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ARTICLE TYPE

A nonsingular fixed-time sliding mode controller for Robot Manipulators in the presence of external perturbations and partially known model

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Abstract

The current contribution introduces a nonsingular fixed-time sliding mode control (SMC) scheme for position and velocity tracking of robot manipulators. The approach avoids singularities by introducing a new sliding surface with the special attribute that the exponent employed to achieve fixed time convergence depends on the tracking error and is smaller than one except when the error is exactly zero whereas the exponent becomes one at zero, which makes the derivative at zero to be well defined. A new theoretical result has been introduced in the form of a lemma to prove this innovative property. Furthermore, model uncertainties are handled by means of a time varying gain given by a polynomial of the powers of the norms of the tracking and velocities errors. The fixed-time convergence is proven employing Lyapunov theory and the result holds globally. Simulation outcomes confirm the developed theory and the advantages of the proposed scheme are shown qualitatively by comparing its performance with well-known equivalent control schemes.

KEY WORDS

Fixed-time stability, robot manipulators, robustness, trajectory tracking.

1 | INTRODUCTION

Despite the highly nonlinear dynamics of robot manipulators, it is in theory rather direct to design model based control laws for position tracking. Some of the main disadvantages in employing the dynamic model of the system is that there can be parameters uncertainties, unmodeled dynamics and the fact that programming nonlinear equations can be demanding. To overcome these drawbacks, one of the most widely employed strategies is robust control, where model uncertainties are compensated by considering them as perturbations to be rejected.

To achieve exact position tracking, sliding mode control (SMC) theory is a well suited methodology¹. In the recent years, there has been a tendency to achieve fixed-time stability². One of the first results in this field is given by Bhat and Bernstein³, where for the first time a rigorous foundation is provided for the theory of finite-time stability of continuous autonomous systems and a closer examination of finite-time stability as a possible objective in control design is motivated. Polyakov⁴ provides some important lemmas and theorems to analyze two types of nonlinear control algorithms for uncertain linear plants.

A possible way to achieve fixed-time stability is the use of time varying gains in the design of controllers and/or observers. There are multiple recent works based on this feature. For instance, Holloway and Krstic⁵ introduce a state observer with time-varying gains for linear systems that tend to infinity as time approaches a prescribed convergence time. The observer is shown to exhibit fixed-time stability with an arbitrary convergence time, which is prescribed by the user irrespective of initial conditions. Zhang *et al.*⁶ present a fixed-time sliding mode control (FxTSMC) for the global fixed-time trajectory tracking of robot manipulators subject to uncertain dynamics and bounded external disturbances. A fixed-time sliding surface is proposed and a singularity-free FxTSMC is constructed. Lyapunov stability theory is employed to prove stability. However, the fixed-time property is given to reach an arbitrarily small region around the origin, while the errors tend to zero asymptotically. Hou *et al.*⁷ introduce a full-order terminal sliding mode surface based on the bi-limit homogeneous property, such that the sliding

motion is finite-time stable independent of the system's initial condition. The control design is applied to servo motor systems. Aldana-Lopez *et al.*⁸ design a differentiator algorithm with fixed-time convergence with an a priori user-defined upper bound by means of time base generators (a class of time-varying gains). Orlov *et al.*⁹ introduce a hybrid differentiator for any time-varying signals, whose second derivative is uniformly bounded. The proposed observer strategy is in successive applications of rescaled and standard super-twisting observers with finite (time-varying and respectively constant) gains. Li *et al.*¹⁰ study the leader-follower consensus problem for feedforward nonlinear time-delay multi-agent systems under a fixed directed topology. To improve performance, bounded time-varying gains are employed. Zheng *et al.*¹¹ consider also the problem of prescribed finite-time leader-following consensus. A composite controller with a time-varying disturbance observer is presented. Stability is analyzed by taking advantage of the properties of barrier Lyapunov functions and time-varying gains. Boukattaya and Gassara¹² introduce a time-varying nonsingular terminal sliding mode (TSM), where the reaching phase existing in conventional TSM control is suppressed to get global robustness of the system against uncertainties and disturbances. An application to n -links rigid robotic manipulators is shown; however, a nominal model is required for implementation. Cui *et al.*¹³ investigate the problem of prescribed time tracking control for Euler-Lagrange systems in the presence of modeling uncertainties, and time-varying state constraints. The goal is achieved by using two proportional-integral (PI)-like control schemes with time-varying gains. However, only pre-specified precision is achieved and not exact tracking. Razmjooei *et al.*¹⁴ present a time-varying chattering-free disturbance observer-based position tracking control law of serial robotic manipulators to track a reference signal in a finite time. Time-varying gains are provided for the convergence of the position tracking error to a neighborhood of zero in a finite time, while exact tracking is not achieved. Rsetam *et al.*¹⁵ propose a robust FxTSMC by using a fixed-time sliding mode observer (FxTSMO) for the trajectory tracking problem of an aerial flexible joint robot attached to drones system. A cascaded fixed-time sliding mode observer (CFxTSMO) is constructed to estimate the unmeasurable variables and lumped disturbances simultaneously in fixed-time, and to reduce the estimation noise. Steeves and Krstic¹⁶ design robust feedback control laws using time-varying backstepping for linear and nonlinear systems, where bounded disturbances (of unknown bound) corrupt the measurements. The particular high-gain control designs are shown to be practically feasible, *i. e.* they achieve particular fixed-time input-to-state stability (ISS) results with respect to the disturbance by uniformly bounded inputs, for linear time invariant (LTI) systems and nonlinear systems modeling robotic manipulators. However, structural conditions are imposed on the disturbances. Hua *et al.*¹⁷ study the global prescribed-time stabilization problem for a class of time-delay nonlinear systems with uncertain parameters by using two time-varying gains with special properties, in which one is introduced into virtual controllers to achieve prescribed-time convergence and the other one is used to construct the Lyapunov-Krasovskii (L-K) functional and Lyapunov function to handle the nonlinear time-delay term and unknown parameters, respectively. Li and Zhang¹⁸ propose a practical prescribed-time controller which combines sliding mode control and time-varying gains to show that the tracking error is guaranteed to be within a prescribed accuracy after a prescribed time, while no exact tracking is obtained. Gu *et al.*¹⁹ propose a novel finite-time variable-gain active disturbance rejection control method for master-slave teleoperated parallel manipulators with disturbances which incorporates error-based variable gains to improve the control performances, such as finite-time variable-gain tracking differentiator, finite-time variable-gain extended state observer, and finite-time variable-gain controller. However, only ultimate boundedness is achieved. Ning *et al.*²⁰ study the adaptive fixed-time control problem for classes of nonlinear cascade systems with parametric uncertainty. A time-varying gain-based control algorithm to ensure that all system states return to the origin in a fixed-time is introduced. A disadvantage of the approach is that the use of some transformations heavily increases the computation burden. Luo *et al.*²¹ propose a control method which does not involve infinite time-varying gains, leading to practical (ultimate boundedness) and global prescribed time tracking control solutions for strict-feedback systems, in spite of both mismatched and non-vanishing uncertainties.

The present contribution introduces a fixed-time controller for robot manipulators where the main contributions are:

1. a new singularity free sliding surface with variable exponent coefficients is introduced;
2. this new sliding surface employs the hyperbolic tangent in an innovative fashion different from current methodologies to be found in the literature, e.g. in the approaches by Yogi *et al.*²⁸ or by Yan *et al.*²⁹ ;
3. a new lemma is provided to show that the use of variable exponent coefficients leads to the same result as using constant exponents smaller than 1, with the advantage that the derivative at zero is well defined;
4. global stability is obtained by using only well-known structural properties of the robot model to create a time-varying gain given by a polynomial of the powers of the norms of the tracking errors, which avoids a more demanding adaptive approach;
5. the few robot model information required for the global stability result does not need to be accurate.

Simulation results confirm the developed theory. The paper is organized as follows. Section 2 provides some basic preliminaries on robot model properties. Section 3 introduces the new control law and the corresponding stability analysis, while simulation results are given in Section 4. The paper concludes in Section 5.

2 | PRELIMINARES AND MOTIVATION

Consider a n -degrees of freedom rigid robot manipulator whose dynamics can be described by²²:

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} + \boldsymbol{\tau}_p, \quad (1)$$

where $\mathbf{q} \in \mathbb{R}^n$ is the vector of generalized joint coordinates, $\mathbf{H}(\mathbf{q}) \in \mathbb{R}^{n \times n}$ is the symmetric positive definite inertia matrix, $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^n$ is the vector of Coriolis and centrifugal torques, $\mathbf{D} \in \mathbb{R}^{n \times n}$ is a symmetric positive semidefinite matrix of joint viscous friction coefficients, $\mathbf{g}(\mathbf{q}) \in \mathbb{R}^n$ is the vector of gravitational torques, $\boldsymbol{\tau}_p \in \mathbb{R}^n$ represents external perturbations, and $\boldsymbol{\tau} \in \mathbb{R}^n$ is the vector of input torques acting at the joints. Assume for simplicity's sake that the robot has revolute joints only and that velocity measurements are available. Some useful model properties are listed below²²:

Property 1. It holds $\lambda_h \|\mathbf{x}\|^2 \leq \mathbf{x}^T \mathbf{H}(\mathbf{q}) \mathbf{x} \leq \lambda_H \|\mathbf{x}\|^2$ for all $\mathbf{q} \in \mathbb{R}^n$, $\mathbf{x} \in \mathbb{R}^n$, and $0 < \lambda_h \leq \lambda_H < \infty$, with $\lambda_h = \min_{\mathbf{q} \in \mathbb{R}^n} \lambda_{\min}(\mathbf{H}(\mathbf{q}))$ and $\lambda_H = \max_{\mathbf{q} \in \mathbb{R}^n} \lambda_{\max}(\mathbf{H}(\mathbf{q}))$. $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalues of a symmetric matrix, respectively. \triangle

Property 2. By using the Christoffel symbols of the first kind to compute $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$, the matrix $\dot{\mathbf{H}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric. \triangle

Property 3. There exists a positive constant $0 < k_c < \infty$ such that $\|\mathbf{C}(\mathbf{q}, \mathbf{x})\| \leq k_c \|\mathbf{x}\|$ holds $\forall \mathbf{x}, \mathbf{q} \in \mathbb{R}^n$. \triangle

Property 4. The vector of the generalized gravitational torques $\mathbf{g}(\mathbf{q})$ satisfies $\|\mathbf{g}(\mathbf{q})\| \leq k_g$, $k_g > 0$. \triangle

Remark 1. In Properties 1, 3 and 4, it is implicitly understood that the bounds λ_H , k_c and k_g , respectively, are the minimal values for which the properties hold. However, it is evident that they hold for larger values as well. This fact is important because the employment of these constants in the appendix is less restrictive than it appears to be at first sight, granted it is always desirable to have accurate estimates available. \triangle

3 | MAIN RESULT

In this section, a control law for position tracking of robot manipulators in the presence of external perturbations will be developed. Given a bounded desired trajectory $\mathbf{q}_d \in \mathbb{R}^n$ with at least bounded first and second derivatives, the control problem consists in achieving that the tracking error

$$\mathbf{e} = \mathbf{q} - \mathbf{q}_d \quad (2)$$

becomes zero in a predefined fixed-time. Define the reference velocity

$$\dot{\mathbf{q}}_r = \dot{\mathbf{q}}_d - \boldsymbol{\Lambda} \mathbf{e} - \mathbf{K}_e \boldsymbol{\tau}_e, \quad (3)$$

where $\mathbf{\Lambda}, \mathbf{K}_e \in \mathbb{R}^{n \times n}$ are diagonal positive definite matrices. The vector $\boldsymbol{\tau}_e \in \mathbb{R}^n$ is inspired by the controllers in^{23, 24}, where for $i = 1, \dots, n$ its i -th element is defined as

$$\tau_{ei} = \text{sign}(e_i) \frac{\tanh(|e_i|^{1/2})}{\tanh(1)} |e_i|^{1/2} \left(1 - 2 \frac{\tanh(e_i^2)}{\tanh(1)}\right) f_{ei}(e_i) \quad (4)$$

$$f_{ei}(e_i) = |e_i|^{\lambda_{ei} \tanh(e_i^2)}, \quad (5)$$

with λ_{ei} the i -th positive elements of the vector $\boldsymbol{\lambda}_e \in \mathbb{R}^n$, satisfying

$$\lambda_{ei} \geq \frac{1/2 \theta_{ei} + 1}{\tanh(1)}, \quad \theta_{ei} > 1. \quad (6)$$

Based on (3) it is possible to define the following new sliding surface

$$\mathbf{s} = \dot{\mathbf{q}} - \dot{\mathbf{q}}_r = \dot{\mathbf{e}} + \mathbf{\Lambda} \mathbf{e} + \mathbf{K}_e \boldsymbol{\tau}_e. \quad (7)$$

The proposed controller is given by

$$\boldsymbol{\tau} = -\|\mathbf{s}\|^2 \mathbf{K}_v \mathbf{s} - (k(t) \mathbf{I} + \mathbf{K}_s) \boldsymbol{\tau}_s, \quad (8)$$

where $k(t) \geq 0$ is a time varying gain designed to deal with model uncertainties, $\mathbf{K}_s, \mathbf{K}_v \in \mathbb{R}^{n \times n}$ are diagonal positive definite matrices, and for $i = 1, \dots, n$ the i -th element τ_{si} of $\boldsymbol{\tau}_s \in \mathbb{R}^n$ is defined as

$$\tau_{si} = \text{sign}(s_i) |s_i|^{\lambda_{si} \tanh(s_i^2)} \equiv \text{sign}(s_i) f_{si}(s_i), \quad (9)$$

with λ_{si} is the i -th positive element of the vector $\boldsymbol{\lambda}_s \in \mathbb{R}^n$, respectively, with

$$\lambda_{si} > \frac{\theta_{si}}{\tanh(1)}, \quad \theta_{si} > 1. \quad (10)$$

From (7) it is possible to get the error dynamics for \mathbf{e} as

$$\dot{\mathbf{e}} = -\mathbf{\Lambda} \mathbf{e} - \mathbf{K}_e \boldsymbol{\tau}_e + \mathbf{s}, \quad (11)$$

while the following error dynamics for \mathbf{s} can be computed by substituting (8) into (1)

$$\mathbf{H}(\mathbf{q}) \dot{\mathbf{s}} = -\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \mathbf{s} - \mathbf{D} \mathbf{s} + \boldsymbol{\tau}_p - \boldsymbol{\tau}_a - \|\mathbf{s}\|^2 \mathbf{K}_v \mathbf{s} - (k(t) \mathbf{I} + \mathbf{K}_s) \boldsymbol{\tau}_s, \quad (12)$$

where

$$\boldsymbol{\tau}_a = \mathbf{H}(\mathbf{q}) \ddot{\mathbf{q}}_r + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}}_r + \mathbf{D} \dot{\mathbf{q}}_r + \mathbf{g}(\mathbf{q}). \quad (13)$$

Lemma 1. Assume that the coefficients λ_{ei} in (6) are chosen as

$$\lambda_{ei} = \frac{3}{2} + \frac{1}{\tanh(1)} \approx 2.8130. \quad (14)$$

Then, the vector $\boldsymbol{\tau}_a$ in (13) is bounded by a polynomial of the form

$$\|\boldsymbol{\tau}_a\| \leq b_0 + b_1 \|\mathbf{e}\| + b_2 \|\mathbf{e}\|^2 + b_3 \|\dot{\mathbf{e}}\| + b_4 \|\dot{\mathbf{e}}\| \|\mathbf{e}\| + b_5 \|\dot{\mathbf{e}}\| \|\mathbf{e}\|^2 + b_6 \|\dot{\mathbf{e}}\| \|\mathbf{e}\|^3 + b_7 \|\dot{\mathbf{e}}\| \|\mathbf{e}\|^4 = b(t), \quad (15)$$

for some positive known coefficients $b_0, b_1, b_2, b_3, b_4, b_5, b_6$, and b_7 which depend exclusively on the constant bounds given in Properties 1 to 4, the bounds for \mathbf{q}_d , and its derivatives and some control parameters. \triangle

The proof of Lemma 1 can be found in the appendix.

Assumption 1. The external perturbation vector lumps also non-model dynamics and is bounded, *i. e.* $\|\tau_p\| \leq \delta$, for all $t \geq t_0$ and some $\delta > 0$. \triangle

Remark 2. Considering an external perturbation to be bounded is a standard assumption, e.g.^{9,11,23,25}, but including in τ_p unknown dynamics is just for simplicity's sake. Indeed, the left hand side of model (1) fully describes the robot dynamics according to the Euler-Lagrange methodology²², with the exception of dissipative effects like friction. However, there is no general model that covers all possible friction effects. In view of the fact that for a physical system any possible friction effect is bounded, Assumption 1 is justified. \triangle

Theorem 1. Consider the closed-loop error dynamics (11)-(12) generated by substituting the control law (8) into the robot dynamics (1). The system is fixed-time stable if Assumption 1 is fulfilled and

$$k(t) \geq b(t) \cdot \frac{1}{\mu_s} \quad (16)$$

$$\lambda_{\min}(\mathbf{K}_s) > \delta \cdot \frac{1}{\mu_s} \quad (17)$$

$$\mu_s = s_m \left(\max_{i \in \{1, \dots, n\}} (\lambda_{s_i} \tanh(s_m^2)) \right) < 1 \quad (18)$$

hold, with s_m in (36) and δ and $b(t)$ defined in Assumption 1 and Lemma 1, respectively. Furthermore

a) s , e and \dot{e} remain bounded for all $t \geq t_0$, and s becomes zero in a fixed-time no larger than

$$T_s \leq T_{s \max} = T_{s1} + T_{s2} = \frac{\lambda_H^2}{\zeta \lambda_h^2} + \frac{2}{\psi} \sqrt{\frac{\lambda_H}{\lambda_h}}, \quad (19)$$

with

$$\zeta = \frac{2\lambda_{\min}(\mathbf{K}_v)}{\lambda_h} \quad (20)$$

$$\psi = 2 \frac{1}{\lambda_h} (\lambda_{\min}(\mathbf{K}_s) \mu_s - \delta). \quad (21)$$

b) e and \dot{e} become zero in a fixed-time no larger than

$$T_e \leq T_{e \max} = T_{s \max} + T_{e1} + T_{e2} = T_{s \max} + \frac{2}{\lambda_{\min}(\mathbf{K}_e)(\theta_e - 1)} + \frac{g_\infty}{2\lambda_{\min}(\mathbf{K}_e)\mu_e}, \quad (22)$$

where λ_h and λ_H are defined in Property 1, θ_e is given in (59), μ_e in (67), and g_∞ is a finite value given in (A7).

Proof. First note that $f_{e_i}(e_i)$ in (5) or $f_{s_i}(s_i)$ in (9) are continuous at $e_i = 0$ and $s_i = 0$, respectively, with $f_{e_i}(0) = f_{s_i}(0) = 1$, so that the right-hand sides of (4) and (9) are locally bounded (see (A34)). Furthermore, for the sake of generality, set

$$f_T(x) = \frac{\tanh(x)}{x}, \quad (23)$$

for $x \in \mathbb{R}$. Since $|x| > \tanh(|x|)$ for $x \neq 0$, then $f_T(x)$ is strictly decreasing²⁶. Furthermore, $f_T(0) = 1$ as can be shown by using L'Hôpital's rule

$$\lim_{x \rightarrow 0} f_T(x) = \lim_{x \rightarrow 0} \frac{\frac{1}{\cosh^2(x)}}{1} = 1. \quad (24)$$

The fact that $f_T(x)$ is decreasing with its maximum at $x = 0$ means that for any $a > 0$ and any $x \in [0, 1]$ one must have

$$\frac{\tanh(ax)}{ax} \geq \frac{\tanh(a)}{a} \Rightarrow \tanh(ax) \geq \tanh(a)x. \quad (25)$$

The steps of the proof are:

a) For the error dynamics given by (12), consider the following positive definite function

$$V(s) = \frac{1}{\lambda_h} s^T \mathbf{H}(q) s, \quad (26)$$

which after Property 1 satisfies

$$\|s\|^2 \leq V(s) \leq \frac{\lambda_H}{\lambda_h} \|s\|^2. \quad (27)$$

This shows that $V(s) : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$ is a continuous radially unbounded function for $s \in \mathbb{R}^n$. The derivative of V along (12) can be shown to satisfy

$$\begin{aligned} \dot{V} &= \frac{2}{\lambda_h} s^T \mathbf{H}(q) \dot{s} + \frac{1}{\lambda_h} s^T \dot{\mathbf{H}}(q) s \\ &= \frac{1}{\lambda_h} s^T \left(\dot{\mathbf{H}}(q) - 2\mathbf{C}(q, \dot{q}) \right) s + \frac{2}{\lambda_h} s^T (\tau_p - \tau_a) - \frac{2}{\lambda_h} s^T \mathbf{D} s - \frac{2}{\lambda_h} s^T (k(t)\mathbf{I} + \mathbf{K}_s) \tau_s - \frac{2}{\lambda_h} \|s\|^2 s^T \mathbf{K}_v s. \end{aligned} \quad (28)$$

By using Property 2, (15), (20), Assumption 1 and the fact that $\mathbf{D} \geq \mathbf{O}$, one has

$$\begin{aligned} \dot{V} &\leq -\frac{2}{\lambda_h} s^T (k(t)\mathbf{I} + \mathbf{K}_s) \tau_s + \frac{2}{\lambda_h} \|s\| (\delta + b(t)) - \frac{2\lambda_{\min}(\mathbf{K}_v)}{\lambda_h} \|s\|^4 \\ &\leq -\frac{2}{\lambda_h} s^T (k(t)\mathbf{I} + \mathbf{K}_s) \tau_s + \frac{2}{\lambda_h} (\delta + b(t)) \sum_{i=1}^n |s_i| - \zeta \|s\|^4, \end{aligned} \quad (29)$$

where $\|s\| \leq |s_1| + \dots + |s_n|$ has been used, while s_i is the i -th element of s for $i = 1, \dots, n$. According to (9), the first two terms in (29) satisfy

$$\begin{aligned} -\frac{2}{\lambda_h} s^T (k(t)\mathbf{I} + \mathbf{K}_s) \tau_s + \frac{2}{\lambda_h} (\delta + b(t)) \sum_{i=1}^n |s_i| &= -\sum_{i=1}^n \frac{2k(t)}{\lambda_h} s_i \text{sign}(s_i) |s_i|^{\lambda_{s_i} \tanh(s_i^2)} - \sum_{i=1}^n \frac{2k_{s_i}}{\lambda_h} s_i \text{sign}(s_i) |s_i|^{\lambda_{s_i} \tanh(s_i^2)} \\ &\quad + \frac{2}{\lambda_h} (\delta + b(t)) \sum_{i=1}^n |s_i|, \end{aligned} \quad (30)$$

where k_{s_i} is the i -th element of the diagonal of \mathbf{K}_s for $i = 1, \dots, n$. This allows to analyze separately the terms

$$-\frac{2k(t)}{\lambda_h} s_i \text{sign}(s_i) |s_i|^{\lambda_{s_i} \tanh(s_i^2)} + \frac{2}{\lambda_h} b(t) |s_i|, \quad (31)$$

and

$$-\frac{2k_{s_i}}{\lambda_h} s_i \text{sign}(s_i) |s_i|^{\lambda_{s_i} \tanh(s_i^2)} + \frac{2}{\lambda_h} \delta |s_i|, \quad (32)$$

for $i = 1, \dots, n$ according to the following two cases:

i) For each of the terms in (32), if $|s_i| > 1$ then

$$-\frac{2k_{s_i}}{\lambda_h} |s_i|^{\lambda_{s_i} \tanh(s_i^2)+1} + \frac{2}{\lambda_h} \delta |s_i| \leq -\frac{2k_{s_i}}{\lambda_h} |s_i|^{\lambda_{s_i} \tanh(s_i^2)+1} + \frac{2}{\lambda_h} \delta |s_i|^{\lambda_{s_i} \tanh(s_i^2)+1} \leq -\frac{2(k_{s_i} - \delta)}{\lambda_h} |s_i|^{\theta_{s_i} + 1} \leq 0, \quad (33)$$

where it has been taken advantage of the fact that $|s_i| \leq |s_i|^{\lambda_{si} \tanh(s_i^2) + 1}$ whenever $|s_i| > 1$, while from (10) it also holds

$$\lambda_{si} \tanh(s_i^2) + 1 > \lambda_{si} \tanh(1) + 1 > \theta_{si} + 1 > 2. \quad (34)$$

After (17) one has $(k_{si} - \delta) > 0$ because $k_{si} \geq \lambda_{\min}(\mathbf{K}_s)$ and $\mu_s < 1$. The same analysis can be made for (31). By taking (16) into account, one gets

$$-\frac{2k(t)}{\lambda_h} s_i \text{sign}(s_i) |s_i|^{\lambda_{si} \tanh(s_i^2)} + \frac{2}{\lambda_h} b(t) |s_i| \leq -\frac{2(k(t) - b(t))}{\lambda_h} |s_i|^{\theta_{si} + 1} \leq 0. \quad (35)$$

ii) If $|s_i| \leq 1$, then for $f_{si}(s_i)$ in (9), it can be shown that $f_{si}(0) = 1$ (see (A34)), while obviously $f_{si}(1) = 1$. This means that $f_{si}(s_i)$ must have a minimum for $|s_i| \in (0, 1)$. To find it, the derivative must be calculated and equaled to zero:

$$\frac{d}{ds_i} f_{si}(s_i) = \frac{d}{ds_i} \exp(\lambda_{si} \tanh(s_i^2) \ln(|s_i|)) = \lambda_{si} f_{si}(s_i) \left(\frac{2s_i \ln(|s_i|)}{\cosh^2(s_i^2)} + \frac{\tanh(s_i^2)}{s_i} \right) \Bigg|_{s_i=s_m} = 0, \quad (36)$$

where $\frac{d}{ds_i} |s_i| = \frac{d}{ds_i} (s_i^2)^{\frac{1}{2}} = \frac{1}{|s_i|} s_i = \text{sign}(s_i)$ has been used. Since $f_{si}(s_i) > 0$, the second term of the product must be zero, which happens at $|s_i| = s_m$ with $s_m \approx 0.5829$. This means that for $s_i \in [-1, 1]$ one has

$$\mu_{si} = s_m^{\lambda_{si} \tanh(s_m^2)} \leq |s_i|^{\lambda_{si} \tanh(s_i^2)} \leq 1, \quad (37)$$

and therefore (32) satisfies

$$-\frac{2k_{si}}{\lambda_h} |s_i| |s_i|^{\lambda_{si} \tanh(s_i^2)} + \frac{2}{\lambda_h} \delta |s_i| \leq -\frac{2k_{si} \mu_{si}}{\lambda_h} |s_i| + \frac{2}{\lambda_h} \delta |s_i| = -\frac{2}{\lambda_h} (k_{si} \mu_{si} - \delta) |s_i| \leq 0, \quad (38)$$

where once again the last inequality holds in view of (17). By taking into account (16), the same analysis can be made for (31) to get for $|s_i| \leq 1$ that

$$-\frac{2k(t)}{\lambda_h} s_i \text{sign}(s_i) |s_i|^{\lambda_{si} \tanh(s_i^2)} + \frac{2}{\lambda_h} b(t) |s_i| \leq -\frac{2}{\lambda_h} (k(t) \mu_{si} - b(t)) |s_i| \leq 0. \quad (39)$$

No matter whether Case i) or Case ii) holds, one has from the previous analysis and (29) that

$$\dot{V} \leq 0, \quad (40)$$

as long as (16)-(17) are satisfied. This shows that $\mathbf{s} \in \mathcal{L}_\infty$ and therefore there exists a positive constant $s_\infty > 0$, such that $\|\mathbf{s}\| \leq s_\infty$ for all $t \geq t_0$. To prove that $\mathbf{e}, \dot{\mathbf{e}} \in \mathcal{L}_\infty$ as well, choose $V_e(\mathbf{e}) = \mathbf{e}^T \mathbf{e}$, whose derivative along (11) satisfies

$$\begin{aligned} \dot{V}_e &= -2\mathbf{e}^T \mathbf{A} \mathbf{e} - 2\mathbf{e}^T \mathbf{K}_e \boldsymbol{\tau}_e + 2\mathbf{e}^T \mathbf{s} \leq -2\lambda_{\min}(\mathbf{A}) \|\mathbf{e}\|^2 + 2\|\mathbf{e}\| s_\infty \\ &\leq -\lambda_{\min}(\mathbf{A}) \|\mathbf{e}\|^2 - \|\mathbf{e}\| (\lambda_{\min}(\mathbf{A}) \|\mathbf{e}\| - 2s_\infty) \\ &\leq -\lambda_{\min}(\mathbf{A}) \|\mathbf{e}\|^2 \leq 0 \end{aligned} \quad \text{for } \|\mathbf{e}\| \geq \frac{2s_\infty}{\lambda_{\min}(\mathbf{A})} = \mu. \quad (41)$$

According to Theorem 4.18 in²⁷, the solutions of (11) remain bounded for all time (they are ultimately bounded in fact). Therefore, since $\mathbf{e} \in \mathcal{L}_\infty$, then $\dot{\mathbf{e}} \in \mathcal{L}_\infty$ from (11).

To show that \mathbf{s} becomes zero in a finite time, note first that from (27) one has

$$V^2 \leq \frac{\lambda_H^2}{\lambda_h^2} \|\mathbf{s}\|^4 \Rightarrow -\|\mathbf{s}\|^4 \leq -\frac{\lambda_h^2}{\lambda_H^2} V^2, \quad (42)$$

so that \dot{V} in (29) satisfies

$$\dot{V} \leq -\frac{2}{\lambda_h} \left(\mathbf{s}^T (k(t) \mathbf{I} + \mathbf{K}_s) \boldsymbol{\tau}_s - (\delta + b(t)) \sum_{i=1}^n |s_i| \right) - \zeta \frac{\lambda_h^2}{\lambda_H^2} V^2 \leq 0. \quad (43)$$

Equation (43) holds for all $t \geq t_0$, and therefore

$$\dot{V} \leq -\zeta \frac{\lambda_h^2}{\lambda_H^2} V^2 \leq 0 \quad (44)$$

must be satisfied as well. Assume that $V(t_0) > 1$. Then, according to Lemma 2 in the appendix, $V(t) \leq 1$ holds from a fixed-time

$$t \geq t_0 + \frac{1}{\zeta \frac{\lambda_h^2}{\lambda_H^2} (2-1)} = t_0 + \frac{\lambda_H^2}{\zeta \lambda_h^2} = t_0 + T_{s1}. \quad (45)$$

Of course, if $V(t_0) \leq 1$ then $T_{s1} = 0$. Since (43) holds for all $t \geq t_0$, then it can be used again with the additional information that for $t \geq t_0 + T_{s1}$ it is known that $V(t) \leq 1$. In fact, if (43) holds, then

$$\dot{V} \leq -\frac{2}{\lambda_h} \left(\mathbf{s}^T (k(t)\mathbf{I} + \mathbf{K}_s) \boldsymbol{\tau}_s - (\delta + b(t)) \sum_{i=1}^n |s_i| \right) \leq 0 \quad (46)$$

holds as well. Note that according to (27), $V(t) \leq 1$ implies that from $t \geq t_0 + T_{s1}$ one must have

$$s_i^2 \leq \|\mathbf{s}\|^2 \leq V(\mathbf{s}) \leq 1 \quad \Rightarrow \quad |s_i| \leq 1 \quad (47)$$

for $i = 1, \dots, n$, where s_i is the i -th element of \mathbf{s} . In view of this fact, *all* the elements s_i fall in the Case ii) described above and the terms of $\dot{V}(t)$ in (46) can be analyzed by taking into account (38)-(39) to get

$$-\frac{2}{\lambda_h} \left(\mathbf{s}^T (k(t)\mathbf{I} + \mathbf{K}_s) \boldsymbol{\tau}_s - (\delta + b(t)) \sum_{i=1}^n |s_i| \right) \leq -\frac{2}{\lambda_h} \sum_{i=1}^n ((k_{si} + k(t))\mu_{si} - (\delta + b(t))) |s_i| \leq -\frac{2}{\lambda_h} (\lambda_{\min}(\mathbf{K}_s)\mu_s - \delta) \sum_{i=1}^n |s_i| \leq 0, \quad (48)$$

where (16)-(18) have been used. By taking into account (21), from (46) and (48) one has

$$\dot{V} \leq -\psi \sum_{i=1}^n |s_i| \leq 0. \quad (49)$$

Since $\sum_{i=1}^n |s_i| = |s_1| + \dots + |s_n| \geq \|\mathbf{s}\|$, then one has

$$\dot{V} \leq -\psi \|\mathbf{s}\| \leq 0. \quad (50)$$

Now, from (27) one has

$$V^{\frac{1}{2}} \leq \sqrt{\frac{\lambda_H}{\lambda_h}} \|\mathbf{s}\| \quad \Rightarrow \quad -\|\mathbf{s}\| \leq -\sqrt{\frac{\lambda_h}{\lambda_H}} V^{\frac{1}{2}}, \quad (51)$$

which means that (50) satisfies

$$\dot{V} \leq -\psi \sqrt{\frac{\lambda_h}{\lambda_H}} V^{\frac{1}{2}} \leq 0. \quad (52)$$

According to Lemma 3 in the appendix, once $V(t_0 + T_{s1}) \leq 1$ then $V(t) \equiv 0$ in a time T_{s2} no larger than

$$T_{s2} = \frac{1}{\psi \sqrt{\frac{\lambda_h}{\lambda_H}} \left(1 - \frac{1}{2}\right)} = \frac{2}{\psi} \sqrt{\frac{\lambda_H}{\lambda_h}}. \quad (53)$$

$T_{s \max}$ in (19) is gotten by considering (45) and (53).

b) In the previous step it was shown that $\mathbf{e}, \dot{\mathbf{e}}, \mathbf{s} \in \mathcal{L}_\infty$ and that \mathbf{s} becomes zero in fixed-time. To show that both \mathbf{e} and $\dot{\mathbf{e}}$ become zero in fixed-time, consider from (11) that once $\mathbf{s} \equiv \mathbf{0}$, the problem is equivalent to analyzing the fixed-time stability of n

independent systems of the form

$$\dot{e}_i = -\lambda_i e_i - k_{ei} \tau_{ei}, \quad (54)$$

where λ_i and k_{ei} are the i -th elements of the diagonals of $\mathbf{\Lambda}$ and \mathbf{K}_{ei} , respectively, for $i = 1, \dots, n$. For each of these subsystems define the positive definite functions

$$V_{ei} = e_i^2, \quad (55)$$

whose derivatives along (54) satisfy

$$\dot{V}_{ei} \leq -2k_{ei} \frac{\tanh(|e_i|^{\frac{1}{2}})}{\tanh(1)} |e_i| |e_i|^{\frac{1}{2}} \left(1 - 2 \frac{\tanh(e_i^2)}{\tanh(1)}\right) f_{ei}(e_i) \quad (56)$$

by using (4). Now, two cases will be considered.

i) If $|e_i| \geq 1$, then by using (5) eq. (56) satisfies

$$\dot{V}_{ei} \leq -2k_{ei} |e_i| |e_i|^{\frac{1}{2}} \left(1 - 2 \frac{\tanh(e_i^2)}{\tanh(1)}\right) |e_i|^{\lambda_{ei} \tanh(e_i^2)} \leq -2k_{ei} |e_i| \left(\lambda_{ei} - \frac{1}{\tanh(1)}\right) \tanh(e_i^2)^{\frac{3}{2}} \leq -2k_{ei} |e_i|^{\frac{\theta_{ei}}{2} \frac{\tanh(e_i^2)}{\tanh(1)} + \frac{3}{2}} \leq -2k_{ei} |e_i|^{\frac{\theta_{ei}+3}{2}} \leq -2k_{ei} V_{ei}^{\frac{\theta_{ei}+3}{4}}, \quad (57)$$

where $\frac{\tanh(|e_i|^{\frac{1}{2}})}{\tanh(1)} \geq 1$, $\frac{\tanh(e_i^2)}{\tanh(1)} \geq 1$, and (6) have been used. By the definition of θ_{ei} one has $\frac{\theta_{ei}+3}{4} > 1$. Hence, according to Lemma 2, $V_{ei} \leq 1$ in a time no larger than

$$T_{e1i} \leq \frac{2}{k_{ei}(\theta_{ei} - 1)}. \quad (58)$$

Define

$$T_{e1} = \frac{2}{\lambda_{\min}(\mathbf{K}_e)(\theta_e - 1)} \quad \text{with} \quad \theta_e = \min_{i \in [1, n]} \theta_{ei} \quad (59)$$

to show that $\|e\| \leq 1$ from $t \geq t_0 + T_s + T_{e1}$.

ii) From $t \geq t_0 + T_s + T_{e1}$ it is guaranteed that $|e_i| \leq 1$ for $i = 1, \dots, n$, so that after (25) one has

$$\frac{\tanh(|e_i|^{\frac{1}{2}})}{\tanh(1)} \geq |e_i|^{\frac{1}{2}}, \quad (60)$$

which shows that (56) satisfies

$$\dot{V}_{ei} \leq -2k_{ei} |e_i|^{\frac{1}{2}} |e_i| |e_i|^{\frac{1}{2}} \left(1 - 2 \frac{\tanh(e_i^2)}{\tanh(1)}\right) |e_i|^{\lambda_{ei} \tanh(e_i^2)} = -2k_{ei} |e_i| \left(2 - \frac{\tanh(e_i^2)}{\tanh(1)}\right) |e_i|^{\lambda_{ei} \tanh(e_i^2)}. \quad (61)$$

As done before for s_i , from (37) it holds

$$\mu_{ei} = s_m^{\lambda_{ei} \tanh(s_m^2)} \leq |e_i|^{\lambda_{ei} \tanh(e_i^2)} \leq 1, \quad (62)$$

for $e_i \in [-1, 1]$. Therefore, (61) satisfies

$$\dot{V}_{ei} \leq -2k_{ei} \mu_{ei} |e_i| \left(2 - \frac{\tanh(e_i^2)}{\tanh(1)}\right) = -2k_{ei} \mu_{ei} V_{ei} \left(1 - \frac{1}{2} \frac{\tanh(e_i^2)}{\tanh(1)}\right). \quad (63)$$

Note that similarly to (60) one must have

$$\frac{\tanh(e_i^2)}{\tanh(1)} \geq e_i^2 = V_{ei} \quad (64)$$

because $e_i^2 \leq 1$. This means that \dot{V}_{ei} satisfies

$$\dot{V}_{ei} \leq -2k_{ei}\mu_{ei}V_{ei}^{(1-\frac{1}{2}V_{ei})}. \quad (65)$$

Therefore, Lemma 4 can be applied to show that $e_i \equiv 0$ in a time no larger than

$$T_{e2i} \leq \frac{g_\infty}{2k_{ei}\mu_{ei}}, \quad (66)$$

with g_∞ in (A7). Define

$$T_{e2} = \frac{g_\infty}{2\lambda_{\min}(\mathbf{K}_e)\mu_e} \quad \text{with} \quad \mu_e = \min_{i \in [1,n]} \mu_{ei} \quad (67)$$

to show that $\|e\| \equiv 0$ in a time no larger than $t_0 + T_s + T_e$, with T_e given in (22) as the sum of T_{e1} in (59) and T_{e2} in (67).

Finally, once $e \equiv 0$ and $s \equiv 0$, then from its definition in (4) $\tau_e \equiv 0$, so that from (11) one also gets $\dot{e} \equiv 0$, which concludes the proof. \square

Remark 3. The proof of Lemma 1 in the appendix shows that the proposed scheme is singularity free. \triangle

Remark 4. The use of the hyperbolic tangent, which is one of the key characteristics of the present approach, is not new in the literature. For example, similar functions to f_{ei} and f_{si} in (5) and (9), respectively, are defined by Yogi *et al.*²⁸. However, they are not employed in the sliding surface, on the contrary to present the work. Also, Yan *et al.*²⁹ apply the hyperbolic tangent to avoid the use of the sign function. Nevertheless, the mathematical analysis yields only to ultimate boundedness, also in clear contrast with the current proposal. \triangle

Remark 5. Many of the fixed-time controllers for robot manipulators actually achieve only exponential asymptotic stability, just as the comparative algorithms used in the simulation section^{6,30}. The fixed-time refers to the time needed for the error to get into an arbitrarily small region around the origin. On the contrary, the present approach achieves fixed-time stability of the tracking errors. The control law needs few information of the robot dynamic model in the form of the polynomial $b(t)$ in (15) to achieve global stability. This few information does not even need to be accurate since it can easily be compensated by increasing the values of the coefficients of $b(t)$. In fact, this trial and error approach avoids the computation of the robot model and is less time consuming. Furthermore, even though a time varying gain is being employed, it does not rely on any adaptive approach, largely simplifying the stability analysis and specially the implementation. The simulation sections shows a better performance with respect to some similar algorithms. \triangle

4 | SIMULATION RESULTS

To test the control law proposed in the previous section, the three Degrees of Freedom robot model described in³¹ is employed (see the reference for details). The simulations are carried out using *Simulink* by MATLAB running in a MacBook Pro with an Apple M1 Chip. An automatic solver selection has been chosen with a maximal step time of 10^{-4} [s]. The desired trajectories have been chosen to be the sum of six different sinusoidal signals for each joint with non related frequencies, as it can be appreciated in Figure 1. See³² for the explicit equations of these trajectories. The algorithms to be implemented are:

1. The globally stable control law given by (8)
2. The algorithm introduced by Zhang *et al.*⁶
3. The algorithm given by Yang and Su³⁰

The schemes in^{6,30} have been chosen because they share almost all the same characteristics as the current proposal, with the exception that the errors tend to zero asymptotically once an ultimate arbitrarily small bound has been reached in a predefined fixed-time. Furthermore, both algorithms include explicitly robot model information, which is assumed to be uncertain. To provide an advantage over the proposed scheme, the models are assumed to be exactly known for the simulations. For the proposed scheme, the gains for the controller given by equations (3)-(4) and (8)-(9) have been chosen as $\Lambda = 100\mathbf{I}$, $\mathbf{K}_e = 150\mathbf{I}$, $\lambda_e = 3\mathbf{I}$, $\lambda_s = 2\mathbf{I}$, $\mathbf{K}_s = \text{diag}\{200 \ 100 \ 35\}$, $\mathbf{K}_v = \text{drag}\{150 \ 30 \ 10\}$, and $k(t) = 1 + \|e\| + \|e\|^2 + \|\dot{e}\| + \|\dot{e}\|\|e\| + \|\dot{e}\|\|e\|^2 + \|\dot{e}\|\|e\|^3 + \|\dot{e}\|\|e\|^4$. As to the algorithms in^{6,30}, the interested reader is referred to these works for details. Both algorithms share the following information $\gamma_1 = 0.0873$, $\gamma_2 = 4.6129$, $\mathbf{H}_0 = \frac{2}{\gamma_1 + \gamma_2}\mathbf{I}$, $\mathbf{C}_0(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$, $\mathbf{D}_0 = \mathbf{D}$, and $\mathbf{g}_0(\mathbf{q}) = \mathbf{g}(\mathbf{q})$. The rest of the gains for the controller by Zhang *et al.*⁶ are $\delta = 0.3$, $\alpha = 0.7$, $r = 1.7$, $\beta = 1.9$, $\mathbf{C}_1 = 3\mathbf{I}$, $\mathbf{C}_2 = 3\mathbf{I}$, $\mathbf{K}_1 = \mathbf{K}_s$, $\mathbf{K}_2 = \mathbf{K}_s$, $\nu_1 = 2.5$, $\nu_2 = 0.5$, $a_0 = 12$, and $a_1 = 2.2$. As to the scheme by Yang and Su³⁰ it was set $\rho_{01} = \rho_{02} = \rho_{03} = 0.25$, $\rho_{\infty 1} = \rho_{\infty 2} = \rho_{\infty 3} = 0.002$, $T_1 = T_2 = T_3 = 1.2$, $l_1 = l_2 = l_3 = 1.5$, $\mathbf{K}_0 = \mathbf{K}_s$, $\mathbf{K}_1 = 0.001\mathbf{I}$, $\mathbf{K}_2 = 0.1\mathbf{I}$, $\sigma = 1.5$, $d_M = 75$, $r = 1.2$, $\delta = 0.01$, $p = 0.5$, and $k = 2$.

For the simulation the robot's initial positions are given by

$$\mathbf{q}(0) = \left[0, 0, 90 \frac{\pi}{180} \right]^T [\text{rad}] \quad \Rightarrow \quad \mathbf{e}(0) = \left[-10 \frac{\pi}{180}, 10 \frac{\pi}{180}, 10 \frac{\pi}{180} \right]^T [\text{rad}], \quad (68)$$

while some perturbations have been added in the form of τ_p as given in eq. (1) (see Figure 4). Figure 1 appears to show a similar performance of all algorithms for trajectory tracking, but in Figure 2 it can be clearly appreciated that the current scheme achieves a faster and more accurate tracking with little chattering. Also, Figure 3 shows a superior performance in velocity tracking. To find out which of the schemes achieves a more precise tracking after the transient response, both in position and velocity, the Root Mean Square Errors (RMSE) are computed as $\text{RMSE}(e) = \sqrt{\frac{1}{m} \sum_{k=1}^m e(k)^2}$, where m is the sample size and e is the error of interest. Since from Figure 1 it is apparent that the three approaches reach the steady state before $t = 2\text{s}$, the RMSE's are computed for $t \in [2, 10]\text{s}$. Table 1 confirms that the proposed scheme obtains a better position tracking, although the comparative algorithm by Zhang *et al.*⁶ also has a very good performance. The proposal by Yang and Su³⁰ shows an acceptable performance but with larger errors in comparison. The result can also be appreciated in Table 2 for the velocity errors, where once again it is confirmed that the controller (8) performs better than the approaches in^{6,30}, despite it is the only scheme which does not use the robot model. Figure 5 clearly reveals that the input torques generated by the current proposal are smoother and smaller in amplitude than those generated by the comparative algorithms. The figure shows the behavior of the input torque at the beginning of the simulation, when huge peaks usually appear for large gains, and then for a representative time lapse of the steady state.

TABLE 1 Position tracking RMSE.

$e[\text{rad}]$	Control (8)	Control by Zhang <i>et al.</i> ⁶	Control by Yang and Su ³⁰
e_1	$8.9358 \cdot 10^{-8}$	$1.7127 \cdot 10^{-6}$	0.0198
e_2	$4.6283 \cdot 10^{-8}$	$4.7257 \cdot 10^{-6}$	0.0025
e_3	$9.2335 \cdot 10^{-8}$	$4.2938 \cdot 10^{-6}$	0.0013

TABLE 2 Velocity tracking RMSE.

$\dot{e}[\text{rad/s}]$	Control (8)	Control by Zhang <i>et al.</i> ⁶	Control by Yang and Su ³⁰
\dot{e}_1	$3.3078 \cdot 10^{-5}$	$4.571 \cdot 10^{-5}$	0.003
\dot{e}_2	$4.9426 \cdot 10^{-5}$	$1.702 \cdot 10^{-4}$	$2.444 \cdot 10^{-4}$
\dot{e}_3	$8.4852 \cdot 10^{-5}$	$6.605 \cdot 10^{-4}$	$6.573 \cdot 10^{-4}$

Remark 6. The sliding surface in (7) is different from that proposed by Moulay *et al.*²⁴. As pointed out by Su³³, the sliding surface given by Moulay *et al.*²⁴ yields to discontinuities of the elements s_i of s as $|e_i| \rightarrow 0$. This is not the case for s in (7) because it is continuous at $e = \mathbf{0}$. This property can be clearly appreciated in Figures 2 and 6, where the behavior of s is shown. \triangle

Remark 7. Just considering conditions (16)-(17) yields to a rather poor gain tuning and, therefore, it is not evident how to tune the different control gains of the current proposal. Some simple empirical rules to achieve a good performance are the following:

1. Choose λ_{si} to satisfy (10) and λ_{ei} as given in (14).
2. Set $\mathbf{K}_e = \mathbf{O}$, where $\mathbf{O} \in \mathbb{R}^{n \times n}$ is the null matrix, and choose Λ in (7) according to previous experience as cutoff frequencies for first order linear stable filters.
3. Set $k(t) = 0$ and $\mathbf{K}_s = \mathbf{O}$ and slowly increase \mathbf{K}_v to compensate gravity terms. Then, slowly increase \mathbf{K}_s to yield \mathbf{s} around zero.
4. Choose $k(t)$ to have the same structure as $b(t)$ in (15) but satisfying (16) to yield \mathbf{s} to zero. If no robot model information is available at all, choose all the coefficient of the polynomial to be 1, multiply it by a constant gain and increase it slowly until a desired performance is reached.
5. Slowly increase \mathbf{K}_e to yield \mathbf{e} and $\dot{\mathbf{e}}$ to zero. It is worth noting that large values of \mathbf{K}_e improve precision, but they also create bigger input peaks. \triangle

Remark 8. As pointed out by Mishra *et al.*³⁴, the fixed-time convergence is independent of the initial conditions. This can be verified by using the relationship (22) to get $T_{e_{\max}} = 207.9035[\text{s}]$. After Figure 2 this value is correct, but it appears to be over-estimated. The reason for this apparent over-estimation is that this value contemplates the rather unrealistic case that $\|e(0)\| = \infty$. Since it is impossible to run a simulation with such a large initial error condition, it has been chosen an error ten times larger per joint by setting

$$\mathbf{q}(0) = \left[-90 \frac{\pi}{180}, 90 \frac{\pi}{180}, -20 \frac{\pi}{180} \right]^T [\text{rad}] \quad \Rightarrow \quad \mathbf{e}(0) = \left[-100 \frac{\pi}{180}, 100 \frac{\pi}{180}, -100 \frac{\pi}{180} \right]^T [\text{rad}], \quad (69)$$

The outcomes are presented in Figure 7, where it can be appreciated that despite the initial errors are larger, they tend to zero as fast as before, i.e. with $T \leq 2[\text{s}]$, thus ratifying the insensitivity of the results regarding the initial conditions. Note that according to Table I in³⁴, this settling time is better than all of the schemes studied in the reference. However, it is worthy to point out that to achieve this insensitivity the input torques must become much bigger. Rather than showing this fact in a figure, it is more interesting to show the time varying gain $k(t)$, i.e. the gain that compensates the unknown dynamics. Recall that it has been chosen to have the same structure of the polynomial $b(t)$ in (15), with all the coefficients set to 1. This means that $k(t)$ tends to a constant value as $\mathbf{e}, \dot{\mathbf{e}} \rightarrow \mathbf{0}$. Figure 8 shows this behavior, where it can be seen that the compensation term becomes larger depending on the initial condition, that it makes look $k(t) \equiv 1$ from the beginning when choosing (68). The conclusion is that for large initial conditions the fixed-time should be chosen larger to avoid huge input torques. \triangle

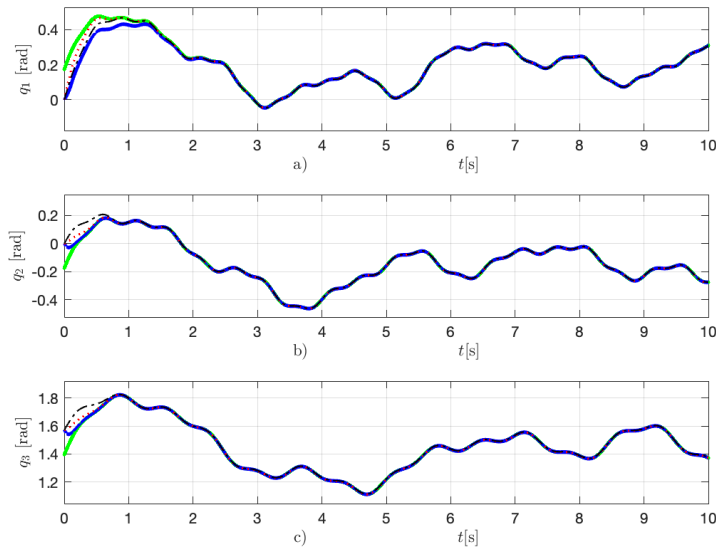


FIGURE 1 Joints positions tracking. Proposed algorithm (8) (—), comparative algorithm by Zhang *et al.*⁶ (⋯⋯⋯), comparative algorithm by Yang and Su³⁰ (- · - ·), and desired trajectories (⋯⋯⋯).

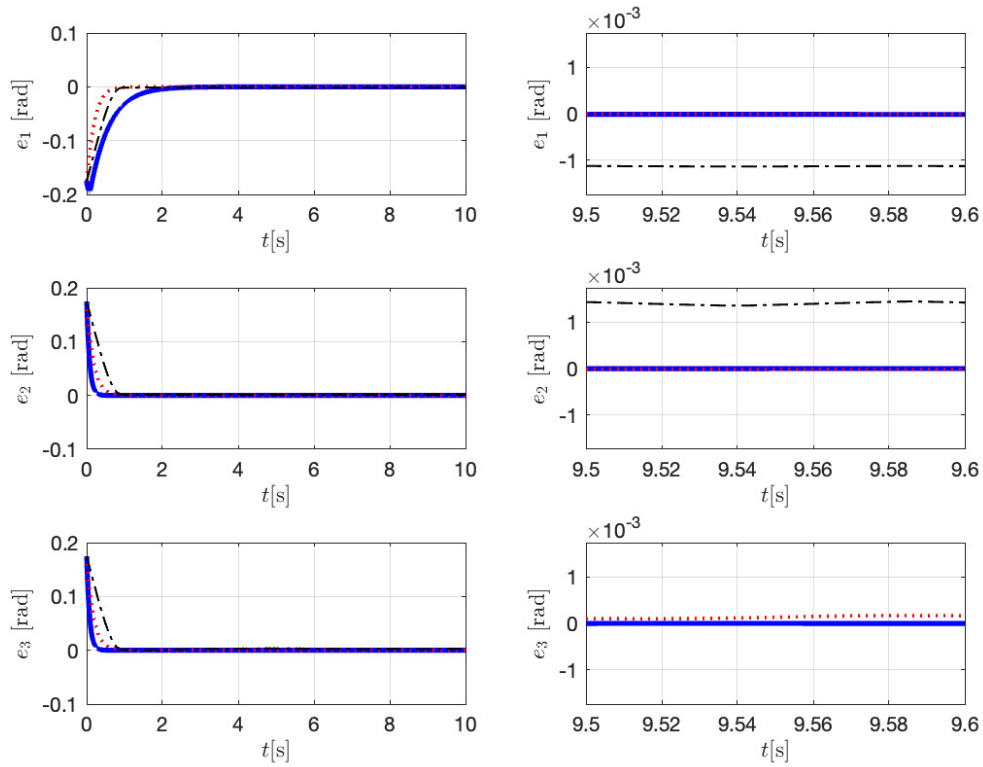


FIGURE 2 Tracking errors. Proposed algorithm (8) (—), comparative algorithm by Zhang *et al.*⁶ (⋯), and comparative algorithm by Yang and Su³⁰ (· · ·). Left column: Behavior for $t \in [0, 10]$ [s]. Right column: Behavior for $t \in [9.5, 9.6]$ [s] and a scale $[-10^{-3}, 10^{-3}]$ [rad].

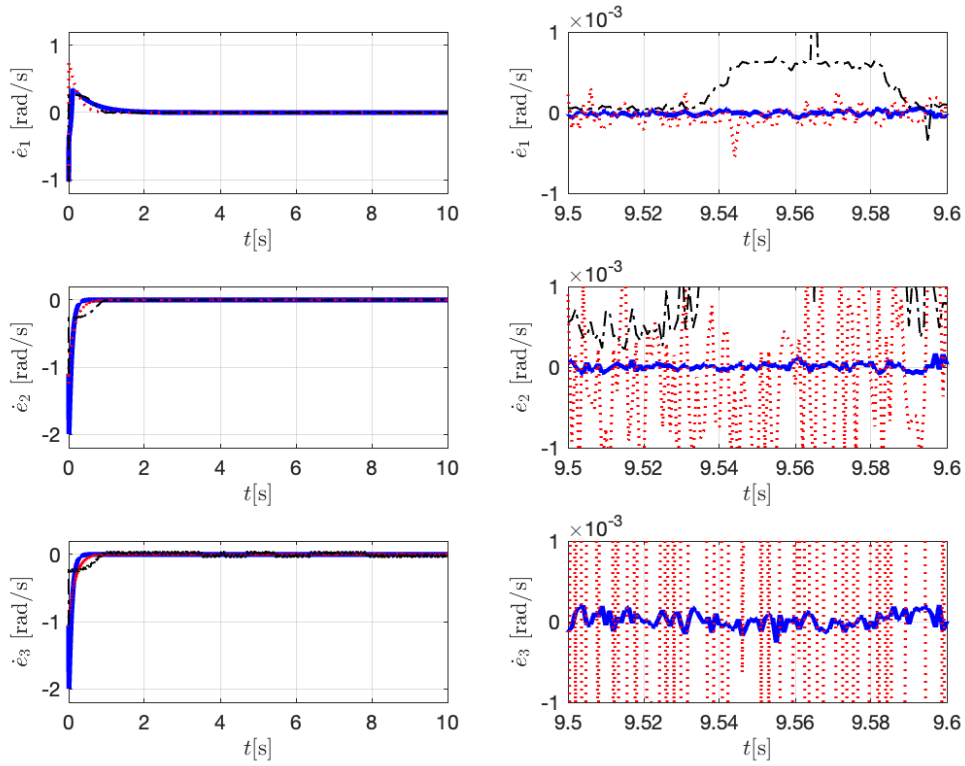


FIGURE 3 Velocity errors. Proposed algorithm (8) (—), comparative algorithm by Zhang *et al.*⁶ (· · · · ·), and comparative algorithm by Yang and Su³⁰ (· - · - ·). Left column: Behavior for $t \in [0, 10][s]$. Right column: Behavior for $t \in [9.5, 9.6][s]$ and a scale $[-10^{-3}, 10^{-3}][\text{rad/s}]$.

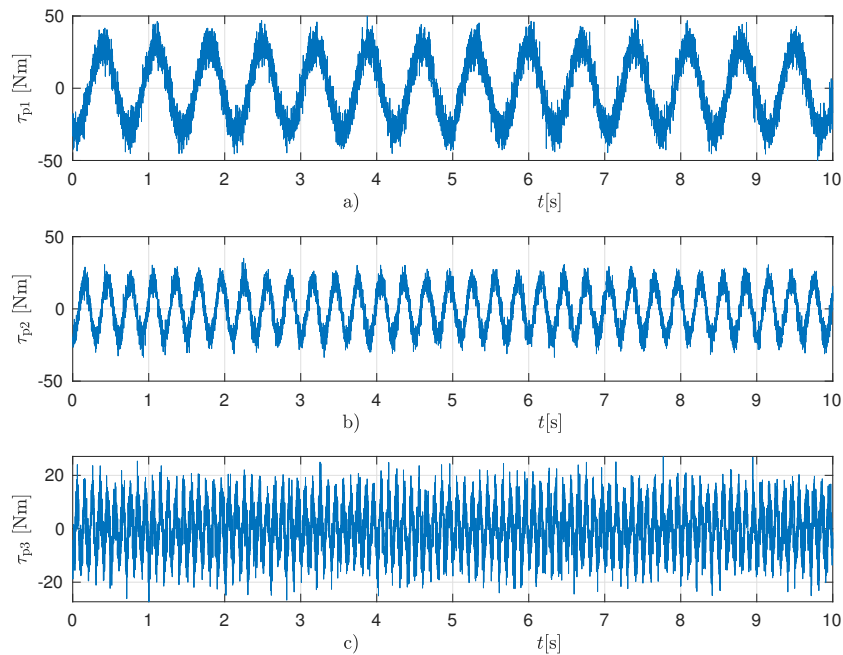


FIGURE 4 Perturbation vector τ_p .

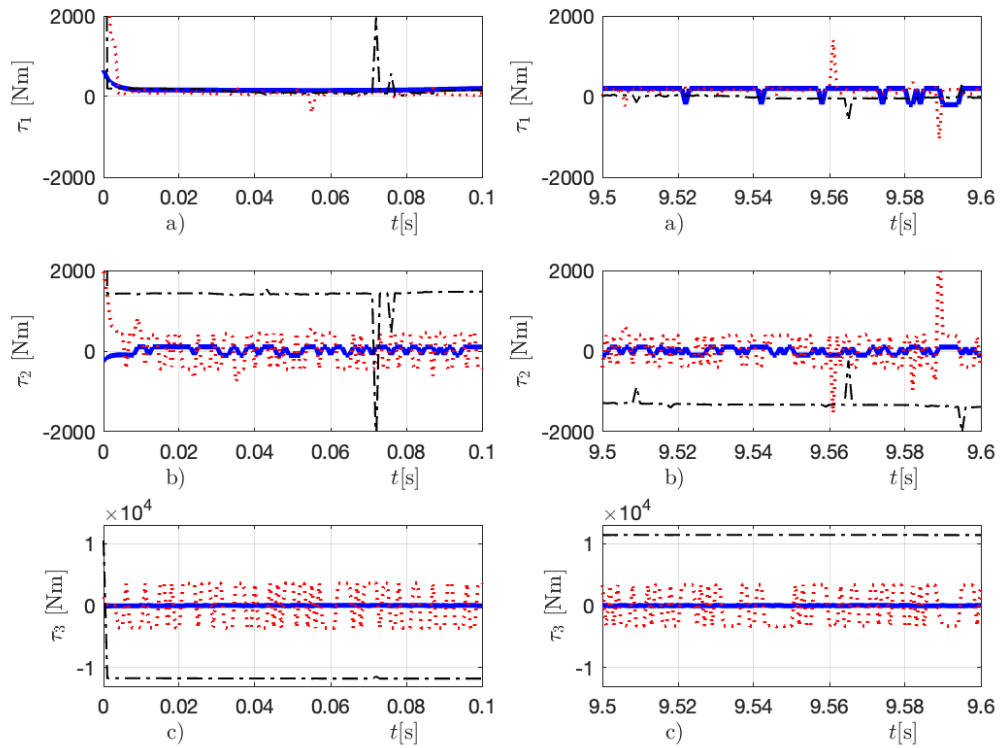


FIGURE 5 Input torques τ . Proposed algorithm (8) (—), comparative algorithm by Zhang *et al.*⁶ (· · · · ·), and comparative algorithm by Yang and Su³⁰ (- · - · -). Left column: Behavior for $t \in [0, 0.1]$ [s]. Right column: Behavior for $t \in [9.5, 9.6]$ [s].

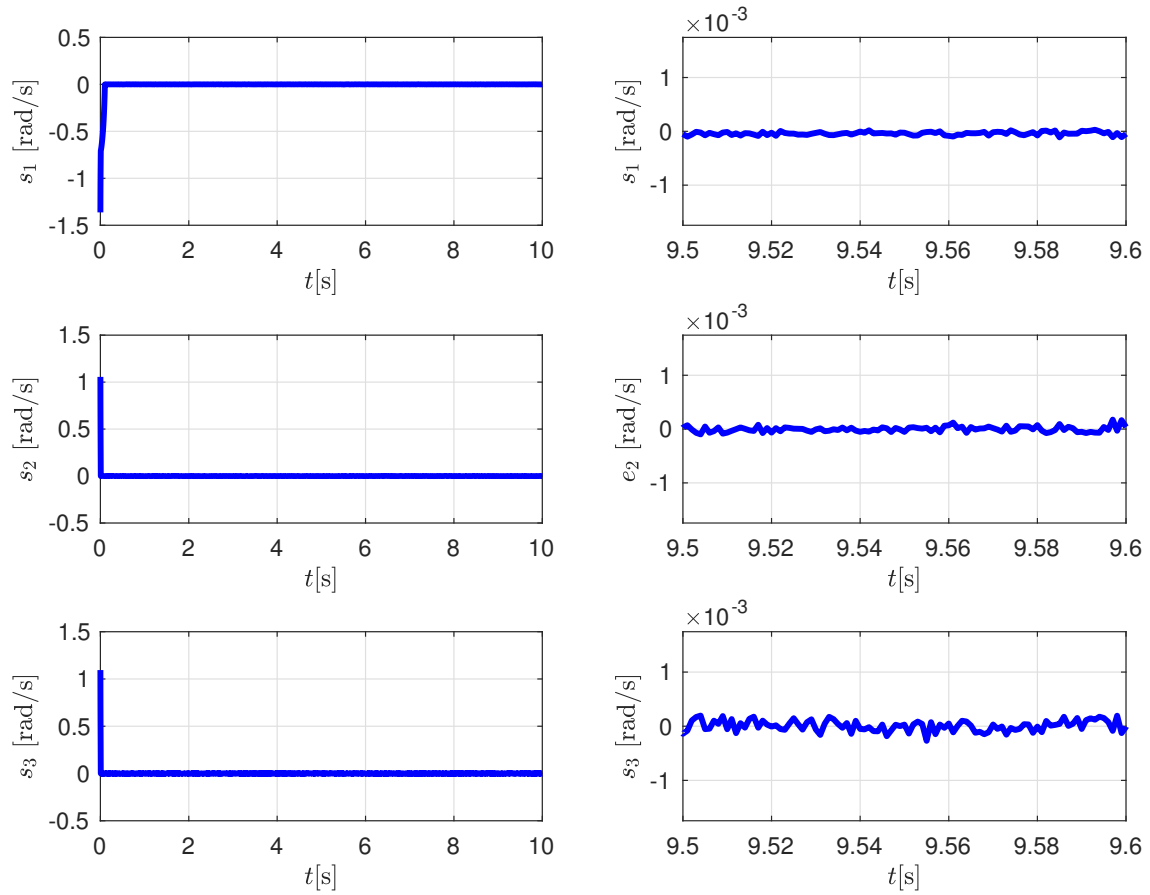


FIGURE 6 Sliding surface s in (7). Left column: Behavior for $t \in [0, 10][s]$. Right column: Behavior for $t \in [9.5, 9.6][s]$ and a scale $[-10^{-3}, 10^{-3}][\text{rad/s}]$.

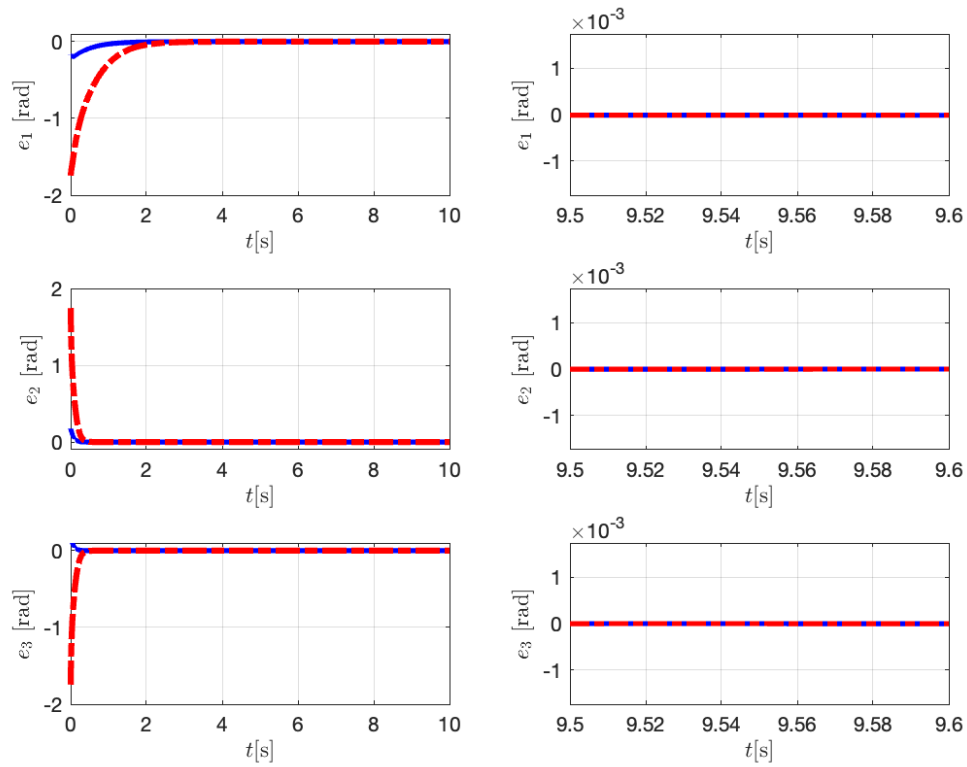


FIGURE 7 Tracking errors. Initial condition (68) (—) and initial condition (69) (· - · - ·) Left column: Behavior for $t \in [0, 10][s]$. Right column: Behavior for $t \in [9.5, 9.6][s]$ and a scale $[-10^{-3}, 10^{-3}][rad]$.

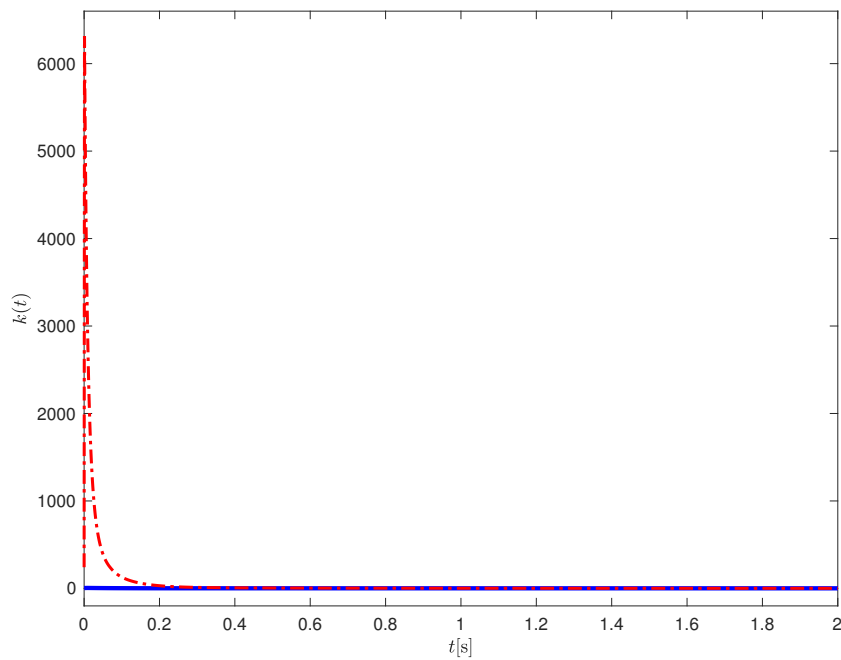


FIGURE 8 Time-varying gain $k(t)$. Initial condition (68) (—) and initial condition (69) (· - · - ·)

5 | CONCLUSIONS

The employment of sliding mode control (SMC) techniques allows to achieve exact position tracking despite a poor knowledge of the dynamic model of a system, while its robustness properties facilitates the rejection of external perturbations. All these characteristics made the employment of SMC very attractive for robot manipulators, which are systems with high nonlinear dynamics and difficult to model accurately. Controllers based on this strategy can yield to zero the tracking errors in finite (but not necessarily tunable) time. In the last years fixed-time stability has become an area of great interest, where a problem to be solved besides model uncertainties and perturbations is the possible appearance of singularities. The current contribution introduced a singularity free fixed-time SMC scheme for position and velocity tracking of robot manipulators in the presence of external perturbations and with only the knowledge of some bounds related to the robot model. By means of a time varying gain in the form of a polynomial of the different powers of the norms of the tracking errors, fixed-time stability was guaranteed where the settling time can be set arbitrarily small independently of the initial conditions, while the result holds globally. To test the performance of the new control law, simulations have been carried out where the outcomes were compared with two recent algorithms with similar properties. The results showed a better performance of the present proposal. As future research an adaptive version of the controller will be developed to avoid any knowledge of the robot model *a priori*.

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FINANCIAL DISCLOSURE

None reported.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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APPENDIX

A COMPLEMENTARY THEORY

A.1 Useful lemmas

Lemma 2. ³⁵ Consider a positive definite continuous radially unbounded function $V(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$, for $\mathbf{x} \in \mathbb{R}^n$ and with $V(\mathbf{x}) = 0$ only for $\mathbf{x} \equiv \mathbf{0}$. If

$$\dot{V}(\mathbf{x}) \leq -\kappa_1 V^q(\mathbf{x}) \quad (\text{A1})$$

for some constants $\kappa_1 > 0$ and $q > 1$, then $V(\mathbf{x}) \leq 1$ in a finite time no larger than $t_0 + T_1$, with $T_1 = \frac{1}{\kappa_1(q-1)}$. \triangle

Lemma 3. Consider a positive definite continuous radially unbounded function $V(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$, for $\mathbf{x} \in \mathbb{R}^n$ and with $V(\mathbf{x}) = 0$ only for $\mathbf{x} \equiv \mathbf{0}$. Assume that $V(\mathbf{x}(t_0)) \leq 1$. If

$$\dot{V}(\mathbf{x}) \leq -\kappa_2 V^p(\mathbf{x}) \quad (\text{A2})$$

for some positive constants κ_2, p with $p < 1$, then $V(\mathbf{x}) \equiv 0$ in a finite time no larger than $t_0 + T_2$, with $T_2 = \frac{1}{\kappa_2(1-p)}$.

Proof. According to the Comparison Lemma²⁷, the solution of (A2) satisfies

$$\begin{aligned} V(\mathbf{x}) &\leq \left(-\kappa_2(1-p)(t-t_0) + V^{1-p}(t_0) \right)^{\frac{1}{1-p}} \\ &\leq \left(-\kappa_2(1-p)(t-t_0) + 1 \right)^{\frac{1}{1-p}}. \end{aligned} \quad (\text{A3})$$

Therefore, at $t = t_0 + T_2$ one has $V(\mathbf{x}) \leq 0$, but since $V(\mathbf{x}) \geq 0$ for all $t \geq t_0$, then necessarily the equality $V(\mathbf{x}) \equiv 0$ holds for all $t \geq t_0 + T_2$, which concludes the proof. \square

Lemma 4. Consider a positive definite continuous radially unbounded function $V(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$, for $\mathbf{x} \in \mathbb{R}^n$ and with $V(\mathbf{x}) = 0$ only for $\mathbf{x} \equiv \mathbf{0}$. Assume that $V(\mathbf{x}(t_0)) \leq 1$. If

$$\dot{V}(\mathbf{x}) \leq -\kappa_3 V^{p(\mathbf{x})}(\mathbf{x}) \quad (\text{A4})$$

for some positive constant κ_3 and where $p(\mathbf{x})$ satisfies

$$p(\mathbf{x}) = 1 - \frac{1}{2}f(\mathbf{x}) \in [0.5, 1] \quad (\text{A5})$$

with $f(\mathbf{x}) \in [0, 1]$ satisfying

$$f(\mathbf{x}) \geq V(\mathbf{x}), \quad (\text{A6})$$

then $V(\mathbf{x}) \equiv 0$ in a finite time no larger than $t_0 + T_3$, with $T_3 = \frac{g_\infty}{\kappa_3}$, where g_∞ is given by

$$g_\infty = \lim_{n \rightarrow \infty} \sum_{i=1}^n g(i), \quad (\text{A7})$$

with

$$g(i) = \left(2^{\frac{1}{2^{i+1}}} - 1 \right) \frac{2^{i+1}}{2^{\frac{i}{2^{i+1}}}}. \quad (\text{A8})$$

Proof. Assume without loss of generality that $V(\mathbf{x}(t_0)) \equiv 1$ and consider that since $p(\mathbf{x}) \in [0.5, 1]$, then $\dot{V}(\mathbf{x})$ satisfies

$$\dot{V}(\mathbf{x}) \leq -\kappa_3 V(\mathbf{x}) \quad (\text{A9})$$

as well. This shows that $V(\mathbf{x}) \rightarrow 0$ and that it is strictly decreasing. In the same manner one has

$$\dot{V}(\mathbf{x}) \leq -\kappa_3 V^{(1-\frac{1}{2}f(\mathbf{x}))} \leq -\kappa_3 V^{(1-\frac{1}{2}V(\mathbf{x}))} \quad (\text{A10})$$

in view of (A6) and the fact that $f(\mathbf{x}) \in [0, 1]$ by assumption. Set now t_1 as the time at which $V(t_1) = \frac{1}{2}$, so that for the time interval $t \in [t_0, t_1]$ one has

$$\dot{V}(\mathbf{x}) \leq -\kappa_3 V^{(1-\frac{1}{2}V(\mathbf{x}))} \leq -\kappa_3 V^{(1-\frac{1}{4})}. \quad (\text{A11})$$

Then, using (A3) with $1 - p = \frac{1}{4}$, the solution of (A11) satisfies

$$V(t_1) \leq \left(-\kappa_3 \frac{1}{4} (t_1 - t_0) + V^{\frac{1}{4}}(t_0) \right)^4, \quad (\text{A12})$$

which in turn implies that

$$t_1 \leq t_0 + \frac{4}{\kappa_3} \left(V^{\frac{1}{4}}(t_0) - V^{\frac{1}{4}}(t_1) \right). \quad (\text{A13})$$

Set now t_2 as the time at which $V(t_2) = \frac{1}{4}$, so that for the time interval $t \in [t_1, t_2]$ one has

$$\dot{V}(\mathbf{x}) \leq -\kappa_3 V^{(1-\frac{1}{2}V(\mathbf{x}))} \leq -\kappa_3 V^{(1-\frac{1}{8})}. \quad (\text{A14})$$

Then, using (A3) with $1 - p = \frac{1}{8}$, the solution of (A14) satisfies

$$V(t_2) \leq \left(-\kappa_3 \frac{1}{8} (t_2 - t_1) + V^{\frac{1}{8}}(t_1) \right)^8, \quad (\text{A15})$$

which in turn implies that

$$t_2 \leq t_1 + \frac{8}{\kappa_3} \left(V^{\frac{1}{8}}(t_1) - V^{\frac{1}{8}}(t_2) \right). \quad (\text{A16})$$

Set now t_3 as the time at which $V(t_3) = \frac{1}{8}$, so that for the time interval $t \in [t_2, t_3]$ one has

$$\dot{V}(\mathbf{x}) \leq -\kappa_3 V^{(1-\frac{1}{2}V(\mathbf{x}))} \leq -\kappa_3 V^{(1-\frac{1}{16})}. \quad (\text{A17})$$

Then, using (A3) with $1 - p = \frac{1}{16}$, the solution of (A17) satisfies

$$V(t_3) \leq \left(-\kappa_3 \frac{1}{16} (t_3 - t_2) + V^{\frac{1}{16}}(t_2) \right)^{16}, \quad (\text{A18})$$

which in turn implies that

$$t_3 \leq t_2 + \frac{16}{\kappa_3} \left(V^{\frac{1}{16}}(t_2) - V^{\frac{1}{16}}(t_3) \right). \quad (\text{A19})$$

Clearly, from (A13), (A16) and (A19) the following sequence emerges

$$t_i \leq t_{i-1} + \frac{2^{i+1}}{\kappa_3} \left(V^{\frac{1}{2^{i+1}}}(t_{i-1}) - V^{\frac{1}{2^{i+1}}}(t_i) \right). \quad (\text{A20})$$

By keeping in mind that it was chosen $V(t_i) = \frac{1}{2^i}$, then one has

$$t_i \leq t_{i-1} + \frac{2^{i+1}}{\kappa_3} \left(\left(\frac{1}{2^{i-1}} \right)^{\frac{1}{2^{i+1}}} - \left(\frac{1}{2^i} \right)^{\frac{1}{2^{i+1}}} \right) = t_{i-1} + \frac{1}{\kappa_3} g(i), \quad (\text{A21})$$

with $g(i)$ given in (A8). Therefore, it is clear that the following sequence arises

$$\begin{aligned} t_1 &\leq t_0 + \frac{1}{\kappa_3} g(1) \\ t_2 &\leq t_1 + \frac{1}{\kappa_3} g(2) \leq t_0 + \frac{1}{\kappa_3} (g(1) + g(2)) \\ t_3 &\leq t_2 + \frac{1}{\kappa_3} g(3) \leq t_0 + \frac{1}{\kappa_3} (g(1) + g(2) + g(3)) \\ &\vdots \end{aligned} \quad (\text{A22})$$

which implies that

$$t_n \leq t_0 + \frac{1}{\kappa_3} \sum_{i=1}^n g(i). \quad (\text{A23})$$

Now, recall that $V(t_i) = \frac{1}{2^i}$, meaning that $\lim_{i \rightarrow \infty} V(t_i) = 0$. Therefore, $V(t)$ becomes zero in a time no larger than

$$T_{\max} \leq t_{\infty} \leq t_0 + T_3, \quad (\text{A24})$$

with

$$T_3 = \frac{1}{\kappa_3} g_{\infty}, \quad (\text{A25})$$

where g_{∞} is given in (A7). □

Remark 9. To the best of the authors' knowledge, there is no closed form solution for g_{∞} in (A7). However, note that the term $2^{\frac{1}{2^{i+1}}}$ in (A8) is tending to 1 with a power $\frac{1}{2^{i+1}}$ which means that $\left(2^{\frac{1}{2^{i+1}}} - 1\right)$ tends to zero faster than $\frac{2^{i+1}}{2^{\frac{1}{2^{i+1}}}} \leq 2^{i+1}$ tends to $+\infty$, so that the limit must exist. g_{∞} can be estimated with a computer. For $n = 60$ one gets $g_{60} = 35.2180$, while for $n = 1000$ one has $g_{1000} = 35.2180$ as well, meaning that this is the value of g_{∞} . △

A.2 Proof of Lemma 1

To show how an estimate of $b(t)$ in (15) can be obtained, consider first that the derivative of (3) can be computed as

$$\ddot{\mathbf{q}}_t = \ddot{\mathbf{q}}_d - \Lambda \dot{\mathbf{e}} - \mathbf{K}_e \dot{\boldsymbol{\tau}}_e. \quad (\text{A26})$$

The derivate $\dot{\boldsymbol{\tau}}_e$ will be computed for each element τ_{ei} in (4). First of all, recall that

$$|e_i| \lambda_{ei} \tanh(e_i^2)^{\frac{1}{2}} \left(1 - 2 \frac{\tanh(e_i^2)}{\tanh(1)}\right) = \exp \left(\left(\lambda_{ei} \tanh(e_i^2) + \frac{1}{2} \left(1 - 2 \frac{\tanh(e_i^2)}{\tanh(1)}\right) \right) \ln(|e_i|) \right). \quad (\text{A27})$$

Therefore, one has

$$\begin{aligned} \dot{\tau}_{ei} &= \text{sign}(e_i) \frac{\tanh(|e_i|^{\frac{1}{2}})}{\tanh(1)} |e_i|^{\lambda_{ei} \tanh(e_i^2) + \frac{1}{2}} \left(1 - 2 \frac{\tanh(e_i^2)}{\tanh(1)}\right) \frac{d}{dt} \left\{ \left(\lambda_{ei} \tanh(e_i^2) + \frac{1}{2} \left(1 - 2 \frac{\tanh(e_i^2)}{\tanh(1)}\right) \right) \ln(|e_i|) \right\} \\ &\quad + |e_i|^{\lambda_{ei} \tanh(e_i^2) + \frac{1}{2}} \left(1 - 2 \frac{\tanh(e_i^2)}{\tanh(1)}\right) \frac{\frac{1}{2} \text{sign}(e_i) |e_i|^{-\frac{1}{2}} \text{sign}(e_i) \dot{e}_i}{\tanh(1) \cosh^2(|e_i|^{\frac{1}{2}})} \\ &= \left\{ \frac{2\lambda_{ei} e_i \dot{e}_i}{\cosh^2(e_i^2)} \ln(|e_i|) - \frac{2e_i \dot{e}_i}{\tanh(1) \cosh^2(e_i^2)} \ln(|e_i|) + \left(\lambda_{ei} \tanh(e_i^2) + \frac{1}{2} \left(1 - 2 \frac{\tanh(e_i^2)}{\tanh(1)}\right) \right) \frac{\dot{e}_i}{e_i} \right\} \tau_{ei} + \frac{|e_i|^{\lambda_{ei} \tanh(e_i^2) + \frac{1}{2}} \left(1 - 2 \frac{\tanh(e_i^2)}{\tanh(1)}\right)}{2|e_i|^{\frac{1}{2}} \tanh(1) \cosh^2(|e_i|^{\frac{1}{2}})} \dot{e}_i. \end{aligned} \quad (\text{A28})$$

Note that it has been used $\frac{d}{dt}|e_i| = \frac{d}{dt}(e_i^2)^{\frac{1}{2}} = \frac{1}{|e_i|} e_i \dot{e}_i = \text{sign}(e_i) \dot{e}_i$, $\frac{d}{dt} \ln(|e_i|) = \frac{1}{|e_i|} \text{sign}(e_i) \dot{e}_i = \frac{1}{e_i} \dot{e}_i$, and $\lim_{e_i \rightarrow 0} e_i \ln(|e_i|) = 0^{32}$.

There are two terms in (A28) which might imply a singularity (keep in mind that $\cosh(0) = 1$). First of all, after (4) one has to analyze

$$\frac{\tau_{ei}}{e_i} = \frac{\tanh(|e_i|^{\frac{1}{2}})}{\tanh(1)} \frac{|e_i|^{\frac{1}{2}} \left(1 - 2 \frac{\tanh(e_i^2)}{\tanh(1)}\right)}{|e_i|} f_{ei}(e_i). \quad (\text{A29})$$

To find out whether there is a singularity or not, one must compute $\lim_{e_i \rightarrow 0} \frac{\tau_{ei}}{e_i}$, where $\frac{\tau_{ei}}{e_i} \geq 0$ for all $e_i \neq 0$. In fact, one can calculate $\lim_{e_i \rightarrow 0} f_{ei}(e_i)$ as in the following. By considering that $f_{ei}(e_i) = |e_i|^{\lambda_{ei} \tanh(e_i^2)} = \exp(\lambda_{ei} \tanh(e_i^2) \ln(|e_i|))$, one must equivalently solve (by using L'Hôpital's rule)

$$\lim_{e_i \rightarrow 0} \tanh(e_i^2) \ln(|e_i|) = \lim_{e_i \rightarrow 0} \frac{\ln\left((e_i^2)^{\frac{1}{2}}\right)}{\coth(e_i^2)} = \lim_{e_i \rightarrow 0} \frac{\frac{1}{|e_i|} \frac{1}{2} (e_i^2)^{-\frac{1}{2}} 2e_i}{-\frac{2e_i}{\sinh^2(e_i^2)}} = - \lim_{e_i \rightarrow 0} \frac{\frac{1}{e_i}}{\frac{8e_i}{(e_i^2 - e^{-e_i^2})^2}} = - \lim_{e_i \rightarrow 0} \frac{(e_i^2 - e^{-e_i^2})^2}{8e_i}, \quad (\text{A30})$$

where $\sinh(x) = \frac{e^x - e^{-x}}{2}$ has been used, with e the Euler's number. Since one has

$$e^{e_i^2} - e^{-e_i^2} = 1 + e_i^2 + \frac{1}{2!}(e_i^2)^2 + \frac{1}{3!}(e_i^2)^3 \dots - \left(1 - e_i^2 + \frac{1}{2!}(e_i^2)^2 - \frac{1}{3!}(e_i^2)^3 \dots\right) = 2e_i^2 + \frac{2}{3!}(e_i^2)^3 + \dots \quad (\text{A31})$$

then it is

$$\frac{e^{e_i^2} - e^{-e_i^2}}{e_i} = 2e_i + \frac{2}{3!}(e_i^{2-\frac{1}{3}})^3 + \dots \quad (\text{A32})$$

Substituting this relationship in (A30) yields

$$\lim_{e_i \rightarrow 0} \tanh(e_i^2) \ln(|e_i|) = 0. \quad (\text{A33})$$

This in turn implies that

$$\lim_{e_i \rightarrow 0} f_{ei}(e_i) = \lim_{e_i \rightarrow 0} \exp(\lambda_{ei} \tanh(e_i^2) \ln(|e_i|)) = 1. \quad (\text{A34})$$

Note that this also shows that $f_{ei}(e_i)$ is continuous at $e_i = 0$ with $f_{ei}(0) = 1$. On the other hand, since the limit of a product equals the product of the limits and $\lim_{e_i \rightarrow 0} f_{ei}(e_i) = 1$, then to compute $\lim_{e_i \rightarrow 0} \frac{\tau_{ei}}{e_i}$ from (A29) one needs now to solve

$$\lim_{e_i \rightarrow 0} \frac{\tanh(|e_i|^{\frac{1}{2}})}{\tanh(1)} \frac{|e_i|^{\frac{1}{2}} \left(1 - 2 \frac{\tanh(e_i^2)}{\tanh(1)}\right)}{|e_i|}. \quad (\text{A35})$$

To simplify this, assume without loss of generality that $|e_i| \leq 1$ and note that from (60) a lower bound of the limit in (A35) can be computed as

$$\lim_{e_i \rightarrow 0} \frac{|e_i|^{1 - \frac{\tanh(e_i^2)}{\tanh(1)}}}{|e_i|} = \lim_{e_i \rightarrow 0} |e_i|^{-\frac{\tanh(e_i^2)}{\tanh(1)}} = 1. \quad (\text{A36})$$

To show this, just keep in mind that as done before for f_{ei} , one can take advantage of the fact that $|e_i|^{-\frac{\tanh(e_i^2)}{\tanh(1)}} = \exp\left(-\frac{\tanh(e_i^2)}{\tanh(1)} \ln(e_i)\right)$ and use (A33) again. This allows to show that the limit

$$\lim_{e_i \rightarrow 0} \frac{\tau_{ei}}{e_i} = 1. \quad (\text{A37})$$

Therefore, there is no singularity. The second term in (A28) to be analyzed is the last one, i.e.

$$\frac{|e_i|^{\lambda_{ei} \tanh(e_i^2) + \frac{1}{2}} \left(1 - 2 \frac{\tanh(e_i^2)}{\tanh(1)}\right)}{2|e_i|^{\frac{1}{2}} \tanh(1) \cosh^2(|e_i|^{\frac{1}{2}})} = \frac{|e_i|^{\lambda_{ei} \tanh(e_i^2)} \cdot |e_i|^{-\frac{\tanh(e_i^2)}{\tanh(1)}}}{2 \tanh(1) \cosh^2(|e_i|^{\frac{1}{2}})}. \quad (\text{A38})$$

By taking into account (A34) and (A36), one gets

$$\lim_{e_i \rightarrow 0} \frac{|e_i|^{\lambda_{ei} \tanh(e_i^2)} \cdot |e_i|^{-\frac{\tanh(e_i^2)}{\tanh(1)}}}{2 \tanh(1) \cosh^2(|e_i|^{\frac{1}{2}})} = \frac{1}{2 \tanh(1)}. \quad (\text{A39})$$

The previous analysis shows that the proposed scheme does not have singularities at $e_i = 0$ for $i = 1, \dots, n$.

Once it has been shown that there is no singularity, the next step consists in finding a bound for $\hat{\tau}_{ei}$ as a function of known variables. From (A28) one has

$$\begin{aligned} |\hat{\tau}_{ei}| &\leq |e_i| \left| \left(\frac{2e_i(\lambda_{ei} \tanh(1) - 1)}{\tanh(1) \cosh^2(e_i^2)} \ln(|e_i|) + \left(\frac{(\lambda_{ei} \tanh(1) - 1) \tanh(e_i^2)}{\tanh(1)} + \frac{1}{2} \right) \frac{1}{e_i} \right) \tau_{ei} + \frac{|e_i|^{\frac{(\lambda_{ei} \tanh(1) - 1) \tanh(e_i^2)}{\tanh(1)} + \frac{1}{2}}}{2|e_i|^{\frac{1}{2}} \tanh(1) \cosh^2(|e_i|^{\frac{1}{2}})} \right| \\ &\leq |e_i| \left| \left(2e_i \lambda_{ei} \ln(|e_i|) |\tau_{ei}| + \left(\lambda_{ei} + \frac{1}{2} \right) \left| \frac{\tau_{ei}}{e_i} \right| \right) + \frac{|e_i|^{\frac{(\lambda_{ei} \tanh(1) - 1) \tanh(e_i^2)}{\tanh(1)}}}{2 \tanh(1)} \right|, \end{aligned} \quad (\text{A40})$$

where in view of (6) it has been taken advantage of the facts that $\lambda_{ei} \tanh(1) > \lambda_{ei} \tanh(1) - 1 > 0$, $\frac{1}{\cosh^2(|e_i|^{\frac{1}{2}})} \leq 1$, $\frac{1}{\cosh^2(e_i^2)} \leq 1$, and $\tanh(e_i^2) \leq 1$. A bound for τ_{ei} in (4) can be found in the following way. Consider two cases. If $|e_i| \leq 1$ then one can rewrite it as

$$\tau_{ei} = \text{sign}(e_i) \frac{\tanh(|e_i|^{\frac{1}{2}})}{\tanh(1)} |e_i|^{\frac{1}{2}} |e_i|^{(\lambda_{ei} - \frac{1}{\tanh(1)}) \tanh(e_i^2)}. \quad (\text{A41})$$

Since $\lambda_{ei} - \frac{1}{\tanh(1)} > 0$ according to (6), then $|e_i|^{(\lambda_{ei} - \frac{1}{\tanh(1)}) \tanh(e_i^2)} \leq 1$ because it shares the same properties as f_{ei} , i.e. $f_{ei}(0) = f_{ei}(1) = 1$ as maximum for $e_i \in [-1, 1]$. But since $\tanh(x)$ is a strictly increasing function, then the maximum of $|\tau_{ei}|$ must be 1 for $|e_i| = 1$ as well, i.e. $|\tau_{ei}| \leq 1$ for $e_i \in [-1, 1]$.

On the other hand, if $|e_i| > 1$, then from (A41) one has

$$|\tau_{ei}| \leq \frac{1}{\tanh(1)} |e_i|^{\frac{1}{2}} |e_i|^{(\lambda_{ei} - \frac{1}{\tanh(1)})}, \quad (\text{A42})$$

because $\tanh(x) < 1$ for $x \geq 0$. Recall that $\lambda_{ei} - \frac{1}{\tanh(1)} > 0$ and it is a known value. The goal of this analysis is to show that a positive constant exists so that $|\tau_{ei}| \leq |e_i|^a$ for some $a > 0$ when $|e_i| > 1$. Therefore, by recalling that $\lambda_{ei} - \frac{1}{\tanh(1)} = \frac{3}{2}$ in view of (14) and by considering the fact $|\tau_{ei}| \leq 1$ for $|e_i| \leq 1$ and (A42), one has

$$|\tau_{ei}| \leq 1 + \frac{1}{\tanh(1)} |e_i|^2. \quad (\text{A43})$$

Now, a bound for $\frac{\tau_{ei}}{e_i}$ in (A29) needs to be calculated. First consider $|e_i| \leq 1$ and recall that it was shown that $\frac{\tau_{ei}}{e_i} = 1$ at $e_i = 0$ and $|e_i| = 1$. Also, recall from (37) that $f_{ei} \leq 1$. Therefore, the problem reduces to finding the maximum value of

$$\frac{\tanh(|e_i|^{\frac{1}{2}})}{\tanh(1)} \frac{|e_i|^{\frac{1}{2}} \left(1 - 2 \frac{\tanh(e_i^2)}{\tanh(1)}\right)}{|e_i|} \leq \frac{|e_i|^{\frac{1}{2}} \left(1 - 2 \frac{\tanh(e_i^2)}{\tanh(1)}\right)}{\tanh(1)|e_i|^{\frac{1}{2}}}, \quad (\text{A44})$$

where the fact that $f_{\Gamma}(x) = \frac{\tanh(x)}{x}$ in (23) has a maximum for $x = 0$ given by 1 has been used. Therefore, one needs to find the maximum of $|e_i|^{-\frac{\tanh(e_i^2)}{\tanh(1)}} = \exp\left\{-\frac{\tanh(e_i^2)}{\tanh(1)} \ln(|e_i|)\right\}$ by computing the derivative:

$$\frac{d}{de_i} |e_i|^{-\frac{\tanh(e_i^2)}{\tanh(1)}} = -\frac{1}{\tanh(1)} |e_i|^{-\frac{\tanh(e_i^2)}{\tanh(1)}} \left(\frac{2e_i \ln(|e_i|)}{\cosh^2(e_i^2)} + \frac{\tanh(e_i^2)}{e_i} \right). \quad (\text{A45})$$

This derivative becomes zero at $|e_i| = s_m \approx 0.5829$ (see (36)), so that one can conclude from (A44) that for $e_i \in [-1, 1]$ it holds

$$\left| \frac{\tau_{ei}}{e_i} \right| \leq \frac{1}{\tanh(1)} |e_i|^{-\frac{\tanh(e_i^2)}{\tanh(1)}} \Big|_{|e_i|=0.5829} = 1.6558. \quad (\text{A46})$$

On the other hand, if $|e_i| > 1$ then from (A29) one can conclude that

$$\left| \frac{\tau_{ei}}{e_i} \right| \leq \frac{1}{\tanh(1)} \frac{1}{|e_i|^{\frac{1}{2} \left(1 + 2 \frac{\tanh(e_i^2)}{\tanh(1)}\right)}} f_{ei}(e_i) \leq \frac{f_{ei}(e_i)}{\tanh(1)}. \quad (\text{A47})$$

From its definition in (5) it is clear that $f_{ei}(e_i) = |e_i|^{\lambda_{ei} \tanh(e_i^2)} \leq |e_i|^{\lambda_{ei}}$, where λ_{ei} is a known value always. Since $\lambda_{ei} \approx 2.8130 < 3$ as given in (14), it can be used for the sake of simplicity $f_{ei}(e_i) \leq |e_i|^{2.8130} \leq |e_i|^3$. Therefore, it can be concluded that

$$\left| \frac{\tau_{ei}}{e_i} \right| \leq 1.6558 + \frac{1}{\tanh(1)} |e_i|^3. \quad (\text{A48})$$

To analyze $|e_i \ln(|e_i|)| = |e_i| \cdot |\ln(|e_i|)|$ in (A40), recall that $\lim_{|e_i| \rightarrow 0} |e_i \ln(|e_i|)| = 0$, while it is obvious that $\lim_{|e_i| \rightarrow \infty} |e_i \ln(|e_i|)| = \infty$. However, $\lim_{|e_i| \rightarrow 0} |\ln(|e_i|)| = \infty$ and therefore it is necessary to find the minimum for $e_i \ln(|e_i|)$ by computing the derivative

$$\frac{d}{de_i} e_i \ln(|e_i|) = \ln(|e_i|) + e_i \frac{1}{e_i} = \ln(|e_i|) + 1 = 0, \quad (\text{A49})$$

so that the minimum takes place at $e_i = \frac{1}{e}$, where e is Euler's number and for which $\frac{1}{e} \ln\left(\left|\frac{1}{e}\right|\right) \approx -0.368$. Since $\ln(1) = 0$, one concludes that $|e_i \ln(|e_i|)| \leq 0.368 = \frac{1}{e}$ for $|e_i| \in [0, 1]$. To analyze the case for $|e_i| \in (1, \infty)$, one only needs to find a bound for $\ln(|e_i|)$. This bound should have the form $\ln(|e_i|) \leq |e_i|^a$ for some positive constant a . To find it, note that equivalently one should have

$$a \ln(|e_i|) \geq \ln(\ln(|e_i|)) \Rightarrow a \geq \frac{\ln(\ln(|e_i|))}{\ln(|e_i|)}, \quad (\text{A50})$$

so that one has to find the maximum of $\frac{\ln(\ln(|e_i|))}{\ln(|e_i|)}$ subject to $|e_i| > 1$. This is the same as computing the maximum for $\frac{\ln(x)}{x}$ subject to $x > 0$. By computing the derivative one has

$$\frac{d}{dx} \frac{\ln(x)}{x} = \frac{x \frac{1}{x} - \ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2} = 0, \quad (\text{A51})$$

which holds for $x = e$, meaning that for $\frac{\ln(\ln(|e_i|))}{\ln(|e_i|)}$ its maximum takes place at $\ln(|e_i|) = e$. After (A50) one has $a \geq \frac{1}{e}$, which means that

$$|e_i \ln(|e_i|)| \leq \frac{1}{e} + |e_i|^2, \quad (\text{A52})$$

where $a = 1$ has been considered for simplicity.

Finally, the factor $|e_i|^{\lambda_{ei}\tanh(e_i^2)}|e_i|^{-\frac{\tanh(e_i^2)}{\tanh(1)}}$ in (A40) can be analyzed. From (A46), the previous analysis about $|e_i|^{\lambda_{ei}\tanh(e_i^2)}$ allows to conclude that $\frac{1}{2\tanh(1)}|e_i|^{\lambda_{ei}\tanh(e_i^2)}|e_i|^{-\frac{\tanh(e_i^2)}{\tanh(1)}} \leq 0.8279$ for $e_i \in [-1, 1]$. On the other hand, under the assumption made about λ_{ei} to get (A48), one gets

$$\frac{|e_i|^{\frac{(\lambda_{ei}\tanh(1)-1)\tanh(e_i^2)}{\tanh(1)}}}{2\tanh(1)} \leq 0.8279 + \frac{|e_i|^3}{2\tanh(1)}. \quad (\text{A53})$$

Therefore, in the end one has from (A40), (A43), (A48), (A52) and (A53)

$$|\dot{\tau}_{ei}| \leq |e_i| \left| \left\{ 2\lambda_{ei} \left(\frac{1}{e} + |e_i|^2 \right) \left(1 + \frac{1}{\tanh(1)} |e_i|^2 \right) + \left(\lambda_{ei} + \frac{1}{2} \right) \left(1.6558 + \frac{1}{\tanh(1)} |e_i|^3 \right) \right\} + 0.8279 + \frac{|e_i|^3}{2\tanh(1)} \right|. \quad (\text{A54})$$

Then by taking into account that $|e_i| \leq \|e\|$ and $|\dot{e}_i| \leq \|\dot{e}\|$, clearly one must have

$$\|\dot{\tau}_e\| \leq \sum_{i=1}^n |\dot{\tau}_{ei}| \leq a'_1 \|\dot{e}\| + a'_2 \|\dot{e}\| \|e\|^2 + a'_3 \|\dot{e}\| \|e\|^3 + a'_4 \|\dot{e}\| \|e\|^4 \quad (\text{A55})$$

$$\|\tau_e\| \leq \sum_{i=1}^n |\tau_{ei}| \leq n + \frac{n}{\tanh(1)} \|e\|^2, \quad (\text{A56})$$

where the definition of the constants a'_1, a'_2, a'_3, a'_4 is obvious and (A43) has been used again. Therefore, the term $\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}}_r$ in (13) can be bounded according to (A26), (A55) and Property 1 as

$$\|\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}}_r\| \leq \lambda_H \{a_d + \lambda_{\bar{\lambda}} \|\dot{e}\| + \lambda_{\bar{e}} \|\dot{\tau}_e\|\} \leq a''_0 + a''_1 \|\dot{e}\| + a''_2 \|\dot{e}\| \|e\|^2 + a''_3 \|\dot{e}\| \|e\|^3 + a''_4 \|\dot{e}\| \|e\|^4, \quad (\text{A57})$$

where $\lambda_{\bar{\lambda}} = \lambda_{\max}(\mathbf{\Lambda})$, $\lambda_{\bar{e}} = \lambda_{\max}(\mathbf{K}_e)$, the definition of the constants $a''_0, a''_1, a''_2, a''_3$ and a''_4 is obvious and $\|\dot{\mathbf{q}}_d\| \leq a_d$ for all $t \geq t_0$ by assumption. For the term $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}_r$ in (13), Property 3, (2), (3), and (A56)

$$\|\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}_r\| \leq k_c (\|\dot{e}\| + v_d) (v_d + \lambda_{\bar{\lambda}} \|e\| + \lambda_{\bar{e}} \|\tau_e\|) \leq a'''_0 + a'''_1 \|e\| + a'''_2 \|\dot{e}\| + a'''_3 \|\dot{e}\| \|e\| + a'''_4 \|\dot{e}\| \|e\|^2, \quad (\text{A58})$$

where the definition of the constants $a'''_0, a'''_1, a'''_2, a'''_3$ and a'''_4 is obvious and $\|\dot{\mathbf{q}}_d\| \leq v_d$ for all $t \geq t_0$ by assumption. Also, one has for $\mathbf{D}\dot{\mathbf{q}}_r$ in (13)

$$\|\mathbf{D}\dot{\mathbf{q}}_r\| \leq \lambda_{\bar{D}} \left(v_d + \lambda_{\bar{\lambda}} \|e\| + \lambda_{\bar{e}} \left(n + \frac{n}{\tanh(1)} \|e\|^2 \right) \right), \quad (\text{A59})$$

where $\lambda_{\bar{D}} = \lambda_{\max}(\mathbf{D})$. By noting that directly from Property 4 one has $\|\mathbf{g}(\mathbf{q})\| \leq k_g$, one can conclude from (A57)-(A59) that (15) holds with a proper definition of the constants b_0, \dots, b_8 .