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PRODUCTS OF POSITIVE RANDOM MATRICES AND MULTI-TYPE BRANCHING PROCESSES IN RANDOM ENVIRONMENTS: MOMENTS AND LARGE DEVIATIONS

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**PRODUCTS OF POSITIVE RANDOM MATRICES AND MULTI-TYPE
BRANCHING PROCESSES IN RANDOM ENVIRONMENTS:
MOMENTS AND LARGE DEVIATIONS**

ION GRAMA, QUANSHENG LIU, AND THI TRANG NGUYEN

ABSTRACT. Motivated primarily by the study of large deviations of multitype branching processes in random environments, we first establish, for products of independent and identically distributed random positive matrices $(M_n)_{n \in \mathbb{Z}}$, a Perron-Frobenius type theorem under the Cramér type changed measure, the stable and mixing convergence of the direction of the random walk $xM_0 \cdots M_n$ (with $x \in \mathbb{R}_+^d$) as $n \rightarrow \infty$, under both the initial probability and the changed measure. We also determine the exact growth rate of the moments of the vector norm $\|xM_0 \cdots M_n\|$, the entrywise L^1 matrix norm $\|M_0 \cdots M_n\|_{1,1}$, and the scalar product $\langle xM_0 \cdots M_n, y \rangle$ for $x, y \in \mathbb{R}_+^d$ with unit norm. As applications, we derive precise large deviation results for the population size $\|Z_n\|$ of n -th generation, for a d -type branching process $Z_n = (Z_n(1), \dots, Z_n(d))$ in an independent and identically distributed random environment, by giving an equivalence of the large deviation probability $\mathbb{P}[\|Z_n\| \geq e^{nq}]$, for $q > 0$ in a suitable range. Additionally, we obtain precise estimation of the moments of $\|Z_n\|$ and those of the j -type population size $Z_n(j)$.

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1. INTRODUCTION AND MAIN RESULTS

1.1. Motivation and main objective. Mainly motivated by the study of precise large deviations for multitype branching processes in random environments, we begin with establishing some limit theorems for products of independent and identically distributed (i.i.d.) random positive $d \times d$ matrices $(M_n)_{n \in \mathbb{Z}}$. This topic is of independent interest, thanks to a large number of applications and interactions with important problems arising in various fields. For important progress on this topic in the last decade, see e.g. Benoit and Quint [7, 8], Buraczewski, Damek, Guivarc'h, Mentemeier [10], and Guivarc'h and Le Page [36]. See also Fernando and Pène [25], Cuny, Dedecker, Merlevède, Peligrad [18], Xiao, Grama and Liu [74, 76, 77] for recent works on large deviations and the convergence rate in central limit theorems.

With the spirit for applications in branching processes, we first establish a Perron-Frobenius type theorem for the products $M_{k,n} = M_k \cdots M_n$, $k \leq n$ of the random matrices (M_n) defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, under the Cramér type changed measure \mathbb{Q}_s^v with s in a suitable interval I_μ of \mathbb{R} , and $v \in \mathcal{S} = \{x \in \mathbb{R}_+^d : \|x\| = 1\}$. For a matrix g , denote by g^T its transpose. Under a simple positivity condition, we prove that there exist two sequences $(u_k), (v_k)$ of elements of $\mathcal{S} \cap \mathbb{R}_+^{*d}$, such that, \mathbb{Q}_s^v a.s. for each fixed $k \in \mathbb{Z}$ as $n \rightarrow \infty$, and for each fixed $n \in \mathbb{Z}$ as $k \rightarrow -\infty$, with $x \cdot g = xg/(\|xg\|)$,

$$x \cdot (M_{k,n})^T - u_k \rightarrow 0, \quad x \cdot M_{k,n} - v_n \rightarrow 0, \quad \text{uniformly for } x \in \mathcal{S}, \quad (1.1)$$

$$\langle xM_{k,n}, y \rangle \sim a_{k,n} \langle u_k, x \rangle \langle v_n, y \rangle, \quad \text{uniformly for } x, y \in \mathbb{R}_+^d \setminus \{0\}, \quad (1.2)$$

where $a_{k,n} = \|M_{k,n}\|_{1,1} = \sum_{i,j=1}^d M_{k,n}(i,j)$ is the entrywise L^1 norm. Moreover, (u_k) and (v_n) satisfy, \mathbb{Q}_s^v -a.s., $u_{k+1} \cdot M_k^T = u_k, v_{k-1} \cdot M_k = v_k, \forall k \in \mathbb{Z}$. These results extend the known ones under the initial probability established by Hennion [37] and in our previous work [30].

We then prove the stable and mixing convergence of the sequences of directions $x \cdot M_{0,n}$ and v_n : for any $m \geq 1$ and \mathbb{R}^m -valued random variable η defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, both sequences $(x \cdot M_{0,n}, \eta)$ and (v_n, η) converge in law, under both the initial measure and the changed measure \mathbb{Q}_s^v , to the product law $\pi_s \otimes \mathcal{L}(\eta)$, where π_s is the unique invariant probability law of the Markov chain $(x \cdot M_{0,n})_{n \geq 0}$ under \mathbb{Q}_s^v , $\mathcal{L}(\eta)$ denotes the law of η . We also give precise description of the growth rate of the moments of the vector-norm $\|xM_{0,n-1}\|$ by proving that, for some function $\kappa(s)$ explicitly defined, the limit

$$m(x, s) = \lim_{n \rightarrow \infty} \frac{\|xM_{0,n-1}\|^s}{\kappa^n(s)} \text{ exists with value in } (0, \infty), \quad \forall x \in \mathcal{S}. \quad (1.3)$$

A similar result is also established for the matrix norm $\|M_{0,n-1}\|_{1,1}$, for the scalar product $\langle xM_{k,n}, y \rangle$ with $x, y \in \mathcal{S}$, and for the spectral radius $\rho_{0,n-1}$ of $M_{0,n-1}$.

The above mentioned results will be applied to study asymptotic properties of multitype branching processes in random environments. A branching process in a random environment is a natural and important extension of the Galton-Watson process. In such a process, the offspring distributions of particles in n -th generation depend on an environment ξ_n at time n .

The process has been first introduced by Smith and Wilkinson [65] for an i.i.d. environment, and by Athreya and Karlin [3, 4] for a stationary and ergodic environment. Fundamental results can be found in the early papers by Athreya and Karlin [3, 4], Kaplin [51] and Tanny [68, 70, 71]. For the development of the theory, we refer to the recent book by Kersting and Vatutin [52].

The process has attracted the attention of many authors over the past two decades, and interest in studying it has grown in recent years, thanks to its numerous applications and interactions with other scientific fields. For the single type case, see e.g. Geiger, Kersting and Vatutin [28] and Afanasyev, Geiger, Kersting, and Vatutin [1] on the survival probability in the subcritical and critical cases, Bansaye and Berestycki [6], Huang and Liu [46], Grama, Liu and Miqueu [29], Buraczewski and Dyszewski [12], and Buraczewski and Damek [13] on large deviations and central limit theorems (with rate of convergence) in the supercritical case. For the multitype case, see e.g. Le Page, Peigné and Pham [60], Vatutin and Dyakonova [72] and Vatutin and Wachtel [73] on the survival probability for critical and subcritical processes, and Grama, Liu and Pin [31, 32] on the Kesten-Stigum theorem and L^p convergence for supercritical processes.

In this paper we consider large deviations for a d -type branching process in an i.i.d. environment, say $Z_n = (Z_n(1), \dots, Z_n(d))$, $n \geq 0$. For the single type case ($d = 1$), moderate and large deviations have been considered in [6, 46, 29, 12]. In particular, precise large deviation for Z_n has been given in [12]. The multitype case ($d > 1$) has been significantly less explored. We are only aware of the Cramér type moderate deviation result established in [34] for $\|Z_n\| = Z_n(1) + \dots + Z_n(d)$, the total population size of generation n . Here we focus on precise large deviations, specifically the Bahadur-Rao type large deviations of $\|Z_n\|$. Under suitable conditions, we prove the precise asymptotic behavior of the form (see Theorem 1.6 below): for $q > 0$ in a suitable range,

$$\mathbb{P}\left(\log \|Z_n\| \geq qn\right) \sim \frac{C(q)}{\sqrt{n}} e^{-I(q)n}, \quad (1.4)$$

where the rate function $I(q)$ and the constant $C(q)$ are given explicitly. We also give the precise growth rate of the moments of $\|Z_n\|$, by proving that the limit

$$Z(s) = \lim_{n \rightarrow \infty} \frac{\|Z_n\|^s}{\kappa^n(s)} \quad \text{exists with value in } (0, \infty). \quad (1.5)$$

A similar result is also established for the j -type population size $Z_n(j)$.

Let us give a precise description of the model. Let $\xi = (\xi_0, \xi_1, \dots)$ be a sequence of independent and identically distributed (i.i.d.) random variables taking values in some measurable space (E, \mathcal{E}) , where ξ_n represents the random environment at time n . Let $d \geq 1$ be an integer. Each realization of ξ_n corresponds to d probability distributions on \mathbb{N}^d :

$$p^r(\xi_n) = \{p_j^r(\xi_n) : j \in \mathbb{N}^d\}, \quad \text{where } p_j^r(\xi_n) \geq 0, \quad \sum_{j \in \mathbb{N}^d} p_j^r(\xi_n) = 1, \quad r = 1, \dots, d.$$

A multitype branching process in the random environment ξ (MBPRE) is a process $Z_n = (Z_n(1), \dots, Z_n(d))$, $n \geq 0$, with values in \mathbb{N}^d , such that for all $n \geq 0$,

$$Z_{n+1} = \sum_{r=1}^d \sum_{l=1}^{Z_n(r)} N_{l,n}^r. \quad (1.6)$$

Here $Z_n(r)$ denotes the number of type r particles of generation n , and the j -th component $N_{l,n}^r(j)$ of $N_{l,n}^r$ is the number of type j offspring of the l -th type r particle of generation n . Conditioned on the environment ξ , the random vectors Z_0 and $N_{l,n}^r$, indexed by $l \geq 1$,

$n \geq 0$ and $1 \leq r \leq d$, are all independent; for all $l \geq 1$, each $N_{l,n}^r$ has the same distribution $p^r(\xi_n)$ depending on the environment ξ_n at time n .

Let $(\Lambda, \mathcal{A}, \mathbb{P}_\xi)$ be the probability space under which the process $(Z_n)_{n \geq 0}$ is defined when the environment ξ is given. The total probability \mathbb{P} can be formulated as

$$\mathbb{P}(d\xi, dy) = \mathbb{P}_\xi(dy)\tau(d\xi), \quad (1.7)$$

defined on the product space $(\Omega, \mathcal{F}) = (E^{\mathbb{N}} \times \Lambda, \mathcal{E}^{\otimes \mathbb{N}} \otimes \mathcal{A})$, where τ denotes the law of the environment sequence ξ (which is a probability measure on $(E^{\mathbb{N}}, \mathcal{E}^{\otimes \mathbb{N}})$); by definition

$$\int_{E^{\mathbb{N}} \times \Lambda} f(\xi, y) \mathbb{P}(d\xi, dy) = \int_{E^{\mathbb{N}}} \left[\int_{\Lambda} f(\xi, y) \mathbb{P}_\xi(dy) \right] \tau(d\xi), \quad (1.8)$$

for any positive and measurable function f (with an abuse of notation, we use the same letter ξ to denote both the environment sequence and its possible values). The probability \mathbb{P}_ξ is usually called quenched law, while the total probability \mathbb{P} is called annealed law.

The quenched law \mathbb{P}_ξ can be considered as the conditional law of \mathbb{P} given the environment ξ : $\mathbb{P}_\xi(\cdot) = \mathbb{P}(\cdot | \xi)$. The expectation with respect to \mathbb{P}_ξ and \mathbb{P} will be denoted respectively by \mathbb{E}_ξ and \mathbb{E} . According to the definition of the model, under \mathbb{P}_ξ , the random vectors $N_{l,n}^r = (N_{l,n}^r(1), \dots, N_{l,n}^r(d))$ are independent and have the probability generating function

$$f_n^r(s) = \mathbb{E}_\xi(s^{N_{l,n}^r}) = \sum_{k \in \mathbb{N}^d} p_k^r(\xi_n) s^k, \quad s = (s_1, \dots, s_d) \in [0, 1]^d \quad (1.9)$$

(which does not depend on l), where by notation, $s^k = \prod_{j=1}^d s_j^{k_j}$ if $s = (s_1, \dots, s_d)$ and $k = (k_1, \dots, k_d)$. Let

$$f_n = (f_n^1, \dots, f_n^d). \quad (1.10)$$

Denote the mean matrix of the offspring distributions of time n by M_n , whose (r, j) -th term is

$$M_n(r, j) = \frac{\partial f_n^r}{\partial s_j}(1) = \mathbb{E}_\xi N_{l,n}^r(j).$$

Notice that the mean matrix M_n just depends on ξ_n (it is of the form $M_n = M(\xi_n)$). Since the environment sequence (ξ_n) is assumed to be i.i.d., so is the sequence of matrices $(M_n)_{n \geq 0}$. We will use the natural filtration $(\mathcal{F}_n)_{n \geq 0}$, where $\mathcal{F}_0 = \sigma\{\xi\}$ and

$$\mathcal{F}_n = \sigma\{\xi, N_{l,j}^r : l \geq 1, 0 \leq j < n, 1 \leq r \leq d\} \quad \text{for } n \geq 1. \quad (1.11)$$

It is well-known that when $\mathbb{E} \log^+ \|M_0\| < +\infty$, then the Lyapunov exponent

$$\gamma := \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\| \quad (1.12)$$

is well-defined, and is equal to the quantity $\inf_{k \geq 1} \frac{1}{k} \mathbb{E} \log \|M_{0,k-1}\|$, which lies in $\{-\infty\} \cup \mathbb{R}$.

The process (Z_n^i) is called supercritical, critical or subcritical, according to $\gamma > 0, = 0$ or < 0 , respectively.

1.2. Notation and conditions. In the sequel, the process Z_n will be noted Z_n^i when $Z_0 = e_i$, which means that the process starts with one initial particle of type i . For any $k, n \in \mathbb{Z}$, let

$$M_{k,n} = M_k \dots M_n \quad \text{and} \quad M_{k,n}^T = (M_{k,n})^T \quad \text{if } k \leq n, \quad M_{k,n} = I_d \quad \text{if } k > n, \quad (1.13)$$

where I_d denotes the $d \times d$ identity matrix, M^T denotes the transpose of M (the notation $M_{k,n}^T$ should not be confused with $M_k^T \dots M_n^T$).

Let \mathbb{R}^d be the d -dimensional Euclidean space. Each element $v \in \mathbb{R}^d$ is written as a row vector. The transpose of $v \in \mathbb{R}^d$ is denoted by v^T . For $1 \leq i \leq d$ denote by $e_i \in \mathbb{R}^d$ the

vector with 1 in the i -th coordinate and 0 elsewhere. The symbol $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^d$ stands for the vector with all coordinates equal to 1. The indicator function of an event E is denoted by $\mathbb{1}_E$. The symbol $\xrightarrow{d(\mathbb{P})}$ denotes the convergence in distribution under \mathbb{P} . For any $x, y \in \mathbb{R}^d$, the scalar product and the L^1 norm in \mathbb{R}^d are defined by

$$\langle x, y \rangle := \sum_{i=1}^d x(i) y(i) \quad \text{and} \quad \|x\| := \sum_{i=1}^d |x(i)|. \quad (1.14)$$

For a matrix or a vector a , we write $a > 0$ to mean that each entry of a is strictly positive. We denote by G the multiplicative semigroup of $d \times d$ matrices with non-negative entries. The subsemigroup of G composed of strictly positive matrices is denoted by G° . Let

$$\mathcal{S} = \{v \in \mathbb{R}_+^d, \|v\| = 1\},$$

where \mathbb{R}_+^d is the positive quadrant of \mathbb{R}^d and denote by \mathcal{S}° the interior of \mathcal{S} . The space \mathcal{S} is referred as the projective space. For any matrix $g \in G$ and any $x \in \mathcal{S}$ we define the action of g on x by setting

$$x \cdot g = \frac{xg}{\|xg\|}, \quad \text{when } xg \neq 0. \quad (1.15)$$

Let us state various conditions to be used latter on. For any $g \in G$, define the operator norm and the iota function:

$$\begin{aligned} \|g\| &= \sup_{\|x\|=1} \|xg\| = \max_{1 \leq i \leq d} \sum_{j=1}^d g(i, j) = \max_{1 \leq i \leq d} \|e_i g\|, \\ \iota(g) &= \inf_{\|x\|=1} \|xg\| = \min_{1 \leq i \leq d} \sum_{j=1}^d g(i, j) = \min_{1 \leq i \leq d} \|e_i g\|, \end{aligned}$$

where $\iota(g) > 0$ for $g \in G^\circ$. We shall also use the entry-wise L^1 -matrix norm: for $g \in G$,

$$\|g\|_{1,1} = \langle \mathbf{1}, \mathbf{1}g \rangle = \sum_{i,j=1}^d g(i, j). \quad (1.16)$$

Let μ be the common law of M_n . Set

$$I_\mu^+ = \{s \geq 0 : \mathbb{E}(\|M_0\|^s) < \infty\} \quad \text{and} \quad I_\mu^- = \{s \leq 0 : \mathbb{E}(\|M_0\|^s) < \infty\}.$$

By Hölder's inequality, both I_μ^+ and I_μ^- are intervals of \mathbb{R} . The interior of a set A is denoted by A° , so that $(I_\mu^+)^\circ$ and $(I_\mu^-)^\circ$ are the interiors of I_μ^+ and I_μ^- respectively.

Denote by $\text{supp } \mu$ the support of the measure μ and by $[\text{supp } \mu]$ the closed semigroup spanned by $\text{supp } \mu$. We will need the following condition introduced by Furstenberg and Kesten [27]:

A1. *There exists a constant $D > 1$ such that for μ -almost every $g \in G$,*

$$0 < \max_{1 \leq i, j \leq d} \langle e_i, e_j g \rangle \leq D \min_{1 \leq i, j \leq d} \langle e_i, e_j g \rangle. \quad (1.17)$$

It can be easily checked that, for any matrix g satisfying (1.17), we have

$$\frac{1}{D} \|g\| \leq \iota(g) \leq \|g\|. \quad (1.18)$$

For any $s \in I_\mu^+ \cup I_\mu^-$, let

$$\kappa(s) = \lim_{n \rightarrow \infty} (\mathbb{E} \|M_{0, n-1}\|^s)^{\frac{1}{n}}, \quad \text{and} \quad \Lambda(s) = \log \kappa(s). \quad (1.19)$$

It is known that the limit above exists (see [11] and [36]), and that the function Λ is convex and analytic on $(I_\mu^+ \cup I_\mu^-)^\circ$. This function plays the same role as the log Laplace transform in the case of sums of i.i.d. real-valued random variables. Introduce the Fenchel-Legendre transform of Λ by

$$\Lambda^*(q) = \sup_{s \in I_\mu^+ \cup I_\mu^-} \{sq - \Lambda(s)\}, \quad q \in \Lambda'(I_\mu^+ \cup I_\mu^-).$$

Notice that if $q = \Lambda'(s)$ for some $s \in (I_\mu^+)^\circ \cup (I_\mu^-)^\circ$, then $\Lambda^*(q) = s\Lambda'(s) - \Lambda(s) > 0$.

We need the following non-arithmeticity condition on μ .

A2. (Non-arithmeticity) *There do not exist $t \in (0, \infty)$, $\theta \in [0, 2\pi)$ and $\varphi : \mathcal{S} \rightarrow \mathbb{R}$ such that*

$$\|vg\|^{it} \varphi(v \cdot g) = e^{i\theta} \varphi(v), \quad \forall g \in [\text{supp } \mu], \quad \forall v \in \text{supp } \nu.$$

It is known that condition **A2** ensures that $\sigma_s^2 = \Lambda''(s) > 0$.

1.3. Main results for products of random matrices. In this subsection, $(M_n)_{n \in \mathbb{Z}}$ stands for an arbitrary sequence of i.i.d. positive random matrices with common law μ , defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any $s \in I_\mu^+ \cup I_\mu^-$, the transfer operator P_s is defined as follows: for any bounded measurable function φ on \mathcal{S} and $v \in \mathcal{S}$,

$$P_s \varphi(v) = \int_{\Gamma_\mu} \|vg\|^s \varphi(v \cdot g) \mu(dg). \quad (1.20)$$

Then the function $\kappa(s)$ defined in (1.19) is an eigenvalue of the operator P_s , with respect to which there is a unique (up to a scaling constant) strictly positive and continuous eigenfunction r_s on \mathcal{S} , and a unique probability eigenmeasure ν_s on \mathcal{S}_ϵ :

$$P_s r_s = \kappa(s) r_s, \quad P_s \nu_s = \kappa(s) \nu_s.$$

Define a new measure \mathbb{Q}_s^v , such that for $v \in \mathcal{S}$, for any $n \geq 0$,

$$\mathbb{Q}_s^v|_{\mathcal{G}_{n-1}} = q_n^s(v, M_{0,n-1}) \mathbb{P}|_{\mathcal{G}_{n-1}}, \quad (1.21)$$

where $\cdot|_{\mathcal{G}_{n-1}}$ denotes the restriction of the measure to $\mathcal{G}_{n-1} = \sigma(M_k, k \leq n-1)$, and

$$q_n^s(v, M_{0,n-1}) = \frac{\|vM_{0,n-1}\|^s r_s(v \cdot M_{0,n-1})}{\kappa(s)^n r_s(v)}.$$

Notice that, while the sequences (u_n) and (v_n) are stationary and ergodic under the original measure \mathbb{P} , they may be non-stationary under the measure \mathbb{Q}_s^v ; also, under \mathbb{Q}_s^v , the sequence of matrices (M_n) is no longer stationary, nor independent. However, under \mathbb{Q}_s^v , we can still establish a Perron-Frobenius type theorem for the products of random matrices (M_n) . Under both the initial probability \mathbb{P} and the changed measure \mathbb{Q}_s^v , we will establish the mixing stable convergence of the sequences of directions $(v_n)_{n \geq 0}$ and $(x \cdot M_{0,n})_{n \geq 0}$, which will be very useful in the estimation of moments and large deviations.

We will need the following positivity condition introduced by Hennion [37]. A matrix is called allowable if every row and every column has at least a strictly positive entry.

A3. M_0 is a.s. allowable and

$$\mathbb{P}(\exists n \geq 0, M_{0,n} > 0) > 0. \quad (1.22)$$

It is known (see [37, Lemma 3.1] or [40, Lemma 2.1]) that (1.22) holds if and only if

$$\mathbb{P}(\exists n \geq 0, M_{0,n} > 0) = 1.$$

Remark 1.1. Remark that condition (1.22) is equivalent to the following condition used in [10] (recall that G° denotes the subsemigroup of G composed of strictly positive matrices):

$$[\text{supp } \mu] \cap G^\circ \neq \emptyset, \quad (1.23)$$

where $[\text{supp } \mu]$ denotes the closed semigroup generated by $\text{supp } \mu$, the support of μ .

To see the equivalence, notice that if $\mathbb{P}(\exists n \geq 0, M_{0,n} > 0) > 0$, then there is $n \geq 0$ such that $\mathbb{P}(M_{0,n} > 0) > 0$. Therefore there is a matrix $g \in \text{supp } M_{0,n}$ (the support of the law of $M_{0,n}$) such that $g > 0$. Since

$$\text{supp } M_{0,n} = \overline{\{g_0 \cdots g_n : g_i \in \text{supp } \mu \ \forall i \in \{0, \dots, n\}\}}, \quad (1.24)$$

where $\overline{\{\cdot\}}$ denotes the closure of the set $\{\cdot\}$, it follows that $g \in [\text{supp } \mu]$ (the equality (1.24) is a consequence of [62, Lemma 2.1] applied to the mapping $f(g_0, \dots, g_n) = g_0 \cdots g_n$). This gives the implication \Rightarrow in (1.23). For the converse implication, assume that $g \in [\text{supp } \mu]$ with $g > 0$. Since

$$[\text{supp } \mu] = \overline{\{g_0 \cdots g_n : n \geq 0, g_i \in \text{supp } \mu \ \forall i \geq 0\}}, \quad (1.25)$$

it follows that there is $n \geq 0$ and $g_0, \dots, g_n \in \text{supp } \mu$ such that $g' := g_0 \cdots g_n > 0$. Again by (1.24), $g' \in \text{supp } M_{0,n}$. Since $g' > 0$, there is a neighborhood $O(g')$ of g' such that $O(g') \subset G^\circ$. Therefore

$$\mathbb{P}(M_{0,n} > 0) \geq \mathbb{P}(M_{0,n} \in O(g')) > 0.$$

This concludes the proof of the implication \Leftarrow in (1.23).

Notice that the equivalence (1.23) remains valid when $[\text{supp } \mu]$ is replaced by the semigroup generated by $\text{supp } \mu$ (without taking the closure).

We sometimes need the following stronger allowability condition than that in **A3**.

A4. *All elements of $[\text{supp } \mu]$ are allowable.*

Our first result is a Perron-Frobenius type theorem for the products of random matrices $M_{k,n} = M_k \cdots M_n$, where $k, n \in \mathbb{Z}, k \leq n$, under the changed measure \mathbb{Q}_s^v .

Theorem 1.2 (Perron-Frobenius type theorem). *Assume condition **A3**. Let $s \in I_\mu^+ \cup I_\mu^-$ and $v \in \mathcal{S}$. Then there are sequences $(u_k)_{k \in \mathbb{Z}}$ and $(v_k)_{k \in \mathbb{Z}}$ of elements of $\mathcal{S} \cap \mathbb{R}_+^{*d}$, such that for each fixed $k \in \mathbb{Z}$ as $n \rightarrow \infty$, and for each fixed $n \in \mathbb{Z}$ as $k \rightarrow -\infty$, \mathbb{Q}_s^v -a.s.*

$$\sup_{x \in \mathcal{S}} \|x \cdot (M_{k,n})^T - u_k\| \rightarrow 0, \quad (1.26)$$

$$\sup_{x \in \mathcal{S}} \|x \cdot M_{k,n} - v_n\| \rightarrow 0, \quad (1.27)$$

$$\sup_{x, y \in \mathbb{R}_+^d \setminus \{0\}} \left| \frac{\langle x M_{k,n}, y \rangle}{a_{k,n} \langle u_k, x \rangle \langle v_n, y \rangle} - 1 \right| \rightarrow 0, \quad (1.28)$$

where $a_{k,n} = \frac{\prod_{j=k}^n \mu_j}{\langle u_k, v_{k-1} \rangle} = \frac{\prod_{j=k}^n \lambda_j}{\langle u_{n+1}, v_n \rangle}$. Moreover, (u_k) and (v_n) satisfy, \mathbb{Q}_s^v -a.s.,

$$u_{k+1} \cdot M_k^T = u_k, \quad v_{k-1} \cdot M_k = v_k, \quad \forall k \in \mathbb{Z}.$$

Note that for $s = 0$, Theorem 1.2 recovers a similar assertion obtained in [30].

Theorem 1.2 is a consequence of Remark 3.5 and Theorem 3.3, where more details, including the relations with the spectral radius $\rho_{k,n}$ of $M_{k,n}$, the eigenvectors $u_{k,n}$ (resp. $v_{k,n}$) of $(M_{k,n})^T$ (resp. $M_{k,n}$), and other possible choices of $a_{k,n}$, are given.

Our second result concerns the mixing stable convergence of the direction of the random walk $(xM_{0,n})_{n \geq 0}$. The result is new in both cases $s = 0$ (under the initial probability \mathbb{P}) and $s \neq 0$ (under the changed measure \mathbb{Q}_s^v).

Theorem 1.3 (Stable and mixing convergence). *Assume condition [A4](#). Let $s \in I_\mu^+ \cup I_\mu^-$ and $v \in \mathcal{S}$. Assume also condition [A3](#) when $s > 0$, and [A1](#) when $s < 0$. Then, under \mathbb{Q}_s^v , for all $x \in \mathcal{S}$, all the three sequences $(x \cdot M_{0,n})$, (v_n) and $(v_{0,n})$, converge mixing stably to π_s : that is, for any $m \geq 1$ and any \mathbb{R}^m -valued random variable η as $n \rightarrow \infty$, we have the following convergence in law under \mathbb{Q}_s^v :*

$$(x \cdot M_{0,n}, \eta) \rightarrow \pi_s \otimes \mathcal{L}_\eta, \quad (v_n, \eta) \rightarrow \pi_s \otimes \mathcal{L}_\eta \quad \text{and} \quad (v_{0,n}, \eta) \rightarrow \pi_s \otimes \mathcal{L}_\eta. \quad (1.29)$$

where π_s is the unique Q_s -invariant measure (cf. (2.15)), \mathcal{L}_η denotes the law of η .

In fact we will prove the stable and mixing convergence of the joint law of $(x \cdot M_{0,n}, v_n, v_{0,n})$: see Theorem 2.9 (under \mathbb{P}) and Theorem 3.7 (under \mathbb{Q}_s^v).

Our third result is on the moments of the products of random matrices.

Theorem 1.4. *Assume [A4](#) and let $s \in I_\mu^- \cup I_\mu^+$. Assume also condition [A3](#) when $s > 0$, and [A1](#) when $s < 0$. Then the spectral radius $\rho_{0,n-1}$ of $M_{0,n-1}$ satisfies*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(\rho_{0,n-1})^s}{\kappa^n(s)} = \int_{\mathcal{S}^2} \frac{\langle u, v \rangle^s}{r_s(v)} \mu_0^s(du) \pi_s(dv), \quad (1.30)$$

where μ_0^s denotes the law of u_0 under $\mathbb{Q}_s^{1/d}$. Moreover, for any $x, y \in \mathcal{S}$, we have

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}\langle x M_{0,n-1}, y \rangle^s}{\kappa^n(s)} = r_s(x) \int_{\mathcal{S}} \frac{\langle v, y \rangle^s}{r_s(v)} \pi_s(dv). \quad (1.31)$$

In particular,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\mathbb{E}\|x M_{0,n-1}\|^s}{\kappa^n(s)} &= r_s(x) \int_{\mathcal{S}} \frac{1}{r_s(v)} \pi_s(dv), \\ \lim_{n \rightarrow \infty} \frac{\mathbb{E}\|M_{0,n-1}\|_{1,1}^s}{\kappa^n(s)} &= \int_{\mathcal{S}} \frac{1}{r_s(v)} \pi_s(dv). \end{aligned}$$

For the proof, see Section 3.4, Theorems 3.8, 3.9 and 3.10.

1.4. Main results for branching processes. We first state our main result about the moments of the MBPRE (Z_n^i) .

Theorem 1.5. *Assume conditions [A4](#), [A1](#) and $s \in I_\mu^+ \setminus \{0\}$. Assume also that when $0 < s \leq 1$, we have $\max_{1 \leq i, j \leq d} \mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)^{1-s}} \log^+ \frac{Z_1^i(j)}{M_0(i,j)}\right) < \infty$ and $\mathbb{E}(\|M_0\|^s \log^+ \|M_0\|) < \infty$, and when $s > 1$, we have $\max_{1 \leq i, j \leq d} \mathbb{E}(Z_1^i(j))^s < \infty$. Then for all $i, j \in \{1, \dots, d\}$,*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}\|Z_n^i\|^s}{\kappa^n(s)} = \mathbb{E}_{\mathbb{Q}_s^{e_i}}(W^i)^s r_s(e_i) \int_{\mathcal{S}} \frac{\pi_s(dx)}{r_s(x)}, \quad (1.32)$$

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(Z_n^i(j))^s}{\kappa^n(s)} = \mathbb{E}_{\mathbb{Q}_s^{e_i}}(W^i)^s r_s(e_i) \int_{\mathcal{S}} \frac{\langle x, e_j \rangle^s}{r_s(x)} \pi_s(dx), \quad (1.33)$$

where the random variable W^i is defined in Section 4.2, with $0 < \mathbb{E}_{\mathbb{Q}_s^{e_i}}(W^i)^s < \infty$.

For the proof, see Section 5. For the single-type case $d = 1$, the result has been proved by Huang and Liu [46, Theorem 1.3]; see also Buraczewski and Dyszewski [12, Lemma 3.1].

We next state our main result about the precise large deviation of the MBPRE (Z_n^i) .

Theorem 1.6. *Let $s \in (I_\mu^+)^{\circ}$ be such that $q = \Lambda'(s) > 0$ and $\sigma_s^2 = \Lambda''(s) > 0$. Assume conditions [A4](#) and [A1](#). Assume also that:*

(1) *When $0 < s < 1$, there is $\delta > 0$ such that*

$$\max_{1 \leq i, j \leq d} \mathbb{E}\left[\left(\mathbb{E}_\xi(Z_1^i(j))^{1+\delta}\right)^{\frac{s}{1+\delta}}\right] < \infty. \quad (1.34)$$

(2) When $s \geq 1$, there is $\delta > 0$ such that

$$\max_{1 \leq i, j \leq d} \mathbb{E} \left[(Z_1^i(j))^{s+\delta} \right] < \infty. \quad (1.35)$$

In the case where $s > 1$, we assume additionally that $\kappa(s) > \kappa(1)$.

Then, for $q = \Lambda'(s)$, we have, with $C(s) = \frac{r_s(e_i)}{s\sigma_s\nu_s(r_s)\sqrt{2\pi}} \mathbb{E}_{\mathbb{Q}_s^{e_i}}(W^i)^s \in (0, \infty)$, as $n \rightarrow \infty$,

$$\mathbb{P} \left(\log \|Z_n^i\| \geq qn \right) \sim \frac{C(s)}{\sqrt{n}} e^{-n\Lambda^*(q)} \quad \text{as } n \rightarrow \infty. \quad (1.36)$$

More generally, for any constant $C > 0$ and any given sequence $(\bar{\delta}_n)_{n \geq 1}$ with $0 \leq \bar{\delta}_n = o(n^{-1/2})$, we have, uniformly in all sequences $(\delta_n)_{n \geq 1}$ such that $\delta_n \leq C\bar{\delta}_n$ for all $n \geq 1$,

$$\mathbb{P} \left(\log \|Z_n^i\| \geq (q + \delta_n)n \right) \sim \frac{C(s)}{\sqrt{n}} e^{-n(\Lambda^*(q) + s\delta_n)} \quad \text{as } n \rightarrow \infty. \quad (1.37)$$

Remark 1.7. In (1.37), we can replace $e^{-n(\Lambda^*(q) + s\delta_n)}$ by $e^{-n\Lambda^*(q + \delta_n)}$, since

$$\Lambda^*(q + \delta_n) = \Lambda^*(q) + s\delta_n + \frac{\delta_n^2}{2\sigma_s^2}(1 + o(1)),$$

uniformly in all sequences $(\delta_n)_{n \geq 1}$ such that $\delta_n \leq C\bar{\delta}_n$ for all $n \geq 1$ (see [74, Lemma 4.1]).

Remark 1.8. Notice that Theorem 1.6 applies to all three cases: supercritical, critical, and subcritical, provided $q > \max(\gamma, 0)$. Note that $\Lambda'(1) \geq 0$ ensures the condition $\kappa(s) > \kappa(1)$ for $s > 1$ is automatically satisfied in all the three cases, except probably in the strongly subcritical case, namely the case where $\Lambda'(1) < 0$.

Remark 1.9. When $d = 1$, the Bahadur-Rao type expansion (1.36) reduces to that of Buraczewski and Dyszewski [12]. Note that in the case where $s < 1$, they assumed the additional condition that the conditional mean M_0 satisfies $\mathbb{E}M_0^{s-1}Z_1 \log^+ Z_1 < \infty$. In fact we can check that this condition is implied by (1.34) that they also assumed in an equivalent form: see Lemma 6.8. The Petrov type expansion (1.36) is new.

1.5. Key ingredients of the proofs and organization of the paper. Let us give a short explanation of the mains ingredients of the proofs.

For the proof of Perron-Frobenius theorem, as in the case under \mathbb{P} , an important step is to use the contraction properties of the Hilbert cross metric d on \mathcal{S} (see (2.1) and (2.2)) and nice properties of the contraction coefficients (see (2.3) and Lemma 3.4), to conclude that (see Remark 3.5)

$$\sup_{y \in \mathcal{S}} d(y \cdot (M_{k,n})^T, u_k) \rightarrow 0 \quad \text{and} \quad \sup_{y \in \mathcal{S}} d(y \cdot M_{k,n}, v_n) \rightarrow 0, \quad \mathbb{Q}_s^v\text{-a.s.} \quad (1.38)$$

For the proof of the stable and mixing convergence under \mathbb{P} and \mathbb{Q}_s^v , we still use the contraction properties of the metric d , together with the spectral gap theory.

Using the measure change technique and the stable and mixing convergence of $(x \cdot M_{0,n}, v_n, v_{0,n})$ under the changed measure, together with the Perron-Frobenius type theorem, we get precise estimation of the moments of the vector norm $\|xM_{0,n}\|$, the matrix norm $\|M_{0,n}\|_{1,1}$, the scalar product $\langle xM_{0,n}, y \rangle$, and the spectral radius $\rho_{0,n}$ of $M_{0,n}$.

Using the stable and mixing convergence of $x \cdot M_{0,n}$ and v_n , also under the changed measure, as well as the L^p convergence results under the changed measure, we get similar estimation for the moments of $\|Z_n\|$ and $Z_n(j)$.

We finally explain the main ingredients of the proof on the precise large deviation of $\|Z_n\|$. Inspired by the approach of Buraczewski and Dyszewski [12] where the single type case was

considered, the starting point of the proof is the following martingale decomposition: for any $1 \leq m \leq n$,

$$Z_n = Z_m M_{m,n-1} + \sum_{k=m+1}^n (Z_k - Z_{k-1} M_{k-1}) M_{k,n-1}$$

(recall that $M_{k,n} = M_k \dots M_n$ if $k \leq n$, and $M_{k,n} = I_d$ the identity $d \times d$ matrix if $k > n$), where the empty sum is taken to be 0, and the summands in $\sum_{k=m+1}^n$ are martingale differences (see Lemma 6.1). With $m = K \lfloor \log n \rfloor$ for a suitable K (large enough), we prove that the sum $\sum_{k=m+1}^n$ above is negligible, so that $\|Z_n\|$ behaves like $\|Z_m M_{m,n-1}\|$, whose precise large deviations can be estimated by conditioning on Z_m and using the Bahadur-Rao type theorem for $x M_{m,n-1}$ with $x \in \mathcal{S}$, established in [74].

In the proof, we mainly use:

1) The Perron-Frobenius type theorem for the products of positive random matrices under the Cramér-type change of measure, that we establish in Section 3.2. Notice that while the sequences (u_n) and (v_n) are stationary and ergodic under the original measure \mathbb{P} , they may be non-stationary under the measure \mathbb{Q}_s^v .

2) The mixing stable convergence of (v_n) and $(x \cdot M_{0,n})$ that we establish in Sections 2.3 and 3.3.

3) Under the changed measure $\mathbb{Q}_s^{e_i}$, the L^p -convergence theorems for the fundamental martingale (W_n^i) , for the normalized j -type population size $Z_n^i(j)$ for each $j \in \{1, \dots, d\}$, and for the total population size $\|Z_n^i\|$ of generation n , and the convergence of the direction $Z_n^i / \|Z_n^i\|$, established in Section 4.

The rest of the paper is organized as follows. In Section 2, we prove the stable and mixing convergence for products of positive random matrices under the initial probability \mathbb{P} . In Section 3, we investigate properties of products of positive random matrices under the changed measure \mathbb{Q}_s^v , where, in particular, we prove the Perron-Frobenius type theorem and the stable and mixing convergence of the direction of $x M_{0,n}$. As application, we determine the exact growth rate of the moments of $\|x M_{0,n}\|$ and $\langle x M_{0,n}, y \rangle$. Section 4 is dedicated to the properties of the branching process Z_n under the changed measure $\mathbb{Q}_s^{e_i}$. We mainly prove the $L^p(\mathbb{Q}_s^{e_i})$ convergence of the fundamental martingale (W_n^i) , the normalized total population size $\|Z_n\| / \mathbb{E}_\xi \|Z_n\|$, as well as the normalized j -type population size $Z_n(j) / \mathbb{E}_\xi Z_n(j)$. We also prove the exponential convergence rate. In addition, the convergence of the direction $Z_n / \|Z_n\|$ is established as well. The exact growth rate of the moments of $\|Z_n\|$ and $Z_n(j)$ are established in Section 5, by using the stable and mixing convergence of $x \cdot M_{0,n}$ and (v_n) , and the L^p convergence under $\mathbb{Q}_s^{e_i}$. These properties are also used in Section 6 for precise large deviations of $\|Z_n\|$.

2. STABLE AND MIXING CONVERGENCE FOR PRODUCTS OF POSITIVE RANDOM MATRICES

Recall that we always assume that the environment $(\xi_k)_{k \in \mathbb{N}}$, is independent identically distributed sequence of random variables with values in (E, \mathcal{E}) . For convenience of applications, we extend it to the double sided sequence. Assume that $M : x \mapsto M(x)$ is a measurable mapping defined on (E, \mathcal{E}) with values in G , so that $M_n := M(\xi_n)$, $n \in \mathbb{Z}$, is an i.i.d. sequence of random elements of G (the set of $d \times d$ non-negative matrices). We notice that any i.i.d. sequence of random elements of G can be written in this form. To see this, it suffices to consider the canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = G^{\mathbb{Z}}$, $\mathbb{P} = \mu^{\otimes \mathbb{Z}}$ with μ the common law of M_n . In this case we have $M_n(\omega) = \omega_n$ for each $\omega = (\omega_k)_{k \in \mathbb{Z}} \in \Omega$, and we can take $\xi = \omega$.

2.1. Preliminaries. Following [37], we equip the projective space \mathcal{S} with the Hilbert cross-ratio metric \mathbf{d} defined as follows: for any $x, y \in \mathcal{S}$,

$$\mathbf{d}(x, y) = \frac{1 - m(x, y)m(y, x)}{1 + m(x, y)m(y, x)}, \quad (2.1)$$

where $m(x, y) = \sup\{\lambda > 0 : \lambda y(i) \leq x(i), \forall i = 1, \dots, d\}$ for $x = (x(1), \dots, x(d)) \in \mathbb{R}^d$ and $y = (y(1), \dots, y(d)) \in \mathbb{R}^d$. By Proposition 3.1 in [37], the distance \mathbf{d} is bounded, $\mathbf{d}(x, y) \leq 1$, for any $x, y \in \mathcal{S}$, and has the important property that for any matrix $g \in G$ the action (1.15) on \mathcal{S} is a contraction with respect to \mathbf{d} , that is, for any g there exists $\mathfrak{c}(g) \leq 1$ such that, for any $x, y \in \mathcal{S}$, it holds

$$\mathbf{d}(x \cdot g, y \cdot g) \leq \mathfrak{c}(g)\mathbf{d}(x, y) \leq \mathfrak{c}(g). \quad (2.2)$$

The number $\mathfrak{c}(g)$ is called contraction coefficient of the matrix g . An explicit calculation of the coefficient $\mathfrak{c}(g)$ in terms of the matrix is performed in [37], where it is shown that

$$\mathfrak{c}(g) = \max_{i, j, k, l \in \{1, \dots, d\}} \frac{|g(k, i)g(l, j) - g(k, j)g(l, i)|}{g(k, i)g(l, j) + g(k, j)g(l, i)}. \quad (2.3)$$

The contraction coefficient satisfies the following properties: $\mathfrak{c}(g) < 1$ iff $g \in G^\circ$, $\mathfrak{c}(g) = \mathfrak{c}(g^T)$ and $\mathfrak{c}(g, g') \leq \mathfrak{c}(g)\mathfrak{c}(g')$ for $g, g' \in G$. It is known that the distance \mathbf{d} satisfies

$$\frac{1}{2}\|x - y\| \leq \mathbf{d}(x, y) \leq 1. \quad (2.4)$$

For any $\epsilon \in (0, 1)$, we denote

$$\mathcal{S}_\epsilon = \{v \in \mathcal{S} : \langle f, v \rangle \geq \epsilon \text{ for all } f \in \mathcal{S}\}.$$

Next we introduce a Banach space which will be used in the sequel. Let $\mathcal{C}(\mathcal{S})$ be the space of complex-valued continuous functions on \mathcal{S} . For any $\varphi \in \mathcal{C}(\mathcal{S})$ and $\gamma > 0$, denote

$$\|\varphi\|_\infty := \sup_{v \in \mathcal{S}} |\varphi(v)| \quad \text{and} \quad [\varphi]_\gamma := \sup_{u, v \in \mathcal{S}} \frac{|\varphi(u) - \varphi(v)|}{\mathbf{d}(u, v)^\gamma},$$

and the Banach space

$$\mathcal{B}_\gamma := \left\{ \varphi \in \mathcal{C}(\mathcal{S}) : \|\varphi\|_\gamma := \|\varphi\|_\infty + [\varphi]_\gamma < \infty \right\}.$$

We shall consider the following weaker version:

A5. *There exists a constant $D > 1$ such that for μ -almost every $g \in G$, and all $1 \leq i \leq d$,*

$$0 < \max_{1 \leq j \leq d} \langle e_i, e_j g \rangle \leq D \min_{1 \leq j \leq d} \langle e_i, e_j g \rangle. \quad (2.5)$$

Clearly condition **A1** implies **A5**. Condition **A5** says that all the entries of each fixed column of the matrix $g \in \text{supp } \mu$ are comparable, while condition **A1** requires that all the entries of g are comparable.

Below we give equivalent formulations of conditions **A1** and **A5**. For any set $B \subset \mathcal{S}$, we denote $B \cdot g = \{v \cdot g : v \in B\}$.

Lemma 2.1. *Assume **A4**. Then: (1) condition **A5** is equivalent to the following statement: there exists a constant $\epsilon \in (0, \frac{\sqrt{2}}{2})$ such that*

$$\mathcal{S} \cdot g \subseteq \mathcal{S}_\epsilon \quad \text{for } \mu\text{-almost every } g \in G, \quad (2.6)$$

*(2) condition **A1** is equivalent to the following more restricted statement: there exists a constant $\epsilon \in (0, \frac{\sqrt{2}}{2})$ such that*

$$\mathcal{S} \cdot g \subseteq \mathcal{S}_\epsilon \quad \text{and} \quad \mathcal{S} \cdot g^T \subseteq \mathcal{S}_\epsilon \quad \text{for } \mu\text{-almost every } g \in G. \quad (2.7)$$

The first assertion is given in [78, Lemma 3.2]. The second one can be proved in the same way, so its proof is omitted.

2.2. Spectral gap theory. In order to state the large deviation results for Z_n^i , we need some spectral properties of transfer operators related to the norm cocycle $\|vM_{0,n}\|$.

For any $s \in I_\mu^+ \cup I_\mu^-$, the transfer operator P_s and the conjugate transfer operator P_s^* are defined as follows: for any bounded measurable function φ on \mathcal{S} and $v \in \mathcal{S}$,

$$P_s\varphi(v) = \int_{\Gamma_\mu} \|vg\|^s \varphi(v \cdot g) \mu(dg), \quad P_s^*\varphi(v) = \int_{\Gamma_\mu} \|vg^T\|^s \varphi(v \cdot g^T) \mu(dg), \quad (2.8)$$

where g^T is the transpose of g . The following conclusions have been known. For simplicity, we just state the results under the condition **A1**, although some of them are valid under much more general conditions.

Lemma 2.2. [10, 75]. *Assume **A4** and let $s \in I_\mu^- \cup I_\mu^+$. Assume also condition **A3** when $s > 0$, and **A1** when $s < 0$. Then $\kappa(s)$ is an eigenvalue of the operator P_s , and for some $\varepsilon > 0$, there is a unique probability eigenmeasure ν_s on \mathcal{S} and a unique (up to a scaling constant) strictly positive and continuous eigenfunction r_s on \mathcal{S} :*

$$P_s r_s = \kappa(s) r_s, \quad P_s \nu_s = \kappa(s) \nu_s.$$

Similarly, there exist a unique probability eigenmeasure ν_s^ on \mathcal{S} and a unique (up to a scaling constant) strictly positive and continuous function r_s^* such that*

$$P_s^* r_s^* = \kappa(s) r_s^*, \quad P_s^* \nu_s^* = \kappa(s) \nu_s^*.$$

With a particular choice of the constant, the eigenfunction r_s (resp. r_s^) and the eigenmeasure ν_s^* (resp. ν_s) are related by*

$$r_s(v) = \int_{\mathcal{S}} \langle v, u \rangle^s \nu_s^*(du), \quad r_s^*(v) = \int_{\mathcal{S}} \langle v, u \rangle^s \nu_s(du), \quad v \in \mathcal{S}, \quad (2.9)$$

which are strictly positive and \bar{s} -Holder continuous with respect to the L^1 norm $\|\cdot\|$, where $\bar{s} = \min\{1, |s|\}$.

In addition, there exists a constant $c_s > 0$ such that for any $n \geq 1$,

$$\kappa(s)^n \leq \mathbb{E}(\|M_{0,n-1}\|^s) \leq c_s \kappa(s)^n \text{ for } s > 0, \quad (2.10)$$

$$c_s \kappa(s)^n \leq \mathbb{E}(\|M_{0,n-1}\|^s) \leq \kappa(s)^n \text{ for } s < 0. \quad (2.11)$$

*Moreover, when **A1** holds, there is some $\varepsilon \in (0, 1)$ such that the measures ν_s and ν_s^* have supports contained in \mathcal{S}_ε .*

The lemma comes from [10, Proposition 3.1] (see also [36]) for $s \in I_\mu^+$, and [75, Proposition 2.2] for $s \in I_\mu^-$. Notice that under the Furstenberg-Kesten condition **A1**, the allowability condition **A4** is equivalent to the condition that the 0 matrix is not in $[\text{supp } \mu]$, which is used in [75, Proposition 2.2].

Remark 2.3. The constant c_s in (2.10) can be taken as follows: $c_s = 1/[\min_{\|g\|=1} \int_{\mathcal{S}} \|xg\|^s \nu_s(dx)]$ for $s \in I_\mu^+$ (see [36, Lemma 2.8], and $c_s = (D^2d)^s$ for $s \in I_\mu^-$ (see the proof of [75, Lemma 2.4]).

For any $s \in I_\mu^+ \cup I_\mu^-$, define the operator Q_s as follows: for any $\varphi \in \mathcal{C}(\mathcal{S})$,

$$Q_s\varphi(v) := \frac{1}{\kappa(s)r_s(v)} P_s(\varphi r_s)(v), \quad v \in \mathcal{S}, \quad (2.12)$$

which is the normalization of the transfer operator P_s . It is not difficult to check that the n -fold iterations of Q_s and P_s satisfy the relation

$$Q_s^n \varphi = \frac{1}{\kappa(s)^n r_s} P_s^n(\varphi r_s). \quad (2.13)$$

Lemma 2.4. [10, 36, 75] *Assume **A4** and let $s \in I_\mu^- \cup I_\mu^+$. Assume also condition **A3** when $s > 0$, and **A1** when $s < 0$. Then uniformly for $\varphi \in \mathcal{C}(\mathcal{S})$,*

$$\lim_{n \rightarrow \infty} Q_s^n \varphi = \pi_s(\varphi), \quad (2.14)$$

where π_s is the unique Q_s -invariant (i.e. $Q_s \pi_s = \pi_s$) probability measure on \mathcal{S} , given by

$$\pi_s(\varphi) = \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)}, \quad \forall \varphi \in \mathcal{C}(\mathcal{S}). \quad (2.15)$$

The result comes from [10, Theorem 3.11] and [36, Theorem 2.6]) for $s \in I_\mu^+$, and [75, Lemmas 2.7 and 2.8] for $s \in I_\mu^-$.

Below we state the spectral gap property of the operator Q_s from [43]. Denote by $\mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ the set of bounded linear operators from \mathcal{B}_γ to \mathcal{B}_γ equipped with the operator norm $\|\cdot\|_{\mathcal{B}_\gamma \rightarrow \mathcal{B}_\gamma}$, by $\rho(A)$ the spectral radius of an operator A acting on \mathcal{B}_γ .

Proposition 2.5. [75, Proposition 2.10] *Assume condition **A1**. Let $s \in I_\mu^+ \cup I_\mu^-$. Then, there exists $\gamma \in (0, 1)$ such that $Q_s \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$, and there exists a constant $a \in (0, 1)$ such that for all $n \geq 1$,*

$$Q_s^n = \Pi_s + N_s^n,$$

where Π_s is an one dimensional projector with $\Pi_s(\varphi)(v) = \pi_s(\varphi)$ for $\varphi \in \mathcal{B}_\gamma$ and $v \in \mathcal{S}$, and N_s satisfies $\Pi_s N_s = N_s \Pi_s = 0$ and $\rho(N_s) < a$.

As a direct consequence of Proposition 2.5 and (2.13), we get the following

Corollary 2.6. [75, Corollary 2.11] *Assume condition **A1**. Let $s \in I_\mu^+ \cup I_\mu^-$. Then, there exists $\gamma \in (0, 1)$ such that $P_s \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$, and $P_s^n = \kappa(s)^n M_s + L_s^n$ for any $n \geq 1$, where $M_s := \nu_s \otimes r_s$ is a rank-one projection on \mathcal{B}_γ defined by $M_s \varphi = \frac{\nu_s(\varphi)}{\nu_s(r_s)} r_s$ for $\varphi \in \mathcal{B}_\gamma$ and P_s^n denotes the n -fold iteration of P_s . Moreover, $M_s L_s = L_s M_s = 0$ and $\rho(L_s) < \kappa(s)$.*

2.3. Stable and mixing convergence of (v_n) and $(x \cdot M_{0,n})$. In this subsection, we consider the stable and mixing convergences of the sequences (v_n) and $(x \cdot M_{0,n})$ ($x \in \mathcal{S}$), for products of i.i.d. non-negative matrices (M_n) . These types of convergence were first introduced by Rényi [64]. We refer to [2] for a nice presentation, and to [24] for more details.

Definition 2.7. *Let X_n be a sequence of random variables with values in a Polish space \mathbb{S} , defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let \mathcal{G} be a sub- σ -field of \mathcal{F} . We say that X_n converges \mathcal{G} -stably to X (or its law), if X is an \mathbb{S} -valued random variable defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ of the original space and if, for any bounded \mathcal{G} -measurable real random variable η and any bounded continuous function $f : \mathbb{S} \rightarrow \mathbb{R}$,*

$$\lim_{n \rightarrow \infty} \mathbb{E}(\eta f(X_n)) = \tilde{\mathbb{E}}(\eta f(X)). \quad (2.16)$$

The stable convergence is called \mathcal{G} -mixing if the above convergence can be improved to the asymptotic independence of X_n and η , in the sense that

$$\lim_{n \rightarrow \infty} \mathbb{E}(\eta f(X_n)) = \mathbb{E}(\eta) \tilde{\mathbb{E}}(f(X)). \quad (2.17)$$

In the case where $\mathcal{G} = \mathcal{F}$, we simply say that $X_n \rightarrow X$ stably, or mixing stably, instead of saying \mathcal{G} -stably, or \mathcal{G} -mixing stably, respectively.

Remark 2.8. It is known, and it can be easily checked that, when X_n are \mathbb{R}^d -valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$, the stable convergence of (X_n) to X defined on some extended space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, is equivalent to the following convergence of the joint law under $\tilde{\mathbb{P}}$: for any $m \geq 1$ and any \mathbb{R}^m -valued random variable η defined on $(\Omega, \mathcal{F}, \mathbb{P})$,

$$(X_n, \eta) \xrightarrow{d(\tilde{\mathbb{P}})} (X, \eta); \quad (2.18)$$

and $X_n \rightarrow X$ mixing stably if and only if (2.18) holds and X and η are $\tilde{\mathbb{P}}$ -independent.

In fact, the implication (2.17) \Rightarrow (2.18) can be seen by considering the convergence of the characteristic function of (X_n, η) ; the converse (2.8) \Rightarrow (2.17) can be seen by considering the approximation of η by its truncating $h_c(\eta)$ at level $c > 0$, where $h_c(\eta) = \eta$ if $|\eta| \leq c$, $h_c(\eta) = c$ if $\eta > c$, and $h_c(\eta) = -c$ if $\eta < -c$, and the passage to the limit while $c \rightarrow \infty$.

As a direct consequence of the equivalent form (2.18), we see that the stable convergence has the stable property that if $X_n \rightarrow X$ stably, then $(X_n, \eta_0) \rightarrow (X, \eta_0)$ stably, for any $m \geq 1$ and \mathbb{R}^m -valued random variable η_0 defined on $(\Omega, \mathcal{F}, \mathbb{P})$, since we can apply (2.18) with η replaced by (η_0, η) .

Under the initial probability \mathbb{P} , as the sequence $(M_k)_{k \in \mathbb{Z}}$ is stationary and ergodic, the sequences (u_n) and (v_n) are also stationary. In particular all the u_n have the same law, and so do all the v_n . From $v_{k-1} \cdot M_k = v_k$, we see that the common law ν of v_k is μ -stationary in the sense that $\nu * \mu = \nu$, where

$$\nu * \mu(B) = \int 1_B(x \cdot g) d\nu(x) d\mu(g), \quad (2.19)$$

for each measurable set $B \subset \mathcal{S}$.

For $k \leq n$, let $\rho_{k,n}$ be the spectral radius of $M_{k,n}$, and $u_{k,n}, v_{k,n} \in \mathbb{R}_+^d$ be eigenvectors of $(M_{k,n})^T$ and $M_{k,n}$ with unit norm, that is,

$$u_{k,n} M_{k,n}^T = \rho_{k,n} u_{k,n}, \quad v_{k,n} M_{k,n} = \rho_{k,n} v_{k,n}, \quad \|u_{k,n}\| = \|v_{k,n}\| = 1. \quad (2.20)$$

The following result shows that each of the sequences $(x \cdot M_{k,n})$, $(v_{k,n})$, (v_n) converges mixing stably, and so does their joint law. As usual we write $\nu(\varphi) = \int_{\mathcal{S}} \varphi(y) \nu(dy)$.

Theorem 2.9. *Assume condition A3. For any real valued continuous function φ on \mathcal{S} and any \mathbb{P} -integrable \mathbb{R} -valued random variable η defined on $(\Omega, \mathcal{F}, \mathbb{P})$, we have for each $k \in \mathbb{Z}$,*

$$\lim_{n \rightarrow \infty} \mathbb{E}(\eta \varphi(x \cdot M_{k,n})) = \mathbb{E}(\eta) \nu(\varphi), \quad \forall x \in \mathcal{S}, \quad (2.21)$$

$$\lim_{n \rightarrow \infty} \mathbb{E}(\eta \varphi(v_n)) = \mathbb{E}(\eta) \nu(\varphi), \quad (2.22)$$

$$\lim_{n \rightarrow \infty} \mathbb{E}(\eta \varphi(v_{k,n})) = \mathbb{E}(\eta) \nu(\varphi). \quad (2.23)$$

In other words, each of the sequences $(x \cdot M_{k,n})$, $(v_{k,n})$ and (v_n) converges mixing stably to ν . Moreover, we have the following convergences of joint laws: for each fixed $k \in \mathbb{Z}$, as $n \rightarrow \infty$,

$$(x \cdot M_{k,n}, v_n, v_{k,n}) \rightarrow (v_\infty, v_\infty, v_\infty) \quad \text{mixing stably}, \quad (2.24)$$

$$(x \cdot M_{k,n}, v_n, v_{k,n}, u_{k,n}) \rightarrow (v_\infty, v_\infty, v_\infty, u_k) \quad \text{stably}, \quad (2.25)$$

where v_∞ is a \mathcal{S} -valued random variable defined on some extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbb{P})$, with law ν , which is independent of u_k .

By Remark 2.8, the conclusion of Theorem 2.9 is equivalent to the following: for any $m \geq 1$ and any \mathbb{R}^m -valued \mathcal{F} -measurable random variable η , we have the following weak convergence of the joint laws: under \mathbb{P} , $\forall x \in \mathcal{S}$, as $n \rightarrow \infty$,

$$(x \cdot M_{0,n}, \eta) \rightarrow \nu \otimes \mathcal{L}_\eta, \quad (v_n, \eta) \rightarrow \nu \otimes \mathcal{L}_\eta \quad \text{and} \quad (v_{0,n}, \eta) \rightarrow \nu \otimes \mathcal{L}_\eta, \quad (2.26)$$

where \mathcal{L}_η denotes the law of η . This implies Theorem 1.3 for the case $s = 0$.

Proof of Theorem 2.9. Step 1. We first establish (2.21) for any \mathcal{G}_m -measurable η with $\mathbb{E}|\eta| < \infty$, for each fixed $m \in \mathbb{Z}, m \geq k$, where $\mathcal{G}_m = \sigma(\{\xi_j, j \leq m\})$, as defined in (3.14). Since η is \mathcal{G}_m -measurable, by the Markov property, we have, for $k \leq m < n$,

$$\mathbb{E}(\eta\varphi(x \cdot M_{k,n})) = \mathbb{E}(\eta\mathbb{E}(\varphi(x \cdot M_{k,n})|\mathcal{G}_m)) = \mathbb{E}(\eta P_0^{n-m}\varphi(x \cdot M_{k,m})), \quad (2.27)$$

where P_0 was defined in (2.8) with $s = 0$, and P_0^k denotes its k -fold iteration. Since

$$\lim_{k \rightarrow \infty} P_0^k \varphi(v) = \lim_{k \rightarrow \infty} \mathbb{E}\varphi(v \cdot M_{0,k-1}) \rightarrow \nu(\phi),$$

from (2.27) and the dominated convergence theorem, we obtain

$$\lim_{n \rightarrow \infty} \mathbb{E}(\eta\varphi(x \cdot M_{k,n})) = \mathbb{E}\eta\nu(\varphi). \quad (2.28)$$

So we have obtained (2.21) when η is \mathcal{G}_m -measurable.

Step 2. We next prove (2.21) for any \mathcal{G} -measurable η with $\mathbb{E}|\eta| < \infty$, where

$$\mathcal{G} := \sigma(\cup_{m \in \mathbb{Z}} \mathcal{G}_m) = \sigma(\xi_j, j \in \mathbb{Z}). \quad (2.29)$$

We will use a monotone class argument. Let $\mathcal{G}^* = \cup_{m \in \mathbb{Z}} \mathcal{G}_m$. It is clear that \mathcal{G}^* is stable under intersection (that is, \mathcal{G}^* is a π -system). Let \mathcal{C} be the class of $B \in \mathcal{G}$ such that (2.21) holds for $\eta = 1_B$. The result proved in Step 1 implies that $\mathcal{G}^* \subset \mathcal{C}$. We check that \mathcal{C} is a monotone class:

- a) $\Omega \in \mathcal{C}$ because $\Omega \in \mathcal{G}^*$ and $\mathcal{G}^* \subset \mathcal{C}$;
- b) if $A, B \in \mathcal{C}$ with $A \subset B$, then by linearity $B \setminus A \in \mathcal{C}$ since $1_{B \setminus A} = 1_B - 1_A$;
- c) if $B_m \in \mathcal{C}$ are increasing, then $B := \cup_{m=1}^{\infty} B_m \in \mathcal{C}$, as shown in the following. We will use the decomposition

$$\mathbb{E}1_B\varphi(x \cdot M_{k,n}) = \mathbb{E}1_{B_m}\varphi(x \cdot M_{k,n}) + \mathbb{E}(1_B - 1_{B_m})\varphi(x \cdot M_{k,n}),$$

and the fact that

$$|\mathbb{E}(1_B - 1_{B_m})\varphi(x \cdot M_{k,n})| \leq \mathbb{E}(1_B - 1_{B_m})\|\varphi\|_{\infty} \rightarrow 0 \text{ as } m \rightarrow \infty,$$

where the last convergence holds by the monotone convergence theorem. Using the above decomposition together with the inequality, taking \limsup as $n \rightarrow \infty$, and then passing to the limit as $m \rightarrow \infty$, we get

$$\limsup_{n \rightarrow \infty} \mathbb{E}1_B\varphi(x \cdot M_{k,n}) \leq \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}1_{B_m}\varphi(x \cdot M_{k,n}) = \lim_{m \rightarrow \infty} \mathbb{E}1_{B_m}\nu(\varphi) = \mathbb{E}1_B\nu(\varphi),$$

where the first equality holds by the result proved in Step 1, and the second one from the sequential continuity of the probability \mathbb{P} . Similarly, we get

$$\liminf_{n \rightarrow \infty} \mathbb{E}1_B\varphi(x \cdot M_{k,n}) \geq \mathbb{E}1_B\nu(\varphi).$$

Therefore

$$\lim_{n \rightarrow \infty} \mathbb{E}1_B\varphi(x \cdot M_{k,n}) = \mathbb{E}1_B\nu(\varphi).$$

This shows that $B \in \mathcal{C}$.

Thus we have proved that \mathcal{C} is a monotone class. Therefore, by the monotone class theorem,

$$\mathcal{C} \supset \mathcal{M}(\mathcal{G}^*) = \sigma(\mathcal{G}^*) = \mathcal{G},$$

where $\mathcal{M}(\mathcal{G}^*)$ denotes the monotone class generated by \mathcal{G}^* . Consequently, $\mathcal{C} = \mathcal{G}$. This means that (2.21) holds for $\eta = 1_B$ for each $B \in \mathcal{G}$.

Then, by linearity, (2.21) holds for each \mathcal{G} -measurable and positive simple function η (which is a linear combination of indicator functions of sets in \mathcal{G} , with positive coefficients). By a similar argument as in c) above, we then conclude that (2.21) holds for each \mathcal{G} -measurable, positive and integrable function η , as such a function is the limit of an increasing

sequence of \mathcal{G} -measurable, positive and simple functions η_m (in fact we just need to replace 1_B and 1_{B^c} by η and η_m , respectively). Again by linearity, this implies that (2.21) holds for each \mathcal{G} -measurable real-valued random variable η with $\mathbb{E}|\eta| < \infty$, since $\eta = \eta_+ - \eta_-$, where η_+ and η_- denote the positive and negative parts of η .

Step 3. We then prove (2.21) for any \mathcal{F} -measurable η with $\mathbb{E}|\eta| < \infty$. To this end we just need to apply the result proved in Step 2 to $\eta_1 = \mathbb{E}(\eta|\mathcal{G})$, and make use of the fact that $M_{k,n}$ is \mathcal{G} -measurable. Indeed, we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}(\eta\varphi(x \cdot M_{k,n})) &= \lim_{n \rightarrow \infty} \mathbb{E}(\mathbb{E}(\eta|\mathcal{G})\varphi(x \cdot M_{k,n})) \\ &= \mathbb{E}(\mathbb{E}(\eta|\mathcal{G}))\nu(\varphi) = \mathbb{E}(\eta)\nu(\varphi). \end{aligned} \quad (2.30)$$

Step 4. We now prove (2.22) and (2.23) about the mixing convergence of (v_n) and $(v_{0,n})$. To this end, consider the modulus of continuity of φ defined by

$$m_\varphi(\varepsilon) = \sup_{x,y \in \mathcal{S}, \|y-x\| \leq \varepsilon} |\varphi(y) - \varphi(x)|, \quad \varepsilon > 0.$$

Then $\lim_{\varepsilon \rightarrow 0} m_\varphi(\varepsilon) = 0$ and, for any $a, b \in \mathcal{S}$,

$$|\varphi(b) - \varphi(a)| \leq m_\varphi(\|b - a\|).$$

Since $\|x \cdot M_{k,n} - v_n\| \rightarrow 0$ a.s., by the dominated convergence theorem, we have

$$|\mathbb{E}[\eta\varphi(x \cdot M_{k,n})] - \mathbb{E}[\eta\varphi(v_n)]| \leq \mathbb{E}[|\eta|m_\varphi(\|x \cdot M_{k,n} - v_n\|)] \rightarrow 0.$$

Therefore (2.21) implies (2.22). Similary, since $\|v_{k,n} - v_n\| \rightarrow 0$ a.s., (2.22) implies (2.23).

Step 5. We prove (2.24) for the mixing stable convergence of $(x \cdot M_{k,n}, v_n, v_{k,n})$. To this end, by Remark 2.8, we just need to prove that, for any $x \in \mathcal{S}$, $m \geq 1$ and any \mathbb{R}^m -valued \mathcal{F} -measurable random variable η ,

$$(x \cdot M_{k,n}, v_n, v_{k,n}, \eta) \xrightarrow{d(\tilde{\mathbb{P}})} (v_\infty, v_\infty, v_\infty, \eta), \quad \text{where } v_\infty \text{ and } \eta \text{ are } \tilde{\mathbb{P}}\text{-independent.} \quad (2.31)$$

This convergence can be easily seen by considering the characteristic function of $(x \cdot M_{k,n}, v_n, v_{k,n}, \eta)$. In fact, for any $(t_1, t_2, t_3, t_4) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^m$, we have, by the dominated convergence theorem, as $n \rightarrow \infty$,

$$\begin{aligned} &|\mathbb{E}e^{i\langle t_1, x \cdot M_{k,n} \rangle} e^{i\langle t_2, v_n \rangle} e^{i\langle t_3, v_{k,n} \rangle} e^{i\langle t_4, \eta \rangle} - \mathbb{E}e^{i\langle t_1, v_n \rangle} e^{i\langle t_2, v_n \rangle} e^{i\langle t_3, v_{k,n} \rangle} e^{i\langle t_4, \eta \rangle}| \\ &\leq \mathbb{E}|\langle t_1, x \cdot M_{0,n} \rangle - \langle t_1, v_n \rangle| \leq \|t_1\| \mathbb{E}\|x \cdot M_{0,n} - v_n\| \rightarrow 0. \end{aligned} \quad (2.32)$$

Similarly,

$$|\mathbb{E}e^{i\langle t_1, v_n \rangle} e^{i\langle t_2, v_n \rangle} e^{i\langle t_3, v_{k,n} \rangle} e^{i\langle t_4, \eta \rangle} - \mathbb{E}e^{i\langle t_1, v_n \rangle} e^{i\langle t_2, v_n \rangle} e^{i\langle t_3, v_n \rangle} e^{i\langle t_4, \eta \rangle}| \rightarrow 0. \quad (2.33)$$

By the mixing convergence of (v_n) , we have

$$\mathbb{E}e^{i\langle t_1, v_n \rangle} e^{i\langle t_2, v_n \rangle} e^{i\langle t_3, v_n \rangle} e^{i\langle t_4, \eta \rangle} \rightarrow \tilde{\mathbb{E}}e^{i\langle t_1, v_\infty \rangle} e^{i\langle t_2, v_\infty \rangle} e^{i\langle t_3, v_\infty \rangle} \mathbb{E}e^{i\langle t_4, \eta \rangle}. \quad (2.34)$$

From (2.32), (2.33) and (2.34), we obtain

$$\mathbb{E}e^{i\langle t_1, x \cdot M_{k,n} \rangle} e^{i\langle t_2, v_n \rangle} e^{i\langle t_3, v_{k,n} \rangle} e^{i\langle t_4, \eta \rangle} \rightarrow \tilde{\mathbb{E}}e^{i\langle t_1, v_\infty \rangle} e^{i\langle t_2, v_\infty \rangle} e^{i\langle t_3, v_\infty \rangle} \mathbb{E}e^{i\langle t_4, \eta \rangle}. \quad (2.35)$$

This gives (2.31), and thus ends the proof of (2.24).

Step 6. We finally prove (2.25) for the stable convergence of $(x \cdot M_{k,n}, v_n, v_{k,n}, u_{k,n})$. To this end, we just need to remark that $u_{k,n} \rightarrow u_k$ a.s., u_k is independent of v_∞ , and use the following stable property of the stable convergence: if X_n are \mathbb{R}^{k_1} -valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which converge stably to X defined on some extended probability space on some extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbb{P})$, and Y_n, Y are \mathbb{R}^{k_2} -valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $Y_n \rightarrow Y$ in probability,

then $(X_n, Y_n) \rightarrow (X, Y)$ stably. (This property can easily be checked by considering the convergence of the joint law of (X_n, Y_n, η) , and by using Levy's theorem, as we did above.) \square

3. PRODUCTS OF POSITIVE RANDOM MATRICES UNDER THE CHANGED MEASURE

In this section we give some properties of the products of i.i.d. random positive matrices under the changed measure. Recall that, as in the previous section, without loss of generality and motivated by applications in the branching process setting, such a sequence is written as $M_n = M(\xi_n), n \in \mathbb{Z}$, where $M : x \mapsto M(x)$ is a measurable mapping defined on (E, \mathcal{E}) with values in G , $\xi = (\xi_n)_{n \in \mathbb{Z}}$ is an i.i.d. sequence of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in the measurable space (E, \mathcal{E}) . Recall also that $\mathcal{G}_n = \sigma(\xi_k, k \leq n)$, for $n \in \mathbb{Z}$.

3.1. Change of measure. The Cramér type change of measure will play an important role. Let $s \in I_\mu^+ \cup I_\mu^-$ and $v \in \mathcal{S}$. Set for $n \geq 0$ and any matrix g ,

$$q_n^s(v, g) = \frac{\|vg\|^s r_s(v \cdot g)}{\kappa(s)^n r_s(v)}, \quad (3.1)$$

so that for $n \geq 0$,

$$q_n^s := q_n^s(v, M_{0,n-1}) = \frac{\|vM_{0,n-1}\|^s r_s(v \cdot M_{0,n-1})}{\kappa(s)^n r_s(v)}. \quad (3.2)$$

Notice that $q_0^s = 1$, since by notation $M_{0,-1}$ stands for the identity matrix I_d .

Lemma 3.1. *Under \mathbb{P} , the sequence $\{q_n^s(v, M_{0,n-1}), \mathcal{G}_{n-1}\}_{n \geq 0}$ is a martingale with mean 1, where $\mathcal{G}_{n-1} = \sigma(\xi_k, k \leq n-1)$.*

Proof. Actually, the mean 1 property that $\int q_n^s(v, M_0 \dots M_{n-1}) d\mathbb{P} = 1 \forall n \geq 1$ comes from the fact that $P_s r_s = \kappa(s) r_s$. We just need to prove that

$$\mathbb{E}(q_{n+1}^s(v, M_0 \dots M_n) | \mathcal{G}_{n-1}) = q_n^s(v, M_0 \dots M_{n-1}) \quad \forall n \geq 0. \quad (3.3)$$

For $n = 0$, Eq. (3.3) holds since, by the independence between M_0 and \mathcal{G}_{-1} and the fact that $P_s r_s = \kappa(s) r_s$, we have

$$\mathbb{E}(q_1^s(v, M_0) | \mathcal{G}_{-1}) = \mathbb{E}(q_1^s(v, M_0)) = 1 = q_0^s(v, I_d). \quad (3.4)$$

For $n \geq 1$, (3.3) follows from (3.4) and the fact that, for any $n \geq 1$,

$$q_{n+1}^s(v, M_0 \dots M_n) = q_n^s(v, M_0 \dots M_{n-1}) q_1^s(v \cdot M_0 \dots M_{n-1}, M_n). \quad (3.5)$$

\square

From Lemma 3.1 and the Kolmogorov extension theorem, there is a unique probability measure \mathbb{Q}_s^v on $\mathcal{G} := \sigma(\cup_{n \in \mathbb{Z}} \mathcal{G}_n) = \sigma(\xi)$, such that for any $n \geq -1$,

$$\mathbb{Q}_s^v |_{\mathcal{G}_n} = q_{n+1}^s(v, M_0 \dots M_n) \mathbb{P} |_{\mathcal{G}_n}, \quad (3.6)$$

where $\mathbb{P} |_{\mathcal{G}_n}$ denotes the restriction of the measure \mathbb{P} to \mathcal{G}_n . In other words, for any non-negative \mathcal{G}_n -measurable random variable Y , we have

$$\mathbb{E}_{\mathbb{Q}_s^v}(Y) = \mathbb{E}\left[q_{n+1}^s(v, M_0 \dots M_n) Y\right]. \quad (3.7)$$

Denote by $\mathbb{E}_{\mathbb{Q}_s^v}$ the corresponding expectation. The following information about the Lyapunov exponent under the changed measure \mathbb{Q}_s^v will be used. For any $s \in I_\mu^+ \cup I_\mu^-$, $v \in \mathcal{S}$ and $z \in \mathbb{R}$ with $s + z \in I_\mu^+ \cup I_\mu^-$, it is easy to prove that

$$\lim_{n \rightarrow \infty} (\mathbb{E}_{\mathbb{Q}_s^v} \|M_0 \cdots M_{n-1}\|^z)^{\frac{1}{n}} \leq \frac{\kappa(s+z)}{\kappa(s)}; \quad (3.8)$$

if condition **A1** holds, then by using the result of [75, Lemma 2.4] we have

$$\kappa_s(z) := \lim_{n \rightarrow \infty} (\mathbb{E}_{\mathbb{Q}_s^v} \|M_0 \cdots M_{n-1}\|^z)^{\frac{1}{n}} = \frac{\kappa(s+z)}{\kappa(s)}. \quad (3.9)$$

The sequence $(\xi_k)_{k \geq 0}$ is no longer stationary, nor independent, under the new measure \mathbb{Q}_s^v . In the following lemma we give some elementary properties of \mathbb{Q}_s^v . For other concerned properties, see Lemma 4.3.

Lemma 3.2. *Let $s \in I_\mu^+ \cup I_\mu^-$ and $v \in \mathcal{S}$. The following properties hold under \mathbb{Q}_s^v :*

- (1) *the sequence $\{\xi_k : k \leq -1\}$ is iid, and has the same law as under \mathbb{P} ;*
- (2) *the two families of random variables $\{\xi_k : k \leq -1\}$ and $\{\xi_k : k \geq 0\}$ are independent;*
- (3) *the process $(v \cdot M_{0,n-1})_{n \geq 0}$ is a Markov chain with the transfer operator Q_s given by (2.12): for any $n \geq 0$ and any non-negative and measurable function φ on \mathcal{S} ,*

$$\mathbb{E}_{\mathbb{Q}_s^v} (\varphi(v \cdot M_{0,n}) | \mathcal{G}_{n-1}) = (Q_s \varphi)(v \cdot M_{0,n-1}). \quad (3.10)$$

Moreover, for any $n \geq 0$,

$$Q_s^{n+1} \varphi(v) = \mathbb{E}_{\mathbb{Q}_s^v} (\varphi(v \cdot M_{0,n})). \quad (3.11)$$

Proof. The proof is straightforward. Indeed, Part 1 comes from the fact that $q_0^s = 1$ so that the law of $(\xi_k)_{k \leq -1}$ is not changed under the new measure. Using Part 1, we can easily check Part 2. The check of (3.10) of Part 3 is also easy. Let us just prove (3.11) using (3.10). In fact, taking the conditional expectation at both sides of (3.10) given \mathcal{G}_{n-2} , we obtain for $n \geq 1$,

$$\mathbb{E}_{\mathbb{Q}_s^v} (\varphi(v \cdot M_{0,n}) | \mathcal{G}_{n-2}) = (Q_s^2 \varphi)(v \cdot M_{0,n-2}). \quad (3.12)$$

Continue in this way, we obtain, for all $k \geq 1$ and $n \geq k - 1$,

$$\mathbb{E}_{\mathbb{Q}_s^v} (\varphi(v \cdot M_{0,n}) | \mathcal{G}_{n-k}) = (Q_s^k \varphi)(v \cdot M_{0,n-k}). \quad (3.13)$$

Applying this with $k = n + 1$, we get the iteration formula (3.11) for Q_s . \square

3.2. Perron-Frobenius theorem under the changed measure. In this section, we will prove a Perron-Frobenius type theorem for products of i.i.d. random positive matrices (M_n) , under the changed measure. It shows that all the conclusions of the Perron-Frobenius type theorem established in [37] and [30] under the initial probability \mathbb{P} still hold under the changed measure \mathbb{Q}_s^v , except the stationarity and ergodicity for the sequences (u_k) , (v_k) , (λ_k) and (μ_k) .

We need the following σ -algebras: for any $k \in \mathbb{Z}$,

$$\mathcal{G}_k = \sigma\{\xi_j, j \leq k\}, \quad \mathcal{G}^k = \sigma\{\xi_j, j \geq k\}. \quad (3.14)$$

As usual, for sequences of real numbers a_n, b_n , we write $a_n \sim b_n$ if $a_n/b_n \rightarrow 1$; for $a_n(\delta), b_n(\delta) \in \mathbb{R}$ depending on some parameter $\delta \in \Delta$, we say that, as $n \rightarrow \infty$,

$$a_n(\delta) \sim b_n(\delta) \text{ uniformly for } \delta \in \Delta \text{ if } \lim_{n \rightarrow \infty} \sup_{\delta \in \Delta} \left| \frac{a_n(\delta)}{b_n(\delta)} - 1 \right| = 0. \quad (3.15)$$

Clearly, this relation \sim with uniformity is an equivalent relation; in particular, if $a_n(\delta) \sim b_n(\delta)$ and $b_n(\delta) \sim c_n(\delta)$, both uniformly for $\delta \in \Delta$, then $a_n(\delta) \sim c_n(\delta)$, also uniformly for $\delta \in \Delta$.

Theorem 3.3 (Perron-Frobenius type theorem). *Assume condition [A3](#). Let $s \in I_\mu^+ \cup I_\mu^-$ and $v \in \mathcal{S}$. Let $\rho_{k,n}$ be the spectral radius of $M_{k,n}$, and $u_{k,n}, v_{k,n} \in \mathbb{R}_+^d$ be the eigenvectors defined in [\(2.20\)](#). Then:*

- (1) *There are sequences $(u_k)_{k \in \mathbb{Z}}$ and $(v_k)_{k \in \mathbb{Z}}$ such that for all k , $u_k > 0$, $v_k > 0$, $\|u_k\| = \|v_k\| = 1$, and that for each fixed $k \in \mathbb{Z}$ as $n \rightarrow \infty$, and for each fixed $n \in \mathbb{Z}$ as $k \rightarrow -\infty$, \mathbb{Q}_s^v -a.s.*

$$u_{k,n} - u_k \rightarrow 0 \quad \text{and} \quad v_{k,n} - v_k \rightarrow 0.$$

Moreover, for all $k \in \mathbb{Z}$, with $\lambda_k = \|u_{k+1} M_k^T\|$ and $\mu_k = \|v_{k-1} M_k\|$, \mathbb{Q}_s^v -a.s.

$$u_{k+1} M_k^T = \lambda_k u_k \quad \text{and} \quad v_{k-1} M_k = \mu_k v_k. \quad (3.16)$$

- (2) *For each fixed $k \in \mathbb{Z}$ as $n \rightarrow \infty$, and for each fixed $n \in \mathbb{Z}$ as $k \rightarrow -\infty$, uniformly for $x, y \in \mathbb{R}_+^d \setminus \{0\}$, \mathbb{Q}_s^v -a.s.*

$$\langle x M_{k,n}, y \rangle \sim a_{k,n} \langle u_{k,n}, x \rangle \langle v_{k,n}, y \rangle \quad (3.17)$$

$$\sim a_{k,n} \langle u_k, x \rangle \langle v_n, y \rangle, \quad (3.18)$$

where $(a_{k,n})$ is any of the following equivalent sequences:

$$\begin{aligned} \text{a) } a_{k,n} &= \frac{\rho_{k,n}}{\langle u_{k,n}, v_{k,n} \rangle}, & \text{b) } a_{k,n} &= \|M_{k,n}\|_{1,1}, \\ \text{c) } a_{k,n} &= \frac{\prod_{j=k}^n \mu_j}{\langle u_k, v_{k-1} \rangle} = \frac{\prod_{j=k}^n \lambda_j}{\langle u_{n+1}, v_n \rangle}. \end{aligned} \quad (3.19)$$

Lemma 3.4. *Assume [A3](#) and $s \in I_\mu^+ \cup I_\mu^-$. For each fixed $k \in \mathbb{Z}$ as $n \rightarrow \infty$, and for each fixed $n \in \mathbb{Z}$ as $k \rightarrow -\infty$, the contraction coefficient of the sequence $(M_{k,n-1})$ satisfies \mathbb{Q}_s^v -a.s.,*

$$\limsup_{n \rightarrow \infty} \frac{\log \mathbf{c}(M_{k,n-1})}{n-k} \leq K_1 \in [-\infty, 0), \quad (3.20)$$

$$\limsup_{k \rightarrow -\infty} \frac{\log \mathbf{c}(M_{k,n-1})}{n-k} \leq K_2 \in [-\infty, 0), \quad (3.21)$$

where K_1, K_2 are constants (which may be $-\infty$). In particular, for each $a \in (e^{K_1}, 1)$, as $n \rightarrow \infty$,

$$\mathbf{c}(M_{0,n-1}) = o(a^n) \quad \mathbb{Q}_s^v\text{-a.s.}$$

Proof. We fix $k \in \mathbb{Z}$, and set $X_i = \log \mathbf{c}(M_i)$. Let $c > 0$ be a constant. Define $X_i^c = X_i$ if $X_i \geq -c$ and $X_i^c = -c$ if $X_i < -c$. That is, $X_i^c = \max(X_i, -c)$. Let

$$S_{k,n-1} = \sum_{i=k}^{n-1} X_i^c, \quad \text{for } n > k. \quad (3.22)$$

We have the following martingale decomposition: for $n > k$,

$$S_{k,n-1} = \sum_{i=k}^{n-1} (X_i^c - \mathbb{E}_{\mathbb{Q}_s^v}(X_i^c | \mathcal{G}_{i-1})) + \sum_{i=k}^{n-1} \mathbb{E}_{\mathbb{Q}_s^v}(X_i^c | \mathcal{G}_{i-1}). \quad (3.23)$$

By [\[79, Corollary 2, p.385\]](#), we can prove that

$$\frac{1}{n-k} \sum_{i=k}^{n-1} (X_i^c - \mathbb{E}_{\mathbb{Q}_s^v}(X_i^c | \mathcal{G}_{i-1})) \xrightarrow{n \rightarrow \infty} 0, \quad \mathbb{Q}_s^v\text{-a.s.} \quad (3.24)$$

We next consider $\frac{1}{n-k} \sum_{i=k}^{n-1} \mathbb{E}_{\mathbb{Q}_s^v}(X_i^c | \mathcal{G}_{i-1})$. For any $A \in \mathcal{G}_{i-1}$, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_s^v}(X_i^c \mathbf{1}_A) &= \mathbb{E}\left(q_{i+1}^s(v, M_{0,i}) X_i^c \mathbf{1}_A\right) \\ &= \mathbb{E}\left(q_i^s(v, M_{0,i-1}) q_1^s(v \cdot M_{0,i-1}, M_i) X_i^c \mathbf{1}_A\right) \\ &= \mathbb{E}\left[q_i^s(v, M_{0,i-1}) \mathbb{E}\left(q_1^s(v \cdot M_{0,i-1}, M_i) X_i^c \mathbf{1}_A \mid \mathcal{G}_{i-1}\right)\right] \\ &= \mathbb{E}_{\mathbb{Q}_s^v}\left[\mathbf{1}_A \mathbb{E}\left(q_1^s(v \cdot M_{0,i-1}, M_i) X_i^c \mid \mathcal{G}_{i-1}\right)\right]. \end{aligned} \quad (3.25)$$

This implies that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_s^v}(X_i^c \mid \mathcal{G}_{i-1}) &= \mathbb{E}\left(q_1^s(v \cdot M_{0,i-1}, M_i) X_i^c \mid \mathcal{G}_{i-1}\right) \\ &= \tilde{q}_1^s(v \cdot M_{0,i-1}), \end{aligned} \quad (3.26)$$

where \tilde{q}_1^s is the function on \mathcal{S} defined by

$$\tilde{q}_1^s(x) = \mathbb{E}(q_1^s(x, M_0) X_0^c), \quad \forall x \in \mathcal{S}.$$

We see that

$$\begin{aligned} \tilde{q}_1^s(x) &= \mathbb{E}(q_1^s(x, M_0) X_0^c) \\ &= \mathbb{E}\left[\frac{\|xM_0\|^s r_s(x \cdot M_0)}{\kappa(s) r_s(x)} X_0^c\right] \leq C_s \mathbb{E}(\|xM_0\|^s X_0^c), \end{aligned} \quad (3.27)$$

where C_s is a constant.

Remark that $\iota(M_0) = \inf_{x \in \mathcal{S}} \|xM_0\| > 0$. It follows from (3.26) and (3.27) that for any $k \leq i \leq n-1$, \mathbb{Q}_s^v -a.s.,

$$\mathbb{E}_{\mathbb{Q}_s^v}(X_i^c | \mathcal{G}_{i-1}) \leq K_1 := \begin{cases} C_s \mathbb{E}(\iota(M_0)^s X_0^c) < 0 & \text{if } s \geq 0, \\ C_s \mathbb{E}(\|M_0\|^s X_0^c) < 0 & \text{if } s < 0. \end{cases} \quad (3.28)$$

This implies that for all $k \in \mathbb{Z}$ and $n > k$,

$$\frac{1}{n-k} \sum_{i=k}^{n-1} \mathbb{E}_{\mathbb{Q}_s^v}(X_i^c | \mathcal{G}_{i-1}) \leq K_1 < 0 \quad \mathbb{Q}_s^v\text{-a.s.} \quad (3.29)$$

Combining (3.23), (3.24) and (3.29), we get that for any fixed $k \in \mathbb{Z}$,

$$\limsup_{n \rightarrow \infty} \frac{S_{k,n-1}}{n-k} \leq K_1, \quad \mathbb{Q}_s^v\text{-a.s.}$$

Since $X_i \leq X_i^c$, this implies that,

$$\limsup_{n \rightarrow \infty} \frac{\sum_{i=k}^{n-1} X_i}{n-k} \leq K_1, \quad \mathbb{Q}_s^v\text{-a.s.}$$

Recall that $X_i = \log \mathbf{c}(M_i)$ and that $c(M_{k,n-1}) \leq \prod_{i=k}^{n-1} c(M_i)$. Therefore

$$\limsup_{n \rightarrow \infty} \frac{\log \mathbf{c}(M_{k,n-1})}{n-k} \leq \limsup_{n \rightarrow \infty} \frac{\sum_{i=k}^{n-1} X_i}{n-k} \leq K_1, \quad \mathbb{Q}_s^v\text{-a.s.}$$

This proves (3.20). For fixed $n \in \mathbb{Z}$ as $k \rightarrow -\infty$, we use an analogous argument as above to get (3.21). \square

Proof of Theorem 3.3. 1. Recall that by Lemma 3.4, under condition A3, for any fixed $k \in \mathbb{Z}$ as $n \rightarrow \infty$, or for any fixed $n \in \mathbb{Z}$ as $k \rightarrow -\infty$, it holds,

$$\mathfrak{c}(M_{k,n}) = \mathfrak{c}((M_{k,n})^T) \rightarrow 0 \quad \mathbb{Q}_s^v\text{-a.s.} \quad (3.30)$$

Consider the event

$$\Omega_1 = \left\{ \lim_{n \rightarrow \infty} \mathfrak{c}((M_{k,n})^T) = 0, \lim_{k \rightarrow -\infty} \mathfrak{c}((M_{k,n})^T) = 0 \right\},$$

whose probability is 1 under the new measure: $\mathbb{Q}_s^v(\Omega_1) = 1$. Note also that, on the event Ω_1 , for any $k \leq n$ in \mathbb{Z} , uniformly in $y, y' \in \mathcal{S}$,

$$\begin{aligned} \frac{1}{2} \|y \cdot (M_{k,n})^T - y' \cdot (M_{k,n})^T\| &\leq d(y \cdot (M_{k,n})^T, y' \cdot (M_{k,n})^T) \\ &\leq \mathfrak{c}((M_{k,n})^T). \end{aligned} \quad (3.31)$$

For any $k \leq n$ in \mathbb{Z} , introduce the set $C_{k,n} = \mathcal{S} \cdot (M_{k,n})^T := \{x \cdot (M_{k,n})^T : x \in \mathcal{S}\}$. From the contraction property (2.2) it follows that

$$\mathcal{S} \cdot M_{n+1}^T \subseteq \mathcal{S}. \quad (3.32)$$

Then applying $(M_{k,n})^T$ to both sides of (3.32) we get that $C_{k,n+1} \subset C_{k,n}$. Moreover, the sets $C_{k,n}$ are compact. Let $C_k = \bigcap_{m=k}^{\infty} C_{k,m} \neq \emptyset$. Since $\lim_{n \rightarrow \infty} \mathfrak{c}((M_{k,n})^T) = 0$, the diameter of C_k in the norm $\|\cdot\|$ is equal to 0. It follows from (3.31) that \mathbb{Q}_s^v -a.s., the set C_k is a singleton, i.e. C_k consists of one single point, which we denote by u_k .

Since $u_k \in C_k \subset C_{k,n}$, there exists $y' \in \mathcal{S}$ such that

$$u_k = y' \cdot (M_{k,n})^T. \quad (3.33)$$

Therefore, by (3.30) and (3.31), we have that, for any $k \leq n$ in \mathbb{Z} , for any $y \in \mathcal{S}$,

$$d(y \cdot (M_{k,n})^T, u_k) = d(y \cdot (M_{k,n})^T, y' \cdot (M_{k,n})^T) \leq \mathfrak{c}((M_{k,n})^T), \quad (3.34)$$

Taking the limit as $n \rightarrow \infty$ we obtain that, for any fixed $k \in \mathbb{Z}$, uniformly in $y \in \mathcal{S}$,

$$\lim_{n \rightarrow \infty} y \cdot (M_{k,n})^T = u_k \quad \mathbb{Q}_s^v\text{-a.s.}, \quad (3.35)$$

which implies that u_k is \mathcal{G}^k -measurable. We note that

$$u_k = \lim_{n \rightarrow \infty} y \cdot (M_k M_{k+1} \dots M_n)^T = \left(\lim_{n \rightarrow \infty} y \cdot (M_{k+1,n})^T \right) \cdot M_k^T = u_{k+1} \cdot M_k^T.$$

Therefore, for all $k \in \mathbb{Z}$, with $\lambda_k = \|u_{k+1} M_k^T\|$, \mathbb{Q}_s^v -a.s., we have $u_{k+1} M_k^T = \lambda_k u_k$. Using the uniform convergence of (3.34), choosing $y = u_{k,n}$ in it, we see that, for any $k \in \mathbb{Z}$,

$$\lim_{n \rightarrow \infty} d(u_{k,n}, u_k) = 0 \quad \mathbb{Q}_s^v\text{-a.s.} \quad (3.36)$$

This implies that for each fixed $k \in \mathbb{Z}$ as $n \rightarrow \infty$,

$$u_{k,n} - u_k \rightarrow 0.$$

We use the same argument as above to prove the existence of the sequence $(v_k)_{k \in \mathbb{Z}}$ such that for all k , $v_k > 0$, $\|v_k\| = 1$, and that for each fixed $k \in \mathbb{Z}$ as $n \rightarrow \infty$,

$$v_{k,n} - v_k \rightarrow 0.$$

2. For any fixed $k \in \mathbb{Z}$ as $n \rightarrow \infty$, we use the following result in [30],

$$\lim_{n \rightarrow \infty} \sup_{x, y \in \mathcal{S}} \left| \frac{\langle y \cdot (M_{k,n})^T, x \rangle}{\langle u_k, x \rangle} - 1 \right| = 0 \quad (3.37)$$

and notice that

$$\langle y \cdot (M_{k,n})^T, x \rangle = \frac{\langle y(M_{k,n})^T, x \rangle}{\|y(M_{k,n})^T\|} = \frac{\langle xM_{k,n}, y \rangle}{\langle \mathbf{1}M_{k,n}, y \rangle}.$$

Therefore, we get uniformly for $x, y \in \mathbb{R}_+^d \setminus \{0\}$, \mathbb{Q}_s^v -a.s.,

$$\frac{\langle xM_{k,n}, y \rangle}{\langle \mathbf{1}M_{k,n}, y \rangle \langle u_k, x \rangle} \rightarrow 1, \quad \text{as } n \rightarrow \infty. \quad (3.38)$$

Substituting $x = v_{k,n}$ into (3.38), we get uniformly for $y \in \mathbb{R}_+^d \setminus \{0\}$, \mathbb{Q}_s^v -a.s.,

$$\frac{\rho_{k,n} \langle v_{k,n}, y \rangle}{\langle \mathbf{1}M_{k,n}, y \rangle \langle u_k, v_{k,n} \rangle} \rightarrow 1, \quad \text{as } n \rightarrow \infty. \quad (3.39)$$

Substituting $y = u_{k,n}$ into (3.38), we get uniformly for $x \in \mathbb{R}_+^d \setminus \{0\}$, \mathbb{Q}_s^v -a.s.,

$$\frac{\rho_{k,n} \langle u_{k,n}, x \rangle}{\langle \mathbf{1}M_{k,n}, u_{k,n} \rangle \langle u_k, x \rangle} = \frac{\langle u_{k,n}, x \rangle}{\langle u_k, x \rangle} \rightarrow 1, \quad \text{as } n \rightarrow \infty. \quad (3.40)$$

Multiplying (3.39) with (3.40), we obtain uniformly for $x, y \in \mathbb{R}_+^d \setminus \{0\}$, \mathbb{Q}_s^v -a.s.,

$$\frac{\rho_{k,n} \langle u_{k,n}, x \rangle \langle v_{k,n}, y \rangle}{\langle u_k, v_{k,n} \rangle \langle \mathbf{1}M_{k,n}, y \rangle \langle u_k, x \rangle} \rightarrow 1, \quad \text{as } n \rightarrow \infty. \quad (3.41)$$

Dividing (3.38) by (3.41), it gives uniformly for $x, y \in \mathbb{R}_+^d \setminus \{0\}$, \mathbb{Q}_s^v -a.s.,

$$\frac{\langle xM_{k,n}, y \rangle}{\rho_{k,n} \frac{\langle u_{k,n}, x \rangle \langle v_{k,n}, y \rangle}{\langle u_k, v_{k,n} \rangle}} \rightarrow 1, \quad \text{as } n \rightarrow \infty. \quad (3.42)$$

Taking $x = v_{k,n}$ in (3.40) implies $\langle u_k, v_{k,n} \rangle \sim \langle u_{k,n}, v_{k,n} \rangle$ as $n \rightarrow \infty$. Therefore, the above convergence implies, uniformly for $x, y \in \mathbb{R}_+^d \setminus \{0\}$, \mathbb{Q}_s^v -a.s.,

$$\langle xM_{k,n}, y \rangle \sim \rho_{k,n} \frac{\langle u_{k,n}, x \rangle \langle v_{k,n}, y \rangle}{\langle u_k, v_{k,n} \rangle}, \quad \text{as } n \rightarrow \infty. \quad (3.43)$$

We now prove that, uniformly for $x, y \in \mathbb{R}_+^d \setminus \{0\}$, \mathbb{Q}_s^v -a.s., as $n \rightarrow \infty$,

$$\langle xM_{k,n}, y \rangle \sim \|M_{k,n}\|_{1,1} \langle u_k, x \rangle \langle v_n, y \rangle. \quad (3.44)$$

Indeed, taking $x = v_{k-1}$ in gives, uniformly for $x, y \in \mathbb{R}_+^d \setminus \{0\}$, \mathbb{Q}_s^v -a.s.,

$$\frac{\langle v_n, y \rangle \mu_k \cdots \mu_n}{\langle u_k, v_{k-1} \rangle \langle \mathbf{1}M_{k,n}, y \rangle} \rightarrow 1, \quad \text{as } n \rightarrow \infty. \quad (3.45)$$

Dividing (3.38) by (3.45), we get, uniformly for $x, y \in \mathbb{R}_+^d \setminus \{0\}$, \mathbb{Q}_s^v -a.s.,

$$\frac{\langle xM_{k,n}, y \rangle}{\langle u_k, x \rangle \langle v_n, y \rangle} \frac{\langle u_k, v_{k-1} \rangle}{\mu_k \cdots \mu_n} \rightarrow 1, \quad \text{as } n \rightarrow \infty. \quad (3.46)$$

Let $x = y = \mathbf{1}$ into (3.46), we obtain, \mathbb{Q}_s^v -a.s.,

$$\|M_{k,n}\|_{1,1} \frac{\langle u_k, v_{k-1} \rangle}{\mu_k \cdots \mu_n} \rightarrow 1, \quad \text{as } n \rightarrow \infty. \quad (3.47)$$

Combining (3.46) and (3.47), we get (3.44).

For any fixed $n \in \mathbb{Z}$ as $k \rightarrow -\infty$, the proof is completely similar as above. This concludes the proof of the theorem. \square

Remark 3.5. The following results will be useful.

- (1) Assume condition **A3** and $s \in I_\mu^+ \cup I_\mu^-$. Then for any fixed $k \in \mathbb{Z}$ as $n \rightarrow \infty$, and for any fixed $n \in \mathbb{Z}$ as $k \rightarrow -\infty$,

$$\sup_{y \in \mathcal{S}} d(y \cdot (M_{k,n})^T, u_k) \rightarrow 0 \quad \text{and} \quad \sup_{y \in \mathcal{S}} d(y \cdot M_{k,n}, v_n) \rightarrow 0, \quad \mathbb{Q}_s^v\text{-a.s.} \quad (3.48)$$

The first result follows from (3.34) and (3.30). The second can be proved similarly.

- (2) For $k, n \in \mathbb{Z}$ with $k \leq n$, the following assertion holds \mathbb{Q}_s^v -a.s.:

$$u_{n+1} M_{k,n}^T = \left(\prod_{j=k}^n \lambda_j \right) u_k \quad \text{and} \quad v_{k-1} M_{k,n} = \left(\prod_{j=k}^n \mu_j \right) v_n. \quad (3.49)$$

3.3. Stable and mixing convergence under the changed measure. Recall that the sequences $(v_n)_{n \geq 0}$ and $(M_n)_{n \geq 0}$ are stationary under the measure \mathbb{P} , but non-stationary under the measure \mathbb{Q}_s^v when $d > 1$. However, under \mathbb{Q}_s^v , we still have the mixing stable convergence of $(v_n)_{n \geq 0}$ and $(x \cdot M_n)_{n \geq 0}$, as in the case under \mathbb{P} (see Theorem 2.9).

We begin with the usual convergence in law under \mathbb{Q}_s^v .

Lemma 3.6. *Assume **A4** and let $s \in I_\mu^- \cup I_\mu^+$, $v \in \mathcal{S}$. Assume also condition **A3** when $s > 0$, and **A1** when $s < 0$. Then under \mathbb{Q}_s^v , for each $x \in \mathcal{S}$ and $k \in \mathbb{Z}$, all the three sequences $(x \cdot M_{k,n})$, (v_n) and $(v_{k,n})$, converge in law to π_s , the unique \mathbb{Q}_s -invariant probability measure given by (2.15).*

Proof. The lemma is a consequence of Lemma 3.2(3) and Lemma 2.4. In fact, by Lemma 2.4 and (3.11), we see that

$$v \cdot M_{0,n} \rightarrow \pi_s \quad \text{in law under} \quad \mathbb{Q}_s^v.$$

From (3.48), we know that $v \cdot M_{0,n} - v_n \rightarrow 0$ \mathbb{Q}_s^v -a.s. So it follows that

$$v_n \rightarrow \pi_s \quad \text{in law under} \quad \mathbb{Q}_s^v.$$

Again from (3.48), we know that for any $x \in \mathcal{S}$ and $k \in \mathbb{Z}$,

$$x \cdot M_{k,n} - v_n \rightarrow 0 \quad \text{and} \quad v_{k,n} - v_n \rightarrow 0 \quad \mathbb{Q}_s^v\text{-a.s.}$$

So the convergence in law of (v_n) to π_s under \mathbb{Q}_s^v implies also that of $(x \cdot M_{k,n})$ and $(v_{k,n})$. \square

Theorem 3.7. *Assume **A4** and let $s \in I_\mu^- \cup I_\mu^+$, $v \in \mathcal{S}$. Assume also condition **A3** when $s > 0$, and **A1** when $s < 0$. Then under \mathbb{Q}_s^v , for each $x \in \mathcal{S}$ and $k \in \mathbb{Z}$, as $n \rightarrow \infty$, all the three sequences $(x \cdot M_{k,n})$, (v_n) and $(v_{k,n})$, converge mixing stably to π_s : that is, for any real valued continuous function φ on \mathcal{S} and any \mathbb{R} -valued random variable η such that $\mathbb{E}_{\mathbb{Q}_s^v}(|\eta|) < \infty$, it holds that for any $k \in \mathbb{Z}$,*

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^v}(\eta \varphi(x \cdot M_{k,n})) = \mathbb{E}_{\mathbb{Q}_s^v}(\eta) \pi_s(\varphi) \quad \forall x \in \mathcal{S}, \quad (3.50)$$

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^v}(\eta \varphi(v_n)) = \mathbb{E}_{\mathbb{Q}_s^v}(\eta) \pi_s(\varphi), \quad (3.51)$$

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^v}(\eta \varphi(v_{k,n})) = \mathbb{E}_{\mathbb{Q}_s^v}(\eta) \pi_s(\varphi), \quad (3.52)$$

where π_s is the unique \mathbb{Q}_s -invariant measure given by (2.15). Moreover, we have the following convergences of the joint laws: under \mathbb{Q}_s^v , for each fixed $k \in \mathbb{Z}$, as $n \rightarrow \infty$,

$$(x \cdot M_{k,n}, v_n, v_{k,n}) \rightarrow \pi_s^3 \quad \text{mixing stably}, \quad (3.53)$$

$$(x \cdot M_{k,n}, v_n, v_{k,n}, u_{k,n}) \rightarrow \pi_s^3 \otimes \mu_k^s \quad \text{stably}, \quad (3.54)$$

where π_s^3 denotes the law image of π_s under the mapping $x \mapsto (x, x, x)$ from \mathcal{S} to \mathcal{S}^3 , and μ_k^s the law of u_k under \mathbb{Q}_s^v .

By Remark 2.8, this theorem implies Theorem 1.3.

Proof of Theorem 3.7. Step 1: proof of (3.50) with $k = 0$ and $x = v$: that is,

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^v} (\eta \varphi(v \cdot M_{0,n})) = \mathbb{E}_{\mathbb{Q}_s^v} (\eta) \pi_s(\varphi) \quad (3.55)$$

for any \mathbb{R} -valued random variable η with $\mathbb{E}_{\mathbb{Q}_s^v} |\eta| < \infty$.

a) We first prove (3.55) for any \mathcal{G}_m -measurable η with $\mathbb{E}_{\mathbb{Q}_s^v} |\eta| < \infty$, for each fixed $m \in \mathbb{Z}$, where $\mathcal{G}_m = \sigma(\{\xi_j, j \leq m\})$, as defined in (3.14). Let $n > m$. Since η is \mathcal{G}_m -measurable, by the Markov property (3.13), we have,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_s^v} (\eta \varphi(v \cdot M_{0,n})) &= \mathbb{E}_{\mathbb{Q}_s^v} (\eta \mathbb{E}_{\mathbb{Q}_s^v} (\varphi(v \cdot M_{0,n}) | \mathcal{G}_m)) \\ &= \mathbb{E}_{\mathbb{Q}_s^v} (\eta Q_s^{n-m} \varphi(v \cdot M_{0,m})). \end{aligned} \quad (3.56)$$

By Lemma 2.4 we know that for any $x \in \mathcal{S}$,

$$\lim_{k \rightarrow \infty} Q_s^k \varphi(x) = \pi_s(\varphi). \quad (3.57)$$

Therefore from (3.56) and the dominated convergence theorem, we get (3.55), for any \mathcal{G}_m -measurable η with $\mathbb{E}_{\mathbb{Q}_s^v} |\eta| < \infty$.

b) Then, with the monotone class argument as in Step 2 of the proof of Theorem 2.9, we can prove that (3.55) holds for any \mathcal{G} -measurable η with $\mathbb{E}_{\mathbb{Q}_s^v} |\eta| < \infty$. Finally, as we did in Step 3 of the proof of Theorem 2.9), by conditioning on \mathcal{G} , we get (3.55) for any \mathcal{F} -measurable η with $\mathbb{E}_{\mathbb{Q}_s^v} |\eta| < \infty$.

Step 2: proof of the mixing convergence of (v_n) , $(x \cdot M_{k,n})$ and $(v_{k,n})$. From (3.55) and the fact that $v \cdot M_{0,n} - v_n \rightarrow 0$ \mathbb{Q}_s^v -a.s., by the same argument as we used in Step 4 of the proof of Theorem 2.9, we get (3.51) about the mixing stable convergence of (v_n) . Similarly, using this convergence, and the fact that

$$x \cdot M_{k,n} - v_n \rightarrow 0 \text{ and } v_{k,n} - v_n \rightarrow 0 \quad \mathbb{Q}_s^v\text{-a.s.},$$

we get (3.50) and (3.52), about the mixing stable convergence of $(x \cdot M_{k,n})$ and $(v_{k,n})$.

Step 3: proof of the convergence of the joint laws. By the same argument as in Steps 5 and 6 of the proof of Theorem 2.9, the mixing stable convergence to π_s of the three sequences $(x \cdot M_{k,n})$, (v_n) and $(v_{k,n})$, and the a.s. convergence of $u_{k,n}$ to u_k , enable us to conclude the mixing stable convergence of $(x \cdot M_{k,n}, v_n, v_{k,n})$ to π_s^3 , the law image of π_s under the mapping $x \mapsto (x, x, x)$ from \mathcal{S} to \mathcal{S}^3 , and the stable convergence of $(x \cdot M_{k,n}, v_n, v_{k,n}, u_{k,n})$ to $\pi_s^3 \otimes \mu_k^s$, where μ_k^s is the law of u_k under \mathbb{Q}_s^v . \square

3.4. Moments for products of random matrices. In this subsection, as applications of the convergence of $v \cdot M_{0,n-1}$ under the new measure \mathbb{Q}_s^v , we give the exact equivalence of the moments of the vector norm $\|v M_{0,n-1}\|$, the matrix norm $\|M_{0,n-1}\|_{1,1}$, and, more generally the scalar product $\langle x M_{0,n-1}, y \rangle$. Recall that π_s is the limit law of v_n under \mathbb{Q}_s^v .

To warm up, we first consider the case for the vector norm $\|x M_{0,n-1}\|$.

Theorem 3.8. *Assume A4 and let $s \in I_\mu^- \cup I_\mu^+$. Assume also condition A3 when $s > 0$, and A1 when $s < 0$. Then, for any $x \in \mathcal{S}$, we have*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E} \|x M_{0,n-1}\|^s}{\kappa^n(s)} = r_s(x) \int_{\mathcal{S}} \frac{1}{r_s(u)} \pi_s(du). \quad (3.58)$$

Proof. By the definition of \mathbb{Q}_s^x , we have

$$\frac{\mathbb{E} \|x M_{0,n-1}\|^s}{\kappa^n(s)} = \mathbb{E}_{\mathbb{Q}_s^x} \left(\frac{r_s(x)}{r_s(x \cdot M_{0,n-1})} \right). \quad (3.59)$$

Since $x \cdot M_{0,n-1}$ converges in law to π_s under \mathbb{Q}_s^x (cf. Lemma 3.6), it follows that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E} \|xM_{0,n-1}\|^s}{\kappa^n(s)} = r_s(x) \int_{\mathcal{S}} \frac{1}{r_s(u)} \pi_s(du).$$

□

We next consider the more general case, i.e., for the scalar product $\langle xM_{0,n-1}, y \rangle$.

Theorem 3.9. *Assume A4 and let $s \in I_\mu^- \cup I_\mu^+$. Assume also condition A3 when $s > 0$, and A1 when $s < 0$. Then for any $x, y \in \mathcal{S}$, we have*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E} \langle xM_{0,n-1}, y \rangle^s}{\kappa^n(s)} = r_s(x) \int_{\mathcal{S}} \frac{\langle u, y \rangle^s}{r_s(u)} \pi_s(du), \quad (3.60)$$

In particular,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E} \|M_{0,n-1}\|_{1,1}^s}{\kappa^n(s)} = \int_{\mathcal{S}} \frac{1}{r_s(u)} \pi_s(du). \quad (3.61)$$

Proof. We have

$$\begin{aligned} \frac{\mathbb{E} \langle xM_{0,n-1}, y \rangle^s}{\kappa^n(s)} &= \mathbb{E} \left[\frac{\|xM_{0,n-1}\|^s \langle xM_{0,n-1}, y \rangle^s}{\kappa^n(s) \|xM_{0,n-1}\|^s} \right] \\ &= \mathbb{E}_{\mathbb{Q}_s^x} \frac{\langle xM_{0,n-1}, y \rangle^s}{\|xM_{0,n-1}\|^s} \frac{r_s(x)}{r_s(x \cdot M_{0,n-1})}. \end{aligned} \quad (3.62)$$

Notice that, by the Perron-Frobenius type theorem (Theorem 3.3), we know that \mathbb{Q}_s^x -a.s.

$$\frac{\langle xM_{0,n-1}, y \rangle}{\|xM_{0,n-1}\|} \sim \langle v_{n-1}, y \rangle, \quad \text{as } n \rightarrow \infty.$$

Since $|\langle v_{n-1}, y \rangle| \leq 1$, it follows that \mathbb{Q}_s^x -a.s.

$$\frac{\langle xM_{0,n-1}, y \rangle}{\|xM_{0,n-1}\|} - \langle v_{n-1}, y \rangle \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

So by the dominated convergence theorem, we get that

$$\mathbb{E}_{\mathbb{Q}_s^x} \left[\left(\frac{\langle xM_{0,n-1}, y \rangle^s}{\|xM_{0,n-1}\|^s} - \langle v_{n-1}, y \rangle^s \right) \frac{r_s(x)}{r_s(x \cdot M_{0,n-1})} \right] \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.63)$$

Combining this with (3.62), we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\mathbb{E} \langle xM_{0,n-1}, y \rangle^s}{\kappa^n(s)} &= \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^x} \left(\langle v_{n-1}, y \rangle^s \frac{r_s(x)}{r_s(x \cdot M_{0,n-1})} \right) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^x} \left[\langle v_{n-1}, y \rangle^s r_s(x) \left(\frac{1}{r_s(x \cdot M_{0,n-1})} - \frac{1}{r_s(v_{n-1})} \right) \right] \\ &\quad + \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^x} \left[\langle v_{n-1}, y \rangle^s \frac{r_s(x)}{r_s(v_{n-1})} \right], \end{aligned} \quad (3.64)$$

provided that the last two limits exist. Their existence will be established below. Using Lemma 2.2 with $\bar{s} = \min\{1, |s|\}$ and (3.48), we have, by the dominated convergence theorem,

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}_s^x} \left[\left\langle v_{n-1}, y \right\rangle^s r_s(x) \left| \frac{1}{r_s(x \cdot M_{0,n-1})} - \frac{1}{r_s(v_{n-1})} \right| \right] \\ & \leq c_s \mathbb{E}_{\mathbb{Q}_s^x} \left(\left| r_s(x \cdot M_{0,n-1}) - r_s(v_{n-1}) \right| \right) \\ & \leq c_s \mathbb{E}_{\mathbb{Q}_s^x} \|x \cdot M_{0,n-1} - v_{n-1}\|^{\bar{s}} \rightarrow 0, \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (3.65)$$

where the constant c_s may change from one line to another.

Combining (3.64) and (3.65) and applying Theorem 3.7, we deduce that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\mathbb{E} \langle x M_{0,n-1}, y \rangle^s}{\kappa^n(s)} &= \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^x} \left[r_s(x) \frac{\langle v_{n-1}, y \rangle^s}{r_s(v_{n-1})} \right] \\ &= r_s(x) \int_{\mathcal{S}} \frac{\langle u, y \rangle^s}{r_s(u)} \pi_s(du). \end{aligned} \quad (3.66)$$

This gives (3.60). Applying (3.60) to $x = y = \frac{1}{d}$, and remarking that $r_s(\frac{1}{d}) = d^{-s}$ (by (2.9)), we get (3.61). So the theorem is proved. \square

The following theorem provides the precise equivalence for the moment of the spectral radius $\rho_{0,n-1}$.

Theorem 3.10. *Assume A4 and let $s \in I_\mu^- \cup I_\mu^+$. Assume also condition A3 when $s > 0$, and A1 when $s < 0$. Then*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(\rho_{0,n-1})^s}{\kappa^n(s)} = \int_{\mathcal{S}^2} \frac{\langle u, v \rangle^s}{r_s(v)} \mu_0^s(du) \pi_s(dv), \quad (3.67)$$

where μ_0^s denotes the law of u_0 under $\mathbb{Q}_s^{\frac{1}{d}}$.

Proof. For any $x \in \mathcal{S}$, by the definition of the changed measure \mathbb{Q}_s^x , we have

$$\begin{aligned} \frac{\mathbb{E}(\rho_{0,n-1})^s}{\kappa^n(s)} &= \mathbb{E} \left[\frac{\|x M_{0,n-1}\|^s}{\kappa^n(s)} \frac{r_s(x \cdot M_{0,n-1})}{r_s(x)} \frac{(\rho_{0,n-1})^s}{\|x M_{0,n-1}\|^s} \frac{r_s(x)}{r_s(x \cdot M_{0,n-1})} \right] \\ &= \mathbb{E}_{\mathbb{Q}_s^x} \left[\frac{(\rho_{0,n-1})^s}{\|x M_{0,n-1}\|^s} \frac{r_s(x)}{r_s(x \cdot M_{0,n-1})} \right]. \end{aligned} \quad (3.68)$$

By part 2 of Theorem 3.3, we have the following equivalence, as $n \rightarrow \infty$,

$$\frac{\rho_{0,n-1}}{\|x M_{0,n-1}\|} \sim \frac{\|M_{0,n-1}\|_{1,1} \langle u_{0,n-1}, v_{0,n-1} \rangle}{\|M_{0,n-1}\|_{1,1} \langle u_0, x \rangle} = \frac{\langle u_{0,n-1}, v_{0,n-1} \rangle}{\langle u_0, x \rangle}. \quad (3.69)$$

Notice that $\rho_{0,n-1} = \|v_{0,n-1} M_{0,n-1}\|$, so that $\frac{\rho_{0,n-1}}{\|x M_{0,n-1}\|}$ lies between two constants thanks to the Furstenberg-Kesten condition A1. Therefore, by the dominated convergence theorem, we get that

$$\mathbb{E}_{\mathbb{Q}_s^x} \left[\left| \left(\frac{\rho_{0,n-1}}{\|x M_{0,n-1}\|} \right)^s - \left(\frac{\langle u_{0,n-1}, v_{0,n-1} \rangle}{\langle u_0, x \rangle} \right)^s \right| \frac{r_s(x)}{r_s(x \cdot M_{0,n-1})} \right] \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.70)$$

From (3.68) and (3.68), it follows that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(\rho_{0,n-1})^s}{\kappa^n(s)} = \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^x} \left[\left(\frac{\langle u_{0,n-1}, v_{0,n-1} \rangle}{\langle u_0, x \rangle} \right)^s \frac{r_s(x)}{r_s(x \cdot M_{0,n-1})} \right], \quad (3.71)$$

provided that the second limit exists. Moreover we know that \mathbb{Q}_s^x -a.s.,

$$\langle u_{0,n-1} - v_{0,n-1} \rangle - \langle u_0, v_{n-1} \rangle \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Using again the dominated convergence theorem, together with the mixing convergence of $(x \cdot M_{0,n-1}, v_{n-1})$ (see Theorem 3.7), it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\mathbb{E}(\rho_{0,n-1})^s}{\kappa^n(s)} &= \lim_{n \rightarrow \infty} \left[\frac{\langle u_0, v_{n-1} \rangle^s}{\langle u_0, x \rangle^s} \frac{r_s(x)}{r_s(x \cdot M_{0,n-1})} \right] \\ &= r_s(x) \tilde{\mathbb{E}}_{\mathbb{Q}_s^x} \left[\frac{\langle u_0, v_\infty \rangle^s}{\langle u_0, x \rangle^s r_s(v_\infty)} \right] =: C_s, \end{aligned} \quad (3.72)$$

where v_∞ is a \mathcal{S} -valued random variable defined on some extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{Q}}_s^x)$ of $(\Omega, \mathcal{F}, \mathbb{Q}_s^x)$, which is independent of u_0 and has law π_s , under $\tilde{\mathbb{Q}}_s^x$. It is interesting to notice that the constant C_s does not depend on the choice of x . Choosing $x = \frac{1}{d}$, we get

$$C_s = \int_{\mathcal{S}^2} \frac{\langle u, v \rangle^s}{r_s(v)} \mu_0^s(du) \pi_s(dv),$$

where μ_0^s denotes the law of u_0 under $\mathbb{Q}_s^{1/d}$. \square

4. THE BRANCHING PROCESS UNDER THE CHANGED MEASURE $\mathbb{Q}_s^{\mathcal{E}^i}$

Let (Z_n^i) be the d-type branching process in an i.i.d. random environment ξ as introduced in the introduction, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, but, for convenience, just as in Section 2, we consider the double sided environment sequence $\xi = (\xi_k : k \in \mathbb{Z})$ instead of the one sided environment sequence initially introduced. Accordingly, we can define the annealed law \mathbb{P} as in (1.7) and (1.8), but with \mathbb{N} therein replaced by \mathbb{Z} .

In this section, we will establish asymptotic properties of the branching process Z_n under the changed measure, about the L^p -convergence of the fundamental martingale (W_n^i) , the normalized j -type population size $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$, the normalized total population size $\|Z_n^i\|/\mathbb{E}_\xi \|Z_n^i\|$, and the convergence of the direction $Z_n^i/\|Z_n^i\|$.

4.1. The environment sequence (ξ_n) under the changed measure. Notice that in the setting of the branching process the change of measure formula (3.7) can be written as follows: for any $n \geq 0$ and \mathcal{G}_n -measurable random variable Y ,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_s^v}(Y) &= \int_{E^{\mathbb{Z}}} \left(\int Y(\xi, y) q_{n+1}^s(v, M_{0,n}) \mathbb{P}_\xi(dy) \right) \tau(d\xi) \\ &= \int_{E^{\mathbb{Z}}} \left[\int Y(\xi, y) \mathbb{P}_\xi(dy) \right] q_{n+1}^s(v, M_{0,n}) \tau(d\xi) \\ &= \int_{E^{\mathbb{Z}}} \int Y(\xi, y) \mathbb{P}_\xi(dy) \tau_s^v(d\xi), \end{aligned}$$

where τ_s^v is the probability measure on $(E^{\mathbb{Z}}, \mathcal{E}^{\otimes \mathbb{Z}})$ such that for all $n \geq 0$,

$$\tau_s^v|_{\mathcal{G}_n} = q_{n+1}^s(v, M_{0,n}) \tau|_{\mathcal{G}_n}, \quad \text{where } \mathcal{G}_n = \sigma(\xi_k : k \in \mathbb{Z}, k \leq n).$$

Inspired by this, we can extend the measure \mathbb{Q}_s^v on $(\Omega, \sigma(\xi))$ defined from (3.7) to a probability measure on (Ω, \mathcal{F}) defined in the same way as in (1.7) and (1.8):

$$\mathbb{Q}_s^v(d\xi, dy) = \mathbb{P}_\xi(dy) \tau_s^v(d\xi), \quad (4.1)$$

which means that for any positive and \mathcal{F} -measurable function f defined the product space $\Omega = E^{\mathbb{N}} \times \Lambda$,

$$\int_{E^{\mathbb{N}} \times \Lambda} f(\xi, y) \mathbb{Q}_s^v(d\xi, dy) = \int_{E^{\mathbb{N}}} \left[\int_{\Lambda} f(\xi, y) \mathbb{P}_\xi(dy) \right] \tau_s^v(d\xi). \quad (4.2)$$

Notice that given the environment ξ , the conditional probability under \mathbb{Q}_s^v coincides with that under \mathbb{P} :

$$\mathbb{Q}_s^v(\cdot|\xi) = \mathbb{P}(\cdot|\xi) = \mathbb{P}_\xi, \quad (4.3)$$

a.s., with respect to both \mathbb{Q}_s^v and \mathbb{P} . Consider the natural filtration:

$$\mathcal{F}_n = \sigma(\xi, N_{l,j}^r, 0 \leq j < n, 1 \leq r \leq d, l \geq 1), \quad n \in \mathbb{Z}, \quad (4.4)$$

$$\mathcal{F}'_n = \sigma(\xi_j, N_{l,j}^r, 0 \leq j < n, 1 \leq r \leq d, l \geq 1), \quad n \in \mathbb{Z}, \quad (4.5)$$

with the convention that $\mathcal{F}_n = \sigma(\xi)$ and $\mathcal{F}'_n = \sigma(\xi_j, j < n)$ if $n \leq 0$. Then

$$\mathcal{G}_{n-1} \subset \mathcal{F}'_n \subset \mathcal{F}_n \quad \forall n \in \mathbb{Z}.$$

It can be easily checked that the restriction of \mathbb{Q}_s^v to \mathcal{F}'_n has the density $q_n^s(v, M_{0,n-1})$ with respect to that of \mathbb{P} :

$$\mathbb{Q}_s^v|_{\mathcal{F}'_n} = q_n^s(v, M_{0,n-1})\mathbb{P}|_{\mathcal{F}'_n} \quad (4.6)$$

In fact we can also define the new measure \mathbb{Q}_s^v starting from this formula. Notice that (4.6) implies (3.7).

The formula (4.1) shows that under the changed measure \mathbb{Q}_s^v , the process $(Z_n)_{n \geq 0}$ is still a branching process in random environment ξ , but with the law of ξ changed to be τ_s^v (compare (4.1) with (1.7)). However, the environment sequence (ξ_n) is no longer i.i.d., and even not stationary, under the new measure \mathbb{Q}_s^v , for $d > 1$. So the usual theory for a branching process in a stationary and ergodic environment does not apply under \mathbb{Q}_s^v . This is one of the remarkable difficulties compared with the single type case $d = 1$. Nevertheless, we will prove that, under the new measure the branching process $(Z_n)_{n \geq 0}$ still behave like a usual branching process in an i.i.d. environment. This is mainly due to the fact that the products of the mean matrices have similar properties under the new measure, as shown in Section 3, and the sub-iid property established in Lemma below.

We give some properties on the products of random matrices $M_{n,n+k} = M_n \cdots M_{n+k}$ under the Furstenberg-Kesten condition **A1** under the changed measure \mathbb{Q}_s^v .

Lemma 4.1. *Assume condition **A1**, and let $s \in I_\mu^+ \cup I_\mu^-$. Then:*

(1) *For all $n \geq 1, k \geq 0$ and $1 \leq i, j, r \leq d$, \mathbb{Q}_s^v -a.s.,*

$$\frac{1}{D} \leq \frac{M_{n,n+k}(i,j)}{M_{n,n+k}(i,r)} \leq D \quad \text{and} \quad \frac{1}{D} \leq \frac{M_{n,n+k}(i,j)}{M_{n,n+k}(r,j)} \leq D; \quad (4.7)$$

$$\frac{1}{D} \|M_{n,n+k}\| \leq \iota(M_{n,n+k}) \leq \|M_{n,n+k}\|. \quad (4.8)$$

(2) *For all $n \geq 0$ and $1 \leq i, j \leq d$, \mathbb{Q}_s^v -a.s.,*

$$u_n(i) \leq D u_n(j) \quad \text{and} \quad \frac{1}{dD} \leq u_n(i) \leq 1; \quad (4.9)$$

(3) *For all $n \geq 1, k \geq 0$ and $1 \leq i, j \leq d$, \mathbb{Q}_s^v -a.s.,*

$$\frac{1}{dD^2} \leq \frac{M_{n,n+k}(i,j)u_{n+k+1}(j)}{\lambda_{n,n+k}u_n(i)} \leq 1. \quad (4.10)$$

Under \mathbb{P} (when $s = 0$), the a.s. assertions (4.7) and (4.9) have been proved in [30, Lemma 3.6]; together with (3.49), they imply (4.10). Also, (4.7) implies (4.8) (just as (1.17) implies (1.18)). The results under \mathbb{Q}_s^v follow from those under \mathbb{P} , by the definition of \mathbb{Q}_s^v .

Remark 4.2. The following consequence of (4.8) will be used several times: under the conditions of Lemma 4.1, we have, \mathbb{Q}_s^v -a.s., for any $k, n \in \mathbb{Z}$ with $k \leq n$, and $x, y \in \mathcal{S}$,

$$\frac{1}{D} \|M_{k,n}\| \leq \|xM_{k,n}\| \leq \|M_{k,n}\|, \quad \|xM_{k,n}\| \leq D \|yM_{k,n}\|, \quad (4.11)$$

and the inequalities remain valid when $M_{k,n}$ is replaced by its transpose $M_{k,n}^T$.

We need the following lemma. Denote

$$\mathcal{G}_{n,n+k} = \sigma(\xi_j, n \leq j \leq n+k) \text{ for } n, k \geq 0, \text{ and } \mathcal{G}_{n,\infty} = \sigma(\xi_j, j \geq n) = \mathcal{G}^n.$$

It is known that $0 < \min_{x \in \mathcal{S}} r_s(x) \leq \max_{x \in \mathcal{S}} r_s(x) < \infty$. Let $R_s \in (0, \infty)$ be defined as

$$R_s := \frac{\max_{x \in \mathcal{S}} r_s(x)}{\min_{x \in \mathcal{S}} r_s(x)}, \quad (4.12)$$

so that

$$r_s(x) \leq R_s r_s(y) \quad \forall x, y \in \mathcal{S}.$$

Recall that under the new measure \mathbb{Q}_s^v , the sequence $(\xi_k)_{k \geq 0}$ is no longer stationary, nor independent. However, the sequence is sub-iid in some sense, as shown in the following lemma.

Lemma 4.3 (Sub-iid property of $(\xi_n)_{n \geq 0}$ under \mathbb{Q}_s^v). *Assume A1. Let $s \in I_\mu^+ \cup I_\mu^-$, and set $C_s = D^{|s|} R_s^2$. Then the following assertions hold:*

- (1) *The sequence $\{\xi_k : k \geq 0\}$ is sub-independent in the sense that for all integers $n, i_1, \dots, i_n \geq 1$, and all measurable functions $f_j : E^{i_j} \rightarrow [0, \infty]$, $1 \leq j \leq n$, we have, with $C_s := D^{|s|} R_s^2$,*

$$\begin{aligned} C_s^{-(n-1)} \prod_{j=1}^n \mathbb{E}_{\mathbb{Q}_s^v} f_j(\xi_0, \dots, \xi_{i_j-1}) &\leq \mathbb{E}_{\mathbb{Q}_s^v} \prod_{j=1}^n f_j(\xi_{i_1+\dots+i_{j-1}}, \dots, \xi_{i_1+\dots+i_{j-1}+i_j-1}) \\ &\leq C_s^{(n-1)} \prod_{j=1}^n \mathbb{E}_{\mathbb{Q}_s^v} f_j(\xi_0, \dots, \xi_{i_j-1}), \end{aligned} \quad (4.13)$$

$$\begin{aligned} C_s^{-2(n-1)} \prod_{j=1}^n \mathbb{E}_{\mathbb{Q}_s^v} f_j(\xi_{i_1+\dots+i_{j-1}}, \dots, \xi_{i_1+\dots+i_{j-1}}) &\leq \mathbb{E}_{\mathbb{Q}_s^v} \prod_{j=1}^n f_j(\xi_{i_1+\dots+i_{j-1}}, \dots, \xi_{i_1+\dots+i_{j-1}}) \\ &\leq C_s^{2(n-1)} \prod_{j=1}^n \mathbb{E}_{\mathbb{Q}_s^v} f_j(\xi_{i_1+\dots+i_{j-1}}, \dots, \xi_{i_1+\dots+i_{j-1}}), \end{aligned} \quad (4.14)$$

where by convention $i_1 + \dots + i_{j-1} = 0$ if $j = 1$.

- (2) *The sequence $\{\xi_k : k \geq 0\}$ is sub-stationary in the sense that for all $n, k \geq 1$, and all measurable functions $f : E^n \rightarrow [0, \infty]$,*

$$C_s^{-1} \mathbb{E}_{\mathbb{Q}_s^v} f(\xi_0, \dots, \xi_{n-1}) \leq \mathbb{E}_{\mathbb{Q}_s^v} f(\xi_k, \dots, \xi_{k+n-1}) \leq C_s \mathbb{E}_{\mathbb{Q}_s^v} f(\xi_0, \dots, \xi_{n-1}). \quad (4.15)$$

- (3) *For each $n \geq 1$ and each measurable and positive function f defined on E^n , we have*

$$C_s^{-2} \mathbb{E}_{\mathbb{Q}_s^v} f(\xi_0, \dots, \xi_{n-1}) \leq \mathbb{E}_{\mathbb{Q}_s^v} (f(\xi_0, \dots, \xi_{n-1}) | \mathcal{G}_{n,\infty}) \leq C_s^2 \mathbb{E}_{\mathbb{Q}_s^v} f(\xi_0, \dots, \xi_{n-1}). \quad (4.16)$$

As a direct consequence, we see that under the new measure \mathbb{Q}_s^v , the sub-iid property holds for any sequence of random matrices (M_k) :

Corollary 4.4. *Assume the hypotheses of Lemma 4.3, and let $M : E \rightarrow G$ be a measurable mapping. Then, under \mathbb{Q}_s^v , the sequence of random matrices $\{M_k = M(\xi_k), k \geq 0\}$ satisfies, for any $n \geq 1, k \geq 0$, and any measurable functions $f : G^n \rightarrow [0, \infty]$ and $g : G^{k+1} \rightarrow [0, \infty]$,*

$$\begin{aligned} C_s^{-1} \mathbb{E}_{\mathbb{Q}_s^v} f(M_0, \dots, M_{n-1}) \mathbb{E}_{\mathbb{Q}_s^v} g(M_0, \dots, M_k) &\leq \mathbb{E}_{\mathbb{Q}_s^v} f(M_0, \dots, M_{n-1}) g(M_n, \dots, M_{n+k}) \\ &\leq C_s \mathbb{E}_{\mathbb{Q}_s^v} f(M_0, \dots, M_{n-1}) \mathbb{E}_{\mathbb{Q}_s^v} g(M_0, \dots, M_k); \end{aligned}$$

$$\begin{aligned} C_s^{-1} \mathbb{E}_{\mathbb{Q}_s^v} f(M_0, \dots, M_{n-1}) &\leq \mathbb{E}_{\mathbb{Q}_s^v} f(M_k, \dots, M_{k+n-1}) \\ &\leq C_s \mathbb{E}_{\mathbb{Q}_s^v} f(M_0, \dots, M_{n-1}); \end{aligned}$$

$$\begin{aligned} C_s^{-2} \mathbb{E}_{\mathbb{Q}_s^v} f(M_0, \dots, M_{n-1}) &\leq \mathbb{E}_{\mathbb{Q}_s^v} (f(M_0, \dots, M_{n-1}) | M_n, M_{n+1}, \dots) \\ &\leq C_s^2 \mathbb{E}_{\mathbb{Q}_s^v} f(M_0, \dots, M_{n-1}). \end{aligned}$$

Proof of Lemma 4.3. Part 1: we first prove (4.13). For $n = 1$, there is nothing to prove. By reduction, we just need to prove the results for $n = 2$. From (4.11) and the cocycle property of q_n^s , we have for $n \geq 1$ and $k \geq 0$,

$$\begin{aligned} q_{n+k+1}^s(v, M_{0,n+k}) &= q_n^s(v, M_{0,n-1}) q_{k+1}^s(v \cdot M_{0,n-1}, M_{n,n+k}) \\ &\leq C_s q_n^s(v, M_{0,n-1}) q_{k+1}^s(v, M_{n,n+k}). \end{aligned} \quad (4.17)$$

Using this and the definition of \mathbb{Q}_s^v , we see that for $n \geq 1, k \geq 0$,

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}_s^v} f_1(\xi_0, \dots, \xi_{n-1}) f_2(\xi_n, \dots, \xi_{n+k}) \\ &= \mathbb{E} q_{n+k+1}^s(v, M_{0,n+k}) f_1(\xi_0, \dots, \xi_{n-1}) f_2(\xi_n, \dots, \xi_{n+k}) \\ &\leq \mathbb{E} D^{|s|} R_s^2 q_n^s(v, M_{0,n-1}) q_{k+1}^s(v, M_{n,n+k}) f_1(\xi_0, \dots, \xi_{n-1}) f_2(\xi_n, \dots, \xi_{n+k}) \\ &= C_s \mathbb{E}_{\mathbb{Q}_s^v} f_1(\xi_0, \dots, \xi_{n-1}) \mathbb{E}_{\mathbb{Q}_s^v} f_2(\xi_0, \dots, \xi_k). \end{aligned}$$

Using the same argument, we obtain the opposite inequality that for $n \geq 0, k \geq 1$,

$$\mathbb{E}_{\mathbb{Q}_s^v} f_1(\xi_0, \dots, \xi_{n-1}) f_2(\xi_n, \dots, \xi_{n+k}) \geq C_s^{-1} \mathbb{E}_{\mathbb{Q}_s^v} f_1(\xi_0, \dots, \xi_{n-1}) \mathbb{E}_{\mathbb{Q}_s^v} f_2(\xi_0, \dots, \xi_k).$$

Thus (4.13) is proved. The assertion (4.14) is a consequence of (4.13) and (4.15) that we will prove below.

Part 2: we next prove (4.15). By the definition of \mathbb{Q}_s^v , the inequality (4.17), and the stationarity of (ξ_n) under the original probability \mathbb{P} , we obtain, for all measurable functions $f : E^n \rightarrow [0, \infty]$,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_s^v} f(\xi_k, \dots, \xi_{k+n-1}) &= \mathbb{E} f(\xi_k, \dots, \xi_{k+n-1}) q_{k+n}^s(v, M_{0,k+n-1}) \\ &\leq C_s \mathbb{E} f(\xi_k, \dots, \xi_{k+n-1}) q_k^s(v, M_{0,k-1}) q_n^s(v, M_{k,k+n-1}) \\ &= C_s \mathbb{E} f(\xi_k, \dots, \xi_{k+n-1}) q_n^s(v, M_{k,k+n-1}) \\ &= C_s \mathbb{E} f(\xi_0, \dots, \xi_{n-1}) q_n^s(v, M_{0,n-1}) \\ &= C_s \mathbb{E}_{\mathbb{Q}_s^v} f(\xi_0, \dots, \xi_{n-1}). \end{aligned}$$

This gives the upper bound in (4.15). The lower bound can be obtained in a similar way.

Part 3: we finally prove (4.16). Let n and f be fixed as in the lemma. We can assume that $\mathbb{E}_{\mathbb{Q}_s^v} f(\xi_0, \dots, \xi_{n-1}) < \infty$, since otherwise there is nothing to prove. We first prove that for each $k \geq 0$,

$$M_k := \mathbb{E}_{\mathbb{Q}_s^v} (f(\xi_0, \dots, \xi_{n-1}) | \mathcal{G}_{n,n+k}) \leq D^{2|s|} R_s^4 \mathbb{E}_{\mathbb{Q}_s^v} f(\xi_0, \dots, \xi_{n-1}). \quad (4.18)$$

To see this, by the sub-iid property of the sequence $(\xi_n)_{n \geq 0}$ under \mathbb{Q}_s^v proved in parts 1 and 2 above, we see that for each $B \in \mathcal{E}^{k+1}$,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_s^v} f(\xi_0, \dots, \xi_{n-1}) \mathbb{1}_{\{(\xi_n, \dots, \xi_{n+k}) \in B\}} &\leq D^{2|s|} R_s^4 \mathbb{E}_{\mathbb{Q}_s^v} f(\xi_0, \dots, \xi_{n-1}) \mathbb{E}_{\mathbb{Q}_s^v} \mathbb{1}_{\{(\xi_n, \dots, \xi_{n+k}) \in B\}} \\ &= D^{2|s|} R_s^4 \mathbb{E}_{\mathbb{Q}_s^v} \left[\mathbb{E}_{\mathbb{Q}_s^v} f(\xi_0, \dots, \xi_{n-1}) \mathbb{1}_{\{(\xi_n, \dots, \xi_{n+k}) \in B\}} \right]. \end{aligned} \quad (4.19)$$

This gives (4.18). From (4.18), passing to the limit as $k \rightarrow \infty$, and using the fact that Levy's martingale (M_k) satisfies

$$M_k \rightarrow \mathbb{E}_{\mathbb{Q}_s^v} (f(\xi_0, \dots, \xi_{n-1}) | \mathcal{G}_{n,\infty}), \text{ a.s. and in } L^1 \text{ under } \mathbb{Q}_s^v,$$

we get the upper bound in (4.16). The lower bound can be obtained similarly. \square

4.2. Non-degeneracy of the martingale limit W^i under the changed measure. For all $1 \leq i \leq d$, set

$$W_n^i := \frac{\langle Z_n^i, u_n \rangle}{\mathbb{E}_\xi \langle Z_n^i, u_n \rangle} = \frac{\langle Z_n^i, u_n \rangle}{\langle e_i M_{0,n-1}, u_n \rangle}, \quad n \geq 0 \quad (4.20)$$

(by convention, $W_0^i = 1$). By (3.49) we see that the sequence (W_n^i) has an alternative expression:

$$W_n^i = \frac{\langle Z_n^i, u_n \rangle}{\lambda_{0,n-1} u_0(i)}, \quad \text{where} \quad \lambda_{0,n-1} = \prod_{j=0}^{n-1} \lambda_j, \quad 1 \leq i \leq d, \quad n \geq 0,$$

with the convention that the empty product is taken to be 1. The following lemma shows that $(W_n^i)_{n \geq 0}$ is a nonnegative martingale, which we call the fundamental martingale of the branching process (Z_n) .

Lemma 4.5. *Assume condition A3. For all $1 \leq i \leq d$, the sequences $(W_n^i, \mathcal{F}_n)_{n \geq 0}$ and $(W_n^i, \mathcal{F}'_n)_{n \geq 0}$ are nonnegative martingales with mean 1 under the laws \mathbb{P}_ξ and $\mathbb{Q}_s^{e_i}$, for $s \in I_\mu^+ \cup I_\mu^-$, and hence converges $\mathbb{Q}_s^{e_i}$ -a.s. to some random variable $W^i \geq 0$ (whose law under $\mathbb{Q}_s^{e_i}$ depends on s).*

The fact that $(W_n^i, \mathcal{F}_n)_{n \geq 0}$ is a martingale under \mathbb{P}_ξ has been established in [31]. It implies that $(W_n^i, \mathcal{F}'_n)_{n \geq 0}$ is also a martingale under \mathbb{P}_ξ . By the change of measure formula (4.1), it can be easily checked that they remain to be martingales under the new measure $\mathbb{Q}_s^{e_i}$. Recall that the function Λ is defined in (1.19).

Proposition 4.6 (Law of large numbers under the changed measure). [10, Theorem 6.1] *Assume A1, $s \in I_\mu^+ \cup I_\mu^-$ and $\mathbb{E}(\|M_0\|^s \log^+ \|M_0\|) < \infty$. Then, for any $v, y \in \mathcal{S}$, we have \mathbb{Q}_s^v -a.s.,*

$$\lim_{n \rightarrow \infty} \frac{\log \|M_{0,n-1}\|}{n} = \lim_{n \rightarrow \infty} \frac{\log \|v M_{0,n-1}\|}{n} = \lim_{n \rightarrow \infty} \frac{\log \langle v M_{0,n-1}, y \rangle}{n} = \Lambda'(s), \quad (4.21)$$

where the derivative function $\Lambda'(s)$ can be rewritten as

$$\Lambda'(s) = \int_{\mathcal{S}} \int_{\Gamma_\mu} \log \|vg\| q_1^s(v, g) \mu(dg) \pi_s(dv).$$

Notice that if we assume $\Lambda'(s) > 0$, then from Proposition 4.6, under \mathbb{Q}_s^v , the process $\{Z_n\}_{n \geq 0}$ is still supercritical in the sense that $\mathbb{E}_\xi \|Z_n^i\| \rightarrow \infty$ \mathbb{Q}_s^v -a.s. However, the environment sequence is not stationary under \mathbb{Q}_s^v , nor independent, so that the known results for a multi-type branching process in an i.i.d. or stationary environment cannot be applied directly.

We will need the following conditions:

H1. For all $1 \leq i \leq d$,

$$\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\frac{\langle Z_1^i, u_1 \rangle}{\lambda_0 u_0(i)} \log^+ \frac{\langle Z_1^i, u_1 \rangle}{\lambda_0 u_0(i)} \right) < +\infty. \quad (4.22)$$

H2. For all $1 \leq i, j \leq d$,

$$\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\frac{Z_1^i(j)}{M_0(i, j)} \log^+ \frac{Z_1^i(j)}{M_0(i, j)} \right) < +\infty. \quad (4.23)$$

Notice that by the definition of $\mathbb{Q}_s^{e_i}$, condition **H2** holds if and only if

$$\mathbb{E} \left(\frac{Z_1^i(j)}{M_0(i, j)^{1-s}} \log^+ \frac{Z_1^i(j)}{M_0(i, j)} \right) < +\infty. \quad (4.24)$$

Remark 4.7. As in [31, pp. 1046-1048, Proof of Lemma A.1], by the convexity of the function $x \mapsto x \log^+ x$, it can be easily checked that, for any $s \in I_\mu^+ \cup I_\mu^-$, **H2** \Rightarrow **H1**.

Theorem 4.8. Assume **A1**, $s \in I_\mu^+ \cup I_\mu^-$, $\Lambda'(s) > 0$, **H2** and $\mathbb{E}(\|M_0\|^s \log^+ \|M_0\|) < \infty$. Then $\mathbb{E}_\xi W^i = 1$, $\mathbb{Q}_s^{e_i}$ a.s.

Proof. The proof is similar to that in [30, Proof of Theorem 4.4] where the case $s = 0$ was treated. In fact, using [30, Lemma 4.5] and writing for $(n, r) \in \mathbb{N} \times \{1, \dots, d\}$,

$$A(n, r) = \mathbb{E}_\xi \left[\frac{\langle N_{1,n}^r, u_{n+1} \rangle}{\lambda_n \langle u_n, e_r \rangle} \min \left\{ \frac{\langle N_{1,n}^r, u_{n+1} \rangle}{\lambda_{0,n}}, 1 \right\} \right]$$

as in [30, Lemma 4.5], we just need to check that

$$\sum_{n=0}^{\infty} \sum_{r=1}^d A(n, r) < \infty, \quad \mathbb{Q}_s^{e_i}\text{-a.s.}$$

To this end, we use the fact that $\frac{1}{n} \log \lambda_{0,n} \rightarrow \Lambda'(s) > 0$, $\mathbb{Q}_s^{e_i}$ -a.s. (due to the identity $\lambda_{0,n} = \|u_{n+1} M_{0,n}^T\|$ and the law of large numbers for $\|x M_{0,n}\|$, see Proposition 4.6), and the sub-stationarity of the sequence $(\xi_n)_{n \geq 0}$ under $\mathbb{Q}_s^{e_i}$, instead of the stationarity under \mathbb{P} used in [30]. \square

4.3. Convergence in probability of $\frac{Z_n^i}{\|Z_n^i\|} - v_{n-1}$ under the changed measure $\mathbb{Q}_s^{e_i}$. We will need the following lemmas.

The first lemma gives the exponential convergence rate in the Perron-Frobenius theorem under the changed measure $\mathbb{Q}_s^{e_i}$. Set

$$\delta = \frac{D^2 - 1}{D^2 + 1} \in (0, 1), \quad (4.25)$$

where D is the constant in **A1**.

Lemma 4.9. Assume the Furstenberg-Kesten condition **A1**. Then there exists a constant $C > 0$ such that for all $1 \leq i, j \leq d$ and $k \geq 0$, with δ defined in (4.25),

$$\sup_{n \geq 0} \left| \frac{M_{n,n+k}(i, j) \langle u_{n,n+k}, v_{n,n+k} \rangle}{\rho_{n,n+k} u_{n,n+k}(i) v_{n,n+k}(j)} - 1 \right| \leq C \delta^k, \quad \mathbb{Q}_s^{e_i}\text{-a.s.} \quad (4.26)$$

The second lemma establishes the exponential convergence rate for $u_{n,n+k}$ and $v_{n,n+k}$, as $k \rightarrow \infty$, uniformly for $n \geq 0$, also under the changed measure:

Lemma 4.10. *Assume Furstenberg-Kesten condition [A1](#). Then there exists a constant $C > 0$ such that for all $k \geq 0$, with δ defined in [\(4.25\)](#),*

$$\sup_{n \geq 0} \|u_{n,n+k} - u_n\| \leq C\delta^k, \quad \mathbb{Q}_s^{e_i}\text{-a.s.} \quad (4.27)$$

and

$$\sup_{n \geq 0} \|v_{0,n+k} - v_{n,n+k}\| \leq C\delta^k, \quad \mathbb{Q}_s^{e_i}\text{-a.s.} \quad (4.28)$$

Lemmas [4.9](#) and [4.10](#) can be proved by following the arguments in [\[54\]](#) and [\[31, Lemma 8.2\]](#), respectively, using the Perron-Frobenius theorem that we established under the changed measure. Since the arguments are quite similar, the proofs are omitted.

The third lemma shows that conditioned on the explosion event, each component $Z_n^i(r)$ of Z_n^i tends to ∞ in probability under the changed measure $\mathbb{Q}_s^{e_i}$.

Lemma 4.11. *Assume [A1](#), $s \in I_\mu^+ \cup I_\mu^-$, $\Lambda'(s) > 0$, and $\mathbb{E}(\|M_0\|^s \log^+ \|M_0\|) < \infty$. Then for all $1 \leq i, r \leq d$, $Z_n(r) \rightarrow \infty$ in probability under $\mathbb{Q}_s^{e_i}$, conditional on the explosion event $E^i = \{\|Z_n^i\| \rightarrow +\infty\}$: for all $K \geq 0$, as $n \rightarrow \infty$,*

$$\mathbb{Q}_s^{e_i}(Z_n^i(r) \leq K, E^i) \rightarrow 0. \quad (4.29)$$

Proof. Under conditions [A1](#), $s \in I_\mu^+ \cup I_\mu^-$, $\Lambda'(s) > 0$, and $\mathbb{E}(\|M_0\|^s \log^+ \|M_0\|) < \infty$, we use [Proposition 4.6](#) we get that for all $K_1 \geq 0$,

$$\mathbb{Q}_s^{e_i}(M_{0,n-1}(i, j) \leq K_1) \xrightarrow{n \rightarrow \infty} 0.$$

Using this and the same argument as in the proof of [\[30, Proposition 5.1\]](#) under the changed measure $\mathbb{Q}_s^{e_i}$, we obtain [\(4.29\)](#). □

We now establish the result for the convergence in probability of the direction $\frac{Z_n^i}{\|Z_n^i\|}$ under the changed measure.

Theorem 4.12. *Assume conditions [A1](#), $s \in I_\mu^+ \cup I_\mu^-$, $\Lambda'(s) > 0$, and $\mathbb{E}(\|M_0\|^s \log^+ \|M_0\|) < \infty$. Then, for all $1 \leq i \leq d$, $\frac{Z_n^i}{\|Z_n^i\|} - v_{n-1} \rightarrow 0$ in probability under the changed measure $\mathbb{Q}_s^{e_i}$, conditional on the event $\{W^i > 0\}$: for all $\varepsilon > 0$, as $n \rightarrow \infty$,*

$$\mathbb{Q}_s^{e_i}\left(\left\|\frac{Z_n^i}{\|Z_n^i\|} - v_{n-1}\right\| > \varepsilon, W^i > 0\right) \rightarrow 0. \quad (4.30)$$

Proof. For any $1 \leq i \leq d$ and for all $n, k \geq 0$, we denote $\vec{Z}_n^i = \frac{Z_n^i}{\|Z_n^i\|}$. By the definition of the branching process, we have for $k \geq 0$,

$$Z_{n+k} = \sum_{r=1}^d \sum_{l=1}^{Z_n(r)} Z_{l,n,k}^r. \quad (4.31)$$

For any $n, k \geq 1$, we calculate the following term:

$$\begin{aligned}
& \frac{\langle u_{n,n+k-1}, v_{n,n+k-1} \rangle}{\rho_{n,n+k-1}} \left\| \frac{Z_{n+k}^i}{\|Z_n^i\|} - \vec{Z}_n^i M_{n,n+k-1} \right\| \\
&= \frac{\langle u_{n,n+k-1}, v_{n,n+k-1} \rangle}{\rho_{n,n+k-1} \|Z_n^i\|} \left\| \sum_{r=1}^d \sum_{l=1}^{Z_n(r)} Z_{l,n,k}^r - Z_n^i M_{n,n+k-1} \right\| \\
&\leq \frac{\langle u_{n,n+k-1}, v_{n,n+k-1} \rangle}{\rho_{n,n+k-1} \|Z_n^i\|} \sum_{j=1}^d \sum_{r=1}^d \left| \sum_{l=1}^{Z_n(r)} (Z_{l,n,k}^r(j) - M_{n,n+k-1}(r, j)) \right| \\
&= \sum_{j=1}^d \sum_{r=1}^d \frac{M_{n,n+k-1}(r, j) \langle u_{n,n+k-1}, v_{n,n+k-1} \rangle}{\rho_{n,n+k-1}} \frac{1}{\|Z_n^i\|} \left| \sum_{l=1}^{Z_n(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r, j)} - 1 \right) \right|. \quad (4.32)
\end{aligned}$$

Notice that $W_n^i \leq \frac{\|Z_n^i\|}{\langle e_i M_{0,n-1}, u_n \rangle}$ and using Proposition 4.6, we get that $\|Z_n^i\| \rightarrow +\infty$ $\mathbb{Q}_s^{e_i}$ -a.s. on $\{W^i > 0\}$. By Lemma 4.11 and the weak law of large number, we get that for all $\varepsilon > 0$,

$$\mathbb{P}_\xi \left(\frac{1}{\|Z_n^i\|} \left| \sum_{l=1}^{Z_n(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r, j)} - 1 \right) \right| > \varepsilon, W^i > 0 \right) \xrightarrow{n \rightarrow +\infty} 0. \quad (4.33)$$

By the dominated convergence theorem and the fact that $\mathbb{Q}_s^{e_i}(d\xi, dx) = \mathbb{P}_\xi(dx) \tau_s^{e_i}(d\xi)$, we get

$$\mathbb{Q}_s^{e_i} \left(\frac{1}{\|Z_n^i\|} \left| \sum_{l=1}^{Z_n(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r, j)} - 1 \right) \right| > \varepsilon, W^i > 0 \right) \xrightarrow{n \rightarrow +\infty} 0. \quad (4.34)$$

Let $C > 0$ be sufficiently large such that (4.26) holds. By (4.26), for any $1 \leq r, j \leq d$, $n \geq 0$ and $k \geq 1$, $\mathbb{Q}_s^{e_i}$ -a.s.,

$$\begin{aligned}
\frac{M_{n,n+k-1}(r, j) \langle u_{n,n+k-1}, v_{n,n+k-1} \rangle}{\rho_{n,n+k-1}} &\leq (1 + C\delta^k) u_{n,n+k-1}(r) v_{n,n+k-1}(j) \\
&\leq 1 + C\delta^k. \quad (4.35)
\end{aligned}$$

Combining (4.32), (4.34) and (4.35) we obtain that for all $\varepsilon > 0$,

$$\mathbb{Q}_s^{e_i} \left(\frac{\langle u_{n,n+k-1}, v_{n,n+k-1} \rangle}{\rho_{n,n+k-1}} \left\| \frac{Z_{n+k}^i}{\|Z_n^i\|} - \vec{Z}_n^i M_{n,n+k-1} \right\| > \varepsilon, W^i > 0 \right) \xrightarrow{n \rightarrow +\infty} 0. \quad (4.36)$$

Moreover, by Lemma 4.9 we have for any $n, k \geq 1$,

$$\begin{aligned}
& \left\| \frac{\vec{Z}_n^i M_{n,n+k-1} \langle u_{n,n+k-1}, v_{n,n+k-1} \rangle}{\rho_{n,n+k-1}} - \langle \vec{Z}_n^i, u_{n,n+k-1} \rangle v_{n,n+k-1} \right\| \\
&\leq \sum_{r=1}^d \sum_{j=1}^d \left| \frac{\vec{Z}_n^i(r) M_{n,n+k-1}(r, j) \langle u_{n,n+k-1}, v_{n,n+k-1} \rangle}{\rho_{n,n+k-1}} - u_{n,n+k-1}(r) v_{n,n+k-1}(j) \vec{Z}_n^i(r) \right| \\
&\leq \sum_{r=1}^d \sum_{j=1}^d u_{n,n+k-1}(r) v_{n,n+k-1}(j) \vec{Z}_n^i(r) \left| \frac{M_{n,n+k-1}(r, j) \langle u_{n,n+k-1}, v_{n,n+k-1} \rangle}{\rho_{n,n+k-1} u_{n,n+k-1}(r) v_{n,n+k-1}(j)} - 1 \right| \\
&\leq \max_{1 \leq r, j \leq d} \left| \frac{M_{n,n+k-1}(r, j) \langle u_{n,n+k-1}, v_{n,n+k-1} \rangle}{\rho_{n,n+k-1} u_{n,n+k-1}(r) v_{n,n+k-1}(j)} - 1 \right| \leq C\delta^k, \quad \mathbb{Q}_s^{e_i}\text{-a.s.} \quad (4.37)
\end{aligned}$$

Let $k_0 \in \mathbb{N}$ be large enough such that $C\delta^{k_0-1} \leq 1/dD$. Then, combining (4.9) and (4.27), we see that for all $1 \leq r \leq d$, $n \geq 1$ and $k \geq k_0$,

$$u_{n,n+k-1}(r) \geq \frac{1}{2dD}, \quad \mathbb{Q}_s^{e_i}\text{-a.s.} \quad (4.38)$$

It follows that for all $n \geq 1$ and $k \geq k_0$,

$$\langle \vec{Z}_n^i, u_{n,n+k-1} \rangle \geq \frac{1}{2dD}, \quad \mathbb{Q}_s^{e_i}\text{-a.s.} \quad (4.39)$$

Let $\varepsilon > 0$. Let $k_1 \in \mathbb{N}$ be such that $2dDC\delta^{k_1} \leq \varepsilon/8$ and $k_1 \geq k_0$. For all $n \geq 0$ and $k \geq k_1$, set

$$Y_{n,k}^i = \frac{\|Z_{n+k}^i\| \langle u_{n,n+k-1}, v_{n,n+k-1} \rangle}{\rho_{n,n+k-1} \langle \vec{Z}_n^i, u_{n,n+k-1} \rangle \|Z_n^i\|}.$$

Notice that

$$Y_{n,k}^i = \|Y_{n,k}^i \vec{Z}_{n+k}^i\|, \quad (4.40)$$

$$Y_{n,k}^i \vec{Z}_{n+k}^i - v_{n,n+k-1} = \frac{1}{\langle \vec{Z}_n^i, u_{n,n+k-1} \rangle} \left[\frac{Z_{n+k}^i \langle u_{n,n+k-1}, v_{n,n+k-1} \rangle}{\rho_{n,n+k-1} \cdot \|Z_n^i\|} - \langle \vec{Z}_n^i, u_{n,n+k-1} \rangle v_{n,n+k-1} \right]. \quad (4.41)$$

Combining (4.36), (4.37), (4.39) and (4.41), we obtain that for all $k \geq k_1$,

$$\mathbb{Q}_s^{e_i} \left(\|Y_{n,k}^i \vec{Z}_{n+k}^i - v_{n,n+k-1}\| > \varepsilon/4, W^i > 0 \right) \xrightarrow{n \rightarrow +\infty} 0. \quad (4.42)$$

We next have for all $k \geq k_1$,

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} \mathbb{Q}_s^{e_i} \left(|Y_{n,k}^i - 1| > \varepsilon/4, W^i > 0 \right) \\ &= \limsup_{n \rightarrow +\infty} \mathbb{Q}_s^{e_i} \left(\left| \|Y_{n,k}^i \vec{Z}_{n+k}^i\| - \|v_{n,n+k-1}\| \right| > \varepsilon/4, W^i > 0 \right) \\ &\leq \limsup_{n \rightarrow +\infty} \mathbb{Q}_s^{e_i} \left(\|Y_{n,k}^i \vec{Z}_{n+k}^i - v_{n,n+k-1}\| > \varepsilon/4, W^i > 0 \right) = 0. \end{aligned} \quad (4.43)$$

For all $n \geq 0, k \geq k_1$, we have the following comparison:

$$\begin{aligned} \|\vec{Z}_{n+k}^i - v_{n,n+k-1}\| &= \|(Y_{n,k}^i - 1)\vec{Z}_{n+k}^i - Y_{n,k}^i \vec{Z}_{n+k}^i + v_{n,n+k-1}\| \\ &\leq \|(Y_{n,k}^i - 1)\vec{Z}_{n+k}^i\| + \|Y_{n,k}^i \vec{Z}_{n+k}^i - v_{n,n+k-1}\| \\ &\leq |Y_{n,k}^i - 1| + \|Y_{n,k}^i \vec{Z}_{n+k}^i - v_{n,n+k-1}\|. \end{aligned} \quad (4.44)$$

From (4.42), (4.43) and (4.44) we have that for all $k \geq k_1$,

$$\begin{aligned} & \mathbb{Q}_s^{e_i} \left(\|\vec{Z}_{n+k}^i - v_{n,n+k-1}\| > \varepsilon/2, W^i > 0 \right) \\ &\leq \mathbb{Q}_s^{e_i} \left(\|Y_{n,k}^i \vec{Z}_{n+k}^i - v_{n,n+k-1}\| > \varepsilon/4, W^i > 0 \right) + \mathbb{Q}_s^{e_i} \left(|Y_{n,k}^i - 1| > \varepsilon/4, W^i > 0 \right) \\ &\xrightarrow{n \rightarrow +\infty} 0. \end{aligned} \quad (4.45)$$

Notice that for any $k_2 \geq 0$,

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} \mathbb{Q}_s^{e_i} \left(\|\vec{Z}_n^i - v_{0,n-1}\| > \varepsilon, W^i > 0 \right) \\ &\leq \limsup_{n \rightarrow +\infty} \mathbb{Q}_s^{e_i} \left(\|\vec{Z}_{n+k_2}^i - v_{n,n+k_2-1}\| > \varepsilon/2, W^i > 0 \right) \\ &+ \limsup_{n \rightarrow +\infty} \mathbb{Q}_s^{e_i} \left(\|v_{0,n+k_2-1} - v_{n,n+k_2-1}\| > \varepsilon/2, W^i > 0 \right). \end{aligned} \quad (4.46)$$

Let $k_2 \geq k_1$ be such that $C\delta^{k_2-1} \leq \varepsilon/2$. Then by (4.28), the second term in the right hand side is 0. Also by (4.45), we get that,

$$\limsup_{n \rightarrow +\infty} \mathbb{Q}_s^{e_i} \left(\|\bar{Z}_n^i - v_{0,n-1}\| > \varepsilon, W^i > 0 \right) = 0. \quad (4.47)$$

By Theorem 3.3, we conclude the theorem. \square

4.4. Convergence in $L^p(\mathbb{Q}_s^{e_i})$ of the martingale (W_n^i) . In this subsection, we prove the convergence in $L^p(\mathbb{Q}_s^{e_i})$ of the fundamental martingale defined in (4.20).

We first consider the convergence in $L^1(\mathbb{Q}_s^{e_i})$. We will use the same conditions as in Theorem 4.8 for the non-degeneracy of W^i .

Theorem 4.13 (Convergence in $L^1(\mathbb{Q}_s^{e_i})$ of (W_n^i)). *Assume **A1**, $s \in I_\mu^+ \cup I_\mu^-$, $\Lambda'(s) > 0$, **H2** and $\mathbb{E}(\|M_0\|^s \log^+ \|M_0\|) < \infty$. Then for all $i = 1, \dots, d$,*

$$W_n^i \xrightarrow{n \rightarrow +\infty} W^i \quad \text{in } L^1(\mathbb{Q}_s^{e_i}).$$

Proof. Since $W_n^i \rightarrow W^i$ a.s. under $\mathbb{Q}_s^{e_i}$ (by Lemma 4.5), and $\mathbb{E}_{\mathbb{Q}_s^{e_i}} W_n^i = \mathbb{E}_{\mathbb{Q}_s^{e_i}} W^i = 1$ (as a consequence of Lemma 4.5), by Scheffé's theorem we conclude that $W_n^i \rightarrow W^i$ in $L^1(\mathbb{Q}_s^{e_i})$. \square

We next consider the convergence in $L^p(\mathbb{Q}_s^{e_i})$ of (W_n^i) , for $p > 1$. Set

$$I_s^- = \{t \leq 0 : \mathbb{E}_{\mathbb{Q}_s^{e_i}} M_0(i, j)^t < +\infty \quad \forall i, j = 1, \dots, d\}. \quad (4.48)$$

Recall that under condition **A1**, the function κ_s defined by (3.9) satisfies $\kappa_s(t) = \frac{\kappa(s+t)}{\kappa(s)}$, so that $\Lambda_s(t) = \log \kappa_s(t)$ is convex (just as $\Lambda(t) = \log \kappa(t)$), $\Lambda_s(0) = 0$, $\Lambda'_s(t) = \Lambda'(s+t)$, and $\Lambda'_s(0) = \Lambda'(s) > 0$. The convexity of Λ , together with the fact that $\Lambda_s(0) = 0$, implies that if $\Lambda_s(t) < 0$ for some $t < 0$, then $\Lambda_s(t') < 0$ for all $t' \in (t, 0)$. It follows that we have the following implication

$$1 < p' < p \quad \text{and} \quad \kappa_s(1-p) < 1 \Rightarrow \kappa_s(1-p') < 1. \quad (4.49)$$

Theorem 4.14. *Assume condition **A1**. Let $p > 1$ be such that $1-p \in I_s^-$. If*

$$\max_{1 \leq i, j \leq d} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\frac{Z_1^i(j)}{M_0(i, j)} \right)^p < +\infty \quad \text{and} \quad \kappa_s(1-p) < 1, \quad (4.50)$$

then

$$W_n^i \xrightarrow{n \rightarrow +\infty} W^i \quad \text{in } L^p(\mathbb{Q}_s^{e_i}) \quad \forall i = 1, \dots, d. \quad (4.51)$$

Moreover, for $\delta = \kappa_s(1-p)$ when $p \in (1, 2]$, and any $\delta \in (\kappa_s(1-p), 1)$ when $p > 2$, there is a constant $C > 0$ such that for all $n \geq 0$,

$$\mathbb{E}_{\mathbb{Q}_s^{e_i}} |W_n^i - W^i|^p \leq C\delta^n. \quad (4.52)$$

Notice that condition (4.50) reads, in terms of the changed measure,

$$\max_{1 \leq i, j \leq d} \mathbb{E} \left(\frac{(Z_1^i(j))^p}{M_0(i, j)^{p-s}} \right) < +\infty \quad \text{and} \quad \kappa(s+1-p) < \kappa(s). \quad (4.53)$$

The following lemma will be used to investigate the convergence in $L^p(\mathbb{Q}_s^{e_i})$.

Lemma 4.15. *Let $(X_k)_{k \geq 1}$ be a sequence of \mathbb{R}^d -valued independent random variables with $\mathbb{E}(X_k) = 0$ for each $k \geq 1$. Then for all $n \geq 1$ and $p > 1$:*

$$\mathbb{E} \left| \sum_{k=1}^n X_k \right|^p \leq d^{p-1} B_p^p n^{(\frac{p}{2}-1)_+} \sum_{k=1}^n \mathbb{E} |X_k|^p,$$

where $B_p = 2 \min \{k^{1/2} : k \in \mathbb{N}, k \geq \frac{p}{2}\}$, and $(\frac{p}{2} - 1)_+ = \max(\frac{p}{2} - 1, 0)$.

For $d = 1$, the result is a direct consequence of the Marcinkiewicz-Zygmund inequality $\mathbb{E}|\sum_{k=1}^n X_k|^p \leq B_p^p \mathbb{E}(\sum_{k=1}^n X_k^2)^{p/2}$ (see [14, p.356, Theorem 2]) and the elementary inequality $(\sum_{k=1}^n x_k)^a \leq n^{(a-1)+} \sum_{k=1}^n x_k^a$ for $a \geq 1$ and $x_k \geq 0$. The conclusion for the case $d > 1$ can be obtained from that for $d = 1$ and the fact that $\|x\|^p \leq d^{p-1} \sum_{i=1}^d |x(i)|^p$, for each $x = (x(1), \dots, x(d)) \in \mathbb{R}^d$.

Proof of Theorem 4.14. The process $(Z_n)_{n \geq 0}$ satisfies the relation

$$Z_{n+k} = \sum_{j=1}^d \sum_{l=1}^{Z_n(j)} Z_{l,n,k}^j, \quad n \geq 0, k \geq 1, \quad (4.54)$$

where $Z_{l,n,k}^j(r)$ denotes the number of the offspring of type r at time $n+k$ of the l -th particle of type j in the generation n ; conditional on the environment ξ , the random vectors $Z_{l,n,k}^j = (Z_{l,n,k}^j(1), \dots, Z_{l,n,k}^j(d))$ indexed by $l \in \mathbb{N}^*$ and $j \in \{1, \dots, d\}$ (for fixed n and k) are independent. Combining $W_n^i = \frac{\langle Z_n^i, u_n \rangle}{\lambda_{0,n-1} u_0(i)}$ and (4.54) we have, for all $n \geq 0, k \geq 0$ and $1 \leq i \leq d$,

$$\begin{aligned} W_{n+k}^i - W_n^i &= \sum_{r=1}^d \frac{u_{n+k}(r)}{\lambda_{0,n+k-1} u_0(i)} \sum_{j=1}^d \sum_{l=1}^{Z_n^i(j)} Z_{l,n,k}^j(r) - W_n^i \\ &= \sum_{j=1}^d \frac{u_n(j)}{\lambda_{0,n-1} u_0(i)} \sum_{l=1}^{Z_n^i(j)} \sum_{r=1}^d \frac{u_{n+k}(r) Z_{l,n,k}^j(r)}{\lambda_{n,n+k-1} u_n(j)} - W_n^i \\ &= \sum_{j=1}^d \frac{u_n(j)}{\lambda_{0,n-1} u_0(i)} \sum_{l=1}^{Z_n^i(j)} (W_{l,n,k}^j - 1), \end{aligned} \quad (4.55)$$

where

$$W_{l,n,k}^j := \frac{\langle Z_{l,n,k}^j, u_{n+k} \rangle}{\lambda_{n,n+k-1} u_n(j)}.$$

Let T be the shift operator of the environment sequence: for $\xi = (\xi_k)$,

$$(T\xi)_k = \xi_{k+1}, \quad k \in \mathbb{Z}, \quad (4.56)$$

and let T^n be its n -fold iteration. It is clear that, given the environment ξ , the random variables $W_{l,n,k}^j$, $l \geq 1$, are i.i.d.; they are independent of $(\xi_k)_{k \leq n-1}$ and Z_n^i , where $(W_{l,n,k}^j)_{k \geq 0}$ is the fundamental martingale associated to the branching process starting with the l -th j -type particle of generation n , with the shifted environment $T^n \xi$.

We divide the proof into 2 steps.

Step 1. We first consider the case $1 < p \leq 2$. Applying (4.55), the convexity of the function $x \mapsto x^p$ (together with the fact that $\sum_{j=1}^d u_n(j) = 1$), Lemma 4.15 and Theorem

3.3, we have, for all $n \geq 0$, $k \geq 1$ and $1 \leq i \leq d$, $\mathbb{Q}_s^{e_i}$ -a.s.,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[|W_{n+k}^i - W_n^i|^p \mid \xi \right] &= \mathbb{E}_\xi |W_{n+k}^i - W_n^i|^p \\
&\leq \mathbb{E}_\xi \left(\sum_{j=1}^d \frac{u_n(j)}{\lambda_{0,n-1} u_0(i)} \left| \sum_{l=1}^{Z_n^i(j)} (W_{l,n,k}^j - 1) \right| \right)^p \\
&\leq \sum_{j=1}^d \frac{u_n(j)}{(\lambda_{0,n-1} u_0(i))^p} \mathbb{E}_\xi \left(\left| \sum_{l=1}^{Z_n^i(j)} (W_{l,n,k}^j - 1) \right| \right)^p \\
&\leq B_p^p \sum_{j=1}^d \frac{u_n(j)}{(\lambda_{0,n-1} u_0(i))^p} \mathbb{E}_\xi Z_n^i(j) \mathbb{E}_\xi |W_{1,n,k}^j - 1|^p \\
&\leq B_p^p \sigma_{n,k}(p) \sum_{j=1}^d \frac{M_{0,n-1}(i, j) u_n(j)}{(\lambda_{0,n-1} u_0(i))^p} \\
&= B_p^p \sigma_{n,k}(p) (\lambda_{0,n-1} u_0(i))^{1-p}, \tag{4.57}
\end{aligned}$$

where

$$\sigma_{n,k}(p) = \max_{1 \leq j \leq d} \mathbb{E}_\xi |W_{1,n,k}^j - 1|^p. \tag{4.58}$$

Using (4.11) and $\lambda_{0,n-1} = \|u_n M_{0,n-1}^T\|$ (which is a consequence of (3.49)), combining with (4.16) and (2.10), we get that for all $z \in I_s^-$, $n \geq 1$, and $1 \leq i \leq d$, $\mathbb{Q}_s^{e_i}$ -a.s.,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_s^{e_i}} (\lambda_{0,n-1}^z \mid T^n \xi) &= \mathbb{E}_{\mathbb{Q}_s^{e_i}} (\|u_n M_{0,n-1}^T\|^z \mid T^n \xi) \\
&\leq D^{-z} \mathbb{E}_{\mathbb{Q}_s^{e_i}} (\|M_{0,n-1}^T\|^z \mid T^n \xi) \\
&\leq D^{-z} D^{2|s|} R_s^4 \mathbb{E}_{\mathbb{Q}_s^{e_i}} \|M_{0,n-1}^T\|^z \\
&\leq c_z C_s \kappa_s(z)^n. \tag{4.59}
\end{aligned}$$

Moreover, from (4.8) and (4.9), we have

$$\langle u_n M_{0,n-1}^T, e_i \rangle \geq \frac{1}{dD} \|M_{0,n-1}(i, \cdot)\| \geq \frac{1}{dD} \iota(M_{0,n-1}) \geq \frac{1}{dD^2} \|M_{0,n-1}\|.$$

Thus, we get that for all $z \in I_s^-$, $n \geq 1$ and $1 \leq i \leq d$, $\mathbb{Q}_s^{e_i}$ -a.s.

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_s^{e_i}} [(\lambda_{0,n-1} u_0(i))^z \mid T^n \xi] &= \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\langle u_n M_{0,n-1}^T, e_i \rangle^z \mid T^n \xi \right] \\
&\leq (dD^2)^{-z} \mathbb{E}_{\mathbb{Q}_s^{e_i}} [\|M_{0,n-1}\|^z] \leq c_z C_s \kappa_s(z)^n. \tag{4.60}
\end{aligned}$$

Then, from (4.57) and (4.60), we get that for all $n \geq 0$, $k \geq 1$ and $1 \leq i \leq d$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_s^{e_i}} |W_{n+k}^i - W_n^i|^p &= \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(|W_{n+k}^i - W_n^i|^p \mid T^n \xi \right) \right] \\
&= \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} (|W_{n+k}^i - W_n^i|^p \mid \xi) \mid T^n \xi \right) \right] \\
&\leq B_p^p \mathbb{E}_{\mathbb{Q}_s^{e_i}} [\sigma_{n,k}(p) \mathbb{E}_{\mathbb{Q}_s^{e_i}} ((\lambda_{0,n-1} u_0(i))^{1-p} \mid T^n \xi)] \\
&\leq B_p^p c_{1-p} C_s \mathbb{E}_{\mathbb{Q}_s^{e_i}} \sigma_{n,k}(p) \kappa_s(1-p)^n, \tag{4.61}
\end{aligned}$$

By (4.10) we have $0 \leq \frac{M_{n,n+k-1}(i,j)u_{n+k}(j)}{\lambda_{n,n+k-1}u_n(i)} \leq 1$ $\mathbb{Q}_s^{e_i}$ a.s. for all $1 \leq i, j \leq d$. So by the inequality $(\sum_{i=1}^d x_i)^p \leq d^{p-1}x_i^p$ for $x_i \geq 0$, and (4.15), we have that for all $n \geq 0, k \geq 1$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_s^{e_i}} \sigma_{n,k}(p) &\leq \mathbb{E}_{\mathbb{Q}_s^{e_i}} \max_{1 \leq j \leq d} \mathbb{E}_{\xi} (W_{1,n,k}^j)^p + 1 \\
&= \mathbb{E}_{\mathbb{Q}_s^{e_i}} \max_{1 \leq j \leq d} \mathbb{E}_{\xi} \left(\frac{\langle Z_{1,n,k}^j, u_{n+k} \rangle}{\lambda_{n,n+k-1} u_n(j)} \right)^p + 1 \\
&= \mathbb{E}_{\mathbb{Q}_s^{e_i}} \max_{1 \leq j \leq d} \mathbb{E}_{\xi} \left(\sum_{l'=1}^d \frac{M_{n,n+k-1}(j,l)u_{n+k}(l)}{\lambda_{n,n+k-1}u_n(j)} \frac{Z_{1,n,k}^j(l')}{M_{n,n+k-1}(j,l')} \right)^p + 1 \\
&\leq \mathbb{E}_{\mathbb{Q}_s^{e_i}} \max_{1 \leq j \leq d} d^{p-1} \sum_{l'=1}^d \mathbb{E}_{\xi} \left(\frac{Z_{1,n,k}^j(l')}{M_{n,n+k-1}(j,l')} \right)^p + 1 \\
&\leq \sum_{j=1}^d \sum_{l'=1}^d d^{p-1} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\frac{Z_{1,n,k}^j(l')}{M_{n,n+k-1}(j,l')} \right)^p + 1 \\
&\leq \sum_{j=1}^d \sum_{l'=1}^d d^{p-1} D^{|s|} R_s^2 \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\frac{Z_{1,0,k}^j(l')}{M_{0,k-1}(j,l')} \right)^p + 1. \tag{4.62}
\end{aligned}$$

Using this for $k = 1$ and the condition (4.50), we get that

$$\sup_{n \geq 0} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \sigma_{n,1}(p) < +\infty. \tag{4.63}$$

Therefore by (4.61), writing $C = B_p [c_{1-p} C_s \sup_{n \geq 0} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \sigma_{n,1}(p)]^{1/p}$, we have, for all $n \geq 0, k \geq 1$ and $1 \leq i \leq d$,

$$\mathbb{E}_{\mathbb{Q}_s^{e_i}} |W_{n+1}^i - W_n^i|^p \leq C \kappa_s (1-p)^n. \tag{4.64}$$

So by the triangular inequality of L^p , it follows that for all $1 \leq i \leq d$,

$$\begin{aligned}
\sup_{n \geq 0} (\mathbb{E}_{\mathbb{Q}_s^{e_i}} (W_n^i)^p)^{1/p} &\leq 1 + \sum_{n=0}^{+\infty} \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} |W_{n+1}^i - W_n^i|^p \right)^{1/p} \\
&\leq 1 + C \sum_{n=0}^{+\infty} \kappa_s (1-p)^{n/p} < +\infty. \tag{4.65}
\end{aligned}$$

Therefore for all $1 \leq i \leq d$, the martingale (W_n^i) is bounded in $L^p(\mathbb{Q}_s^{e_i})$, so that it converges in $L^p(\mathbb{Q}_s^{e_i})$. Notice that (4.52) is a consequence of (4.64). So the proof of the theorem is finished for $1 < p \leq 2$.

Step 2. We then consider the case $p > 2$. In the following $C > 0$ will be a constant which may depend on s and which may differ from line to line. Applying (4.55), the inequality $(\sum_{j=1}^d x_j)^p \leq d^{p-1} \sum_{j=1}^d x_j^p$ for $x_j \geq 0$, and Lemma 4.15, for all $n \geq 0, k \geq 1$ and $1 \leq i \leq d$,

$\mathbb{Q}_s^{e_i}$ -a.s. we have

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}_s^{e_i}}(|W_{n+k}^i - W_n^i|^p \mid \xi) \\
&= \mathbb{E}_\xi |W_{n+k}^i - W_n^i|^p \\
&\leq d^{p-1} B_p^p \sum_{j=1}^d \left(\frac{u_n(j)}{\lambda_{0,n-1} u_0(i)} \right)^p \mathbb{E}_\xi (Z_n^i(j))^{p/2} \mathbb{E}_\xi |W_{1,n,k}^j - 1|^p \\
&\leq C \sigma_{n,k}(p) \sum_{j=1}^d (u_n(j))^{p/2} \mathbb{E}_\xi \left(\frac{u_n(j) Z_n^i(j)}{\lambda_{0,n-1} u_0(i)} \right)^{p/2} (\lambda_{0,n-1} u_0(i))^{-p/2} \\
&\leq C \sigma_{n,k}(p) \left(\sum_{j=1}^d u_n(j) \right) \mathbb{E}_\xi (W_n^i)^{p/2} (\lambda_{0,n-1} u_0(i))^{-p/2} \\
&= C \sigma_{n,k}(p) \mathbb{E}_\xi (W_n^i)^{p/2} (\lambda_{0,n-1} u_0(i))^{-p/2}, \tag{4.66}
\end{aligned}$$

with $\sigma_{n,k}(p)$ defined in (4.58). Let $j_p \in \mathbb{N}$ be the unique integer such that $1 < \frac{p}{2^{j_p}} \leq 2$. For all $n \geq 0$, $1 \leq i \leq d$ and $1 \leq j \leq j_p$, define

$$a_{n,j}^i(p) := (\lambda_{0,n-1} u_0(i))^{p/2^j - p} \mathbb{E}_\xi (W_n^i)^{p/2^j}. \tag{4.67}$$

From (4.66), we obtain that for all $n \geq 0$, $k \geq 1$ and $1 \leq i \leq d$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_s^{e_i}} |W_{n+k}^i - W_n^i|^p &= \mathbb{E}_{\mathbb{Q}_s^v} \left[\mathbb{E}_{\mathbb{Q}_s^{e_i}} (|W_{n+k}^i - W_n^i|^p \mid T^n \xi) \right] \\
&\leq C \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\sigma_{n,k}(p) \mathbb{E}_{\mathbb{Q}_s^{e_i}} (a_{n,1}^i(p) \mid T^n \xi) \right]. \tag{4.68}
\end{aligned}$$

To prove (4.51), it is enough to show that there exists a constant $C_1 > 0$ (which may depend on s) such that for all $n \geq 0$, $1 \leq i \leq d$, $1 \leq j \leq j_p$ and $\delta > \kappa_s(1-p)^{1/p}$,

$$\delta^{-n} (\mathbb{E}_{\mathbb{Q}_s^{e_i}} (a_{n,j}^i(p) \mid T^n \xi))^{1/p} \leq C_1 \quad \mathbb{Q}_s^{e_i}\text{-a.s.} \tag{4.69}$$

In fact, combining (4.68) with $k=1$ and (4.69) for $j=1$, we see that for all $\delta > \kappa_s(1-p)^{1/p}$, all $1 \leq i \leq d$, and $n \geq 0$,

$$\mathbb{E}_{\mathbb{Q}_s^{e_i}} |W_{n+1}^i - W_n^i|^p \leq C_2 \delta^{np}, \tag{4.70}$$

with $C_2 = CC_1 \sup_{n \geq 0} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \sigma_{n,1}(p) < +\infty$ by (4.63) which is still valid for $p > 2$ (since (4.62) remains valid for $p > 2$). By the triangular inequality in L^p , it follows that

$$\sup_{n \geq 0} (\mathbb{E}_{\mathbb{Q}_s^{e_i}} (W_n^i)^p)^{1/p} \leq 1 + C_2 \sum_{n=0}^{+\infty} \delta^n. \tag{4.71}$$

Taking $\delta \in (\kappa_s(1-p)^{1/p}, 1)$, we deduce that the martingale (W_n^i) is bounded in $L^p(\mathbb{Q}_s^{e_i})$, for all $1 \leq i \leq d$. Hence, the sequence (W_n^i) converges in $L^p(\mathbb{Q}_s^{e_i})$. Since (4.70) implies (4.52), the proof of the theorem is finished for $p > 2$, provided that (4.69) holds.

It remains to prove (4.69) under (4.50). We will prove (4.69) by induction on j . First consider the case $j = j_p$. By definition of j_p we have $1 < p/2^{j_p} \leq 2$. So, by the triangular inequality in $L^{p/2^{j_p}}$ under $\mathbb{Q}_s^{e_i}$ conditional on ξ and (4.57), it follows that for all $1 \leq i \leq d$

and $n \geq 0$, $\mathbb{Q}_s^{e_i}$ -a.s.,

$$\begin{aligned}
[a_{n,j_p}^i(p)]^{2^{j_p}/p} &\leq (\lambda_{0,n-1}u_0(i))^{1-2^{j_p}} \\
&\quad + (\lambda_{0,n-1}u_0(i))^{1-2^{j_p}} \sum_{l=0}^{n-1} (\mathbb{E}_\xi |W_{l+1}^i - W_l^i|^{p/2^{j_p}})^{2^{j_p}/p} \\
&\leq (\lambda_{0,n-1}u_0(i))^{1-2^{j_p}} \\
&\quad + C \sum_{l=0}^{n-1} \left[\sigma_{l,1} \left(\frac{p}{2^{j_p}} \right) \right]^{2^{j_p}/p} (\lambda_{0,l-1}u_0(i))^{2^{j_p}/p-2^{j_p}} \lambda_{l,n-1}^{1-2^{j_p}}. \tag{4.72}
\end{aligned}$$

Taking the $L^{p/2^{j_p}}$ norm under $\mathbb{Q}_s^{e_i}$ conditional on $T^n\xi$, and using the triangular inequality in $L^{p/2^{j_p}}$ and the inequality (4.60), we obtain that for all $1 \leq i \leq d$ and $n \geq 0$, $\mathbb{Q}_s^{e_i}$ -a.s.,

$$\begin{aligned}
&(\mathbb{E}_{\mathbb{Q}_s^{e_i}}(a_{n,j_p}^i(p) \mid T^n\xi))^{2^{j_p}/p} \\
&\leq (\mathbb{E}_{\mathbb{Q}_s^{e_i}}((\lambda_{0,n-1}u_0(i))^{p/2^{j_p}-p} \mid T^n\xi))^{2^{j_p}/p} \\
&\quad + C \sum_{l=0}^{n-1} \left\{ \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[(\lambda_{0,l-1}u_0(i))^{1-p} \mid T^l\xi \right] \sigma_{l,1} \left(\frac{p}{2^{j_p}} \right) \lambda_{l,n-1}^{p/2^{j_p}-p} \right) \mid T^n\xi \right] \right\}^{2^{j_p}/p} \\
&\leq C \kappa_s \left(\frac{p}{2^{j_p}} - p \right)^{\frac{n2^{j_p}}{p}} + C \sum_{l=0}^{n-1} \left\{ \kappa_s (1-p)^l \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\left(\sigma_{l,1} \left(\frac{p}{2^{j_p}} \right) \lambda_{l,n-1}^{p/2^{j_p}-p} \right) \mid T^n\xi \right] \right\}^{2^{j_p}/p}. \tag{4.73}
\end{aligned}$$

Notice that if $1 \leq j \leq j_p$, then we have $1-p < \frac{p}{2^j} - p < -\frac{p}{2}$. Since κ_s is log-convexe on I_s^- , we obtain that

$$\max_{1 \leq j \leq j_p} \left\{ \kappa_s \left(\frac{p}{2^j} - p \right) \right\} \leq \max \left\{ \kappa_s(1-p), \kappa_s \left(-\frac{p}{2} \right) \right\} =: \delta_c(p)^p < 1, \tag{4.74}$$

where the last inequality holds by (4.49).

We now deal with the second term in (4.73), by calculating first the conditional expectation under $\mathbb{Q}_s^{e_i}$ given $T^{l+1}\xi$. By the triangular inequalities of $L^{p/2^j}$ under $\mathbb{Q}_s^{e_i}$ conditional on ξ , and inequalities (4.59), it holds that for all $l \geq 0$ and $1 \leq j \leq j_p$, $\mathbb{Q}_s^{e_i}$ -a.s.,

$$\begin{aligned}
&\left\{ \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\sigma_{l,1} \left(\frac{p}{2^j} \right) \lambda_l^{p/2^j-p} \mid T^{l+1}\xi \right) \right\}^{2^j/p} \\
&= \left\{ \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\max_{1 \leq r \leq d} \mathbb{E}_\xi |W_{1,l,1}^r - 1|^{p/2^j} \lambda_l^{p/2^j-p} \mid T^{l+1}\xi \right) \right\}^{2^j/p} \\
&\leq \left\{ \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\max_{1 \leq r \leq d} \mathbb{E}_\xi (W_{1,l,1}^r)^{p/2^j} \lambda_l^{p/2^j-p} \mid T^{l+1}\xi \right) \right\}^{2^j/p} + (\mathbb{E}_{\mathbb{Q}_s^{e_i}}(\lambda_l^{p/2^j-p} \mid T^{l+1}\xi))^{2^j/p} \\
&\leq \left\{ \sum_{r=1}^d \mathbb{E}_{\mathbb{Q}_s^{e_i}} [\mathbb{E}_\xi (W_{1,l,1}^r)^{p/2^j} \lambda_l^{p/2^j-p} \mid T^{l+1}\xi] \right\}^{2^j/p} + C \kappa_s \left(\frac{p}{2^j} - p \right)^{2^j/p}.
\end{aligned}$$

From inequality (4.74), we have $\kappa_s \left(\frac{p}{2^j} - p \right) < 1$. So from the above inequality, and the triangular inequality of $L^{p/2^j}$ under $\mathbb{Q}_s^{e_i}$ conditional on $T^{l+1}\xi$, we see that for all $l \geq 0$ and

$1 \leq j \leq j_p$, $\mathbb{Q}_s^{e_i}$ -a.s.,

$$\begin{aligned}
& \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\sigma_{l,1} \left(\frac{p}{2^j} \right) \lambda_l^{p/2^j - p} \mid T^{l+1}\xi \right) \right)^{2^j/p} \\
& \leq d^{2^j/p} \max_{1 \leq r \leq d} \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\lambda_l^{-p} \mathbb{E}_\xi (\lambda_l W_{1,l,1}^r)^{p/2^j} \mid T^{l+1}\xi \right] \right)^{2^j/p} + C \\
& \leq C \max_{1 \leq r \leq d} \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\lambda_l^{-p} \mathbb{E}_\xi (\lambda_l W_{1,l,1}^r \mathbb{1}_{\{\lambda_l W_{1,l,1}^r \leq 1\}})^{p/2^j} \mid T^{l+1}\xi \right] \right)^{2^j/p} \\
& + C \max_{1 \leq r \leq d} \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\lambda_l^{-p} \mathbb{E}_\xi (\lambda_l W_{1,l,1}^r \mathbb{1}_{\{\lambda_l W_{1,l,1}^r > 1\}})^{p/2^j} \mid T^{l+1}\xi \right] \right)^{2^j/p} + C \\
& \leq C \max_{1 \leq r \leq d} \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\lambda_l^{1-p} \mathbb{E}_\xi W_{1,l,1}^r \mid T^{l+1}\xi \right] \right)^{2^j/p} + C \max_{1 \leq r \leq d} \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left((W_{1,l,1}^r)^p \mid T^{l+1}\xi \right) \right)^{2^j/p} + C,
\end{aligned}$$

where the last step holds since $\mathbb{1}_{\{\lambda_l W_{1,l,1}^r \leq 1\}} \leq (\lambda_l W_{1,l,1}^r)^{-a}$ for $a = 1 - \frac{2^j}{p} > 0$, and $\mathbb{1}_{\{\lambda_l W_{1,l,1}^r > 1\}} \leq (\lambda_l W_{1,l,1}^r)^b$ for $b = 2^j - 1 > 0$.

Notice that $(W_{1,l,k}^r)_{k \geq 0}$ is the fundamental martingale associated with the branching process starting with the 1-st r -type particle in generation l , with the shifted environment $T^l \xi$. In particular we have $\mathbb{E}_\xi W_{1,l,1}^r = 1$ a.s. Therefore, applying again (4.59), (4.74), it follows that for all $l \geq 0$ and $1 \leq j \leq j_p$, $\mathbb{Q}_s^{e_i}$ -a.s.,

$$\begin{aligned}
& \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\sigma_{l,1} \left(\frac{p}{2^j} \right) \lambda_l^{p/2^j - p} \mid T^{l+1}\xi \right) \right)^{2^j/p} \\
& \leq C \max_{1 \leq r \leq d} \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} (\lambda_l^{1-p} \mid T^{l+1}\xi) \right)^{2^j/p} + C \max_{1 \leq r \leq d} \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left((W_{1,l,1}^r)^p \mid T^{l+1}\xi \right) \right)^{2^j/p} + C \\
& \leq C \kappa_s (1-p)^{2^j/p} + C \max_{1 \leq r \leq d} \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left((W_{1,l,1}^r)^p \mid T^{l+1}\xi \right) \right)^{2^j/p} + C \\
& \leq C + C \max_{1 \leq r \leq d} \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left((W_{1,l,1}^r)^p \mid T^{l+1}\xi \right) \right)^{2^j/p}. \tag{4.75}
\end{aligned}$$

Then, since $0 \leq \frac{M_l(r,j)u_{l+1}(j)}{\lambda_l u_l(r)} \leq 1$ $\mathbb{Q}_s^{e_i}$ -a.s., by the triangular inequality of L^p , together with (4.15) and (4.16), we have for all $1 \leq r \leq d$ and $l \geq 0$, $\mathbb{Q}_s^{e_i}$ -a.s.,

$$\begin{aligned}
\left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left((W_{1,l,1}^r)^p \mid T^{l+1}\xi \right) \right)^{1/p} & = \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\left(\frac{\langle Z_{1,l,1}^r(j), u_{l+1} \rangle}{\lambda_l u_l(r)} \right)^p \mid T^{l+1}\xi \right) \right)^{1/p} \\
& = \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\left(\sum_{j=1}^d \frac{M_l(r,j)u_{l+1}(j)}{\lambda_l u_l(r)} \frac{Z_{1,l,1}^r(j)}{M_l(r,j)} \right)^p \mid T^{l+1}\xi \right) \right)^{1/p} \\
& \leq \sum_{j=1}^d \left[\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\left(\frac{Z_{1,l,1}^r(j)}{M_l(r,j)} \right)^p \mid T^{l+1}\xi \right) \right]^{1/p} \\
& \leq \sum_{j=1}^d D^{3|s|/p} R_s^{6/p} \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\left(\frac{Z_1^r(j)}{M_0(r,j)} \right)^p \right) \right)^{1/p} < +\infty. \tag{4.76}
\end{aligned}$$

Putting together (4.75) and (4.76), we get that for all $l \geq 0$ and $1 \leq j \leq j_p$, $\mathbb{Q}_s^{e_i}$ -a.s.,

$$\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\sigma_{l,1} \left(\frac{p}{2^j} \right) \lambda_l^{p/2^j - p} \mid T^{l+1}\xi \right) \leq C. \tag{4.77}$$

Therefore, for all $n \geq 0$ and $0 \leq l \leq n-1$, $\mathbb{Q}_s^{e_i}$ -a.s. (the value of the constant C may change from line to line),

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\sigma_{l,1} \left(\frac{p}{2^{j_p}} \right) \lambda_{l,n-1}^{p/2^{j_p}-p} \mid T^n \xi \right] &= \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\sigma_{l,1} \left(\frac{p}{2^{j_p}} \right) \lambda_l^{p/2^{j_p}-p} \mid T^{l+1} \xi \right) \right) \lambda_{l+1,n-1}^{p/2^{j_p}-p} \mid T^n \xi \right] \\ &\leq C \mathbb{E}_{\mathbb{Q}_s^{e_i}} (\lambda_{l+1,n-1}^{p/2^{j_p}-p} \mid T^n \xi) \\ &\leq C \left[\kappa_s \left(\frac{p}{2^{j_p}} - p \right) \right]^{n-1-l} \\ &\leq C \delta_c(p)^{(n-1-l)p}, \end{aligned}$$

where the last two inequalities hold by (4.59) and (4.74). Combining this with (4.73) and (4.74), we obtain that for all $1 \leq i \leq d$ and $n \geq 0$, $\delta > \delta_c(p)$, $\mathbb{Q}_s^{e_i}$ -a.s.,

$$\begin{aligned} (\mathbb{E}_{\mathbb{Q}_s^{e_i}} (a_{n,j_p}^i(p) \mid T^n \xi))^{2^{j_p}/p} &\leq C \delta_c(p)^{n2^{j_p}} + C \sum_{l=0}^{n-1} (\delta_c(p)^{lp} \delta_c(p)^{(n-1-l)p})^{2^{j_p}/p} \\ &= C \delta_c(p)^{n2^{j_p}} + C n \delta_c(p)^{(n-1)2^{j_p}} \leq C \delta^{n2^{j_p}}. \end{aligned}$$

So (4.69) holds for $j = j_p$.

Now suppose that (4.69) holds for $j+1 \leq j_p$ with $j \geq 1$. We will prove that it still holds for j . By recurrence this will prove that (4.69) holds for all $j = 1, \dots, j_p$. Since $j+1$ satisfies (4.69), for all $n \geq 0$, $1 \leq i \leq d$, and $\delta > \delta_c(p)$,

$$\delta^{-n} (\mathbb{E}_{\mathbb{Q}_s^{e_i}} (a_{n,j+1}^i(p) \mid T^n \xi))^{1/p} \leq C \quad \mathbb{Q}_s^{e_i}\text{-a.s.} \quad (4.78)$$

By the definition of j_p we have $p/2^j > 2$. As in the proof of (4.73), but applying (4.66) instead of (4.57), we obtain that for all $1 \leq i \leq d$ and $n \geq 0$, $\mathbb{Q}_s^{e_i}$ -a.s.,

$$\begin{aligned} &(\mathbb{E}_{\mathbb{Q}_s^{e_i}} (a_{n,j}^i(p) \mid T^n \xi))^{2^j/p} \\ &\leq C \kappa_s \left(\frac{p}{2^j} - p \right)^{n2^j/p} + C \times \\ &\quad \sum_{l=0}^{n-1} \left\{ \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\sigma_{l,1} \left(\frac{p}{2^j} \right) \mathbb{E}_\xi (W_l^i)^{\frac{p}{2^{j+1}}} (\lambda_{0,l-1} u_0(i))^{\frac{p}{2^{j+1}}-p} \lambda_{l,n-1}^{p/2^j-p} \mid T^n \xi \right] \right\}^{2^j/p} \\ &= C \kappa_s \left(\frac{p}{2^j} - p \right)^{n2^j/p} + C \sum_{l=0}^{n-1} \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\sigma_{l,1} \left(\frac{p}{2^j} \right) \lambda_{l,n-1}^{p/2^j-p} \mathbb{E}_{\mathbb{Q}_s^{e_i}} (a_{l,j+1}^i(p) \mid T^l \xi) \mid T^n \xi \right] \right)^{2^j/p}. \end{aligned} \quad (4.79)$$

Combining this with the recurrence hypothesis (4.78), together with (4.74), (4.77) and (4.59), we obtain that for all $n \geq 0$, $1 \leq i \leq d$, $\delta > \delta_c(p)$,

$$\begin{aligned} (\mathbb{E}_{\mathbb{Q}_s^{e_i}} (a_{n,j}^i(p) \mid T^n \xi))^{2^j/p} &\leq C \delta_c(p)^{n2^j} + C \times \\ &\quad \sum_{l=0}^{n-1} \delta^{l2^j} \left\{ \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\sigma_{l,1} \left(\frac{p}{2^j} \right) \lambda_l^{p/2^j-p} \mid T^{l+1} \xi \right] \lambda_{l+1,n-1}^{p/2^j-p} \mid T^n \xi \right] \right\}^{2^j/p} \\ &\leq C \delta_c(p)^{n2^j} + C \sum_{l=0}^{n-1} \delta^{l2^j} \kappa_s \left(\frac{p}{2^j} - p \right)^{(n-1-l)2^j/p} \\ &\leq C \delta^{n2^j}. \end{aligned} \quad (4.80)$$

So (4.69) also holds for j . Therefore, by recurrence, we have proved that (4.69) holds for all $j = 1, \dots, j_p$.

This ends the proof of theorem.

□

4.5. Convergence in $L^p(\mathbb{Q}_s^{e_i})$ of the normalized j -type population size $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$. In this section, we will establish the theorems about the convergence in $L^p(\mathbb{Q}_s^{e_i})$ of $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$, the normalized population size of type j .

We first consider the convergence in $L^1(\mathbb{Q}_s^{e_i})$ of the normalized j -type population size $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$.

Theorem 4.16 (Convergence in L^1 under $\mathbb{Q}_s^{e_i}$ of $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$). *Assume **A1**, $s \in I_\mu^+ \cup I_\mu^-$, $\Lambda'(s) > 0$, **H2** and $\mathbb{E}(\|M_0\|^s \log^+ \|M_0\|) < \infty$. Then for all $i, j = 1, \dots, d$,*

$$\bar{Z}_n^i(j) := \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \xrightarrow{n \rightarrow +\infty} W^i \quad \text{in } L^1(\mathbb{Q}_s^{e_i}).$$

Proof. We first prove that

$$\bar{Z}_n^i(j) \xrightarrow[n \rightarrow +\infty]{\mathbb{Q}_s^{e_i}} W^i. \quad (4.81)$$

Indeed, by (4.10) in Lemma 4.1, for all $1 \leq i, j \leq d$, $n \geq 1$, $\mathbb{Q}_s^{e_i}$ -a.s., we have

$$W_n^i = \sum_{j=1}^d \frac{M_{0,n-1}(i,j)u_n(j)}{\lambda_{0,n-1}u_0(i)} \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \geq \frac{1}{dD^2} \bar{Z}_n^i(j).$$

Therefore, $\bar{Z}_n^i(j) \rightarrow 0$ $\mathbb{Q}_s^{e_i}$ -a.s. on the event $\{W^i = 0\}$. We next consider on the event $\{W^i > 0\}$. By using part 2 of Theorem 3.3, we have the following estimation for all $1 \leq i, j \leq d$, as $n \rightarrow +\infty$, $\mathbb{Q}_s^{e_i}$ -a.s.,

$$\begin{aligned} \frac{1}{W_n^i} \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} &= \frac{\lambda_{0,n-1}u_0(i)}{M_{0,n-1}(i,j)} \frac{Z_n^i(j)}{\langle Z_n^i, u_n \rangle} \\ &\sim \frac{\langle u_n, v_{n-1} \rangle}{v_{n-1}(j)} \frac{Z_n^i(j)}{\langle Z_n^i, u_n \rangle} \\ &\sim \frac{1}{v_{n-1}(j)} \frac{Z_n^i(j)}{\|Z_n^i\|} \sum_{r=1}^d \frac{Z_n^i(r)u_n(r)}{\langle Z_n^i, u_n \rangle} v_{n-1}(r) \frac{\|Z_n^i\|}{Z_n^i(r)}. \end{aligned} \quad (4.82)$$

By Theorem 4.12, we get that for all $1 \leq i, j \leq d$,

$$\frac{1}{W_n^i} \bar{Z}_n^i(j) \xrightarrow[n \rightarrow +\infty]{\mathbb{Q}_s^{e_i}} 1.$$

This implies that

$$\bar{Z}_n^i(j) - W_n^i = \frac{\bar{Z}_n^i(j) - W_n^i}{W_n^i} W_n^i = \left(\frac{\bar{Z}_n^i(j)}{W_n^i} - 1 \right) W_n^i \xrightarrow[n \rightarrow +\infty]{\mathbb{Q}_s^{e_i}} 0. \quad (4.83)$$

Hence, we get (4.81). Combining this and the fact that $\mathbb{E}_{\mathbb{Q}_s^{e_i}} \bar{Z}_n^i(j) = \mathbb{E}_{\mathbb{Q}_s^{e_i}} W^i = 1$, by Scheffé's theorem, we see that $\bar{Z}_n^i(j) \rightarrow W^i$ in $L^1(\mathbb{Q}_s^{e_i})$. □

We next present the following theorem concerning the convergence in $L^p(\mathbb{Q}_s^{e_i})$ ($p > 1$) of the normalized j -type population size $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$.

Theorem 4.17. *Assume **A1**. Let $p > 1$ be such that $1 - p \in I_s^-$. If (4.50) holds, then*

$$\bar{Z}_n^i(j) \xrightarrow{n \rightarrow +\infty} W^i \quad \text{in } L^p(\mathbb{Q}_s^{e_i}) \quad \text{for any } 1 \leq i, j \leq d.$$

Moreover, there exist $\bar{\delta} \in (0, 1)$ and a constant $C > 0$ such that for all $n \geq 0$, $1 \leq i, j \leq d$,

$$\mathbb{E}_{\mathbb{Q}_s^{e_i}} |\bar{Z}_n^i(j) - W^i|^p \leq C \bar{\delta}^n. \quad (4.84)$$

We begin by establishing some preliminary results concerning the products of random matrices $M_{n,n+k}$ under the new measure $\mathbb{Q}_s^{e_i}$.

For $1 \leq i \leq d$, let $(\Pi_n^i)_{n \geq 0}$ be the sequence of random matrices in G such that for all $1 \leq j, r \leq d$,

$$\Pi_0^i(j, r) := \delta_{i,r}, \quad \Pi_n^i(j, r) := \frac{M_{0,n-1}(i, r)M_n(r, j)}{M_{0,n}(i, j)}, \quad n \geq 1.$$

By definition all the entries of the i -th column of Π_0^i are equal to 1, the others are 0; each Π_n^i is a stochastic matrix. For $n, k \geq 0$ let

$$\Pi_{n+k,n}^i := \Pi_{n+k}^i \cdots \Pi_n^i$$

be the products of the matrices Π_n^i . Clearly each $\Pi_{n+k,n}^i$ is a non-negative stochastic random matrix.

The following lemma concerns the convergence of the products $\Pi_{n+k,n}^i$ of random matrices and their exponential rate as $k \rightarrow +\infty$ under the change of measure $\mathbb{Q}_s^{e_i}$, which will be useful for proving of the L^p convergence of the normalized population size $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$.

Lemma 4.18. *Assume condition A1. Then for all $n \geq 0$ and $1 \leq i \leq d$, as $k \rightarrow +\infty$, the sequence $(\Pi_{n+k,n}^i)_{k \geq 0}$ converges $\mathbb{Q}_s^{e_i}$ -a.s. to some random matrix $\Pi_{\infty,n}^i$ such that:*

(1) for all $1 \leq j, r \leq d$,

$$\Pi_{\infty,0}^i(j, r) := \Pi_0^i(j, r), \quad \Pi_{\infty,n}^i(j, r) := \frac{M_{0,n-1}(i, r)u_n(r)}{\lambda_{0,n-1}u_0(i)}, \quad n \geq 1;$$

(2) there exist constants $C > 0$ and $\delta \in (0, 1)$ such that for all $k \geq 0$ and $1 \leq i \leq d$,

$$\sup_{n \geq 0} \|\Pi_{n+k,n}^i - \Pi_{\infty,n}^i\| \leq C \delta^k \quad \mathbb{Q}_s^{e_i}\text{-a.s.} \quad (4.85)$$

Proof. (1) It is easy to see that for all $n \geq 1$, $k \geq 0$ and $1 \leq i, j, r \leq d$,

$$\Pi_{n+k,n}^i(j, r) = \frac{M_{0,n-1}(i, r)M_{n,n+k}(r, j)}{M_{0,n+k}(i, j)}. \quad (4.86)$$

Combining (4.86) and Theorem 3.3, we deduce that for all $n \geq 1$ and $1 \leq i, j, r \leq d$, $\mathbb{Q}_s^{e_i}$ -a.s. as $k \rightarrow +\infty$,

$$\Pi_{n+k,n}^i(j, r) \sim \frac{M_{0,n-1}(i, r)}{a_{n,n+k}u_n(r)v_{n+k}(j)} a_{0,n+k}u_0(i)v_{n+k}(j), \quad (4.87)$$

where $a_{n,n+k} = \frac{\prod_{j=n}^{n+k} \lambda_j}{\langle u_{n+k+1}, v_{n+k} \rangle}$ and $a_{0,n+k} = \frac{\prod_{j=0}^{n+k} \lambda_j}{\langle u_{n+k+1}, v_{n+k} \rangle}$. Therefore, we get that for all $n \geq 1$ and $1 \leq i, j, r \leq d$, as $k \rightarrow +\infty$,

$$\Pi_{n+k,n}^i(j, r) \rightarrow \Pi_{\infty,n}^i(j, r) = \frac{M_{0,n-1}(i, r)u_n(r)}{\lambda_{0,n-1}u_0(i)} \quad \mathbb{Q}_s^{e_i}\text{-a.s.} \quad (4.88)$$

Hence, as $k \rightarrow \infty$, $\mathbb{Q}_s^{e_i}$ -a.s., $\Pi_{k,0}^i = \Pi_{k,1}^i \Pi_0^i = \Pi_{\infty,0}^i$, where

$$\Pi_{\infty,0}^i(j, r) = \sum_{l=1}^d \Pi_{\infty,1}^i(j, l) \Pi_0^i(l, r) = \sum_{l=1}^d \Pi_{\infty,1}^i(j, l) \delta_{l,r} = \delta_{i,r}, \quad 1 \leq j, r \leq d.$$

(2) By (4.7) in Lemma 4.1, we have, for all $n \geq 1$, $1 \leq i, j, r \leq d$,

$$\frac{1}{\Pi_n^i(j, r)} = \frac{M_{0,n}(i, j)}{M_{0,n-1}(i, r)M_n(r, j)} = \sum_{l=1}^d \frac{M_{0,n-1}(i, l)M_n(l, j)}{M_{0,n-1}(i, r)M_n(r, j)} \leq dD^2 \quad \mathbb{Q}_s^{e_i}\text{-a.s.}$$

Hence,

$$\Pi_n^i(j, r) \geq \frac{1}{dD^2} \quad \mathbb{Q}_s^{e_i}\text{-a.s.} \quad (4.89)$$

Since $(\Pi_n^i)_{n \geq 0}$ is a sequence of positive stochastic matrices satisfying (4.89), by [66, Theorem 4.19], there exist two constants $C > 0$ and $\delta \in (0, 1)$ such that for all $k \geq 0$ and $1 \leq i \leq d$,

$$\sup_{n \geq 0} \|\Pi_{n+k, n}^i - \Pi_{\infty, n}^i\| \leq C\delta^k, \quad \mathbb{Q}_s^{e_i}\text{-a.s.} \quad (4.90)$$

This concludes the proof of Lemma 4.18. \square

Proof of Theorems 4.17. For all $n \geq 0$ and $1 \leq i, j \leq d$, recall that

$$\bar{Z}_n^i(j) = \frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i, j)}.$$

Now we prove that (4.50) is sufficient for the convergence in $L^p(\mathbb{Q}_s^{e_i})$ of $\bar{Z}_n^i(j)$, $1 \leq i, j \leq d$. Assume (4.50). By the definition of the branching process (Z_n^i) , we have the following decomposition: for all $1 \leq i, j \leq d$ and $n, k \geq 1$,

$$\begin{aligned} \bar{Z}_{n+k}^i(j) &= \sum_{r=1}^d \frac{M_{n,n+k-1}(r, j)}{M_{0,n+k-1}(i, j)} \sum_{l=1}^{Z_n^i(r)} \frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r, j)} \\ &= \sum_{r=1}^d \frac{M_{n,n+k-1}(r, j)}{M_{0,n+k-1}(i, j)} Z_n^i(r) \\ &\quad + \sum_{r=1}^d \frac{M_{n,n+k-1}(r, j)}{M_{0,n+k-1}(i, j)} \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r, j)} - 1 \right). \end{aligned} \quad (4.91)$$

Combining (4.91) and (4.86), we get that for all $1 \leq i, j \leq d$ and $n, k \geq 1$,

$$\begin{aligned} \bar{Z}_{n+k}^i(j) &= \sum_{r=1}^d \Pi_{n+k-1, n}^i(j, r) \bar{Z}_n^i(r) \\ &\quad + \sum_{r=1}^d \frac{\Pi_{n+k-1, n}^i(j, r)}{M_{0,n-1}(i, r)} \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r, j)} - 1 \right) \\ &= \langle \bar{Z}_n^i(\Pi_{n+k-1, n}^i)^\top, e_j \rangle + R_{n,k}^i(j), \end{aligned} \quad (4.92)$$

with

$$R_{n,k}^i(j) := \sum_{r=1}^d \frac{\Pi_{n+k-1, n}^i(j, r)}{M_{0,n-1}(i, r)} \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r, j)} - 1 \right).$$

Notice that by the definition of W_n^i and that of $\Pi_{\infty, n}^i$ (cf. Lemma 4.18 (1)),

$$W_n^i = \sum_{r=1}^d \Pi_{\infty, n}^i(j, r) \bar{Z}_n^i(r) = \langle \bar{Z}_n^i(\Pi_{\infty, n}^i)^\top, e_j \rangle \quad (4.93)$$

for any $1 \leq i, j \leq d$. Using (4.92) and (4.93), together with the triangular inequality in L^p under $\mathbb{Q}_s^{e_i}$, we obtain that for all $1 \leq i, j \leq d$ and $n, k \geq 1$,

$$\begin{aligned}
& \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} |\bar{Z}_{n+k}^i(j) - W^i|^p \right)^{1/p} \\
&= \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} |\langle \bar{Z}_n^i(\Pi_{n+k-1,n}^i)^T, e_j \rangle - W^i + R_{n,k}^i(j)|^p \right)^{1/p} \\
&= \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} |W_n^i - W^i + \langle \bar{Z}_n^i(\Pi_{n+k-1,n}^i - \Pi_{\infty,n}^i)^T, e_j \rangle + R_{n,k}^i(j)|^p \right)^{1/p} \\
&\leq \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} |W_n^i - W^i|^p \right)^{1/p} + \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \|\bar{Z}_n^i(\Pi_{n+k-1,n}^i - \Pi_{\infty,n}^i)^T\|^p \right)^{1/p} \\
&\quad + \max_{1 \leq j \leq d} \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} |R_{n,k}^i(j)|^p \right)^{1/p} \\
&= J_1^i(n) + J_2^i(n, k) + J_3^i(n, k). \tag{4.94}
\end{aligned}$$

In the following $C > 0$ will be a constant which may depend on p and s which may differ from line to line.

Control of $J_1^i(n)$. By condition (4.50) and Theorem 4.14 we get that there exists $\delta_1 \in (0, 1)$ such that for all $n \geq 1$ and $1 \leq i \leq d$,

$$J_1^i(n) = \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} |W_n^i - W^i|^p \right)^{1/p} \leq C \delta_1^n. \tag{4.95}$$

Control of $J_2^i(n, k)$. Applying the relation (4.85) of Lemma 4.18, we get that there exists $\delta_2 \in (0, 1)$ such that for all $n, k \geq 1$ and $1 \leq i \leq d$,

$$J_2^i(n, k) = \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \|\bar{Z}_n^i(\Pi_{n+k-1,n}^i - \Pi_{\infty,n}^i)^T\|^p \right)^{1/p} \leq C \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \|\bar{Z}_n^i\|^p \right)^{1/p} \delta_2^k. \tag{4.96}$$

By (4.10) in Lemma 4.1, we know that, under **A1**, we have for all $n \geq 0$ and $1 \leq i, j \leq d$

$$\frac{1}{dD^2} \leq \frac{M_{0,n-1}(i, j)u_n(j)}{\lambda_{0,n-1}u_0(i)} \leq 1 \quad \mathbb{Q}_s^{e_i}\text{-a.s.}$$

Therefore we obtain that for all $n \geq 0$ and $1 \leq i, j \leq d$,

$$\begin{aligned}
\frac{Z_n^i(j)}{M_{0,n-1}(i, j)} &\leq dD^2 \frac{M_{0,n-1}(i, j)u_n(j)}{\lambda_{0,n-1}u_0(i)} \frac{Z_n^i(j)}{M_{0,n-1}(i, j)} \\
&= dD^2 \frac{Z_n^i(j)u_n(j)}{\lambda_{0,n-1}u_0(i)} \leq dD^2 W_n^i. \tag{4.97}
\end{aligned}$$

Combining (4.97) and Theorem 4.14, and using condition (4.50), we obtain that, for all $1 \leq i \leq d$,

$$\sup_{n \geq 0} \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} \|\bar{Z}_n^i\|^p \right)^{1/p} \leq d^2 D^2 \sup_{n \geq 0} \left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} (W_n^i)^p \right)^{1/p} < +\infty. \tag{4.98}$$

This, together with (4.96), implies that for all $n, k \geq 1$ and $1 \leq i \leq d$,

$$J_2^i(n, k) \leq C \delta_2^k. \tag{4.99}$$

Control of $J_3^i(n, k)$ for $1 < p \leq 2$. Assume that $1 < p \leq 2$. Using the convexity of the function $x \mapsto x^p$ (together with $\sum_{r=1}^d \Pi_{n+k-1,n}^i(j, r) = 1$) and Lemma 4.15, for all $n, k \geq 1$

and $1 \leq i, j \leq d$, $\mathbb{Q}_s^{e_i}$ -a.s., we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_s^{e_i}}(|R_{n,k}^i(j)|^p | \xi) &= \mathbb{E}_\xi |R_{n,k}^i(j)|^p \\
&\leq \mathbb{E}_\xi \left(\sum_{r=1}^d \frac{\Pi_{n+k-1,n}^i(j,r)}{M_{0,n-1}(i,r)} \left| \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right) \right| \right)^p \\
&\leq \mathbb{E}_\xi \left(\sum_{r=1}^d \frac{\Pi_{n+k-1,n}^i(j,r)}{M_{0,n-1}(i,r)^p} \left| \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right) \right|^p \right) \\
&\leq B_p^p \sum_{r=1}^d \frac{\Pi_{n+k-1,n}^i(j,r)}{M_{0,n-1}(i,r)^p} \mathbb{E}_\xi Z_n^i(r) \mathbb{E}_\xi \left| \frac{Z_{1,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right|^p \\
&= B_p^p \sum_{r=1}^d \Pi_{n+k-1,n}^i(j,r) M_{0,n-1}(i,r)^{1-p} \mathbb{E}_\xi \left| \frac{Z_{1,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right|^p \\
&\leq B_p^p \bar{\sigma}_{n,k}(p) \max_{1 \leq r \leq d} M_{0,n-1}(i,r)^{1-p},
\end{aligned}$$

where

$$\bar{\sigma}_{n,k}(p) = \max_{1 \leq r, j \leq d} \mathbb{E}_\xi \left| \frac{Z_{1,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right|^p.$$

So, using Lemma 4.3 we get that for all $n, k \geq 1$ and $1 \leq i \leq d$,

$$\begin{aligned}
J_3^i(n, k)^p &\leq \max_{1 \leq j \leq d} \mathbb{E}_{\mathbb{Q}_s^{e_i}} |R_{n,k}^i(j)|^p \\
&= \max_{1 \leq j \leq d} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(|R_{n,k}^i(j)|^p \mid T^n \xi \right) \right] \\
&\leq B_p^p C_s \mathbb{E}_{\mathbb{Q}_s^{e_i}} \bar{\sigma}_{0,k}(p) \sum_{r=1}^d \mathbb{E}_{\mathbb{Q}_s^{e_i}} [M_{0,n-1}(i,r)^{1-p}]. \tag{4.100}
\end{aligned}$$

By (4.98) we have

$$\sup_{k \geq 0} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \bar{\sigma}_{0,k}(p) \leq d^2 \max_{1 \leq r, j \leq d} \sup_{k \geq 0} \mathbb{E}_{\mathbb{Q}_s^{e_i}} (|\bar{Z}_k^r(j) - 1|^p) < +\infty. \tag{4.101}$$

Therefore, putting together the relations (4.100) and (4.101), we get that for $n, k \geq 1$ and $1 \leq i \leq d$,

$$J_3^i(n, k)^p \leq C \kappa_s (1-p)^n \tag{4.102}$$

(recall that the value of C may change from line to line by our convention).

Control of $J_3^i(n, k)$ for $p \geq 2$. Assume that $p \geq 2$. Similar to the preceding case, by the convexity of $x \mapsto x^p$ (together with $\sum_{r=1}^d \Pi_{n+k-1,n}^i(j,r) = 1$) and Lemma 4.15, for all

$n, k \geq 1$ and $1 \leq i, j \leq d$, $\mathbb{Q}_s^{e_i}$ -a.s.,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_s^{e_i}}(|R_{n,k}^i(j)|^p|\xi) &= \mathbb{E}_\xi |R_{n,k}^i(j)|^p \\
&\leq \mathbb{E}_\xi \left(\sum_{r=1}^d \frac{\Pi_{n+k,n+1}^i(j,r)}{M_{0,n-1}(i,r)} \left| \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right) \right| \right)^p \\
&\leq B_p^p \sum_{r=1}^d \frac{\Pi_{n+k,n+1}^i(j,r)}{M_{0,n-1}(i,r)^p} \mathbb{E}_\xi (Z_n^i(r))^{p/2} \mathbb{E}_\xi \left| \frac{Z_{1,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right|^p \\
&\leq B_p^p \bar{\sigma}_{n,k}(p) \sum_{r=1}^d \Pi_{n+k,n+1}^i(j,r) \mathbb{E}_\xi (\bar{Z}_n^i(r))^{p/2} M_{0,n-1}(i,r)^{-p/2} \\
&\leq B_p^p \bar{\sigma}_{n,k}(p) \max_{1 \leq r \leq d} \left\{ \mathbb{E}_\xi (\bar{Z}_n^i(r))^{p/2} M_{0,n-1}(i,r)^{-p/2} \right\}. \tag{4.103}
\end{aligned}$$

Notice that (4.101) still holds when $p \geq 2$. Therefore, using (4.101), we obtain that for all $n, k \geq 1$ and $1 \leq i \leq d$,

$$\begin{aligned}
J_3^i(n, k)^p &\leq B_p^p C_s \mathbb{E}_{\mathbb{Q}_s^{e_i}} \bar{\sigma}_{0,k}(p) \sum_{1 \leq r \leq d} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\mathbb{E}_\xi (\bar{Z}_n^i(r))^{p/2} M_{0,n-1}(i,r)^{-p/2} \right] \\
&\leq C \sum_{r=1}^d \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\mathbb{E}_\xi (\bar{Z}_n^i(r))^{p/2} M_{0,n-1}(i,r)^{-p/2} \right]. \tag{4.104}
\end{aligned}$$

Using (4.7) in Lemma 4.1, for all $n \geq 1$ and $1 \leq i, r \leq d$, $\mathbb{Q}_s^{e_i}$ -a.s., we have

$$\begin{aligned}
M_{0,n-1}(i, r) &\geq \frac{1}{dD} \|M_{0,n-1}(i, \cdot)\| \geq \frac{1}{dD} \langle M_{0,n-1}(i, \cdot), u_n \rangle \\
&= \frac{1}{dD} \lambda_{0,n-1} u_0(i).
\end{aligned}$$

Combining this with (4.104) and (4.97), we get that for $p > 2$, $n, k \geq 1$ and $1 \leq i \leq d$,

$$J_3^i(n, k)^p \leq C \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\mathbb{E}_\xi (W_n^i)^{p/2} (\lambda_{0,n-1} u_0(i))^{-p/2} \right] = C \mathbb{E}_{\mathbb{Q}_s^{e_i}} a_{n,1}^i(p),$$

where $a_{n,1}^i(p)$ is defined in (4.67) with $j = 1$. This, together with (4.69) (which holds under condition (4.50)), implies that there exists $\delta_3 \in (0, 1)$ such that for all $n, k \geq 1$ and $1 \leq i \leq d$,

$$J_3^i(n, k)^p \leq C \delta_3^n. \tag{4.105}$$

Combining (4.94), (4.95), (4.99), (4.102) and (4.105), we obtain that for all $n, k \geq 1$ and $1 \leq i, j \leq d$,

$$\left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} |\bar{Z}_{n+k}^i(j) - W^i|^p \right)^{1/p} \leq C(\delta_1^n + \delta_2^k + \delta_3^n).$$

Applying this inequality with n replaced by $\lfloor n/2 \rfloor$ (the integral part of $n/2$) and taking $k = n - \lfloor n/2 \rfloor$, we see that for all $n \geq 1$ and $1 \leq i, j \leq d$,

$$\left(\mathbb{E}_{\mathbb{Q}_s^{e_i}} |\bar{Z}_n^i(j) - W^i|^p \right)^{1/p} \leq C(\delta_1^{n/2} + \delta_2^{n/2} + \delta_3^{n/2}) \leq C \bar{\delta}^n,$$

with $\bar{\delta} = \max\{\delta_1^{1/2}, \delta_2^{1/2}, \delta_3^{1/2}\} < 1$. Therefore, for any $1 \leq i, j \leq d$ the normalized population size $\bar{Z}_n^i(j)$ converges in $L^p(\mathbb{Q}_s^{e_i})$ to W^i as $n \rightarrow \infty$. This concludes the proof of Theorem 4.17. \square

4.6. Convergence in $L^p(\mathbb{Q}_s^{e_i})$ of the normalized norm $\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|}$. We now deduce from Theorems 4.17 and 4.16 the convergence in $L^p(\mathbb{Q}_s^{e_i})$ of $\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|}$, the normalized total population size of generation n .

Theorem 4.19. *Assume **A1**, $s \in I_\mu^+ \cup I_\mu^-$.*

(1) *If $\Lambda'(s) > 0$, **H2** holds and $\mathbb{E}(\|M_0\|^s \log^+ \|M_0\|) < \infty$, then we have for any $1 \leq i \leq d$,*

$$\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} \rightarrow W^i \text{ in } L^1(\mathbb{Q}_s^{e_i}), \quad \text{as } n \rightarrow +\infty. \quad (4.106)$$

(2) *Let $p > 1$ be such that $\max_{1 \leq i, j \leq d} \mathbb{E} M_0(i, j)^{s+1-p} < \infty$. If (4.50) holds, then for any $1 \leq i \leq d$,*

$$\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} \rightarrow W^i \text{ in } L^p(\mathbb{Q}_s^{e_i}), \quad \text{as } n \rightarrow +\infty. \quad (4.107)$$

Moreover, there exist $\delta \in (0, 1)$ and a constant $C > 0$ such that for all $n \geq 0$, $1 \leq i \leq d$,

$$\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left| \frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} - W^i \right|^p \leq C\delta^n. \quad (4.108)$$

Proof. For any $1 \leq i \leq d$, we have the decomposition

$$\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} - W^i = \sum_{j=1}^d \frac{M_{0,n-1}(i, j)}{\|M_{0,n-1}(i, \cdot)\|} \left(\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} - W^i \right). \quad (4.109)$$

Therefore, applying (4.109), and using the inequality for $p \geq 1$, $|\sum_{j=1}^d x_j|^p \leq d^{p-1} \sum_{j=1}^d |x_j|^p$, then there exist $\delta \in (0, 1)$ and a constant $C > 0$, we have for any $1 \leq i \leq d$,

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left| \frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} - W^i \right|^p \\ &= \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left| \sum_{j=1}^d \frac{M_{0,n-1}(i, j)}{\|M_{0,n-1}(i, \cdot)\|} \left(\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} - W^i \right) \right|^p \\ &\leq d^{p-1} \left(\sum_{j=1}^d \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left| \frac{M_{0,n-1}(i, j)}{\|M_{0,n-1}(i, \cdot)\|} \left(\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} - W^i \right) \right|^p \right) \\ &\leq d^{p-1} \left(\sum_{j=1}^d \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left| \frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} - W^i \right|^p \right). \end{aligned} \quad (4.110)$$

If $p = 1$, using this and Theorem 4.16, it follows (4.106). If $p > 1$, using (4.110) and Theorem 4.17, we get (4.107) and (4.108). \square

5. MOMENTS OF $\|Z_n^i\|$ AND $Z_n^i(j)$

In this subsection, as applications of the convergence in $L^s(\mathbb{Q}_s^{e_i})$ of $\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|}$, $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$ and of moments for products of random matrices, we establish the exact equivalence of the moments of $\|Z_n^i\|$ and $Z_n^i(j)$.

Theorem 5.1. *Assume conditions [A4](#), [A1](#) and $s \in I_\mu^+ \setminus \{0\}$. Additionally, when $s > 1$, we assume $\mathbb{E}(Z_1^i(j))^s < \infty$ for any $1 \leq i, j \leq d$; when $0 < s \leq 1$, we assume*

$$\max_{1 \leq i, j \leq d} \mathbb{E} \left(\frac{Z_1^i(j)}{M_0(i, j)^{1-s}} \log^+ \frac{Z_1^i(j)}{M_0(i, j)} \right) < \infty \quad \text{and} \quad \mathbb{E}(\|M_0\|^s \log^+ \|M_0\|) < \infty.$$

Then we have for all $i, j \in \{1, \dots, d\}$,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}\|Z_n^i\|^s}{\kappa^n(s)} = \mathbb{E}_{\mathbb{Q}_s^{e_i}} (W^i)^s r_s(e_i) \int_{\mathcal{S}} \frac{\pi_s(dx)}{r_s(x)}. \quad (5.1)$$

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(Z_n^i(j))^s}{\kappa^n(s)} = \mathbb{E}_{\mathbb{Q}_s^{e_i}} (W^i)^s r_s(e_i) \int_{\mathcal{S}} \frac{\langle x, e_j \rangle^s}{r_s(x)} \pi_s(dx). \quad (5.2)$$

Proof. Part 1: we first prove (5.1). By the definition of $\mathbb{Q}_s^{e_i}$, we have

$$\begin{aligned} \frac{\mathbb{E}\|Z_n^i\|^s}{\kappa^n(s)} &= \mathbb{E} \left(\frac{\|e_i M_{0, n-1}\|^s}{\kappa^n(s)} \frac{\|Z_n^i\|^s}{\|\mathbb{E}_\xi Z_n^i\|^s} \right) \\ &= \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\frac{\|Z_n^i\|^s}{\|\mathbb{E}_\xi Z_n^i\|^s} \frac{r_s(e_i)}{r_s(e_i \cdot M_{0, n-1})} \right) \\ &= \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\left(\frac{\|Z_n^i\|^s}{\|\mathbb{E}_\xi Z_n^i\|^s} - (W^i)^s \right) \frac{r_s(e_i)}{r_s(e_i \cdot M_{0, n-1})} \right] + \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[(W^i)^s \frac{r_s(e_i)}{r_s(e_i \cdot M_{0, n-1})} \right]. \end{aligned} \quad (5.3)$$

By Theorem 4.19, $\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} \rightarrow W^i$ in $L^s(\mathbb{Q}^{e_i})$ when $s > 1$, and in $L^1(\mathbb{Q}^{e_i})$ when $s \in (0, 1]$. Since the function r_s lies between two positive constants, it follows that

$$\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\left(\frac{\|Z_n^i\|^s}{\|\mathbb{E}_\xi Z_n^i\|^s} - (W^i)^s \right) \frac{r_s(e_i)}{r_s(e_i \cdot M_{0, n-1})} \right] \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (5.4)$$

since the family of the random variables under the expectation is uniformly integrable. Therefore, from (5.3), we obtain that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}\|Z_n^i\|^s}{\kappa^n(s)} = \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[(W^i)^s \frac{r_s(e_i)}{r_s(e_i \cdot M_{0, n-1})} \right] = \mathbb{E}_{\mathbb{Q}_s^{e_i}} (W^i)^s r_s(e_i) \int_{\mathcal{S}} \frac{\pi_s(dx)}{r_s(x)}, \quad (5.5)$$

where the last equality holds by the mixing convergence of $(e_i \cdot M_{0, n-1})$ (see Lemma 3.7). Hence, (5.1) is proved.

Part 2: we next prove (5.2). We have the following calculation

$$\begin{aligned} \frac{\mathbb{E}(Z_n^i(j))^s}{\kappa^n(s)} &= \mathbb{E} \left(\frac{M_{0, n-1}(i, j)^s}{\kappa^n(s)} \frac{(Z_n^i(j))^s}{(\mathbb{E}_\xi Z_n^i(j))^s} \right) \\ &= \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\frac{(Z_n^i(j))^s}{(\mathbb{E}_\xi Z_n^i(j))^s} \frac{r_s(e_i)}{r_s(e_i \cdot M_{0, n-1})} \frac{M_{0, n-1}(i, j)^s}{\|e_i M_{0, n-1}\|^s} \right) \\ &= \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\left(\frac{(Z_n^i(j))^s}{(\mathbb{E}_\xi Z_n^i(j))^s} - (W^i)^s \right) \frac{r_s(e_i)}{r_s(e_i \cdot M_{0, n-1})} \frac{M_{0, n-1}(i, j)^s}{\|e_i M_{0, n-1}\|^s} \right] \\ &\quad + \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[(W^i)^s \frac{r_s(e_i)}{r_s(e_i \cdot M_{0, n-1})} \frac{M_{0, n-1}(i, j)^s}{\|e_i M_{0, n-1}\|^s} \right]. \end{aligned} \quad (5.6)$$

By F-K condition [A1](#), we have the bound

$$\frac{1}{(dD)^s} \leq \frac{M_{0,n-1}(i,j)^s}{\|e_i M_{0,n-1}\|^s} \leq D^s.$$

By Theorems [4.16](#) and [4.17](#), we have $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} \rightarrow W^i$ in $L^s(\mathbb{Q}^{e_i})$ when $s > 1$, and in $L^1(\mathbb{Q}^{e_i})$ when $s \in (0, 1]$. Hence, the family of $\left(\left(\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}\right)^s\right)$ is uniformly integrable. Since the function r_s lies between two positive constants, we get that

$$\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\left(\frac{(Z_n^i(j))^s}{(\mathbb{E}_\xi Z_n^i(j))^s} - (W^i)^s \right) \frac{r_s(e_i)}{r_s(e_i \cdot M_{0,n-1})} \frac{M_{0,n-1}(i,j)^s}{\|e_i M_{0,n-1}\|^s} \right] \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Therefore, from [\(5.6\)](#) we derive

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\mathbb{E}(Z_n^i(j))^s}{\kappa^n(s)} &= \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[(W^i)^s \frac{r_s(e_i)}{r_s(e_i \cdot M_{0,n-1})} \frac{M_{0,n-1}(i,j)^s}{\|e_i M_{0,n-1}\|^s} \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[(W^i)^s \frac{r_s(e_i)}{r_s(e_i \cdot M_{0,n-1})} \left(\frac{M_{0,n-1}(i,j)^s}{\|e_i M_{0,n-1}\|^s} - (v_{n-1}(j))^s \right) \right] \\ &\quad + \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[(W^i)^s \frac{r_s(e_i)}{r_s(e_i \cdot M_{0,n-1})} (v_{n-1}(j))^s \right], \end{aligned} \quad (5.7)$$

provided that the limit exists. By [\(3.18\)](#), we see that

$$\begin{aligned} \left| \frac{M_{0,n-1}(i,j)^s}{\|e_i M_{0,n-1}\|^s} - (v_{n-1}(j))^s \right| &\leq C_s \left| \frac{M_{0,n-1}(i,j)}{\|e_i M_{0,n-1}\|} - v_{n-1}(j) \right|^s \\ &\rightarrow C_s \left| \frac{a_{0,n-1} u_0(i) v_{n-1}(j)}{a_{0,n-1} u_0(i) \|v_{n-1}\|} - v_{n-1}(j) \right|^s \\ &\rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence, by the dominated convergence theorem we get that

$$\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[(W^i)^s \frac{r_s(e_i)}{r_s(e_i \cdot M_{0,n-1})} \left(\frac{M_{0,n-1}(i,j)^s}{\|e_i M_{0,n-1}\|^s} - (v_{n-1}(j))^s \right) \right] \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (5.8)$$

Therefore, from [\(5.7\)](#), provided that the limit exists, it follows that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(Z_n^i(j))^s}{\kappa^n(s)} = \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[(W^i)^s \frac{r_s(e_i)}{r_s(e_i \cdot M_{0,n-1})} (v_{n-1}(j))^s \right]. \quad (5.9)$$

Using again the dominated convergence theorem, we also obtain that

$$\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[(W^i)^s (v_{n-1}(j))^s \left(\frac{r_s(e_i)}{r_s(e_i \cdot M_{0,n-1})} - \frac{r_s(e_i)}{r_s(v_{n-1})} \right) \right] \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (5.10)$$

Combining [\(5.9\)](#) and [\(5.10\)](#), it implies that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\mathbb{E}(Z_n^i(j))^s}{\kappa^n(s)} &= \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left((W^i)^s r_s(e_i) \frac{\langle v_{n-1}, e_j \rangle^s}{r_s(v_{n-1})} \right) \\ &= \mathbb{E}_{\mathbb{Q}_s^{e_i}} (W^i)^s r_s(e_i) \int_{\mathcal{S}} \frac{\langle x, e_j \rangle^s}{r_s(x)} \pi_s(dx), \end{aligned} \quad (5.11)$$

the last step holds by Theorem [3.7](#). So this gives [\(5.2\)](#). \square

6. PRECISE LARGE DEVIATIONS

In this section we will prove Theorem 1.6 on the precise large deviation of $\|Z_n^i\|$. We start with the following decomposition by telescoping: for any $1 \leq m \leq n$,

$$\begin{aligned} Z_n &= Z_m M_{m,n-1} + \sum_{k=m+1}^n (Z_k - Z_{k-1} M_{k-1}) M_{k,n-1} \\ &=: Z_m M_{m,n-1} + D_{m,n} \end{aligned} \quad (6.1)$$

(recall that $M_{k,n} = M_k \dots M_{n-1}$ if $k \leq n$, $M_{k,n} =$ the identity matrix if $k > n$, and the empty sum is taken to be 0). Notice that the summands in $\sum_{k=m+1}^n$ are martingale differences by the following lemma, which can be easily checked.

Lemma 6.1. [30]

(1) For any $n \geq 0$, we have

$$\mathbb{E}_\xi[Z_{n+1} | \mathcal{F}_n] = Z_n M_n. \quad (6.2)$$

(2) For fixed $n \geq 1$, let

$$W_{k,n} = Z_k M_{k,n-1} \quad \text{for } 0 \leq k < n, \quad W_{n,n} = Z_n. \quad (6.3)$$

Then $\{W_{k,n} : 0 \leq k \leq n\}$ is a \mathbb{R}_+^d -valued martingale under \mathbb{P}_ξ w.r.t. the filtration $\{\mathcal{F}_k : 0 \leq k \leq n\}$.

We will show that, with a suitable choice of $m = m_n$, for precise large deviations of Z_n , the reminder $D_{m,n}$ in the decomposition (6.1) is negligible, and Z_n behaves like $Z_m M_{m,n-1}$. To this end, we will make use of the following elementary bounds: for $0 < u < t$,

$$\mathbb{P}(\|Z_n\| \geq t) \leq \mathbb{P}(\|Z_m M_{m,n-1}\| \geq t - u) + \mathbb{P}(\|D_{m,n}\| \geq u), \quad (6.4)$$

$$\mathbb{P}(\|Z_n\| \geq t) \geq \mathbb{P}(\|Z_m M_{m,n-1}\| \geq t + u) - \mathbb{P}(\|D_{m,n}\| \geq u), \quad (6.5)$$

and we will need the estimation of the moments of $\|Z_n^i\|$, and that of $\|Z_k - Z_{k-1} M_{k-1}\|$ appearing in $D_{m,n}$. Recall that Theorem 5.1 gives a precise estimation of the moments of $\|Z_n^i\|$ any order $\alpha > 0$. Below we give an upper bound uniform for α in an interval.

Lemma 6.2. Assume the hypotheses of Theorem 1.6. Then, for any $s > 0$, there exist constants $\delta_0 > 0$ and $c > 0$ (which may depend on s) such that for all $\alpha \in [s, s + \delta_0]$, $i \in \{1, \dots, d\}$ and $n \geq 1$,

$$\mathbb{E}[\|Z_n^i\|^\alpha] \leq \begin{cases} c\kappa(\alpha)^n & \text{if } s < 1 \text{ and } \alpha \in [s, 1], \\ c n^\alpha \kappa(\alpha)^n & \text{if } s \geq 1. \end{cases} \quad (6.6)$$

Moreover, there exist some constants $\theta_0 \in (0, \kappa(s))$ and $C \in (0, \infty)$ such that for any $n \geq 1$,

$$\mathbb{E} \left(\left\| Z_n^i - Z_{n-1}^i M_{n-1} \right\|^s \right) \leq C \theta_0^n. \quad (6.7)$$

Proof. Part 1: we first consider the case $s < 1$. Take any $\alpha \in [s, 1]$. Since the function $x \mapsto x^\alpha$ is concave on $[0, \infty)$, the conditional Jensen inequality and (2.10) entails

$$\begin{aligned} \mathbb{E}[\|Z_n^i\|^\alpha] &= \mathbb{E} \left[\mathbb{E}_\xi(\|Z_n^i\|^\alpha) \right] \leq \mathbb{E} \left[(\mathbb{E}_\xi \|Z_n^i\|)^\alpha \right] \\ &= \mathbb{E} \left[\|\mathbb{E}_\xi Z_n^i\|^\alpha \right] = \mathbb{E} \left[\|e_i M_{0,n-1}\|^\alpha \right] \leq \|e_i\|^\alpha \mathbb{E}[\|M_{0,n-1}\|^\alpha] \\ &\leq (D^2 d)^s \kappa(\alpha)^n \leq (D^2 d)^{s+\delta} \kappa(\alpha)^n, \end{aligned} \quad (6.8)$$

which proves (6.6). We now prove (6.7). In the following, for simplicity we just write Z_n instead of Z_n^i . We need the following decomposition:

$$Z_n - Z_{n-1}M_{n-1} = \sum_{r=1}^d \sum_{l=1}^{Z_{n-1}(r)} \left(N_{l,n-1}^r - M_{n-1}(r, \cdot) \right), \quad (6.9)$$

where under \mathbb{P}_ξ , the family of random variables $\{N_{l,n-1}^r - M_{n-1}(r, \cdot) : r, l \geq 1\}$ are i.i.d. with zero mean, and independent of \mathcal{F}_{n-1} , and Z_{n-1} is \mathcal{F}_{n-1} -measurable. Let $0 < \varepsilon \leq s/2$. Then $\kappa(s - \varepsilon) < \kappa(s)$ because $\Lambda(s)$ is strictly increasing in $s \in I_\mu^+$, remarking that for such $s > 0$, $\Lambda'(s) > 0$. Applying the inequality in Lemma 4.15 with $p = \frac{s}{s-\varepsilon} \leq 2$ (because $\varepsilon \leq s/2$), writing $\mathbb{E}_{\xi, n-1}(\cdot) = \mathbb{E}_\xi(\cdot | \mathcal{F}_{n-1})$ for the conditional expectation under \mathbb{P}_ξ given \mathcal{F}_{n-1} , we have, for some constant $c = c(d, p) > 0$ whose value may change from line to line,

$$\begin{aligned} \mathbb{E}_{\xi, n-1} \left(\left\| Z_n - Z_{n-1}M_{n-1} \right\|^{\frac{s}{s-\varepsilon}} \right) &\leq d^{p-1} \sum_{r=1}^d \mathbb{E}_{\xi, n-1} \left(\left\| \sum_{l=1}^{Z_{n-1}(r)} \left(N_{l,n-1}^r - M_{n-1}(r, \cdot) \right) \right\|^{\frac{s}{s-\varepsilon}} \right) \\ &\leq c \sum_{r=1}^d Z_{n-1}(r) \mathbb{E}_\xi \left(\left\| N_{1,n-1}^r - M_{n-1}(r, \cdot) \right\|^{\frac{s}{s-\varepsilon}} \right) \\ &\leq c \|Z_{n-1}\| \sum_{r=1}^d \mathbb{E}_\xi \left(\left\| N_{1,n-1}^r - M_{n-1}(r, \cdot) \right\|^{\frac{s}{s-\varepsilon}} \right). \end{aligned}$$

Therefore, by Jensen's inequality,

$$\begin{aligned} &\mathbb{E} \left(\|Z_n - Z_{n-1}M_{n-1}\|^s \right) \\ &\leq \mathbb{E} \left[\left(\mathbb{E}_{\xi, n-1} \left(\|Z_n - Z_{n-1}M_{n-1}\|^{\frac{s}{s-\varepsilon}} \right) \right)^{s-\varepsilon} \right] \\ &\leq c \mathbb{E} \left[\|Z_{n-1}\|^{s-\varepsilon} \cdot \left(\sum_{r=1}^d \mathbb{E}_\xi \left(\|N_{1,n-1}^r - M_{n-1}(r, \cdot)\|^{\frac{s}{s-\varepsilon}} \right) \right)^{s-\varepsilon} \right] \\ &= c \mathbb{E} \|Z_{n-1}\|^{s-\varepsilon} \cdot \mathbb{E} \left[\left(\sum_{r=1}^d \mathbb{E}_\xi \left(\|Z_1^r - M_0(r, \cdot)\|^{\frac{s}{s-\varepsilon}} \right) \right)^{s-\varepsilon} \right], \quad (6.10) \end{aligned}$$

where the last equality can be seen by first conditioning on ξ and by using the i.i.d. property of the sequence (ξ_n) . The last expectation in (6.10) is finite by the condition $\max_{1 \leq i, j \leq d} \mathbb{E} M_0(i, j)^s < \infty$ and the hypothesis (1.34), for $\varepsilon > 0$ small enough such that $p = \frac{s}{s-\varepsilon} \leq 1 + \delta$. Combining (6.10) with (6.8) we get (6.7).

We next consider the case $s \geq 1$. Below we will prove that: for any $\beta \in [1, s + \delta]$ (with δ as in (1.35)) we have

$$\mathbb{E}[\|Z_n\|^\beta] \leq \begin{cases} d(\beta)n^\beta \kappa(\beta)^n, & \text{if } \kappa(\beta) > \kappa(1), \\ d(\beta)n^\beta \kappa(1)^n, & \text{if } \kappa(\beta) \leq \kappa(1), \end{cases} \quad (6.11)$$

for some continuous function d on $[1, s + \delta]$. Note that (6.11) together with the condition $\kappa(s) > \kappa(1)$ entails (6.6).

For any $n \geq 1$, we have the following decomposition:

$$Z_n = Z_0 M_{0, n-1} + \sum_{k=1}^n (Z_{n-k+1} - Z_{n-k} M_{n-k}) M_{n-k+1, n-1}. \quad (6.12)$$

By (6.12) and (2.10), together with the inequality $(\sum_{k=1}^{n+1} x_k)^\beta \leq (n+1)^{\beta-1} x_k^\beta$ for $x_k \geq 0$, we see that there exists a constant $c_\beta > 0$ such that

$$\begin{aligned} \mathbb{E}[\|Z_n\|^\beta] &\leq (n+1)^{\beta-1} \left(\mathbb{E}\|Z_0 M_{0,n-1}\|^\beta + \sum_{k=1}^n \mathbb{E}\|Z_{n-k+1} - Z_{n-k} M_{n-k}\|^\beta \mathbb{E}\|M_{n-k+1,n-1}\|^\beta \right) \\ &\leq (n+1)^{\beta-1} \left(c_\beta \kappa(\beta)^n + \sum_{k=1}^n \mathbb{E}\|Z_{n-k+1} - Z_{n-k} M_{n-k}\|^\beta c_\beta \kappa(\beta)^k \right) \\ &= c_\beta (n+1)^{\beta-1} \left(\kappa(\beta)^n + \sum_{k=1}^n \mathbb{E}\|Z_{n-k+1} - Z_{n-k} M_{n-k}\|^\beta \kappa(\beta)^k \right). \end{aligned} \quad (6.13)$$

Using the decomposition (6.9) and Lemma 4.15 yields

$$\mathbb{E}[\|Z_n - Z_{n-1} M_{n-1}\|^\beta] \leq C_\beta \mathbb{E} \left[\left(\sum_{r=1}^d \|Z_1^r - M_0(r, \cdot)\| \right)^\beta \right] \mathbb{E}[\|Z_{n-1}\|^{\beta^*}], \quad (6.14)$$

where $\beta^* = \frac{\beta}{2} \vee 1$. Combining (6.13) and (6.14), we obtain the following estimate

$$\mathbb{E}[\|Z_n\|^\beta] \leq c_\beta (n+1)^{\beta-1} \left(\kappa(\beta)^n + c(\beta) \sum_{k=1}^n \mathbb{E}[\|Z_{n-k}\|^{\beta^*}] \kappa(\beta)^k \right), \quad (6.15)$$

where $c(\beta) = C_\beta \mathbb{E} \left[\left(\sum_{r=1}^d \|Z_1^r - M_0(r, \cdot)\| \right)^\beta \right]$.

The above estimate is the key step in the proof of (6.11). From now we proceed by induction on m such that $\beta \in (2^m, 2^{m+1}]$.

Assume first that $\beta \in [1, 2]$, then $\beta^* = 1$ and $\mathbb{E}[\|Z_{n-k}\|^{\beta^*}] = \mathbb{E}[\|Z_{n-k}\|] \leq \mathbb{E}\|M_0 \dots M_{n-k-1}\| \leq c_1 \kappa(1)^{n-k}$, where c_1 is a constant. If $\kappa(\beta) > \kappa(1)$, from (6.15) we have

$$\begin{aligned} \mathbb{E}[\|Z_n\|^\beta] &\leq c_\beta (n+1)^{\beta-1} \left(\kappa(\beta)^n + c(\beta) c_1 \sum_{k=1}^n \kappa(1)^{n-k} \kappa(\beta)^k \right) \\ &\leq c_\beta (n+1)^{\beta-1} \left(\kappa(\beta)^n + c(\beta) c_1 \kappa(\beta)^n n \right) \\ &= c_\beta (n+1)^{\beta-1} \left(1 + c(\beta) c_1 n \right) \kappa(\beta)^n. \end{aligned} \quad (6.16)$$

Therefore we obtain (6.11) when $\beta \in [1, 2]$ and $\kappa(\beta) > \kappa(1)$.

If $\beta \in [1, 2]$ and $\kappa(\beta) \leq \kappa(1)$, then again by (6.15),

$$\begin{aligned} \mathbb{E}[\|Z_n\|^\beta] &\leq c_\beta (n+1)^{\beta-1} \left(\kappa(1)^n + c(\beta) c_1 \kappa(1)^n n \right) \\ &= c_\beta (n+1)^{\beta-1} \left(1 + c(\beta) c_1 n \right) \kappa(1)^n. \end{aligned} \quad (6.17)$$

Hence, we also get (6.11) when $\beta \in [1, 2]$ and $\kappa(\beta) \leq \kappa(1)$.

We have therefore proved that (6.11) holds when $\beta \in [1, 2]$.

Assume now that $\beta \in (2^m, 2^{m+1}]$ for $m \geq 1$. We consider two cases. If $\kappa(\beta) > \kappa(1)$, then by convexity $\kappa(\beta/2) < \kappa(\beta)$ and by the induction hypothesis we have

$$\mathbb{E}[\|Z_{n-k}\|^{\beta/2}] \leq d(\beta/2) (n-k)^{\beta/2} \max\{\kappa(1), \kappa(\beta/2)\}^{n-k}. \quad (6.18)$$

Since $\beta^* = \beta/2$, combining the above inequality with (6.15), we obtain

$$\begin{aligned}
\mathbb{E}[\|Z_n\|^\beta] &\leq c_\beta(n+1)^{\beta-1} \left(\kappa(\beta)^n + c(\beta) \sum_{k=1}^n d(\beta/2)(n-k)^{\beta/2} \max\{\kappa(1), \kappa(\beta/2)\}^{n-k} \kappa(\beta)^k \right) \\
&\leq c_\beta(n+1)^{\beta-1} \left(\kappa(\beta)^n + c(\beta)d(\beta/2)\kappa(\beta)^n \sum_{k=1}^n \left(\frac{\max\{\kappa(1), \kappa(\beta/2)\}}{\kappa(\beta)} \right)^{n-k} (n-k)^{\beta/2} \right) \\
&\leq c_\beta(n+1)^{\beta-1} \left(\kappa(\beta)^n + c(\beta)d(\beta/2)\kappa(\beta)^n n^{\beta/2+1} \right) \\
&= c_\beta(n+1)^{\beta-1} \left(1 + c(\beta)d(\beta/2)n^{\beta/2+1} \right) \kappa(\beta)^n. \tag{6.19}
\end{aligned}$$

This gives (6.11).

If $\kappa(\beta) \leq \kappa(1)$, in this case $\kappa(\beta/2) < \kappa(1)$ and by the induction hypothesis we have

$$\mathbb{E}[\|Z_{n-k}\|^{\beta/2}] \leq d(\beta/2)(n-k)^{\beta/2} \kappa(1)^{n-k}. \tag{6.20}$$

Combining this with (6.15), we get

$$\begin{aligned}
\mathbb{E}[\|Z_n\|^\beta] &\leq c_\beta(n+1)^{\beta-1} \left(\kappa(\beta)^n + c(\beta) \sum_{k=1}^n d(\beta/2)(n-k)^{\beta/2} \kappa(1)^{n-k} \kappa(\beta)^k \right) \\
&\leq c_\beta(n+1)^{\beta-1} \left(\kappa(1)^n + c(\beta)d(\beta/2)\kappa(1)^n \sum_{k=1}^n (n-k)^{\beta/2} \right) \\
&\leq c_\beta(n+1)^{\beta-1} \left(\kappa(1)^n + c(\beta)d(\beta/2)\kappa(1)^n n^{\beta/2+1} \right) \\
&= c_\beta(n+1)^{\beta-1} \left(1 + c(\beta)d(\beta/2)n^{\beta/2+1} \right) \kappa(1)^n. \tag{6.21}
\end{aligned}$$

This implies (6.11) for this case. Hence, (6.11) is proved.

From (6.14) we obtain that

$$\mathbb{E}[\|Z_n - Z_{n-1}M_{n-1}\|^s] \leq C_s \mathbb{E}[\|Z_{n-1}\|^{s^*}] \mathbb{E} \left[\left(\sum_{r=1}^d \|Z_1^r - M_0(r, \cdot)\| \right)^s \right], \tag{6.22}$$

where $s^* = \frac{s}{2} \vee 1$. Combining (1.35), (6.6) and (6.22), we get (6.7). \square

Since now, we set

$$m = m_n = K \lfloor \log n \rfloor, \tag{6.23}$$

where $\lfloor a \rfloor$ denotes the integer part of $a \in \mathbb{R}$, $K > 0$ is a constant which will be chosen large enough. The following lemma gives an estimation of the deviation of $D_{m,n}$.

Lemma 6.3. *Assume hypotheses of Theorem 1.6. Then there exist some constants $b_s \in (0, \infty)$ and $\theta \in (0, \kappa(s))$, such that for any $t > 0$ and $n > 0$, with m defined in (6.23),*

$$\mathbb{P}(\|D_{m,n}\| \geq t) \leq \frac{b_s}{t^s} \left(\frac{\theta}{\kappa(s)} \right)^m \kappa(s)^n.$$

Proof. In the following we assume that n is large enough, say $n \geq n_0$, such that $0 < m < n$. (For $n \leq n_0$, the inequality holds clearly by taking b_s large enough.) For any $t > 0$, we have

$$\begin{aligned} & \mathbb{P}(\|D_{m,n}\| \geq t) \\ & \leq \mathbb{P}\left(\sum_{k=m+1}^n \|(Z_k - Z_{k-1}M_{k-1})M_{k,n-1}\| \geq \sum_{k=m+1}^n \frac{t}{2k^2}\right) \\ & \leq \sum_{k=m+1}^n \mathbb{P}\left(\|(Z_k - Z_{k-1}M_{k-1})M_{k,n-1}\| \geq \frac{t}{2k^2}\right). \end{aligned}$$

By the Markov inequality, using Lemma 6.2 and (2.10), there exist some constants $b_s \in (0, \infty)$ (whose value can change from line to line) and $\theta_0 \in (0, \kappa(s))$ such that, for any $n > 0$,

$$\begin{aligned} & \sum_{k=m+1}^n \mathbb{P}\left(\|(Z_k - Z_{k-1}M_{k-1})M_{k,n-1}\| \geq \frac{t}{2k^2}\right) \\ & \leq \sum_{k=m+1}^n \mathbb{E}(\|(Z_k - Z_{k-1}M_{k-1})M_{k,n-1}\|^s) \cdot \left(\frac{t}{2k^2}\right)^{-s} \\ & \leq \sum_{k=m+1}^n \mathbb{E}(\|Z_k - Z_{k-1}M_{k-1}\|^s) \mathbb{E}(\|M_{k,n-1}\|^s) \left(\frac{t}{2k^2}\right)^{-s} \\ & \leq b_s \sum_{k=m+1}^n \theta_0^k \kappa(s)^{n-k} k^{2s} t^{-s} \\ & = b_s \kappa(s)^n t^{-s} \sum_{k=m+1}^n \left(\frac{\theta_0}{\kappa(s)}\right)^k k^{2s} \\ & \leq b_s \kappa(s)^n t^{-s} \left(\frac{\theta}{\kappa(s)}\right)^m, \end{aligned}$$

where $\theta \in (\theta_0, \kappa(s))$. This gives the conclusion. \square

Now we shall analyse the quantity $\|Z_m M_{m,n-1}\|$ in the decomposition (6.1). We have, for any $u, t \geq 0$,

$$\begin{aligned} & \mathbb{P}\left(\|Z_m M_{m,n-1}\| \geq e^t\right) \\ & = \mathbb{P}\left(\left\|\frac{Z_m}{\|Z_m\|} M_{m,n-1}\right\| \|Z_m\| \geq e^t\right) \\ & = \mathbb{P}\left(\left\|\frac{Z_m}{\|Z_m\|} M_{m,n-1}\right\| \|Z_m\| \geq e^t, \|Z_m\| \leq e^u\right) \\ & \quad + \mathbb{P}\left(\left\|\frac{Z_m}{\|Z_m\|} M_{m,n-1}\right\| \|Z_m\| \geq e^t, \|Z_m\| > e^u\right) \\ & =: J_1(t, u) + J_2(t, u). \end{aligned} \tag{6.24}$$

We will prove that, with suitable $t = t(n)$ and $u = u(n)$, the term J_2 is small (see Lemma 6.4), and the term J_1 gives the main contribution.

Fix $s \in (I_\mu^+)^{\circ}$. Let $q = \Lambda'(s) > 0$. Let $\{\delta_n\}_{n \geq 1}$ be any sequence of real numbers satisfying

$$|\delta_n| \leq C \bar{\delta}_n, \quad \forall n \geq 1,$$

where $C > 0$ is a constant, $\{\bar{\delta}_n\}_{n \geq 1}$ is an arbitrary given sequence of real numbers such that $0 \leq \bar{\delta}_n = o(n^{-1/2})$. We will first focus on estimating $J_{2,n}$, we have the following lemma:

Lemma 6.4. *Under the hypotheses of Theorem 1.6, we have, with m defined in (6.23), as $n \rightarrow \infty$,*

$$J_{2,n} := \mathbb{P} \left(\|Z_m^i M_{m,n-1}\| \geq e^{n(q+\delta_n)}, \|Z_m^i\| > e^{mq+m^\beta} \right) = o \left(\frac{e^{-n\Lambda^*(q)-sn\delta_n}}{\sqrt{n}} \right). \quad (6.25)$$

Proof. Using

$$\left\{ \frac{\|Z_m^i\|}{e^{mq+m^\beta}} > 1 \right\} = \bigcup_{l=0}^{\infty} \left\{ \frac{\|Z_m^i\|}{e^{mq+m^\beta}} \in (e^l, e^{l+1}] \right\}, \quad (6.26)$$

and the independence of Z_m^i and $M_{m,n-1}$, we get

$$\begin{aligned} J_{2,n} &\leq \sum_{l \geq 0} \mathbb{P} \left(\left\| \frac{Z_m^i}{\|Z_m^i\|} M_{m,n-1} \right\| \geq \frac{e^{n(q+\delta_n)}}{\|Z_m^i\|}, \frac{\|Z_m^i\|}{e^{mq+m^\beta}} \in (e^l, e^{l+1}] \right) \\ &\leq \sum_{l \geq 0} \mathbb{P} \left(\left\| \frac{Z_m^i}{\|Z_m^i\|} M_{m,n-1} \right\| \geq e^{n(q+\delta_n)-mq-m^\beta-l-1}, \frac{\|Z_m^i\|}{e^{mq+m^\beta}} \in (e^l, e^{l+1}] \right) \\ &\leq \sum_{l \geq 0} \mathbb{P} \left(\|M_{m,n-1}\| \geq e^{n(q+\delta_n)-mq-m^\beta-l-1}, \frac{\|Z_m^i\|}{e^{mq+m^\beta}} \in (e^l, e^{l+1}] \right) \\ &\leq \sum_{l \geq 0} \mathbb{P} \left(\|M_{m,n-1}\| \geq e^{n(q+\delta_n)-mq-m^\beta-l-1} \right) \mathbb{P} \left(\|Z_m^i\| > e^{mq+m^\beta} e^l \right). \end{aligned} \quad (6.27)$$

Recall that the Fenchel-Legendre transform Λ^* of $\Lambda(s) = \log \kappa(s)$ satisfies, for $q = \Lambda'(s)$ with $s > 0$,

$$\Lambda^*(q) = sq - \Lambda(s).$$

Using the Markov inequality, the first term of (6.27) can be bounded as follows:

$$\begin{aligned} &\mathbb{P} \left(\|M_{m,n-1}\| \geq e^{n(q+\delta_n)-mq-m^\beta-l-1} \right) \\ &\leq \mathbb{E} \left[\|M_{m,n-1}\|^s \right] \times e^{-snq-sn\delta_n+msq+m^\beta s+s(l+1)} \\ &\leq c_s \kappa(s)^{n-m} \times e^{-snq-sn\delta_n+msq+m^\beta s+s(l+1)} \\ &= c_s e^{-n\Lambda^*(q)} e^{-\Lambda(s)m-sn\delta_n+msq+sm^\beta+sl}. \end{aligned} \quad (6.28)$$

By Lemma 6.2, there exist some constants δ_0 and $c_0 > 0$ such that for any $\alpha \in [s, s + \delta_0]$,

$$\mathbb{E}[\|Z_n^i\|^\alpha] \leq c_0 n^\alpha \kappa(\alpha)^n. \quad (6.29)$$

Let $\varepsilon \in (0, \delta_0)$ and recall the Taylor expansion: $\Lambda(s + \varepsilon) = \Lambda(s) + q\varepsilon + \frac{\varepsilon^2}{2} \Lambda''(\alpha)$, for some $\alpha \in [s, s + \varepsilon]$. Taking $c = c(s, \delta) = \max(\sup_{a \in [s, s+\delta]} \Lambda''(a), c_0)$, and using again Markov's inequality for the second term in (6.27), we get, for any $\varepsilon \in (0, \delta_0)$,

$$\begin{aligned} \mathbb{P} \left(\|Z_m^i\| > e^{qm+m^\beta} e^l \right) &\leq \mathbb{E}[\|Z_m^i\|^{s+\varepsilon}] e^{-q(s+\varepsilon)m-(s+\varepsilon)m^\beta-(s+\varepsilon)l} \\ &\leq c(s, \delta) m^{s+\delta} [\kappa(s + \varepsilon)]^m e^{-q(s+\varepsilon)m-(s+\varepsilon)m^\beta-(s+\varepsilon)l} \\ &= c(s, \delta) m^{s+\delta} e^{m\Lambda(s+\varepsilon)} e^{-q(s+\varepsilon)m-(s+\varepsilon)m^\beta-(s+\varepsilon)l} \\ &\leq c m^{s+\delta} \kappa(s)^m e^{cm\varepsilon^2} e^{-qsm-(s+\varepsilon)m^\beta-(s+\varepsilon)l}. \end{aligned} \quad (6.30)$$

Combining (6.28) and (6.30) we obtain

$$\begin{aligned} J_{2,n} &\leq \sum_{l \geq 0} c_s c m^{s+\delta} e^{-n\Lambda^*(q)} e^{-\Lambda(s)m - sn\delta_n + msq + sm^\beta + sl} \cdot \kappa(s)^m e^{cm\varepsilon^2} e^{-qsm - (s+\varepsilon)m^\beta} e^{-(s+\varepsilon)l} \\ &\leq c_s c m^{s+\delta} \frac{e^{-n\Lambda^*(q)}}{1 - e^{-\varepsilon}} e^{-sn\delta_n + cm\varepsilon^2 - \varepsilon m^\beta}. \end{aligned} \quad (6.31)$$

Taking $\varepsilon = (\log n)^{-1/2}$, we get

$$J_{2,n} = o\left(\frac{e^{-n\Lambda^*(q)} - sn\delta_n}{\sqrt{n}}\right).$$

□

We next handle the term

$$J_{1,n} := \mathbb{P}\left(\|Z_m^i M_{m,n-1}\| \geq e^{n(q+\delta_n)}, \|Z_m^i\| \leq e^{mq+m^\beta}\right). \quad (6.32)$$

Lemma 6.5. *Under the hypotheses of Theorem 1.6, we have, with m defined in (6.23), as $n \rightarrow \infty$,*

$$J_{1,n} \sim \frac{r_s(e_i)}{s\sigma_s\sqrt{2\pi n}} e^{-n\Lambda^*(q) - sn\delta_n} \left(\frac{\mathbb{E}\left(\bar{r}_s\left(\frac{Z_m^i}{\|Z_m^i\|}\right) \|Z_m^i\|^s\right)}{\kappa(s)^m r_s(e_i)} + o(1) \right). \quad (6.33)$$

Proof. Set $L_n = \lfloor e^{mq+m^\beta} \rfloor$, with m defined in (6.23). Using the independence of Z_m^i and $M_{m,n-1}$, we rewrite $J_{1,n}$ as follows:

$$\begin{aligned} J_{1,n} &= \sum_{z: 1 \leq \|z\| \leq L_n} \mathbb{P}\left(\|Z_m^i M_{m,n-1}\| \geq e^{n(q+\delta_n)}, Z_m^i = z\right) \\ &= \sum_{z: 1 \leq \|z\| \leq L_n} \mathbb{P}\left(\left\|\frac{z}{\|z\|} M_{m,n-1}\right\| \geq e^{n(q+\delta_n - \frac{\log\|z\|}{n})}\right) \mathbb{P}(Z_m^i = z) \\ &= \sum_{z: 1 \leq \|z\| \leq L_n} \mathbb{P}\left(\left\|\frac{z}{\|z\|} M_{m,n-1}\right\| \geq e^{(n-m)(q+\delta_n - \frac{\log\|z\|}{n}) + m(q+\delta_n - \frac{\log\|z\|}{n})}\right) \\ &\quad \times \mathbb{P}(Z_m^i = z) \\ &= \sum_{z: 1 \leq \|z\| \leq L_n} \mathbb{P}\left(\left\|\frac{z}{\|z\|} M_{m,n-1}\right\| \geq e^{(n-m)(q+l_n)}\right) \mathbb{P}(Z_m^i = z), \end{aligned} \quad (6.34)$$

where l_n is defined by

$$l_n = l_n(z) = \frac{mq}{n-m} + \frac{n\delta_n}{n-m} - \frac{\log\|z\|}{n-m}, \quad (6.35)$$

which satisfies

$$\sup_{\|z\| \in [1, L_n]} |l_n| = o(n^{-1/2}).$$

By the large deviation result from [74, Theorem 2.1 and Eq. (2.4)], we have,

$$\begin{aligned} &\mathbb{P}\left(\left\|\frac{z}{\|z\|} M_{m,n-1}\right\| \geq e^{(n-m)(q+l_n)}\right) \\ &= \bar{r}_s\left(\frac{z}{\|z\|}\right) \frac{1 + \varepsilon_{1,n}(z)}{s\sigma_s\sqrt{2\pi(n-m)}} \exp(-(n-m)\Lambda^*(q+l_n)), \end{aligned} \quad (6.36)$$

where $\bar{r}_s = \frac{r_s}{\nu_s(r_s)}$, $\sup_{\|z\| \geq 1} |\varepsilon_{1,n}(z)| \rightarrow 0$, and

$$\Lambda^*(q + l_n) = \Lambda^*(q) + sl_n + \frac{l_n^2}{2\sigma_s^2}(1 + \varepsilon_{2,n}(z)) \quad \text{with} \quad \sup_{\|z\| \in [1, L_n]} |\varepsilon_{2,n}(z)| \rightarrow 0, \quad (6.37)$$

uniformly in all sequences $(\delta_n)_{n \geq 1}$ such that $\delta_n \leq C\bar{\delta}_n$ for all $n \geq 1$ (see [74, Lemma 4.1]). As $\frac{n}{n-m} \rightarrow 1$ and $l_n^2(n-m) \rightarrow 0$, it follows that

$$\begin{aligned} & \mathbb{P} \left(\left\| \frac{z}{\|z\|} M_{m,n-1} \right\| \geq e^{(n-m)(q+l_n)} \right) \\ &= \bar{r}_s \left(\frac{z}{\|z\|} \right) \frac{1 + \varepsilon_{3,n}(z)}{s\sigma_s\sqrt{2\pi n}} \exp \left(- (n-m)(\Lambda^*(q) + sl_n) \right), \end{aligned} \quad (6.38)$$

where $\sup_{\|z\| \in [1, L_n]} |\varepsilon_{3,n}(z)| \rightarrow 0$. Remark that, with the definition of l_n , and the fact that $\Lambda^*(q) = sq - \Lambda(s)$, we have

$$(n-m)(\Lambda^*(q) + sl_n) = n(\Lambda^*(q) + s\delta_n) + m\Lambda(s) - s \log \|z\|. \quad (6.39)$$

From (6.34), (6.38) and (6.39) we deduce that the term $J_{1,n}$ defined by (6.32) satisfies

$$J_{1,n} = \sum_{z: 1 \leq \|z\| \leq L_n} \frac{1 + \varepsilon_{5,n}(z)}{s\sigma_s\sqrt{2\pi n}} e^{-n\Lambda^*(q) - sn\delta_n - m\Lambda(s)} \bar{r}_s \left(\frac{z}{\|z\|} \right) \|z\|^s \mathbb{P} \left(Z_m^i = z \right),$$

where $\sup_{\|z\| \in [1, L_n]} |\varepsilon_{5,n}(z)| \rightarrow 0$. Therefore for any $\varepsilon \in (0, 1)$, there is $n_0 = n_0(\varepsilon) > 0$ such that, for all $n \geq n_0$, $J_{1,n}$ lies between $(1 \pm \varepsilon)J'_{1,n}$, where

$$J'_{1,n} = \frac{1}{s\sigma_s\sqrt{2\pi n}} e^{-n\Lambda^*(q) - sn\delta_n} \kappa(s)^{-m} \mathbb{E} \left(\bar{r}_s \left(\frac{Z_m^i}{\|Z_m^i\|} \right) \|Z_m^i\|^s \mathbf{1}_{\{1 \leq \|Z_m^i\| \leq L_n\}} \right). \quad (6.40)$$

For the last expectation, we have

$$\begin{aligned} & \mathbb{E} \left(\bar{r}_s \left(\frac{Z_m^i}{\|Z_m^i\|} \right) \|Z_m^i\|^s \mathbf{1}_{\{1 \leq \|Z_m^i\| \leq L_n\}} \right) \\ &= \mathbb{E} \left(\bar{r}_s \left(\frac{Z_m^i}{\|Z_m^i\|} \right) \|Z_m^i\|^s \right) - \mathbb{E} \left(\bar{r}_s \left(\frac{Z_m^i}{\|Z_m^i\|} \right) \|Z_m^i\|^s \mathbf{1}_{\{\|Z_m^i\| > L_n\}} \right), \end{aligned} \quad (6.41)$$

where the second term is $o(\kappa(s)^m)$, as will be seen in Lemma 6.6 below. Therefore, (6.40) and (6.41) imply the desired result (6.33). \square

Lemma 6.6. *Assume hypotheses of Theorem 1.6. The following assertion holds, with m defined in (6.23) and as $n \rightarrow \infty$:*

$$\mathbb{E} \left(\|Z_m^i\|^s \mathbf{1}_{\{\|Z_m^i\| > L_n\}} \right) = o(\kappa(s)^m). \quad (6.42)$$

Proof. Using again the decomposition (6.26), we get

$$\begin{aligned} & \mathbb{E} \left(\|Z_m^i\|^s \mathbf{1}_{\{\|Z_m^i\| > e^{qm+m^\beta}\}} \right) \\ & \leq \sum_{l \geq 0} \mathbb{E} \left(\|Z_m^i\|^s \mathbf{1}_{\{e^{qm+m^\beta} e^l < \|Z_m^i\| \leq e^{qm+m^\beta} e^{l+1}\}} \right) \\ & \leq \sum_{l \geq 0} e^{sqm+sm^\beta} e^{s(l+1)} \mathbb{P} \left(\|Z_m^i\| > e^{qm+m^\beta} e^l \right). \end{aligned} \quad (6.43)$$

Combining (6.30) and (6.43) we know that there is a constant $c = c(s, \delta)$ such that for any $\varepsilon \in (0, \delta)$,

$$\begin{aligned}
& \mathbb{E} \left(\|Z_m^i\|^s \mathbf{1} \left\{ \|Z_m^i\| > e^{qm+mb^\beta} \right\} \right) \\
& \leq \sum_{l \geq 0} e^{sqm+sm^\beta} e^{s(l+1)} \cdot cm^{s+\delta} \kappa(s)^m e^{cm\varepsilon^2} e^{-qsm-(s+\varepsilon)m^\beta} e^{-(s+\varepsilon)l} \\
& = \sum_{l \geq 0} e^{-\varepsilon l} cm^{s+\delta} \kappa(s)^m e^{cm\varepsilon^2 - \varepsilon m^\beta} \\
& = \frac{1}{1 - e^{-\varepsilon}} cm^{s+\delta} \kappa(s)^m e^{cm\varepsilon^2 - \varepsilon m^\beta}.
\end{aligned} \tag{6.44}$$

Taking $\varepsilon = (\log n)^{-\eta}$ with $\eta > 1 - \beta$, we obtain (6.42). \square

We now prove that the expectation in (6.33) converges to a constant.

Lemma 6.7. *Assume conditions of Theorem 1.6. Then*

$$C_Z := \lim_{n \rightarrow \infty} \frac{\mathbb{E} \left(\bar{r}_s \left(\frac{Z_n^i}{\|Z_n^i\|} \right) \|Z_n^i\|^s \right)}{\kappa(s)^n r_s(e_i)} = \frac{1}{\nu_s(r_s)} \mathbb{E}_{\mathbb{Q}_s^{e_i}} (W^i)^s \in (0, \infty). \tag{6.45}$$

To prove Lemma 6.7, we need the following technical result:

Lemma 6.8.

(1) *For any $\delta > 0, \eta \in (0, 1)$, there is a constant $C = C(\delta, \eta) > 0$ such that for any \mathbb{R}_+ -valued random variable,*

$$E(X \log^+ X) \leq C (EX)^{1-\eta} (E(X^{1+\delta}))^\eta. \tag{6.46}$$

(2) *Assume A1. When $0 < s < 1$, the condition (1.34) implies condition H2. When $s \geq 1$, the condition (1.35) also implies condition H2.*

Proof. (1) By Hölder's inequality $E|X_1 X_2| \leq (E|X_1|^p)^{1/p} (E|X_2|^q)^{1/q}$ with $p = 1/(1 - \eta)$ and $q = 1/\eta$, and the elementary inequality $(\log^+ x) \leq Cx^{\delta\eta}$ for some constant $C = C(\delta, \eta)$ and all $x \geq 0$, we get

$$\begin{aligned}
E(X \log^+ X) &= E(X^{1-\eta} \cdot X^\eta \log^+ X) \leq (EX)^{1-\eta} (E(X(\log^+ X)^{1/\eta}))^\eta \\
&\leq C (EX)^{1-\eta} (E(X^{1+\delta}))^\eta.
\end{aligned} \tag{6.47}$$

(2) Using (1) for $X = \frac{Z_1^i(j)}{M_0(i,j)}$ under the law \mathbb{P}_ξ , we obtain for all $1 \leq i, j \leq d$,

$$\mathbb{E}_\xi \left[\frac{Z_1^i(j)}{M_0(i,j)} \log^+ \frac{Z_1^i(j)}{M_0(i,j)} \right] \leq C \left[\mathbb{E}_\xi \left(\frac{Z_1^i(j)}{M_0(i,j)} \right)^{1+\delta} \right]^\eta, \tag{6.48}$$

Taking expectation at both sides, we get that for all $1 \leq i, j \leq d$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\frac{Z_1^i(j)}{M_0(i,j)} \log^+ \frac{Z_1^i(j)}{M_0(i,j)} \right) &\leq C \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\mathbb{E}_\xi \left(\frac{Z_1^i(j)}{M_0(i,j)} \right)^{1+\delta} \right]^\eta \\
&\leq CC_s \mathbb{E} \left[M_0(i,j)^{s-(1+\delta)\eta} \left(\mathbb{E}_\xi (Z_1^i(j))^{1+\delta} \right)^\eta \right],
\end{aligned} \tag{6.49}$$

where C_s is a constant, and the last step holds by A1.

When $0 < s \leq 1$, taking $\eta = \frac{s}{1+\delta}$, we obtain that for all $1 \leq i, j \leq d$,

$$\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\frac{Z_1^i(j)}{M_0(i,j)} \log^+ \frac{Z_1^i(j)}{M_0(i,j)} \right) \leq CC_s \mathbb{E}[(\mathbb{E}_\xi(Z_1^i(j))^{1+\delta})^{\frac{s}{1+\delta}}], \quad (6.50)$$

So when $s \in (0, 1)$, (1.34) implies **H2**, and when $s = 1$, (1.35) also implies **H2**.

When $s > 1$, then (1.35) implies $\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\frac{Z_1^i(j)}{M_0(i,j)} \right)^s < \infty$, so that **H2** holds. \square

Proof of Lemma 6.7. By the definition of the measure $\mathbb{Q}_s^{e_i}$, we have

$$\begin{aligned} \mathbb{E} \left(\bar{r}_s \left(\frac{Z_n^i}{\|Z_n^i\|} \right) \|Z_n^i\|^s \right) &= \mathbb{E} \left(\bar{r}_s \left(\frac{Z_n^i}{\|Z_n^i\|} \right) \left(\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} \right)^s \|\mathbb{E}_\xi Z_n^i\|^s \right) \\ &= \mathbb{E} \left(\bar{r}_s \left(\frac{Z_n^i}{\|Z_n^i\|} \right) \left(\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} \right)^s e^{s \log \|e_i M_{0,n-1}\|} \right) \\ &= \kappa(s)^n r_s(e_i) \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\frac{\bar{r}_s \left(\frac{Z_n^i}{\|Z_n^i\|} \right)}{r_s(e_i M_{0,n-1})} \left(\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} \right)^s \right). \end{aligned}$$

Dividing by $\kappa(s)^n r_s(e_i)$ we get

$$\frac{\mathbb{E} \left(\bar{r}_s \left(\frac{Z_n^i}{\|Z_n^i\|} \right) \|Z_n^i\|^s \right)}{\kappa(s)^n r_s(e_i)} = \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\frac{\bar{r}_s \left(\frac{Z_n^i}{\|Z_n^i\|} \right)}{r_s(e_i \cdot M_{0,n-1})} \left(\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} \right)^s \right). \quad (6.51)$$

We will use Theorem 4.19 to prove that

$$\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} \rightarrow W^i \text{ in } L^s(\mathbb{Q}_s^{e_i}) \text{ when } s > 1, \text{ and in } L^1(\mathbb{Q}_s^{e_i}) \text{ when } s \in (0, 1]. \quad (6.52)$$

To this end we just need to prove that the conditions of Theorem 4.19 are satisfied under those of Theorem 1.6. This is easily seen by Lemma 6.8. Hence, $\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} \xrightarrow{\mathbb{Q}_s^{e_i}} W^i$ in probability as $n \rightarrow +\infty$. Therefore, the convergence of $\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|}$ in $L^s(\mathbb{Q}_s^{e_i})$ is equivalent to the uniform integrability of $\left(\left(\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} \right)^s \right)$, so that $\left(\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} \right)^s$ converges in $L^1(\mathbb{Q}_s^{e_i})$:

$$\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left| \left(\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} \right)^s - (W^i)^s \right| \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (6.53)$$

Using (6.53) and the fact that the function r_s lies between two positive constants, from (6.51) we get

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E} \left(\bar{r}_s \left(\frac{Z_n^i}{\|Z_n^i\|} \right) \|Z_n^i\|^s \right)}{\kappa(s)^n r_s(e_i)} = \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\frac{\bar{r}_s \left(\frac{Z_n^i}{\|Z_n^i\|} \right)}{r_s(e_i \cdot M_{0,n-1})} (W^i)^s \right), \quad (6.54)$$

provided that the last limit exists. The existence of this limit will be established below. We now prove that in the last expression, $\bar{r}_s \left(\frac{Z_n^i}{\|Z_n^i\|} \right) / r_s(e_i \cdot M_{0,n-1})$ can be replaced by $\bar{r}_s(v_{n-1}) / r_s(v_{n-1})$. Using the identity $\frac{a}{b} - \frac{c}{d} = \frac{(a-c)d + c(d-b)}{bd}$ and the fact that r_s and \bar{r}_s lie

between two positive constants, we see that for some constant $c = c(s) > 0$,

$$\begin{aligned} & \left| \frac{\bar{r}_s\left(\frac{Z_n^i}{\|Z_n^i\|}\right)}{r_s(e_i \cdot M_{0,n-1})} - \frac{\bar{r}_s(v_{n-1})}{r_s(v_{n-1})} \right| \\ & \leq c \left(\left| \bar{r}_s\left(\frac{Z_n^i}{\|Z_n^i\|}\right) - \bar{r}_s(v_{n-1}) \right| + |r_s(v_{n-1}) - r_s(e_i \cdot M_{0,n-1})| \right). \end{aligned} \quad (6.55)$$

Therefore, for some constant $c = c(s)$ whose value can change from line to line,

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left| \frac{\bar{r}_s\left(\frac{Z_n^i}{\|Z_n^i\|}\right)}{r_s(e_i \cdot M_{0,n-1})} - \frac{\bar{r}_s(v_{n-1})}{r_s(v_{n-1})} \right| (W^i)^s \\ & \leq c \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\left| \bar{r}_s\left(\frac{Z_n^i}{\|Z_n^i\|}\right) - \bar{r}_s(v_{n-1}) \right| (W^i)^s \right] + c \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[|r_s(e_i \cdot u_0^T v_{n-1}) - r_s(e_i \cdot M_{0,n-1})| (W^i)^s \right] \\ & \leq c \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\left\| \frac{Z_n^i}{\|Z_n^i\|} - v_{n-1} \right\|^{\bar{s}} (W^i)^s \right] + c \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\|v_{n-1} - e_i \cdot M_{0,n-1}\|^{\bar{s}} (W^i)^s \right] \\ & \leq c \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\left\| \frac{Z_n^i}{\|Z_n^i\|} - v_{n-1} \right\|^{\bar{s}} (W^i)^s \mathbf{1}_{(W^i > 0)} \right] + c \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left[\|v_{n-1} - e_i \cdot M_{0,n-1}\|^{\bar{s}} (W^i)^s \right], \end{aligned} \quad (6.56)$$

where the last inequality holds since r_s and \bar{r}_s are \bar{s} -Holder continuous with respect to the the L^1 norm $\|\cdot\|$ (by Lemma 2.2). By Theorem 4.12 we see that for all $\varepsilon > 0$, as $n \rightarrow +\infty$,

$$\mathbb{Q}_s^{e_i} \left(\left\| \frac{Z_n^i}{\|Z_n^i\|} - v_{n-1} \right\| > \varepsilon, W^i > 0 \right) \rightarrow 0. \quad (6.57)$$

We now also see that by (3.48), under the changed measure $\mathbb{Q}_s^{e_i}$,

$$e_i \cdot M_{0,n-1} - v_{n-1} \xrightarrow[n \rightarrow +\infty]{} 0 \quad \mathbb{Q}_s^{e_i} \text{ a.s.} \quad (6.58)$$

Moreover, we know that,

$$\mathbb{E}_{\mathbb{Q}_s^{e_i}} (W^i)^s < \infty. \quad (6.59)$$

Combining (6.56)-(6.59), and using the dominated convergence theorem, we get

$$\mathbb{E}_{\mathbb{Q}_s^{e_i}} \left| \frac{\bar{r}_s\left(\frac{Z_n^i}{\|Z_n^i\|}\right)}{r_s(e_i \cdot M_{0,n-1})} - \frac{\bar{r}_s(v_{n-1})}{r_s(v_{n-1})} \right| (W^i)^s \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (6.60)$$

Therefore we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\frac{\bar{r}_s\left(\frac{Z_n^i}{\|Z_n^i\|}\right)}{r_s(e_i \cdot M_{0,n-1})} (W^i)^s \right) &= \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\frac{\bar{r}_s(v_{n-1})}{r_s(v_{n-1})} (W^i)^s \right) \\ &= \frac{1}{\nu_s(r_s)} \mathbb{E}_{\mathbb{Q}_s^{e_i}} (W^i)^s. \end{aligned} \quad (6.61)$$

the second equality holds by Theorem 3.7. With (6.54), this implies (6.45). \square

We are now ready to finish the proof Theorem 1.6 on the precise large deviation of $\|Z_n^i\|$.

Proof of Theorem 1.6. We begin with the decomposition (6.1). Notice that for any (δ_n) with $\delta_n \leq C\bar{\delta}_n$ for all $n \geq 1$, and any $\varepsilon \in (0, 1]$, we have

$$\mathbb{P}[\|Z_n^i\| \geq e^{nq+n\delta_n}] \leq \mathbb{P}[\|Z_m^i M_{m,n-1}\| \geq e^{nq+n\delta_n-\varepsilon}] + \mathbb{P}[\|D_{m,n}\| \geq e^{nq+n\delta_n}(1-e^{-\varepsilon})]. \quad (6.62)$$

We first deal with the second term in the right-hand side. From Lemma 6.3, choosing $t = \delta e^{nq+n\delta_n}$ with $\delta > 0$, and setting $\eta := \frac{\theta}{\kappa(s)} < 1$, we obtain, with $m = K \lfloor \log n \rfloor$ and $K > \frac{1}{-2 \log \eta}$,

$$\begin{aligned} \mathbb{P}[\|D_{m,n}\| \geq \delta e^{nq+n\delta_n}] &\leq c\delta^{-s}\eta^m e^{-n\Lambda^*(q)-sn\delta_n} \\ &= \frac{\delta^{-s} e^{-n\Lambda^*(q)-sn\delta_n}}{\sqrt{n}} o(1). \end{aligned} \quad (6.63)$$

We next deal with the first term in the right-hand side of (6.62). From (6.33) in Lemma 6.5, and Lemma 6.7, we know that the term $J_{1,n}$ defined by (6.32) satisfies

$$J_{1,n} = \frac{C_Z r_s(e_i)}{s\sigma_s \sqrt{2\pi n}} e^{-n\Lambda^*(q)-sn\delta_n} (1 + o(1)) \quad \text{as } n \rightarrow \infty. \quad (6.64)$$

For the estimation of $J_{2,n}$ defined by (6.25), combining Lemma 6.4 and (6.64), we obtain that, for any constant $C > 0$, uniformly in (δ_n) with $\delta_n \leq C\bar{\delta}_n$ for all $n \geq 1$,

$$\mathbb{P}\left(\left\|\left\|Z_m^i M_{m,n-1}\right\|\right\| \geq e^{n(q+\delta_n)}\right) = \frac{C_Z r_s(e_i)}{s\sigma_s \sqrt{2\pi n}} e^{-n\Lambda^*(q)-sn\delta_n} (1 + o(1)) \quad \text{as } n \rightarrow \infty. \quad (6.65)$$

From (6.65) (applied to $\delta'_n = \delta_n - \varepsilon/n$), and to the constant $C + C_0$, where C_0 satisfies $1/n \leq C_0\bar{\delta}_n$ for all $n \geq 1$), we see that for any constant $C > 0$, uniformly in $\varepsilon \in (0, 1]$ and (δ_n) with $\delta_n \leq C\bar{\delta}_n$ for all $n \geq 1$, as $n \rightarrow \infty$,

$$\mathbb{P}[\|Z_m^i M_{m,n-1}\| \geq e^{nq+n\delta_n-\varepsilon}] = \frac{C_Z r_s(e_i) e^{s\varepsilon}}{s\sigma_s \sqrt{2\pi n}} e^{-n\Lambda^*(q)-sn\delta_n} (1 + o(1)). \quad (6.66)$$

We now prove the upper bound of the desired large deviation probability. Combining (6.62), (6.63) and (6.66) yields that for any $\varepsilon' > 0$, there is $n_0 = n_0(\varepsilon')$ (which may depend also on the given constant $C > 0$ and the initial sequence $(\bar{\delta}_n)$, such that for all $n \geq n_0$ and any $\varepsilon \in (0, 1]$,

$$\sqrt{n} e^{n\Lambda^*(q)} e^{sn\delta_n} \mathbb{P}[\|Z_n^i\| \geq e^{nq+n\delta_n}] \leq \frac{C_Z r_s(e_i)}{s\sigma_s \sqrt{2\pi}} e^{s\varepsilon} (1 + \varepsilon') + (1 - e^{-\varepsilon})^{-s} \varepsilon'. \quad (6.67)$$

For the lower bound, we can use a similar argument: using the inequality that for any $\varepsilon \in (0, 1]$,

$$\mathbb{P}[\|Z_n^i\| \geq e^{nq+n\delta_n}] \geq \mathbb{P}[\|Z_m^i M_m \cdots M_{n-1}\| \geq e^{nq+n\delta_n+\varepsilon}] - \mathbb{P}[\|D_{m,n}\| \geq e^{nq+n\delta_n}(e^\varepsilon - 1)].$$

we conclude that, for any $\varepsilon' > 0$, there is $n_0 = n_0(\varepsilon')$ such that for all $n \geq n_0$ and any $\varepsilon \in (0, 1]$,

$$\sqrt{n} e^{n\Lambda^*(q)} e^{sn\delta_n} \mathbb{P}[\|Z_n^i\| \geq e^{nq+n\delta_n}] \geq \frac{C_Z r_s(e_i)}{s\sigma_s \sqrt{2\pi}} e^{-s\varepsilon} (1 - \varepsilon') - (e^\varepsilon - 1)^{-s} \varepsilon'. \quad (6.68)$$

As $\varepsilon \in (0, 1]$ and $\varepsilon' > 0$ are arbitrary, from (6.67) and (6.68), we conclude that, uniformly in (δ_n) with $\delta_n \leq C\bar{\delta}_n$ for all $n \geq 1$, as $n \rightarrow \infty$,

$$\begin{aligned} \mathbb{P}[\|Z_n^i\| \geq e^{nq+n\delta_n}] &\sim \frac{C_Z r_s(e_i)}{s\sigma_s \sqrt{2\pi n}} e^{-n\Lambda^*(q) - s n \delta_n} \\ &= \frac{C(s)}{\sqrt{n}} e^{-n\Lambda^*(q)}, \end{aligned} \quad (6.69)$$

where, by Lemma 6.7, the constant $C(s) \in (0, \infty)$ has the expression

$$\begin{aligned} C(s) &= \frac{r_s(e_i)}{s\sigma_s \sqrt{2\pi}} \lim_{m \rightarrow \infty} \frac{\mathbb{E} \left(\bar{r}_s \left(\frac{Z_m^i}{\|Z_m^i\|} \right) \|Z_m^i\|^s \right)}{\kappa(s)^m r_s(e_i)} \\ &= \frac{r_s(e_i)}{s\sigma_s \nu_s(r_s) \sqrt{2\pi}} \mathbb{E}_{\mathbb{Q}_s^{e_i}} (W^i)^s. \end{aligned} \quad (6.70)$$

□

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