BERRY-ESSEEN'S BOUND, MODERATE DEVIATIONS AND HARMONIC MOMENTS FOR FIXED TYPE POPULATION OF A SUPERCRITICAL MULTI-TYPE BRANCHING PROCESS IN A RANDOM ENVIRONMENT

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QUANSHENG LIU AND THI TRANG NGUYEN

Abstract. Let $Z_n^i = (Z_n^i(1), \cdots, Z_n^i(d))$, $n \geqslant 0$, be a d-type supercritical branching process in an independent and identically distributed random environment $\xi = (\xi_0, \xi_1, \cdots)$, starting with one initial particle of type $i \in \{1, \cdots, d\}$. We study asymptotic properties of $Z_n^i(j)$, the j-type population size of generation n, for each $j \in \{1, \cdots, d\}$, as n goes to infinity. For $Z_n^i(j)$ we establish a Berry-Esseen type bound for the rate of convergence in the central limit theorem and a moderate deviation principle. As an important ingredient of the proofs, we also demonstrate the existence of the harmonic moments (which is of independent interest) of the normalized population size $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$, uniformly in $n \geqslant 1$, where $\mathbb{E}_\xi$ stands for the conditional expectation given the environment ξ.

1. Introduction

A branching process in a random environment (BPRE) is a natural and important generalisation of the Galton-Watson process, where the reproduction law varies according to a random environment indexed by time. It was introduced in Smith and Wilkinson [20] to model the growth of a population in an unknown exogenous environment. This process has attracted the attention of many authors in the last two decades, see for example the recent book by Kersting and Vatutin [18] and many references therein. The interest of study of such processes is growing in recent years, thanks to a large number of applications and interactions to other scientific fields.

A fundamental result in probability theory is the Central Limit Theorem (CLT), which provides the asymptotic normality of appropriately scaled sums of random variables. In the context of branching processes, the CLT describes the asymptotic distribution of the population size after a large number of generations. The Berry-Esseen bound goes further by providing a quantitative bound on the rate of convergence to the normal distribution. This bound measures how closely the distribution of the population size approximates a Gaussian distribution as the number of generations increases.

For a single-type supercritical branching process (Z_n) in a random environment, the law of large numbers on $\log Z_n$ has been established by Tanny [21]. The central limit theorem and the rate of convergence in the Gaussian approximation have been studied in [17], [14], and [4]. Large and moderate deviations have been investigated in [17, 3, 14, 5], among others. We also refer to [17] and [8] for the study of harmonic moments which play an important role in the study of limit theorems, particularly when comparing the branching process with the associated random walk.

For a multitype supercritical branching process (Z_n^i) in a random environment, starting with one initial particle of type i, Grama, Liu and Pin [9, 13] established the Berry-Esseen's

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bound, harmonic moments and Cramér type moderate deviation expansion for a suitable norming of $\|Z_n^i\|$, the total population size of generation n.

In this work, we will establish a Berry-Esseen type bound and a moderate deviation principle for $Z_n^i(j)$, the j-type population size of generation n. As a central element of the proofs, we will prove the existence of harmonic moments of the normalized population size $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$, uniformly in n, where $\mathbb{E}_\xi$ denotes the conditional expectation given the environment ξ.

Let us give a quick presentation of the results. For an integer $d \geqslant 1$, consider a d-type branching process $Z_n = (Z_n(1), \cdots, Z_n(d))$, $n \geqslant 0$, in an independent and identically distributed (i.i.d.) random environment $\xi = (\xi_0, \xi_1, \cdots)$. We will write Z_n^i for Z_n when the process starts with one initial particle of type i, that is, when $Z_0 = e_i$, where e_i is the vector in $\mathbb{R}^d$ with 1 in the i-th place and 0 elsewhere. Let M_n be the random matrix of the conditioned means of the offspring distribution of n-th generation given the environment, that is

$$M_n(i,j) = \mathbb{E}_\xi[Z_{n+1}(j) \mid Z_n = e_i], \quad 1 \leqslant i, j \leqslant d,$$

where $\mathbb{E}_\xi$ denotes the conditional expectation given the environment ξ. Define the product matrix $M_{0,n} = M_0 \cdots M_n$, and the associated Lyapunov exponent

$$\gamma := \lim_{n\to+\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\|,$$

where $\|M_{0,n-1}\|$ is the operator norm of the matrix $M_{0,n-1}$ associated with the L^1 norm of $\mathbb{R}^d$.

Recently, the asymptotic behavior of (Z_n^i) in the supercritical case $\gamma > 0$ has been studied in [12], where a strong law of large numbers is proved: under appropriate conditions, it holds that on the explosion event $\{\|Z_n^i\| \to +\infty\}$,

$$\lim_{n\to+\infty} \frac{1}{n} \log Z_n^i(j) = \gamma \quad \text{a.s.} \quad \forall i, j = 1, \cdots, d. \tag{1.1}$$

The first objective in this paper is to establish a Berry-Esseen type theorem on the rate of convergence in the central limit theorem for $\log Z_n^i(j)$: we will show that

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{\log Z_n^i(j) - n\gamma}{\sigma\sqrt{n}} \leqslant x \right) - \Phi(x) \right| \leqslant \frac{C}{\sqrt{n}}, \tag{1.2}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$ is the standard normal distribution function,

$$\sigma^2 = \lim_{n\to+\infty} \frac{1}{n} \mathbb{E}[(\log \|xM_{0,n-1}\| - n\gamma)^2]$$

is the asymptotic variance which is independent of x and $C > 0$ is a constant.

The second objective of this paper is to establish a moderate deviation principle for $Z_n^i(j)$. Let $\{a_n\}_{n\geqslant 1}$ be a sequence of positive numbers satisfying

$$\frac{a_n}{n} \to 0 \quad \text{and} \quad \frac{a_n}{\sqrt{n}} \to \infty \quad \text{as } n \to \infty.$$

Under suitable conditions, we will show that, for any measurable subset B of $\mathbb{R}$,

$$\begin{aligned} -\inf_{x\in B^o} \frac{x^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\left(\frac{\log Z_n^i(j) - n\gamma}{a_n} \in B \right) \\ &\leqslant \limsup_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\left(\frac{\log Z_n^i(j) - n\gamma}{a_n} \in B \right) \leqslant -\inf_{x\in\bar{B}} \frac{x^2}{2\sigma^2}, \end{aligned} \tag{1.3}$$

where B^o denotes the interior of B, and $\bar{B}$ its closure.

An important ingredient in our proofs is the existence of harmonic moments of the normalized population size $\bar{Z}_n^i(j) := Z_n^i(j)/M_{0,n-1}(i,j)$, uniformly in n: that is, for some $a > 0$, $\sup_{n\geqslant 1} \mathbb{E}[(\bar{Z}_n^i(j))^{-a}] < \infty$. Proving this is the third objective of the paper. The result is of independent interest, and can be useful in establishing other limit theorems, especially on moderate and large deviations.

The uniform result for the existence of harmonic moments of $\bar{Z}_n^i(j)$ allows us to do a precise comparison between $Z_n^i(j)$ and the entries $M_{0,n-1}(i,j)$ of products of random matrices. Its proof is based on a recurrence relation satisfied by the quenched Laplace transform of $\bar{Z}_n^i(j)$.

2. Notation and main results

For $d \geqslant 1$, let $\mathbb{R}^d$ be the d-dimensional space of vectors. We equip $\mathbb{R}^d$ with the scalar product and the L^1-norm respectively defined by

$$\langle x, y\rangle := \sum_{i=1}^{d} x(i)\, y(i) \quad \text{and} \quad \|x\| := \sum_{i=1}^{d} |x(i)|, \quad x, y \in \mathbb{R}^d.$$

Recall that $\mathbb{R}_+^d$ is the non-negative quadrant. Let $\mathcal{S} = \{x \in \mathbb{R}_+^d : \|x\| = 1\}$. For each $1 \leqslant i \leqslant d$, e_i will be the d-dimensional vector with 1 in the i-th place and 0 elsewhere. Let $\mathbf{0} = (0, \cdots, 0) \in \mathbb{R}^d$ be the vector with all coordinates equal to 0. Denote by $\mathbb{N} = \{0, 1, \cdots\}$ the set of non-negative integers. Set $\mathbb{1}_A$ for the indicator of an event A.

We denote by G the multiplicative semigroup of $d \times d$ matrices with non-negative entries. The subsemigroup of G with strictly positive matrices is denoted by G°. For $M \in G$, we define the operator norm with respect to L^1 vectorial norm by

$$\|M\| := \sup_{x\in\mathcal{S}} \|xM\| = \max_{1\leqslant i\leqslant d} \sum_{1\leqslant j\leqslant d} M(i,j) = \max_{1\leqslant i\leqslant d} \|e_i M\|, \quad M \in \mathcal{M}_d(\mathbb{R}),$$

where $M(i,j)$ denotes the (i,j)-th entry of M. (As the vectors $x \in \mathbb{R}^d$ are represented in the row form, the action of a matrix M on x is denoted xM, to be consistent with the matrix multiplication; the operator norm $\|M\|$ defined here corresponds to the induced L^1-norm of the transpose M^T defined with column vectors, since $\sup_{\|x\|=1} \|xM\| = \sup_{\|x\|=1} \|M^T x^T\| = \sup_{\|y\|=1} \|M^T y\|$.) In addition we equip $\mathbb{R}^d$ and $\mathcal{M}_d(\mathbb{R})$ with the L^∞-norms:

$$\|x\|_\infty := \max_{1\leqslant i\leqslant d} |x(i)|, \quad x \in \mathbb{R}^d;$$
$$\|M\|_\infty := \sup_{\|x\|_\infty=1} \|xM\|_\infty = \max_{1\leqslant j\leqslant d} \sum_{1\leqslant i\leqslant d} M(i,j) = \|M^T\|, \quad M \in \mathcal{M}_d(\mathbb{R}).$$

For a matrix or a vector X, we write $X > 0$ when all the entries of X are strictly positive.

Now we give a precise definition of the multi-type branching process in random environment (MBPRE). The environment $\xi = (\xi_n)_{n\geqslant 0}$ is a sequence of independent and identically distributed (i.i.d.) random variables taking values in an abstract space $\mathbb{X}$. To each realization of ξ_n correspond d probability distributions on $\mathbb{N}^d$ identified by the probability generating functions

$$f_n^r(s) = \sum_{k_1,\cdots,k_d=0}^{\infty} p_{k_1,\cdots,k_d}^r(\xi_n) s_1^{k_1} \cdots s_d^{k_d}, \quad s = (s_1, \ldots, s_d) \in [0,1]^d, \tag{2.1}$$

where $1 \leqslant r \leqslant d$. The d-type branching process $Z_n = (Z_n(1), \cdots, Z_n(d))$, $n \geqslant 0$, in the random environment ξ is a process with values in $\mathbb{N}^d$ such that for all $n \geqslant 0$,

$$Z_{n+1} = \sum_{r=1}^{d} \sum_{l=1}^{Z_n(r)} N_{l,n}^r. \tag{2.2}$$

where $N_{l,n}^r = (N_{l,n}^r(1), \cdots, N_{l,n}^r(d))$ is a random vector whose j-th component $N_{l,n}^r(j)$ represents the offspring of type j at time $n+1$ of the l-th particle of type r in generation n, and $Z_n(j)$ is the total number of particles of type j in generation n. Conditioned on the environment ξ, the random vectors Z_0 and $N_{l,n}^r$ indexed by $l \geqslant 1$, $n \geqslant 0$ and $1 \leqslant r \leqslant d$, are all independent of each other, and each $N_{l,n}^r$ has the same law $p^r(\xi_n) = \{p_k^r(\xi_n) : k \in \mathbb{N}^d\}$ whose probability generating function is f_n^r. In the sequel, we write Z_n^i for Z_n when $Z_0 = e_i$, that is, when the process starts with one initial particle of type i.

Denote by $\mathbb{P}_\xi$ the quenched law, i.e. the probability under which the process is defined when the environment ξ is given. Let τ be the law of ξ. The total probability $\mathbb{P}$ of (Z_n), usually called annealed law, is defined by $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$. Denote by $\mathbb{E}_\xi$ and $\mathbb{E}$ the corresponding expectation with respect to $\mathbb{P}_\xi$ and $\mathbb{P}$. With our notation,

$$f_n^r(s) = \mathbb{E}_\xi\Big(\prod_{j=1}^{d} s_j^{N_{l,n}^r(j)}\Big), \quad s = (s_1, \ldots, s_d) \in [0,1]^d$$

is the quenched probability generating function of $N_{l,n}^r$. Let

$$f_n = (f_n^1, \cdots, f_n^d). \tag{2.3}$$

For $n \geqslant 0$, let M_n be the $d \times d$ random matrix whose (i,j)-th entry $M_n(i,j)$ is the conditioned mean of the number of children of type j produced by a particle of type i at time n:

$$M_n(i,j) := \frac{\partial f_n^i}{\partial s_j}(\mathbf{1}) = \mathbb{E}_\xi\big[Z_{n+1}(j) | Z_n = e_i\big],$$

where $\frac{\partial f}{\partial s_j}(\mathbf{1})$ denotes the left derivative at $\mathbf{1}$ of a d-dimensional probability generating function f with respect to s_j. Since the sequence of the environments (ξ_n) is i.i.d., the sequence of the mean matrices (M_n) is also i.i.d.. For $0 \leqslant k \leqslant n$, denote by

$$M_{k,n} := M_k \cdots M_n,$$

the product of the mean matrices $M_k, \ldots, M_n$; by convention $M_{k,n}$ denotes the identity matrix when $k > n$. It follows that, for $n \geqslant 0$ and $1 \leqslant i, j \leqslant d$,

$$\mathbb{E}_\xi Z_n^i(j) = M_{0,n-1}(i,j). \tag{2.4}$$

For $0 \leqslant k \leqslant n$, let $\rho_{k,n}$ be the spectral radius of $M_{k,n}$. By the Perron-Frobenius theorem, $\rho_{k,n}$ is an eigenvalue of $M_{k,n}$, and there exists a non negative eigenvector $u_{k,n}$ associated to $\rho_{k,n}$ with $\|u_{k,n}\| = 1$. Under conditions, the limit

$$u_k := \lim_{n\to\infty} u_{k,n} \tag{2.5}$$

exists a.s., with $u_k > 0$ and $\|u_k\| = 1$; moreover the sequence (u_k) satisfies the relation

$$u_{k+1} M_k^T = \lambda_k u_k, \tag{2.6}$$

where $\lambda_k = \|u_{k+1} M_k^T\|$ are positive random scalars called the pseudo-spectral radii of the random matrices (M_k). Set $\lambda_{k,n} = \lambda_k \cdots \lambda_n$. By iteration of (2.6), we obtain

$$u_{n+1} M_{k,n}^T = \lambda_{k,n} u_k, \quad n, k \geqslant 0, \quad \text{with } M_{k,n}^T = (M_{k,n})^T. \tag{2.7}$$

Then the following sequence constitues a martingale:

$$W_0^i = 1, \quad W_n^i = \frac{\langle Z_n^i, u_n\rangle}{\lambda_{0,n-1}u_0(i)} = \frac{\langle Z_n^i, u_n\rangle}{\langle e_i M_{0,n-1}, u_n\rangle}, \quad n \geqslant 1. \tag{2.8}$$

Throughout the paper, we assume that M_0 is allowable (every row and column contains a strictly positive element), and that the following positivity property holds:

$$\mathbb{P}\Big(\exists n \geqslant 1 \text{ such that } M_{0,n-1} > 0\Big) > 0. \tag{2.9}$$

By the results of Hennion, under condition (2.9) there exists the random vectors u_k and the random scalars λ_k defined by (2.5) and (2.6), which satisfy the relation (2.7) and that (u_k) and (λ_k) are stationary ergodic; moreover, we know that the sequence (W_n^i) defined in (2.8) is a non-negative martingale under the measure $\mathbb{P}_\xi$ and $\mathbb{P}$, w.r.t. the filtration

$$\mathcal{F}_0 = \sigma(\xi), \quad \mathcal{F}_n = \sigma(\xi, N_{l,k}^r(j), 0 \leqslant k \leqslant n-1, 1 \leqslant r, j \leqslant d, l \geqslant 1) \text{ for } n \geqslant 1.$$

Let $W^i := \lim_{n\to+\infty} W_n^i$ be the a.s. limit of the martingale (W_n^i).

It is well known that, under the following moment condition

$$\mathbb{E}\log^+ \|M_0\| < +\infty, \tag{2.10}$$

the Lyapunov exponent γ of the sequence of matrices $(M_n)_{n\geqslant 0}$ exists, with

$$\gamma := \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}\log\|M_{0,n-1}\| = \inf_{n\geqslant 1}\frac{1}{n}\mathbb{E}\log\|M_{0,n-1}\|.$$

Moreover, Furstenberg and Kesten established a strong law of large numbers for $\log\|M_{0,n-1}\|$:

$$\lim_{n\to+\infty}\frac{1}{n}\log\|M_{0,n-1}\| = \gamma \quad \mathbb{P}\text{-a.s.}$$

According to the values of the Lyapunov exponent γ, we have the following classification: a MBPRE is supercritical if $\gamma > 0$, critical if $\gamma = 0$, and subcritical if $\gamma < 0$. In this article, the process (Z_n) is assumed to be supercritical, i.e. $\gamma > 0$.

Under the supercritical condition $\gamma > 0$, we know by [10, Theorem 2.6 and Corollary 2.8] that the condition

$$\mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)}\log^+\frac{Z_1^i(j)}{M_0(i,j)}\right) < +\infty \quad \forall 1 \leqslant i,j \leqslant d \tag{2.11}$$

is sufficient for the non-degeneracy of each W^i (in the sense that $\mathbb{P}(W^i > 0) > 0$), with

$$\mathbb{E}_\xi W^i = 1 \quad \text{and} \quad \mathbb{P}_\xi(W^i > 0) = \mathbb{P}_\xi(\|Z_n^i\| \underset{n\to\infty}{\to} \infty) = 1 - q^i(\xi) > 0, \quad \text{a.s.} \tag{2.12}$$

where $q^i(\xi) = \mathbb{P}_\xi(Z_n^i \to 0)$ is the quenched probability of extinction of the process (Z_n^i). By Sheffé's theorem, the property that $\mathbb{E}W^i = 1$ is equivalent to the L^1-convergence of W_n^i.

Denote

$$\bar{Z}_n^i(j) := \frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i,j)}.$$

For $n \geqslant 0$, define the row vector $p_0(\xi_n)$ and the matrix $P_1(\xi_n)$, whose components are

$$p_0(\xi_n)(i) = f_n^i(\mathbf{0}) \quad \text{and} \quad P_1(\xi_n)(i,j) = \frac{\partial f_n^i}{\partial s_j}(\mathbf{0}), \quad 1 \leqslant i,j \leqslant d. \tag{2.13}$$

Then, for $1 \leqslant i,j \leqslant d$,

$$p_0(\xi_n)(i) = \mathbb{P}_{T^n\xi}(\|Z_1^i\| = 0) \text{ and } P_1(\xi_n)(i,j) = \mathbb{P}_{T^n\xi}(Z_1^i = e_j),$$

where T^n is the n-fold iteration of the translation T defined as

$$T\xi = (\xi_1, \xi_2, \cdots) \quad \text{if} \quad \xi = (\xi_0, \xi_1, \cdots).$$

Throughout the paper, we will assume the following condition:

H1. *The vector* $p_0(\xi_0) = (f_0^1(\mathbf{0}), \dots, f_0^d(\mathbf{0}))$ *satisfies*

$$p_0(\xi_0) = \mathbf{0} \quad \mathbb{P}\text{-}a.s. \tag{2.14}$$

The condition **H1** means that each individual of the population gives birth to at least one child, so that $q^i(\xi) = 0$ a.s. When (2.12) holds, this implies that $\|Z_n^i\| \to +\infty$ a.s. as $n \to +\infty$. When **H1** holds, the process (Z_n^i) is called increasing since in this case $\|Z_n^i\|$ is increasing a.s.

We introduce the following condition introduced by Furstenberg and Kesten [7].

H2. *There exists a constant* $D > 1$ *such that* $\mathbb{P}$*-a.s.,*

$$1 \leqslant \frac{\max_{1\leqslant i,j\leqslant d} M_0(i,j)}{\min_{1\leqslant i,j\leqslant d} M_0(i,j)} \leqslant D.$$

We introduce the following assumption :

H3. *There exist constants* $p \in (1,2]$, $A > A_1 > 1$ *and* $A_2 > 0$ *such that for any* $1 \leqslant i, j \leqslant d$, $\mathbb{P}$*-a.s.*

$$A_2 \leqslant M_0(i,j), \quad A_1 \leqslant \|M_0(i,\cdot)\| \quad \textit{and} \quad \mathbb{E}_\xi(Z_1^i(j)^p) \leqslant A^p.$$

It is clear that **H3** implies the conditions (2.9), (2.10), (2.11) and $\gamma > 0$. From **H3** we have also that for all $1 \leqslant i, j \leqslant d$,

$$M_0(i,j) \leqslant A \quad \mathbb{P}\text{-a.s.} \tag{2.15}$$

Under the condition **H3**, it implies that

$$1 \leqslant \frac{\max_{1\leqslant i,j\leqslant d} M_0(i,j)}{\min_{1\leqslant i,j\leqslant d} M_0(i,j)} \leqslant \frac{A}{A_2}. \tag{2.16}$$

Therefore, **H3** implies **H2** with $D = \frac{A}{A_2}$.

Now we study the existence of harmonic moments for $\bar{Z}_n^i(j)$ under the boundedness condition **H3**. We shall also use the following condition:

H4. *There exist a constant* $a_0 > 0$ *such that* $\mathbb{E}[(Z_1^i(j))^{-a_0}] < +\infty$ *for all* $1 \leqslant i, j \leqslant d$.

The following theorem concerns the existence of harmonic moments of $\bar{Z}_n^i(j)$ uniformly in n, which is quite useful in applications. It will be used in the proofs of the Berry-Esseen bound and moderate deviations for $\log Z_n^i(j)$.

Theorem 2.1 (Harmonic moments)**.** *Assume conditions* ***H1****,* ***H3****,* ***H4*** *and* $\gamma > 0$*. Then there exists* $a > 0$ *such that* $\sup_{n\geqslant 1} \mathbb{E}(\bar{Z}_n^i(j))^{-a} < +\infty$ *for all* $1 \leqslant i, j \leqslant d$.

Notice that **H4** is necessary since $\sup_{n\geqslant 1} \mathbb{E}(\bar{Z}_n^i(j))^{-a} < +\infty$ implies $\mathbb{E}(\bar{Z}_1^i(j))^{-a} < +\infty$. Notice also that, since $\bar{Z}_n^i(j)) \to W^i$ a.s., by Fatou's lemma, $\sup_{n\geqslant 1} \mathbb{E}(\bar{Z}_n^i(j))^{-a} < +\infty$ implies $\mathbb{E}(W^i)^{-a} < \infty$. For the harmonic moments of W^i, sufficient conditions have been given in [9]. We mention that the uniform finiteness of the harmonic moments of $\bar{Z}_n^i(j)$ is more delicate compared with that of the martingale W_n^i where each type particle contributes.

Now we formulate a Berry-Esseen type theorem for $\log Z_n^i(j)$, for all $1 \leqslant i, j \leqslant d$. It has been proved in [22, Proposition 3.15] that if for some $\varepsilon > 0$ and all $i, j \in \{1, \cdots, d\}$, $\mathbb{E}M_0(i,j)^\varepsilon + \mathbb{E}M_0(i,j)^{-\varepsilon} < \infty$, then the asymptotic variance σ^2

$$\sigma^2 = \lim_{n\to+\infty} \frac{1}{n} \mathbb{E}[(\log\langle xM_{0,n-1}, e_i\rangle - n\gamma)^2]$$

exists with value in $[0, \infty)$, uniformly in $x \in \mathcal{S}$ (in fact this was proved in [22] under weaker conditions). For the rate of convergence we need the following assumption :

H5. *The asymptotic variance σ^2 satisfies*

$$\sigma^2 > 0.$$

Theorem 2.2 (Berry-Esseen bound)**.** *Assume conditions **H1**, **H3**, **H4**, $\gamma > 0$ and **H5**. Then there exists a constant $C > 0$ such that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i, j \leqslant d$,*

$$\left| \mathbb{P}\left(\frac{\log Z_n^i(j) - n\gamma}{\sigma\sqrt{n}} \leqslant x \right) - \Phi(x) \right| \leqslant \frac{C}{\sqrt{n}},$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$ is the standard normal distribution function.

A similar result for the norm $\|Z_n^i\|$ has been established in [9].

Our last result is a moderate deviation principle for $\log Z_n^i(j)$.

Theorem 2.3 (Moderate Deviation Principle)**.** *Assume conditions **H1**, **H3**, **H4** and $\gamma > 0$ and **H5**. Let $\{a_n\}_{n\geqslant 1}$ be a sequence of positive numbers satisfying*

$$\frac{a_n}{n} \to 0 \quad \text{and} \quad \frac{a_n}{\sqrt{n}} \to \infty \quad \text{as } n \to \infty.$$

Then for any measurable subset B of $\mathbb{R}$,

$$\begin{aligned} -\inf_{x\in B^o} \frac{x^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\left(\frac{\log Z_n^i(j) - n\gamma}{a_n} \in B \right) \\ &\leqslant \limsup_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\left(\frac{\log Z_n^i(j) - n\gamma}{a_n} \in B \right) \leqslant -\inf_{x\in \bar{B}} \frac{x^2}{2\sigma^2}, \end{aligned} \tag{2.17}$$

where B^o denotes the interior of B, and $\bar{B}$ its closure.

3. Harmonic moments of $\bar{Z}_n^i(j)$

3.1. Auxiliary results.

Lemma 3.1. *([9, Lemma 3.1]) Assume condition (2.11) and $\gamma > 0$. Then for all $n \geqslant 0$, $1 \leqslant i \leqslant d$ and any convex function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$,*

$$\lim_{n\to+\infty} \mathbb{E}_\xi \phi(W_n^i) = \sup_{n\geqslant 0} \mathbb{E}_\xi \phi(W_n^i) = \mathbb{E}_\xi \phi(W^i), \tag{3.1}$$

and

$$\lim_{n\to+\infty} \mathbb{E}\phi(W_n^i) = \sup_{n\geqslant 0} \mathbb{E}\phi(W_n^i) = \mathbb{E}\phi(W^i). \tag{3.2}$$

Lemma 3.2. *([19, Lemma 4.4]) Let X be a positive random variable, and $a > 0$. We have the following assertions:*

(1) if $\mathbb{E}X^{-a} < +\infty$ then $\mathbb{E}e^{-tX} = O_{t\to+\infty}(t^{-a})$;
(2) if $\mathbb{E}e^{-tX} = O_{t\to+\infty}(t^{-a})$ then $\mathbb{E}X^{-b} < +\infty$ for all $b \in (0, a)$;
(3) $\mathbb{E}e^{-tX} = O_{t\to+\infty}(t^{-a})$ if and only if $\mathbb{P}(X \leqslant x) = O_{x\to 0}(x^a)$.

For all $n \geqslant 0$, $1 \leqslant i, j \leqslant d$, let

$$\phi_{\xi,n}^{i,j}(t) = \mathbb{E}_\xi e^{-t\bar{Z}_n^i(j)} \quad \text{and} \quad \phi_n^{i,j}(t) = \mathbb{E}\phi_{\xi,n}^{i,j}(t) = \mathbb{E}e^{-t\bar{Z}_n^i(j)}, \quad t \geqslant 0,$$

be respectively the quenched and annealed Laplace transform of $\bar{Z}_n^i(j)$. Denote by

$$\phi_{\xi,n}(t) = (\phi_{\xi,n}^1(t), \cdots, \phi_{\xi,n}^d(t)) \quad \text{and} \quad \phi_n(t) = (\phi_n^1(t), \cdots, \phi_n^d(t)), \quad t \geqslant 0.$$

Let T be the shift operator of the environment sequence:

$$T\xi = (\xi_1, \xi_2, \cdots) \quad \text{if} \quad \xi = (\xi_0, \xi_1, \cdots),$$

and let T^n be its n-fold iteration.

The following lemma gives the functional equations and inequalities that the quenched Laplace transforms $\phi^{i,j}_{\xi,n}$ satisfy. For a matrix M, we denote its i-th row by $M(i,\cdot)$, that is, $M(i,\cdot) = e_i M$.

Lemma 3.3. *Assume condition **H2**. Then for all $n \geqslant 1$, $i,j = 1,\cdots,d$, the quenched Laplace transform $\phi^{i,j}_{\xi,n}$ of $\bar{Z}^i_n(j)$ satisfies, for all $t \geqslant 0$,*

$$\phi^{i,j}_{\xi,n+1}(t) = f^i_0\Big(\phi^{1,j}_{T\xi,n}\Big(t\frac{M_{1,n}(1,j)}{M_{0,n}(i,j)}\Big),\cdots,\phi^{d,j}_{T\xi,n}\Big(t\frac{M_{1,n}(d,j)}{M_{0,n}(i,j)}\Big)\Big) \tag{3.3}$$

$$\leqslant f^i_0\Big(\phi^{1,j}_{T\xi,n}\Big(\frac{t}{D\|M_0(1,\cdot)\|}\Big),\cdots,\phi^{d,j}_{T\xi,n}\Big(\frac{t}{D\|M_0(d,\cdot)\|}\Big)\Big) \tag{3.4}$$

$$\leqslant f^i_0\Big(\phi^{1,j}_{T\xi,n}\Big(\frac{t}{D\|M_0\|}\Big),\cdots,\phi^{d,j}_{T\xi,n}\Big(\frac{t}{D\|M_0\|}\Big)\Big). \tag{3.5}$$

Proof. Conditioned on the environment ξ, the random vectors $N^r_{l,n,k} = (N^r_{l,n,k}(1),\cdots,N^r_{l,n,k}(d))$, with $N^r_{l,n,k}(j)$ denoting the offspring of type j at time $n+k$ of the l-th r-type particle of generation n, are independent and have the same probability generating function $f^r_n \circ f_{n+1}\circ\cdots\circ f_{n+k-1}$, where f^r_n and f_n are defined in (2.1) and (2.3). The process $(Z_n)_{n\geqslant 0}$ satisfies the following relation

$$Z_{n+k} = \sum_{r=1}^{d}\sum_{l=1}^{Z_n(r)} N^r_{l,n,k}, \quad n\geqslant 0, k\geqslant 1. \tag{3.6}$$

From (3.6) and Furstenberg-Kesten condition **H2**, we get that for all $n\geqslant 1$ and $1\leqslant i,j\leqslant d$,

$$\begin{aligned}
\bar{Z}^i_{n+1}(j) &= \sum_{r=1}^{d}\sum_{l=1}^{Z^i_1(r)} \frac{N^r_{l,1,n}(j)}{M_{0,n}(i,j)} \\
&= \sum_{r=1}^{d}\sum_{l=1}^{Z^i_1(r)} \frac{N^r_{l,1,n}(j)}{M_{1,n}(r,j)}\frac{M_{1,n}(r,j)}{M_{0,n}(i,j)} \\
&= \sum_{r=1}^{d}\sum_{l=1}^{Z^i_1(r)} \bar{Z}^r_{l,1,n}(j)\frac{M_{1,n}(r,j)}{\sum_{l=1}^d M_0(i,l)M_{1,n}(l,j)} \\
&\geqslant \sum_{r=1}^{d}\sum_{l=1}^{Z^i_1(r)} \bar{Z}^r_{l,1,n}(j)\frac{1}{D\|M_0(i,\cdot)\|}.
\end{aligned} \tag{3.7}$$

Taking the Laplace transform and using the independence under $\mathbb{P}_\xi$ of the random variables $\bar{Z}^r_{l,1,n}(j)$ and $Z^i_1(r)$ for $l\geqslant 0$ and $1\leqslant r\leqslant d$, we get that for all $n\geqslant 1$ and $1\leqslant i,j\leqslant d$, $t\geqslant 0$,

$$\begin{aligned}
\phi^{i,j}_{\xi,n+1}(t) &\leqslant \mathbb{E}_\xi\Big[\prod_{r=1}^{d}\prod_{l=1}^{Z^i_1(r)} \mathbb{E}_\xi\big[e^{-\frac{t}{D\|M_0(i,\cdot)\|}\bar{Z}^r_{l,1,n}(j)}\big]\Big] \\
&= \mathbb{E}_\xi\Big[\prod_{r=1}^{d}\Big(\phi^{r,j}_{T\xi,n}\Big(\frac{t}{D\|M_0(i,\cdot)\|}\Big)\Big)^{Z^i_1(r)}\Big] \\
&= f^i_0\Big(\phi^{1,j}_{T\xi,n}\Big(\frac{t}{D\|M_0(1,\cdot)\|}\Big),\cdots,\phi^{d,j}_{T\xi,n}\Big(\frac{t}{D\|M_0(d,\cdot)\|}\Big)\Big).
\end{aligned} \tag{3.8}$$

Since $\max_{1\leqslant r\leqslant d}\|M_0(r,\cdot)\| = \|M_0\|$, this gives the conclusion of the lemma. □

Lemma 3.4. *Let $(X_k)_{k\in\mathbb{N}^*}$ be a sequence of i.i.d. random centered variables. Then for all $n\in\mathbb{N}^*$ and $p>1$:*

$$\mathbb{E}\Big|\sum_{k=1}^{n} X_k\Big|^p \leqslant \begin{cases}(B_p)^p\mathbb{E}|X_k|^p n, & \text{if } 1<p\leqslant 2,\\ (B_p)^p\mathbb{E}|X_k|^p n^{\frac{p}{2}}, & \text{if } p>2,\end{cases}$$

where $B_p = 2\min\{k^{1/2} : k\in\mathbb{N}, k\geqslant \frac{p}{2}\}$.

3.2. **Existence of the annealed harmonic moments $\mathbb{E}(\bar{Z}_n^i(j))^{-a}$.** The aim of this section is to prove Theorem 2.1 which gives the existence of the harmonic moments $\mathbb{E}(\bar{Z}_n^i(j))^{-a}$ for a small $a>0$, under the boundedness condition **H3**. We will prove the following theorem.

Theorem 3.5. *Assume conditions **H1**, **H3**, **H4** and $\gamma>0$. Then there exist two constants $a>0$ and $C>0$ such that for all $n\geqslant 1$, $1\leqslant i,j\leqslant d$, $t>0$, and $x>0$,*

$$\phi_n^{i,j}(t)\leqslant \frac{C}{t^a}. \tag{3.9}$$

$$\mathbb{P}(\bar{Z}_n^i(j)\leqslant x)\leqslant Cx^a \quad \text{and} \quad \mathbb{E}(\bar{Z}_n^i(j))^{-a}\leqslant C. \tag{3.10}$$

Clearly Theorem 2.1 follows from Theorem 3.5.

For the proof of Theorem 3.5, we shall need the following technical lemma about the decay late of functions which satisfy a recurrent relation.

Lemma 3.6. *Consider a sequence of functions $\psi_n:(0,\infty)\to[0,\infty)$, $n\geqslant 0$. Assume that there exist a random variable $A>0$ and some constants $\alpha\in(0,1)$, $C\geqslant 0$, $a>0$ such that for all $n\geqslant 0$, and all $t>0$,*

$$\psi_{n+1}(t)\leqslant \alpha\mathbb{E}\psi_n(At)+\frac{C}{t^a}. \tag{3.11}$$

(1) If $\alpha\mathbb{E}A^{-a}<1$ and ψ_0 is bounded on $(0,\infty)$, then for all $t>0$,

$$\limsup_{n\to\infty}\psi_n(t)\leqslant \frac{C_1}{t^a}, \quad \text{where} \quad C_1=\frac{C}{1-\alpha\mathbb{E}(A^{-a})}.$$

(2) If $\alpha\mathbb{E}A^{-a}<1$ and $\psi_0(t)\leqslant \frac{C_0}{t^a}$ for some constant $C_0>0$ and all $t>0$, then for all $t>0$,

$$\sup_{n\geqslant 0}\psi_n(t)\leqslant \frac{C_2}{t^a}, \quad \text{where} \quad C_2=C_0+\frac{C}{1-\alpha\mathbb{E}(A^{-a})}.$$

Proof. Let $\{A_i\}_{i\geqslant 0}$ be independent copies of A. By induction, for all $n\geqslant 1$ and all $t>0$,

$$\psi_n(t)\leqslant \alpha^n\mathbb{E}\psi_0(t\prod_{k=0}^{n-1}A_k)+C\Big[1+\sum_{k=0}^{n-1}(\alpha\mathbb{E}(A^{-a}))^k\Big]t^{-a}. \tag{3.12}$$

Therefore taking the limsup as $n\to\infty$, or taking the supremum on $n\geqslant 1$, we get the desired results. □

For the proof of Theorem 3.5, we will need the following preliminary result about the quenched moments $\mathbb{E}_\xi(\bar{Z}_n^i(j))^p$ and the control of the quenched Laplace transform $\phi_{\xi,n}^{i,j}(t)=\mathbb{E}_\xi e^{-t\bar{Z}_n^i(j)}$, for all $n\geqslant 1$ and $1\leqslant i,j\leqslant d$. For all $k\geqslant 0$ and $p>1$, let

$$\theta_k(p)=\theta_k(p,\xi)=\theta(p,\xi_k)=\max_{1\leqslant r,j\leqslant d}\mathbb{E}_\xi\Big|\frac{N_{1,k}^r(j)}{M_k(r,j)}-1\Big|^p \tag{3.13}$$

(recall that $N_{1,k}^r$ has law $p^r(\xi_k)$ under $\mathbb{P}_\xi$). Then $\theta_0(p,\xi)=\max_{1\leqslant r,j\leqslant d}\mathbb{E}_\xi\big|\frac{Z_1^r(j)}{M_0(r,j)}-1\big|^p$ and $\theta_k(p,\xi)=\theta_0(p,T^k\xi)$.

Lemma 3.7.

*(1) Assume Furstenberg-Kesten condition **H2**. Then $\mathbb{P}$-a.s., for all $n \geqslant 1$ and $1 \leqslant i, j \leqslant d$, with $\theta_k(p)$ defined in (3.13),*

$$\Big(\mathbb{E}_\xi(\bar{Z}_n^i(j))^p\Big)^{1/p} \leqslant dD^2\left(1 + B_p \sum_{k=0}^{n-1} \theta_k(p)^{\frac{1}{p}} \max_{1\leqslant r\leqslant d} M_{0,k-1}(i,r)^{\frac{1-p}{p}}\right). \tag{3.14}$$

*(2) Assume conditions **H1** and **H3**. Then there exist two constants $\beta \in (0,1)$ and $t_0 > 0$ such that for all $n \geqslant 1$, $1 \leqslant i, j \leqslant d$ and $t \geqslant t_0$,*

$$\phi_{\xi,n}^{i,j}(t) \leqslant \beta \quad \mathbb{P}\text{-a.s.} \tag{3.15}$$

Proof. We first prove (3.14). Recall that (see (2.7)) for $n \geqslant 1$, $u_n M_{0,n-1}^T = \lambda_{0,n-1} u_0$. Under **H2**, by [11, (4.5)], we know that for all $n \geqslant 1$ and $1 \leqslant i, j \leqslant d$,

$$\frac{1}{dD^2} \leqslant \frac{M_{0,n-1}(i,j)u_n(j)}{\lambda_{0,n-1}u_0(i)} \leqslant 1 \quad \mathbb{P}\text{-a.s.}$$

Therefore we obtain that for all $n \geqslant 1$ and $1 \leqslant i, j \leqslant d$,

$$\begin{aligned}\bar{Z}_n^i(j) = \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} &\leqslant dD^2 \frac{M_{0,n-1}(i,j)u_n(j)}{\lambda_{0,n-1}u_0(i)} \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \\ &= dD^2 \frac{Z_n^i(j)u_n(j)}{\lambda_{0,n-1}u_0(i)} \leqslant dD^2 W_n^i. \end{aligned} \tag{3.16}$$

Using the decomposition $W_n^i = \sum_{k=0}^{n-1}(W_{k+1}^i - W_k^i)$ and the triangular inequality in L^p under $\mathbb{P}_\xi$, we see that for all $n \geqslant 1$,

$$\Big(\mathbb{E}_\xi[(W_n^i)^p]\Big)^{1/p} \leqslant \Big(1 + \sum_{k=0}^{n-1}\Big(\mathbb{E}_\xi|W_{k+1}^i - W_k^i|^p\Big)^{1/p}\Big). \tag{3.17}$$

By [9, (3.8)] we have for all $k \geqslant 0$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s., with $\theta_k(p)$ defined in (3.13),

$$\mathbb{E}_\xi|W_{k+1}^i - W_k^i|^p \leqslant B_p^p \theta_k(p) \max_{1\leqslant j\leqslant d} (M_{0,k-1}(i,j))^{1-p}. \tag{3.18}$$

Combining (3.16), (3.17) and (3.18), we obtain (3.14).

Now, we assume conditions **H1** and **H3**. We prove the second part of the lemma. We know that **H3** implies **H2**, then under **H3**, we also have (3.14). Using the triangular inequality in L^p under $\mathbb{P}_\xi$ and condition **H3**, for $n \geqslant 0$ we have

$$\theta_n(p) \leqslant \max_{1\leqslant r,j\leqslant d}\left(\frac{[\mathbb{E}_\xi(N_{1,n}^r(j))^p]^{1/p}}{M_n(i,j)} + 1\right)^p \leqslant \Big(\frac{A}{A_2} + 1\Big)^p \ \mathbb{P}\text{-a.s.} \tag{3.19}$$

Using [9, (3.11)] that $M_{0,n-1}(i,r) \geqslant A_2 A_1^{n-1}$ and (3.19), we obtain that for all $1 \leqslant i, j \leqslant d$ and $n \geqslant 0$,

$$\Big(\mathbb{E}_\xi(\bar{Z}_n^i(j))^p\Big)^{1/p} \leqslant C. \tag{3.20}$$

Since $x \to (e^{-x} - 1 + x)/x^p$ is a positive bounded function on $\mathbb{R}_+^*$, it follows that there exists a constant $C_1 > 0$ such that for all $x > 0$,

$$e^{-x} \leqslant 1 - x + C_1 x^p.$$

Combining this with (3.20), we see that for all $1 \leqslant i, j \leqslant d$ and $n \geqslant 1$, and $t \geqslant 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\phi_{\xi,n}^{i,j}(t) = \mathbb{E}_\xi e^{-t\bar{Z}_n^i(j)} &\leqslant \mathbb{E}_\xi(1 - t\bar{Z}_n^i(j) + C_1 t^p(\bar{Z}_n^i(j))^p) \\ &\leqslant 1 - Ct + CC_1 t^p. \end{aligned} \tag{3.21}$$

Let $h(t) = 1 - Ct + CC_1 t^p$, $t \geqslant 0$. We observe that the minimal value of $h(t)$ is $\beta = h(t_0)$, where $t_0 := (pC_1)^{1/(1-p)}$, and we have

$$\beta = 1 - C(pC_1)^{1/(1-p)} + \frac{C}{p}(pC_1)^{1/(1-p)} = 1 - \Big(1 - \frac{1}{p}\Big)Ct_0. \tag{3.22}$$

So $\beta \in (0,1)$. Since the quenched Laplace transform $\phi^{i,j}_{\xi,n}$ is decreasing on $\mathbb{R}_+$, we conclude from (3.21) that for all $n \geqslant 1$, $1 \leqslant i,j \leqslant d$, $t \geqslant t_0$,

$$\phi^{i,j}_{\xi,n}(t) \leqslant \phi^{i,j}_{\xi,n}(t_0) \leqslant h(t_0) = \beta \quad \mathbb{P}\text{-a.s.} \tag{3.23}$$

This concludes the proof of Lemma 3.7. □

We begin with the following result for the existence of the harmonic moments $\mathbb{E}(\bar{Z}^i_n(j))^{-a}$ for each $n \geqslant 1$.

Proposition 3.8. *Assume* ***H2****. Let $a > 0$ be such that $\mathbb{E}[\|M_0\|^a] < \infty$ and $\mathbb{E}[(\bar{Z}^i_1(j))^{-a}] < \infty$ for all $i,j \in \{1,\cdots,d\}$. Then for all $n \geqslant 1$ and $1 \leqslant i,j \leqslant d$,*

$$\mathbb{E}(\bar{Z}^i_n(j))^{-a} < +\infty. \tag{3.24}$$

Proof. We will prove (3.24) by induction on $n \geqslant 1$. For $n = 1$, (3.24) holds under the given condition on $\bar{Z}^i_1(j)$. Assume that (3.24) holds for some $n \geqslant 1$ and all $1 \leqslant i,j \leqslant d$. We will prove that (3.24) also holds for $n+1$. By (3.7), we have the following decomposition: for $n \geqslant 1$ and $i,j \in \{1,\cdots,d\}$,

$$\begin{aligned}\bar{Z}^i_{n+1}(j) &= \sum_{r=1}^{d}\sum_{l=1}^{Z^i_1(r)} \frac{N^r_{l,1,n}(j)}{M_{1,n}(r,j)}\frac{M_{1,n}(r,j)}{M_{0,n}(i,j)} \\ &\geqslant \sum_{r=1}^{d}\sum_{l=1}^{Z^i_1(r)} \frac{N^r_{l,1,n}(j)}{M_{1,n}(r,j)}\frac{1}{D\|M_0\|}.\end{aligned} \tag{3.25}$$

Notice that from the condition on $\bar{Z}^i_1(j)$, we have $Z^i_1(j) > 0$ a.s. for all $i,j \in \{1,\cdots,d\}$. Therefore, from (3.25) we get that for all $n \geqslant 1$, $i,r \in \{1,\cdots,d\}$, and $a > 0$,

$$\begin{aligned}\mathbb{E}(\bar{Z}^i_{n+1}(j))^{-a} &\leqslant D^a\mathbb{E}\Big[\Big(\frac{N^r_{1,1,n}(j)}{M_{1,n}(r,j)}\Big)^{-a}\|M_0\|^a\Big] \\ &= D^a\mathbb{E}[(\bar{Z}^r_n(j))^{-a}]\ \mathbb{E}[\|M_0\|^a].\end{aligned} \tag{3.26}$$

So by the recurrence hypothesis and the condition on $\bar{Z}^i_1(j)$, we see that (3.24) still holds for $n+1$ and all $i,j \in \{1,\cdots,d\}$. Therefore, by induction, (3.24) holds for all $n \geqslant 1$ and $i,j \in \{1,\cdots,d\}$.

□

We now prove the uniform finiteness of the harmonic moments of $\bar{Z}^i_n(j)$, that is, the existence of some $a > 0$ such that $\sup_{n\geqslant 1}\mathbb{E}(\bar{Z}^i_n(j))^{-a} < \infty$, which is much more delicate than for a fixed $n \geqslant 1$.

Proof of Theorem 3.5. We first notice that the implication (3.9) $\Rightarrow$ (3.10) holds by Lemma 3.2. Therefore, it remains to show (3.9) under the given conditions. Assume conditions **H1**, **H3**, **H4** and $\gamma > 0$. For every environment ξ and every integer $n \geqslant 0$ and $j = 1,\cdots,d$, set

$$\phi^{\cdot,j}_{\xi,n}(t) = (\phi^{1,j}_{\xi,n}(t),\cdots,\phi^{d,j}_{\xi,n}(t)), \quad \text{with} \quad \phi^{i,j}_{\xi,n}(t) = \mathbb{E}_\xi e^{-t\bar{Z}^i_n(j)},\ \ i = 1,\cdots,d,\ \ t \geqslant 0. \tag{3.27}$$

Let ϕ_n^j be the function on $\mathbb{R}_+$ defined by

$$\phi_n^j(t)=\mathbb{E}\|\phi_{\xi,n}^{\cdot,j}(t)\|_\infty=\mathbb{E}\max_{1\leqslant r\leqslant d}\phi_{\xi,n}^{r,j}(t),\quad t\geqslant 0. \tag{3.28}$$

Notice that for all $t\geqslant 0$,

$$\|\phi_n^{\cdot,j}(t)\|_\infty=\|\mathbb{E}\phi_{\xi,n}^{\cdot,j}(t)\|_\infty\leqslant\mathbb{E}\|\phi_{\xi,n}^{\cdot,j}(t)\|_\infty=\phi_n^j(t).$$

Therefore (3.9) holds if we prove that there exist constants $C,a>0$ such that for all $n\geqslant 1$ and $t>0$,

$$\phi_n^j(t)\leqslant Ct^{-a}. \tag{3.29}$$

Now we prove (3.29). In the following, for all statements involving the environment ξ, the validity is for almost every environment, though this is not explicitly stated for simplicity.

From Lemma 3.3 we obtain that $\phi_{\xi,n}^{\cdot,j}(t)$ satisfies the following equation: for all $n\geqslant 0$, $1\leqslant j\leqslant d$ and $t\geqslant 0$,

$$\phi_{\xi,n+1}^{\cdot,j}(t)\leqslant f_0\Big(\phi_{T\xi,n}^{\cdot,j}\Big(\frac{t}{D\|M_0\|}\Big)\Big). \tag{3.30}$$

For $n\geqslant 0$, denote by $Q_1(\xi_n)$ the positive random matrix whose entries are, for $1\leqslant i,j\leqslant d$,

$$Q_1(\xi_n)(i,j)=\mathbb{P}_\xi(\|Z_{n+1}\|\geqslant 2,Z_{n+1}(j)\geqslant 1,Z_{n+1}(r)=0,r<j|Z_n=e_i).$$

It is clear that for $n\geqslant 0$, $Q_1(\xi_n)$ depends only of ξ_n and that the events $\{Z_{n+1}(j)\geqslant 1,Z_{n+1}(r)=0\ \forall r<j\}$, $1\leqslant j\leqslant d$, constitute a partition of $\{\|Z_{n+1}\|\geqslant 1\}$. Hence

$$\|Q_1(\xi_n)(i,\cdot)\|=\sum_{j=1}^d Q_1(\xi_n)(i,j)=\mathbb{P}_\xi(\|Z_{n+1}\|\geqslant 2|Z_n=e_i)\quad\forall 1\leqslant i\leqslant d.$$

By **H1** and the fact that $\|P_1(\xi_n)(i,\cdot)\|=\mathbb{P}_\xi(\|Z_{n+1}\|=1|Z_n=e_i)$, we get that for all $n\geqslant 0$ and $1\leqslant i\leqslant d$,

$$\|P_1(\xi_n)(i,\cdot)+Q_1(\xi_n)(i,\cdot)\|=1, \tag{3.31}$$

which means that $P_1(\xi_n)+Q_1(\xi_n)$ is a stochastic matrix, and $\|P_1(\xi_n)+Q_1(\xi_n)\|=1$. Then, by definition of the matrix $Q_1(\xi_0)$ and using again **H1** and the partition $\{Z_{n+1}(j)\geqslant 1,Z_{n+1}(r)=0\ \forall r<j\}$ $(1\leqslant j\leqslant d)$ of $\{\|Z_{n+1}\|\geqslant 1\}$, we see that for all $s=(s_1,\cdots,s_d)\in[0,1]^d$ and $1\leqslant i\leqslant d$, we have

$$\begin{aligned}f_0^i(s)&=\sum_{j=1}^d\mathbb{P}_\xi(Z_1^i=e_j)s_j+\sum_{k\in\mathbb{N}^d,\|k\|\geqslant 2}\mathbb{P}_\xi(Z_1^i=k)s_1^{k(1)}\cdots s_d^{k(d)}\\&\leqslant\sum_{j=1}^d P_1(\xi_0)(i,j)s_j+\|s\|_\infty\sum_{j=1}^d Q_1(\xi_0)(i,j)s_j\\&=\langle s([P_1(\xi_0)+\|s\|_\infty Q_1(\xi_0)])^T,e_i\rangle.\end{aligned}$$

In terms of the vector form, it reads for all $s=(s_1,\cdots,s_d)\in[0,1]^d$,

$$f_0(s)\leqslant s([P_1(\xi_0)+\|s\|_\infty Q_1(\xi_0)])^T. \tag{3.32}$$

This, together with (3.30), implies that for all $n\geqslant 0$, $1\leqslant j\leqslant d$ and $t\geqslant 0$,

$$\phi_{\xi,n+1}^{\cdot,j}(t)\leqslant\phi_{T\xi,n}^{\cdot,j}\Big(\frac{t}{D\|M_0\|}\Big)\Big(P_1(\xi_0)+\Big\|\phi_{T\xi,n}^{\cdot,j}\Big(\frac{t}{D\|M_0\|}\Big)\Big\|_\infty Q_1(\xi_0)\Big)^T. \tag{3.33}$$

Taking the norm in (3.33), we get for all $n \geqslant 0$ and $t \geqslant 0$,

$$\|\phi^{\cdot,j}_{\xi,n+1}(t)\|_\infty \leqslant \Big\|\phi^{\cdot,j}_{T\xi,n}\Big(\frac{t}{D\|M_0\|}\Big)\Big(P_1(\xi_0) + \Big\|\phi^{\cdot,j}_{T\xi,n}\Big(\frac{t}{D\|M_0\|}\Big)\Big\|_\infty Q_1(\xi_0)\Big)^T\Big\|_\infty. \tag{3.34}$$

Now, by Lemma 3.7, we know that there exist two constants $\beta \in (0,1)$ and $t_0 > 0$ such that $\|\phi^{\cdot,j}_{\xi,n}(t)\|_\infty \leqslant \beta$ for all $t \geqslant t_0$. Using the condition $\|M_0\| \leqslant (dA)$, this implies that for all $n \geqslant 1$, $1 \leqslant j \leqslant d$,

$$\Big\|\phi^{\cdot,j}_{T\xi,n}\Big(\frac{t}{D\|M_0\|}\Big)\Big\|_\infty \leqslant \beta, \quad \forall t \geqslant t_0(dDA). \tag{3.35}$$

From (3.35), (3.34), and the condition that the sequence (ξ_n) is i.i.d., we see that for all $n \geqslant 1$ and $t \geqslant t_0(dDA)$,

$$\begin{aligned}
\phi^j_{n+1}(t) &= \mathbb{E}\Big[\|\phi^{\cdot,j}_{\xi,n+1}(t)\|_\infty\Big] \\
&\leqslant \mathbb{E}\Big[\Big\|\phi^{\cdot,j}_{T\xi,n}\Big(\frac{t}{D\|M_0\|}\Big)\Big(P_1(\xi_0) + \beta Q_1(\xi_0)\Big)^T\Big\|_\infty\Big] \\
&\leqslant \mathbb{E}\Big\{\Big\|\big(P_1(\xi_0) + \beta Q_1(\xi_0)\big)^T\Big\|_\infty \mathbb{E}\Big[\Big\|\phi^{\cdot,j}_{T\xi,n}\Big(\frac{t}{D\|M_0\|}\Big)\Big\|_\infty \mid \xi_0\Big]\Big\} \\
&= \mathbb{E}\Big[\Big\|P_1(\xi_0) + \beta Q_1(\xi_0)\Big\|_\infty \phi^j_n\Big(\frac{t}{D\|M_0\|}\Big)\Big].
\end{aligned} \tag{3.36}$$

We will write this inequality in the form (3.11) in order to use Lemma 3.6. Set

$$\alpha := \mathbb{E}\Big[\Big\|P_1(\xi_0) + \beta Q_1(\xi_0)\Big\|\Big],$$

and let A be a positive random variable whose distribution is determined by the following expectation: for all measurable functions h: $\mathbb{R}_+ \to \mathbb{R}_+$,

$$\mathbb{E}h(A) = \frac{1}{\alpha}\mathbb{E}\Big[\Big\|P_1(\xi_0) + \beta Q_1(\xi_0)\Big\|h\Big(\frac{1}{D\|M_0\|}\Big)\Big].$$

Then we can rewrite (3.36) as follows: for all $n \geqslant 1$ and $t \geqslant t_0(dDA)$,

$$\phi^j_{n+1}(t) \leqslant \alpha\mathbb{E}[\phi^j_n(At)]. \tag{3.37}$$

Let $a > 0$ and $C = (t_0(dDA))^a$. Since $\phi^j_{n+1}(t) \leqslant 1$, it follows that for all $n \geqslant 1$ and $t > 0$,

$$\phi^j_{n+1}(t) \leqslant \alpha\mathbb{E}[\phi^j_n(At)] + \frac{C}{t^a}. \tag{3.38}$$

We will prove that

$$\alpha < 1 \quad \text{and} \quad \alpha\mathbb{E}A^{-a} < 1 \tag{3.39}$$

for some $a > 0$. For the proof of (3.39), recall that from the condition on $\bar{Z}^i_1(j)$, we have $Z^i_1(j) \geqslant 1$ a.s. for all $i, j \in \{1, \cdots, d\}$, so that $\|Z^i_1\| \geqslant d$ a.s. Thus

$$\|P_1(\xi_0)\| = \max_{1\leqslant i\leqslant d}\|P_1(\xi_0)(i,\cdot)\| = \max_{1\leqslant i\leqslant d}\mathbb{P}_\xi(\|Z^i_1\| = 1) = 0.$$

Since $\|P_1(\xi_0) + Q_1(\xi_0)\| = 1$, it follows that

$$\begin{aligned}
\alpha = \mathbb{E}\Big[\Big\|P_1(\xi_0) + \beta Q_1(\xi_0)\Big\|\Big] &= \mathbb{E}\Big[\beta(P_1(\xi_0) + Q_1(\xi_0)) + (1-\beta)P_1(\xi_0)\Big] \\
&\leqslant \beta + (1-\beta)\mathbb{E}\|P_1(\xi_0)\| = \beta < 1.
\end{aligned} \tag{3.40}$$

By the definition of A, we have

$$\alpha\mathbb{E}A^{-a} = \mathbb{E}\Big[(D\|M_0\|)^a\|P_1(\xi_0) + \beta Q_1(\xi_0)\|\Big].$$

Recall that $\|M_0\| \leqslant dA$. So by the dominated convergence theorem we get that

$$\lim_{a\to 0} \alpha \mathbb{E}A^{-a} = \alpha < 1.$$

Therefore for some $a_1 > 0$ and all $a \in (0, a_1]$, we have $\alpha \mathbb{E}A^{-a} < 1$. This ends the proof of (3.39).

Letting $a = \min(a_0, a_1)$, with a_0 defined in condition **H4**, we see that for some constant $C_1 > 0$ and all $t > 0$, $\phi_1^j(t) \leqslant C_1/t^a$. Using this together with (3.38), (3.39) and Lemma 3.6, we see that for all $t > 0$,

$$\sup_{n\geqslant 1} \phi_n(t) \leqslant \frac{C_2}{t^a}, \quad \text{where} \quad C_2 = C_1 + \frac{C}{1 - \alpha\mathbb{E}(A^{-a})}.$$

This ends the proof of Theorem 3.5. □

4. Berry-Esseen bound for $\log Z_n^i(j)$.

In this section, we prove Theorem 2.2 which gives a Berry-Esseen bound for the logarithm of $\log Z_n^i(j)$, for any $1 \leqslant i, j \leqslant d$.

First, we formulate the following lemma giving the convergence in L^1 of $\log \bar{Z}_n^i(j)$ to $\log W^i$ with an exponential rate, for all $1 \leqslant i, j \leqslant d$.

Lemma 4.1. *Assume the conditions **H1**, **H3**, **H4** and $\gamma > 0$. Then there exists two constants $C > 0$ and $\delta \in (0, 1)$ such that for all $n \geqslant 1$ and $1 \leqslant i, j \leqslant d$,*

$$\mathbb{E}|\log \bar{Z}_n^i(j) - \log W^i| \leqslant C.\delta^n. \tag{4.1}$$

Proof. For any $n \geqslant 1$ and $1 \leqslant i, j \leqslant d$, set

$$R_n^{i,j} := \frac{\bar{Z}_n^i(j)}{W^i} - 1.$$

Then, for all $n \geqslant 1$ and $1 \leqslant i, j \leqslant d$ we have

$$\log \bar{Z}_n^i(j) - \log W^i = \log(1 + R_n^{i,j}). \tag{4.2}$$

Let $K \in (0, 1)$ be a constant. From (4.2) we get that for all $n \geqslant 1$ and $1 \leqslant i, j \leqslant d$,

$$\begin{aligned}\mathbb{E}|\log \bar{Z}_n^i(j) - \log W^i| &= \mathbb{E}|\log(1 + R_n^{i,j})\mathbb{1}_{\{R_n^{i,j}\geqslant -K\}}| + \mathbb{E}|\log(1 + R_n^{i,j})\mathbb{1}_{\{R_n^{i,j}<-K\}}| \\ &:= I_1(n) + I_2(n).\end{aligned} \tag{4.3}$$

Now we control the two terms $I_1(n)$ and $I_2(n)$.

Control of $I_1(n)$. Let $\varepsilon \in (0, 1]$, the function $x \mapsto |x|^{-\varepsilon}\log(1+x)$ is bounded on $[-K, +\infty)$, so for all $n \geqslant 1$, $\varepsilon = a$ and $1 \leqslant i, j \leqslant d$ we get

$$I_1(n) = \mathbb{E}|\log(1 + R_n^{i,j})\mathbb{1}_{\{R_n^{i,j}\geqslant -K\}}| \leqslant C\mathbb{E}|R_n^{i,j}|^a. \tag{4.4}$$

We see that

$$\mathbb{E}|R_n^{i,j}|^a = \mathbb{E}\Big|\frac{\bar{Z}_n^i(j)}{W^i} - 1\Big|^a \leqslant \mathbb{E}\Big|\bar{Z}_n^i(j) - W^i\Big|^a \mathbb{E}(W^i)^{-a}. \tag{4.5}$$

By applying the result from [11] for $p > 1$, there exist two constants $C > 0$ and $\delta \in (0, 1)$ such that

$$\mathbb{E}\Big|\bar{Z}_n^i(j) - W^i\Big|^a \leqslant \mathbb{E}\Big|\bar{Z}_n^i(j) - W^i\Big| \leqslant \Big(\mathbb{E}\Big|\bar{Z}_n^i(j) - W^i\Big|^p\Big)^{1/p} \leqslant C\delta^n. \tag{4.6}$$

Combining this with $\mathbb{E}(W^i)^{-a} \leqslant C$, we get

$$\mathbb{E}|R_n^{i,j}|^a \leqslant C\delta^n. \tag{4.7}$$

Therefore, $I_1(n) \leqslant C\delta^n$.

Control of $I_2(n)$. Using the inequality $|\log x|^2 \leqslant C(x + x^{-a})$ for $x > 0$ and Theorem 2.1, we have for all $n \geqslant 1$, $1 \leqslant i, j \leqslant d$,

$$\begin{aligned}\sup_{n\geqslant 0}\Big(\mathbb{E}|\log(1+R_n^{i,j})|^2\Big)^{1/2} &= \sup_{n\geqslant 0}\Big(\mathbb{E}|\log \bar{Z}_n^i(j) - \log W_n^i|^2\Big)^{1/2}\\ &\leqslant \sup_{n\geqslant 0}\Big(\mathbb{E}|\log \bar{Z}_n^i(j)|^2\Big)^{1/2} + \sup_{n\geqslant 0}\Big(\mathbb{E}|\log W_n^i|^2\Big)^{1/2}\\ &\leqslant C\sup_{n\geqslant 0}\Big(\mathbb{E}(\bar{Z}_n^i(j)) + \mathbb{E}(\bar{Z}_n^i(j))^{-a}\Big) + C\sup_{n\geqslant 0}\Big(\mathbb{E}(W_n^i) + \mathbb{E}(W_n^i)^{-a}\Big)\\ &< +\infty. \end{aligned} \tag{4.8}$$

Applying Cauchy-Schwarz's inequality, Markov's inequality and combining with (4.7)-(4.8), we get that for all $n \geqslant 1$, $1 \leqslant i, j \leqslant d$,

$$\begin{aligned} I_2(n) &= \mathbb{E}|\log(1+R_n^{i,j})\mathbb{1}_{\{R_n^{i,j}<-K\}}|\\ &\leqslant \Big(\mathbb{E}|\log(1+R_n^{i,j})|^2\Big)^{1/2}\Big(\mathbb{E}\mathbb{1}_{\{R_n^{i,j}<-K\}}\Big)^{1/2}\\ &\leqslant \Big[\sup_{k\geqslant 0}\Big(\mathbb{E}|\log(1+R_k^{i,j})|^2\Big)^{1/2}\Big]\Big[\mathbb{P}\Big(|R_n^{i,j}|>K\Big)\Big]^{1/2}\\ &\leqslant C\Big(\mathbb{E}|R_n^{i,j}|^a\Big)^{1/2} \leqslant C\delta^{n/2}. \end{aligned} \tag{4.9}$$

Thus, we conclude that that there exist two constants $C > 0$ and $\delta \in (0, 1)$ such that

$$\mathbb{E}|\log \bar{Z}_n^i(j) - \log W^i| \leqslant C.\delta^n.$$

□

Below we recall the Berry-Esseen bound for $\log\langle yM_{0,n-1}, e_j\rangle$, for any $y \in \mathcal{S}$ and $1 \leqslant j \leqslant d$, established in [22, Theorem 2.1]. It will be used in proving the Berry-Esseen theorem for $\log Z_n^i(j)$, for any $1 \leqslant i, j \leqslant d$.

Lemma 4.2. [22] *Assume that for some* $\varepsilon > 0$ *and all* $i, j \in \{1, \cdots, d\}$, $\mathbb{E}M_0(i,j)^{\varepsilon} + \mathbb{E}M_0(i,j)^{-\varepsilon} < \infty$ *and assume also* ***H5****. Then there exists a constant* $C > 0$ *such that for all* $n \geqslant 1$, $y \in \mathcal{S}$ *and* $x \in \mathbb{R}$,

$$\left|\mathbb{P}\left(\frac{\log\langle yM_{0,n-1}, e_j\rangle - n\gamma}{\sigma\sqrt{n}} \leqslant x\right) - \Phi(x)\right| \leqslant \frac{C}{\sqrt{n}}.$$

The following lemma provides inequalities concerning the concentration of the joint distribution of $(\log Z_n^i(j), \log M_{0,n-1}(i,j))$. It reveals that $\log Z_n^i(j)$ and $\log M_{0,n-1}(i,j)$ behave similarly with large probability.

Lemma 4.3. *Assume conditions* ***H1****,* ***H3****,* ***H4****,* $\gamma > 0$ *and* ***H5****. Then there exists a constant* $C > 0$ *such that for all* $x \in \mathbb{R}$,

$$\mathbb{P}\left(\frac{\log Z_n^i(j) - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log M_{0,n-1}(i,j) - n\gamma}{\sigma\sqrt{n}} > x\right) \leqslant \frac{C}{\sqrt{n}} \tag{4.10}$$

and

$$\mathbb{P}\left(\frac{\log Z_n^i(j) - n\gamma}{\sigma\sqrt{n}} > x, \frac{\log M_{0,n-1}(i,j) - n\gamma}{\sigma\sqrt{n}} \leqslant x\right) \leqslant \frac{C}{\sqrt{n}} \tag{4.11}$$

Proof. We will only give a proof of (4.10), since the other inequality (4.11) can be proved by similar arguments. For $n \geqslant 1$, $0 \leqslant m \leqslant n$ and $y, z \in \mathbb{R}_+^d \setminus \{0\}$ and $1 \leqslant i, j \leqslant d$, set

$$S_{m,n}^{y,z} := \frac{\log\langle yM_{m,n-1}, z\rangle - (n-m)\gamma}{\sigma\sqrt{n}} \quad \text{and} \quad L_{m,n}^{i} := \frac{\log W_m^i}{\sigma\sqrt{n}}, \quad L_{n,n}^{i,j} := \frac{\log \bar{Z}_n^i(j)}{\sigma\sqrt{n}}.$$

In the following we take $m := m(n) = \lfloor\sqrt{n}\rfloor$, where $\lfloor x\rfloor$ is the integer part of x; $C > 0$ will be a constant which may depend on p and η, and which may differ from line to line. By Markov's inequality, we get that there exist constants $\delta_1, \delta_2 \in (0,1)$ such that for all $n \geqslant 1$ and $1 \leqslant i, j \leqslant d$,

$$\begin{aligned}
\mathbb{P}\Big(|L_{n,n}^{i,j} - L_{m,n}^{i}| > \frac{1}{\sqrt{n}}\Big) &\leqslant \sqrt{n}\,\mathbb{E}|L_{n,n}^{i,j} - L_{m,n}^{i}| \\
&= \frac{1}{\sigma}\,\mathbb{E}|\log \bar{Z}_n^i(j) - \log W_m^i| \\
&\leqslant \frac{1}{\sigma}\,\mathbb{E}|\log \bar{Z}_n^i(j) - \log W^i| + \frac{1}{\sigma}\,\mathbb{E}|\log W_m^i - \log W^i| \quad \leqslant C(\delta_1^n + \delta_2^m),
\end{aligned}$$

the last step holds by using [9, Lemma 5.1] and Lemma 4.1.

Since $\delta_1^n + \delta_2^m = o(\frac{1}{\sqrt{n}})$ as $n \to +\infty$, this implies that for all $n \geqslant 1$ and $1 \leqslant i, j \leqslant d$,

$$\mathbb{P}\Big(|L_{n,n}^{i,j} - L_{m,n}^{i}| > \frac{1}{\sqrt{n}}\Big) \leqslant \frac{C}{\sqrt{n}}. \tag{4.12}$$

Thus we obtain that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i, j \leqslant d$,

$$\begin{aligned}
&\mathbb{P}\Big(\frac{\log Z_n^i(j) - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log M_{0,n-1}(i,j) - n\gamma}{\sigma\sqrt{n}} > x\Big) \\
=\ &\mathbb{P}\Big(S_{0,n}^{e_i,e_j} + L_{n,n}^{i,j} \leqslant x, S_{0,n}^{e_i,e_j} > x\Big) \\
\leqslant\ &\mathbb{P}\Big(S_{0,n}^{e_i,e_j} + L_{m,n}^{i} \leqslant x + \frac{1}{\sqrt{n}}, S_{0,n}^{e_i,e_j} > x\Big) \\
&+\mathbb{P}\Big(|L_{n,n}^{i,j} - L_{m,n}^{i}| > \frac{1}{\sqrt{n}}\Big) \\
\leqslant\ &\mathbb{P}\Big(S_{0,n}^{e_i,e_j} + L_{m,n}^{i} \leqslant x + \frac{1}{\sqrt{n}}, S_{0,n}^{e_i,e_j} > x\Big) + \frac{C}{\sqrt{n}}.
\end{aligned} \tag{4.13}$$

Recall that for $y \in \mathcal{S}$ and $M \in G^\circ$, we denote by $y \cdot M := \frac{yM}{\|yM\|}$ the projective action of M on G°. Then, for $y \in \mathcal{S}$ the process

$$X_0^y = y, \quad \text{and} \quad X_n^y = y \cdot M_{0,n-1}, \quad n \geqslant 1,$$

is a Markov chain on $\mathcal{S}$. Notice that for all $n \geqslant 1$ and $1 \leqslant i, j \leqslant d$, we have the decomposition

$$\begin{aligned}
S_{0,n}^{e_i,e_j} &= \frac{\log\langle (e_iM_{0,m})M_{m+1,n-1}, e_j\rangle - n\gamma}{\sigma\sqrt{n}} \\
&= \frac{\log\|e_iM_{0,m}\| + \log\langle (e_i \cdot M_{0,m})M_{m+1,n-1}, e_j\rangle - n\gamma}{\sigma\sqrt{n}} \\
&= \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i,\mathbb{1}} + S_{m+1,n}^{X_{m+1}^{e_i},e_j}.
\end{aligned} \tag{4.14}$$

Moreover, by (2.6) we have that for all $n \geqslant 0$,

$$1 \geqslant \min_{1\leqslant r\leqslant d} u_n(r) = \min_{1\leqslant r\leqslant d} \frac{\langle M_n(r,\cdot), u_{n+1}\rangle}{\|u_{n+1}M_n^T\|} \geqslant \min_{1\leqslant r,u\leqslant d} \frac{M_n(r,u)}{\|M_n\|}. \tag{4.15}$$

We have the two following inequalities: for all $n \geqslant 1$ and $1 \leqslant i, j \leqslant d$, $\mathbb{P}$-a.s.,

$$\log \|Z_n^i\| \leqslant \log \|M_{0,n-1}(i,\cdot)\| + \log W_n^i - \min_{1\leqslant r\leqslant d} \log u_n(r), \tag{4.16}$$

$$\log \|Z_n^i\| \geqslant \log \|M_{0,n-1}(i,\cdot)\| + \log W_n^i + \min_{1\leqslant r\leqslant d} \log u_n(r). \tag{4.17}$$

Therefore, putting together the relations (4.13)-(4.16), we obtain that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i, j \leqslant d$,

$$\begin{aligned} &\mathbb{P}\Big(\frac{\log Z_n^i(j) - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log M_{0,n-1}(i,j) - n\gamma}{\sigma\sqrt{n}} > x\Big) \\ \leqslant\ &\mathbb{P}\Big(\sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i,\mathbb{1}} + S_{m+1,n}^{X_{m+1}^{e_i},e_j} + B_{m,n}^i \leqslant x + \frac{1}{\sqrt{n}}, \\ &\sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i,\mathbb{1}} + S_{m+1,n}^{X_{m+1}^{e_i},e_j} > x\Big) + \frac{C}{\sqrt{n}}, \end{aligned} \tag{4.18}$$

with

$$B_{m,n}^i := \frac{1}{\sigma\sqrt{n}} \log \frac{\|Z_m^i\|}{\|M_{0,m-1}(i,\cdot)\|} + \frac{1}{\sigma\sqrt{n}} \min_{1\leqslant r,u\leqslant d} \log \frac{M_m(r,u)}{\|M_m\|}.$$

Denote by $\nu_{m,n}^i$ the joint law of $\Big(X_{m+1}^{e_i}, \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i,\mathbb{1}}, B_{m,n}^i\Big)$ on $\mathcal{S} \times \mathbb{R} \times \mathbb{R}$. For $y \in \mathcal{S}$ and $x \in \mathbb{R}$, set

$$G_{m,n}^{y,e_j}(x) = \mathbb{P}(S_{m,n}^{y,e_j} \leqslant x).$$

Since $S_{m+1,n}^{y,e_j}$ is independent of $X_{m+1}^{e_i}$, $S_{0,m+1}^{e_i,\mathbb{1}}$ and $B_{m,n}^i$ for any $y \in \mathcal{S}$, we obtain from (4.18) that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i, j \leqslant d$,

$$\begin{aligned} &\mathbb{P}\Big(\frac{\log Z_n^i(j) - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log M_{0,n-1}(i,j) - n\gamma}{\sigma\sqrt{n}} > x\Big) \\ \leqslant\ &\int \mathbb{P}\Big(S_{m+1,n}^{y,e_j} + s + t \leqslant x + \frac{1}{\sqrt{n}}, S_{m+1,n}^{y,e_j} + s > x\Big) \nu_{m,n}^i(dy, ds, dt) + \frac{C}{\sqrt{n}} \\ =\ &\int \mathbb{1}_{\{t\leqslant \frac{1}{\sqrt{n}}\}} \Big[G_{m+1,n}^{y,e_j}\Big(x - s - t + \frac{1}{\sqrt{n}}\Big) - G_{m+1,n}^{y,e_j}(x-s)\Big] \nu_{m,n}^i(dy, ds, dt) \\ &+ \frac{C}{\sqrt{n}}. \end{aligned} \tag{4.19}$$

The random matrices M_n, $n \geqslant 0$, are i.i.d., so for $n \geqslant 1$, $y \in \mathcal{S}$ and $x \in \mathbb{R}$ we have

$$G_{m+1,n}^{y,e_j}(x) = \mathbb{P}\Big(\frac{\log\langle yM_{0,n-m-1}, e_j\rangle - (n-m-1)\gamma}{\sigma\sqrt{n}} \leqslant x\Big) = G_{0,n-m-1}^{y,e_j}(a_n x),$$

with $a_n = \sqrt{\frac{n}{n-m-1}}$. Notice that $a_n = (1 - \frac{m+1}{n})^{-1/2} = 1 + O(\frac{m}{n}) = 1 + O(\frac{1}{\sqrt{n}})$ as $n \to +\infty$. Therefore, applying the Berry-Esseen bound of Lemma 4.2, we get that for all $n \geqslant 1$, $y \in \mathcal{S}$ and $x \in \mathbb{R}$,

$$\begin{aligned} |G_{m+1,n}^{y,e_j}(x) - \Phi(a_n x)| &= |G_{0,n-m-1}^{y,e_j}(a_n x) - \Phi(a_n x)| \\ &\leqslant \frac{C}{\sqrt{n-m-1}} \\ &= \frac{Ca_n}{\sqrt{n}} \leqslant \frac{C}{\sqrt{n}}. \end{aligned} \tag{4.20}$$

Moreover, using the mean value theorem on the function $t \mapsto \Phi(tx)$ with $t \geqslant 1$, we obtain that for all $n \geqslant 1$ and $x \in \mathbb{R}$,

$$|\Phi(a_n x) - \Phi(x)| \leqslant |a_n - 1| \sup_{t \geqslant 1} |x\Phi'(tx)| \leqslant \frac{C}{\sqrt{n}} \sup_{z \in \mathbb{R}} |z\Phi'(z)|. \tag{4.21}$$

It is clear that $z \mapsto |z\Phi'(z)|$ is a bounded function on $\mathbb{R}$. Combining this with the inequalities (4.19)-(4.21), we deduce that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} &\mathbb{P}\Big(\frac{\log Z_n^i(j) - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log M_{0,n-1}(i,j) - n\gamma}{\sigma\sqrt{n}} > x\Big) \\ \leqslant\ & \int \mathbb{1}_{\{t \leqslant \frac{1}{\sqrt{n}}\}} \Big|\Phi\Big(x - s - t + \frac{1}{\sqrt{n}}\Big) - \Phi(x - s)\Big| \nu_{m,n}^i(dy, ds, dt) + \frac{C}{\sqrt{n}}. \end{aligned} \tag{4.22}$$

By the mean value theorem and the fact that $\sup_{x \in \mathbb{R}} |\Phi'(x)| \leqslant 1$, for all $x, z \in \mathbb{R}$ we have

$$|\Phi(x + z) - \Phi(x)| \leqslant |z|.$$

This, together with (4.22), implies that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i, j \leqslant d$,

$$\begin{aligned} &\mathbb{P}\Big(\frac{\log Z_n^i(j) - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log M_{0,n-1}(i,j) - n\gamma}{\sigma\sqrt{n}} > x\Big) \\ \leqslant\ & \int \mathbb{1}_{\{t \leqslant \frac{1}{\sqrt{n}}\}} \Big|\frac{1}{\sqrt{n}} - t\Big| \nu_{m,n}^i(dy, ds, dt) + \frac{C}{\sqrt{n}} \\ \leqslant\ & \int |t| \nu_{m,n}^i(dy, ds, dt) + \frac{C}{\sqrt{n}} \\ =\ & \mathbb{E}|B_{m,n}^i| + \frac{C}{\sqrt{n}}. \end{aligned} \tag{4.23}$$

By definition of $B_{m,n}^i$, combining with (4.15)-(4.16), we get that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s.,

$$\begin{aligned} \sigma\sqrt{n}|B_{m,n}^i| &\leqslant \Big|\log \frac{\|Z_m^i\|}{\|M_{0,m-1}(i,\cdot)\|}\Big| + \min_{1 \leqslant r,u \leqslant d} |\log M_m(r,u)| + |\log \|M_m\||| \\ &\leqslant |\log W_m^i| + |\log \|M_m\||| + \max_{1 \leqslant r,u \leqslant d} \Big(|\log u_m(u)| + |\log M_m(r,u)|\Big) \\ &\leqslant |\log W_m^i| + 2|\log \|M_m\||| + 2 \max_{1 \leqslant r,u \leqslant d} |\log M_m(r,u)|. \end{aligned} \tag{4.24}$$

By [9, Lemma 5.1] we have $\sup_{n \geqslant 0} \mathbb{E}|\log W_n^i| < +\infty$ for any $1 \leqslant i \leqslant d$. Moreover, from condition **H3** and the inequality $|\log x| \leqslant C(x^\eta + x^{-\eta})$ for $x > 0$, it holds that $\mathbb{E}|\log \|M_0\|| < +\infty$ and $\mathbb{E}|\log M_0(r,u)| < +\infty$, $1 \leqslant r, u \leqslant d$. Therefore, taking expectation in (4.24), this implies that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}|B_{m,n}^i| \leqslant \frac{C}{\sqrt{n}}. \tag{4.25}$$

Hence, (4.10) follows from (4.23) and (4.25). This concludes the proof of Lemma 4.3. □

Proof of Theorem 2.2. For $x \in \mathbb{R}$ we write

$$\begin{aligned}
&\mathbb{P}\Big(\frac{\log Z_n^i(j) - n\gamma}{\sigma\sqrt{n}} \leqslant x\Big) \\
=\ &\mathbb{P}\Big(\frac{\log Z_n^i(j) - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log M_{0,n-1}(i,j) - n\gamma}{\sigma\sqrt{n}} \leqslant x\Big) \\
&+\mathbb{P}\Big(\frac{\log Z_n^i(j) - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log M_{0,n-1}(i,j) - n\gamma}{\sigma\sqrt{n}} > x\Big) \\
=\ &\mathbb{P}\Big(\frac{\log M_{0,n-1}(i,j) - n\gamma}{\sigma\sqrt{n}} \leqslant x\Big) \\
&-\mathbb{P}\Big(\frac{\log Z_n^i(j) - n\gamma}{\sigma\sqrt{n}} > x, \frac{\log M_{0,n-1}(i,j) - n\gamma}{\sigma\sqrt{n}} \leqslant x\Big) \\
&+\mathbb{P}\Big(\frac{\log Z_n^i(j) - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log M_{0,n-1}(i,j) - n\gamma}{\sigma\sqrt{n}} > x\Big).
\end{aligned}$$

By Lemma 4.3, we get that there exists a constant $C > 0$ such that for all $x \in \mathbb{R}$,

$$\Big|\mathbb{P}\Big(\frac{\log Z_n^i(j) - n\gamma}{\sigma\sqrt{n}} \leqslant x\Big) - \mathbb{P}\Big(\frac{\log M_{0,n-1}(i,j) - n\gamma}{\sigma\sqrt{n}} \leqslant x\Big)\Big| \leqslant \frac{C}{\sqrt{n}}. \tag{4.26}$$

Combining (4.26) with Lemma 4.2, we obtain the Berry-Esseen bound for $\log Z_n^i(j)$. This concludes the proof of Theorem 2.2. □

5. Moderate deviation principle for $\log Z_n^i(j)$

In this section we prove the moderate deviation principle (MDP) for $\log Z_n^i(j)$, using Theorem 2.1 about the uniform finiteness of the harmonic moments of $\bar{Z}_n^i(j)$ and the MDP for the products of random matrices established in [23].

Proof of Theorem 2.3. From the moderate deviation principle for $\log M_{0,n-1}(i,j)$ established in [23, Theorem 2.7], we know that Theorem 2.3 holds if we can prove the following exponential equivalence (see Theorem 4.2.13 in [6]): for any $\delta > 0$,

$$\lim_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\Big(\frac{|\log Z_n^i(j) - \log M_{0,n-1}(i,j)|}{a_n} > \delta\Big) = -\infty. \tag{5.1}$$

To see this, we take $a \in (0,1]$ such that $C = \sup_n \mathbb{E}(\bar{Z}_n^i(j))^{-a} < \infty$ (this is possible by Theorem 2.1), and use and the elementary inequality that $e^{a|\log x|} \leqslant x^a + x^{-a}$ for $x > 0$. Thus, by the Markov inequality, we have for any $\delta > 0$,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\Big(\frac{|\log Z_n^i(j) - \log M_{0,n-1}(i,j)|}{a_n} > \delta\Big) \\
&= \limsup_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\Big(\frac{|\log \bar{Z}_n^i(j)|}{a_n} > \delta\Big) \\
&= \limsup_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\big(e^{a|\log \bar{Z}_n^i(j)|} > e^{a\delta a_n}\big) \\
&\leqslant \limsup_{n\to\infty} \frac{n}{a_n^2} \log \left(e^{-a_n a\delta} \mathbb{E}(\bar{Z}_n^i(j)^a + \bar{Z}_n^i(j)^{-a})\right) \\
&\leqslant \limsup_{n\to\infty} \frac{-na\delta}{a_n} + \limsup_{n\to\infty} \frac{n}{a_n^2} \log(1+C),
\end{aligned} \tag{5.2}$$

where the last step holds since $\mathbb{E}\bar{Z}_n^i(j) = 1$ and $\mathbb{E}(\bar{Z}_n^i(j))^{-a} \leqslant C$. In (5.2), the first lim sup is $-\infty$ due to the condition that $n/a_n \to \infty$; the second lim sup is equal to 0 because $n/a_n^2 \to 0$. Therefore (5.1) follows from (5.2). This ends the proof of Theorem 2.3.

□

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