



**HAL**  
open science

# Teaching and learning proof in mathematics at university: new perspectives in education with proof assistants?

Cécile Ouvrier-Bufferet

## ► To cite this version:

Cécile Ouvrier-Bufferet. Teaching and learning proof in mathematics at university: new perspectives in education with proof assistants?. 2024. hal-04838823

**HAL Id: hal-04838823**

**<https://hal.science/hal-04838823v1>**

Preprint submitted on 15 Dec 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Teaching and learning proof in mathematics at university: new perspectives in education with proof assistants?

Cécile Ouvrier-Bufferet

Paris-Est Créteil University, Laboratoire de Didactique André Revuz

**Abstract.** This article (*work in progress*) investigates the potential of proof assistants (PAs) in the teaching and learning of mathematical proofs at the university level, with a focus on addressing students' difficulties during the transition from secondary to tertiary education. This transition involves a significant shift in students' relationship to mathematical proofs. PAs appear as additional teaching tools with pedagogical and didactic potential. They can provide immediate feedback, facilitate structured proof writing, and enhance students' engagement with formalization and formal reasoning. However, careful considerations regarding interface design, feedback mechanisms and instrumental geneses are presented. This article concludes by emphasizing the need for further research and development in PA-based proof education, particularly in bridging the gap between argumentation and formal proof, and in adapting PAs to different student needs, in a collaborative research perspective.

## 1 Introduction

Proof is fundamental to mathematical activity. A diversity of epistemological perspectives exists among researchers in mathematics. This diversity influences teaching practices, particularly at the university level. The secondary-tertiary transition is characterized by a change in requirements and a shift in the relationship to mathematical concepts and proving processes, which become more complex and formal, requiring acculturation to the practices of mathematicians (e.g. Balacheff, 2008; Dawkins & Weber, 2017; Selden, 2012). This is clearly an epistemological shift, from an active stance of the practitioner to a theoretical stance following given norms of inference within a mathematical theory (as recalled by Balacheff & Boy de la Tour, 2019). Students often report considerable difficulties with the logic and formalism required to construct proofs and to write demonstrations (Gueudet & Vandebrouck, 2022; Selden, 2012; Selden & Selden, 2003).

Furthermore, the need to develop new approaches to teaching proofs that incorporate Interactive Theorem Provers / Proof Assistants (hereafter referred to as PAs) is clearly demonstrated and asserted by Balacheff and Boy de la Tour (2019) and Hanna and Knipping (2020). PAs are used by mathematicians and not for educational purposes, so if they are to be used in an educational context, didactic research needs to be developed. They are free open-source software that mechanically verify a proof, making the logic and the formalism visible (e.g. Coq, LEVN, Isabelle etc.). As Tran Minh et al. (2024, forthcoming) point out in their survey of PAs for teaching, “The growing role of formalization in mathematics and computer science (the most famous example being the formalization and proof of the four-color theorem (...)) has at the same time stimulated the use of proof assistants in education, especially for teaching mathematics, logic, and computer science” (p. 1) and the development of PAs themselves. Recently used in teaching with graphical interfaces (e.g. DEDUCTION (point-and-click) or Verbose LEVN (in a very controlled natural language) for LEVN and Edukera for Coq), their use at the university level seems to be considered. Initial didactical *a priori* analyses (e.g. Bartzia et al., 2022) and promising experiments (e.g. Kerjean et al., 2022) exist, but how can such results be generalized? What are tangible benefits of integrating PAs in the teaching of proof? And, from an inclusive educational perspective, how can PAs support the learning of proof for students with special needs?

This article thus aims to contribute to the difficult learning of proof by synthesizing the difficulties of students in this area at the beginning of the university level and by exploring the potential of interactive environments such as PAs. The perspective of inclusive education is kept in mind in this contribution.

## 2 Students' difficulties in learning proof

The most commonly used definitions of “proof” in mathematics have similar features (e.g. Balacheff, 1988; Stylianides, 2016): a proof refers to an explanation that is accepted by a given community at a given time. This social dimension of exploratory proving is important, as pointed out by Yackel & Cobb (1996) with the socio-mathematical norms where the criteria for acceptability of arguments are negotiated in a classroom. In a proof, the explanations have a specific form: they are successive (true) statements following specific deductive rules. Note that Balacheff (1987, 2008) distinguishes between pragmatic and intellectual proofs. It brings tools to analyze students' proving processes, the way the students formulate mathematical statements, how they build, structure and validate their proving processes and the writing of a proof. In this article, both proof as a process (exploring a problem, searching, formulating mathematical statements and proving) and proof as a product (writing) are considered.

International research syntheses (e.g., Gueudet & Vandebrouck, 2022; Selden, 2012; Stylianides, 2016) identify students' multifaceted and stable difficulties in the learning of mathematical proofs and their underlying causes, especially at the transition from secondary to tertiary education: an insufficient knowledge of mathematical concepts and theorems, a lack of meta-knowledge about proof, difficulties in conceptualizing abstract mathematical notions, but also the use of inappropriate reasoning, frequently based on everyday logic rather than formal mathematical reasoning (Selden & Selden, 2003; Selden et al., 2010; Selden, 2012). In addition, students struggle to combine semantic, syntactic, and pragmatic approaches (e.g. Deloustal et al., 2020), which leads to difficulties in proof writing. Furthermore, with the exception of Weber's (2001) qualitative findings, there is limited understanding of how students validate their proofs, both during construction and reading. The secondary-tertiary transition requires students to acculturate to the practices of professional mathematicians, often leading to a significant change in the didactic contract (Brousseau, 1997). Other factors contributing to this difficult transition include "the disjointedness of reform efforts", a concern raised by Harel and Fuller (2009) that remains relevant today.

A recent state-of-art regarding international research findings on the topic of mathematics education and Mathematical Learning Disabilities or Difficulties (MLD) over ten years shows that there is no specific study with university students, nor a longitudinal study from secondary to tertiary levels (Deruaz et al., 2020<sup>1</sup>). This article also points out the strong need for a collaboration between mathematics education, cognitive science and special education, exploring both math disorder, learning disabilities and severe difficulties in mathematics, and remedial interventions, support devices and scaffolding. Such a systematic state-of-art shows the lack of research on the teaching and the learning of proof at the university level from the perspective of inclusive education.

In this context, where students in high school and university have difficulties finding, constructing and writing (valid) proofs, what are the potentials of PAs to change this situation? Are PAs interesting as a mathematical digital technology in the context of inclusive education?

## 3 Potentials of Proof Assistants in education

In mathematics education, the teaching and the learning of proof is a recurring topic of study for which several approaches and models exist (e.g. the synthesis of Balacheff, 2024; Hanna & De Villiers, 2012). Balacheff and Boy de la Tour (2019) and Hanna et al. (2023) provide their main features as well as the ongoing debates, with the background of TEL<sup>2</sup> environments. Bearing in mind the distance between teaching mathematics and being a professional mathematician, it is necessary to distinguish the students' learning and understanding process which consists in learning proving processes from argumentation to proof, on the one hand, and the students' structuration of a formal written proof on the other hand. Balacheff and Boy de la Tour (2019) highlight the issues which are crucial in mathematics education: “(i) bridging the gap between argumentation and proof, (ii) facilitating the transition from problem solving to proving, and (iii) scaffolding the transition from empirical validation to the mathematical norms at play in the classroom.” (p. 358).

To what extend may PAs improve the learning of proof and proving processes in those perspectives? What are the suitable future developments of PAs for education?

---

<sup>1</sup> State-of-art extended from 2007 to 2021 in Peteers et al. (2023).

<sup>2</sup> Technology Enhanced Learning.

### 3.1. First results of experiments

Research in this area is still in its early stages, with mostly isolated promising case studies available (e.g. Avigad, 2019; Kerjean et al., 2022; Thoma & Iannone, 2022).

Considering those case studies, several hypotheses can be made about the perspectives with PAs in teaching and learning proofs:

- The PAs may have effects on students' memorization and wording of mathematical statements: definitions, properties and theorems are always at hand, the students have to read, understand and use them appropriately, and, perhaps, memorize them in a better way in such a digital environment. Here, the place and the role of the paper-and-pencil environment has to be questioned.
- The PAs can have an impact on students' manipulation of formal statements. Here, the importance of the nature of feedback and validation appears, because there are different ways in PAs to automate the manipulation of formal statements (to click or to write mathematical statements). The didactic contract (Brousseau, 1997) may also have effects.
- And then, the PAs may impact students' conception of proof and proving.

Generally, it seems that students develop proving habits or technics with PAs which are modern tools in mathematics (Thomas & Iannone, 2022), but a lot of questions appear from an educational point of view (e.g. Kerjean et al., 2022), from the design of the feedback to the management by the teacher (see the following sub-sections).

### 3.2. The interface and the nature of the feedback

Balacheff and Boy de la Tour (2019) emphasize the necessity to carefully think about the interface because “the interface is the space in which they [the students] experience and learn” (p. 355).

PAs share features with other digital technologies where the design can help all students including those with special needs: for example, colors and size of texts in Chromium Library can be adapted for blind or visually impaired people. As underscored by Stöger et al. (2022), the decomposition of terms into proper sub-terms *via* keyboard and Braille display for blind or visually impaired people is notable. In the specific cases of PAs, it seems like two other aspects can support the learning for MLD students:

- The automatic deductions/steps – such as mathematical calculations – allowed by the PAs,
- And the fact that the writing part (for example with DEDUCTION point-and-click) can be reduced.

It can be considered as interesting for some special needs (dyspraxia for instance). But a didactical analysis has to be done in order to identify the key stages absolutely necessary to learn and understand proof, on the one hand, and the “dispensable” steps, on the other hand.

Independently of special needs, but still in an inclusive perspective to improve the learning for all students, another potential advantage of PAs can be underscored.

The mathematical context is always present in PAs. Students can request mathematical definitions, properties, theorems that are implemented in the context. The PAs then make available the knowledge which is embedded in the proof. The proof state (current goal and hypothesis) is also available at any point in the proof. The PAs provide immediate – but binary – feedback on the mathematical validity of a proof step.

The organization of the writing in PAs (indentation, colors with different meanings, etc.) gives a conventional structuration of the proof's writing, it is not neutral for the learning and should be studied: various representations exist in PAs, but also in the practices of teachers in high-schools and university.

Some PAs can display a proof tree. Such an alternative representation of a proof has several advantages for teaching and learning: students can identify the development of their proof, what they have done, what is the new goal at this point, the whole structure of the proof. They then have an intermediary tool to write a proof in a paper-and-pencil environment. Moreover, as underscored by Hanna et al. (2023), tactics of Lean can help students to decompose a proof into intermediary goals (example: “rw” means “rewrite” tells Lean to do a replacement). These possibilities offered by PAs should unquestionably be studied and developed in a collaborative work between mathematicians, mathematics educators and computer scientists.

From the teacher's point of view, other interests of environments like PAs are the potential replays of students' proofs, but also the tracking of students' improvements in reasoning – provided that data are collected in a longitudinal study and analyzed from a didactic point of view. It would be interesting if the teacher could select, define and adapt specific feedback in order to differentiate the learning, taking into account the diagnosed difficulties of her/his students.

Again, these features highlight potentials of PAs from a pedagogical and didactical perspective, but many developments remain to be done:

- The design of the feedback – configurable by the teacher (if possible) – so that students can interpret and use it mathematically to check and modify their ongoing writing of a proof.
- The development of textual and/or visual and/or sound feedback, but also representations of the proving process (such as proof trees with their variants depending on students’ needs), in PAs, in order to promote students’ autonomy in controlling and structuring the proofs they produce.
- The nature of the interface itself: should it be close to an intuitive approach of the proof or, on the contrary deliberately far away in order to engage students in a formal environment? Hanna et al. (2023) remind that

The art of shifting between more and less formal versions of the same mathematical thought is not easy for instructors to demonstrate, and is certainly not taught simply by presenting students with formal definitions and telling them that these are what university-level mathematics is about. (p.105 quoting Selden, 2012).

Where are the more urgent needs in education? Is it possible to develop a progressive continuous interface, with or without AI adaptations?

### ***3.3 Learning to prove or learning to write proofs with PAs?***

As pointed out above (section 2), the concept of proof contains several features generating difficulties for students at the secondary-tertiary transition. This concept is “ambiguous between the logical, textual, and psychosocial aspects” (Hanna et al., 2023, p. 105).

PAs make logic and formalism more visible and allow manipulation of formal statements. They require students to think with symbols and to take responsibility for the validity (i.e. syntactic correctness) of the use of mathematical statements and inferences. In fact, using a PA “forces students to come up with an overall strategy for the proof” (Hanna et al., 2023, p. 108) and then to choose steps to implement the idea of their proof in the PA, that is to introduce mathematical objects, to identify hypotheses, rules of inference, to use deductions, and so on. It forces students to make explicit inference rules and their instantiations which are often implicit in students’ practices. It also brings a common structure and a shared language for writing a proof, which is invariant depending on the mathematical content. All those features are interesting for education and, in particular, at the transition from secondary to tertiary education. However, currently, PAs cannot autonomously manage the necessary transition between the heuristics and the structures of reasonings involved in argumentation and formal proofs. Balacheff and Boy de Tour (2019) offer an interesting perspective:

We can also see proof assistants as systems that translate (some form of) mathematical proofs into formal proofs. We may thus examine the possibility of a reverse translation—from formal proofs to the kind of proofs that mathematicians write, and that we can read. This kind of translation would ideally leave aside all the gory details that readers do not need to know and provide only the essence of the proof. (p. 353).

These perspectives emphasize the need of new research in education about the dialectic between argumentation, proof and formal proof, the articulation between pen-and-paper environment and PAs, and the place and the role of the teacher in such a dialectic (see various examples in Hanna et al., 2019 for several TEL, including some PAs).

## **4 Discussion and perspectives**

Mathematics education researchers agree that PAs may be promising (e.g. Hanna et al., 2019, 2023; see also Hanna & Yan, 2021), as an “additional instructional tool in undergraduate mathematics teaching” (Hanna et al., 2023, p. 108), particularly during the problematic secondary-tertiary transition. PAs may enhance students’ ability to construct valid proofs and could potentially integrate with traditional methods to teach and learn mathematical proofs. In this perspective, several research questions need to be investigated in mathematics education with a didactical approach focusing on PAs’ development and their potential benefits for student learning. Collaborative research between mathematicians, mathematics educators, and computer scientists will require careful consideration of pedagogical, didactical, and inclusive factors. Based on the previous sections, key areas for future research are developed below.

#### ***4.1. Development of PAs for education: interfaces and feedback potentials***

The fast-acting development of PAs by computer scientists goes over the didactical work required to define appropriate feedback, select and design suitable exercises taking into account well-known difficulties of students in learning new mathematical contents, and study the articulation between PAs and paper-and-pencil methods. The question of the form, the place and the role of the interfaces and the feedback remain fundamental as above-mentioned.

Research should also explore the development of a non-uniform representation of proof in PAs. Indeed, several visual or schematic proofs exist depending on cultural practices in different countries (e.g. two-columns proofs, flowchart proofs, proof trees, etc.). The impact of such representations of relationships and properties between mathematical objects on students' understanding may be different in digital environments.

In addition, what types of PAs, manipulation techniques, and feedback mechanisms are most effective?

Moreover, it is not transparent that the students will be able to transfer skills developed in and with PAs to other mathematical problems and contexts.

#### ***4.2 Classroom implementation of PAs (high-school and university): new skills and knowledge? The features of instrumental knowledge***

Research must address the costs, and the schemes associated with using PAs in classrooms. It is quite clear that time and effort are required for both teachers and students to learn new environments with specific uses, rules and languages.

The challenge of implementing PAs in high-school and university is twofold: to combine the use of PAs with paper-and-pencil methods and to balance automated and manual proof skills. Then, key research question and analysis' axis include:

- How should teachers use the articulation between formal proofs with PAs and paper-and-pencil proofs in order to develop students' skills in searching and writing a proof?
- How can PAs enhance students' understanding of mathematical statements?

From the mathematics education point of view, a necessary step consists in studying the process of PAs' appropriation both by students and teachers, considering the time required and the epistemological need to reconcile different approaches to proof (in mathematics, algorithmic, logic, computer science) with secondary and tertiary teaching practices. For instance, in the perspective of teaching proof in mathematics, it means designing situations and their didactical analysis to engage students to interpret feedback from a mathematical point of view about proof (example: verifying theorem hypotheses before application).

However, the introduction of a digital artifact into the learner's environment induces a series of effects that could not be anticipated and analyzed with a purely didactic approach. It requires the use of theoretical frameworks compatible with didactic engineering, such as the instrumental approach<sup>3</sup> (Artigue, 2002), following on from the work of Rabardel (1995) in cognitive ergonomics. For the researcher in didactic, it means identifying the prerequisite knowledge for an effective use of PAs: what is the "elementary" knowledge for first uses of PAs? It also means studying more complex knowledge (instrumented action schemes), which is necessary for solving the problem and which have to be combined with mathematical knowledge. So, the questions are: what do students have to learn in such an environment before taking advantage of PAs? What kind of skills and knowledge (i.e. instrumental knowledge) should the teacher institutionalize in PA-based experimental approaches to proving?

Such an instrumental approach, studying schemes in a digital environment, is necessary for the implementation of PAs in education. It also brings tools to take charge of students with special needs and to make evolve their schemes in PAs.

---

<sup>3</sup> In particular, "instrumentation concerns the emergence and development of utilization and instrumented action schemes" (Rabardel & Beguin, 2005, p. 444) (assimilation of the artifact) and "Instrumentalization may be defined as a process in which the subject enriches the artifact's properties" (ibid.).

### 4.3 Collaborative learning and autonomy development

The importance of sociomathematical norms of argumentation (e.g. Yackel & Cobb, 1996) is pointed out by Balacheff (2024) from the perspective of learning proof in mathematics. Then, research should explore pedagogical strategies to incorporate collaborative learning in the PA environment. It also means rethinking the balance between the requirement for students' independence with their need for support, especially for those with special needs who often require the help of an assistant.

### 4.4 The development of automatisms and a risk of uniformity?

PAs offer a step-by-step guidance, with potential trial and errors strategies. It is important to develop strategies to prevent meaningless automatisms and maintain a global vision of proof structure. So the necessary investigation about the development of students' automatisms in proof writing and their potential benefits and disadvantages. And then, the researchers should bear in mind the necessity to avoid the risk of uniformity in students' written proofs (a kind of standardizing writing of proofs) in paper-and-pencil environment and to still analyze individual thought processes and reasoning of students in open mathematical problems and situations in order to track their difficulties.

### 4.5 Preserving students' heuristic approaches

To develop didactical situations in order to combine PA use with paper-and-pencil environment remains necessary to preserve students' proving activity (individual and social), empirical (a mode of reasoning which is still used by undergraduates), pragmatical and intellectual.

Balacheff and Boy de la Tour (2019) outline what a TEL environment such as PAs (called Automatic Theorem Provers – ATP – in their synthesis) could look like, taking into account previous results in mathematics education:

(...) First, it would facilitate identifying different types of activity and enabling navigation among them. A problem-solving space (e.g., a mathematical microworld) could be identified and associated with a formulation space (e.g., a digital notebook for expressing and freely linking mathematical statements) and with a validation space (for expressing a proof in line with the curriculum; e.g., as a two column proof, flowchart proof, graph, or paragraphs). Second, the ATP would be initialized with axioms and theorems chosen by the teacher or stipulated by the curriculum. This initialization would create a theoretical reference framework in line with Mariotti's model of theorem (statement, proof, theory) and be easily accessible to students. It would give students a foundation for understanding that they are proving a statement within a theory. (Balacheff & Boy de la Tour, 2019, p. 361).

Several theoretical frameworks and various methods can be used to design and analyze the implementation of PAs in classrooms in order to study the impact of such environments on students' proof learning processes, as above-mentioned.

To address all these questions, ongoing research is using methods such as questionnaires, student interviews, and didactical engineering implementations with PAs. These efforts aim at better understanding students' current conceptions of proof and how PAs impact those conceptions and proof-related performance. This is the aim of the APPAM project<sup>4</sup> started in January 2024: to fill the gap in research in France on the teaching and the learning of proof, at the transition from secondary to higher education. It proposes a multidisciplinary approach, combining tools, results and methods from mathematics, didactics, and computer science. The aim is to develop and evaluate a new approach to teaching proof, based on the adaptation and the integration of PAs at the beginning of university. Three objectives structure this project: to characterize the difficulties of learners and the existing uses of PAs in France; to analyze PAs in a learning perspective; to experiment and evaluate the benefits of PAs at the beginning of university (see preliminary results of a pre-experiment with freshmen in Sisco & Ouvrier-Buffer, a & b, forthcoming). The methods used to achieve these objectives mobilize internationally validated epistemological, didactic, and cognitive frameworks on proof and proving, and combine quantitative and qualitative approaches.

---

<sup>4</sup> APPAM: « Les assistants de preuve pour les apprentissages mathématiques » ("Proof assistants for mathematics learning") - <https://appam.icube.unistra.fr/>

## Funding

APPAM project, funded by ANR (French National Research Agency) - ANR-23-CE38-0009

## References

- Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *Journal of Computers for Mathematical Learning*, 7(3), 245–274. <https://doi.org/10.1023/A:1022103903080>
- Avigad J. (2019). Learning logic and proof with an interactive theorem prover. In G. Hanna, D.A. Reid, M. de Villiers (Eds.), *Proof technology in mathematics research and teaching* (pp. 277–290). Springer.
- Balacheff, N. (1987). Processus de preuve et situations de validation [Proof processes and validation situations]. *Educational Studies in Mathematics*, 18, 147–176. <https://doi.org/10.1007/BF00314724>
- Balacheff N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216–235). Hodder & Stoughton.
- Balacheff, N. (2008). The role of the researcher's epistemology in mathematics education: an essay on the case of proof. *ZDM International Journal on Mathematics Education*, 40(3), 501–512.
- Balacheff, N. (2024). Situations pour l'apprentissage de la preuve en mathématiques. État de la recherche et questions ouvertes [Situations for the learning of proof: A review of research and open questions]. *Recherches en Didactique des Mathématiques. Numéro spécial*, 15–59. <https://doi.org/10.46298/rdm.12905>
- Balacheff, N. & Boy de la Tour, T. (2019). Proof technology and learning in mathematics: common issues and perspectives. In G. Hanna, D.A. Reid, M. de Villiers (Eds.), *Proof technology in mathematics research and teaching* (pp. 349–365). Springer.
- Bartzia, E., Meyer, A., & Narboux, J. (2022). Proof assistants for undergraduate mathematics and computer science education: Elements of a priori analysis. In R. Hochmuth (Ed.), *INDRUM 2022 Proceedings* (pp. 253–262), Leibniz University Hannover, Germany.
- Brousseau, G. (1997). *Theory of the Didactical Situations in Mathematics*. Kluwer Academic Publishers.
- Dawkins, P.C., & Weber, K. (2017). Values and norms of proof for mathematicians and students. *Educational Studies in Mathematics*, 95, 123–142. <https://doi.org/10.1007/s10649-016-9740-5>
- Deloustal-Jorrand V., Gandit M., Mesnil Z. & Da Ronch M. (2020). Utilisation de l'articulation entre les points de vue syntaxique et sémantique dans l'analyse d'un cours sur le raisonnement [Using the articulation between syntactic and semantic viewpoints in analyzing a course about reasoning]. In T. Hausberger, M. Bosch, & F. Chellougui (Eds.), *INDRUM 2020 Proceedings* (pp. 378–387). University of Carthage, Tunisia.
- Deruaz, M., Dias, T., Gardes, M.-L., Gregorio, F., Ouvrier-Buffet, C., Peteers, F., & Robotti, E. (2020). Exploring MLD in mathematics education: Ten years of research. *The Journal of Mathematical Behavior*, 60, 1–17. <https://doi.org/10.1016/j.jmathb.2020.100807>
- Gueudet, G. & Vandebrouck, F. (2022). Transition secondaire-supérieur : Ce que nous apprend la recherche en didactique des mathématiques [The secondary-tertiary transition: What research in mathematics education tells us]. *EpiDEMES*, 1. <https://doi.org/10.46298/epidemmes-7486>
- Hanna, G., & Knipping, C. (2020). Proof in mathematics education, 1980–2020: An overview. *Journal of Educational Research in Mathematics, Special Issue*, 1–13.
- Hanna, G., & De Villiers, M. (Eds.). (2012). *Proof and proving in mathematics education*. Springer.
- Hanna, G., & Yan, X. (2021). Opening a discussion on teaching proof with automated theorem provers. *For the Learning of Mathematics*, 41(3), 42–46.
- Hanna G., Larvor, B.P., & Yan X. (K.) (2023). Human-Machine Collaboration in the Teaching of Proof. *Journal of Humanistic Mathematics*, 13(1), 99–117.
- Hanna, G., Reid, D.A., & De Villiers, M. (2019). *Proof technology in mathematics research and teaching*. Springer.
- Harel, G., & Fuller, E. (2009). Current contributions toward comprehensive perspectives on the learning and teaching of proof. In D.A. Stylianou, M.L. Blanton, & E.J. Knuth (Eds.), *Teaching and learning proof across the grades – A K-16 perspective* (pp. 355–370). Routledge.
- Kerjean, M., Le Roux, F., Massot, P., Mesnil, Z., Modeste, S., Narboux, J. & Rousselin, P. (2022). Utilisation des assistants de preuves pour l'enseignement en L1 [Using proof assistants for teaching in Licence 1]. *La Gazette de la société mathématique de France*, 174, 35–55.
- Peteers, F., Lemrich, C., Gregorio, F., & Gardes, M.-L. (2023). Cartographie de la recherche en Mathematics Education de 2007 à 2021 sur les troubles et difficultés d'apprentissage en mathématiques [Mapping research in mathematics education from 2007 to 2021 on learning disorders and difficulties in mathematics]. *Revue de Mathématiques pour l'École*, 240, 3–15. <https://doi.org/10.26034/vd.rm.2023.4229>
- Rabardel, P. (1995). *Les hommes et les technologies, approche cognitive des instruments*



- contemporains* [Humans and technologies: A cognitive approach to contemporary instruments]. Armand Colin.
- Rabardel, P., & Beguin, P. (2005). Instrument mediated activity: from subject development to anthropocentric design. *Theoretical Issues in Ergonomics Science*, 6(5), 429–461. <https://doi.org/10.1080/14639220500078179>
- Selden, A. (2012). Transitions and proof and proving at tertiary level. In G. Hanna & M. De Villiers (Eds.), *Proof and proving in mathematics education* (pp. 391–422). Springer.
- Selden, A., McKee, K., & Selden, J. (2010). Affect, behavioral schemas, and the proving process. *International Journal of Mathematical Education in Science and Technology*, 41(2), 199–215. <https://doi.org/10.1080/00207390903388656>
- Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Mathematics Education Research*, 34, 4–36.
- Sisco, C. & Ouvrier-Bufferet, C. (a) (2025, forthcoming). Is it possible to assess the impact of proof assistants on undergraduate students' proofs? Preliminary results. *14th Congress of the European Society for Research in Mathematics Education*, Bolzano, Italy.
- Sisco, C. & Ouvrier-Bufferet, C. (b) (2025, forthcoming). La preuve en mathématiques : difficultés d'étudiants et apports potentiels des assistants de preuve [Proof in mathematics: Students' difficulties and potential contributions of proof assistants]. *Colloque Espace Mathématique Francophone EMF 2025*, Montréal, Canada.
- Stöger, B., Miesenberger, K., Neuper, W., Wenzel, M., Neumayr, T. (2022). Designing an inclusive and accessible mathematical learning environment based on a theorem prover. In Miesenberger, K., Kouroupetroglou, G., Mavrou, K., Manduchi, R., Covarrubias Rodriguez, & M., Penáz, P. (Eds.), *Computers Helping People with Special Needs: ICCHP-AAATE* (Vol 13341, pp. 47–55). Springer, Cham. [https://doi.org/10.1007/978-3-031-08648-9\\_7](https://doi.org/10.1007/978-3-031-08648-9_7)
- Stylianides, A.J. (2016). *Proving in the elementary mathematics classroom*. Oxford University Press.
- Thoma, A., & Iannone, P. (2022). Learning about proof with the theorem prover LEAN: The abundant numbers task. *International Journal of Research in Undergraduate Mathematics Education*, 8, 64–93. <https://doi.org/10.1007/s40753-021-00140-1>
- Tran Minh, F., Gonnord, L., & Narboux, J. (2024, forthcoming). Proof assistants for teaching: a survey. *TheEdu Congress*, May 2024, Nancy, France.
- Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, 48, 101–119. <https://doi.org/10.1023/A:1015535614355>
- Yackel, E., & Cobb, P. (1996). Socio-mathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27(4), 458–477.