqs.(A.2) and (A.3) in Appendix A. Further using (12a) for the height  $H$ , we find

$$H = \frac{F}{2P\sqrt{P}} \left( \sqrt{P}\ell - 2 \tan \frac{\sqrt{P}\ell}{2} \right) \quad (22a)$$

$$\Delta = F^2 \left\{ \frac{\sqrt{P}\ell [2 + \cos \sqrt{P}\ell] - 3 \sin \sqrt{P}\ell}{4P^2 \sqrt{P} (\cos \sqrt{P}\ell + 1)} + \frac{1/2 - \ell}{8P^2} \frac{1 - \cos \sqrt{P}\ell}{1 + \cos \sqrt{P}\ell} \right\} \quad (22b)$$

$$P = \left( \frac{\pi}{1/2 - \ell} \right)^2 \quad (22c)$$

211 This branch starts at point  $C_1$  (see Figure 2) when  $\ell =$   
212  $1/3$ ,  $\ell' = 1/6$ ,  $P = 36\pi^2$ ,  $F = 72 \sqrt{2} \pi^2 \sqrt{\Delta}$ , and  $H =$   
213  $(\sqrt{2}/3) \sqrt{\Delta}$  and ends at point  $D_1$  when  $\ell = 1/4$ ,  $\ell' =$   
214  $1/4$ ,  $P = 16\pi^2$ ,  $F = 0$ , and  $H = H_{\Delta}/2$ .

When  $n \geq 1$ , we have  $2n\ell + 2\ell' = 1$  where the value of  $\ell'$  is still given by Eq. (A.1) in Appendix A. We have  $F = 2nf$  and  $\Delta = 1 - 2[nx(\ell) + x(\ell')]$  where  $x(\ell)$  is given by Eq. (12b) and  $x(\ell')$  by Eqs. (A.2) and (A.3) in Appendix A. As in Section 4.1 and 4.2, we use  $H_u = nH$ ,  $P_u = P/n^2$ , and  $F_u = F/n^3$  to write the generalization of Eqs. (22) to any  $n \geq 1$  value

$$H_u = \frac{F_u}{2P_u \sqrt{P_u}} \left( \sqrt{P_u} \ell_u - 2 \tan \frac{\sqrt{P_u} \ell_u}{2} \right) \quad (23a)$$

$$\Delta = F_u^2 \left\{ \frac{\sqrt{P_u} \ell_u [2 + \cos \sqrt{P_u} \ell_u] - 3 \sin \sqrt{P_u} \ell_u}{4P_u^2 \sqrt{P_u} (\cos \sqrt{P_u} \ell_u + 1)} + \frac{1/2 - \ell_u}{8P_u^2} \frac{1 - \cos \sqrt{P_u} \ell_u}{1 + \cos \sqrt{P_u} \ell_u} \right\} \quad (23b)$$

$$P_u = \frac{1}{n^2} \left( \frac{\pi}{1/2 - \ell_u} \right)^2 \quad (23c)$$

215 where we have introduced  $\ell_u = n\ell$ . The explicit pres-  
216 ence of the Atlas number  $n$  in Eq. (23c) implies that,  
217 contrary to what was found in Sections (4.1) and (4.2),  
218 there is no for-all- $n$  collapse of the solution branches  
219 for the Hanging-Fold states, as can be seen in Figures 7  
220 and 8. The  $n \geq 1$  branches start when  $\ell = 1/(1 + 2n)$ ,  
221  $\ell' = 1/(2 + 4n)$ ,  $P = 4\pi^2(1 + 2n)^2$ ,  $F = 8\pi^2 \sqrt{\Delta} \sqrt{2/3} (1 +$   
222  $2n)^{5/2} \sqrt{n}$ , and  $H = \sqrt{2}\Delta / \sqrt{3n(1 + 2n)}$  and ends when  
223  $\ell = \ell' = 1/(2 + 2n)$ ,  $P = 4\pi^2(1 + n)^2$ ,  $F = 0$ , and  
224  $H = H_{\Delta}/(1 + n)$ .

225 We recall that the description of the sequence of the  
226 different states we have made in this section 4 holds both  
227 when  $H$  is constant and  $\Delta$  increased or when  $\Delta$  is con-  
228 stant and  $H$  decreased: The important quantity defining  
229 the state of the system is the compaction ratio  $H_{\Delta}/H$ .



## 5. The continuous $n$ approximation: The cellular model

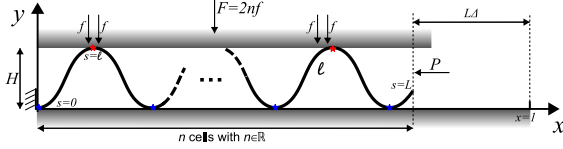


Figure 9: In the cellular model, the system is the repetition of the Arch solution with  $L = 2n\ell$ . As the Atlas number  $n$  takes non-integer values, the boundary condition at  $s = L$  is unsatisfied. The importance of this defect decreases as  $n$  gets large and is in fact already minor for small  $n$  values.

As the compaction ratio  $H_\Delta/H = 2\sqrt{\Delta}/(\pi H)$  is increased, the system repeatedly visits the Point-Contact state with an increasing Atlas number  $n$ . When  $n$  becomes large, the precise boundary conditions at the rod extremities become less important, that is the integer character of  $n$  is no longer critical and one could approximate the system's response with a model where  $n$  is a non-integer quantity, see Figure 9. We look for the equilibrium of such a cellular system by minimizing the total potential energy

$$E_{\text{tot}} = 2nE_\kappa + 2nx(\ell)P + y(\ell)F \quad (24)$$

where we work in the dead loading case (i.e. for prescribed  $P$  and  $F$ ) as it yields simpler calculations than the (equivalent) rigid loading case (i.e. for prescribed  $\Delta$  and  $H$ ). We use Eqs. (12) with  $2n\ell = 1$  and  $F = 2nf$ , and find

$$E_{\text{tot}}(z) = P + \frac{F^2}{2P^2} z \left[ \mu_\kappa(z) - \mu_x(z) + \mu_y(z) \right] \quad (25a)$$

$$\text{with } \mu_\kappa(z) = \frac{z - \sin z}{1 + \cos z} \quad (25b)$$

$$\mu_x(z) = \frac{z(2 + \cos z) - 3 \sin z}{1 + \cos z} \quad (25c)$$

$$\mu_y(z) = 2z - 4 \frac{\sin z}{1 + \cos z} \quad (25d)$$

with  $z = \sqrt{P}\ell$ . The minimum is reached for  $z = z_e$  such that  $z_e = \tan z_e$ , that is  $z_e \simeq 4.5$ . For this solution, we

have  $\mu_e = \mu_x(z_e) = \mu_y(z_e) = 2\mu_\kappa(z_e) \simeq 14$  and

$$P = \frac{z_e \mu_e}{2} \frac{\Delta}{H^2} \simeq 31 \frac{\Delta}{H^2} \quad (26a)$$

$$F = \frac{z_e \mu_e}{2} \frac{\Delta^2}{H^3} \simeq 31 \frac{\Delta^2}{H^3} \quad (26b)$$

$$n = \sqrt{\frac{\mu_e}{8z_e}} \frac{\sqrt{\Delta}}{H} \simeq 0.62 \frac{\sqrt{\Delta}}{H} \quad (26c)$$

$$2nE_\kappa = \frac{z_e \mu_e}{4} \frac{\Delta^2}{H^2} \simeq 16 \frac{\Delta^2}{H^2} \quad (26d)$$

Eq. (26c) is to be compared to Eq. (2.19) of Tzokova (2020) Phd dissertation, which also introduced a continuous approach based on the vertical force  $F$  peaks. The total curvature energy of the cellular system,  $2nE_\kappa$ , is plotted in Figure 10 and compared to the total curvature energy of the different states introduced in Section 4. In this Figure 10, we see that the cellular model is approximately a convexification of the energy of the 'exact' system: Non-integer  $n$  values of the cellular model appear as approximate phase transitions between the integer- $n$  states of the 'exact' system.

Using the scalings introduced in Section 4, we can rewrite these quantities as

$$P_u = \frac{P}{n^2} = 4z_e^2 \simeq 81 \text{ and } F_u = \frac{F}{n^3} = \frac{8\sqrt{2}z_e^{5/2}}{\sqrt{\mu_e}} \sqrt{\Delta} \simeq 130 \sqrt{\Delta} \quad (27)$$

(See Appendix B for the different other ways to express these quantities). These quantities are plotted in Figures 5, 6, 7, 8, and 10 as black lines. Another important quantity is the maximum value of the deflection angle  $\theta$  which happens at the inflection point  $s = \ell/2$ . In this continuous  $n$  model, we have

$$\theta_{\text{max}} = \frac{\cos(z_e/2) - 1}{\cos(z_e/2)} \frac{F_u}{P_u} \simeq 1.3 \frac{F_u}{P_u} \simeq 2.1 \sqrt{\Delta} \quad (28)$$

which near the value found for the Point-Contact state, see Eq. (18), and therefore yields the same limit  $\Delta < 0.06$  to stay in the small angle,  $\theta_{\text{max}} < 0.5$ , regime.

In Figure 11 we plot the vertical force  $F$  as a function of the horizontal force  $P$ , with properly rescaled axes. As explained earlier, for Point-Contact and Extended-Contact states this  $(P, F)$  diagram does not depend on the values of  $\Delta$  or  $n$ . Moreover, we see that the relation is almost linear, with a slope value approximately given by the cellular model:  $\frac{130}{81-4\pi^2} \simeq 3.1$ . Additionally, we plot the numerical results of the fully nonlinear system (1)-(6) for Point-Contact and Extended-Contact solutions to measure how much the nonlinearities cause a deviation from the virtually linear  $(P, F)$  relation, see

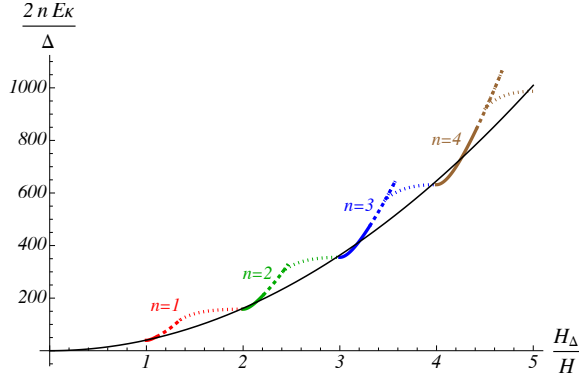


Figure 10: Curvature energy of the system as a function of the compaction ratio  $H_{\Delta}/H$ . Point-Contact states (continuous line), Extended-Contact states (dashed line, only  $k = 1$ ), and Hanging-Fold states (dotted line) are shown for  $n = 1$  to  $n = 4$ . The black line corresponds to the continuous- $n$  model, Eq. (26d), and is approximately a convexification of the energy of the system.

gray dots in Figure 11 and (Wang et al., 2025). This linear relation implies that

$$FH \sim P\Delta \quad (29)$$

246 which means that the work done by the horizontal and 271  
247 vertical loads are comparable. 272

## 248 6. Global phase diagram

249 In a typical experiment, one imposes the value of  $\Delta$  271  
250 and  $H$  and observes the state of the system. In Figure 272  
251 12, we plot the limit of the ranges of the existence 273  
252 of the different states for  $n = 1$  and  $n = 2$  and 274  
253 see that a clear arrangement materializes, linked to the 275  
254 compaction ratio  $H_{\Delta}/H$  (for simplicity reasons, we only 276  
255 consider Extended-Contact states with their flat region 277  
256 in  $k = 1$  piece). Nevertheless, as shown in Figure 13, 278  
257 where  $nH$  is plotted as a function of  $\sqrt{\Delta}$ , this arrangement 279  
258 is not conserved as  $n$  is increased: point  $A_{n+1}$  280  
259 (of slope  $\frac{2}{\pi} \frac{n}{n+1}$ ) moves relatively to point  $C_n$  (of slope 281  
260  $\sqrt{\frac{2n}{2n+1}}/\sqrt{3}$ ) and crosses it as  $n \simeq 7.1$ . At a larger  $n$  282  
261 value ( $n \simeq 9.7$ ), the same point  $A_{n+1}$  even crosses point 283  
262  $B_n$  (of slope  $1/\sqrt{3}$ ). And at  $n \simeq 16.8$  it is point  $A_{n+2}$  that 284  
263 crosses  $C_n$ . Eventually, for any fixed integer  $j$ , when 285  
264  $n \rightarrow \infty$  the arrangement is such that  $A_{n+j}$  is reached be- 286  
265 fore  $B_n$  (in terms of compaction ratio  $H_{\Delta}/H$  values). In 287  
266 contrast, the points  $B_n$  and  $C_n$  keep their relative posi- 288  
267 tioning:  $B_n > C_n > B_{n+1} > C_{n+1} > \dots > B_{n+j} > C_{n+j}$ , 289  
268 for any  $n$  and  $j$ . Overall, we conclude that for  $n < 8$  (and 290  
269 under the assumption  $k = 1$ ), we will observe sequen- 291  
270 tially the three states: Point-Contact, Extended-Contact, 292

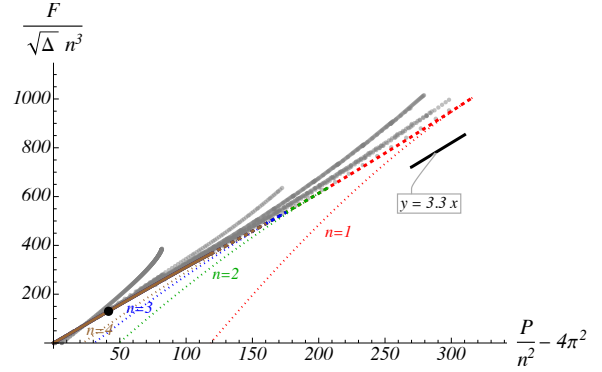


Figure 11: Evolution of the horizontal  $P$  and the vertical  $F$  forces as  $H$  is decreased or  $\Delta$  is increased. Point-Contact state (continuous line), Extended-Contact state (dashed line, only  $k = 1$ ), and Hanging-Fold state (dotted line). For the Point-Contact and Extended-Contact states the plot does not depend on the Atlas number  $n$ . The single black point at  $(41, 130)$  corresponds to the response of the cellular model. The black line, with slope 3.3, is drawn as a guide to the eye to show the roughly linear behavior of the Point-Contact and Extended-Contact states. Gray dots correspond to numerical solutions of the nonlinear system (1)-(6) where only Point-Contact and Extended-Contact states are shown, a total of 1298 points with  $H \in (0.02, 0.15)$  and  $\Delta \in (0.006, 0.4)$ .

271 and finally Hanging-Fold, before switching to the  $n + 1$  272  
273 mode. In contrast, for  $n \geq 8$ , we will not observe the 274  
275 Hanging-Fold state and will switch directly from the  $n^{\text{th}}$  276  
277 Extended-Contact state to the  $n+1^{\text{th}}$  Point-Contact state. 278

279 Note that all these arrangements depend on the value 280  
281 of  $k$  taken for the number of fragmented pieces of the 282  
283 flat region in the Extended-Contact state (here  $k = 1$ ), 284  
285 with  $k_{\text{max}} = 2n + 1$ . In Figure 14 we plot a high  $n$  part 286  
287 of the bifurcation diagram, with  $k = 3$ , and show the 288  
289 behavior of the system will exhibit multistability and 290  
291 hysteresis. We consider we have loaded the system by 292  
293 increasing  $H_{\Delta}/H$  up to the point  $U_0$  lying at the end of 294  
295 the  $k = 3$  segment of the 25<sup>th</sup> Extended-Contact branch. 296  
297 Upon increasing  $H_{\Delta}/H$  further, and if  $k$  does not in- 298  
299 crease, the system will jump on the point  $U_1$  on the 26<sup>th</sup> 300  
301 Point-Contact branch. We then start unloading. The 302  
303 system will reach point  $U_2$  on the 25<sup>th</sup> Hanging-Fold 304  
305 branch and, upon further decrease of  $H_{\Delta}/H$ , will jump 306  
307 toward on of the points  $U_3, U_4$ , or  $U_5$ . Resuming the 308  
309 loading will close the hysteresis loop. 310

## 311 7. Discussion and Conclusion

312 We have detailed the different equilibrium configura- 313  
314 tions a planar beam may adopt when buckled in a rect- 315  
316 angular cavity (Section 4). We have shown that they are 317  
318 all based on an Arch solution that spans from the lower 319  
320 to the upper wall (Section 3). For each solution type, 321

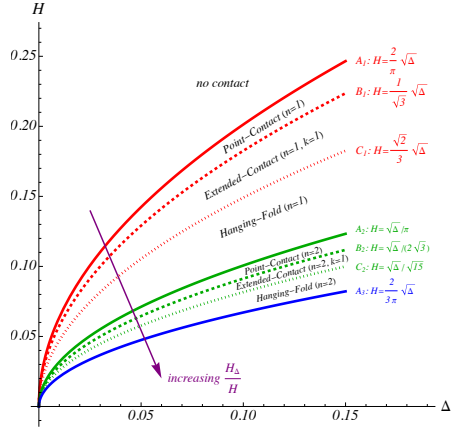


Figure 12: An attempt to draw a global bifurcation diagram for the Constrained Euler Buckling problem. Here, we give the state of the system for given values of  $H$  and  $\Delta$ , and for Atlas numbers  $n = 1$  and  $n = 2$ . A clear arrangement of the different solutions appears, but this arrangement does not hold for any  $n$ , as is shown in Figure 13.

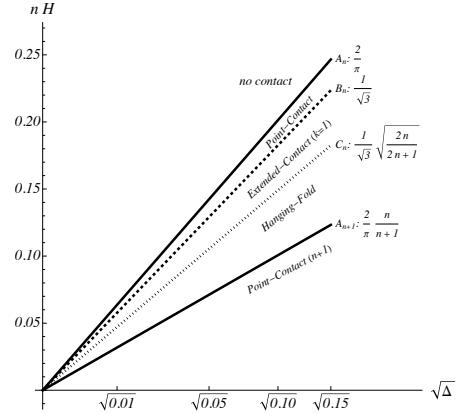


Figure 13: The relative positioning of the curves in Figure 12 does not hold for any  $n$ . The curves  $B_n$  and  $C_n$  keep their relative positioning, but the curves  $A_{n+j}$  all eventually cross  $B_n$  and  $C_n$ . For example,  $A_{n+1}$  crosses  $C_n$  at  $n \approx 7.1$  and  $B_n$  at  $n \approx 9.7$ .

297 we have given their range of existence in terms of the  
 298 imposed horizontal ( $\Delta$ ) and vertical ( $H$ ) displacements.  
 299 We have explained how the global system is a repetition  
 300 of the same solution and have introduced the Atlas number  
 301  $n$  which counts the repeats. We have used rescaled  
 302 quantities to illustrate how the vertical ( $F$ ) and horizontal  
 303 ( $P$ ) applied forces vary as functions of the imposed  
 304 displacements  $H$  and  $\Delta$ , and we have shown that the  
 305 state of the system during loading is entirely character-  
 306 ized by the compaction ratio  $H_\Delta/H = (2/\pi) \sqrt{\Delta}/H$ . Finally,  
 307 we have introduced a cellular model that yields  
 308 an averaged response of the system as the loads are in-  
 309 creased. The cellular model also clearly illustrates why  
 310 it is much harder to compress and squash a beam inside  
 311 a cavity than in the unconstrained case, see Figure 15.

312 All the above results are based on the von Karman  
 313 kinematic approximation and are then only valid as long as  
 314  $H$  and  $\Delta$  are small enough. As  $H > 0.2$  and/or  $\Delta >$   
 315  $0.1$ , the system enters the full nonlinear regime and, for  
 316 example, the exact collapse seen in Figure 7 and 8 is no  
 317 longer valid, Ying-Yang solutions may appear, and the  
 318 forces  $F$  and  $P$  (together with the number  $n$  of repeats)  
 319 will reach maximum values before decreasing (Deboeuf  
 320 et al., 2024). The global bifurcation diagram, Figures  
 321 12 and 13, gets more complicated and depends on the  
 322 loading history due to the presence of large hysteresis.  
 323 These features will be the subject of a subsequent report.

## 324 Acknowledgments

325 It is a pleasure to thank Paul Grandgeorge, Eytan  
 326 Katzav, and Benoit Roman for numerous discussions.

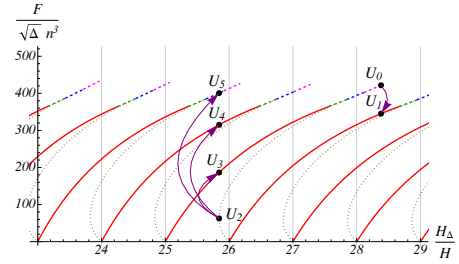


Figure 14: Hysteresis and multistability at large Atlas number. Starting on point  $U_1$  and decreasing the compaction ratio  $H_\Delta/H = (2/\pi) \sqrt{\Delta}/H$  will cause the system to visit point  $U_2$  then jump on either points  $U_3$ ,  $U_4$ , or  $U_5$ . Then increasing the compaction ratio will bring the system to  $U_0$  where it will jump on  $U_1$  to close a hysteresis loop. Point-Contact states are plotted as continuous red lines, Extended-Contact states with  $k = 1$  as dashed green lines, Extended-Contact states with  $k = 2$  as dashed blue lines, Extended-Contact states with  $k = 3$  as dashed magenta lines, and Hanging-Fold states as dotted brown lines.

## 327 Appendix A. The planar elastica solution in the 328 hanging-fold state

We here compute some basic quantities attached to the Hanging-fold state. In this state, part of the system does not touch both walls. The rod equilibrium in this part of the system is then a planar elastica solution, see Figure A.16, with no vertical component in its inner force. We therefore have to integrate system (8) with  $n_y(s) \equiv 0$ , and the boundary conditions (2a) together with  $\theta(\ell') = 0$ . The solution of  $\theta''(s) = -P\theta(s)$  with  $\theta(0) = 0$  is  $\theta(s) = c \sin \sqrt{P}s$ . Enforcing  $\theta(\ell') = 0$  yields

$$\sqrt{P} \ell' = \pi \quad (\text{A.1})$$

Additionally, the continuity of the bending moment at the contact point ( $s = 0$  in Figure A.16) implies that

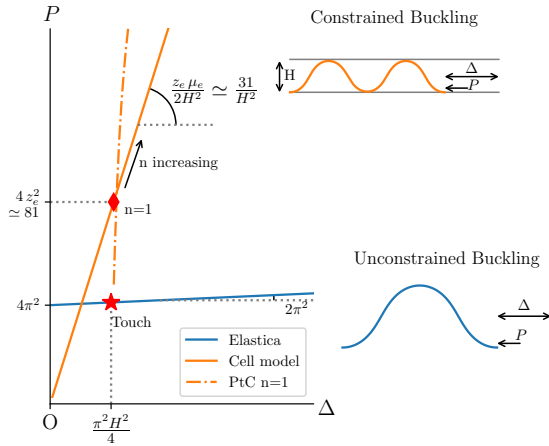


Figure 15: Buckling an elastica inside a tunnel is much harder than in the unconstrained setup. The horizontal force needed to compress an elastic rod increases much more rapidly in the constrained case (orange) than in the unconstrained case (blue). Note that we have considered the fully nonlinear version of the equilibrium equations (1), (2), (4), (5), (6) for the unconstrained case, where  $P(\Delta) = 4\pi^2 + 2\pi^2 \Delta + \dots$  instead of  $P(\Delta) = 4\pi^2$  in the von Karman approximation, see for example [Bazant and Cedolin \(2010, Fig. 1.26a\)](#).

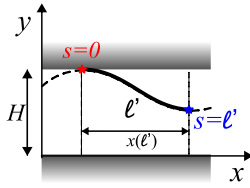


Figure A.16: . The part of the system of the Hanging-fold state where the rod does not touch both walls. The rod equilibrium is a planar elastica solution with an unknown length  $2\ell'$  and an internal force  $n_x = -P$  and  $n_y = 0$ .

$\theta'(s = 0) = c \sqrt{P}$  equals the value given in Eq. (13). This continuity constraint yields

$$c = \frac{f}{P} \tan \frac{\sqrt{P}\ell}{2} \quad (\text{A.2})$$

Finally, we can compute the horizontal extent of the solution as

$$x(\ell') = \int_0^{\ell'} 1 - (1/2)\theta^2(s) ds = \ell' \left(1 - \frac{c^2}{4}\right) \quad (\text{A.3})$$

## Appendix B. The cellular model formula in all variables

The solution of the cellular model (Section 5) may be expressed using different pairs of variables. We list in Table B.1 the different possibilities.

	$P$	$F$	$2nE_k$	–
$(\Delta, H)$	$31 \frac{\Delta}{H^2}$	$31 \frac{\Delta^2}{H^3}$	$16 \frac{\Delta^2}{H^2}$	$n = 0.62 \frac{\sqrt{\Delta}}{H}$
$(\Delta, n)$	$81 n^2$	$130 n^3 \sqrt{\Delta}$	$40 \Delta n^2$	$H = 0.62 \frac{\sqrt{\Delta}}{n}$
$(H, n)$	$81 n^2$	$208 H n^4$	$104 H^2 n^4$	$\Delta = 2.6 H^2 n^2$

Table B.1: Formulas for the forces  $P$  and  $F$ , the bending energy  $2nE_k$ , the end shortening  $\Delta$ , the height  $H$ , and the Atlas number  $n$  for the cellular model, as functions of different pairs of variables.

## References

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