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Modeling of a non-Newtonian thin film passing a thin porous medium

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Abstract

This theoretical study deals with asymptotic behavior of a coupling between a thin film of fluid and an adjacent thin porous medium. We assume that the size of the microstructure of the porous medium is given by a small parameter $0 < \varepsilon \ll 1$, the thickness of the thin porous medium is defined by a parameter $0 < h_\varepsilon \ll 1$, and the thickness of the thin film is defined by a small parameter $0 < \eta_\varepsilon \ll 1$, where h_ε and η_ε are devoted to tend to zero when $\varepsilon \rightarrow 0$. In this paper, we consider the case of a non-Newtonian fluid governed by the incompressible Stokes equations with power law viscosity of flow index $r \in (1, +\infty)$, and we prove that there exists a critical regime, which depends on r , between ε , η_ε and h_ε . More precisely, in this critical regime given by $h_\varepsilon \approx \eta_\varepsilon^{\frac{2r-1}{r-1}} \varepsilon^{-\frac{r}{r-1}}$, we prove that the effective flow when $\varepsilon \rightarrow 0$ is described by a 1D Darcy law coupled with a 1D Reynolds law.

AMS classification numbers: 74Q10, 76A05, 76A20, 76S05.

Keywords: Homogenization, non-Newtonian fluid, thin film, thin porous medium; Reynolds equation.

1 Introduction

In this paper, we consider a incompressible viscous 2D non-Newtonian fluid in a domain D_ε composed by two parts in contact: a periodic thin porous medium Ω_ε with characteristic size of the pores $0 < \varepsilon \ll 1$ and thickness of the domain $0 < h_\varepsilon \ll 1$, and a thin film I_ε with thickness $0 < \eta_\varepsilon \ll 1$, where h_ε and η_ε are devoted to tend to zero when $\varepsilon \rightarrow 0$ (see Figure 1 for more details). Drilling and hydraulic fracturing fluids used in the oil industry are usually non-Newtonian liquids. Therefore during well drilling or hydraulic fracturing operations, the non-Newtonian drilling muds or hydraulic fluids will infiltrate into permeable formations surrounding the wellbore, which may seriously damage the formation. The rheological behavior of drilling muds, cement slurries and hydraulic fracturing fluids is often described by a power-law model (see Cloud and Clark [26], Shah [39]). The importance of modeling flow of non-Newtonian fluids from the wellbore into the surrounding formations has been recognized in the industry.

One way to study this problem is to use the homogenization theory, which has been applied to the study of perforated materials for a long time (see for instance classical studies of Allaire [1], Sanchez-Palencia [38] and Tartar [42] in the case of Newtonian fluids, or Bourgeat *et al.* [18] and Bourgeat and Mikelić [21] in the case of non-Newtonian fluids with viscosity given by the power law or the Carreau law). The question of a porous medium in contact with a thin film with properties different from those of the rest of the material has been the subject of many studies previously.

Let us make a recollection of some previous results in relation to the objective of this paper. Bourgeat *et al.* [20] considered the asymptotic behavior of the solution of the 2D Newtonian Stokes system in a porous medium with thickness of order one, with characteristic size of the pores ε and containing a thin fissure of thickness η_ε , with η_ε devoted to tend to zero with ε . It was proved the existence of a critical regime given by

$$\eta_\varepsilon \approx \varepsilon^{\frac{2}{3}}, \tag{1.1}$$

where the coupling effect appears and the effective flow behaves like a 2D Darcy flow in the porous medium coupled with a 1D Reynolds problem. We refer to Bourgeat *et al.* [17] for preliminary results on this problem, and to Zhao and Yao [46, 47] for the extension of this result to the non-stationary Stokes case and the Navier-Stokes case, respectively. We also refer to Bourgeat [19] and Bourgeat and Tapiéro [22] for a similar problem but for the Laplace equation.

Moreover, this problem was also generalized in Anguiano and Suárez-Grau [10] to the case of a non-Newtonian Stokes flow with viscosity given by the power law with flux index r satisfying $1 < r < +\infty$, where the critical regime is now given by

$$\eta_\varepsilon \approx \varepsilon^{\frac{r}{2r-1}}, \tag{1.2}$$

which agrees with (1.1) for $r = 2$. In this case, the effective flow behaves like a 2D nonlinear Darcy flow in the porous medium coupled with a 1D nonlinear Reynolds problem. We also refer to Anguiano [4] for the case of a non-stationary non-Newtonian flow in a porous medium containing a thin fissure, where in the critical regime (1.2), the flow is described by a time-dependent non-linear Reynolds problem coupling the effects of the porous medium with those of the free part.

On the other hand, the derivation of effective laws for fluids in porous domains with small thickness (the so-called *thin porous medium*) is attracting much attention, see for instance Almqvist *et al.* [2], Anguiano [9], Anguiano and Bunoiu [5], Anguiano *et al.* [6, 7], Anguiano and Suárez-Grau [8, 11, 12, 13, 14], Fabricius *et al.* [30, 29], Fabricius and Gahn [28], Forslund *et al.* [31], Jouybari and Lundström [33], Mei and Vernescu [35], Suárez-Grau [40, 41] or Zhengan and Hongxing [48]. A thin porous medium can be defined as a bounded domain confined between two parallel plates with a distance h_ε , perforated by periodically distributed obstacles of size ε , with h_ε devoted to tend to zero when $\varepsilon \rightarrow 0$.

In this context, the modeling of a Newtonian flow in a thin porous medium and an adjacent thin film flow described by Figure 1, which is the domain we are interested in this paper, was considered in Bayada *et al.* [15]. Thus, considering three positive and small parameters ε , h_ε and η_ε (where h_ε and η_ε are devoted to tend to zero), where ε is the size of the microstructure, h_ε is the thickness of the thin porous medium and η_ε is the thickness of the thin film, it was proved the existence of a critical regime between these parameters given by

$$h_\varepsilon \approx \eta_\varepsilon^3 \varepsilon^{-2}, \tag{1.3}$$

and an effective modified Reynolds equation (a 1D Darcy problem coupled with a 1D Reynolds problem) was derived. Observe that if the thickness of the porous medium $h_\varepsilon \equiv 1$, then the critical regime (1.3) coincides with that critical one given in (1.1). We also refer to Anguiano and Suárez-Grau [9] for the

derivation of a coupled Darcy–Reynolds equation for a fluid flow in a thin porous medium including a fissure, where the microstructure of thin porous medium is a collection of small cylinders (see Anguiano [3] for the non-stationary case).

The goal of this paper is to generalize the result described in [15] to the case of a non-Newtonian fluid with a viscosity described by the power law with flow index $r \in (1, +\infty)$, which is important for industrial applications as described above. We prove that there exists a critical regime between these parameters given by

$$h_\varepsilon \approx \eta_\varepsilon^{\frac{2r-1}{r-1}} \varepsilon^{-\frac{r}{r-1}}, \quad (1.4)$$

and we derive an effective nonlinear limit problem, i.e. a modified nonlinear Reynolds problem coupling the effects of the thin porous medium (1D nonlinear Darcy problem) and the thin film (1D nonlinear Reynolds problem) for the limit pressure (see Theorem 6.3 for more details). We observe that if $r = 2$, then the critical regime (1.4) coincides with the critical one given in (1.3). Also, if the thickness of the porous medium $h_\varepsilon \equiv 1$, then the critical regime (1.4) coincides with the critical one given in (1.2). To prove this result, we first derive global *a priori* estimates of the velocity and pressure and also, particular *a priori* estimates in both media, which let us find the critical regime (1.4). Then, in this critical regime, we study independently the asymptotic behavior in the thin porous medium and in the thin film. Finally, we deduce that the pressure is continuous in the interface and derive the coupled effective problem for the limit pressure. We have introduced the following novelties in this work with respect to previous studies to study the asymptotic behavior in the thin porous medium: a new restriction operator $\mathcal{R}_r^\varepsilon$ to extend the pressure in the thin porous medium to the thin domain without microstructure (see Subsection 3.3), and a new version of the unfolding method (for classical versions see Cioranescu *et al.* [24, 25]) to capture the effects of the microstructure of the thin porous medium (see Subsection 4.1). All this is combined with dimension reduction techniques and monotonicity arguments to be able to pass to the limit when $\varepsilon \rightarrow 0$ and so, to derive the modified Reynolds equation.

We think that this theoretical study provides a quite complete description of the asymptotic behavior of generalized Newtonian fluids with power law viscosity through a thin film passing a thin porous medium, which provides a model amenable for the numerical simulations. For this reason, we believe that it could also prove useful in the engineering practice as well.

Finally, we comment the structure of the paper. In Section 2 we introduce the domain considered and the statement of the problem. In Section 3, we derive *a priori* estimates of the velocity and pressure. Thanks to the estimates, we find the critical regime (1.4). Assuming the critical regime, in Section 4 we study the asymptotic behavior of the problem in the thin porous medium (extending the pressure by duality arguments using the restriction operator $\mathcal{R}_r^\varepsilon$ and using the version of the unfolding method to capture the effects of the microstructure), and in Section 5, we study the asymptotic behavior of the problem in the thin film (which differs a little bit from the classical study of the asymptotic behavior of a non-Newtonian fluid in a thin domain). Finally, in Section 6 we deduce that the pressure is continuous in the interface of both media and we derive the modified Reynolds problem coupling the effects of the thin porous medium and the thin film, which is given Theorem 6.3. We finish the paper with a conclusion section and a list of references.

2 Formulation of the problem and preliminaries

2.1 Geometrical setting

Let $\omega \in (-1/2, 1/2) \subset \mathbb{R}$. We consider three positive and small parameters ε , h_ε and η_ε (where h_ε and η_ε are devoted to tend to zero when $\varepsilon \rightarrow 0$) satisfying the following relation

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{h_\varepsilon} = 0, \quad (\text{i.e. } \varepsilon \ll h_\varepsilon), \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\eta_\varepsilon} = 0, \quad (\text{i.e. } \varepsilon \ll \eta_\varepsilon). \quad (2.5)$$

We consider $D_\varepsilon \subset \mathbb{R}^2$ to be an open set of the following form

$$D_\varepsilon = \Omega_\varepsilon \cup \Sigma \cup I_\varepsilon,$$

where Ω_ε is a thin porous medium and I_ε is a thin layer without obstacles (see Figure 1). Moreover, Σ is the interface between the thin porous medium and the thin film and is defined by

$$\Sigma = \omega \times \{x_2 = 0\}.$$

Below, we describe subdomains Ω_ε and I_ε :

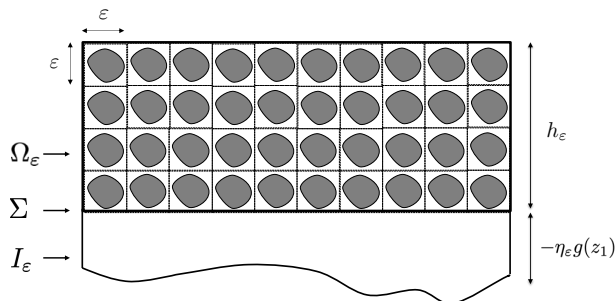


Figure 1: View of the domain D_ε .

- To describe the thin porous medium Ω_ε , we consider the parameters ε and h_ε satisfying (2.5). We consider a thin layer of height h_ε which is perforated by ε -periodic distributed obstacles of size ε . The thin layer without microstructure is denoted by Q_ε , i.e.

$$Q_\varepsilon = \omega \times (0, h_\varepsilon). \quad (2.6)$$

Let us now give a better description of the microstructure of the thin layer. We denote $Y = (-1/2, 1/2)^2$ the unitary cube in \mathbb{R}^2 as the reference cell and T an open connected subset of Y with a smooth boundary ∂T such that $\bar{T} \subset Y$. We denote $Y_f = Y \setminus \bar{T}$. Thus, for $k \in \mathbb{Z}^2$, each cell $Y_{k,\varepsilon} = \varepsilon k + \varepsilon Y$ is similar to the unit cell Y rescaled to size ε and $T_{k,\varepsilon} = \varepsilon k + \varepsilon T$ is similar to T rescaled to size ε . We denote $Y_{f_{k,\varepsilon}} = Y_{k,\varepsilon} \setminus \bar{T}_{k,\varepsilon}$ (see Figure 2).

We denote by $\tau(\bar{T}_{k,\varepsilon})$ the set of all translated images of $\bar{T}_{k,\varepsilon}$, i.e. the set $\tau(\bar{T}_{k,\varepsilon})$ represents the obstacles in \mathbb{R}^2 .

The thin porous media Ω_ε is defined by (see Figure 1)

$$\Omega_\varepsilon = Q_\varepsilon \setminus \bigcup_{k \in \mathcal{K}_\varepsilon} \bar{T}_{k,\varepsilon}, \quad (2.7)$$

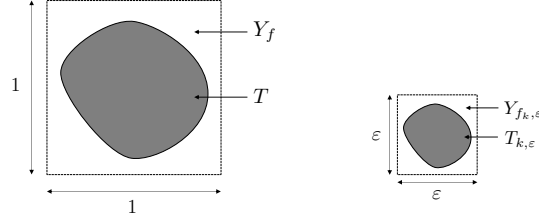


Figure 2: View of the reference cell Y (left) and the rescaled cell $Y_{k,\epsilon}$ (right).

where $\mathcal{K}_\epsilon := \{k \in \mathbb{Z}^3 : Y_{k,\epsilon} \cap Q_\epsilon \neq \emptyset\}$. By construction, Ω_ϵ is a periodically perforated channel with obstacles of the same size as the period. We make the assumption that the obstacles $\tau(\bar{T}_{k,\epsilon})$ do not intersect the boundary ∂Q_ϵ . We denote by T_ϵ the set of all the obstacles contained in Ω_ϵ . Then, T_ϵ is a finite union of obstacles, i.e.

$$T_\epsilon = \bigcup_{k \in \mathcal{K}_\epsilon} \bar{T}_{k,\epsilon}.$$

As usual when we deal with thin domains, we will use the dilatation in the variable x_2 given by

$$z_1 = x_1, \quad z_2 = \frac{x_2}{h_\epsilon}, \quad \forall x \in \Omega_\epsilon. \quad (2.8)$$

Then, we define the rescaled porous media $\tilde{\Omega}_\epsilon$ by (see Figure 3)

$$\tilde{\Omega}_\epsilon = \{z = (z_1, z_2) \in \mathbb{R}^2 : (z_1, h_\epsilon z_2) \in \Omega_\epsilon\}. \quad (2.9)$$

We also introduce the rescaled sets $\tilde{Y}_{k,\epsilon}$ by (see Figure 3)

$$\tilde{Y}_{k,\epsilon} = \{z \in \mathbb{R}^2 : (z_1, \epsilon z_2) \in Y_{k,\epsilon}\},$$

and, in the same way, we define the rescaled fluid part $\tilde{Y}_{f_{k,\epsilon}}$, the rescaled solid part $\tilde{T}_{k,\epsilon}$ of $\tilde{Y}_{k,\epsilon}$ and the union of rescaled obstacles \tilde{T}_ϵ . Finally, we denote by Ω the domain with fixed height without microstructure, i.e.

$$\Omega = \omega \times (0, 1).$$

- To describe the thin layer I_ϵ , we consider the positive and small parameter η_ϵ satisfying (2.5). We define I_ϵ as follows

$$I_\epsilon = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \in \omega, g_\epsilon(x_1) < x_2 < 0\}, \quad (2.10)$$

where the function g_ϵ is given by

$$g_\epsilon(x_1) = -\eta_\epsilon g(x_1), \quad \forall x_1 \in \omega.$$

We define the lower boundary by

$$\Gamma_g^\epsilon = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \in \omega, x_2 = g_\epsilon(x_1)\}.$$

Moreover, the following assumptions concerning the function g are made:

$$g \in C(\omega), \quad 0 < a \leq g(x_1) \leq b, \quad \forall x_1 \in \omega \quad (\text{with } a, b > 0). \quad (2.11)$$

As before, to rescale I_ε in a set with fixed thickness, we will use the dilatation in the variable x_2 given by

$$z_1 = x_1, \quad z_2 = \frac{x_2}{\eta_\varepsilon}, \quad \forall x \in I_\varepsilon. \quad (2.12)$$

Then, we define the rescaled domain \tilde{I}_1 by

$$\tilde{I}_1 = \{z = (z_1, z_2) \in \mathbb{R}^2 : z_1 \in \omega, -g(z_1) < z_2 < 0\}, \quad (2.13)$$

and the rescaled lower boundary by

$$\Gamma_g = \{z = (z_1, z_2) \in \mathbb{R}^2 : z_1 \in \omega, z_2 = -g(z_1)\}.$$

Finally, we define the domain with microstructure by

$$\Lambda_\varepsilon = Q_\varepsilon \cup \Sigma \cup I_\varepsilon,$$

the rescaled domain with microstructure (see Figure 3) by

$$\tilde{D}_\varepsilon = \tilde{\Omega}_\varepsilon \cup \Sigma \cup \tilde{I}_1,$$

and the whole rescaled domain without microstructure by

$$D = \Omega \cup \Sigma \cup \tilde{I}_1.$$

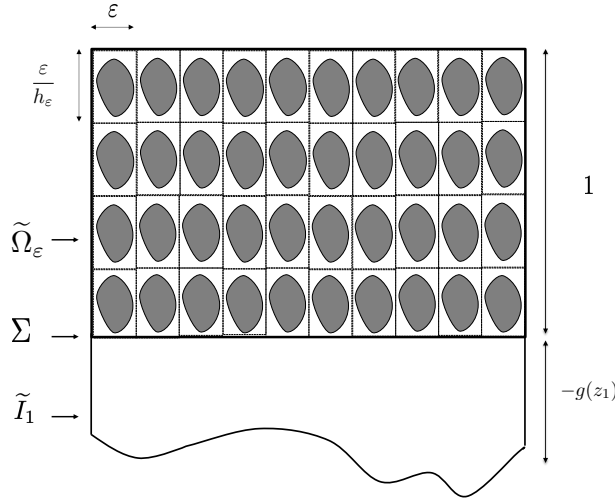


Figure 3: View of the rescaled domain \tilde{D}_ε .

2.2 Some notation

Let us consider a vectorial function $\varphi = (\varphi_1, \varphi_2)$ and a scalar function ψ . We have denoted by $\mathbb{D} : \mathbb{R}^2 \rightarrow \mathbb{R}_{\text{sym}}^2$ the symmetric part of the velocity gradient, that is

$$\mathbb{D}[\varphi] = \frac{1}{2}(D\varphi + (D\varphi)^T) = \begin{pmatrix} \partial_{x_1}\varphi_1 & \frac{1}{2}(\partial_{x_1}\varphi_2 + \partial_{x_2}\varphi_1) \\ \frac{1}{2}(\partial_{x_1}\varphi_2 + \partial_{x_2}\varphi_1) & \partial_{x_2}\varphi_2 \end{pmatrix},$$

and also, we have used the following operators

$$\Delta\varphi = \partial_{x_1}^2\varphi + \partial_{x_2}^2\varphi, \quad \operatorname{div}(\varphi) = \partial_{x_1}\varphi_1 + \partial_{x_2}\varphi_2, \quad \nabla\psi = (\partial_{x_1}\psi, \partial_{x_2}\psi)^t.$$

For a vectorial function $\tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2)$ and a scalar function $\tilde{\psi}$ obtained respectively from φ and ψ by using the rescaling (2.8) in the set Ω_ε , we will denote

$$\partial_{z_1}[\tilde{\varphi}] = \frac{1}{2}(\partial_{z_1}\tilde{\varphi} + (\partial_{z_1}\tilde{\varphi})^T) = \begin{pmatrix} \partial_{z_1}\tilde{\varphi}_1 & \frac{1}{2}\partial_{z_1}\tilde{\varphi}_2 \\ \frac{1}{2}\partial_{z_1}\tilde{\varphi}_2 & 0 \end{pmatrix}, \quad \partial_{z_2}[\tilde{\varphi}] = \begin{pmatrix} 0 & \frac{1}{2}\partial_{z_2}\tilde{\varphi}_1 \\ \frac{1}{2}\partial_{z_2}\tilde{\varphi}_1 & \partial_{z_2}\tilde{\varphi}_2 \end{pmatrix},$$

and then,

$$\mathbb{D}_{h_\varepsilon}[\tilde{\varphi}] = \partial_{z_1}[\tilde{\varphi}] + h_\varepsilon^{-1}\partial_{z_2}[\tilde{\varphi}] = \begin{pmatrix} \partial_{z_1}\tilde{\varphi}_1 & \frac{1}{2}(\partial_{z_1}\tilde{\varphi}_2 + h_\varepsilon^{-1}\partial_{z_2}\tilde{\varphi}_1) \\ \frac{1}{2}(\partial_{z_1}\tilde{\varphi}_2 + h_\varepsilon^{-1}\partial_{z_2}\tilde{\varphi}_1) & h_\varepsilon^{-1}\partial_{z_2}\tilde{\varphi}_2 \end{pmatrix}. \quad (2.14)$$

Also, we define the operators Δ_{h_ε} , D_{h_ε} , ∇_{h_ε} and $\operatorname{div}_{h_\varepsilon}$ as follows

$$\begin{aligned} (D_{h_\varepsilon}\tilde{\varphi})_{i,1} &= \partial_{x_1}\tilde{\varphi}_i, & (D_{h_\varepsilon}\tilde{\varphi})_{i,2} &= h_\varepsilon^{-1}\partial_{z_2}\tilde{\varphi}_i \text{ for } i = 1, 2, \\ \Delta_{h_\varepsilon}\tilde{\varphi} &= \partial_{z_1}^2\tilde{\varphi} + h_\varepsilon^{-2}\partial_{z_2}^2\tilde{\varphi}, & \operatorname{div}_{h_\varepsilon}(\tilde{\varphi}) &= \partial_{z_1}\tilde{\varphi}_1 + h_\varepsilon^{-1}\partial_{z_2}\tilde{\varphi}_2, \\ \nabla_{h_\varepsilon}\tilde{\psi} &= (\partial_{x_1}\tilde{\psi}, h_\varepsilon^{-1}\partial_{z_2}\tilde{\psi})^t. \end{aligned}$$

Similarly, we define the operators $\mathbb{D}_{\eta_\varepsilon}$, $\Delta_{\eta_\varepsilon}$, D_{η_ε} , $\nabla_{\eta_\varepsilon}$ and $\operatorname{div}_{\eta_\varepsilon}$ by using rescaling (2.12) in the set I_ε . The definitions are analogous to the operators $\mathbb{D}_{h_\varepsilon}$, Δ_{h_ε} , D_{h_ε} , ∇_{h_ε} and $\operatorname{div}_{h_\varepsilon}$, just replacing h_ε by η_ε .

Let $C_{\text{per}}^\infty(Y)$ be the space of infinitely differentiable functions in \mathbb{R}^2 that are Y -periodic. By $L_{\text{per}}^r(Y)$ (resp. $W_{\text{per}}^{1,r}(Y)$) we denote its completion in the norm $L^r(Y)$ (resp. $W^{1,r}(Y)$). We denote by $L_0^{r'}$ the space of functions of $L^{r'}$ with null integral and by $L_{0,\text{per}}^{r'}(Y)$ the space of functions in $L_{\text{per}}^{r'}(Y)$ with zero mean value.

We denote by $:$ the full contraction of two matrices, i.e. for $A = (a_{ij})_{1 \leq i,j \leq 3}$ and $B = (b_{ij})_{1 \leq i,j \leq 2}$, we have $A : B = \sum_{i,j=1}^2 a_{ij}b_{ij}$. The canonical basis in \mathbb{R}^2 is denoted by $\{e_1, e_2\}$.

Finally, we denote by O_ε a generic real sequence, which tends to zero with ε and can change from line to line, and by C a generic positive constant which also can change from line to line.

2.3 Statement of the problem

Let us consider a sequence $(u_\varepsilon, p_\varepsilon) \in W_0^{1,r}(D_\varepsilon)^2 \times L_0^{r'}(D_\varepsilon)$, $1 < r < +\infty$, which satisfies

$$\begin{cases} -\nu \operatorname{div}(|\mathbb{D}[u_\varepsilon]|^{r-2}\mathbb{D}[u_\varepsilon]) + \nabla p_\varepsilon = f & \text{in } D_\varepsilon, \\ \operatorname{div}(u_\varepsilon) = 0 & \text{in } D_\varepsilon, \end{cases} \quad (2.15)$$

and the boundary condition

$$u_\varepsilon = 0 \quad \text{on } \partial T_\varepsilon \cup \partial \Lambda_\varepsilon, \quad (2.16)$$

where $\nu > 0$ is the consistency, and $r' = r/(r-1)$ is the conjugate exponent of r . We assume

$$f = (f_1(x_1), 0) \quad \text{with} \quad f_1 \in L^{r'}(\omega), \quad (2.17)$$

which is usual when we deal with thin domains. Since the thickness of the domain is small, then the vertical component of the force can be neglected, and moreover, the force can be considered independent of the vertical variable.

Our aim is to describe the asymptotic behavior of the velocity u_ε and the pressure p_ε of the fluid as ε tends to zero and identify an homogenized model coupling the effects of the thickness and the microgeometry of the domain. To do this, we will use the equivalent weak variational formulation of (2.15)–(2.16), which is the following one: find $u_\varepsilon \in W_0^{1,r}(D_\varepsilon)^2$ and $p_\varepsilon \in L_0^{r'}(D_\varepsilon)$ such that

$$\begin{aligned} \nu \int_{D_\varepsilon} |\mathbb{D}[u_\varepsilon]|^{r-2} \mathbb{D}[u_\varepsilon] : \mathbb{D}[\varphi] \, dx - \int_{D_\varepsilon} p_\varepsilon \operatorname{div}(\varphi) \, dx &= \int_{D_\varepsilon} f \cdot \varphi \, dx, \quad \forall \varphi \in W_0^{1,r}(D_\varepsilon)^2, \\ \int_{D_\varepsilon} \operatorname{div}(u_\varepsilon) \psi \, dx &= 0 \quad \forall \psi \in L^{r'}(D_\varepsilon). \end{aligned} \quad (2.18)$$

It is well known (see for instance the classical theory [43]) that, for every $\varepsilon > 0$, problem (2.15)–(2.16) has a unique weak solution $(u_\varepsilon, p_\varepsilon) \in W_0^{1,r}(D_\varepsilon)^2 \times L_0^{r'}(D_\varepsilon)$.

In order to find the limit problem when ε tends to zero, it is necessary to obtain a priori estimates in fixed domains (with respect to ε), so we introduce the rescaling given by (2.8) for the thin porous media and (2.12) for the thin film, that is

$$\begin{cases} z_1 = x_1, & z_2 = \frac{x_2}{h_\varepsilon} & \text{if } (x_1, x_2) \in \Omega_\varepsilon, \\ z_1 = x_1, & z_2 = \frac{x_2}{\eta_\varepsilon} & \text{if } (x_1, x_2) \in I_\varepsilon. \end{cases} \quad (2.19)$$

Using this rescaling, we can define $\tilde{u}_\varepsilon \in W_0^{1,r}(\tilde{D}_\varepsilon)^2$ and $\tilde{p}_\varepsilon \in L_0^{r'}(\tilde{D}_\varepsilon)$ by

$$\begin{cases} \tilde{u}_\varepsilon(z) = u_\varepsilon(z_1, h_\varepsilon z_2), & \tilde{p}_\varepsilon(z) = p_\varepsilon(z_1, h_\varepsilon z_2) & \text{if } z \in \tilde{\Omega}_\varepsilon, \\ \tilde{u}_\varepsilon(z) = u_\varepsilon(z_1, \eta_\varepsilon z_2), & \tilde{p}_\varepsilon(z) = p_\varepsilon(z_1, \eta_\varepsilon z_2) & \text{if } z \in \tilde{I}_1, \end{cases} \quad (2.20)$$

so the rescaled weak variational formulation is the following: find $\tilde{u}_\varepsilon \in W_0^{1,r}(\tilde{D}_\varepsilon)^2$, $\tilde{p}_\varepsilon \in L_0^{r'}(\tilde{D}_\varepsilon)$ such that

$$\begin{aligned} \nu \int_{\tilde{\Omega}_\varepsilon} h_\varepsilon |\mathbb{D}_{h_\varepsilon}[\tilde{u}_\varepsilon]|^{r-2} \mathbb{D}_{h_\varepsilon}[\tilde{u}_\varepsilon] : \mathbb{D}_{h_\varepsilon}[\tilde{\varphi}] \, dz + \nu \int_{\tilde{I}_1} \eta_\varepsilon |\mathbb{D}_{\eta_\varepsilon}[\tilde{u}_\varepsilon]|^{r-2} \mathbb{D}_{\eta_\varepsilon}[\tilde{u}_\varepsilon] : \mathbb{D}_{\eta_\varepsilon}[\tilde{\varphi}] \, dz \\ - \int_{\tilde{\Omega}_\varepsilon} h_\varepsilon \tilde{p}_\varepsilon \operatorname{div}_{h_\varepsilon}(\tilde{\varphi}) \, dz - \int_{\tilde{I}_1} \eta_\varepsilon \tilde{p}_\varepsilon \operatorname{div}_{\eta_\varepsilon}(\tilde{\varphi}) \, dz \\ = \int_{\tilde{\Omega}_\varepsilon} h_\varepsilon f \cdot \tilde{\varphi} \, dz + \int_{\tilde{I}_1} \eta_\varepsilon f \cdot \tilde{\varphi} \, dz, \quad \forall \tilde{\varphi} \in W_0^{1,r}(\tilde{D}_\varepsilon)^2, \\ \int_{\tilde{\Omega}_\varepsilon} h_\varepsilon \operatorname{div}_{h_\varepsilon}(\tilde{u}_\varepsilon) \tilde{\psi} \, dz + \int_{\tilde{I}_1} \eta_\varepsilon \operatorname{div}_{\eta_\varepsilon}(\tilde{u}_\varepsilon) \tilde{\psi} \, dz = 0, \quad \forall \tilde{\psi} \in L^{r'}(\tilde{D}_\varepsilon). \end{aligned} \quad (2.21)$$

Moreover, to study the behavior of the velocity in the two media, we introduce the following notation for velocity:

$$u_\varepsilon = v_\varepsilon + \mathcal{U}_\varepsilon,$$

where

- v_ε denotes the velocity in the thin porous medium, extended by zero to the thin domain I_ε , i.e.

$$v_\varepsilon(x) = \begin{cases} u_\varepsilon(x) & \text{if } x \in \Omega_\varepsilon, \\ 0 & \text{if } x \in I_\varepsilon. \end{cases} \quad (2.22)$$

We denote by \tilde{v}_ε the dilated velocity in $\tilde{\Omega}_\varepsilon$ obtained from v_ε by using the change of variables (2.8). Then, \tilde{v}_ε satisfies the following equality

$$\begin{aligned} \nu \int_{\tilde{\Omega}_\varepsilon} |\mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon]|^{r-2} \mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon] : \mathbb{D}_{h_\varepsilon}[\tilde{\varphi}] dz - \int_{\tilde{\Omega}_\varepsilon} \tilde{p}_\varepsilon \operatorname{div}_{h_\varepsilon}(\tilde{\varphi}) dz &= \int_{\tilde{\Omega}_\varepsilon} f \cdot \tilde{\varphi} dz, \quad \forall \tilde{\varphi} \in W_0^{1,r}(\tilde{\Omega}_\varepsilon)^2, \\ \int_{\tilde{\Omega}_\varepsilon} \operatorname{div}_{h_\varepsilon}(\tilde{v}_\varepsilon) \tilde{\psi} dz &= 0, \quad \forall \tilde{\psi} \in L^{r'}(\tilde{\Omega}_\varepsilon). \end{aligned} \quad (2.23)$$

- \mathcal{U}_ε denotes the velocity in the thin film I_ε , extended by zero to the thin porous medium Ω_ε , i.e.

$$\mathcal{U}_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in \Omega_\varepsilon, \\ u_\varepsilon(x) & \text{if } x \in I_\varepsilon. \end{cases} \quad (2.24)$$

We denote by $\tilde{\mathcal{U}}_\varepsilon$ the dilated velocity in \tilde{I}_1 obtained from \mathcal{U}_ε by using the change of variables (2.12). Then $\tilde{\mathcal{U}}_\varepsilon$ satisfies the following equality

$$\begin{aligned} \nu \int_{\tilde{I}_1} |\mathbb{D}_{\eta_\varepsilon}[\tilde{\mathcal{U}}_\varepsilon]|^{r-2} \mathbb{D}_{\eta_\varepsilon}[\tilde{\mathcal{U}}_\varepsilon] : \mathbb{D}_{\eta_\varepsilon}[\tilde{\varphi}] dz - \int_{\tilde{I}_1} \tilde{p}_\varepsilon \operatorname{div}_{\eta_\varepsilon}(\tilde{\varphi}) dz &= \int_{\tilde{I}_1} f \cdot \tilde{\varphi} dz, \quad \forall \tilde{\varphi} \in W_0^{1,r}(\tilde{I}_1)^2, \\ \int_{\tilde{I}_1} \operatorname{div}_{\eta_\varepsilon}(\tilde{\mathcal{U}}_\varepsilon) \tilde{\psi} dz &= 0 \quad \forall \tilde{\psi} \in L^{r'}(\tilde{I}_1). \end{aligned} \quad (2.25)$$

3 A priori estimates

3.1 Some technical estimates

Let us begin with the Poincaré and Korn inequalities in the thin porous medium Ω_ε .

Lemma 3.1 (Poincaré and Korn inequalities). *For every $\varphi \in W^{1,r}(\Omega_\varepsilon)^2$, $1 < r < +\infty$, with $\varphi = 0$ on $\partial\Omega_\varepsilon \setminus \Sigma$, there exists a positive constant C , independent of ε , such that,*

$$\|\varphi\|_{L^r(\Omega_\varepsilon)^2} \leq C\varepsilon \|D\varphi\|_{L^r(\Omega_\varepsilon)^{2 \times 2}}, \quad (3.26)$$

$$\|D\varphi\|_{L^r(\Omega_\varepsilon)^{2 \times 2}} \leq C \|\mathbb{D}[\varphi]\|_{L^r(\Omega_\varepsilon)^{2 \times 2}}. \quad (3.27)$$

Proof. We observe that Ω_ε can be divided in small cubes of lateral and vertical length ε . We consider the periodic cell Y_f and have a Friedrichs inequality

$$\int_{Y_f} |\varphi|^r dz \leq C \int_{Y_f} |D\varphi|^r dz, \quad (3.28)$$

for every $\varphi(z) \in W^{1,r}(Y_f)^2$ such that $\varphi = 0$ on ∂T , where the constant C depends only on Y_f . Then, for every $k \in \mathbb{Z}^2$, by the change of variable

$$k + z = \frac{x}{\varepsilon}, \quad dz = \frac{dx}{\varepsilon^2}, \quad \partial_z = \varepsilon \partial_x, \quad (3.29)$$

we rescale (3.28) from Y_f to $Y_{f_k, \varepsilon}$. This yields that, for every function $\varphi(x) \in W^{1,r}(Y_{f_k, \varepsilon})^2$, one has

$$\int_{Y_{f_k, \varepsilon}} |\varphi|^r dx \leq C \varepsilon^r \int_{Y_{f_k, \varepsilon}} |D\varphi|^r dx,$$

with the same constant C as in (3.28). Summing previous inequality for every $k \in \mathcal{K}_\varepsilon$, we get (3.26).

Finally, Korn's inequality (3.27) follows from the classical Korn inequality, see [23].

□

Next, we give an useful estimate in the thin film I_ε .

Lemma 3.2. *For every function $\varphi \in W_0^{1,r}(D_\varepsilon)^2$, with $1 < r < \infty$, there exists a constant $C > 0$ independent of ε , such that,*

$$\|\varphi\|_{L^r(I_\varepsilon)^2} \leq C \eta_\varepsilon^{\frac{1}{2}} (\eta_\varepsilon + \varepsilon)^{\frac{1}{2}} \|\mathbb{D}[\varphi]\|_{L^r(D_\varepsilon)^{2 \times 2}}. \quad (3.30)$$

Proof. Because the thickness of I_ε (see for instance [37]), we have

$$\|\varphi\|_{L^r(I_\varepsilon)^2} \leq C \eta_\varepsilon \|D\varphi\|_{L^r(I_\varepsilon)^{2 \times 2}}. \quad (3.31)$$

Next, if we choose a point $t \in T_\varepsilon$, which is close to the point $x \in I_\varepsilon$, then, we have

$$|\varphi(x) - \varphi(t)| = |D\varphi(\xi)(x - t)| \leq (\varepsilon + \eta_\varepsilon) |D\varphi|.$$

Since $\varphi(t) = 0$ because $t \in T_\varepsilon$, we have

$$\|\varphi\|_{L^r(I_\varepsilon)^2} \leq C(\varepsilon + \eta_\varepsilon) \|D\varphi\|_{L^r(I_\varepsilon)^{2 \times 2}}.$$

Multiplying the above inequality with inequality (3.31), we get

$$\|\varphi\|_{L^r(I_\varepsilon)^2} \leq C \eta_\varepsilon^{\frac{1}{2}} (\varepsilon + \eta_\varepsilon)^{\frac{1}{2}} \|Dv\|_{L^r(I_\varepsilon)^{2 \times 2}} \leq C \eta_\varepsilon^{\frac{1}{2}} (\varepsilon + \eta_\varepsilon)^{\frac{1}{2}} \|Dv\|_{L^r(D_\varepsilon)^{2 \times 2}}, \quad (3.32)$$

and from the classical Korn inequality in I_ε , we obtain the estimate (3.30). □

3.2 Estimates for velocity

We derive the estimates for velocity in the whole domain D_ε for u_ε , and also, in the sets Ω_ε and I_ε for v_ε and \mathcal{U}_ε respectively.

Lemma 3.3. *There exists a constant $C > 0$ independent of ε , such that if $u_\varepsilon \in W_0^{1,r}(D_\varepsilon)^2$, with $1 < r < +\infty$, is the solution of problem (2.15)–(2.16), it holds*

$$\|v_\varepsilon\|_{L^r(\Omega_\varepsilon)^2} \leq C \left(\eta_\varepsilon^{\frac{2r-1}{r}} \varepsilon^{r-1} + h_\varepsilon^{\frac{r-1}{r}} \varepsilon^r \right)^{\frac{1}{r-1}}, \quad (3.33)$$

$$\|\mathcal{U}_\varepsilon\|_{L^r(I_\varepsilon)^2} \leq C \eta_\varepsilon^{1+\frac{2r-1}{r(r-1)}} + \varepsilon^{\frac{1}{r-1}} \eta_\varepsilon h_\varepsilon^{\frac{1}{2}} + \eta_\varepsilon^{\frac{1}{2}} \varepsilon^{\frac{1}{2}+\frac{1}{r-1}} h_\varepsilon^{\frac{1}{2}}, \quad (3.34)$$

$$\|\mathbb{D}[u_\varepsilon]\|_{L^r(D_\varepsilon)^{2 \times 2}} \leq C \left(\eta_\varepsilon^{\frac{2r-1}{r}} + h_\varepsilon^{\frac{r-1}{r}} \varepsilon \right)^{\frac{1}{r-1}}, \quad (3.35)$$

$$\|Du_\varepsilon\|_{L^r(D_\varepsilon)^{2 \times 2}} \leq C \left(\eta_\varepsilon^{\frac{2r-1}{r}} + h_\varepsilon^{\frac{r-1}{r}} \varepsilon \right)^{\frac{1}{r-1}}. \quad (3.36)$$

Proof. Using u_ε as test function in (2.18), we have

$$\nu \|\mathbb{D}[u_\varepsilon]\|_{L^r(D_\varepsilon)^{2 \times 2}}^r = \int_{D_\varepsilon} f \cdot u_\varepsilon \, dx. \quad (3.37)$$

Using the Hölder inequality and the assumption of f given in (2.17), we obtain that there exists a constant $C > 0$ such that

$$\int_{D_\varepsilon} f \cdot u_\varepsilon \, dx \leq C \left(\eta_\varepsilon^{\frac{1}{r'}} \|u_\varepsilon\|_{L^r(I_\varepsilon)^2} + h_\varepsilon^{\frac{1}{r'}} \|u_\varepsilon\|_{L^r(\Omega_\varepsilon)^2} \right),$$

and by inequalities (3.26), (3.27) and (3.30), we have

$$\begin{aligned} \int_{D_\varepsilon} f \cdot u_\varepsilon \, dx &\leq C \left(\eta_\varepsilon^{\frac{1}{r'}} \eta_\varepsilon^{\frac{1}{2}} (\varepsilon + \eta_\varepsilon)^{\frac{1}{2}} + h_\varepsilon^{\frac{1}{r'}} \varepsilon \right) \|\mathbb{D}[u_\varepsilon]\|_{L^r(D_\varepsilon)^{2 \times 2}} \\ &\leq C \left(\eta_\varepsilon^{\frac{1}{r'}+1} + \eta_\varepsilon^{\frac{1}{r'}+\frac{1}{2}} \varepsilon^{\frac{1}{2}} + h_\varepsilon^{\frac{1}{r'}} \varepsilon \right) \|\mathbb{D}[u_\varepsilon]\|_{L^r(D_\varepsilon)^{2 \times 2}}. \end{aligned}$$

Therefore, from equation (3.37), we get

$$\|\mathbb{D}[u_\varepsilon]\|_{L^r(D_\varepsilon)^{2 \times 2}} \leq C \left(\eta_\varepsilon^{\frac{1}{r'}+1} + \eta_\varepsilon^{\frac{1}{r'}+\frac{1}{2}} \varepsilon^{\frac{1}{2}} + h_\varepsilon^{\frac{1}{r'}} \varepsilon \right)^{\frac{1}{r-1}}.$$

Since $\varepsilon \ll \eta_\varepsilon$, then $\eta_\varepsilon^{\frac{1}{r'}+\frac{1}{2}} \varepsilon^{\frac{1}{2}} \ll \eta_\varepsilon^{\frac{1}{r'}+1}$ and so, the term $\eta_\varepsilon^{\frac{1}{r'}+\frac{1}{2}} \varepsilon^{\frac{1}{2}}$ can be dropped. Then, taking into account that $1/r' + 1 = (2r - 1)/r$ and $1/r' = (r - 1)/r$, we get estimate (3.35). From the classical Korn inequality, we have inequality (3.36). Applying inequality (3.26) together with inequality (3.36), we obtain inequality (3.33).

Finally, applying inequalities (3.30) and (3.35), we get

$$\begin{aligned} \|\mathcal{U}_\varepsilon\|_{L^r(I_\varepsilon)^2} &\leq C(\eta_\varepsilon + \eta_\varepsilon^{\frac{1}{2}} \varepsilon^{\frac{1}{2}}) \left(\eta_\varepsilon^{\frac{2r-1}{r(r-1)}} + h_\varepsilon^{\frac{1}{r}} \varepsilon^{\frac{1}{r-1}} \right) \\ &= C \left(\eta_\varepsilon^{1+\frac{2r-1}{r(r-1)}} + \varepsilon^{\frac{1}{r-1}} \eta_\varepsilon^{\frac{1}{r}} h_\varepsilon^{\frac{1}{r}} + \eta_\varepsilon^{\frac{1}{2}} \eta_\varepsilon^{\frac{2r-1}{r(r-1)}} \varepsilon^{\frac{1}{2}} + \eta_\varepsilon^{\frac{1}{2}} \varepsilon^{\frac{1}{2}+\frac{1}{r-1}} h_\varepsilon^{\frac{1}{r}} \right). \end{aligned}$$

Since $\varepsilon \ll \eta_\varepsilon$, then $\eta_\varepsilon^{\frac{1}{2}} \eta_\varepsilon^{\frac{2r-1}{r(r-1)}} \varepsilon^{\frac{1}{2}} \ll \eta_\varepsilon^{1+\frac{2r-1}{r(r-1)}}$ and so, the term $\eta_\varepsilon^{\frac{1}{2}} \eta_\varepsilon^{\frac{2r-1}{r(r-1)}} \varepsilon^{\frac{1}{2}}$ can be dropped, and inequality (3.34) holds. \square

As consequence, by using the change of variables (2.8) in the thin porous medium Ω_ε and (2.12) in the thin film I_ε , we derive the estimates for the rescaled velocities.

Corollary 3.4. *There exists a constant $C > 0$ independent of ε , such that we have the following estimates depending on the media:*

– In the porous media $\tilde{\Omega}_\varepsilon$, we have

$$\|\tilde{v}_\varepsilon\|_{L^r(\tilde{\Omega}_\varepsilon)^2} \leq C \left(\eta_\varepsilon^{\frac{2r-1}{r}} \varepsilon^{r-1} h_\varepsilon^{-\frac{r-1}{r}} + \varepsilon^r \right)^{\frac{1}{r-1}}, \quad (3.38)$$

$$\|\mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon]\|_{L^r(\tilde{\Omega}_\varepsilon)^{2 \times 2}} \leq C \left(\eta_\varepsilon^{\frac{2r-1}{r}} h_\varepsilon^{-\frac{r-1}{r}} + \varepsilon \right)^{\frac{1}{r-1}}, \quad (3.39)$$

$$\|D_{h_\varepsilon} \tilde{v}_\varepsilon\|_{L^r(\tilde{\Omega}_\varepsilon)^{2 \times 2}} \leq C \left(\eta_\varepsilon^{\frac{2r-1}{r}} h_\varepsilon^{-\frac{r-1}{r}} + \varepsilon \right)^{\frac{1}{r-1}}. \quad (3.40)$$

– In the free media \tilde{I}_1 , we have

$$\|\tilde{\mathcal{U}}_\varepsilon\|_{L^r(\tilde{I}_1)^2} \leq C \left(\eta_\varepsilon^{\frac{r}{r-1}} + \varepsilon^{\frac{1}{r-1}} h_\varepsilon^{\frac{1}{r}} \eta_\varepsilon^{\frac{r-1}{r}} + \varepsilon^{\frac{r+1}{2(r-1)}} h_\varepsilon^{\frac{1}{r}} \eta_\varepsilon^{\frac{r-2}{2r}} \right), \quad (3.41)$$

$$\|\mathbb{D}_{\eta_\varepsilon}[\tilde{\mathcal{U}}_\varepsilon]\|_{L^r(\tilde{I}_1)^{2 \times 2}} \leq C \left(\eta_\varepsilon + h_\varepsilon^{\frac{r-1}{r}} \varepsilon \eta_\varepsilon^{-\frac{r-1}{r}} \right)^{\frac{1}{r-1}}, \quad (3.42)$$

$$\|D_{\eta_\varepsilon} \tilde{\mathcal{U}}_\varepsilon\|_{L^r(\tilde{I}_1)^{2 \times 2}} \leq C \left(\eta_\varepsilon + h_\varepsilon^{\frac{r-1}{r}} \varepsilon \eta_\varepsilon^{-\frac{r-1}{r}} \right)^{\frac{1}{r-1}}. \quad (3.43)$$

Proof. Estimates for dilated velocity (3.38)–(3.40) in $\tilde{\Omega}_\varepsilon$ are obtained directly from (3.33), (3.35) and (3.36) by applying the change of variables (2.8), just taking into account that

$$\begin{aligned} \|v_\varepsilon\|_{L^r(\Omega_\varepsilon)^2} &= h_\varepsilon^{\frac{1}{r}} \|\tilde{v}_\varepsilon\|_{L^r(\tilde{\Omega}_\varepsilon)^2}, \quad \|\mathbb{D}v_\varepsilon\|_{L^r(\Omega_\varepsilon)^{2 \times 2}} = h_\varepsilon^{\frac{1}{r}} \|\mathbb{D}_{h_\varepsilon} \tilde{v}_\varepsilon\|_{L^r(\tilde{\Omega}_\varepsilon)^{2 \times 2}}, \\ \|Dv_\varepsilon\|_{L^r(\Omega_\varepsilon)^{2 \times 2}} &= h_\varepsilon^{\frac{1}{r}} \|D_{h_\varepsilon} \tilde{v}_\varepsilon\|_{L^r(\tilde{\Omega}_\varepsilon)^{2 \times 2}}. \end{aligned}$$

Similarly, estimates for dilated velocity (3.41)–(3.43) in \tilde{I}_1 are obtained directly from (3.34), (3.35) and (3.36) by applying the change of variables (2.12), just taking into account that

$$\begin{aligned} \|\mathcal{U}_\varepsilon\|_{L^r(I_\varepsilon)^2} &= \eta_\varepsilon^{\frac{1}{r}} \|\tilde{\mathcal{U}}_\varepsilon\|_{L^r(\tilde{I}_1)^2}, \quad \|\mathbb{D}[\mathcal{U}_\varepsilon]\|_{L^r(I_\varepsilon)^{2 \times 2}} = \eta_\varepsilon^{\frac{1}{r}} \|\mathbb{D}_{\eta_\varepsilon}[\tilde{\mathcal{U}}_\varepsilon]\|_{L^r(\tilde{I}_1)^{2 \times 2}}, \\ \|D\mathcal{U}_\varepsilon\|_{L^r(I_\varepsilon)^{2 \times 2}} &= \eta_\varepsilon^{\frac{1}{r}} \|D_{\eta_\varepsilon} \tilde{\mathcal{U}}_\varepsilon\|_{L^r(\tilde{I}_1)^{2 \times 2}}. \end{aligned}$$

\square

3.3 Estimates for pressure in the porous part

Next, we derive a priori estimates for the pressure in the porous part. To do this, we need to extend the pressure to the whole thin film Q_ε (which also depends on ε). To do this, we generalize a result from [15, Lemma 3.3] (see also [40, Lemma 5.3]) by introducing a restriction operator $\mathcal{R}_r^\varepsilon$ from $W_0^{1,r}(Q_\varepsilon)^2$ into $W_0^{1,r}(\Omega_\varepsilon)^2$, $1 < r < +\infty$. We remark that in the case $r = 2$, this restriction operator $\mathcal{R}_2^\varepsilon$ agrees with the one defined in [15, Lemma 3.3] and in [40, Lemma 5.3].

Lemma 3.5. *There exists a (restriction) operator $\mathcal{R}_r^\varepsilon$ acting from $W_0^{1,r}(Q_\varepsilon)^2$ into $W_0^{1,r}(\Omega_\varepsilon)^2$ such that*

1. $\mathcal{R}_r^\varepsilon \varphi = \varphi$, if $\varphi \in W_0^{1,r}(\Omega_\varepsilon)^2$.
2. $\operatorname{div}(\mathcal{R}_r^\varepsilon \varphi) = 0$ in Ω_ε , if $\operatorname{div}(\varphi) = 0$ on Q_ε .
3. For every $\varphi \in W_0^{1,r}(Q_\varepsilon)^3$, there exists a positive constant C , independent of φ and ε , such that

$$\|\mathcal{R}_r^\varepsilon \varphi\|_{L^r(\Omega_\varepsilon)^2} + \varepsilon \|D\mathcal{R}_r^\varepsilon \varphi\|_{L^r(\Omega_\varepsilon)^{2 \times 2}} \leq C \left(\|\varphi\|_{L^r(Q_\varepsilon)^2} + \varepsilon \|D\varphi\|_{L^r(Q_\varepsilon)^{2 \times 2}} \right). \quad (3.44)$$

Proof. Let us consider the linear map \mathcal{R}_r constructed in [21, Lemma 1.1] from $W^{1,r}(Y)^2 \rightarrow W_T^{1,r}(Y_f)^2$, $1 < r < +\infty$, where $W_T^{1,r}(Y_f)^2 = \{\varphi \in W^{1,r}(Y_f)^2 : \varphi = 0 \text{ on } T\}$, such that

$$\|\mathcal{R}_r \varphi\|_{W^{1,r}(Y_f)^2} \leq C \|\varphi\|_{W^{1,r}(Y)^2}, \quad (3.45)$$

and $\mathcal{R}_r \varphi$ coincides with φ if φ is zero on T (i.e. if $\varphi \in W_T^{1,r}(Y_f)^2$) and $\operatorname{div}(\varphi) = 0$ implies $\operatorname{div}(\mathcal{R}_r \varphi) = 0$. Then, $\mathcal{R}_r^\varepsilon$ is defined by applying \mathcal{R}_r to each $Y_{k,\varepsilon}$. Consequently, the two first items are satisfied.

Finally, we will prove the third item. From (3.45), by the change of variables (3.29), as in Lemma 3.1, we have

$$\int_{Y_{f_k,\varepsilon}} |\mathcal{R}_r^\varepsilon \varphi|^r dx + \varepsilon^r \int_{Y_{f_k,\varepsilon}} |D\mathcal{R}_r^\varepsilon \varphi|^r dx \leq C \left(\int_{Y_{k,\varepsilon}} |\varphi|^r dx + \varepsilon^r \int_{Y_{k,\varepsilon}} |D\varphi|^r dx \right).$$

and so, summing previous inequality for every $k \in \mathcal{K}_\varepsilon$, we deduce (3.44). \square

Lemma 3.6. *Setting $\tilde{\mathcal{R}}_r^\varepsilon \tilde{\varphi} = \mathcal{R}_r^\varepsilon \varphi$ for any $\tilde{\varphi} \in W_0^{1,r}(\Omega)^2$, $1 < r < +\infty$, where $\tilde{\varphi}$ is obtained from φ by using the change of variables (2.8), and $\mathcal{R}_r^\varepsilon$ is defined in Lemma 3.5, we have the following estimates:*

$$\|\tilde{\mathcal{R}}_r^\varepsilon \tilde{\varphi}\|_{L^r(\tilde{\Omega}_\varepsilon)^2} \leq C \|\tilde{\varphi}\|_{W_0^{1,r}(\Omega)^2}, \quad \|D_{h_\varepsilon} \tilde{\mathcal{R}}_r^\varepsilon \tilde{\varphi}\|_{L^r(\tilde{\Omega}_\varepsilon)^{2 \times 2}} \leq C \varepsilon^{-1} \|\tilde{\varphi}\|_{W_0^{1,r}(\Omega)^2}. \quad (3.46)$$

Proof. Applying the change of variables (2.8) to estimates (3.44) and taking into account that $\varepsilon \ll h_\varepsilon$, we get

$$\begin{aligned} \|\tilde{\mathcal{R}}_r^\varepsilon \tilde{\varphi}\|_{L^r(\tilde{\Omega}_\varepsilon)^2} + \varepsilon \|D_{h_\varepsilon} \tilde{\mathcal{R}}_r^\varepsilon \tilde{\varphi}\|_{L^r(\tilde{\Omega}_\varepsilon)^{2 \times 2}} &\leq C \left(\|\tilde{\varphi}\|_{L^r(\Omega)^2} + \varepsilon \|D_{h_\varepsilon} \tilde{\varphi}\|_{L^r(\Omega)^{2 \times 2}} \right) \\ &\leq C \left(\|D\tilde{\varphi}\|_{L^r(\Omega)^{2 \times 2}} + \varepsilon h_\varepsilon^{-1} \|D\tilde{\varphi}\|_{L^r(\Omega)^{2 \times 2}} \right) \\ &\leq C \|\tilde{\varphi}\|_{W_0^{1,r}(\Omega)^2}, \end{aligned}$$

which implies estimates (3.46). \square

Denoting by p_ε^1 the restriction to Ω_ε of the overall pressure p_ε , with the additive constant being determined by $\int_{\Omega_\varepsilon} p_\varepsilon^1 dx = 0$, we give the existence of an extended pressure to Q_ε by duality arguments.

Lemma 3.7. *There exists an extension $P_\varepsilon^1 \in L_0^{r'}(Q_\varepsilon)$ of the pressure p_ε^1 . Moreover, defining the dilated and extended pressure $\tilde{P}_\varepsilon^1 \in L_0^{r'}(\Omega)$ obtained from P_ε^1 by using the change of variables (2.8), then there exists a positive constant C independent of ε , such that*

$$\|\tilde{P}_\varepsilon^1\|_{L^{r'}(\Omega)} \leq C \left(h_\varepsilon^{-\frac{r-1}{r}} \eta_\varepsilon^{\frac{2r-1}{r}} \varepsilon^{-1} + 1 \right), \quad \|\nabla_{h_\varepsilon} \tilde{P}_\varepsilon^1\|_{W^{-1,r'}(\Omega)^2} \leq C \left(h_\varepsilon^{-\frac{r-1}{r}} \eta_\varepsilon^{\frac{2r-1}{r}} \varepsilon^{-1} + 1 \right). \quad (3.47)$$

Proof. We divide the proof in two steps.

Step 1. Extension of p_ε^1 to Q_ε . Using the restriction operator $\mathcal{R}_r^\varepsilon$ given in Lemma 3.5, we introduce F_ε in $W^{-1,r'}(Q_\varepsilon)^2$ in the following way

$$\langle F_\varepsilon, \varphi \rangle_{W^{-1,r'}(Q_\varepsilon)^2, W_0^{1,r}(Q_\varepsilon)^2} = \langle \nabla p_\varepsilon, \mathcal{R}_r^\varepsilon \varphi \rangle_{W^{-1,r'}(\Omega_\varepsilon)^2, W_0^{1,r}(\Omega_\varepsilon)^2}, \quad \text{for any } \varphi \in W_0^{1,r}(Q_\varepsilon)^2, \quad (3.48)$$

and compute the right hand side of (3.48) by using in (2.18), which gives

$$\langle F_\varepsilon, \varphi \rangle_{W^{-1,r'}(Q_\varepsilon)^2, W_0^{1,r}(Q_\varepsilon)^2} = -\nu \int_{\Omega_\varepsilon} |\mathbb{D}[v_\varepsilon]|^{r-2} \mathbb{D}[v_\varepsilon] : \mathbb{D}[\mathcal{R}_r^\varepsilon \varphi] dx + \int_{\Omega_\varepsilon} f \cdot (\mathcal{R}_r^\varepsilon \varphi) dx. \quad (3.49)$$

Using Corollary 3.4 for fixed ε , we see that it is a bounded functional on $W_0^{1,r}(Q_\varepsilon)$ (see Step 2 of the proof), and in fact $F_\varepsilon \in W^{-1,r'}(Q_\varepsilon)^3$. Moreover, $\text{div}(\varphi) = 0$ implies $\langle F_\varepsilon, \varphi \rangle = 0$, and the DeRham theorem gives the existence of $P_\varepsilon^1 \in L_0^{r'}(Q_\varepsilon)$ with $F_\varepsilon = \nabla P_\varepsilon^1$.

Step 2. Estimates for dilated and extended pressure. Consider \tilde{P}_ε^1 obtained from P_ε^1 by using the change of variables (2.8). By using the Nečas inequality (see for instance [23]) for $\tilde{P}_\varepsilon^1 \in L_0^{r'}(\Omega)$, then

$$\|\tilde{P}_\varepsilon^1\|_{L^{r'}(\Omega)} \leq C \|\nabla_z \tilde{P}_\varepsilon^1\|_{W^{-1,r'}(\Omega)^2} \leq C \|\nabla_{h_\varepsilon} \tilde{P}_\varepsilon^1\|_{W^{-1,r'}(\Omega)^2},$$

and thus, to prove (3.47), it is enough to prove the second estimate in (3.47) for $\nabla_{h_\varepsilon} \tilde{P}_\varepsilon^1$.

Let us prove it. For any $\tilde{\varphi} \in W_0^{1,r}(\Omega)^2$, using the change of variables (2.8), we have

$$\begin{aligned} \left\langle \nabla_{h_\varepsilon} \tilde{P}_\varepsilon^1, \tilde{\varphi} \right\rangle_{W^{-1,r'}(\Omega)^2, W_0^{1,r}(\Omega)^2} &= - \int_{\Omega} \tilde{P}_\varepsilon^1 \text{div}_{h_\varepsilon}(\tilde{\varphi}) dz \\ &= -h_\varepsilon^{-1} \int_{Q_\varepsilon} P_\varepsilon^1 \text{div}(\varphi) dx = h_\varepsilon^{-1} \langle \nabla P_\varepsilon^1, \varphi \rangle_{W^{-1,r'}(Q_\varepsilon)^2, W_0^{1,r}(Q_\varepsilon)^2}. \end{aligned}$$

Then, using the identification (3.49) of F_ε , we get

$$\left\langle \nabla_{h_\varepsilon} \tilde{P}_\varepsilon^1, \tilde{\varphi} \right\rangle_{W^{-1,r'}(\Omega)^2, W_0^{1,r}(\Omega)^2} = h_\varepsilon^{-1} \left(-\nu \int_{\Omega_\varepsilon} |\mathbb{D}[v_\varepsilon]|^{r-2} \mathbb{D}[v_\varepsilon] : \mathbb{D}[\mathcal{R}_r^\varepsilon \varphi] dx + \int_{\Omega_\varepsilon} f \cdot (\mathcal{R}_r^\varepsilon \varphi) dx \right),$$

and applying the change of variables (2.8), we get

$$\left\langle \nabla_{h_\varepsilon} \tilde{P}_\varepsilon^1, \tilde{\varphi} \right\rangle_{W^{-1,r'}(\Omega)^2, W_0^{1,r}(\Omega)^2} = -\nu \int_{\tilde{\Omega}_\varepsilon} |\mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon]|^{r-2} \mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon] : \mathbb{D}_{h_\varepsilon}[\tilde{\mathcal{R}}_r^\varepsilon \tilde{\varphi}] dz + \int_{\tilde{\Omega}_\varepsilon} f \cdot (\tilde{\mathcal{R}}_r^\varepsilon \tilde{\varphi}) dz. \quad (3.50)$$

Let us now estimate the right-hand side of (3.50). From the Hölder inequality and using estimates for \tilde{v}_ε in (3.38)–(3.40), assumption of f given in (2.17) and estimates of the dilated restricted operator (3.46), we obtain

$$\begin{aligned} \left| \int_{\tilde{\Omega}_\varepsilon} |\mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon]|^{r-2} \mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon] : \mathbb{D}_{h_\varepsilon}[\tilde{\mathcal{R}}_r^\varepsilon \tilde{\varphi}] dz \right| &\leq C \|\mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon]\|_{L^r(\tilde{\Omega}_\varepsilon)^{2 \times 2}}^{r-1} \|D_{h_\varepsilon} \tilde{\mathcal{R}}_r^\varepsilon \tilde{\varphi}\|_{L^r(\tilde{\Omega}_\varepsilon)^{2 \times 2}} \\ &\leq C \left(h_\varepsilon^{-\frac{r-1}{r}} \eta_\varepsilon^{\frac{2r-1}{r}} + \varepsilon \right) \|D_{h_\varepsilon} \tilde{\mathcal{R}}_r^\varepsilon \tilde{\varphi}\|_{L^r(\tilde{\Omega}_\varepsilon)^{2 \times 2}} \\ &\leq C \left(h_\varepsilon^{-\frac{r-1}{r}} \eta_\varepsilon^{\frac{2r-1}{r}} \varepsilon^{-1} + 1 \right) \|\tilde{\varphi}\|_{W_0^{1,r}(\Omega)^2}, \end{aligned}$$

$$\left| \int_{\tilde{\Omega}_\varepsilon} f \cdot (\tilde{\mathcal{R}}_r^\varepsilon \tilde{\varphi}) dz \right| \leq C \|\tilde{\mathcal{R}}_r^\varepsilon \tilde{\varphi}\|_{L^r(\tilde{\Omega}_\varepsilon)^2} \leq C \|\tilde{\varphi}\|_{W_0^{1,r}(\Omega)^2},$$

which together with (3.50) gives

$$\left| \left\langle \nabla_{h_\varepsilon} \tilde{P}_\varepsilon, \tilde{\varphi} \right\rangle_{W^{-1,r'}(\Omega)^2, W_0^{1,r}(\Omega)^2} \right| \leq C \left(h_\varepsilon^{-\frac{r-1}{r}} \eta_\varepsilon^{\frac{2r-1}{r}} \varepsilon^{-1} + 1 \right) \|\tilde{\varphi}\|_{W_0^{1,r}(\Omega)^2}.$$

This implies the second estimate given in (3.47), which concludes the proof. \square

3.4 Estimates for pressure in the free part

Now, we obtain estimates of the pressure in the thin film. We denote by \tilde{P}_ε^2 the restriction to \tilde{I}_1 of the overall pressure \tilde{p}_ε , i.e.

$$\tilde{P}_\varepsilon^2(z) = \tilde{p}_\varepsilon(z) - \tilde{c}_\varepsilon \quad \text{if } z \in \tilde{I}_1, \quad (3.51)$$

with the additive constant \tilde{c}_ε determined by

$$\tilde{c}_\varepsilon = \frac{1}{|\tilde{I}_1|} \int_{\tilde{I}_1} \tilde{p}_\varepsilon dz. \quad (3.52)$$

Lemma 3.8. *There exists a positive constant C independent of ε , such that*

$$\|\tilde{P}_\varepsilon^2\|_{L^{r'}(\tilde{I}_1)} \leq C \left(1 + h_\varepsilon^{-\frac{r-1}{r}} \varepsilon \eta_\varepsilon^{-\frac{2r-1}{r}} \right), \quad \|\nabla_{\eta_\varepsilon} \tilde{P}_\varepsilon^2\|_{W^{-1,r'}(\tilde{I}_1)^2} \leq C \left(1 + h_\varepsilon^{-\frac{r-1}{r}} \varepsilon \eta_\varepsilon^{-\frac{2r-1}{r}} \right). \quad (3.53)$$

Proof. Let us first remark that, by using the Nečas inequality (see for instance [23]) for $\tilde{P}_\varepsilon^2 \in L^{r'}(\tilde{I}_1)$, then

$$\|\tilde{P}_\varepsilon^2\|_{L^{r'}(\tilde{I}_1)} \leq C \|\nabla_z \tilde{P}_\varepsilon^2\|_{W^{-1,r'}(\tilde{I}_1)^2} \leq C \|\nabla_{\eta_\varepsilon} \tilde{P}_\varepsilon^2\|_{W^{-1,r'}(\tilde{I}_1)^2},$$

and thus, to prove (3.53), it is enough to prove the second estimate in (3.53) for $\nabla_{\eta_\varepsilon} \tilde{P}_\varepsilon^2$.

Let us prove it. For any $\tilde{\varphi} \in W_0^{1,r}(\tilde{I}_1)^2$, using the change of variables (2.12), we have

$$\left\langle \nabla_{\eta_\varepsilon} \tilde{P}_\varepsilon^2, \tilde{\varphi} \right\rangle_{W^{-1,r'}(\Omega)^2, W_0^{1,r}(\Omega)^2} = -\nu \int_{\tilde{I}_1} |\mathbb{D}_{\eta_\varepsilon}[\tilde{\mathcal{U}}_\varepsilon]|^{r-2} \mathbb{D}_{\eta_\varepsilon}[\tilde{\mathcal{U}}_\varepsilon] : \mathbb{D}_{\eta_\varepsilon}[\tilde{\varphi}] dz + \int_{\tilde{I}_1} f \cdot \tilde{\varphi} dz. \quad (3.54)$$

Let us now estimate the right-hand side of this equality:

– From the Hölder inequality and using estimates for $\tilde{\mathcal{U}}_\varepsilon$ in (3.41)–(3.43), we get

$$\begin{aligned} \left| -\nu \int_{\tilde{I}_1} |\mathbb{D}_{\eta_\varepsilon}[\tilde{\mathcal{U}}_\varepsilon]|^{r-2} \mathbb{D}_{\eta_\varepsilon}[\tilde{\mathcal{U}}_\varepsilon] dz \right| &\leq C \|\mathbb{D}_{\eta_\varepsilon}[\tilde{\mathcal{U}}_\varepsilon]\|_{L^r(\tilde{I}_1)^{2 \times 2}}^{r-1} \|D_{\eta_\varepsilon} \tilde{\varphi}\|_{L^r(\tilde{I}_1)^{2 \times 2}} \\ &\leq C \left(\eta_\varepsilon + h_\varepsilon^{\frac{r-1}{r}} \varepsilon \eta_\varepsilon^{-\frac{r-1}{r}} \right) \|D_{\eta_\varepsilon} \tilde{\varphi}\|_{L^r(\tilde{I}_1)^{2 \times 2}} \\ &\leq C \left(1 + h_\varepsilon^{\frac{r-1}{r}} \varepsilon \eta_\varepsilon^{-\frac{2r-1}{r}} \right) \|\tilde{\varphi}\|_{W_0^{1,r}(\tilde{I}_1)^2}, \end{aligned}$$

where, in the last inequality, we have used

$$\|D_{\eta_\varepsilon} \tilde{\varphi}\|_{L^r(\tilde{I}_1)^{2 \times 2}} \leq C \eta_\varepsilon^{-1} \|\tilde{\varphi}\|_{W_0^{1,r}(\tilde{I}_1)^2}. \quad (3.55)$$

– Applying the change of variables (2.12) to inequality (3.32), we get

$$\|\tilde{\varphi}\|_{L^r(\tilde{I}_1)^2} \leq C \eta_\varepsilon^{\frac{1}{2}} (\varepsilon + \eta_\varepsilon)^{\frac{1}{2}} \|D_{\eta_\varepsilon} \tilde{\varphi}\|_{L^r(\tilde{I}_1)^{2 \times 2}},$$

and from the Hölder inequality and using this inequality, assumption of f given in (2.17) and (3.55), we have

$$\begin{aligned} \left| \int_{\tilde{I}_1} f \cdot \tilde{\varphi} dz \right| &\leq C \|\tilde{\varphi}\|_{L^r(\tilde{I}_1)^2} \leq C \eta_\varepsilon^{\frac{1}{2}} (\eta_\varepsilon + \varepsilon)^{\frac{1}{2}} \|D_{\eta_\varepsilon} \tilde{\varphi}\|_{L^r(\tilde{I}_1)^{2 \times 2}} \\ &\leq C \eta_\varepsilon^{-\frac{1}{2}} (\eta_\varepsilon + \varepsilon)^{\frac{1}{2}} \|\tilde{\varphi}\|_{W_0^{1,r}(\tilde{I}_1)^2} \leq C \left(1 + \eta_\varepsilon^{-\frac{1}{2}} \varepsilon^{\frac{1}{2}} \right) \|\tilde{\varphi}\|_{W_0^{1,r}(\tilde{I}_1)^2}. \end{aligned}$$

Previous estimates together with (3.54) gives

$$\left| \left\langle \nabla_{\eta_\varepsilon} \tilde{P}_\varepsilon^2, \tilde{\varphi} \right\rangle_{W^{-1,r'}(\tilde{I}_1)^2, W_0^{1,r}(\tilde{I}_1)^2} \right| \leq C \left(1 + h_\varepsilon^{\frac{r-1}{r}} \varepsilon \eta_\varepsilon^{-\frac{2r-1}{r}} + \eta_\varepsilon^{-\frac{1}{2}} \varepsilon^{\frac{1}{2}} \right) \|\tilde{\varphi}\|_{W_0^{1,r}(\tilde{I}_1)^2}.$$

Since $\eta_\varepsilon \gg \varepsilon$, then the term $\eta_\varepsilon^{-\frac{1}{2}} \varepsilon^{\frac{1}{2}} \ll 1$ and it can be dropped. This implies the second estimate given in (3.53), which concludes the proof. \square

Remark 3.9. *In view of estimates of the velocity and the pressure given in the previous section, there is a critical case when*

$$h_\varepsilon \approx \eta_\varepsilon^{\frac{2r-1}{r-1}} \varepsilon^{-\frac{r}{r-1}} \quad \text{with} \quad \lim_{\varepsilon \rightarrow 0} \frac{h_\varepsilon}{\eta_\varepsilon^{\frac{2r-1}{r-1}} \varepsilon^{-\frac{r}{r-1}}} = \lambda, \quad 0 < \lambda < +\infty, \quad (3.56)$$

where the pressure has the same order of magnitude in the porous medium and in the free film. From now on, we focus our study in this case, which is the most interesting one.

4 Critical case: problem in the thin porous medium

In this section, we study the asymptotic behavior of the fluid in the thin porous part assuming the critical regime (3.56). Under this assumption, the estimates given in Corollary 3.4 and Lemma 3.7 read as follow

$$\|\tilde{v}_\varepsilon\|_{L^r(\tilde{\Omega}_\varepsilon)^2} \leq C \varepsilon^{\frac{r}{r-1}}, \quad \|\mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon]\|_{L^r(\tilde{\Omega}_\varepsilon)^{2 \times 2}} \leq C \varepsilon^{\frac{1}{r-1}}, \quad \|D_{h_\varepsilon} \tilde{v}_\varepsilon\|_{L^r(\tilde{\Omega}_\varepsilon)^{2 \times 2}} \leq C \varepsilon^{\frac{1}{r-1}}, \quad (4.57)$$

$$\|\tilde{P}_\varepsilon^1\|_{L^{r'}(\Omega)} \leq C, \quad \|\nabla_{h_\varepsilon} \tilde{P}_\varepsilon^1\|_{W^{-1,r'}(\Omega)^2} \leq C.$$

To describe the behavior of the solution in the microstructure associated to $\tilde{\Omega}_\varepsilon$, we introduce an adaptation of the unfolding method (for classical versions see [24, 25]), which is related with the change of variables applied in [15] (see also [40]) to study the porous part in the Newtonian case of modeling of a thin film passing a thin porous media.

4.1 Adaptation of the unfolding method

This version of the unfolding method consists of dividing the domain $\tilde{\Omega}_\varepsilon$ into squares of horizontal length ε and vertical length $\varepsilon/h_\varepsilon$. In order to apply the version of the unfolding method, we need the following notation: for $k \in \mathbb{Z}^2$, we define $\kappa : \mathbb{R}^2 \rightarrow \mathbb{Z}^2$ by

$$\kappa(x) = k \iff x \in Y_{k,1}. \quad (4.58)$$

Remark that κ is well defined up to a set of zero measure in \mathbb{R}^2 , which is given by $\cup_{k \in \mathbb{Z}^2} \partial Y_{k,1}$. Moreover, for every $\varepsilon, h_\varepsilon > 0$, we have

$$\kappa\left(\frac{x}{\varepsilon}\right) = k \iff x \in Y_{k,\varepsilon} \quad \text{which is equivalent to} \quad \kappa\left(\frac{z_1}{\varepsilon}, \frac{h_\varepsilon z_2}{\varepsilon}\right) = k \iff z \in \tilde{Y}_{k,\varepsilon}.$$

Definition 4.1. Let $\tilde{\varphi}$ be in $L^s(\tilde{\Omega}_\varepsilon)^2$, $1 \leq s < +\infty$, and $\tilde{\psi}$ be in $L^{s'}(\Omega)$, $1/s + 1/s' = 1$. We define the functions $\hat{\varphi}_\varepsilon \in L^s(\mathbb{R}^2 \times Y_f)^2$ and $\hat{\psi}_\varepsilon \in L^{s'}(\mathbb{R}^2 \times Y)$ by

$$\hat{\varphi}_\varepsilon(z, y) = \tilde{\varphi}\left(\varepsilon \kappa\left(\frac{z_1}{\varepsilon}, \frac{h_\varepsilon z_2}{\varepsilon}\right) \cdot e_1 + \varepsilon y_1, \frac{\varepsilon}{h_\varepsilon} \kappa\left(\frac{z_1}{\varepsilon}, \frac{h_\varepsilon z_2}{\varepsilon}\right) \cdot e_2 + \frac{\varepsilon}{h_\varepsilon} y_2\right), \quad \text{a.e. } (z, y) \in \mathbb{R}^2 \times Y_f, \quad (4.59)$$

$$\hat{\psi}_\varepsilon(z, y) = \tilde{\psi}\left(\varepsilon \kappa\left(\frac{z_1}{\varepsilon}, \frac{h_\varepsilon z_2}{\varepsilon}\right) \cdot e_1 + \varepsilon y_1, \frac{\varepsilon}{h_\varepsilon} \kappa\left(\frac{z_1}{\varepsilon}, \frac{h_\varepsilon z_2}{\varepsilon}\right) \cdot e_2 + \frac{\varepsilon}{h_\varepsilon} y_2\right), \quad \text{a.e. } (z, y) \in \mathbb{R}^2 \times Y, \quad (4.60)$$

assuming $\tilde{\varphi}$ (resp. $\tilde{\psi}$) is extended by zero outside $\tilde{\Omega}_\varepsilon$ (resp. Ω), where the function κ is defined by (4.58).

Remark 4.2. The restrictions of $\hat{\varphi}_\varepsilon$ to $\tilde{Y}_{k,\varepsilon} \times Y_f$ (resp. $\hat{\psi}_\varepsilon$ to $\tilde{Y}_{k,\varepsilon} \times Y$) does not depend on z , while as a function of y it is obtained from $\tilde{\varphi}$ (resp. from $\tilde{\psi}$) by using the change of variables

$$y_1 = \frac{z_1 - \varepsilon k_1}{\varepsilon}, \quad y_2 = \frac{h_\varepsilon z_2 - \varepsilon k_2}{\varepsilon}, \quad (4.61)$$

which transforms $\tilde{Y}_{f,k,\varepsilon}$ into Y_f (resp. $\tilde{Y}_{k,\varepsilon}$ into Y).

Next, we give some properties of the unfolded functions.

Proposition 4.3. Let $\tilde{\varphi}$ be in $W^{1,s}(\tilde{\Omega}_\varepsilon)^2$, $1 \leq s < +\infty$, and $\tilde{\psi}$ be in $L^{s'}(\Omega)$, $1/s + 1/s' = 1$. Then, we have

$$\|\hat{\varphi}_\varepsilon\|_{L^s(\mathbb{R}^2 \times Y_f)^2} = \|\tilde{\varphi}\|_{L^s(\tilde{\Omega}_\varepsilon)^2}, \quad \|\partial_{y_1} \hat{\varphi}_\varepsilon\|_{L^s(\mathbb{R}^2 \times Y_f)^2} = \varepsilon \|\partial_{z_1} \tilde{\varphi}\|_{L^s(\tilde{\Omega}_\varepsilon)^2}, \quad \|\partial_{y_2} \hat{\varphi}_\varepsilon\|_{L^s(\mathbb{R}^2 \times Y_f)^2} = \frac{\varepsilon}{h_\varepsilon} \|\partial_{z_2} \tilde{\varphi}\|_{L^s(\tilde{\Omega}_\varepsilon)^2},$$

$$\|\hat{\psi}_\varepsilon\|_{L^{s'}(\mathbb{R}^2 \times Y)} = \|\tilde{\psi}\|_{L^{s'}(\Omega)}.$$

Proof. We will only make the proof for $\widehat{\varphi}_\varepsilon$. The procedure for $\widehat{\psi}_\varepsilon$ is similar, so we omit it. Taking into account the definition (4.59) of $\widehat{\varphi}_\varepsilon$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2 \times Y_f} |\partial_{y_1} \widehat{\varphi}_\varepsilon(z, y)|^s dz dy &= \sum_{k \in \mathbb{Z}^2} \int_{\widetilde{Y}_{k, \varepsilon}} \int_{Y_f} |\partial_{y_1} \widehat{\varphi}_\varepsilon(z, y)|^s dy dz \\ &= \sum_{k \in \mathbb{Z}^2} \int_{\widetilde{Y}_{k, \varepsilon}} \int_{Y_f} |\partial_{y_1} \widetilde{\varphi}(\varepsilon k' + \varepsilon y_1, \varepsilon h_\varepsilon^{-1} k_2 + \varepsilon h_\varepsilon^{-1} y_2)|^s dy dz. \end{aligned}$$

We observe that $\widetilde{\varphi}$ does not depend on z , then we can deduce

$$\int_{\mathbb{R}^2 \times Y_f} |\partial_{y_1} \widehat{\varphi}_\varepsilon(z, y)|^s dz dy = \frac{\varepsilon^2}{h_\varepsilon} \sum_{k \in \mathbb{Z}^2} \int_{Y_f} |\partial_{y_1} \widetilde{\varphi}(\varepsilon k_1 + \varepsilon y_1, \varepsilon h_\varepsilon^{-1} k_2 + \varepsilon h_\varepsilon^{-1} y_2)|^s dy.$$

By the change of variables (4.61), we obtain

$$\int_{\mathbb{R}^2 \times Y_f} |\partial_{y_1} \widehat{\varphi}_\varepsilon(z, y)|^s dz dy = \varepsilon^s \sum_{k \in \mathbb{Z}^2} \int_{\widetilde{Y}_{f, \varepsilon}} |\partial_{y_1} \widetilde{\varphi}(z)|^s dz = \varepsilon^s \int_{\widetilde{\Omega}_\varepsilon} |\partial_{x_1} \widetilde{\varphi}(z)|^s dz.$$

Thus, we get the property for $\partial_{y_1} \widehat{\varphi}_\varepsilon$.

Similarly, we have

$$\int_{\mathbb{R}^2 \times Y_f} |\partial_{y_2} \widehat{\varphi}_\varepsilon(z, y)|^s dz dy = \frac{\varepsilon^2}{h_\varepsilon} \sum_{k \in \mathbb{Z}^2} \int_{Y_f} |\partial_{y_2} \widetilde{\varphi}(\varepsilon k_1 + \varepsilon y_1, \varepsilon h_\varepsilon^{-1} k_2 + \varepsilon h_\varepsilon^{-1} y_2)|^s dy.$$

By the change of variables (4.61) we obtain

$$\int_{\mathbb{R}^2 \times Y_f} |\partial_{y_2} \widehat{\varphi}_\varepsilon(z, y)|^s dz dy = \frac{\varepsilon^s}{h_\varepsilon^s} \sum_{k \in \mathbb{Z}^2} \int_{\widetilde{Y}_{f, \varepsilon}} |\partial_{z_2} \widetilde{\varphi}(z)|^s dz = \frac{\varepsilon^s}{h_\varepsilon^s} \int_{\widetilde{\Omega}_\varepsilon} |\partial_{z_2} \widetilde{\varphi}(z)|^s dz,$$

so the the property for $\partial_{y_2} \widehat{\varphi}_\varepsilon$ is proved. Finally, reasoning analogously we deduce

$$\int_{\mathbb{R}^2 \times Y_f} |\widehat{\varphi}_\varepsilon(z, y)|^s dz dy = \int_{\widetilde{\Omega}_\varepsilon} |\widetilde{\varphi}(z)|^s dz,$$

and the property for $\widehat{\varphi}_\varepsilon$ holds. \square

Lemma 4.4. *We assume that the parameters $\varepsilon, \eta_\varepsilon$ and h_ε satisfy (2.5) and (3.56). We define the unfolded velocity \widehat{v}_ε from the dilated velocity $\widetilde{v}_\varepsilon$ by means of (4.59) and the unfolded pressure $\widehat{P}_\varepsilon^1$ from the dilated and extended pressure $\widetilde{P}_\varepsilon^1$ by means of (4.60). Then, there exists a constant $C > 0$ independent of ε , such that \widehat{v}_ε and $\widehat{P}_\varepsilon^1$ satisfy*

$$\|\widehat{v}_\varepsilon\|_{L^r(\mathbb{R}^2 \times Y_f)^2} \leq C \varepsilon^{\frac{r}{r-1}}, \quad \|D_y \widehat{v}_\varepsilon\|_{L^r(\mathbb{R}^2 \times Y_f)^{2 \times 2}} \leq C \varepsilon^{\frac{r}{r-1}}, \quad (4.62)$$

$$\|\widehat{P}_\varepsilon^1\|_{L^{r'}(\mathbb{R}^2 \times Y)} \leq C. \quad (4.63)$$

Proof. Estimates (4.62) and (4.63) easily follow from Proposition 4.3, with $s = r$ and $s' = r'$, and estimates of velocity in $\widetilde{\Omega}_\varepsilon$ and estimate of the pressure in Ω given in (4.57). \square

4.2 Convergences of velocity and pressure

From now on, we denote by \tilde{V}_ε the extension by zero of \tilde{v}_ε to the whole domain Ω (the velocity is zero in the obstacles). Then, estimates given in (4.57) remain valid for the extension \tilde{V}_ε , which is divergence free too. Here, we obtain some compactness results concerning the behavior of the sequence $(\tilde{V}_\varepsilon, \tilde{P}_\varepsilon^1)$ and $(\hat{v}_\varepsilon, \hat{P}_\varepsilon^1)$.

Lemma 4.5. *For a subsequence of ε still denoted by ε , there exist:*

– $v \in W^{1,r}(0, 1; L^r(\omega)^2)$, with $v_2 \equiv 0$ and $v_1 = 0$ on $\{z_2 = 1\}$, such that

$$\varepsilon^{-\frac{r}{r-1}} \tilde{V}_\varepsilon \rightharpoonup v \quad \text{in } W^{1,r}(0, 1; L^r(\omega)^2), \quad (4.64)$$

$$\partial_{z_1} \left(\int_0^1 v_1(z) dz_2 \right) = 0 \quad \text{in } \omega, \quad \left(\int_0^1 v_1(z) dz_2 \right) n = 0 \quad \text{on } \partial\omega. \quad (4.65)$$

– $\hat{v} \in L^r(\mathbb{R}^2; W_{\text{per}}^{1,r}(Y_f)^2)$, with \hat{v}_2 independent of z_2 and $\hat{v} = 0$ in $\Omega \times T$ and in $(\mathbb{R}^2 \setminus \Omega) \times Y_f$, such that

$$\varepsilon^{-\frac{r}{r-1}} \hat{v}_\varepsilon \rightharpoonup \hat{v} \quad \text{in } L^r(\mathbb{R}^2; W^{1,r}(Y_f)^2), \quad \varepsilon^{-\frac{r}{r-1}} D_y \hat{v}_\varepsilon \rightharpoonup D_y \hat{v} \quad \text{in } L^r(\mathbb{R}^2 \times Y_f)^{2 \times 2}. \quad (4.66)$$

$$\text{div}_y \hat{v}(z, y) = 0 \quad \text{in } \mathbb{R}^2 \times Y_f. \quad (4.67)$$

Moreover, the following relation between v and \hat{v} holds

$$v(z) = \int_{Y_f} \hat{v}(z, y) dy \quad \text{a.e. in } \Omega, \quad \text{i.e.} \quad \int_{Y_f} \hat{v}(z, y) dy = v_1(z) \quad \text{and} \quad \int_{Y_f} \hat{v}_2(z, y) dy = 0. \quad (4.68)$$

Proof. We divide the proof in two parts:

– We start with the extended velocity \tilde{V}_ε . From the first and second estimate in (4.57), we get the existence of $v \in W^{1,r}(0, 1; L^r(\omega)^2)$ such that, up to a subsequence, it holds

$$\varepsilon^{-\frac{r}{r-1}} \tilde{V}_\varepsilon \rightharpoonup v \quad \text{in } W^{1,r}(0, 1; L^r(\omega)^2). \quad (4.69)$$

This implies

$$\varepsilon^{-\frac{r}{r-1}} \partial_{z_1} \tilde{V}_{\varepsilon,1} \rightharpoonup \partial_{z_1} v_1 \quad \text{in } W^{1,r}(0, 1; W^{-1,r'}(\omega)^2). \quad (4.70)$$

Since $\text{div}_{h_\varepsilon}(\tilde{V}_\varepsilon) = 0$ in Ω , multiplying by $h_\varepsilon \varepsilon^{-\frac{r}{r-1}}$ we obtain

$$h_\varepsilon \varepsilon^{-\frac{r}{r-1}} \partial_{z_1} \tilde{V}_{\varepsilon,1} + \varepsilon^{-\frac{r}{r-1}} \partial_{z_2} \tilde{V}_{\varepsilon,2} = 0 \quad \text{in } \Omega, \quad (4.71)$$

which, combined with (4.70), implies that $\varepsilon^{-\frac{r}{r-1}} \partial_{z_2} \tilde{V}_{\varepsilon,2}$ is bounded in $W^{1,r}(0, 1; W^{-1,r'}(\omega)^2)$ and tends to zero. Also from (4.69), we have that $\varepsilon^{-\frac{r}{r-1}} \partial_{z_2} \tilde{V}_{\varepsilon,2}$ tends to $\partial_{z_2} v_2$ in $L^r(\Omega)$. From the uniqueness of the limit, we have that $\partial_{z_2} v_2 = 0$, which implies that v_2 is independent of z_2 . Moreover, the continuity of the trace applications from the space of functions $\tilde{\varphi}$ such that $\|\tilde{\varphi}\|_{L^r}$ and $\|\partial_{z_2} \tilde{\varphi}\|_{L^r}$ to $L^r(\omega \times \{1\})$ implies $v = 0$ on $z_2 = \{1\}$. From this boundary condition and since v_2 does not depend on z_2 , we deduce $v_2 \equiv 0$. This completes the proof of (4.64).

Next, by considering $\tilde{\varphi} \in \mathcal{D}(\omega)$ as test function in the divergence condition $\operatorname{div}_{h_\varepsilon} \tilde{V}_\varepsilon = 0$ in Ω , we get

$$\int_{\Omega} \left(\partial_{z_1} \tilde{V}_{\varepsilon,1} \tilde{\varphi} + h_\varepsilon^{-1} \partial_{z_2} \tilde{V}_{\varepsilon,2} \tilde{\varphi} \right) dz = 0,$$

which, after integration by parts and multiplication by $\varepsilon^{-\frac{r}{r-1}}$, gives

$$\int_{\Omega} \varepsilon^{-\frac{r}{r-1}} \tilde{V}_{\varepsilon,1} \partial_{z_1} \tilde{\varphi} dz = 0.$$

Passing to the limit by using convergence (4.64), we deduce

$$\int_{\Omega} v_1 \partial_{z_1} \tilde{\varphi} dz = 0,$$

and, since $\tilde{\varphi}$ does not depend on z_2 , we obtain the following divergence condition (4.65).

- Now we focus on the velocity \hat{v}_ε . From estimates of \hat{v}_ε given in (4.62) we have the existence of $\hat{v} \in L^r(\mathbb{R}^2; W_{\text{per}}^{1,r}(Y_f)^2)$ satisfying, up to a subsequence, convergences (4.66). Taking into account that \hat{v}_ε vanishes on $\mathbb{R}^2 \times T$, we deduce that \hat{v} also vanishes on $\mathbb{R}^2 \times T$. Moreover, by construction \hat{v}_ε is zero outside $\tilde{\Omega}_\varepsilon$ and so, \hat{v} vanishes on $(\mathbb{R}^2 \setminus \Omega) \times Y_f$.

Since $\operatorname{div}_{h_\varepsilon}(\hat{v}_\varepsilon) = 0$ in $\tilde{\Omega}_\varepsilon$, by applying the change of variables (4.61) we get

$$\varepsilon^{-1} \operatorname{div}_y(\hat{v}_\varepsilon) = 0 \quad \text{in } \mathbb{R}^2 \times Y_f.$$

Multiplying by $\varepsilon^{-\frac{1}{r-1}}$ and passing to the limit by using convergence (4.66), we deduce $\operatorname{div}_y(\hat{v}) = 0$ in $\mathbb{R}^2 \times Y_f$, i.e. property (4.67).

It remains to prove that \hat{v} is periodic in y . This follows by passing to the limit in the equality

$$\varepsilon^{-\frac{r}{r-1}} \hat{v}_\varepsilon \left(z + \varepsilon \mathbf{e}_1, -\frac{1}{2}, y_2 \right) = \varepsilon^{-\frac{r}{r-1}} \hat{v}_\varepsilon \left(z, \frac{1}{2}, y_2 \right),$$

which is a consequence of definition (4.59). Passing to the limit, this shows

$$\hat{v} \left(z, -\frac{1}{2}, y_2 \right) = \hat{v} \left(z, \frac{1}{2}, y_2 \right),$$

and then is proved the periodicity of \hat{v} with respect to y_1 . To prove the periodicity with respect to y_2 , we consider

$$\varepsilon^{-\frac{r}{r-1}} \hat{v}_\varepsilon \left(z + \frac{\varepsilon}{h_\varepsilon} \mathbf{e}_2, y_1, -\frac{1}{2} \right) = \varepsilon^{-\frac{r}{r-1}} \hat{v}_\varepsilon \left(z, y_1, \frac{1}{2} \right),$$

and passing to the limit we have

$$\hat{v} \left(z, y_1, -\frac{1}{2} \right) = \hat{v} \left(z, y_1, \frac{1}{2} \right),$$

which shows the periodicity with respect to y_2 .

Finally, relation (4.68) follows from Proposition 4.3 with $s = 1$, which gives

$$\int_{\Omega} v(z) dz = \int_{\Omega \times Y_f} \hat{v}(z, y) dz dy = \int_{\Omega} \left(\int_{Y_f} \hat{v}(z, y) dy \right) dz.$$

From relation (4.68) and since $v_2 \equiv 0$, it holds that $\int_{Y_f} \hat{v}_2 dy = 0$.

□

Lemma 4.6. *For a subsequence of ε still denoted by ε , there exists $p^1 \in L_0^{r'}(\omega)$ independent of z_2 , such that*

$$\tilde{P}_\varepsilon^1 \rightarrow p^1 \quad \text{in } L^{r'}(\Omega), \quad (4.72)$$

$$\hat{P}_\varepsilon^1 \rightarrow p^1 \quad \text{in } L^{r'}(\mathbb{R}^2 \times Y). \quad (4.73)$$

Proof. Taking into account the first estimate of the pressure in (4.57), we deduce that there exist $p^1 \in L^{r'}(\Omega)$ such that, up to a subsequence,

$$\tilde{P}_\varepsilon^1 \rightharpoonup p^1 \quad \text{in } L^{r'}(\Omega). \quad (4.74)$$

From convergence (4.74) we deduce that $\partial_{z_2} \tilde{P}_\varepsilon^1$ also converges to $\partial_{z_2} p^1$ in $W^{-1,r'}(\Omega)$. Also, from the second estimate of the pressure in (4.57), we can deduce that $\partial_{z_2} \tilde{P}_\varepsilon^1$ converges to zero in $W^{-1,r'}(\Omega)$. By the uniqueness of the limit, then we obtain $\partial_{z_2} p^1 = 0$ and so p^1 is independent of z_2 . Since \tilde{P}_ε^1 has null mean value in Ω , then p^1 has null mean value in ω .

Next, following [21] adapted to the case of a thin layer, we prove that the convergence of the pressure is in fact strong. Let $\tilde{w}_\varepsilon, \tilde{w}$ be in $W_0^{1,r}(\Omega)^2$ such that

$$\tilde{w}_\varepsilon \rightharpoonup \tilde{w} \quad \text{in } W_0^{1,r}(\Omega)^2. \quad (4.75)$$

Then, as p^1 only depends on z_1 , we have

$$\begin{aligned} & \left| \langle \nabla_z \tilde{P}_\varepsilon^1, \tilde{w}_\varepsilon \rangle_{W^{-1,r'}(\Omega)^2, W_0^{1,r}(\Omega)^2} - \langle \nabla_z p^1, \tilde{w} \rangle_{W^{-1,r'}(\Omega)^2, W_0^{1,r}(\Omega)^2} \right| \\ & \leq \left| \langle \nabla_z \tilde{P}_\varepsilon^1, \tilde{w}_\varepsilon - \tilde{w} \rangle_{W^{-1,r'}(\Omega)^2, W_0^{1,r}(\Omega)^2} \right| + \left| \langle \nabla_z (\tilde{P}_\varepsilon^1 - p^1), \tilde{w} \rangle_{W^{-1,r'}(\Omega)^2, W_0^{1,r}(\Omega)^2} \right|. \end{aligned}$$

On the one hand, using convergence (4.74), we have

$$\left| \langle \nabla_z (\tilde{P}_\varepsilon^1 - p^1), \tilde{w} \rangle_{W^{-1,r'}(\Omega)^2, W_0^{1,r}(\Omega)^2} \right| = \left| \int_\Omega (\tilde{P}_\varepsilon^1 - p^1) \operatorname{div}_z \tilde{w} \, dz \right| \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

On the other hand, from (3.50) and proceeding similarly to the proof of Lemma 3.7, we have

$$\begin{aligned} \left| \langle \nabla_z \tilde{P}_\varepsilon^1, \tilde{w}_\varepsilon - \tilde{w} \rangle_{W^{-1,r'}(\Omega)^2, W_0^{1,r}(\Omega)^2} \right| & \leq \left| \langle \nabla_{h_\varepsilon} \tilde{P}_\varepsilon^1, \tilde{w}_\varepsilon - \tilde{w} \rangle_{W^{-1,r'}(\Omega)^2, W_0^{1,r}(\Omega)^2} \right| \\ & \leq C (\|\tilde{w}_\varepsilon - \tilde{w}\|_{L^r(\Omega)^2} + \varepsilon \|D_{h_\varepsilon}(\tilde{w}_\varepsilon - \tilde{w})\|_{L^r(\Omega)^{2 \times 2}}) \\ & \leq C (\|\tilde{w}_\varepsilon - \tilde{w}\|_{L^r(\Omega)^2} + \varepsilon h_\varepsilon^{-1} \|D_z(\tilde{w}_\varepsilon - \tilde{w})\|_{L^r(\Omega)^{2 \times 2}}). \end{aligned}$$

The right-hand side of the previous inequality tends to zero as $\varepsilon \rightarrow 0$, by virtue of relation (2.5), (4.75) and the Rellich theorem. This implies that $\nabla_z \tilde{P}_\varepsilon^1 \rightarrow \nabla_z p^1 = (\partial_{z_1} p^1, 0)^t$ strongly in $W^{-1,r'}(\Omega)^3$, which together the classical Nečas inequality implies the strong convergence of the pressure \tilde{P}_ε^1 given in (4.72). Finally, the strong convergence of \hat{P}_ε^1 given in (4.73) follows from [25, Proposition 1.9-(ii)] and the strong convergence of \tilde{P}_ε^1 given in (4.72).

□

4.3 Average velocity in the porous medium

We deduce an expression for the average limit velocity in the thin porous medium.

Theorem 4.7. *Consider the pair of limit functions (\widehat{v}, p^1) given in Lemmas 4.5 and 4.6. Defining the average velocity by*

$$V_{av}(z_1) = \int_0^1 \int_{Y_f} \widehat{v}(z, y) dy dz_2,$$

we have

$$V_{av,1}(z_1) = \frac{\mu}{\nu^{r'-1}} \left| f_1(z_1) - \frac{d}{dz_1} p^1(z_1) \right|^{r'-2} \left(f_1(z_1) - \frac{d}{dz_1} p^1(z_1) \right), \quad V_{av,2} \equiv 0, \quad \text{in } \omega, \quad (4.76)$$

and from (4.65) and $\int_0^1 v_1(z) dz_2 = V_{av,1}(z_1)$, we have

$$\partial_{z_1}(V_{av,1}(z_1)) = 0, \quad \text{in } \omega, \quad V_{av,1} \cdot n = 0 \quad \text{on } \partial\omega, \quad (4.77)$$

where $\mu \in \mathbb{R}$ is the permeability defined by

$$\mu = \int_{Y_f} \widehat{w}(y) \cdot e_1 dy, \quad (4.78)$$

where $(\widehat{w}, \widehat{q}) \in W_{\text{per}}^{1,r}(Y)^2 \times L_{0,\text{per}}^{r'}(Y)$, $1 < r < +\infty$, is the unique solution of the auxiliary problem

$$\begin{cases} -\text{div}_y (|\mathbb{D}_y[\widehat{w}]|^{r-2} \mathbb{D}_y[\widehat{w}]) + \nabla_y \widehat{q} = e_1 & \text{in } Y_f, \\ \text{div}_y \widehat{w} = 0 & \text{in } Y_f, \\ \widehat{w} = 0 & \text{on } T. \end{cases} \quad (4.79)$$

Remark 4.8. *As is pointed in [21, Remark 8], we observe that we have derived a Darcy law (4.76) identical the usual filtration law used in standard engineering treatment (see for instance Wu et al [44, p. 140]). We point out that the version of the unfolding method and the restriction operator introduced in this paper are powerfull tools that could be used to study the asymptotic behavior of different type of (two dimensional or three dimensional) fluids in a thin porous medium defined by Ω_ε .*

Proof of Theorem 4.7. We divide the proof in three steps.

Step 1. Variational formulation for $(\widehat{v}_\varepsilon, \widehat{P}_\varepsilon^1)$. Let us first write the variational formulation satisfied by the functions $(\widehat{u}_\varepsilon, \widehat{P}_\varepsilon^1)$ in order to pass to the limit. According to Lemma 4.5, we consider $\widetilde{\varphi}_\varepsilon(z) = (\widehat{\varphi}_1(z_1, z_2, z_1/\varepsilon, h_\varepsilon z_2/\varepsilon), \widehat{\varphi}_2(z_1, z_1/\varepsilon, h_\varepsilon z_2/\varepsilon))$, as test function in (2.23) where $\widehat{\varphi}(z, y) = (\widehat{\varphi}_1(z, y), \widehat{\varphi}_2(z_1, y)) \in \mathcal{D}(\Omega; C_{\text{per}}^\infty(Y)^2)$ with $\widehat{\varphi}(z, y) = 0$ in $\Omega \times T$ and $(\mathbb{R}^2 \setminus \Omega) \times Y$ (thus, $\widetilde{\varphi}_\varepsilon(z) \in W_0^{1,r}(\Omega_\varepsilon)^2$). Then, we have

$$\nu \int_{\widetilde{\Omega}_\varepsilon} S_r(\mathbb{D}_{h_\varepsilon}[\widetilde{v}_\varepsilon]) : \mathbb{D}_{h_\varepsilon} \widetilde{\varphi}_\varepsilon dz + \langle \nabla_{h_\varepsilon} \widetilde{p}_\varepsilon, \widetilde{\varphi}_\varepsilon \rangle_{W^{-1,r'}(\widetilde{\Omega}_\varepsilon)^2, W_0^{1,r}(\widetilde{\Omega}_\varepsilon)^2} = \int_{\widetilde{\Omega}_\varepsilon} f_1(\widetilde{\varphi}_\varepsilon)_1 dz, \quad (4.80)$$

where, for simplicity, we have denoted by $S_r : \mathbb{R}_{\text{sym}}^{2 \times 2} \rightarrow \mathbb{R}_{\text{sym}}^{2 \times 2}$ the r -Laplace operator, i.e. S_r is defined by

$$S_r(\xi) = |\xi|^{r-2} \xi, \quad \forall \xi \in \mathbb{R}_{\text{sym}}^{2 \times 2}, \quad 1 < r < +\infty. \quad (4.81)$$

Taking into account the extension of the pressure, we get

$$\langle \nabla_{h_\varepsilon} \tilde{p}_\varepsilon, \tilde{\varphi}_\varepsilon \rangle_{W^{-1,r'}(\tilde{\Omega}_\varepsilon)^2, W_0^{1,r}(\tilde{\Omega}_\varepsilon)^2} = \langle \nabla_{h_\varepsilon} \tilde{P}_\varepsilon^1, \tilde{\varphi}_\varepsilon \rangle_{W^{-1,r'}(\Omega)^2, W_0^{1,r}(\Omega)^2} = - \int_{\tilde{\Omega}} \tilde{P}_\varepsilon^1 \operatorname{div}_{h_\varepsilon}(\tilde{\varphi}_\varepsilon) dz,$$

and then, the variational formulation (4.80) reads

$$\nu \int_{\tilde{\Omega}_\varepsilon} S_r(\mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon]) : \mathbb{D}_{h_\varepsilon} \tilde{\varphi}_\varepsilon dz - \int_{\tilde{\Omega}} \tilde{P}_\varepsilon^1 \operatorname{div}_{h_\varepsilon}(\tilde{\varphi}_\varepsilon) dz = \int_{\tilde{\Omega}_\varepsilon} f_1(\tilde{\varphi}_\varepsilon)_1 dz. \quad (4.82)$$

Taking into account the definition of $\tilde{\varphi}_\varepsilon$, we have

$$\partial_{z_1} \tilde{\varphi}_{\varepsilon,1}(z) = \partial_{z_1} \hat{\varphi}_1 + \varepsilon^{-1} \partial_{y_1} \hat{\varphi}_1, \quad h_\varepsilon^{-1} \partial_{z_1} \tilde{\varphi}_{\varepsilon,1}(z) = h_\varepsilon^{-1} \partial_{z_1} \hat{\varphi}_1 + \varepsilon^{-1} \partial_{y_2} \hat{\varphi}_1,$$

$$\partial_{z_1} \tilde{\varphi}_{\varepsilon,2}(z) = \partial_{z_1} \hat{\varphi}_2 + \varepsilon^{-1} \partial_{y_1} \hat{\varphi}_2, \quad h_\varepsilon^{-1} \partial_{z_2} \tilde{\varphi}_{\varepsilon,1}(z) = \varepsilon^{-1} \partial_{y_2} \hat{\varphi}_2,$$

which can be written as follows

$$\mathbb{D}_{h_\varepsilon}[\tilde{\varphi}_\varepsilon(z)] = \mathbb{D}_{h_\varepsilon}[\hat{\varphi}] + \varepsilon^{-1} \mathbb{D}_y[\hat{\varphi}], \quad \operatorname{div}_{h_\varepsilon}(\tilde{\varphi}_\varepsilon(z)) = \partial_{z_1} \hat{\varphi}_1 + \varepsilon^{-1} \operatorname{div}_y(\hat{\varphi}).$$

Then, we have that (4.82) reads as follows

$$\begin{aligned} & \nu \int_{\tilde{\Omega}_\varepsilon} S_r(\mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon]) : \mathbb{D}_{h_\varepsilon}[\hat{\varphi}] dz + \nu \varepsilon^{-1} \int_{\tilde{\Omega}_\varepsilon} S_r(\mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon]) : \mathbb{D}_y[\hat{\varphi}] dz \\ & - \int_{\tilde{\Omega}} \tilde{P}_\varepsilon^1 \partial_{z_1} \hat{\varphi}_1 dz - \varepsilon^{-1} \int_{\tilde{\Omega}} \tilde{P}_\varepsilon^1 \operatorname{div}_y(\hat{\varphi}) dz = \int_{\tilde{\Omega}_\varepsilon} f_1 \hat{\varphi}_1 dz. \end{aligned} \quad (4.83)$$

Applying Hölder's inequality and taking into account estimates (4.57) and $\varepsilon \ll h_\varepsilon$ given in (2.5), we get

$$\left| \nu \int_{\tilde{\Omega}_\varepsilon} S_r(\mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon]) : \mathbb{D}_{h_\varepsilon}[\hat{\varphi}] dz \right| \leq C\varepsilon \|D_{h_\varepsilon} \hat{\varphi}\|_{L^r(\tilde{\Omega}_\varepsilon)^{2 \times 2}} \leq C\varepsilon \|D_z \hat{\varphi}\|_{L^r(\tilde{\Omega}_\varepsilon)^{2 \times 2}} \leq C\varepsilon h_\varepsilon^{-1} \rightarrow 0,$$

and taking into account that $\varepsilon^{-r} S_r(\mathbb{D}_y[\hat{v}_\varepsilon]) = S_r(\varepsilon^{-\frac{r}{r-1}} \mathbb{D}_y[\hat{v}_\varepsilon])$, by the change of variables given in Remark 4.2, we obtain

$$\begin{aligned} & \nu \int_{\Omega \times Y_f} S(\varepsilon^{-\frac{r}{r-1}} \mathbb{D}_y[\hat{v}_\varepsilon]) : \mathbb{D}_y \hat{\varphi} dz dy - \int_{\Omega \times Y} \hat{P}_\varepsilon^1 \partial_{z_1} \hat{\varphi}_1 dz dy \\ & - \varepsilon^{-1} \int_{\Omega \times Y} \hat{P}_\varepsilon^1 \operatorname{div}_y(\hat{\varphi}) dx' dz_3 dy = \int_{\Omega \times Y_f} f_1 \hat{\varphi}_1 dz dy + O_\varepsilon. \end{aligned} \quad (4.84)$$

Step 2. Passing to the limit. Now, we want to prove that the pair of limit functions (\hat{v}, p^1) given in Lemmas 4.5 and 4.6, satisfies the following two pressure limit system

$$\left\{ \begin{array}{l} -\nu \operatorname{div}_y(|\mathbb{D}_y[\hat{v}]|^{r-2} \mathbb{D}_y[\hat{v}]) + \nabla_y \hat{\pi} = (f_1(z_1) - \partial_{z_1} p^1(z_1)) e_1 \quad \text{in } Y_f, \\ \operatorname{div}_y(\hat{v}) = 0 \quad \text{in } Y_f, \\ \hat{v} = 0 \quad \text{on } T \quad \text{for a.e. } z \in \Omega, \\ \partial_{z_1} \left(\int_0^1 \int_{Y_f} \hat{v}_1(z, y) dy dz_2 \right) = 0 \quad \text{in } \omega, \\ \left(\int_0^1 \int_{Y_f} \hat{v}_1(z, y) dy dz_2 \right) n = 0 \quad \text{on } \partial\omega, \\ (\hat{v}, \hat{\pi}) \text{ is } Y\text{-periodic,} \end{array} \right. \quad (4.85)$$

which has a unique solution $(\widehat{v}, p^1, \widehat{\pi}) \in L^r(\Omega; W_{\text{per}}^{1,r}(Y_f)^2) \times (L_0^{r'}(\omega) \cap W^{1,r'}(\omega)) \times L^{r'}(\Omega; L_{0,\text{per}}^{r'}(Y_f))$.

To do this, we consider $\widehat{w}(z, y) = (\widehat{w}_1(z, y), \widehat{w}_2(z_1, y)) \in \mathcal{D}(\Omega; W_{\text{per}}^{1,r}(Y)^3)$, such that $\widehat{w} = 0$ in $\Omega \times T$ and $\text{div}_y(\widehat{w}) = 0$ in $\Omega \times Y_f$. Thus, we consider the following test function in (4.84):

$$\widehat{\varphi}_\varepsilon(z, y) = \widehat{w}(z, y) - \varepsilon^{-\frac{r}{r-1}} \widehat{v}_\varepsilon.$$

So we have

$$\nu \int_{\Omega \times Y_f} S_r(\varepsilon^{-\frac{r}{r-1}} \mathbb{D}_y[\widehat{v}_\varepsilon]) : \mathbb{D}_y[\widehat{\varphi}_\varepsilon] dzdy - \int_{\Omega \times Y} \widehat{P}_\varepsilon^1 \partial_{z_1} \widehat{\varphi}_{\varepsilon,1} dzdy = \int_{\Omega \times Y_f} f_1 \widehat{\varphi}_{\varepsilon,1} dzdy + O_\varepsilon.$$

which is equivalent to

$$\begin{aligned} & \nu \int_{\Omega \times Y_f} \left(S_r(\mathbb{D}_y[\widehat{w}]) - S_r(\varepsilon^{-\frac{r}{r-1}} \mathbb{D}_y[\widehat{v}_\varepsilon]) \right) : \mathbb{D}_y[\widehat{\varphi}_\varepsilon] dzdy - \nu \int_{\Omega \times Y_f} S_r(\mathbb{D}_y[\widehat{w}]) : \mathbb{D}_y[\widehat{\varphi}_\varepsilon] dzdy \\ & + \int_{\Omega \times Y} \widehat{P}_\varepsilon^1 \partial_{z_1} \widehat{\varphi}_{\varepsilon,1} dzdy = - \int_{\Omega \times Y_f} f_1 \widehat{\varphi}_{\varepsilon,1} dzdy + O_\varepsilon. \end{aligned}$$

Since S_r is monotone, i.e.

$$\left(S_r(\mathbb{D}_y[\widehat{w}]) - S_r(\varepsilon^{-\frac{r}{r-1}} \mathbb{D}_y[\widehat{v}_\varepsilon]) \right) : (\mathbb{D}_y[\widehat{w}(z, y)] - \varepsilon^{-\frac{r}{r-1}} \mathbb{D}_y[\widehat{v}_\varepsilon]) \geq 0,$$

we can deduce

$$\nu \int_{\Omega \times Y_f} S_r(\mathbb{D}_y[\widehat{w}]) : \mathbb{D}_y[\widehat{\varphi}_\varepsilon] dzdy - \int_{\Omega \times Y} \widehat{P}_\varepsilon^1 \partial_{z_1} \widehat{\varphi}_{\varepsilon,1} dzdy \geq \int_{\Omega \times Y_f} f_1 \widehat{\varphi}_{\varepsilon,1} dzdy + O_\varepsilon.$$

Passing to the limit by using convergences (4.66) and (4.73), we obtain

$$\nu \int_{\Omega \times Y_f} S_r(\mathbb{D}_y[\widehat{w}]) : \mathbb{D}_y[\widehat{w} - \widehat{v}] dzdy - \int_{\Omega \times Y} p^1 \partial_{z_1} (\widehat{w}_1 - \widehat{v}_1) dzdy \geq \int_{\Omega \times Y_f} f_1 (\widehat{w}_1 - \widehat{v}_1) dzdy.$$

From Minty's Lemma [31, Chapter 3, Lemma 1.2], then previous inequality is equivalent to the following variational formulation

$$\nu \int_{\Omega \times Y_f} S_r(\mathbb{D}_y[\widehat{v}]) : D_y \widehat{w} dzdy - \int_{\Omega \times Y} p^1 \partial_{z_1} \widehat{w}_1 dzdy = \int_{\Omega \times Y_f} f_1 \widehat{w}_1 dzdy. \quad (4.86)$$

By density, this equality holds for every function in the Banach space \mathcal{W} defined by

$$\mathcal{W} = \left\{ \widehat{w}(z, y) \in L^r(\Omega; W_{\text{per}}^{1,r}(Y)^2) : \text{div}_y \widehat{w}(z, y) = 0 \text{ in } \Omega \times Y_f, \quad \widehat{w}(z, y) = 0 \text{ in } \Omega \times T \right\}.$$

Reasoning as in [21], by integration by parts, the variational formulation (4.86) is equivalent to the system (4.85), where $\widehat{\pi}$ arises as a Lagrange multiplier of the incompressibility constraint $\text{div}_y(\widehat{w})$ in $\Omega \times Y_f$. From [21, Theorem 8] (whose proof is similar to the proof of [21, Theorem 2]), it holds the uniqueness of solution $(\widehat{v}, p^1, \widehat{\pi}) \in L^r(\Omega; W_{\text{per}}^{1,r}(Y_f)^2) \times (L_0^{r'}(\omega) \cap W^{1,r'}(\omega)) \times L^{r'}(\Omega; L_{0,\text{per}}^{r'}(Y_f))$ and so, the whole sequence converges.

Step 3. Average velocity. We will deduce the expression for velocity V_{av} given in (4.76). To do this, let us define the local problems which are useful to eliminate the variable y of the homogenized problem (4.85).

Let us separate variables y and z in the homogenized problem (4.85) satisfied by $(\widehat{v}, p^1, \widehat{\pi})$. It is easy to check that $(\widehat{v}, \widehat{\pi})$ can be written as follow

$$\begin{aligned}\widehat{v}(z, y) &= \frac{1}{\nu^{r'-1}} \left| f_1(z_1) - \frac{d}{dz_1} p^1(z_1) \right|^{r'-2} \left(f_1(z_1) - \frac{d}{dz_1} p^1(z_1) \right) \widehat{w}(y), \\ \widehat{\pi}(z, y) &= \left(f_1(z_1) - \frac{d}{dz_1} p^1(z_1) \right) \widehat{q}(y),\end{aligned}$$

where $(\widehat{w}, \widehat{q})$ is the unique solution of problem (4.79).

Finally, taking into account the definition of V_{av} and relation (4.68), we have

$$\begin{aligned}V_{av,1}(z_1) &= \frac{1}{\nu^{r'-1}} \left| f_1(z_1) - \frac{d}{dz_1} p^1(z_1) \right|^{r'-2} \left(f_1(z_1) - \frac{d}{dz_1} p^1(z_1) \right) \int_{Y_f} \widehat{w}(y) \cdot e_1 dy \\ V_{av,2}(z_1) &\equiv 0,\end{aligned}$$

and from relation (4.68), we have (4.77). Taking into account the definition of μ given in (4.78), gives expression (4.76). □

5 Critical case: problem in the thin film

As in the previous section, we assume the relation between the parameters in the critical regime (3.56). From this and $\varepsilon \ll \eta_\varepsilon$ given in (2.5), we deduce that the last two terms in the estimate of $\widetilde{\mathcal{U}}_\varepsilon$ given in (3.41) satisfy

$$\varepsilon^{\frac{1}{r-1}} h_\varepsilon^{\frac{1}{r}} \eta_\varepsilon^{\frac{r-1}{r}} \approx \varepsilon^{\frac{1}{r-1}} \eta_\varepsilon^{\frac{r-1}{r}} \eta_\varepsilon^{\frac{2r-1}{r(r-1)}} \varepsilon^{-\frac{1}{r-1}} \ll \eta_\varepsilon^{\frac{1}{r-1} + \frac{r-1}{r} + \frac{2r-1}{r(r-1)} - \frac{1}{r-1}} = \eta_\varepsilon^{\frac{r}{r-1}},$$

and

$$\varepsilon^{\frac{r+1}{2(r-1)}} h_\varepsilon^{\frac{1}{r}} \eta_\varepsilon^{\frac{r-2}{2r}} \approx \varepsilon^{\frac{r+1}{2(r-1)}} \eta_\varepsilon^{\frac{r-2}{2r}} \eta_\varepsilon^{\frac{2r-1}{r(r-1)}} \varepsilon^{-\frac{1}{r-1}} \ll \eta_\varepsilon^{\frac{r+1}{2(r-1)} + \frac{r-2}{2r} + \frac{2r-1}{r(r-1)} - \frac{1}{r-1}} = \eta_\varepsilon^{\frac{r}{r-1}}.$$

Then, estimates (3.41)–(3.43) for $\widetilde{\mathcal{U}}_\varepsilon$ and (3.53) for $\widetilde{P}_\varepsilon^2$ in \widetilde{I}_1 read as follows

$$\|\widetilde{\mathcal{U}}_\varepsilon\|_{L^r(\widetilde{I}_1)^2} \leq C \eta_\varepsilon^{\frac{r}{r-1}}, \quad \|\mathbb{D}_{\eta_\varepsilon}[\widetilde{\mathcal{U}}_\varepsilon]\|_{L^r(\widetilde{I}_1)^{2 \times 2}} \leq C \eta_\varepsilon^{\frac{1}{r-1}}, \quad \|D_{\eta_\varepsilon} \widetilde{\mathcal{U}}_\varepsilon\|_{L^r(\widetilde{I}_1)^{2 \times 2}} \leq C \eta_\varepsilon^{\frac{1}{r-1}}, \quad (5.87)$$

$$\|\widetilde{P}_\varepsilon^2\|_{L^{r'}(\widetilde{I}_1)} \leq C, \quad \|\nabla_{\eta_\varepsilon} \widetilde{P}_\varepsilon^2\|_{W^{-1,r'}(\widetilde{I}_1)^2} \leq C. \quad (5.88)$$

5.1 Convergences of velocity and pressure

Using estimates (5.87) and (5.88) and compactness, we prove the following lemma.

Lemma 5.1. *For a subsequence of ε still denoted by ε , there exist:*

– $\mathcal{U} \in W^{1,r}(-g(z_1), 0; L^r(\omega)^2)$, with $\mathcal{U}_2 \equiv 0$ and $\mathcal{U}_1 = 0$ on $\Sigma \cup \Gamma_g$, such that

$$\eta_\varepsilon^{-\frac{r}{r-1}} \tilde{\mathcal{U}}_\varepsilon \rightharpoonup \mathcal{U} \quad \text{in } W^{1,r}(-g(z_1), 0; L^r(\omega)^2), \quad (5.89)$$

– $p^2 \in L_0^{r'}(\tilde{I}_1) \cap W^{1,r'}(\omega)$ independent of z_2 , such that

$$\tilde{P}_\varepsilon^2 \rightharpoonup p^2 \quad \text{in } L^{r'}(\tilde{I}_1). \quad (5.90)$$

Proof. From estimates (5.87), there exist $\mathcal{U} \in W^{1,r}(-g(z_1), 0; L^r(\omega)^2)$ such that convergence (5.89) holds.

Let $\tilde{\varphi} \in C_0^\infty(\tilde{I}_1)$, then

$$\eta_\varepsilon^{-\frac{1}{r-1}} \int_{\tilde{I}_1} \left(\partial_{z_1} \tilde{\mathcal{U}}_{\varepsilon,1} + \eta_\varepsilon^{-1} \partial_{z_2} \tilde{\mathcal{U}}_{\varepsilon,2} \right) \tilde{\varphi} dz = -\eta_\varepsilon^{-\frac{1}{r-1}} \int_{\tilde{I}_1} \tilde{\mathcal{U}}_{\varepsilon,1} \partial_{z_1} \tilde{\varphi} dz - \eta_\varepsilon^{-\frac{r}{r-1}} \int_{\tilde{I}_1} \tilde{\mathcal{U}}_{\varepsilon,2} \partial_{z_2} \tilde{\varphi} dz.$$

Taking the limit $\varepsilon \rightarrow 0$, we get

$$\int_{\tilde{I}_1} \mathcal{U}_2 \partial_{z_2} \tilde{\varphi} dz = 0,$$

so that \mathcal{U}_2 does not depend on z_2 .

Since \mathcal{U} , $\partial_{z_2} \mathcal{U} \in L^r(\tilde{I}_1)^2$, the traces $\mathcal{U}(z_1, -g(z_1))$, $\tilde{\mathcal{U}}(z_1, 0)$ are well defined in $L^r(\omega)^2$. The proof of $\mathcal{U}(z_1, -g(z_1)) = 0$ straightforward from the boundary condition $\tilde{\mathcal{U}}_\varepsilon(z_1, -g(z_1)) = 0$. Next, we prove that $\mathcal{U}(z_1, 0) = 0$. Proceeding similarly to the proof of Lemma 3.2 (but now choosing a point $\beta_{z_1} \in T_\varepsilon$ which is close to the point $\alpha_{z_1} \in \Sigma$), then we have

$$|\tilde{\mathcal{U}}_\varepsilon(\alpha_{z_1})| = |\tilde{u}_\varepsilon(\alpha_{z_1}) - \tilde{u}_\varepsilon(\beta_{z_1})| = |D\tilde{u}_\varepsilon(\xi)(\alpha_{z_1} - \beta_{z_1})| \leq \varepsilon |D_{h_\varepsilon} \tilde{u}_\varepsilon|.$$

Since $\tilde{u}_\varepsilon(\beta_{z_1}) = 0$ because $\beta_{z_1} \in T_\varepsilon$, we have

$$\|\tilde{\mathcal{U}}_\varepsilon\|_{L^r(\Sigma)^2} \leq C\varepsilon \|D_{h_\varepsilon} \tilde{u}_\varepsilon\|_{L^r(\tilde{\Omega}_\varepsilon)^{2 \times 2}}.$$

Then, taking into account estimate (4.57) in the critical case, we have

$$\|\eta_\varepsilon^{-\frac{r}{r-1}} \tilde{\mathcal{U}}_\varepsilon\|_{L^r(\Sigma)^2} \leq C\varepsilon \eta_\varepsilon^{-\frac{r}{r-1}} \varepsilon^{\frac{1}{r-1}} = C(\varepsilon/\eta_\varepsilon)^{\frac{r}{r-1}},$$

which tends to zero as ε to zero, because $\varepsilon \ll \eta_\varepsilon$. This implies that $\tilde{\mathcal{U}}(z_1, 0) = 0$. Consequently, $\tilde{\mathcal{U}}_2 \equiv 0$, which finishes the proof of convergence (5.89).

Next, estimate (5.88)₁ implies the existence of $p^2 \in L^{r'}(\tilde{I}_1)$ such that convergence (5.90) holds. Similarly to the proof of Lemma 4.6, from estimate (5.88)₂, we deduce that p_2 does not depend on z_2 . Since \tilde{P}^2 has null mean value in \tilde{I}_1 , then p^2 also has null mean value in \tilde{I}_1 .

Now, by considering $\tilde{\varphi} \in \mathcal{D}(\omega)$ as test function in the divergence condition $\operatorname{div}_{\eta_\varepsilon}(\tilde{\mathcal{U}}_\varepsilon) = 0$ in \tilde{I}_1 , we get

$$\int_{\tilde{I}_1} \left(\partial_{z_1} \tilde{\mathcal{U}}_{\varepsilon,1} \tilde{\varphi} + \eta_\varepsilon^{-1} \partial_{z_2} \tilde{\mathcal{U}}_{\varepsilon,2} \tilde{\varphi} \right) dz = 0,$$

which, after integration by parts and multiplication by $\eta_\varepsilon^{-\frac{r}{r-1}}$, gives

$$\int_{\tilde{I}_1} \eta_\varepsilon^{-\frac{r}{r-1}} \tilde{\mathcal{U}}_{\varepsilon,1} \partial_{z_1} \tilde{\varphi} dz = 0.$$

Passing to the limit by using convergence (5.89), we deduce

$$\int_{\tilde{I}_1} \mathcal{U}_1 \partial_{z_1} \tilde{\varphi} dz = 0,$$

and, since φ does not depend on z_2 , we obtain the following divergence condition

$$\partial_{z_1} \left(\int_{-g(z_1)}^0 \mathcal{U}_1(z) dz_2 \right) = 0 \quad \text{in } \omega, \quad \left(\int_{-g(z_1)}^0 \mathcal{U}_1(z) dz_2 \right) n = 0 \quad \text{on } \partial\omega. \quad (5.91)$$

By using convergences (5.89) and (5.90), we refer to [37, Propositions 3.1 and 3.2] in order to identify the effective system

$$\left\{ \begin{array}{l} -\partial_{z_2} (|\partial_{z_2} \mathcal{U}_1|^{r-2} \partial_{z_2} \mathcal{U}_1) = \frac{2^{\frac{r}{2}}}{\nu} (f_1(z_1) - \partial_{z_1} p^2(z_1)) \quad \text{in } \tilde{I}_1, \\ \partial_{z_1} \left(\int_{-g(z_1)}^0 \mathcal{U}_1(z) dz_2 \right) = 0 \quad \text{in } \omega, \\ \left(\int_{-g(z_1)}^0 \mathcal{U}_1(z) dz_2 \right) n = 0 \quad \text{on } \partial\omega, \\ \mathcal{U}_1 = 0 \quad \text{on } \Sigma \cup \Gamma_g. \end{array} \right. \quad (5.92)$$

Furthermore, we have that $p^2 \in W^{1,r'}(\omega) \cap L_0^{r'}(\tilde{I}_1)$ according to [37, Proposition 3.3]. This concludes the proof. \square

5.2 Average velocity in the thin film

Theorem 5.2. *Consider (\mathcal{U}, p^2) given in Lemma 5.1. Then, we have that the average velocity*

$$\mathcal{V}_{av}(z_1) = \frac{1}{g(z_1)} \int_{-g(z_1)}^0 \mathcal{U}(z) dz_2,$$

is given by

$$\left\{ \begin{array}{l} \mathcal{V}_{av,1}(z_1) = \frac{g(z_1)^{r'}}{2^{\frac{r'}{2}} (r' + 1) \nu^{r'-1}} \left| f_1(z_1) - \frac{d}{dz_1} p^2(z_1) \right|^{r'-2} \left(f_1(z_1) - \frac{d}{dz_1} p^2(z_1) \right) \quad \text{in } \omega, \\ \mathcal{V}_{av,2} \equiv 0. \end{array} \right. \quad (5.93)$$

Proof. Since $\mathcal{U}_2 \equiv 0$, it is only necessary to obtain the expression of \mathcal{U}_1 , which satisfies problem (5.92)_{1,3}. We remark that this problem is formally an ordinary differential equation in the variable z_2 , with parameter $z_1 \in \omega$. The resolution of (5.92) is similar to [37, Proposition 3.4] (see also [30, Lemma 6.3]), so we omit it. □

6 Critical case: a generalized Reynolds limit equation

The conclusion of the previous two sections is that for any sequence of solutions $(\tilde{v}_\varepsilon, \tilde{P}_\varepsilon^1)$ and $(\tilde{\mathcal{U}}_\varepsilon, \tilde{P}_\varepsilon^2)$ and letting $\varepsilon \rightarrow 0$, we can extract subsequences, still denoted by the same symbol, and find functions $(v, p^1) \in W^{1,r}(0, 1; L^r(\omega)^2) \times W^{1,r'}(\omega)$ and $(\mathcal{U}, p^2) \in W^{1,r}(-g(z_1), 0; L^r(\omega)^2) \times W^{1,r'}(\omega)$ such that

$$\begin{aligned} \varepsilon^{-\frac{r}{r-1}} \tilde{V}_\varepsilon &\rightharpoonup v = (v_1, 0) \quad \text{in } W^{1,r}(0, 1; L^r(\omega)^2), & \tilde{P}_\varepsilon^1 &\rightarrow p^1 \quad \text{in } L^{r'}(\Omega), \\ \eta_\varepsilon^{-\frac{r}{r-1}} \tilde{\mathcal{U}}_\varepsilon &\rightharpoonup \mathcal{U} = (\mathcal{U}_1, 0) \quad \text{in } W^{1,r}(-g(z_1), 0; L^r(\omega)^2), & \tilde{P}_\varepsilon^2 &\rightarrow p^2 \quad \text{in } L^{r'}(\tilde{\Gamma}_1). \end{aligned} \quad (6.94)$$

Moreover, functions $(V_{av}, p^1), (\mathcal{V}_{av}, p^2)$, with $V_{av} = \int_0^1 v(z) dz_2$ and $\mathcal{V}_{av} = g(z_1)^{-1} \int_{-g(z_1)}^0 \mathcal{U}(z) dz_2$, necessarily satisfy the following equations in ω

$$\begin{aligned} V_{av,1}(z_1) &= \frac{\mu}{\nu^{r'-1}} \left| f_1(z_1) - \frac{d}{dz_1} p^1(z_1) \right|^{r'-2} \left(f_1(z_1) - \frac{d}{dz_1} p^1(z_1) \right), & V_{av,2} &\equiv 0, \\ \mathcal{V}_{av,1}(z_1) &= \frac{g(z_1)^{r'}}{2^{\frac{r'}{2}} (r' + 1) \nu^{r'-1}} \left| f_1(z_1) - \frac{d}{dz_1} p^2(z_1) \right|^{r'-2} \left(f_1(z_1) - \frac{d}{dz_1} p^2(z_1) \right), & \mathcal{V}_{av,2} &\equiv 0, \end{aligned} \quad (6.95)$$

with μ defined in (4.78).

Next, we find the connection between the functions p^1 and p^2 , i.e. the coupling effects between the solution in the porous and free media.

Lemma 6.1. *We assume that the parameters $\varepsilon, \eta_\varepsilon$ and h_ε satisfy (2.5) and (3.56). Let $p^1 \in W^{1,r'}(\omega)$ and $p^2 \in W^{1,r'}(\omega)$ be such that (6.94) and (6.95) hold. Then, we have*

$$\begin{aligned} &\frac{\mu}{\nu^{r'-1}} \int_\omega \left| f_1(z_1) - \frac{d}{dz_1} p^1(z_1) \right|^{r'-2} \left(f_1(z_1) - \frac{d}{dz_1} p^1(z_1) \right) \frac{d}{dz_1} \psi dz_1 \\ &+ \frac{1}{\lambda 2^{\frac{r'}{2}} (r' + 1) \nu^{r'-1}} \int_\omega g(z_1)^{r'} \left| f_1(z_1) - \frac{d}{dz_1} p^2(z_1) \right|^{r'-2} \left(f_1(z_1) - \frac{d}{dz_1} p^2(z_1) \right) \frac{d}{dz_1} \psi dz_1 = 0, \end{aligned} \quad (6.96)$$

for every $\psi \in W^{1,r'}(\omega)$.

Proof. Choosing $\psi \in W^{1,r'}(\omega)$ as test function in (2.21)₂, putting $\tilde{u}_\varepsilon = 0$ in the solid part (recalling that \tilde{V}_ε is the extension by zero of \tilde{v}_ε to the whole Ω) and integrating by parts, we get

$$\int_\Omega h_\varepsilon \tilde{V}_{\varepsilon,1} \partial_{z_1} \psi(z_1) dz + \int_{\tilde{\Gamma}_1} \eta_\varepsilon \tilde{\mathcal{U}}_{\varepsilon,1} \partial_{z_1} \psi(z_1) dz = 0. \quad (6.97)$$

Multiplying (6.97) by $\varepsilon^{-\frac{r}{r-1}} h_\varepsilon^{-1}$, we have

$$\int_{\Omega} \varepsilon^{-\frac{r}{r-1}} \tilde{V}_{\varepsilon,1} \partial_{z_1} \psi(z_1) dz + \int_{\tilde{I}_1} \varepsilon^{-\frac{r}{r-1}} h_\varepsilon^{-1} \eta_\varepsilon \tilde{\mathcal{U}}_{\varepsilon,1} \partial_{z_1} \psi(z_1) dz = 0,$$

which can be rewritten as follows

$$\int_{\Omega} \varepsilon^{-\frac{r}{r-1}} \tilde{V}_{\varepsilon,1} \partial_{z_1} \psi(z_1) dz + \varepsilon^{-\frac{r}{r-1}} \eta_\varepsilon^{\frac{2r-1}{r-1}} h_\varepsilon^{-1} \int_{\tilde{I}_1} \eta_\varepsilon^{-\frac{r}{r-1}} \tilde{\mathcal{U}}_{\varepsilon,1} \partial_{z_1} \psi(z_1) dz = 0.$$

From convergences (6.94) and relation (3.56), passing to the limit as $\varepsilon \rightarrow 0$, we deduce

$$\int_{\Omega} v_1 \partial_{z_1} \psi(z_1) dz + \lambda^{-1} \int_{\tilde{I}_1} \mathcal{U}_1 \partial_{z_1} \psi(z_1) dz = 0.$$

Then, since ψ does not depend on z_2 , we have

$$\int_{\omega} V_{av,1} \partial_{z_1} \psi(z_1) dz_1 + \lambda^{-1} \int_{\omega} g(z_1) \mathcal{V}_{av,1} \partial_{z_1} \psi(z_1) dz_1 = 0.$$

Taking into account (6.95), this is the equation (6.96). \square

In the following result, we are going to prove the relation between the pressures p^1 and p^2 , i.e. the continuity of the pressure in Σ .

Lemma 6.2. *We assume that the parameters $\varepsilon, \eta_\varepsilon$ and h_ε satisfy (2.5) and (3.56). Let p^1 and p^2 be the limit pressures from expression (6.94). Then, there exists $c^* \in \mathbb{R}$ such that*

$$p^1 = p^2 + c^*. \quad (6.98)$$

Proof. For any $\tilde{\varphi} \in \mathcal{D}(\omega)$, we define $(\tilde{\phi}, \tilde{\psi}) \in W^{1,r}(\Omega) \times W^{1,r}(\tilde{I}_1)$ such that

$$\tilde{\phi} = 0 \quad \text{on } \tilde{\Omega} \setminus \Sigma, \quad \tilde{\psi} = 0 \quad \text{on } \partial \tilde{I}_1 \setminus \Sigma, \quad \tilde{\phi} = \tilde{\psi} = \tilde{\varphi} \quad \text{on } \Sigma.$$

Let us define the following global test function in \tilde{D}_ε given by

$$\tilde{w}_\varepsilon(z) = \begin{cases} \tilde{\phi}(z) (\tilde{\mathcal{R}}_r^\varepsilon e_2)(z) & \text{in } \tilde{\Omega}_\varepsilon, \\ \tilde{\psi}(z) e_2 & \text{in } \tilde{I}_1, \end{cases} \quad (6.99)$$

where $\tilde{\mathcal{R}}_r^\varepsilon$ is the restriction operator defined in Lemma 3.6. We observe that $\tilde{\mathcal{R}}_r^\varepsilon e_2$ tends to its Y -average $\int_Y (\mathcal{R}_r e_2)(y) dy$ (where the restriction operator \mathcal{R}_r is defined in the proof of Lemma 3.5), and $\tilde{\mathcal{R}}_r^\varepsilon(e_2)_2$ tends to 1 in $L^r(\Omega)$.

Now, we take \tilde{w}_ε as test function in the system (2.21), and we obtain

$$\begin{aligned} & \nu \int_{\tilde{\Omega}_\varepsilon} h_\varepsilon S_r(\mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon]) : \mathbb{D}_{h_\varepsilon}[\tilde{w}_\varepsilon] dz + \nu \int_{\tilde{I}_1} \eta_\varepsilon S_r(|\mathbb{D}_{\eta_\varepsilon}[\tilde{u}_\varepsilon]|) : \mathbb{D}_{\eta_\varepsilon}[\tilde{w}_\varepsilon] dz \\ & - \int_{\tilde{\Omega}_\varepsilon} h_\varepsilon \tilde{p}_\varepsilon \operatorname{div}_{h_\varepsilon}(\tilde{w}_\varepsilon) dz - \int_{\tilde{I}_1} \eta_\varepsilon \tilde{p}_\varepsilon \operatorname{div}_{\eta_\varepsilon}(\tilde{w}_\varepsilon) dz \\ & = \int_{\tilde{\Omega}_\varepsilon} h_\varepsilon f \cdot \tilde{w}_\varepsilon dz + \int_{\tilde{I}_1} \eta_\varepsilon f \cdot \tilde{w}_\varepsilon dz. \end{aligned} \quad (6.100)$$

From Hölder's inequality, estimates (3.46) and (4.57), and $\varepsilon \ll h_\varepsilon$, we deduce

$$\begin{aligned}
 & \left| \nu \int_{\tilde{\Omega}_\varepsilon} h_\varepsilon S_r(|\mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon]|) : \mathbb{D}_{h_\varepsilon}[\tilde{w}_\varepsilon] dz \right| \\
 &= \left| \nu h_\varepsilon \int_{\tilde{\Omega}_\varepsilon} S_r(|\mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon]|) : (\nabla_{h_\varepsilon} \tilde{\phi}(z) \cdot (\mathcal{R}_r^\varepsilon e_2)(z)) dz + \nu h_\varepsilon \int_{\tilde{\Omega}_\varepsilon} S_r(|\mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon]|) : \mathbb{D}_{h_\varepsilon}[(\tilde{\mathcal{R}}_r^\varepsilon e_2)(z)] \tilde{\phi}(z) dz \right| \\
 &\leq Ch_\varepsilon \left(\|\mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon]\|_{L^r(\tilde{\Omega}_\varepsilon)^{2 \times 2}}^{r-1} \|\nabla_{h_\varepsilon} \tilde{\phi}\|_{L^r(\Omega)^2} + \|\mathbb{D}_{h_\varepsilon}[\tilde{v}_\varepsilon]\|_{L^r(\tilde{\Omega}_\varepsilon)^{2 \times 2}}^{r-1} \|D_{h_\varepsilon} \tilde{\mathcal{R}}_r^\varepsilon e_2\|_{L^r(\Omega)^2} \right) \\
 &\leq Ch_\varepsilon (\varepsilon h_\varepsilon^{-1} + \varepsilon \varepsilon^{-1}) \leq Ch_\varepsilon,
 \end{aligned} \tag{6.101}$$

and by using estimates (5.87), we deduce

$$\begin{aligned}
 & \left| \nu \int_{\tilde{I}_1} \eta_\varepsilon S_r(|\mathbb{D}_{\eta_\varepsilon}[\tilde{\mathcal{U}}_\varepsilon]|) : \mathbb{D}_{\eta_\varepsilon}[\tilde{w}_\varepsilon] dz \right| = \left| \nu \int_{\tilde{I}_1} S_r(|\mathbb{D}_{\eta_\varepsilon}[\tilde{\mathcal{U}}_\varepsilon]|) : \left(\eta_\varepsilon \partial_{z_1} [\tilde{\psi}(z) e_2] + \partial_{z_2} [\tilde{\psi}(z) e_2] \right) dz \right| \\
 &\leq C(\eta_\varepsilon + 1) \|\mathbb{D}_{\eta_\varepsilon}[\tilde{\mathcal{U}}_\varepsilon]\|_{L^r(\tilde{I}_1)^{2 \times 2}}^{r-1} \leq C\eta_\varepsilon.
 \end{aligned} \tag{6.102}$$

From the unfolding change of variables (4.61) and $\operatorname{div}_y(\tilde{\mathcal{R}}_r^\varepsilon e_2) = 0$ in Y , we deduce that

$$\begin{aligned}
 & \int_{\tilde{\Omega}_\varepsilon} h_\varepsilon \tilde{p}_\varepsilon \operatorname{div}_{h_\varepsilon}(\tilde{w}_\varepsilon) dz = \int_{\tilde{\Omega}_\varepsilon} h_\varepsilon \tilde{P}_\varepsilon^1 \operatorname{div}_{h_\varepsilon}(\tilde{\phi}(z) (\tilde{\mathcal{R}}_r^\varepsilon e_2)(z)) dz \\
 &= \int_{\tilde{\Omega}_\varepsilon} h_\varepsilon \tilde{P}_\varepsilon^1 \nabla_{h_\varepsilon} \tilde{\phi}(z) \cdot (\tilde{\mathcal{R}}_r^\varepsilon e_2)(z) dz + \int_{\tilde{\Omega}_\varepsilon \times Y} h_\varepsilon \varepsilon^{-1} \tilde{P}_\varepsilon^1 \tilde{\phi}(z) \operatorname{div}_y(\tilde{\mathcal{R}}_r^\varepsilon e_2) dz dy \\
 &= \int_{\tilde{\Omega}_\varepsilon} h_\varepsilon \tilde{P}_\varepsilon^1 \nabla_{h_\varepsilon} \tilde{\phi}(z) \cdot (\tilde{\mathcal{R}}_r^\varepsilon e_2)(z) dz,
 \end{aligned} \tag{6.103}$$

and from estimate (4.63), we have

$$\left| \int_{\tilde{\Omega}_\varepsilon} h_\varepsilon \tilde{P}_\varepsilon^1 \nabla_{h_\varepsilon} \tilde{\phi}(z) \cdot (\tilde{\mathcal{R}}_r^\varepsilon e_2)(z) dz \right| \leq C.$$

From the definition of \tilde{P}_ε^2 given in (3.51) and \tilde{c}_ε given in (3.52), we have that

$$\begin{aligned}
 & \eta_\varepsilon \int_{\tilde{I}_1} \tilde{p}_\varepsilon \operatorname{div}_{\eta_\varepsilon}(\tilde{w}_\varepsilon) dz = \eta_\varepsilon \int_{\tilde{I}_1} (\tilde{p}_\varepsilon - \tilde{c}_\varepsilon) \operatorname{div}_{\eta_\varepsilon}(\tilde{w}_\varepsilon) dz + \eta_\varepsilon \tilde{c}_\varepsilon \int_{\tilde{I}_1} \operatorname{div}_{\eta_\varepsilon}(\tilde{w}_\varepsilon) dz \\
 &= \int_{\tilde{I}_1} \tilde{P}_\varepsilon^2 \partial_{z_2} \tilde{\psi}(z) dz + \tilde{c}_\varepsilon \int_{\tilde{I}_1} \partial_{z_2} \tilde{\psi}(z) dz,
 \end{aligned} \tag{6.104}$$

and from estimate (5.88), we have

$$\left| \int_{\tilde{I}_1} \tilde{P}_\varepsilon^2 \partial_{z_2} \tilde{\psi}(z) dz \right| \leq C.$$

Taking into account that that f is given by (2.17) and \tilde{w}_ε given in (6.99), we deduce

$$\int_{\tilde{\Omega}_\varepsilon} h_\varepsilon f \cdot \tilde{w}_\varepsilon dz = 0, \quad \text{and} \quad \int_{\tilde{I}_1} \eta_\varepsilon f \cdot \tilde{w}_\varepsilon dz = 0. \tag{6.105}$$

From (6.101)–(6.105) and the fact that $\eta_\varepsilon \ll 1$ and $h_\varepsilon \ll 1$, we deduce that $|\tilde{c}_\varepsilon| \leq C$ and so there exists c^* such that \tilde{c}_ε tends to c^* . Moreover, we deduce that (6.100) reads as follow

$$\int_{\Omega} h_\varepsilon \tilde{P}_\varepsilon^1 \nabla_{h_\varepsilon} \tilde{\phi}(z) \cdot (\tilde{\mathcal{R}}_r^\varepsilon e_2)(z) dz + \int_{\tilde{I}_1} \tilde{P}_\varepsilon^2 \partial_{z_2} \tilde{\psi}(z) dz + \tilde{c}_\varepsilon \int_{\tilde{I}_1} \partial_{z_2} \tilde{\psi}(z) dz + O_\varepsilon = 0. \quad (6.106)$$

Passing to the limit when $\varepsilon \rightarrow 0$, from strong convergence of \tilde{P}_ε^1 given in (4.72) and convergence $\tilde{\mathcal{R}}_r^\varepsilon e_2$ to 1 in the first term, convergence of \tilde{P}_ε^2 given in (5.90) in the second term and convergence of \tilde{c}_ε to c^* in the third term, we get

$$\int_{\Omega} p^1(z_1) \partial_{z_2} \tilde{\phi}(z) dz + \int_{\tilde{I}_1} p^2(z_1) \partial_{z_2} \tilde{\psi}(z) dz + c^* \int_{\tilde{I}_1} \partial_{z_2} \tilde{\psi}(z) dz = 0.$$

Since p^1 and p^2 do not depend on z_2 , this can be written as follows

$$\int_{\omega} p^1(z_1) \left(\int_0^1 \partial_{z_2} \tilde{\phi}(z) dz_2 \right) dz_1 + \int_{\omega} p^2(z_1) \left(\int_{-g(z_1)}^0 \partial_{z_2} \tilde{\psi}(z) dz_2 \right) dz_1 + c^* \int_{\omega} \int_{-g(z_1)}^0 \partial_{z_2} \tilde{\psi}(z) dz_2 dz_1 = 0,$$

and integrating with respect to z_2 , by taking into account that $\tilde{\phi}(z_1, 1) = \tilde{\psi}(z_1, -g(z_1)) = 0$, we get

$$- \int_{\omega} p^1(z_1) \tilde{\phi}(z_1, 0) dz_1 + \int_{\omega} p^2(z_1) \tilde{\psi}(z_1, 0) dz_1 + c^* \int_{\omega} \tilde{\psi}(z_1, 0) dz_1 = 0,$$

and taking into account that $\tilde{\phi} = \tilde{\psi} = \tilde{\varphi}$ on Σ , then we deduce

$$- \int_{\omega} p^1(z_1) \tilde{\varphi}(z_1) dz_1 + \int_{\omega} (p^2(z_1) + c^*) \tilde{\varphi}(z_1) dz_1 = 0,$$

for any $\tilde{\varphi} \in \mathcal{D}(\omega)$, which implies that equation (6.98) holds. □

We have already proved the convergence of \tilde{P}_ε^1 to p^1 in Ω and \tilde{P}_ε^2 to p^2 in \tilde{I}_1 . Let us define the following pressure in D by

$$p^* = \begin{cases} p^1 & \text{in } \Omega, \\ p^2 + c^* & \text{in } \tilde{I}_1. \end{cases} \quad (6.107)$$

Next, we give the main result of this paper.

Theorem 6.3. *We assume that the parameters $\varepsilon, \eta_\varepsilon$ and h_ε satisfy (2.5) and (3.56). Then, the asymptotic pressure p^* defined in (6.107) is the unique solution of the generalized Reynolds equation:*

Find $p^* \in W^{1,r'}(\omega) \cap L_0^{r'}(\omega)$ such that

$$\int_{\omega} \left(\frac{\mu}{\nu^{r'-1}} + \frac{g(z_1)^{r'}}{\lambda 2^{\frac{r'}{2}} (r'+1) \nu^{r'-1}} \right) \left| f_1(z_1) - \frac{d}{dz_1} p^*(z_1) \right|^{r'-2} \left(f_1(z_1) - \frac{d}{dz_1} p^*(z_1) \right) \frac{d}{dz_1} \psi dz_1 = 0, \quad (6.108)$$

for every $\psi \in W^{1,r'}(\omega)$.

Moreover, the average velocity field in the free media is given by

$$\begin{cases} \mathcal{V}_{av,1}(z_1) = \frac{g(z_1)^{r'}}{2^{\frac{r'}{2}}(r'+1)\nu^{r'-1}} \left| f_1(z_1) - \frac{d}{dz_1} p^*(z_1) \right|^{r'-2} \left(f_1(z_1) - \frac{d}{dz_1} p^*(z_1) \right) \\ \mathcal{V}_{av,2} \equiv 0 \end{cases} \quad \text{in } \omega,$$

and the average velocity field in the porous media is given by

$$\begin{cases} V_{av,1}(z_1) = \frac{\mu}{\nu^{r'-1}} \left| f_1(z_1) - \frac{d}{dz_1} p^*(z_1) \right|^{r'-2} \left(f_1(z_1) - \frac{d}{dz_1} p^*(z_1) \right) \\ V_{av,2} \equiv 0, \end{cases} \quad \text{in } \omega,$$

with $\mu > 0$ defined by

$$\mu = \int_{Y_f} |\mathbb{D}_y[\widehat{w}]|^r dy, \quad (6.109)$$

where $(\widehat{w}, \widehat{q}) \in W_{\text{per}}^{1,r}(Y)^2 \times L'_{0,\text{per}}(Y)$, $1 < r < +\infty$, is the unique solution of the auxiliary problem

$$\begin{cases} -\text{div}_y (|\mathbb{D}_y[\widehat{w}]|^{r-2} \mathbb{D}_y[\widehat{w}]) + \nabla_y \widehat{q} = e_1 & \text{in } Y_f, \\ \text{div}_y \widehat{w} = 0 & \text{in } Y_f. \\ \widehat{w} = 0 & \text{in } T. \end{cases} \quad (6.110)$$

Proof. All the results presented in the theorem are consequences of the previous results. In particular, from equation (6.96), (6.98) and (6.107), we obtain the variational formulation for the limit pressure (6.108).

Let us prove that the uniqueness of solution up to an additive constant of (6.108). The proof relies on standard monotonicity arguments. Let us first introduce some notation and properties. Thanks to (2.11), we have

$$G(z_1) := \frac{\mu}{\nu^{r'-1}} + \frac{g(z_1)^{r'}}{\lambda 2^{\frac{r'}{2}}(r'+1)\nu^{r'-1}} \geq \frac{\mu}{\nu^{r'-1}} + \frac{a^{r'}}{\lambda 2^{\frac{r'}{2}}(r'+1)\nu^{r'-1}} := C_{\mu,\nu,r'}^{\lambda,a}, \quad (6.111)$$

with $C_{\mu,\nu,r'}^{\lambda,a} > 0$ a constant. For $1 < r' < +\infty$, we define the r' -Laplace operator $A_{r'}(\xi) = |\xi|^{r'-2}\xi$, $\forall \xi \in \mathbb{R}$, which is strongly monotone in the following sense (see for instance [16, 27]):

- If $1 < r' < 2$, then there exists $\alpha_1 > 0$ such that

$$\int_{\omega} (A_{r'}(u) - A_{r'}(v))(u - v) dz_1 \geq \alpha_1 \frac{\|u - v\|_{L^{r'}(\omega)}^2}{(\|u\|_{L^{r'}(\omega)} + \|v\|_{L^{r'}(\omega)})^{2-r'}}, \quad (6.112)$$

- If $r' \geq 2$, then there exists $\alpha_2 > 0$ such that

$$\int_{\omega} (A_{r'}(u) - A_{r'}(v))(u - v) dz_1 \geq \alpha_2 \|u - v\|_{L^{r'}(\omega)}^{r'}. \quad (6.113)$$

Let us now suppose that (6.108) has two solutions $p, q \in W^{1,r'}(\omega)$. Then, subtracting the corresponding variational equations and taking $\psi(z_1) = q(z_1) - p(z_1)$ as test function, we get

$$\int_{\omega} G(z_1) \left\{ A_{r'} \left(f_1(z_1) - \frac{d}{dz_1} p(z_1) \right) - A_{r'} \left(f_1(z_1) - \frac{d}{dz_1} q(z_1) \right) \right\} \frac{d}{dz_1} (q(z_1) - p(z_1)) dz_1 = 0,$$

or, equivalently,

$$\int_{\omega} G(z_1) \left\{ A_{r'} \left(f_1(z_1) - \frac{d}{dz_1} p(z_1) \right) - A_{r'} \left(f_1(z_1) - \frac{d}{dz_1} q(z_1) \right) \right\} \left\{ \left(f_1 - \frac{d}{dz_1} p \right) - \left(f_1 - \frac{d}{dz_1} q \right) \right\} dz_1 = 0.$$

Taking into account (6.111), it holds

$$\begin{aligned} & C_{\mu,\nu,r'}^{\lambda,a} \int_{\omega} \left\{ A_{r'} \left(f_1 - \frac{d}{dz_1} p \right) - A_{r'} \left(f_1 - \frac{d}{dz_1} q \right) \right\} \left\{ \left(f_1 - \frac{d}{dz_1} p \right) - \left(f_1 - \frac{d}{dz_1} q \right) \right\} dz_1 \\ & \leq \int_{\omega} G(z_1) \left\{ A_{r'} \left(f_1 - \frac{d}{dz_1} p \right) - A_{r'} \left(f_1 - \frac{d}{dz_1} q \right) \right\} \left\{ \left(f_1 - \frac{d}{dz_1} p \right) - \left(f_1 - \frac{d}{dz_1} q \right) \right\} dz_1 = 0, \end{aligned}$$

and then,

$$C_{\mu,\nu,r'}^{\lambda,a} \int_{\omega} \left\{ A_{r'} \left(f_1 - \frac{d}{dz_1} p \right) - A_{r'} \left(f_1 - \frac{d}{dz_1} q \right) \right\} \left\{ \left(f_1 - \frac{d}{dz_1} p \right) - \left(f_1 - \frac{d}{dz_1} q \right) \right\} dz_1 \leq 0. \quad (6.114)$$

By respectively using the monotonicity properties (6.112) for $1 < r' < 2$ and (6.113) for $r' \geq 2$ applied to the left-hand side of (6.114), we deduce for $r' \in (1, +\infty)$ that

$$\left\| \frac{d}{dz_1} (p - q) \right\|_{L^{r'}(\omega)} = 0.$$

Thus, we get that $\partial_{z_1}(p(z_1) - q(z_1)) = 0$ in ω and so, the solution of (6.108) is only determined up to an additive constant. Taking into account (6.107) and Lemma 4.6, which says that $p^1 \in L_0^{r'}(\omega)$, a supplementary constraint has to be added to obtain p^* , namely $p^* \in W^{1,r'}(\omega) \cap L_0^{r'}(\omega)$ is the unique solution of (6.108).

Finally, we just observe that multiplying equation (6.110) by \widehat{w} , integrating in Y_f and taking into account that $\widehat{w} = 0$ on T , we deduce that the permeability constant defined by (4.78) also satisfies

$$\mu = \int_{Y_f} \widehat{w} \cdot e_1 dy = \int_{Y_f} |\mathbb{D}_y[\widehat{w}]|^r dy,$$

which is (6.109). □

7 Conclusions

Main result. In this paper, we consider an incompressible viscous stationary 2D non-Newtonian fluid in a domain composed by two parts in contact: a periodic thin porous medium Ω_ε with characteristic size of the pores $0 < \varepsilon \ll 1$ and thickness of the domain $0 < h_\varepsilon \ll 1$, and a thin film I_ε with thickness $0 < \eta_\varepsilon \ll 1$, where h_ε and η_ε are devoted to zero when $\varepsilon \rightarrow 0$. The interface between Ω_ε and I_ε is defined by $\Sigma = \omega \times \{x_2 = 0\}$. More precisely, we consider the case of a non-Newtonian fluid governed by the

incompressible Stokes equations with power law viscosity of flow index $r \in (1, +\infty)$, and we prove that there exists a critical regime between these parameters given by

$$h_\varepsilon \approx \eta_\varepsilon^{\frac{2r-1}{r-1}} \varepsilon^{-\frac{r}{r-1}}, \quad \text{i.e.} \quad \frac{h_\varepsilon}{\eta_\varepsilon^{\frac{2r-1}{r-1}} \varepsilon^{-\frac{r}{r-1}}} \rightarrow \lambda \in (0, +\infty),$$

where the pressure has the same order of magnitude in the porous medium and in the free film (with a continuity relation of their limits through Σ) and is described by a modified Reynolds equation, coupling the effects of the thin porous medium (1D nonlinear Darcy problem with permeability $\mu > 0$ given by (6.109)) and the thin film (1D nonlinear Reynolds problem), which given by

$$\begin{cases} -\frac{d}{dz_1} \left[\left(\frac{\mu}{\nu^{r'-1}} + \frac{g(z_1)^{r'}}{\lambda 2^{\frac{r'}{2}} (r'+1) \nu^{r'-1}} \right) \left| f_1(z_1) - \frac{d}{dz_1} p^*(z_1) \right|^{r'-2} \left(f_1(z_1) - \frac{d}{dz_1} p^*(z_1) \right) \right] = 0 & \text{in } \omega, \\ \left(\frac{\mu}{\nu^{r'-1}} + \frac{g(z_1)^{r'}}{\lambda 2^{\frac{r'}{2}} (r'+1) \nu^{r'-1}} \right) \left| f_1(z_1) - \frac{d}{dz_1} p^*(z_1) \right|^{r'-2} \left(f_1(z_1) - \frac{d}{dz_1} p^*(z_1) \right) n = 0 & \text{on } \partial\omega, \end{cases}$$

where $\nu > 0$ is the consistency of the fluid, r' is the conjugate exponent of r satisfying $1/r + 1/r' = 1$, function f_1 is the external force and function g is such that its graph defines the lower boundary of the thin film (both functions defined in ω).

Novelties in the techniques. We point out that the version of the unfolding method and the restriction operator, introduced in this paper to study the asymptotic behavior of the fluid in the thin porous medium Ω_ε , are powerful tools that could be used to derive lower-dimensional macroscopic laws for different type of (two dimensional or three dimensional) non-Newtonian fluids in a thin porous medium.

Future improvements. Using the present study as a starting point, various improvements can be proposed. The first one is the generalization of the asymptotic study, which leads to the coupled Darcy–Reynolds equation, to a truly (stationary or non-stationary) nonlinear 3D Navier-Stokes system (and not only Stokes system). Another possible way is regarding the boundary conditions. To avoid technical difficulties connected with non-homogeneous boundary conditions for velocity (or pressure in some cases), we have considered a flow with no-slip condition on the exterior boundary of the domain. To derive a more general limit problem, we remark that, with some technical efforts, this model could be adapted to periodic boundary conditions on the lateral boundaries, to the case of a non-Newtonian fluid with injection as in [36], or to stress (Neumann) boundary condition on the lateral boundary as in [28, 29, 30].

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Conflict of interest

The authors confirm that there is no conflict of interest to report.

Data availability statement

No new data/code were created or analyzed in this study.

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