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# PARABOLIC RESTRICTIONS AND DOUBLE DEFORMATIONS OF WEIGHT MULTIPLICITIES

CÉDRIC LECOUEVEY

ABSTRACT. We introduce some  $(p, q)$ -deformations of the weight multiplicities for the representations of any simple Lie algebra  $\mathfrak{g}$  over the complex numbers. This is done by associating the indeterminate  $q$  to the positive roots of a parabolic subsystem of  $\mathfrak{g}$  and the indeterminate  $p$  to the remaining positive roots. When  $p = q$ , we so recover the usual Lusztig analogues of weight multiplicities. We then study the positivity of the coefficients in these double deformations and prove in particular a conjecture by Lee.

## 1. INTRODUCTION

In this note, we introduce a double deformation of the weight multiplicities for the representations of any simple Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$  depending of two indeterminates  $p$  and  $q$ . The deformation is constructed from the  $(p, q)$ -Kostant partition function defined by the series

$$\prod_{\alpha \in R_+ \setminus \overline{R}_+} \frac{1}{1 - pe^\alpha} \prod_{\alpha \in \overline{R}_+} \frac{1}{1 - qe^\alpha} = \sum_{\beta \in Q_+} P_{p,q}(\beta) e^\beta.$$

Here  $R_+$  is the set of positive roots of  $\mathfrak{g}$  and  $\overline{R}_+$  the set of positive roots corresponding to a parabolic subsystem, that is the set of positive roots associated to a Levi subalgebra  $\overline{\mathfrak{g}}$  of  $\mathfrak{g}$ . For any dominant weight  $\nu$  and any weight  $\mu$  ( $\nu$  and  $\mu$  are weights for  $\mathfrak{g}$ ), we so get polynomials  $K_{\nu,\mu}(p, q)$  in  $\mathbb{Z}[p, q]$  such that  $K_{\nu,\mu}(q, q)$  coincides with the ordinary Lusztig  $q$ -analogue [15]. In Section 2, we give a decomposition of the polynomial  $K_{\nu,\mu}(p, q)$  in terms of  $p$ -branching multiplicities corresponding to the restriction from  $\mathfrak{g}$  to  $\overline{\mathfrak{g}}$  and ordinary Lusztig  $q$ -weight multiplicities for  $\overline{\mathfrak{g}}$  (Theorem 3.1). This permits to study the positivity of the coefficients of the polynomials  $K_{\nu,\mu}(p, q)$  in Section 3 where we establish that  $K_{\nu,\mu}(1, q)$  and  $K_{\nu,\mu}(0, q)$  belong to  $\mathbb{N}[q]$  when  $\nu$  and  $\mu$  are dominant weights. This proves in particular Conjecture 1.4 proposed by Lee in [14]. More generally, we also show that one can relax the hypothesis for  $\mu$  to be dominant and prove, by using arguments due to Panyushev [18], that the polynomial  $K_{\nu,\mu}(p+1, q+1)$  then belongs to  $\mathbb{N}[p, q]$  (Theorem 4.3). In Section 4, we get some positivity results for the stabilized versions of the polynomials  $K_{\nu,\mu}(p, q)$ . In the classical types and for the ordinary Lusztig  $q$ -analogues, such stabilized versions were defined in [10] and [11] and proved to coincide with one-dimensional sums in [13]. The stabilization was then obtained by translating the dominant weights  $\nu$  and  $\mu$  by a multiple of the fundamental weight  $\omega_n$ . Here we generalize the construction in both directions: first this is done for our two parameters deformations and next we get stabilized polynomials from translations by an arbitrary sum of distinct fundamental weights (Theorem 5.3). Finally, in Section 5, we propose some  $(p, q)$ -analogues of Hall-Littlewood polynomials and examine the question of the combinatorial description of the double deformation  $K_{\nu,\mu}(p, q)$  of weight multiplicities.

## 2. DOUBLE DEFORMATION OF WEIGHT MULTIPLICITIES

In this Section, we recall some classical results on the representation theory of the Lie algebras over  $\mathbb{C}$ . We refer the reader to [1], [4], or [7] for a detailed exposition. Consider such a finite-dimensional simple algebra  $\mathfrak{g}$  and assume its root system  $R$  is realized in the Euclidean space  $E$ . Its Dynkin diagram is indexed by  $I$  and we denote as usual by

- $W$  its Weyl group,
- $R_+$  and  $S$  its subsets of positive and simple roots,
- $Q$  its root lattice and  $Q_+$  the cone generated by the positive roots,
- $P$  its weight lattice and  $P_+$  the cone of dominant weights generated by the fundamental weights  $\omega_i, i \in I$ ,
- $V(\nu)$  the simple  $\mathfrak{g}$ -module of highest weight  $\nu \in P_+$ ,
- $\langle \cdot, \cdot \rangle$  the usual Euclidean scalar product on  $E$ ,
- $\leq$  the dominance order on  $P$  such that  $\gamma \leq \mu$  if and only if  $\mu - \gamma \in Q_+$ .

Now, let  $\bar{I}$  be a subset of  $I$  and denote by  $\bar{R}$  the associated parabolic root system with Weyl group  $\bar{W}$ , set of positive roots  $\bar{R}_+$  and set of dominant weights  $\bar{R}_+$ . Recall that each right coset  $\bar{W}w$  admits a unique minimal length element and write  $U$  for the set of these minimal length elements. Then each  $w$  in  $W$  decomposes in a unique way on the form  $W = \bar{w}u$  with  $u \in U$  and  $\bar{w} \in \bar{W}$ . To each  $\bar{\lambda} \in \bar{P}_+$  is associated a simple  $\bar{\mathfrak{g}}$ -module  $\bar{V}(\bar{\lambda})$ .

Let  $\bar{E} = \text{span}(\bar{R}_+)$  the linear real subspace of  $E$  generated by the elements in  $\bar{R}_+$  and set  $\bar{E}^\perp = \{x \in E \mid \langle x, y \rangle = 0 \ \forall y \in \bar{E}\}$  the orthogonal of  $\bar{E}$  in  $E$ . Each vector  $v$  in  $E$  decomposes uniquely on the form

$$(1) \quad v = \bar{v} + v^\perp$$

where  $\bar{v} \in \bar{E}$  and  $v^\perp \in \bar{E}^\perp$ . Observe that the elements of  $\bar{E}^\perp$  are fixed under the action of  $\bar{W}$ . Also since  $\bar{E}$  is stabilized by  $\bar{W}$ , we must have  $\bar{w}(R_+ \setminus \bar{R}_+) = R_+ \setminus \bar{R}_+$ , that is the set  $R_+ \setminus \bar{R}_+$  is invariant under the action of  $\bar{W}$ . Nevertheless, each root in  $R_+ \setminus \bar{R}_+$  is not fixed by the action of  $\bar{W}$ : the action of an element of  $\bar{W}$  will permute the roots in  $R_+ \setminus \bar{R}_+$ .

Consider  $q$  and  $p$  two formal indeterminates and introduce the  $p, q$ -Kostant partitions  $P_{p,q}$  defined from the expansion of the formal series

$$(2) \quad \prod_{\alpha \in R_+ \setminus \bar{R}_+} \frac{1}{1 - pe^\alpha} \prod_{\alpha \in \bar{R}_+} \frac{1}{1 - qe^\alpha} = \sum_{\beta \in Q_+} P_{p,q}(\beta) e^\beta.$$

When  $p = q$ , this is the ordinary  $q$ function partition (see [8]). Given two dominant weights  $\nu$  and  $\mu$  in  $P_+$ , set

$$(3) \quad K_{\nu,\mu}(p, q) = \sum_{w \in W} \varepsilon(w) P_{p,q}(w(\lambda + \rho) - \mu - \rho)$$

where  $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$ . When  $p = q$ , the polynomial  $K_{\nu,\mu}(q, q)$  is an ordinary Lusztig  $q$ -analogue.

**Remark 2.1.** We have  $w(\lambda + \rho) - \mu - \rho = w(\lambda + \rho) - (\lambda + \rho) + (\lambda - \mu)$  and  $(\lambda + \rho) - w(\lambda + \rho)$  belongs to  $Q_+$ . Also for any weight  $\gamma \in P$ , we also have  $P_{p,q}(\gamma) \neq 0$  only if  $\gamma \in Q_+$ . Therefore, the polynomial  $K_{\nu,\mu}(p, q)$  is nonzero only if  $\mu \leq \lambda$ .

Let us now define the partition function  $\hat{P}_p$  from the expansion of the formal series

$$\prod_{\alpha \in R_+ \setminus \bar{R}_+} \frac{1}{1 - pe^\alpha} = \sum_{\gamma \in \hat{Q}_+} \hat{P}_p(\gamma) e^\gamma$$

where  $\hat{Q}_+ = \sum_{\alpha \in R_+ \setminus \bar{R}_+} \mathbb{N}\alpha$ . Since we have  $\bar{w}(R_+ \setminus \bar{R}_+) = R_+ \setminus \bar{R}_+$  for any  $\bar{w} \in \bar{W}$ , the cone  $\hat{Q}_+$  is also stable under the action of  $\bar{W}$ . Moreover, for any  $\bar{w} \in \bar{W}$  and any  $\gamma \in \hat{Q}_+$ , we have  $\hat{P}_p(\bar{w}(\gamma)) = \hat{P}_p(\gamma)$ . Given a dominant weight  $\nu \in P_+$  and a parabolic dominant weight  $\bar{\lambda} \in \bar{P}_+$  define the polynomial

$$b_{\nu, \bar{\lambda}}(p) = \sum_{w \in W} \varepsilon(w) \hat{P}_p(w(\nu + \rho) - \bar{\lambda} - \rho).$$

Then, it is classical to deduce from the Weyl character formula (see for example [5]) that  $b_{\nu, \bar{\lambda}}(1)$  gives the multiplicity of  $\bar{V}(\bar{\lambda})$  in the restriction of  $V(\nu)$  from  $\mathfrak{g}$  to  $\bar{\mathfrak{g}}$ . Observe that in general, the polynomial  $b_{\nu, \bar{\lambda}}(p)$  does not belong to  $\mathbb{N}[p]$ . We nevertheless mention the following conjecture which, as far as we are aware, is only partially proved (in particular by Broer [2] in type  $A$ ). We also refer to [19], [20] and the references therein for more details and results on this problem.

**Conjecture 2.2.** *The polynomial  $b_{\nu, \bar{\lambda}}(p)$  has nonnegative integer coefficients at least when  $\bar{\lambda} \in P_+$ , that is when  $\bar{\lambda}$  is a dominant weight for  $\mathfrak{g}$  (and not only for  $\bar{\mathfrak{g}}$ ).*

Write  $\bar{P}_q$  for the usual partition function associated to the root system  $\bar{R}$  defined by

$$\prod_{\alpha \in \bar{R}_+} \frac{1}{1 - qe^\alpha} = \sum_{\delta \in \bar{Q}_+} \bar{P}_q(\delta) e^\delta.$$

Given two dominant weights  $\bar{\lambda}$  and  $\bar{\mu}$  in  $\bar{P}_+$ , we can consider the Lusztig  $q$ -analogue for the subalgebra  $\bar{\mathfrak{g}}$

$$\bar{K}_{\bar{\lambda}, \bar{\mu}}(q) = \sum_{\bar{w} \in \bar{W}} \varepsilon(\bar{w}) \bar{P}_q(\bar{w}(\bar{\lambda} + \bar{\rho}) - \bar{\mu} - \bar{\rho}).$$

In fact, we can enlarge the definition by replacing the dominant weight  $\bar{\lambda}$  by any weight  $\bar{\kappa} \in \bar{P}$ . Then we get  $\bar{K}_{\bar{\kappa}, \bar{\mu}}(q) = 0$  or there exists  $\bar{w}$  in  $\bar{W}$  and  $\bar{\lambda} \in \bar{P}_+$  such that

$$\bar{w}^{-1}(\bar{\lambda} + \bar{\rho}) - \bar{\rho} = \bar{\kappa} \text{ and } \bar{K}_{\bar{\kappa}, \bar{\mu}}(q) = \varepsilon(\bar{w}) \bar{K}_{\bar{\lambda}, \bar{\mu}}(q).$$

Finally, by elementary properties of formal series, we must have for any  $\beta \in Q_+$

$$(4) \quad P_{p,q}(\beta) = \sum_{\gamma \in \hat{Q}_+, \delta \in \bar{Q}_+ | \gamma + \delta = \beta} \hat{P}_p(\gamma) \bar{P}_q(\delta) = \sum_{\gamma \in \hat{Q}_+} \hat{P}_p(\gamma) \bar{P}_q(\beta - \gamma).$$

We also recall the Weyl character formula. For each dominant weight  $\nu$  in  $P_+$ , the character of  $V(\nu)$  is the polynomial in  $\mathbb{Z}^W[P]$  such that

$$s_\nu = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\nu + \rho) - \rho}}{\prod_{\alpha \in R_+} (1 - e^{-\alpha})}.$$

In particular for  $\nu = 0$ , we get the Weyl denominator formula

$$\prod_{\alpha \in R_+} (1 - e^{-\alpha}) = \sum_{w \in W} \varepsilon(w) e^{w(\rho) - \rho}.$$

We shall also use the notation  $a_\gamma = \sum_{w \in W} \varepsilon(w) e^{w(\gamma)}$  for any  $\gamma \in P$ . Then  $s_\lambda = \frac{a_{\lambda + \rho}}{a_\rho}$ .

## 3. DECOMPOSITION OF THE DOUBLE DEFORMATION

In this section, we establish a decomposition theorem for our double deformations of weight multiplicities. When  $p = q = 1$ , it just means that one can obtain the restriction of any representation of  $\mathfrak{g}$  to its Cartan subalgebra  $\mathfrak{h}$  by performing first its restriction to  $\overline{\mathfrak{g}}$  and next the restriction from  $\overline{\mathfrak{g}}$  to  $\mathfrak{h}$ .

**Theorem 3.1.** *For any dominant weights  $\lambda$  and  $\mu$  in  $P_+$  we have the decomposition*

$$K_{\nu,\mu}(p, q) = \sum_{\overline{\lambda} \in \overline{P}_+} b_{\nu, \overline{\lambda} + \mu^\perp}(p) \overline{K}_{\overline{\lambda}, \overline{\mu}}(q)$$

where  $\overline{\lambda} + \mu^\perp$  belongs to  $\overline{P}_+$ .

*Proof.* By definition of the polynomial  $K_{\nu,\mu}(p, q)$  and by using (4), we obtain

$$K_{\nu,\mu}(p, q) = \sum_{u \in U} \varepsilon(u) \sum_{\gamma \in \hat{Q}_+} \hat{P}_p(\gamma) \sum_{\overline{w} \in \overline{W}} \varepsilon(\overline{w}) \overline{P}_q(\overline{w}u(\nu + \rho) - \mu - \rho - \gamma)$$

where we have decomposed each  $w$  in  $W$  on the form  $w = \overline{w}u$  with  $u \in U$  and  $\overline{w} \in \overline{W}$ . We can now apply decompositions of type (1) to get  $\rho = \overline{\rho} + \rho^\perp$  and  $\rho = \overline{\mu} + \mu^\perp$ . By using that  $\overline{w}(\rho^\perp) = \rho^\perp$  and  $\overline{w}(\mu^\perp) = \mu^\perp$ , we obtain

$$\begin{aligned} \overline{w}(u(\nu + \rho)) - \mu - \rho - \gamma &= \overline{w}[u(\nu + \rho) - \mu^\perp - \rho^\perp - \overline{w}^{-1}(\gamma)] - \overline{\rho} - \overline{\mu} = \\ &= \overline{w}[(u(\nu + \rho) - \mu^\perp - \rho^\perp - \overline{w}^{-1}(\gamma) - \overline{\rho}) + \overline{\rho}] - \overline{\rho} - \overline{\mu} = \\ &= \overline{w}[(u(\nu + \rho) - \mu^\perp - \rho - \overline{w}^{-1}(\gamma)) + \overline{\rho}] - \overline{\rho} - \overline{\mu}. \end{aligned}$$

This gives

$$\overline{P}_q(\overline{w}u(\nu + \rho) - \mu - \rho - \gamma) = \overline{P}_q(\overline{w}[(u(\nu + \rho) - \rho - \mu^\perp - \overline{w}^{-1}(\gamma)) + \overline{\rho}] - \overline{\rho} - \overline{\mu}).$$

Therefore, we have

$$\sum_{\overline{w} \in \overline{W}} \varepsilon(\overline{w}) \overline{P}_q(\overline{w}u(\nu + \rho) - \mu - \rho - \gamma) = \overline{K}_{u(\nu + \rho) - \rho - \mu^\perp - \overline{w}^{-1}(\gamma), \overline{\mu}}(q).$$

This gives

$$K_{\nu,\mu}(p, q) = \sum_{u \in U} \varepsilon(u) \sum_{\gamma \in \hat{Q}_+} \hat{P}_p(\gamma) \overline{K}_{u(\nu + \rho) - \rho - \mu^\perp - \overline{w}^{-1}(\gamma), \overline{\mu}}(q).$$

Now, since  $\hat{Q}_+$  is stable under the action of  $\overline{W}$ , we can set  $\gamma' = \overline{w}^{-1}(\gamma)$  in the second sum and get

$$K_{\nu,\mu}(p, q) = \sum_{u \in U} \varepsilon(u) \sum_{\gamma' \in \hat{Q}_+} \hat{P}_p(\gamma') \overline{K}_{u(\nu + \rho) - \rho - \mu^\perp - \gamma', \overline{\mu}}(q).$$

for  $\hat{P}_p(\gamma') = \hat{P}_p(\gamma)$ . For each nonzero polynomial  $\overline{K}_{u(\nu + \rho) - \rho - \mu^\perp - \gamma', \overline{\mu}}(q)$  there exists  $\overline{w}$  in  $\overline{W}$  and  $\overline{\lambda} \in \overline{P}_+$  such that

$$\overline{w}^{-1}(\overline{\lambda} + \overline{\rho}) - \overline{\rho} = u(\nu + \rho) - \rho - \mu^\perp - \gamma' \text{ and } \overline{K}_{u(\nu + \rho) - \rho - \mu^\perp - \gamma', \overline{\mu}}(q) = \varepsilon(\overline{w}) \overline{K}_{\overline{\lambda}, \overline{\mu}}(q).$$

We then have

$$\gamma' = u(\nu + \rho) - \rho - \mu^\perp - \overline{w}^{-1}(\overline{\lambda} + \overline{\rho}) + \overline{\rho} = u(\nu + \rho) - \rho^\perp - \mu^\perp - \overline{w}^{-1}(\overline{\lambda} + \overline{\rho}).$$

By gathering all the contributions corresponding to the same dominant weight  $\bar{\lambda}$ , we thus obtain

$$K_{\nu,\mu}(p, q) = \sum_{\bar{\lambda} \in \bar{P}_+} \left( \sum_{u \in U} \varepsilon(u) \sum_{\bar{w} \in \bar{W}} \varepsilon(\bar{w}) \hat{P}_p \left( u(\nu + \rho) - \rho^\perp - \mu^\perp - \bar{w}^{-1}(\bar{\lambda} + \bar{\rho}) \right) \right) \bar{K}_{\bar{\lambda}, \bar{\mu}}(q).$$

Recall that for any  $\eta \in \hat{Q}_+$  and any  $\bar{w} \in \bar{W}$ , we have  $\hat{P}_p(\bar{w}(\eta)) = \hat{P}_p(\eta)$ . We also have  $\bar{w}(\rho^\perp) = \rho^\perp$  and  $\bar{w}(\mu^\perp) = \mu^\perp$  and thus

$$\begin{aligned} \hat{P}_p \left( u(\nu + \rho) - \rho^\perp - \mu^\perp - \bar{w}^{-1}(\bar{\lambda} + \bar{\rho}) \right) &= \hat{P}_p \left( \bar{w}u(\nu + \rho) - \rho^\perp - \mu^\perp - \bar{\lambda} - \bar{\rho} \right) \\ &= \hat{P}_p \left( \bar{w}u(\nu + \rho) - \mu^\perp - \bar{\lambda} - \rho \right). \end{aligned}$$

This permits to write

$$\begin{aligned} (5) \quad K_{\nu,\mu}(p, q) &= \sum_{\bar{\lambda} \in \bar{P}_+} \left( \sum_{u \in U} \sum_{\bar{w} \in \bar{W}} \varepsilon(\bar{w}u) \hat{P}_p \left( \bar{w}u(\nu + \rho) - \mu^\perp - \bar{\lambda} - \rho \right) \right) \bar{K}_{\bar{\lambda}, \bar{\mu}}(q) = \\ &= \sum_{\bar{\lambda} \in \bar{P}_+} \left( \sum_{w \in W} \varepsilon(w) \hat{P}_p \left( w(\nu + \rho) - \mu^\perp - \bar{\lambda} - \rho \right) \right) \bar{K}_{\bar{\lambda}, \bar{\mu}}(q) = \sum_{\bar{\lambda} \in \bar{P}_+} b_{\nu, \bar{\lambda} + \mu^\perp}(p) \bar{K}_{\bar{\lambda}, \bar{\mu}}(q) \end{aligned}$$

as claimed. It is straightforward to check that  $\bar{\lambda} + \mu^\perp$  belongs to  $\bar{P}_+$  since  $\mu^\perp$  belongs to  $\bar{E}^\perp$  and thus  $\langle \bar{\lambda} + \mu^\perp, \alpha_i \rangle = \langle \bar{\lambda}, \alpha_i \rangle \geq 0$  for any  $i \in \bar{I}$ .  $\square$

#### 4. SOME POSITIVITY RESULTS ON THE DOUBLE DEFORMATION

We examine in this section, the nonnegativity of the coefficients of the polynomials  $K_{\nu,\mu}(p, q)$  whose coefficients belong a priori to  $\mathbb{Z}$ . We start by the following easy corollary of our Theorem 3.1.

**Corollary 4.1.** *Consider  $\lambda$  and  $\mu$  two dominant weights for  $\mathfrak{g}$ . We have*

$$\begin{aligned} K_{\nu,\mu}(1, q) &= \sum_{\bar{\lambda} \in \bar{P}_+} b_{\nu, \bar{\lambda} + \mu^\perp}(1) \bar{K}_{\bar{\lambda}, \bar{\mu}}(q), \\ K_{\nu,\mu}(0, q) &= \bar{K}_{\bar{\nu}, \bar{\mu}}(0, q), \\ K_{\nu,\mu}(p, 0) &= b_{\nu,\mu}(p), \\ K_{\nu,\mu}(q, q) &= K_{\nu,\mu}(q). \end{aligned}$$

*In particular, the polynomials  $K_{\nu,\mu}(1, q)$ ,  $K_{\nu,\mu}(0, q)$  and  $K_{\nu,\mu}(q, q)$  have nonnegative integer coefficients. The polynomial  $K_{\nu,\mu}(p, 0)$  has nonnegative coefficients in the case where Conjecture 2.2 holds.*

*Proof.* We apply the decomposition in Theorem 3.1 with  $p = 1$ . Then the integers  $b_{\nu, \bar{\lambda} + \mu^\perp}(1)$  are nonnegative because they are branching coefficients and the polynomials  $\bar{K}_{\bar{\lambda}, \bar{\mu}}(q)$  have nonnegative integer coefficients because they are Lusztig  $q$ -analogues for the subalgebra  $\bar{\mathfrak{g}}$ . This gives

$$K_{\nu,\mu}(1, q) = \sum_{\bar{\lambda} \in \bar{P}_+} b_{\nu, \bar{\lambda} + \mu^\perp}(1) \bar{K}_{\bar{\lambda}, \bar{\mu}}(q) \in \mathbb{N}[q].$$

Now when  $p = 0$ , we have  $\hat{P}_p(\gamma) = \delta_{1,\gamma}$  for any  $\gamma \in \hat{Q}_+$ . Therefore for any  $\bar{\lambda} \in \bar{P}_+$ , we get  $b_{\nu, \bar{\lambda} + \mu^\perp}(0) \neq 0$  if and only if  $\nu = \bar{\lambda} + \mu^\perp$ . This means that  $\bar{\lambda} = \bar{\nu}$  and  $\mu^\perp = \nu^\perp$ . In this case we get the equality  $b_{\nu, \bar{\nu} + \nu^\perp}(0) = b_{\nu, \nu}(0) = 1$ . This therefore gives

$$K_{\nu, \mu}(0, q) = \bar{K}_{\bar{\nu}, \bar{\mu}}(q).$$

When  $q = 0$ , we have  $\bar{K}_{\bar{\lambda}, \bar{\mu}}(0) \neq 0$  only when  $\bar{\lambda} = \bar{\mu}$  and then  $\bar{K}_{\bar{\lambda}, \bar{\mu}}(0) = 1$ . Thus by Theorem 3.1, we get

$$K_{\nu, \mu}(p, 0) = b_{\nu, \bar{\mu} + \mu^\perp}(p) = b_{\nu, \mu}(p)$$

which has nonnegative coefficient in the case where Conjecture 2.2 holds.

The last identity  $K_{\nu, \mu}(q, q) = K_{\nu, \mu}(q)$  follows directly from the definition of our polynomials  $K_{\nu, \mu}(p, q) = K_{\nu, \mu}(q)$ .  $\square$

**Remark 4.2.** When  $p = 1$ ,  $\mathfrak{g}$  is one of the classical Lie algebras of type  $B_n, C_n, D_n$  and  $\bar{\mathfrak{g}}$  its Levi subalgebra of type  $A_{n-1}$  obtained by removing the node  $n$  in the Dynkin diagram, the positivity of the polynomials  $K_{\nu, \mu}(1, q)$  was recently conjecture by Lee in [14].

The following result is obtained by adapting arguments due to Panyushev in [18] for the classical one parameter Lusztig  $q$ -analogue. Set

$$\Delta = \frac{\prod_{\alpha \in R_+} (1 - e^{-\alpha})}{\prod_{\alpha \in R_+ \setminus \bar{R}_+} (1 - (p+1)e^{-\alpha}) \prod_{\alpha \in \bar{R}_+} (1 - (q+1)e^{-\alpha})}$$

**Theorem 4.3.**

- (1) We have  $\Delta = \sum_{\mu \in Q_+} K_{0, \mu}(p+1, q+1)e^\mu$  where each polynomial  $K_{0, \mu}(p+1, q+1)$  belongs to  $\mathbb{N}[p, q]$ .
- (2) Given a dominant weight  $\nu \in P_+$  and any weight  $\mu \in P$ , the polynomial  $K_{\nu, \mu}(p+1, q+1)$ , regarded as a polynomial in  $\mathbb{Z}[p, q]$ , has nonnegative coefficients.

*Proof.* We resume the notation of Section 2. First, there should exist some polynomials  $m_\mu(p+1, q+1) \in \mathbb{Z}[p+1, q+1], \mu \in Q_+$  such that

$$(6) \quad \Delta = \frac{\prod_{\alpha \in R_+ \setminus \bar{R}_+} (1 - e^{-\alpha})}{\prod_{\alpha \in R_+ \setminus \bar{R}_+} (1 - (p+1)e^{-\alpha})} \times \frac{\prod_{\alpha \in \bar{R}_+} (1 - e^{-\alpha})}{\prod_{\alpha \in \bar{R}_+} (1 - (q+1)e^{-\alpha})} = \sum_{\mu \in Q_+} m_\mu(p+1, q+1)e^\mu.$$

Recall that we have

$$\prod_{\alpha \in R_+} (1 - e^{-\alpha}) = \sum_{w \in W} \varepsilon(w) e^{w(\rho) - \rho} \quad \text{and} \quad \prod_{\alpha \in R_+ \setminus \bar{R}_+} \frac{1}{1 - pe^{-\alpha}} \prod_{\alpha \in \bar{R}_+} \frac{1}{1 - qe^{-\alpha}} = \sum_{\beta \in Q_+} P_{p, q}(\beta) e^{-\beta}.$$

This gives

$$\begin{aligned} \Delta &= \sum_{\beta \in Q_+} \sum_{w \in W} \varepsilon(w) P_{p+1, q+1}(\beta) e^{w(\rho) - \rho - \beta} = \\ &= \sum_{\mu \in Q_+} \sum_{w \in W} \varepsilon(w) P_{p+1, q+1}(w(\rho) - \rho - \mu) e^\mu = \sum_{\mu \in Q_+} K_{0, \mu}(p+1, q+1) e^\mu \end{aligned}$$

which permits to conclude that  $m_\mu(p+1, q+1) = K_{0, \mu}(p+1, q+1)$  for any  $\mu \in Q_+$ . We also have for any positive root  $\alpha$

$$\frac{1 - e^{-\alpha}}{1 - (p+1)e^{-\alpha}} = 1 + \sum_{k \geq 0} (p+1)^k e^{(k+1)\alpha}.$$

Therefore the formal expansion of  $\Delta$  should have nonnegative integer coefficients as claimed. Now introduce the generating series

$$u_\nu := \sum_{\mu \in P} K_{\nu, \mu}(p+1, q+1)e^\mu.$$

We have

$$\begin{aligned} u_\nu &= \sum_{\mu \in P} \sum_{w \in W} \varepsilon(w) P_{p+1, q+1}(w(\nu + \rho) - \mu - \rho) e^\mu = \sum_{w \in W} \varepsilon(w) \sum_{\gamma \in P} P_{p+1, q+1}(\gamma) e^{w(\nu + \rho) - \gamma - \rho} = \\ &= \sum_{w \in W} \varepsilon(w) \left( \sum_{\gamma \in P} P_{p+1, q+1}(\gamma) e^{-\gamma} \right) e^{w(\nu + \rho) - \rho} = \frac{e^{-\rho} \sum_{w \in W} \varepsilon(w) e^{w(\nu + \rho)}}{\prod_{\alpha \in R_+ \setminus \bar{R}_+} (1 - (p+1)e^{-\alpha}) \prod_{\alpha \in \bar{R}_+} (1 - (q+1)e^{-\alpha})} \\ &= \Delta \frac{a_{\nu + \rho}}{a_\rho} = \Delta s_\nu \end{aligned}$$

where we set  $\gamma = w(\nu + \rho) - \mu - \rho$  in the second equality. Now we get by using Assertion 1

$$\begin{aligned} u_\nu &= \sum_{\mu \in P} K_{\nu, \mu}(p+1, q+1)e^\mu = \sum_{\beta \in Q_+} K_{0, \beta}(p+1, q+1)e^\beta \times \sum_{\gamma \in P} K_{\nu, \gamma} e^\gamma = \\ &= \sum_{\beta, \gamma} K_{\nu, \gamma} K_{0, \beta}(p+1, q+1)e^{\gamma + \beta} = \sum_{\mu \in P} \sum_{\beta \in Q_+} K_{\nu, \mu - \beta} K_{0, \beta}(p+1, q+1)e^\mu. \end{aligned}$$

This gives

$$(8) \quad K_{\nu, \mu}(p+1, q+1) = \sum_{\beta \in Q_+} K_{\nu, \mu - \beta} K_{0, \beta}(p+1, q+1) \in \mathbb{N}[p]$$

because the coefficients  $K_{\nu, \mu - \beta}$  are nonnegative integers and we have already proved that the polynomials  $K_{0, \beta}(p+1, q+1)$  belongs to  $\mathbb{N}[p, q]$ .  $\square$

## 5. STABILIZED VERSION OF THE DOUBLE DEFORMATION

We resume the notation of Section 2. In particular, we recall the decomposition  $\rho = \bar{\rho} + \rho^\perp$ . Since  $\rho = \sum_{i \in I} \omega_i$  and  $\bar{\rho} = \sum_{i \in \bar{I}} \omega_i$ , we must have

$$\rho^\perp = \sum_{i \in I \setminus \bar{I}} \omega_i.$$

The goal of this section is to prove that, once the two dominant weights  $\nu$  and  $\mu$  are fixed, the sequence polynomials

$$K_{\nu + k\rho^\perp, \mu + k\rho^\perp}(p, q), k \in \mathbb{N}$$

stabilizes and its limit has interesting positivity properties under a natural hypothesis inspired by the classical Lie algebras setting of types  $B_n, C_n, D_n$  and their parabolic restrictions to type  $A_{n-1}$  exploited in [10] and [11]. We will first need the following lemma.

**Lemma 5.1.** *Consider a dominant weight  $\nu$  in  $P_+$  and a weight  $\mu \in P$  such that  $\mu \leq \nu$ . Then, there exists a nonnegative integer  $k_0$  such that*

$$w(\nu + \rho + k\rho^\perp) - (\mu + \rho + k\rho^\perp) \notin Q_+$$

for any  $k \geq k_0$  and any  $w \in W \setminus \bar{W}$ .



*Proof.* Consider  $w \in W \setminus \overline{W}$ . Since  $\rho^\perp$  belongs to  $P_+$  as a sum of fundamental weights, we must have  $\rho^\perp - w(\rho^\perp) \in Q_+$ , that is  $\rho^\perp - w(\rho^\perp)$  is a nonnegative sum of positive roots in  $R_+$ . Moreover,  $w(\rho^\perp) \neq \rho^\perp$  because  $w \in W \setminus \overline{W}$  and the stabilizer of  $\rho^\perp$  under the action of  $W$  should be  $\overline{W} = \langle s_i \mid i \in \overline{I} \rangle$ . Therefore, we can set  $\rho^\perp - w(\rho^\perp) = \beta_w$  with  $\beta_w \in Q_+ \setminus \{0\}$ . Now, we get for any nonnegative integer  $k$

$$w(\nu + \rho + k\rho^\perp) - (\mu + \rho + k\rho^\perp) = w(\nu + \rho) - (\nu + \rho) + (\nu - \mu) - k\beta_w.$$

We have  $w(\nu + \rho) - (\nu + \rho) < 0$ . Moreover,  $\nu - \mu$  does not depend on  $k$  and  $\beta_w > 0$ . There thus exists an integer  $k_w$  such that  $w(\nu + \rho) - (\nu + \rho) + (\nu - \mu) - k\beta_w \notin Q_+$  for  $k \geq k_w$  sufficiently large. We are done by considering  $k_0 = \max\{k_w \mid w \in W\}$  (recall  $W$  is finite).  $\square$

**Proposition 5.2.** *For any  $\nu \in P_+$  and any  $\overline{\lambda} \in \overline{P}_+$ , there exists a nonnegative integer  $k_0$  such that for any  $k \geq k_0$*

$$b_{\nu+k\rho^\perp, \overline{\lambda}+k\rho^\perp}(p) = b_{\lambda+k_0\rho^\perp, \overline{\lambda}+k_0\rho^\perp}(p) = \sum_{\overline{w} \in \overline{W}} \varepsilon(\overline{w}) \hat{P}_p(\overline{w}(\nu + \overline{\rho}) - (\overline{\lambda} + \overline{\rho})).$$

*In particular,  $b_{\nu, \overline{\lambda}}^{\text{stab}}(p) := b_{\nu+k\rho^\perp, \overline{\lambda}+k\rho^\perp}(p)$  does not depend on  $k$  for  $k$  sufficiently large.*

*Proof.* By Lemma 5.1, there exists a nonnegative integer  $k_0$  such that for any  $k \geq k_0$  and any  $w \in W \setminus \overline{W}$ , we have  $w(\nu + \rho + k\rho^\perp) - (\rho + \overline{\lambda} + k\rho^\perp) \notin Q_+$ . Then, we obtain

$$\begin{aligned} b_{\nu+k\rho^\perp, \overline{\lambda}+k\rho^\perp}(p) &= \sum_{w \in W} \varepsilon(w) \hat{P}_p(w(\nu + \overline{\rho} + k\rho^\perp) - (\overline{\lambda} + \overline{\rho} + k\rho^\perp)) = \\ &= \sum_{\overline{w} \in \overline{W}} \varepsilon(\overline{w}) \hat{P}_p(\overline{w}(\nu + \overline{\rho} + k\rho^\perp) - (\overline{\lambda} + \overline{\rho} + k\rho^\perp)) = \sum_{\overline{w} \in \overline{W}} \varepsilon(\overline{w}) \hat{P}_p(\overline{w}(\nu + \overline{\rho}) - (\overline{\lambda} + \overline{\rho})) \end{aligned}$$

because  $\overline{w}(\rho^\perp) = \rho^\perp$  for any  $\overline{w}$  in  $\overline{W}$ .  $\square$

**Theorem 5.3.** *For any dominant weights  $\nu$  and  $\mu$  in  $P_+$  there exists a nonnegative integer  $k_0$  such that, for any  $k \geq k_0$  we have the decomposition*

$$K_{\nu+k\rho^\perp, \mu+k\rho^\perp}(p, q) = \sum_{\overline{\lambda} \in \overline{P}_+} b_{\nu, \overline{\lambda}+\mu^\perp}^{\text{stab}}(p) \overline{K}_{\overline{\lambda}, \overline{\mu}}(q).$$

*Proof.* By using Lemma 5.1 and the definition (3) of  $K_{\nu+k\rho^\perp, \mu+k\rho^\perp}(p, q)$ , we get

$$K_{\nu+k\rho^\perp, \mu+k\rho^\perp}(p, q) = \sum_{\overline{w} \in \overline{W}} \varepsilon(\overline{w}) P_{p, q}(\overline{w}(\lambda + \rho) - \mu - \rho).$$

Indeed, by using the integer  $k_0$  of Lemma 5.1, for  $k \geq k_0$ , we have for any  $w \in W \setminus \overline{W}$

$$w(\lambda + \rho + k\rho^\perp) - \mu - \rho - k\rho^\perp \notin Q_+$$

and then  $P_{p, q}(\overline{w}(\nu + \rho + k\rho^\perp) - \mu - \rho - k\rho^\perp) = 0$ . In the remaining cases

$$\overline{w}(\nu + \rho + k\rho^\perp) - \mu - \rho - \rho^\perp = \overline{w}(\nu + \rho) - \mu - \rho$$

since  $\overline{w}(\rho^\perp) = \rho^\perp$ . Now, we can argue exactly as in the proof of Theorem 3.1 (but by replacing the set  $U$  by  $\{id\}$ ). The equality (5) is then replaced by

$$K_{\nu, \mu}(p, q) = \sum_{\overline{\lambda} \in \overline{P}_+} \left( \sum_{\overline{w} \in \overline{W}} \varepsilon(\overline{w}) \hat{P}_p(\overline{w}(\nu + \rho) - \mu^\perp - \overline{\lambda} - \rho) \right) \overline{K}_{\overline{\lambda}, \overline{\mu}}(q) = \sum_{\overline{\lambda} \in \overline{P}_+} b_{\nu, \overline{\lambda}+\mu^\perp}^{\text{stab}}(p) \overline{K}_{\overline{\lambda}, \overline{\mu}}(q). \quad \square$$

From now on, we shall need the following additional hypothesis.

**Hypothesis 5.4.** We assume in this section that  $\langle \rho^\perp, \alpha \rangle = c$  is constant for any  $\alpha \in R_+ \setminus \overline{R}_+$  (i.e. does not depend on the choice of  $\alpha$  in  $R_+ \setminus \overline{R}_+$ ).

**Remark 5.5.**

- (1) Observe that our condition is in particular satisfied for the classical root system of type  $C_n$  or  $D_n$  when  $\overline{I} = \{1, \dots, n-1\}$  is the set of labels of the parabolic subsystem of type  $A_{n-1}$ . Then for any  $\alpha \in R_+ \setminus \overline{R}_+$ , we have  $\langle \rho^\perp, \alpha \rangle = \langle \omega_n, \alpha \rangle = 2$  in type  $C_n$  and  $\langle \rho^\perp, \alpha \rangle = \langle \omega_n, \alpha \rangle = 1$  in type  $D_n$ .
- (2) It is also satisfied for the parabolic subsystems of type  $B_{n-1}, C_{n-1}$  or  $D_{n-1}$  inside the root systems of type  $B_n$  or  $D_n$ , respectively. We then indeed get  $\langle \rho^\perp, \alpha \rangle = \langle \omega_1, \alpha \rangle = 1$  for any  $\alpha \in R_+ \setminus \overline{R}_+$ .
- (3) Another case where the hypothesis is satisfied is for  $\mathfrak{g}$  of type  $A_{n-1}$  and  $\overline{\mathfrak{g}}$  the Levi subalgebra obtained by removing the  $k$ -th node in the Dynkin diagram of  $\mathfrak{g}$ . In this case, we obtain  $\langle \rho^\perp, \alpha \rangle = \langle \omega_k, \alpha \rangle = 1$  for any  $\alpha \in R_+ \setminus \overline{R}_+$ .
- (4) As in [13], it could be also interesting to extend the stabilization phenomenon to deformations of the Lusztig analogues defined by using weight functions on  $p$  and  $q$  depending on the length of the root considered. When  $p = q$ , this is in particular unavoidable for equating the one-dimensional sums for affine root systems with  $q$ -analogues of weight multiplicities. For simplicity, we do not pursue in this direction here.

Consider  $\gamma$  in  $\hat{Q}_+$  and a decomposition  $\gamma = \sum_{\alpha \in R_+ \setminus \overline{R}_+} a_\alpha \alpha$  where the  $a_\alpha$ 's are nonnegative integers. Then

$$L(\gamma) := \frac{1}{c} \langle \rho^\perp, \gamma \rangle = \frac{1}{c} \sum_{\alpha \in R_+ \setminus \overline{R}_+} a_\alpha \langle \rho^\perp, \alpha \rangle = \sum_{\alpha \in R_+ \setminus \overline{R}_+} a_\alpha$$

by Hypothesis 5.4. This shows that the number of positive roots in  $R_+ \setminus \overline{R}_+$  is imposed by  $L(\gamma)$ . Therefore, we have  $\hat{P}_p(\gamma) = p^{L(\gamma)} \hat{P}_1(\gamma)$  for any  $\gamma$  in  $\hat{Q}_+$ . Similarly, we get from Proposition 5.2

$$b_{\nu, \bar{\lambda}}^{\text{stab}}(q) = \sum_{\overline{w} \in \overline{W}} \varepsilon(\overline{w}) \hat{P}_p(\overline{w}(\nu + \overline{\rho}) - (\bar{\lambda} + \overline{\rho})).$$

For any  $\overline{w} \in \overline{W}$

$$\begin{aligned} L(\gamma) &= \langle \rho^\perp, \overline{w}(\nu + \overline{\rho}) - (\bar{\lambda} + \overline{\rho}) \rangle = \langle \rho^\perp, \overline{w}(\nu + \overline{\rho}) \rangle - \langle \rho^\perp, \bar{\lambda} + \overline{\rho} \rangle = \\ &= \langle \overline{w}^{-1}(\rho^\perp), \nu + \overline{\rho} \rangle - \langle \rho^\perp, \bar{\lambda} + \overline{\rho} \rangle = \langle \rho^\perp, \nu + \overline{\rho} \rangle - \langle \rho^\perp, \bar{\lambda} + \overline{\rho} \rangle = \langle \rho^\perp, \nu - \bar{\lambda} \rangle \end{aligned}$$

because  $\rho^\perp$  is stabilized by  $\overline{W}$ . Therefore

$$b_{\nu, \bar{\lambda}}^{\text{stab}}(p) = p^{L(\nu - \bar{\lambda})} b_{\nu, \bar{\lambda}}^{\text{stab}}(1) \text{ for any } \nu \in P_+ \text{ and any } \bar{\lambda} \in \overline{P}_+.$$

This proves that  $b_{\nu, \bar{\lambda}}^{\text{stab}}(q)$  has nonnegative integer coefficients. We then get the following Corollary of Theorem 5.3.

**Corollary 5.6.** *Under Hypothesis 5.4, for any dominant weights  $\nu$  and  $\mu$ , the polynomial  $K_{\nu, \mu}^{\text{stab}}(p, q)$  belongs to  $\mathbb{N}[p, q]$ .*

## 6. FURTHER DIRECTIONS

**6.1. Some analogues of the Hall-Littlewood polynomials.** Recall that the character ring  $\mathbb{Z}[p, q]^W[P]$  of  $\mathfrak{g}$  can be endowed with a scalar product  $\langle \cdot, \cdot \rangle_{(p, q)}$  such that  $\langle s_\nu, s_\mu \rangle_{(p, q)} = \delta_{\nu, \mu}$  for any dominant weights  $\nu$  and  $\mu$  in  $P_+$ . For any  $\mu$  in  $P_+$ , one can then define the polynomial

$$Q'_\mu(p, q) = \frac{1}{a_\rho} \sum_{w \in W} \varepsilon(w) w \left( \frac{e^{\mu + \rho}}{\prod_{\alpha \in R_+ \setminus \bar{R}_+} (1 - pe^\alpha) \prod_{\alpha \in \bar{R}_+} (1 - qe)} \right).$$

**Proposition 6.1.** *We have*

$$Q'_\mu(p, q) = \sum_{\nu \in P_+ | \mu \leq \nu} K_{\nu, \mu}(p, q) s_\nu.$$

*In particular*  $\{Q'_\mu(p, q) \mid \mu \in P_+\}$  *is a*  $\mathbb{Z}$ -*basis of*  $\mathbb{Z}[p, q]^W[P]$

*Proof.* By using the expansion (2) of the  $(p, q)$ -Kostant partition, we get

$$Q'_\mu(p, q) = \sum_{\beta \in Q_+} \frac{1}{a_\rho} \sum_{w \in W} \varepsilon(w) e^{w(\mu + \rho + \beta)} = \sum_{\beta \in Q_+} P_{p, q}(\beta) \frac{a_{\mu + \beta + \rho}}{a_\rho} = \sum_{\beta \in Q_+} P_{p, q}(\beta) s_{\mu + \beta}.$$

Now, for any  $\beta \in Q_+$  we have  $s_{\mu + \beta} = 0$  or there exists an element  $w$  in  $W$  and a dominant weight  $\nu$  such that  $\nu + \rho = w^{-1}(\mu + \beta) + \rho$  and  $s_{\mu + \beta} = \varepsilon(w) s_\nu$ . Then, we have  $\beta = w(\nu + \rho) - \rho - \mu$ . By gathering the contributions of the same character  $s_\nu$ , one obtains

$$Q'_\mu(p, q) = \sum_{w \in W} \varepsilon(w) P_{p, q}(w(\nu + \rho) - \rho - \mu) s_\nu = \sum_{\nu \in P_+ | \mu \leq \nu} K_{\nu, \mu}(p, q) s_\nu$$

as expected because  $K_{\nu, \mu}(p, q)$  is nonzero only when  $\mu \leq \nu$ .  $\square$

We can then define the basis  $\{P_\nu(p, q) \mid \nu \in P_+\}$  as the dual basis of  $\{Q'_\mu(p, q) \mid \mu \in P_+\}$  for the scalar product  $\langle \cdot, \cdot \rangle_{(p, q)}$ . We then have

$$s_\nu = \sum_{\mu \in P_+} K_{\nu, \mu}(p, q) P_\mu(p, q).$$

When  $p = q$ , the polynomials  $P_\mu(q)$  are the Hall-Littlewood polynomials. They are deeply connected with the theory of affine Hecke algebras (see [17] for a detailed review). In particular, by using the Satake isomorphism, one can prove that the polynomials  $K_{\nu, \mu}(q, q) = K_{\nu, \mu}(q)$  are affine Kazhdan-Lusztig polynomials. For the double deformations introduced here, the connection with the theory of affine Hecke algebras cannot be obtained similarly. There indeed exist unequal parameters affine Hecke algebras constructed from weight functions on the set of positive roots. Nevertheless, these weight functions should be constant on the roots which belong to the same orbit under the action of  $W$ . This is not always the case in our setting since  $\bar{R}_+$  is not stable under the action of  $W$ .

**6.2. Combinatorial description of the double deformation.** A difficult question in the theory of Lusztig  $q$ -analogues of weight multiplicities consists in their combinatorial description. In type  $A_{n-1}$  this was achieved by Lascoux and Schützenberger in [9]. For the classical types, the description is only partially known. We refer the interested reader to [14], [12] and the references therein. In view to Theorem 3.1, the combinatorial description of the polynomials  $K_{\nu, \mu}(p, q)$  reduces to the combinatorial description of the polynomials  $b_{\nu, \bar{\lambda} + \mu^\perp}(p)$  and  $\bar{K}_{\bar{\lambda}, \bar{\mu}}(q)$ . For  $p = 1$ , the branching coefficient  $b_{\nu, \bar{\lambda} + \mu^\perp}(1)$  can be easily described thanks to crystal graph theory or Littelmann path theory since both are compatible with parabolic restrictions. Recall

that to each dominant weight  $\nu$ , is associated a crystal  $B(\nu)$  which can be regarded as the combinatorial skeleton of the representation  $V(\lambda)$ . This is an oriented graph where the arrows are described in terms of the crystal operators  $\tilde{f}_i$  and  $\tilde{e}_i$  with  $i \in I$ . Each vertex  $b$  in  $B(\lambda)$  has a weight  $\text{wt}(b) \in P$  so that

$$s_\lambda = \sum_{b \in B(\lambda)} e^{\text{wt}(b)}.$$

In particular the cardinality of  $B(\lambda)$  is equal to the dimension of  $V(\lambda)$ . Also  $B(\lambda)$  has a unique source vertex  $b_\lambda$  which is the unique vertex killed by all the crystal operators  $\tilde{e}_i, i \in I$ . The multiplicity  $b_{\nu, \bar{\lambda} + \mu^\perp}(1)$  is then equal to the cardinality of the set  $H_{\bar{\lambda} + \mu^\perp} = \{b \in B(\lambda) \mid \tilde{e}_i(b) = 0, i \in \bar{I}\}$ . This thus reduces the combinatorial description of the polynomials  $K_{\nu, \mu}(1, q)$  to that of the polynomials  $\bar{K}_{\bar{\lambda}, \bar{\mu}}(q)$ , i.e. to that of the ordinary Lusztig  $q$ -analogues.

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