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Máire Ní Ríordáin, Kirstin Erath

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Máire Ní Ríordáin, Kirstin Erath. Proceedings of the Sixteenth ERME Topic Conference on Language and Social Interaction in Heterogeneous Mathematics Classrooms. Sixteenth ERME Topic Conference on Language and Social Interaction in Heterogeneous Mathematics Classrooms, 2024, 978-1-7384754-1-4. hal-04833321

HAL Id: hal-04833321

<https://hal.science/hal-04833321v1>

Submitted on 12 Dec 2024

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Sixteenth ERME Topic Conference on Language and Social Interaction in Heterogeneous Mathematics Classrooms



Editors: Máire Ní Ríordáin, Kirstin Erath

**Martin Luther University Halle-Wittenberg, Institute for
Mathematics, Germany**

Date of Publication: December 2024



Editors

Máire Ní Ríordáin, University College Cork, Ireland

Kirstin Erath, Martin Luther University Halle-Wittenberg, Germany

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Publisher:

ERME / HAL Archive

ISBN: 978-1-7384754-1-4

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Recommended citation:

Surname, F. (2024). Title of paper. In M. Ní Ríordáin, and K. Erath (Eds.), *Proceedings of the Sixteenth ERME Topic Conference on Language and Social Interaction in Mathematics Classrooms* (pp. xx– yy). ERME / HAL Archive.

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Editorial ETC16 Language and Social Interaction in Heterogeneous Classrooms

Máire Ní Ríordáin¹ and Kirstin Erath²

¹University College Cork, Ireland; maire.niriordain@ucc.ie

²Martin Luther University Halle-Wittenberg, Germany; kirstin.erath@mathematik.uni-halle.de

Introduction

The ETC16 conference, titled “Language and Social Interaction in Heterogeneous Mathematics Classrooms,” took place at Martin Luther University in Halle, Germany, in 2024. This event marked the fourth topic conference of the Mathematics and Language Thematic Working Group (TWG) of ERME, following previous conferences in Dresden, Germany (ETC4, 2018), Montpellier, France (ETC7, 2020), and Oxford, UK (ETC12, 2022). These conferences consistently attract researchers from the ERME community and beyond who are dedicated to exploring various dimensions of language in mathematics education. Key areas of focus include, but are not limited to, interaction processes, multilingualism and language as a resource for teaching and learning, the development of academic language, and classroom discursive practices. The 2024 conference was hosted by Kirstin Erath, reflecting her longstanding commitment to advancing research on language in mathematics education, particularly through her contributions to TWG9, “Mathematics and Language,” now “Interaction, Communication and Language in Mathematics Education”, within the CERME conference series.

The International Program Committee (IPC) for ETC16 comprised the following members: Máire Ní Riordáin (Ireland, Chair), Kirstin Erath (Germany, Chair of the Local Organizing Committee), Marie Therese Farrugia (Malta), Ingólfur Gíslason (Iceland), Jenni Ingram (UK), Núria Planas (Spain), Frode Rønning (Norway), Dilan Şahin-Gür (Germany), and Alexander Schüler-Meyer (Netherlands). The local organizing team included Kirstin Erath, Anna-Marietha Vogler, Ronny Becker, Jessica Mähner, Christoph Aumann, and Maximilian Büttner. The Early Career Researcher Day was organised by Dilan Şahin-Gür, with additional support from Jessica Mähner. We extend our gratitude to both the IPC and the local organizing team for their efforts in bringing this conference to fruition. Furthermore, we sincerely thank the ERME board and its president, Carl Winsløw, for their approval and support for this event.

The primary purpose of ETC16, "Language and Social Interaction in Heterogeneous Mathematics Classrooms," is to advance the understanding of how language and social interaction shape the teaching and learning of mathematics in diverse and multilingual educational settings. The conference builds upon a strong tradition of ERME Topic Conferences and seeks to provide a focused platform for researchers, educators, and teacher educators to explore the multifaceted role of language as a resource in mathematics classrooms. By addressing themes such as social interaction and discourse, multilingualism, and the integration of language considerations in professional development and teacher education, ETC16 aimed to foster a deeper academic dialogue and collaboration. It emphasized the importance of viewing heterogeneity in classroom settings not as a challenge to

overcome but as a valuable resource for enhancing educational outcomes. Additionally, the conference was committed to supporting early career researchers by providing opportunities for engagement, networking, and scholarly contribution. With a blend of plenary sessions, paper presentations, thematic discussions, and interactive formats like poster presentations, ETC16 aimed to facilitate the exchange of ideas, critical reflection, and the co-creation of knowledge, ensuring that the field of mathematics education continues to evolve in ways that are inclusive and responsive to the complexities of linguistic and cultural diversity. With these developments in mind, the subthemes (ST) of ETC16 were identified to be:

- ST1:** Teaching and learning mathematics with a focus on social interaction and discourse;
- ST2:** Various languages and language varieties as resources for teaching and learning mathematics;
- ST3:** Professional Development and Initial Teacher Education with a focus on language or social interaction in heterogeneous classrooms

Altogether, we received 25 submissions (three design poster, six posters, and 16 paper proposals). After reviewing the submissions, the IPC noticed that there were a small number of papers on professional development and initial teacher education but more on mathematics, language and interaction in early years / kindergarten. Responding to this observation, the IPC decided to refocus the third subtheme but at the same time to connect the submitted papers on professional development and initial teacher education by arranging presentations in the same slot. The survey of the accepted paper, poster, and design poster shows a much-appreciated diversity towards age, mathematical topic, (country) contexts, theories, methods, and research interest, all connected by the overarching conference theme “Language and Social Interaction in Heterogeneous Mathematics Classrooms”.

Developments in Research on Language in the Mathematics Classroom

ETC16 reflects the ongoing evolution of research themes and theoretical approaches established in prior conferences, particularly ETC12. Building on the groundwork laid in Oxford (Schüler-Meyer, Ingram, & Erath, 2023), where teacher education and language-responsive pedagogy emerged as prominent themes, ETC16 continues to extend the scope and depth of inquiry into language's role in mathematics education. At ETC12, research emphasized equipping teachers to notice and support student discourse practices and literacy in mathematics classrooms (Prediger, Pöhler-Friedrich, & Şahin-Gür, 2023). This focus has deepened at ETC16 through diverse studies addressing multilingualism and discourse as resources for mathematical understanding.

Notably, ETC16 demonstrates a progression from investigating teacher noticing to actively integrating linguistic diversity into classroom practices. While ETC12 highlighted theoretical advancements and practical strategies for fostering teacher expertise in language-responsive teaching (Seago, Neumeyer DePiper, & Knotts, 2023), ETC16 expands this lens to consider how translanguaging, multilingual resources, and classroom discourse strategies contribute to equitable and effective teaching. Furthermore, ETC16 reflects increased theoretical diversity, embracing frameworks that address the intersectionality of language, culture, and cognition in mathematics education.

The keynote speakers at ETC16 made significant contributions to advancing the understanding of language-responsive practices in mathematics education within diverse and multilingual contexts. Dr.

Máire Ní Ríordáin's keynote highlighted the potential of language-responsive mathematics teaching to support learners with additional educational needs in a content and language integrated learning context. Through an innovative case study in Irish medium education, her research shed light on the pedagogical strategies needed to cater to diverse learners, emphasizing the integration of mathematical content and language development. Prof. Núria Planas's keynote explored the role of translation and translanguaging moves in mathematics teaching talk, drawing on rich data from bilingual classrooms in Barcelona. Her work provided actionable insights into how linguistically responsive teaching practices can enhance mathematical reasoning and discourse, while advocating for translanguaging as a critical framework in multilingual education. Together, these keynotes underscored the transformative potential of language-responsive approaches in fostering equity and inclusivity in mathematics classrooms, inspiring further research and practice.

Even though not focusing on professional development or initial teacher education directly, many contributions of ETC16 also highlight the growing emphasis on supporting teacher educators to integrate research findings into professional development. This builds on ETC12's foundational efforts to explore language as both a challenge and a resource in mathematics classrooms (Schüler-Meyer, Ingram, & Erath, 2023), solidifying the trajectory toward developing robust, theory-informed practices sensitive to the complexities of linguistic diversity.

We are deeply grateful for the contributions of all participants, whose research, reflections and constructive reviews have enriched our understanding of these critical issues. Their work ensures that the Mathematics and Language TWG remains at the forefront of addressing the intricate interplay between language and mathematics learning in increasingly diverse educational contexts.

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Cognitive roles' characterisation of the Inside-Out Model in Habermas' rationality perspective: the cases of Pest and Blogger

Giovannina Albano¹, Cristina Coppola¹ and Angela Donatiello¹

¹University of Salerno, Italy; galbano@unisa.it; ccoppola@unisa.it; adonatiello@unisa.it

In this paper we would like to give a characterisation of the cognitive roles played during a problem-solving activity in algebra in terms of rationality, both on the plane of knowledge and meta-knowledge. We analyse the online discursive interactions emerging in data of a pilot study, involving 26 students attending 10th grade, using the lens of the new characterisation, taking into account the Habermas' rationality perspective and the Balacheff's viewpoint about the social dimension of proof.

Keywords: Rationality, problem solving, discursive interactions, cognitive roles.

Introduction

This paper aims to give a characterisation of the cognitive roles devised by Albano, Coppola and Dello Iacono (2021) in terms of Habermas rationalities. In that work, the authors identified five key functions that are active in the mind of a mathematician engaged in a problem-solving activity. Such functions became embodied characters of a digital storytelling, allowing the students to play the story as its characters acting according to the cognitive roles. This study aims to provide an *a priori* characterisation of cognitive roles, based on Habermas' rationality construct and the Rational Mathematical Template (RMT) model (Boero & Azrou, 2023) and so to bring to light the emerging rationalities in the role-playing of the Inside-Out model, according to Habermas' rationality construct. Here we focus only on two cognitive roles, Pest and Blogger, that are strictly connected to communicative rationality. Subsequently, we would like to present a preliminary analysis of a pilot experiment, using such an analysis lens that can serve to further refine the *a priori* analysis made. In this frame, we wonder:

RQ1: How do emerging rationalities in the proving process allow for the characterisation of cognitive roles with particular reference to Pest and Blogger?

RQ2: How does such characterisation allow the recognition of the different rationalities in discursive interactions in problem solving situations?

Conceptual background

The 'inside-out' cognitive roles

According to the so-called 'inside-out' model (Albano, et al. 2021), we refer to various cognitive functions that come into play in problem solving. Each of them corresponds to a character's role played by each student during a collaborative problem-solving activity in a digital story-problem:

- Promoter, i.e. prospecting function, is in charge of the exploration phase needed periodically when questions arise for continuing the problem-solving process;
- Boss, i.e. planning function, is devoted to keep the thread of the resolution process with respect to the previous outcomes and the goal to be achieved; moreover, she takes care of the group engagement and collaborative mood;
- Blogger, i.e. publishing function, is in charge of sharing the resolution process and outcomes with other, according to the standards of the mathematical community discourse;
- Pest, i.e. validating function, responsible for the validation of the group's work by means of continuous checks of the ongoing solving process and partial outcomes;

- Guru, i.e. knowledge and wisdom function, represents the external resources we can turn outside.

Each person should experience within herself all the above cognitive roles during problem solving. In order to internalise these, students are required to experience all of them, at two different levels: as actors or as onlookers.

Communication and Proof

The occurrence of internal functions of the Inside-Out Model recalls the construct identified by Habermas as Reflexive Character (Habermas, 1998, p.310). Reflection with oneself necessarily develops in a dialogical form, seeing oneself as another than oneself, in the form of an inner discourse constituted within the communicative act. This approach can be considered as foundational to the structure of the Inside-Out model, as the roles represent parts of one and the same person, who thus constructs an inner discourse of a dialogical type, fostering self-reflection. Indeed, Habermas considers argumentative praxis as “a reflexive form of communicative action” (Habermas, 1998, p. 311). This perspective sets the conditions for a cognitive approach to reflection, that is, as a move away from the egocentric position towards welcoming other’s perspective, an act that Habermas recognises as an intrinsic structure of the argumentative process. From the perspective of Habermas (1998), an opinion is rational when it is possible to motivate it in a given context; an action is rational when it is aim-oriented and is achieved by means that are intentionally chosen and implemented; a communication is rational when the use of language is oriented towards understanding, i.e. the illocutionary aim of making the interlocutor understand one’s own communication content is achieved by consciously choosing the means to make communication effective. The three components of rationality -epistemic, teleological and communicative- are closely interconnected and, in a learning interaction aimed at producing a mathematical proof, enable students to develop a communication space in which there is a common understanding-oriented horizon, as also clarified in the following reference.

Communicative rationality is expressed in the unifying force of speech oriented towards understanding, which secures for the participating speakers an intersubjectively shared lifeworld, thereby securing at the same time the horizon within which everyone can refer to one and the same objective world. (Habermas, 1998, p. 315)

In the act of producing a mathematical proof, the communicative action of the Inside-Out Model takes place first as communication with oneself, then as internal communication within a group that develops its own language, intermediate between the everyday language and the formalised language of mathematics, and finally as external communication, towards the Culture. In this perspective we can find Balacheff’s social dimension of proof. Balacheff makes a distinction between explanation, proof and mathematical proof. The explanation is seen as research of certainty and clarification, making the truth of a proposition understandable to an interlocutor, even to oneself, using everyday language (Balacheff, 2010). This inner discourse fosters the proving process. The transition from explanation to proof refers to a social process in which the discourse that warrants the validity of a statement is accepted by a restricted community, which uses its own language. The status of a proof is not definitive but can evolve over time and its aim is research for understanding. The mathematical proof is a highly codified discourse for communication organised according to specific rules, accepted by the Culture (Balacheff, 2010). From the point of view of the epistemic rationality, Urhan and

Bülbul (2023a; 2023b) consider the categorisation made by Boero and Morselli (2010) who subdivide the epistemic component into two sub-components or types of requirements, modelling requirements (MR) and systemic requirements (SR). The SR sub-component concerns the correct handling of representations within the same semiotic system, whereas the MR component investigates the coherence between the different representations used and the problem posed, as well as the accuracy of the theoretical justifications given by the students, relating to the individual steps of the proving or problem-solving process and the conversion between different semiotic systems. In the $ck\phi$ model, Balacheff (2010) describes the Conceptions as a quadruple (Problems, Operators, Representation and Control Structure) in Vergnaud's (2009) point of view, the Knowing as the characterisation of a dynamic set of conceptions and the Concept as a piece of a mathematical theory, not reduced to a formal definition, but with its own history and practices validated by specific cultural contexts. For Balacheff, the control structure (i.e. the fourth component of the quadruple defining Conceptions of the $ck\phi$ model) is a monitoring of the student-milieu system that guarantees the conception's coherence. It makes reference to the behaviours and the tools put in place during the process, such as making choices, making decisions, giving feedback, expressing judgement and checking the advancement of a problem-solving process on the knowledge level. On the meta-knowledge level, Boero and Azrou (2023) used the RMT model as a theoretical construct to analyse the students' behaviours in mathematical activities. It consists of a couple (rational process; mathematical object) in which the mathematical object can be a mathematical proof, while the rational process aims to produce an instance on the mathematical object. In this perspective mathematical proof is considered as a concept, according to Vergnaud's point of view (2009), and the logical structure of proof as a set of operational invariants (Boero & Azrou, 2023). Thus, the awareness of students during the proving process plays an important role on the meta-knowledge level and into the relationship between the two levels. In the development of a mathematical proof, Durand-Guerrier (2008) emphasises the importance of the relationship between the syntactic and semantic aspect of a proving process, to foster the connection between everyday language and mathematical language in the interpretation of students' answers about the validity of a statement within a well-defined theoretical system.

Methodology

According to the theoretical perspectives presented, an *a priori* analysis was carried out aimed to characterise the cognitive roles of the Inside-Out model. In this paper we present Pest and Blogger. Subsequently, through the lens of the characterisation, a set of protocols of a group of students participating in a pilot study carried out as a curricular activity in January-February 2020, was analysed. The group consisted of 26 students, about 14 years old, attending the 10th grade (the second year of an Italian Liceo Scientifico Scienze Applicate). During the experimentation, a researcher played the role of Guru and the classroom teacher was present as an observer. At the first time, the roles are assigned by the researcher or teacher, but in the next scenes the students have to turn to assume another role. The empirical data were collected in a classroom with a PC workstation in which students and researcher could talk only into the Moodle chats and not speak to each other. The mathematical problem the students had to face involved algebraic thinking and was: *Choose four consecutive natural numbers, multiply the two intermediate numbers, multiply the two extremes, and subtract the results. What do you get?* (Mellone & Tortora, 2015). We intend a proving activity as a type of problem-solving activity, because in this cooperative task the students explore a situation,

have to develop a conjecture and have to produce a proof of the statement created. From the perspective of Habermas' (1998) rationality construct, for the development of the *a priori* analysis on cognitive roles we took into account the RMT model, (Azrou & Boero, 2023), as well as the analyses carried out on students' difficulty in the proof and proving process by Urhan and Bülbül, (2023a; 2023b). This led us to look at the roles from different levels: *knowledge level*, *meta-knowledge level* and *interplay between the levels*. The terms *knowledge level* and *meta-knowledge level* were originally conceptualised by Boero and Azrou (2023) to denote, respectively, an action using mathematical concepts and schemes and an action using the knowledge of the structure of mathematical theories and of formal logic. Moreover, the *interplay between the levels* concerns the relationship between the other two levels. With reference to the *knowledge level*, the *a priori* analysis is operationalised through the criteria identified by Urhan and Bülbül (2023a; 2023b), which are inherent to the different components of rationality. The authors utilise these criteria in specific areas of mathematics (algebraic, geometric, differential analysis), detailing the criteria used for the analysis of each individual task proposed to the students in relation to the epistemological domain in which the problems arise. In the *a priori* analysis, we decided to search among the aforementioned criteria for some invariants that might be independent of the specific mathematical domain involved. For the knowledge level, in particular for the role of Pest, Balacheff's cKç model was taken into account to refer to the control structure of the proving process underway. In this paper, the word 'checking' is used in two different ways: *checking as control*, on the knowledge level, taking into account Balacheff's point of view (Balacheff, 2010) and *checking as awareness*, on the meta-knowledge level, in line with Habermas' point of view (Habermas, 1998). For the meta-knowledge level and for the relationship between the two levels, we refer mainly to the work of Boero and Azrou (2023), as well as to Balacheff's (2010) construct of the social dimension of proof. Furthermore, the logical aspects of the proof structure with reference to a specific field of theory, the focus on the validity of a statement within a specific theoretical context rather than on truth, and the distinction between syntactic and semantic proofs will be taken into account (Durand-Guerrier, 2008).

Characterisation of the cognitive roles: the cases of Pest and Blogger

For the building of the characterisation (Figure 1), the indicators are partly taken and partly re-elaborated from the works of Urhan and Bülbül (2023a; 2023b), Balacheff (2010), Boero and Azrou (2023) and Durand-Guerrier (2008).

Rationality			
Roles	Knowledge level	Meta - knowledge level	Interplay between levels
Pest	<p>Validation of statements from accepted premises and using correct reasoning rules (Habermas, 1998)</p> <p>Checking the requirements of truth of statements and validity of reasoning. Checking the consistency of the patterns of use of the concepts, procedures and conceptual properties used and the correctness of the representations of the concepts - Epistemic Rationality. Checks the lack of operation mistakes while structuring the process. (Urhan & Bülbul, 2023a)</p> <p>Checking on constructions mathematically correct and valid representations. Checking the justify of all the steps of reasoning. Checking the soundness of all reasoning steps. (Urhan & Bülbul, 2023b)</p>	<p>Meta-knowledge of proof as a concept (according to Vergnaud's conceptual field theory) and reference of validity of proof to the epistemological assumptions of a theory. (Boero & Azrou, 2023)</p> <p>Checking the logical aspects of the chosen demonstrative schemes and their consistency in relation to the reference theory (in particular also with regard to semantics). (Balacheff 2010; Boero & Azrou, 2023; Durand-Guerrier, 2008; Habermas, 1998)</p> <p>Checking the accuracy of deductive chains and symbolic and verbal representations - Communicative rationality</p>	<p>Checking the ambiguity of the statements made and the quality of the communication of logical steps.</p> <p>In the transition from knowledge level to meta-knowledge and vice versa, checking for epistemic lacks in the misinterpretation of an algebraic transformation or calculation or choices that do not correspond to the identified demonstrative scheme</p> <p>Interplay between semantic and syntactic points of view in the proving process. (Durand-Guerrier, 2008)</p>
Blogger	<p>Writing correct and valid explanations regarding the steps taken during the process. Using mathematical language and symbols correctly and in accordance with the aim, thereby producing formal proof. (Urhan & Bülbul, 2023a)</p> <p>Communicating with others in a valid and comprehensible way, and hence activating the writer-interpreter dynamics. Using the symbolic language of mathematics properly, facilitating the reading and understanding of problem solving by verbal and/or textual explanations. Providing the reasons for the correctness of the drawings and the transitions within and/or between the representations (Urhan & Bülbul, 2023b)</p>	<p>Internal role in him/herself: Effectiveness of communication with herself in the proving process, producing an explanation of the validity of a statement from the subject's own perspective. (Balacheff, 2010; Boero & Azrou, 2023)</p> <p>Internal role in the group: Effectiveness of communication with others, to achieve the status of proof, accepted and valued by a community. (Balacheff, 2010).</p> <p>External role for the Culture: Effectiveness of communication with the Culture, in the formal writing of the mathematical proof, taking cultural format constraints into account. (Balacheff, 2010; Boero & Azrou, 2023)</p>	<p>Use of symbolic and verbal representations concerning implications and deductive chains. (Boero & Azrou, 2023)</p>

Figure1: Roles' characterisation of Pest and Blogger

An analysis of chat protocols in the light of the characterisation provided

We present the analysis of the synchronous interactions in the textual chats integrated in Moodle Platform, produced by students playing the Inside-Out Model. In the following, we present excerpts and we look for the characterisations connected to the cognitive functions associated with the roles of Pest and Blogger. Since we observed that students did not always 'keep' their roles and, for the purposes of this study, we are interested in observing the emergence of the rationalities in students' interactions, in the excerpts we indicate as Sn any student (regardless the specific role played) and as G the Guru. The analysis is made using Figure 1 as a lens to interpret the interactions.

Scene 1

- 33 S3: Could $(n+1*n+2)-(n+3*n+4)$ also be OK?
- 34 S3: Wait, I made a mistake in the writing
- 35 S3: $(n*n+1)-(n+2*n+3)$
- 37 S3: I made a mistake again

- 55 S1: the result is always 2
 60 S1: $(n^2 * n^3) - (n^1 * n^4) = 2$ is this OK?
 63 S2: [...] I don't think so because it looks like you multiply n by 2,3 or 4
 68 S3: $(n+1)*(n+2)-(n)*(n+3)$ can you see if that's OK too?!
 94 G: Can you explain it better?
 96 S2: we thought that by taking four consecutive numbers and multiplying the extremes and the averages and subtracting the multiplication of the extremes from the multiplication of the averages, the result will always be 2

Here we can find the checking on the effectiveness of communication with herself and with the others, that for us characterises a meta-knowledge level of Blogger's role (33 - 37). In particular, S1 (60), to clarify her ideas on the positions of the elements to be multiplied, uses the formalism that makes use of indices, but which naturally moves away from a coded writing of consecutive numbers. However, this moment is essential in intrapersonal communication, to activate a process of clarification and control of the communication itself. The epistemic component is thus ensured by S2 (63) who activates the checking on constructions mathematically correct and valid representations and, in the transition from knowledge level to meta-knowledge, checking for epistemic lacks in the misinterpretation of an algebraic transformation that do not correspond to the identified proving scheme. This enables the change in the communication of S3 within the group, writing in the formal mathematical language (68). Finally, S1 checks the truth requirements of statements and validity of reasoning (55) and S2 is involved in the effectiveness of communication towards others (96) by explaining in words to the teacher the process followed by the group. In Scene 2, students play with the same task, trying to produce a written conjecture, using the words available in a digital App.

Scene 2

- 119 S1: Do we also add x not equal to 0?
 120 S4: I know, but we need the not equal sign
 122 S3: do I write x not equal to 0, in the email?
 127 S1: you could maybe tell him that the not equal sign was missing in the app so we wrote it in words

Here a communicative need emerges in which the limitations of the digital tool let the students to perform a semiotic system conversion, moving from formal language to that of verbal language. In Scene 3, the students perform the same task again, proving the previous statement.

Scene 3

- 51 S4: Gianmaria (*i.e. Guru*) told me: 'who can tell me that the first member is always equal to 2, whatever x I take?'
 52 S1: let's take an example
 54 S4: $3*4-2*5$
 57 S4: Gianmaria said we have already tried examples, now we have to show that it works without substituting any value for x
 60 S1: if you do the computation without substituting a value for x you get 2
 61 S3: so if we try to carry out the equation with one unknown without substituting any number for x, we get 2
 62 S1: Exactly

- 67 S2: so the formula is true because even if you try infinite examples, the result will always be 2 and also because if you want to carry out the equation without substituting x , the result is always 2
- 85 G: what have you done materially to arrive at saying that the result of $(x+1)*(x+2)-x*(x+3)$ is always 2?
- 87 S4: we previously multiplied $(x+1)*(x+2)$
- 89 S4: and it got $x^2 + 3x + 2$
- 92: S4: then we multiplied $x*(x+3)$
- 94 S4: $-x^2 - 3x$
- 97 S4: because the minus in front changes the signs inside the parenthesis
- 103 G: can we say which property allows us to say that $x*(x+3) = x^2 + 3x$?
- 108 S4: x the monomial is multiplied by both members of the polynomial
- 114 S1: distributive property

S1 (60) checks the consistency of the usage schemes of concepts and procedures and (114) checks conceptual properties used during the proving process. Furthermore, S1 validates statements from accepted premises (62), while S4 works on checking the logical aspects of the chosen proving schemes and their consistency in the reference theory, in particular also with regard to semantics (87 - 97; 108). Furthermore, S3, S2 and S4 use verbal representations concerning implications, such as for example “so” (61, 67) and causal relationships, found by the use of the “because” (67, 97).

Results and conclusions

With respect to RQ1, we produced an *a priori* analysis visible in Figure 1, in which the rationalities involved in the epistemic and communicative dimensions are associated with the roles of Pest and Blogger, according to the theoretical frameworks presented. This characterisation then became for us an analytical lens with which to interpret the discursive interactions in chats in order to recognise, in the pilot, the rationalities mentioned above. With respect to RQ2, data analysis supports the *a priori* analysis, which gave rise to two further refinements in the Blogger’s role, added in this paper: 1) for the role of the Blogger at the meta-knowledge level, we distinguish between inner communication - seen as an inner dialogue- and external communication -aimed at understanding- according to Habermas (1998) and Boero and Azrou (2023); 2) through Balacheff’s social dimension of proof (2010), the act of constructing a proof takes place as a social act, especially for the Blogger role, so adding an intermediate passage through a restricted community, with its own language still evolving. Moreover, we also find in the data Pest’s epistemic control. In fact, Pest checks that the formulas produced are well formed, that the algebraic transformations used are correct, that the statements produced are correct, in line with Urhan and Bülbül’s point of view (2023a; 2023b). In addition, we found that she addresses the epistemology of mathematics, checking the ambiguity of the statements made, on the one hand and the quality of communication of logical steps on the other (Boero & Azrou, 2023). Pest’s rationality is thus characterised by both an epistemic and communicative component. This type of process was also made possible by the fact that the figure of the cultural mediator Guru was introduced in the 2020 pilot (T in the data), as announced in Albano et al. (2020) and which will be the object of further future developments in the next experimentation. With reference to Scene 3, providing students with a limited set of mathematical symbols in a digital App activated a communicative need, due to which students were induced to perform semiotic system conversion, moving from the semiotic system of formal language to that of verbal language, i.e. a transition

between the representations, in line with Urhan and Bülbül' s indicators (2023b), present in the Figure 1. Finally, let us clarify a substantial difference between the typical cognitive functions of the human brain and the cognitive roles played by the characters in the Inside-Out model, as although each role is associated with a characteristic function, it happens that sometimes the cognitive functions predominantly associated with one role are played or experienced by other characters. This depends on the dynamic development of the cognitive process and opens up new future developments, also in the field of neuroscience, taking into account new theories of brain flexibility.

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Quality criteria of individual prompts in mathematics classrooms

Melanie Ansteeg¹ and Johanna Heitzer¹

¹RWTH Aachen University, Germany; melanie.ansteeg@matha.rwth-aachen.de;
johanna.heitzer@matha.rwth-aachen.de

As a core practice, prompting has a significant influence on students' learning processes. It is an essential feature of teaching and affects the depth of engagement with the content through the quality of interactions between teachers and students (Lipowsky & Rzejak 2021). According to Mason (2020), prompts have a significant influence on learners' perceptions of mathematics. Together with Watson (1998), he presents a series of prompts for mathematical thinking. However, there is little literature on assessing the quality of prompts specific to mathematics classrooms. Furthermore, there are hardly any considerations on how to improve teachers' prompting. This article aims to fill this gap by proposing quality criteria for individual math-specific prompts. They are presented for discussion to examine if these criteria can serve as a guide for (prospective) teachers seeking to improve their prompting strategies.

Keywords: Prompting, individualized instruction, dialogic learning, teacher education

Introduction

The success of teaching and learning highly depends on engaging communication, typically cultivated through interaction and knowledge exchange. However, the significance of prompts, which frequently propel students forward in their learning journey, is often underestimated. Prompting is an essential skill for educators as it initiates, guides and enhances learning processes in ways suited for each student's distinctive learning path. Yet, dealing with the diverse needs of multiple learners simultaneously and managing the complexities of prompting pose substantial challenges for many teachers.

This article focuses on prompts for individual students. First of all, the term 'prompt' is defined in more detail and contextualized in the concept of dialogic learning. This will set the stage for deriving and discussing the quality criteria of prompts. These criteria will then be applied to analyze three sample prompts. The aim is to evaluate the suitability of these criteria for assessing the quality of a prompt. The criteria appear to be beneficial for both practicing and prospective teachers and are intended to assist them in enhancing the quality of their prompts. The criteria have already been presented to and evaluated by (prospective) teachers as well as teacher trainers and are also shared here with the intention of using the ensuing discussion to further refine them.

Providing good prompts based on the concept of dialogic learning

The concept of dialogic learning seems to be particularly appropriate to serve as a basis for an in-depth examination of prompts, as it illustrates how individually a prompt can be created. This offers both an opportunity and a challenge. In the following, a definition of the term 'prompt' will be provided, followed by an introduction to the principles of dialogic learning according to Ruf and Gallin (1998).

What is a prompt?

This article is based on the following definition of a prompt (cf. Ansteeg 2023, p. 534):

A **prompt** is a subject-oriented response to a learner’s contribution, purposed to assist the learner’s independent and individual advancement in exploring the relevant content.

The emphasis on subject orientation excludes purely motivational prompts such as “Keep it up!”, and thus makes the large field of possible prompts somewhat smaller. Nevertheless, the development of prompts remains a multifaceted process influenced by a variety of factors. As a focus is necessary, components such as accompanying gestures or facial expressions are not considered here.

This definition strongly resembles the concept of scaffolding, where learners are given support without undermining their responsibility for independently overcoming their difficulties (Hermkes et al., 2018, p. 153). But in contrast to the contingent-shift-principle, on which scaffolding is based (ibid., p. 148), the aim of this article is not to formulate a rule for giving prompts, but rather to present criteria by which the quality of a prompt can be assessed.

Basically, two objectives can be attained with subject-oriented prompts: The prompt can assist the teacher in gaining a deeper insight into the learner’s current stage in the learning process, or it can encourage the learner to advance in their learning. Often, a more comprehensive evaluation of the learner’s progress is required to formulate an applicable prompt for further learning. However, it is not unusual for a single prompt to be suitable both for diagnosing the learner’s comprehension and stimulating their advancement, thereby enhancing their understanding of the content.

Principles of dialogic learning

Starting with an assignment, dialogic learning leads the students into a learning phase in which they discover the content largely independently. Wherever possible, the assignment should trigger an urge of production in the learners, in which they approach the core idea through their own exploration and construction (Ruf & Gallin, 1998, p. 127). A core idea is understood to be the individual approach or the core content of a topic (Klimke & Lutz-Westphal, 2018, p. 977). In their learning process, learners move closer and closer to the core idea, which is expressed by an iconic spiral in Figure 1.

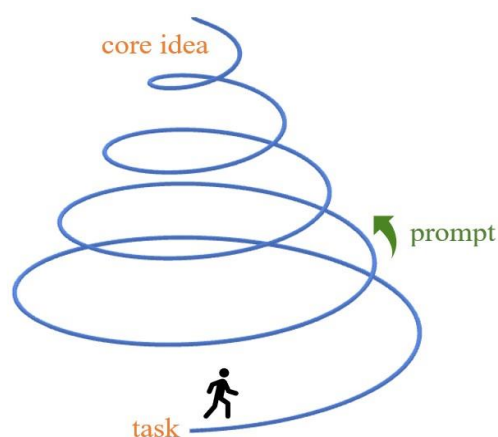


Figure 1: Learning spiral as approach to the core idea (cf. Ansteeg, 2023, p. 534)

On their individual learning path, it is possible for learners to reach dead ends, go astray or stall. Even if this is part of the learning process and should generally be allowed, the teacher can intervene with prompts and get the learner on a promising track. It should be taken into account that a prompt always represents an intervention in the learning process. For this reason, it is important to consider whether it is needed and how it can provide support without disrupting the exploration phase. After all, giving students the space to experience the content also plays an important role in developing the personal relation to mathematics.

Not only wrong paths may require teachers to give prompts. Prompts are also suitable for deepening content and consciously moving away from the straight path so that the content can be discovered more closely or from different perspectives. Furthermore, consistent reference to the contributions of the learners plays a decisive role in dialogic learning. While the teacher's contribution is usually considered a learning offer, it is now the learner's contribution that is seen as an offer which provides a basis for further working processes. This creates a completely different appreciation of the learner's ideas. Of course, there are also challenges associated with this, as it can be difficult to empathize with the students' ideas. However, this should not prevent teachers from trying again and again.

Characteristics of math-specific prompts

Due to its stringent and logical structure, the subject of mathematics has the advantage that learners can often judge the correctness of their contributions themselves, making it easier for them to discover content independently (Heiter & Weigand, 2015, pp. 5–6). Self-exploration requires the learners' positive attitude towards mathematics (ibid., p. 2). However, the two-valued logic and the intolerance towards calculation and reasoning errors can make it difficult for learners to find an approach to mathematics (ibid., p. 5). In order to provide them with a good basis for their discoveries, it is essential to build up solid basic ideas of the content.

The structure of mathematics brings with it a number of didactic specifics that offer good starting points for finding prompts. These include hierarchies in the acquisition of concepts and knowledge, heuristic strategies and, in particular, typical mathematical principles such as the operational one. This leads to certain analogies between the topic of prompts and other math didactics topics such as the characteristics of good tasks or the guidance in problem-solving according to Pólya.

A particular challenge in providing prompts is the appropriate intensity or openness of a prompt. In mathematics lessons, as it is a very abstract subject, there may be greater tendency than in other subjects to explain content and teach small-step procedures. This seems to enable all learners to keep up with the lesson and supposedly allows them to progress quickly through the material. However, this puts learners in a rather passive role and deprives them of the opportunity to make their own discoveries.

The level of personal opinion seems to be less present in mathematics lessons, as most tasks have a definitive solution. Questions that ask for the learner's opinion have a tendency of being superficial, e.g. when asking which calculation method learners prefer. In the end, there is usually a *correct* solution because one way of calculating seems smarter than the other. The dialogic learning approach can counteract this deficiency and is reflected in the following quality criteria.

Quality criteria of prompts

Based on the concept of dialogic learning, five quality criteria of prompts have already been derived in earlier work (Ansteeg, 2023, p. 534). They are slightly revised in this article and supplemented by two further characteristics that have proven to be useful in the assessment of prompts. Note however that quality criteria are always situation-dependent, which is why any collection of criteria may not be seen as extensive.

A good prompt...

- is easy to understand and adapted to the learner. (Clarity)
- ties in with the learner's ideas and concepts, even if these are inconsistent or incorrect. (Connection)
- leads the learner directly into production and thus (back) into independent exploration of the content. (Production)
- is of appropriate intensity. (Intensity)
- emphasizes understanding over results and is therefore sustainable. (Sustainability)
- is aware of the student's resources. (Awareness)
- enables the learner to assess their progress in the learning process and the correctness of their contribution. (Feedback)

Clarity: This is a very basic characteristic and it quickly becomes clear that it is necessary: if the learners do not understand the prompt, they will not be able to follow it. In addition to the use of technical terms, attention should also be paid to using appropriate language for the learners. Younger learners, for example, may not be able to classify irony or provocative statements.

Connection: A prompt should clearly indicate that it uses the students' contribution as a basis for further progress and not merely as an opportunity to intervene with a prompt. Otherwise it does not ensure that a link can be established between the prompt and the previous ideas from the contribution. Furthermore, a prompt should not only build on the learners' ideas but also be tailored to their level of learning. Connection thus refers to the students' contributions themselves and to their stage in the learning process. It can be compared to contingency which is one of the three key characteristics of scaffolding (Van de Pol, 2012, p. 34).

Production: Not every student activity is sufficient to trigger an urge of production. Students may also become active upon describing what is already clear to them. Explaining their own approach in a simple manner may help students to recognize mistakes or to release a blockage by subsequently coming up with new approaches (Mason et al., 2010, pp. 103–104). This in turn leads to production. The point of production is to let students explore, apprehend and understand content for themselves.

Recognizing a mistake is not the only way for learners to get into a new impetus of production. Also in cases of correct answers, it is beneficial to introduce prompts to deepen the understanding of the content. Mason et al. (2010) consider results to be truly understood if they can be embedded in a broader context and recommend, for instance, altering certain prerequisites and reflecting during a review phase on which analogous questions could be solved in a similar manner (pp. 39–42). It is important for the impetus of production that learners have an idea of how they can make progress.

Whether the idea is the right one is not decisive. We all know that when doing mathematics, you cannot always have the right idea straight away. Acquiring negative knowledge is also beneficial.

Intensity: Intensity expresses the extent to which the prompt is effective in terms of content, i.e. how far learners progress in the learning process. It can also be described by the level of learning output. If the intensity of the prompt is too high, important experiences in the learning process might be missed out. On the other hand, if the intensity is too low, the need for a follow-up prompt is likely.

Sustainability: Sustainability here means the long-term availability of what has been learned, which should be applicable permanently. This goes hand in hand with the fact that when providing prompts, understanding the content is more important than having the correct result written down on paper in the end. For instance, if the teacher explains the revealed solution while there is still a possibility for learners to find it themselves it undermines their solution-oriented reasoning and does not align with the idea of scaffolding (Hermkes et al. 2018, p. 150).

Awareness: Awareness is required with regard to the resources of both teachers and learners. This means, that a prompt should aim for the greatest possible progress in terms of content in the least possible amount of time. According to the concept of scaffolding, a prompt should provide a ‘tailored’ support that is neither overly instructional, thereby replacing students’ knowledge construction activities, nor too minimal, thereby taken as unsatisfactory by the learners (Hermkes et al. 2018, pp. 147-148). It should be noted that interventions in the learning process can also be disruptive factors and should therefore be chosen with care. Care must also be taken with regard to student’s emotions: Being made aware of a mistake is not always pleasant and should be initiated cautiously.

Feedback: Learners should be given the opportunity to assess their current status in the learning process. This feedback can sometimes be indirect, for example if the teacher draws a conclusion that is obviously incorrect. ("Consequently, the plane lands underneath the earth’s surface.") Direct feedback immediately shows the learners whether their previous work is correct ("That's not quite right yet. Try to ..."). If learners are not shown where they stand when the prompt is given, this can lead to confusion and frustration.

Investigation of prompts: 3 examples

The example of a student’s answer shown in Figure 2 was presented to students and trainee teachers with the task of noting down a prompt to the student. The student was asked to convert to the unit given in parentheses. This is a very guided task, which is certainly not suitable for meeting the principles of dialogic learning, as it is more likely to be used for assessment or practice purposes rather than provoke learning. However, this is meant to show how complex prompting can be even with such relatively straightforward tasks and that a dialogical starting point is not necessarily required to apply the quality criteria.

4 Gib in der Einheit an, die in der Klammer steht.

a) 5,38 m (km)

5,38 m = 5380 km

Figure 2: Example of a student contribution

(Task: Convert to the unit given in parentheses; 1 km = 1000 m)

To complete the task, the student must already have an idea of quantities (especially lengths) and be able to distinguish between units of measurement (such as metres and kilometres) and numbers. Required basic skills and knowledge include the decimal system and notion, the place value system, as well as dealing with powers of ten.

In the following, three prompts from prospective teachers are examined using the quality criteria described above in order to check the extent to which the criteria can be used to assess the quality of prompts. For this purpose, a general classification of the prompt is provided before the individual criteria are examined in more detail.

Example 1: “If the unit becomes larger, the measured value must become smaller.”

This prompt is a statement that contains a (partial) explanation. However, it does not fully argue why the claimed statement is true. There are technical terms used in this short prompt, which the student must be familiar with. In addition to this, it is hard for young students to process this prompt when it is only said verbally. Clarity may therefore be limited. In addition, it is not responding to the student’s contribution. The teacher is giving a general rule that could have been given to any student. The triggering of an impetus of production is also questionable as there is no direct request. It is very likely that the student will not be able to do anything with the prompt at first, so that no production momentum is triggered, and a follow-up prompt is necessary. This means that the prompt cannot be very sustainable. As it is a short comment, it is supposedly efficient. However, efficiency also means that the prompt is successful (using few resources). Although the prompt in example 1 uses few time resources, the chances of success are not very high due to the factors already described. Furthermore, the teacher does not give any direct feedback on the correctness of the result. The student must conclude from the statement itself that they made a mistake. Due to the impersonal address, the student could also simply ignore the prompt. Finally, the intensity of the prompt is also questionable. Assuming that the mistake was not caused by carelessness, the student has not yet fully grasped the conversion of quantities. Receiving the rule alone may not help the student, given that the rule is not very precise as it does not even address the factors at which value and unit are supposed to change.

Example 2: “Please stand up and walk five metres forward. – And now walk 5380 kilometres forward. – Is that possible?”

Here, two requests are coupled with a short follow-up question. The prompt is formulated simply so that it should be easy to understand. It is linked to the student’s contribution as the incorrect result is ‘worked on’ and a comparison with the initial value is encouraged. Production is stimulated by building on the student’s everyday knowledge in form of walking the length of the route in the classroom. It is to be expected that the student will have no difficulty in walking the 5 m and will then be taken aback by the 5380 km. The idea of a distance of 5380 km should then make them wonder and realize that the two distances are not the same length. Compared to the prompt in example 1, this prompt is more sustainable. The focus lies on realizing the error and building up appropriate ideas of the quantities. In terms of time resources, the prompt from example 2 is somewhat more time-consuming, as the student first has to stand up and then walk the distance (at least mentally). The teacher has to wait until the student has finished and can only then ask about the 5380 km. The teacher should also allow some time for reflection before asking the final question. This means that although the prompt requires more time, it enables the development of suitable ideas of quantities

and is therefore intensive. Especially if the student is able to apply the idea of visualizing content and linking mathematics and reality to future tasks. This prompt does not provide any feedback on the correctness of the answer. Feedback may be seen as hidden in the fact that the teacher gives a prompt at all, which perhaps (unfortunately) occurs less frequently with a correct solution.

Example 3: “But 1 kilometre is 1000 metres, isn't it? Then, how many metres are 5380 km?”

The prompt is formulated in a way that is easy to understand. At first glance, the suggestive question may not seem very well chosen – in principle, the student could only confirm the statement. Instead, the question "How many metres fit into 1 kilometre?" combined with the follow-up question "How many meters fit into 5380 km?" could be a good alternative. If the student then compares this result with their previous calculation, it enables them to notice the error. However, the leading question also assumes that the learner already knows the facts. In this respect, it is adapted to the learner and at the same time ensures a common starting point. Similar to example 2, the student's answer is also used here by picking up on the incorrect result in the second part of the prompt. The incorrect reply is considered from a different perspective by reversing the direction of the question and thus doublechecking the result. Since multiplication is often easier for learners than division, this approach seems promising in terms of allowing the student to recognize the error and leading into production. The intensity is even greater if the prompt causes the student to derive the general strategy of approaching a task from a different perspective in order to check the result. Even if it is not absolutely necessary for the teacher to wait for the answer to the second question, this is recommended for reasons of appreciation towards the student and the opportunity to reflect on the prompt. There is no direct feedback on the student's previous work. However, the "but" and the "then" indicate that the student's work is still incorrect.

Discussion and conclusion

When examining the three sample prompts, the quality criteria made a significant contribution to ensuring that the analysis could be carried out in a targeted manner. They are therefore appropriate for classifying prompts and assessing their quality. It has also become clear that the criteria set a very high standard for the prompts. It does not seem realistic to be able to fulfill all the criteria with a single prompt. However, this is also not necessary. It is important that teachers are aware of the goals they (can) pursue with their prompts and of the aspects playing a role in achieving these goals. To this end, it is helpful to consider the strengths and weaknesses of a prompt using the quality criteria, which can provide teachers with precise suggestions on how to improve their use of prompts. The criteria are practicable enough to support teachers in the reflective use of prompts, as they provide them with points of reference that they can use as a guide when giving and reflecting on prompts.

However, the above investigation of three sample prompts also has limitations. There is a single example situation which covers some of the expected occasions for prompting but not all. In the exemplary situation, the discovery of an error induces a prompt. Remember however, that prompts may also be helpful in situations where no error occurred, e.g. to provide further perspectives. Moreover, the error in the exemplary situation is more likely to be located in a training phase rather than in a phase of new content exploration. Besides, the example does not provide details about the situation, the learner's previous experiences, etc. In practice, the assessment is certainly easier when placed in the respective context, as it allows to take into account all influencing factors.

It thus seems promising to observe teachers during class, then reflect on their prompts on the basis of the student's reactions and with the help of the quality criteria, and finally discuss possible improvements. A concept for a seminar which raises awareness for the topic and improves their prompting has already been developed and tested (see Ansteeg in print). It is available on request.

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How to successfully participate in mathematics classrooms? Identifying and differentiating situational and normative expectations

Christoph Aumann¹ and Kirstin Erath¹

¹Martin Luther University Halle-Wittenberg, Germany; christoph.aumann@mathematik.uni-halle.de

Learning mathematics can be conceptualised as a special form of socialisation process. For successful participation, it is not only important to activate or acquire subject-specific knowledge, but also to act in an accepted manner. What is accepted and what is not is regulated by social, mathematical and sociomathematical norms that are negotiated in interaction. In this paper, we take on an interactionist perspective on mathematics classrooms and learning mathematics. We identify situational and normative expectations in classroom interaction and show that the concept of expectations is useful for drawing conclusions about underlying sociomathematical norms by qualitative analysis of specific classroom situations.

Keywords: Expectations, sociomathematical norms, classroom interaction, participation.

The role of participation for learning processes

The learning processes of pupils are the core of teaching and learning in mathematics classrooms. In contrast to cognitive psychology, learning can generally be seen as a form of *participation* in social practices (Cobb & Bauersfeld, 1995; Sfard, 2008). The idea of learning-as-acquisition is contrasted with that of learning-as-participation (Sfard, 2008). In the context of mathematics classrooms, participation in the respective social processes is therefore fundamental for students' learning processes. As cognitively demanding content is often addressed in phases of moderated class discussion (Nührenbörger & Schwarzkopf, 2010), participation in these class discussions is of particular importance for learning in the classroom. However, it is not always clear how to participate in these conversations in a way that is accepted, i.e. matches the interactionally established (sociomathematical) norms. In order to address this question, it is first necessary to examine the processes by which such norms are negotiated or communicated.

Norms and expectations in mathematics classrooms

As previously stated, class discussions are particularly intriguing from an interactionist and ethnomethodological perspective. This is because – in these perspectives – individuals construct their own reality through *interaction*, with regard to the meanings that objects or situations have for them (Blumer, 1986; Garfinkel, 1967). Through interaction, the individual interpretations of a situation are displayed to each other, with the result that an interpretation that is regarded as shared can be formed (Krummheuer, 1983). The interpretation negotiated in this way forms the basis for further interaction and thus represents the starting point for successful learning processes, since, from an interactionist perspective, learning takes place in interaction. In the classroom, negotiation processes often take place in class discussions, as interactions in these class discussions are more or less accessible to everyone. The extent to which individuals engage in respective conversation can vary. In addition to the speaker and hearer, there may be other recipients whose involvement in the conversation may be more or less pronounced and may fluctuate over time (Krummheuer & Brandt, 2001). Nevertheless,

active participation in the form of verbal contributions represents a distinctive form of participation, as this is where the realisation of thoughts takes place (Vygotskij et al., 1962). One's own interpretation of the situation is brought into the public sphere and thus forms a part of the negotiation process about its meaning.

Mathematics classrooms can be understood not merely in terms of a social system (Parsons, 1951), but in particular as a process of socialisation itself (Voigt, 1994). A crucial part of a socialisation process is the establishment of the corresponding *norms* of this social system (Parsons, 1951). Pupils and teachers are thereby already part of other social systems where they are socialized to the norms and practices that apply or are negotiated in these systems. These systems can include, for example, primary school classrooms, the parental home, other subject specific classrooms or the sports club. This is reflected in the classroom insofar as the interactants participate in a manner, that is influenced by their earlier socialisation (Jorgensen et al., 2014). Following transitions such as that from primary school to secondary school, the norms of the new social system are not yet established. Instead, they must first be negotiated and established. This establishment of norms occurs anew in the form of negotiation processes about the shared norms in each classroom, with each teacher and in each subject. In the classroom, the negotiation processes about shared norms are often limited in the sense that the teacher usually represents the normative authority that decides on the correctness of statements in case of doubt (Fiehler, 2017). In addition to social norms, which are normative descriptions of social behaviour, *sociomathematical norms* are particularly relevant for mathematics lessons. These describe “normative aspects of mathematical discussions that are specific to students' mathematical activity” (Yackel & Cobb, 1996, p. 458). Both types of norms can shape participation in classroom practices as they can provide guidelines for accepted behaviour. However, the norms themselves often stay implicit and are not made explicit by the teacher or the students.

The defining characteristics of norms is their relative situational stability, meaning that norms cannot be identified in one specific situation but as “repeatedly observed characteristics of practices” (Ingram et al., 2023, p. 2). Consequently, the question arises how to empirically identify norms in social interactions such as mathematics classrooms. Our approach initially moves away from norms and thus a long-term perspective and focusses first on specific sequences in classroom discussions, mostly bounded by a question that is discussed and answered or a task that is reviewed. We assume that in each sequence, all interactants have certain *expectations* for this situation. These expectations are based on individual experiences and represent ideas about how the situation should be interpreted and what kind of (inter)action is appropriate (Garfinkel, 1967). If the expectations of the interactants are sufficiently compatible, the interaction can run smoothly. However, if divergent expectations meet, conflict can arise, which can be resolved by negotiating a consensus. Expectations can, but do not always have to be based on a norm (Ohlhus, 2017). In order to identify norms, it is therefore necessary to identify the expectations of the interactants in several sequences: If certain expectations or practices occur regularly across situations, an underlying norm can be inferred and the expectation takes on a normative character. Buttlar (2017) has already shown that expectations of a normative nature can be conveyed through feedback processes from the teacher in the subjects of language and basic science education in the early years of primary school. It is therefore important for pupils to decipher these teacher expectations in order to successfully participate in these classrooms.

Based on the presented theoretical considerations, the question arises, to which extent the expectations of the interactants can also be identified in sequences of classroom discussions from secondary mathematics classrooms. A particular focus is placed on those expectations of sociomathematical nature, as these represent a special feature of mathematics classrooms. Furthermore, it is of interest to what extent the concept of expectations is suitable for analysing socialisation processes in mathematics classrooms.

Methods

Methodologically, this work follows the interpretative tradition of classroom research in everyday mathematics lessons. The data used in this study was obtained from a video study conducted in a fifth-grade classroom at a German upper-secondary school (Gymnasium) in a medium-sized city. The first four weeks of regular mathematics lessons after the transition to secondary school were recorded using video technology, resulting in a data corpus comprising of 20 lessons of video material, which were all recorded in their entirety. The focus was on the classroom setting, so that the entire class can always be seen in the image.

The sequence selected for presentation in this paper is about a ritual at the beginning of a mathematics lesson, called “daily exercise”, and its correction. The teacher regularly orally poses up to ten tasks at the beginning of the lesson on the subject matter of the current topic, as well as on topics from further back. These tasks are then to be completed in written form by the pupils in a short space of time and then compared. We selected this situation because the teacher explicitly announced that this format will be graded in the future. Accordingly, this format is relevant to academic success, particularly in this mathematics classroom. This paper focuses on two specific tasks – task one and task four – of the daily exercise from the second mathematics lesson in the new school, which are similar in many ways and yet different. We chose them because one of the pupils indicated the need for clarification during the correction of those tasks. The selected sequence was transcribed and then analysed qualitatively on the basis of interactional analysis (Schütte et al., 2019).

Empirical insights

The teacher initiates the lesson with a daily exercise. Two pupils complete the tasks on the blackboard, while the rest of the class works in their exercise books. The teacher reads out the tasks one after the other, allowing a brief interval between each one for the pupils to attempt to solve them.

1 Teacher: [...] So that you don't have to ask twice, I'll write three numbers on the board. [...] The numbers are forty-eight, nineteen and twenty-nine. [...] You are to use these numbers to write down a correct minus task. [...]
[...]

In line #1, the teacher begins the task with the note “so that you don't have to ask twice”. This can be interpreted in two ways. Firstly, it can be seen as a kind of service provided by the teacher to save the students work in terms of asking questions. Secondly, it can also be seen as an indication that asking questions is not a behaviour desired by the teacher during daily exercise. Thus, the three numbers are written down on the board. This also emphasises the importance of these three numbers for the solution process, as the written record is more binding. The teacher then formulates the task, namely that the pupils should “use these three numbers to write down a correct minus task” (#1). At this

point, it is still unclear whether only these three numbers should be used or whether they should only occur and can be supplemented by others. Furthermore, the teacher emphasises the “correctness of the minus task” as a benchmark for assessing the solutions. Serving the teacher's idea of correctness, whatever it may be, therefore seems to be the teacher's expectation of the result of the pupil's activity. Further tasks are then set, but apart from the following, these are not relevant here and are therefore omitted.

3 Teacher: We are on task four, there are again three numbers to choose from. I'll write them down again, which ones you can use. The four, the two and the seven, and the task is: form the smallest possible number from these three [*added and a little quieter:*] then digits.

[...]

The use of “again” by the teacher in line #3 establishes a parallel between this fourth task and the previous situation from line #1. By explicitly mentioning the three numbers, it is likely that she is referring to the task set at the beginning. This parallel can be recognized not only in relation to these three numbers, but also in their written form. Interpretatively, a certain comparability of the situations could therefore be seen here. The task is to form the smallest number. The addition “then digits” is introduced at a later point in the task and in a more subdued manner, indicating that the numbers are to be utilized as digits. Consequently, the initial interpretation of the task is that, as in the preceding task, numbers should be connected in a manner that yields the smallest possible result. However, the addition clarifies that the parallel to the preceding task is limited to this point, as the numbers are now to be treated as digits rather than operands.

Subsequently, further tasks are presented, and the solutions are evaluated by two students in the assessment phase. The teacher then requests that the class verifies the solutions, initiating a social negotiation process regarding the solutions. In the following, Merle refers to the fourth task and the solution “ $7 - 4 - 2 = 1$ ” on the blackboard.

8 Merle Yes, um, I noticed a mistake. We should form the smallest possible number and not seven minus four minus two.
 9 Teacher What did you write down?
 10 Merle I wrote down two hundred and forty-seven.
 11 Teacher And two hundred and forty-seven is correct. From the three I gave you, you should sort them so that it is the smallest possible. [...] That's how it was meant to be. So basically, we have a mistake here. [...] Merle, what else did you want to add?

Merle points out a solution presented on the board, that she identifies as a mistake. While the solution on the board is “ $7 - 4 - 2 = 1$ ”, Merle marks it as incorrect. Only upon prompting, Merle presents her answer, 247, as a concatenation of the three digits. This could indicate that she is uncertain about the correctness of her solution. In her answer, the teacher validates Merle's objection and explicitly explains how the task was intended. It becomes evident that the criterion for correctness is not merely the mathematical correctness of the solution, but rather the alignment with the teacher's expectations. The mathematically correct result of the subtraction task is not only not acknowledged, it is also explicitly marked as a mistake by the teacher.

Merle then adds her thoughts on the task in line #1, which is answered by $48 - 19 = 29$ on the board:

- 12 Merle I've got all the numbers here in this minus task, so forty-eight minus twenty-nine minus nineteen equals zero.
- 13 Teacher It just says: Write down a correct minus problem with the numbers, and if you've found a correct one - yours is also correct, yes - I'll accept that too. Great, nice, you've found a second solution. Alexandra?

Immediately afterwards, Merle adds another option to the first task, although this time the blackboard solution “ $48 - 19 = 29$ ” is not marked as incorrect by her. Merle has found what she considers to be another way to form a “correct minus problem” using the numbers 48, 19 and 29 than is shown on the blackboard. To do this, she used a zero in addition to the numbers mentioned and written down and used the three given numbers as operands for the subtraction. The fact that she wants to express the possibility publicly in front of the teacher can either be interpreted as her being unsure about the validity of her solution or that she wants her alternative solution to be recognised and honoured separately. In this case, as before, the teacher is the evaluating authority. She takes this role by repeating the task and emphasising its openness or ambiguity (line #13). Furthermore, the teacher explicitly praises Merle's solution and marks it as equally correct.

- 14 Alexandra Um, would it also be okay if we have forty-eight minus twenty-nine equals nineteen?
- 15 Teacher Did you just listen carefully to what I said to her? What do you think my answer will be now? Yes, or no?
- 16 Alexandra Yes.
- 17 Teacher You see, that's exactly what we do. So, one thing: there is not always just one solution in maths. [...]

After the teacher explicitly praises Merle's proposed solution, Alexandra asks whether her solution is also accepted. It is unclear whether she also wants her solution to be recognised or whether she is actually unsure about its validity. One reason could be that at this point Alexandra is not yet able to decipher the teacher's expectations. In response, the teacher does not validate Alexandra's proposed solution directly but refers to her previous statement and obviously sees her own judgement as predictable, which is why she demands this from Alexandra. The choice of words and the undertone allow the interpretation that the teacher sees this demand as rather superfluous and unnecessary. Maybe this connection between Alexandra's question and the answer given to Merle by the teacher lets Alexandra understand what the teacher did expect from the solutions to this task, maybe it was a lucky guess. However, Alexandra answers her own question with the result, that her solution is valid and accepted. Finally, the teacher emphasises again that there is not always just one solution in mathematics, which is probably intended to explain the many possible solutions to this task.

Discussion and conclusion

The analysis of the sequence shows how the concept of expectations can occur in classroom interactions. Firstly, in the phase of solving the tasks, the pupils have time to prepare the solution according to their expectations. This is done on the basis of how they interpret and assess the situation leading to an expectation of what is to do in this situation and what is appropriate for a solution. Then, in the phase of public comparison of the solutions, these expectations are brought together, and a conflict arises, as Merle's expectation of the correctness of the task is in contrast to what was offered as a solution on the board. Specifically, it is about the question of which mathematical actions the task requires, or what is declared valid. These are therefore divergent expectations of a

sociomathematical nature that can be thought of in analogy to sociomathematical norms (Yackel & Cobb, 1996). In the corresponding negotiation process, the teacher takes the lead and evaluates Merle's solution as correct and in line with her expectations ("That's how it was meant to be", #11). The deviating solution, on the other hand, is marked as an error. Since errors in performance assessment contexts are associated with lower grades, it is therefore important for students to avoid such errors. Finding out exactly what the problem was or what the teacher expects is therefore highly relevant for successful further participation in lessons.

In the fourth task, several proposed solutions are now presented. Once again, this phase begins with a statement from Merle in which she compares the solution on the board with her solution and recognises a difference. The fact that she does not mark a mistake this time can be an indicator for two things. Firstly, it could be, that despite the difference between her solution and the one on the board there is no contradiction to her expectation of the correctness of this task. That would mean that she is aware that different solutions can be correct at the same time and both solutions fulfil her expectations of correctness and appropriateness. Secondly, it seems possible that she is no longer sure whether her expectation of the correctness of the task is congruent with the teacher's expectation and would like to be certain about this. This could then be part of a socialisation process in which Merle tries to decipher the teacher's expectations in order to be able to act accordingly in the future. In any case, both Merle and the teacher do not mark the solution presented on the board as an error. Instead, the teacher emphasises that the solutions, given on the board and by Merle, are different but still meet the teacher's expectations of correctness (#13). Here, reference is made to the sociomathematical aspect of difference (Yackel & Cobb, 1996). In addition, the teacher points out that only the information provided by the task is of interest; if it is ambiguous, several solutions are possible. The result is that mathematically different but correct solutions are accepted. The scope of that acceptance is thereby quite vague, as it still depends on how the teacher assesses the ambiguity.

While in the first task the alternative solution of forming a subtraction task from the numbers that results in a smaller value than the expected 247 is labelled as a mistake, an alternative solution is accepted in the fourth task. Both the addition of an arithmetic operation in the first task and the addition of another number to the fixed numbers in the fourth task are not explicitly allowed. Nevertheless, both situations are assessed very differently. In the fourth task, the teacher refers to the general nature of mathematics (#17). The associated expectation that several solutions can also be valid therefore appears to have a cross-situational scope. It can therefore be ascribed a normative character, as it seems likely that mathematically different solutions will also be accepted in principle in other situations. But why is a mistake marked in the first situation? Obviously, in the eyes of the teacher, the solution using the subtraction task is not a valid alternative to the expected solution of concatenation. Her expectation of a valid solution is that the numbers are to be used as digits. The teacher's explanation is very situation-specific as she explains how this task was intended (#11). The scope of this expectation, that the concatenation is to be formed here, is therefore more likely to be interpreted situationally. In this situation, the teacher obviously attaches greater importance to this situational expectation than to the normative expectation that different solutions are possible in principle. In the fourth task, there does not appear to be a strong situational expectation of a very specific subtraction task, as the general normative expectation of possible alternative solutions applies

here. The basic acceptance of different solutions therefore appears to be part of the normative foil (Ohlhus, 2017) of this situation. However, there are limitations if the teacher applies other situational expectations, as in task one. For the students, it is therefore not only a matter of deciphering the teacher's expectations in order to be able to participate in the lesson in an accepted manner, but also of distinguishing between the situational and normative relevance of the expectations in order to be able to participate in the lesson again in future situations in line with expectations. This involves recognising what is expected across all situations, but also the exception when other expectations, which may run counter to the rule, represent the requirement in specific situations.

Following on from Buttlar (2017), expectations were also identified in upper-secondary school mathematics classroom. We found first empirical evidence that allows a distinction to be made between situational and normative expectations, which differ in terms of the scope of validity. This made it possible to identify parts of the normative background against which the lessons might develop. In the present sequence it was shown that in concrete situations situational expectations can take precedence over normative ones without the normative expectation losing its validity. This identification of expectations represents a crucial part of the socialisation process in mathematics lessons, as this is where the demands for accepted participation in lessons are negotiated (Parsons, 1951; Voigt, 1994). In further situations, the extent to which the interplay of situational and normative expectations develops and how the pupils behave in accordance with the expectations, i.e., how the individual socialisation processes proceed, still needs to be investigated. The expectations approach appears to be promising for further research in order to better understand classroom discourse and socialisation processes and to be able to work out the normative expectations by analysing further situations and thus draw conclusions about the norms underlying the expectations.

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Mathematics argumentation in multilingual classrooms: Opportunities and obstacles for students and teachers

Giuseppe Bianco¹, Benedetto Di Paola¹ and Giovanni Giuseppe Nicosia²

¹Department of Mathematics and Computer Science, University of Palermo, Italy;
giuseppe.bianco08@community.unipa.it benedetto.dipaola@unipa.it

²ISGEm, RSDDM, Collège Notre Dame - Morez Hauts de Bienne, France. gg.nicosia@gmail.com

This contribution presents a case study about an experience in an 8th-grade multilingual classroom that included Chinese and Italian speaking students. We analyse the transition from a student's L1 (Chinese) to her L2 (Italian) in a mathematical argumentation. The theoretical perspectives are based on Bishop's multicultural research and on linguistic studies by Cummins on the link between L1 and L2. Finally, we discuss the actions of the teacher and student regarding their perception of linguistic differences as an obstacle or resource in a mathematical argumentation.

Keywords: Mathematics education, L2 vs L1, multilingual classrooms, argumentation.

Introduction

In contemporary classrooms we can see the co-presence of pupils with different languages and of different cultural heritage. Educators are often not aware of the deep differences behind these various diversities, and, moreover, are not always able to promote these differences as a value to be shared with all the class (Council of Europe, 2022). The construction of realistic classroom-knowledge, that is rooted in the meaningful experience of the pupils of the class itself, is difficult. This meaningfulness is subjective and connected with the culture and community in which the students and teachers are embedded. Indeed, a key point is the institutional power by adults: the culture of the teacher, which may be different to the culture of students, is the dominant one, in particular in relation to the students whose families come from other countries. In this case we see a dissociation between the official culture, transmitted and institutionalised by the teacher, and the “real” culture of the classroom, constituted also by the students. Moreover, behind a pedagogical distance between the various cultures involved, and rooted in the social system of values, there is also a disciplinary distance. On one hand, there is in-school mathematics supported by teacher, exams and education system; on the other hand, there is out-of-school mathematics, represented by families, and usually undervalued. Here, we will deepen the dialectics between these key factors, with a special focus on a case study, analysed both in its linguistic (and then cultural) aspects, and its educational ones.

From culture to language

In this paper, we will not present a particular definition of *culture*. Rather, we will assume that in every comprehensive definition of culture are included some common *actions* such as sharing, mediating, symbolizing, practising, believing, knowing, communicating and *concepts* such as tools, artefacts, community, heritage, behaviour, history, and society (Bram et al., 1953). Language can arguably be seen as part of culture (being the prominent sign-based activity of humans), while each language is a lens to see and to explain a world, so it encloses cultural elements and works as a cultural tool, as stated by Sapir: “Language is primarily a cultural or social product and must be understood

as such” (Bram et al., 1953, p. 230). In didactical situations that include many cultures and languages, the main purpose is to create learning opportunities for the non-native speaking students. This may sometimes lead to the idea that the dominant language (useful for an inter-cultural communication) and the dominant culture should be transmitted (*acculturation*). In an acculturation-driven education system, language diversity can be seen as an obstacle, as cited polemically in Cummins (2000, p. 4): “bilingualism shuts doors” and “monolingual education opens doors to the wider world”. Conversely, in an intercultural-driven school system language is seen as a goal and a medium (Council of Europe, 2022). We align with this latter view and think that valuing the different languages can be an opportunity to give meaning to the students’ different way of thinking, even in connection with their family cultures. In this perspective, the focus is on the individual: we see a shift even in the terminology, from multilingual (compresence in the same *area* of more than one language) to bilingualism/plurilingualism (compresence in the same *person* of more than one language). Here, we discuss a didactical experience in a multilingual context with Chinese and Italian students, with the aim to address the following research problem: *How can a mathematical argumentation produced in a non-dominant language be an opportunity for a non-native speaking student (8th-grade)?* Focusing on both the student and teacher viewpoints, the present case study deepens the research question: *What happen in the passage from a mathematical argumentation in a native language to a non-native language?*

Theoretical perspectives

To consider cultures as a key aspect in mathematics education, the pivotal point is to bring the idea of culture from the background to the foreground, from an environment setting to a research topic. This issue emerged during the globalisation and the rise of “new” cultures, emancipated by the western colonisers (Bishop, 1991); then, educators started to explore the background of their students, and their cultures. Bishop (1991) considers mathematics to be a product of society, a result of a process of *enculturation*. Far from being inherent to the individual human being, mathematics is always filtered by some *cultural interfaces* that intervene, variously, in its teaching. Since society has a significant influence on the characteristics that a community accepts/rejects as mathematical, interfaces are socio-culturally rooted. Thus, teachers should try to understand the mathematical practices by their students, to share and open the knowledge to all the class (Bianco et al., 2023).

Second language acquisition: from L1 to L2

With the aim of deepening the relationship between dominant and non-dominant languages, we refer to the work on L1 and L2 by Jim Cummins. Cummins (2000) makes a distinction between BICS (“basic interpersonal communicative skills”), the multimodal, expressive, concrete language used in daily life and CALP (“cognitive academic language proficiency”), the abstract, context reduced, and specialized language used in classroom. What is relevant to our concern are the basic mechanics which occur in a second language (L2) acquisition by a yet proficient speaker in another language, her/his mother tongue (L1). According to Cummins, the proficiency in two languages is linked, and working on students’ L1, can promote both L1 and L2: “students who have developed literacy in their L1 will tend to make stronger progress in acquiring literacy in L2” (Cummins, 2000, p. 173). This fact is encapsulated in dual-iceberg metaphor of bilingual proficiency and Interdependence

Hypothesis (Cummins, 1981): even if non-native speaking students show a different superficial proficiency in their L1 and L2, there is an underlying core common to both L1 and L2 (CUP – “common underlying proficiency”). According to the Interdependence Hypothesis, the CUP is shared by L1 and L2 and involves CALP aspects (it is “under” BICS “level”). This means deep linguistic and meta-linguistic skills – such as working *on* syntax – can be transferred from L1 to L2, because are already common. Thus, educators should value and work on this deep core in L1 (what is already known) to strengthen the emergent aspects (what “should” be known) even in L2 (Figure 1). The move from L1 to L2 will be the pivotal point of our case study. In particular, we assume the student perspective: L1 will be the Chinese language, L2 the Italian one.

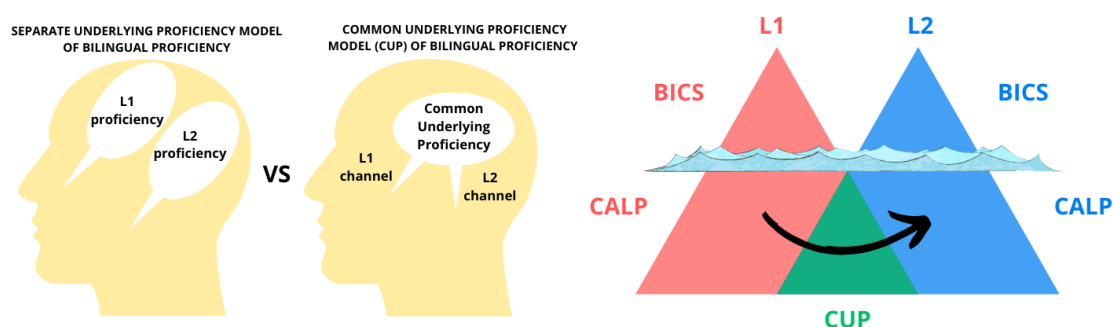


Figure 1: Separate vs common underlying proficiency and dual-iceberg by Cummins (1981), p. 23–24.

Some notes on the Chinese language structure and syntax

For the benefit of the international reader, we will outline some linguistic aspects relating to the Standard Chinese language (Ross et al., 2014). The Chinese language is very different to the Italian language, with respect to language family (Sino-Tibetan vs Indo-European), writing (idiographic vs alphabetic), and phonology (tonal vs non-tonal). A key aspect of the Chinese language is the direct relation between its atomic elements: syllable (sound), character (written sign), morpheme (meaning). For our study, the most relevant aspect is the grammar and, in particular, the syntactic structure. In Chinese there are no declinations (nor plurals), no conjugations and no articles, so all the meaning and non-ambiguity of a sentence lies in the clear and modular syntactic structure. The structure of a Chinese (transitive) sentence is always “Subject + Verb + Object”. This makes Chinese a head-initial language. This is like Italian syntax except that in Italian, Indirect Objects come at the end of the sentence, while in Chinese they occur after the Subject and before the Verb. A second fact is that all modifiers precede the noun they modify, and this makes Chinese a head-final language: the adjective and the possessive pronoun come before the noun, the adverb before the verb. In contrast, in Italian modifiers follow the modified. An example is given by the possessive/genitive pattern of the Chinese language, marked by the particle 的 (de) which means “of”. The use of this particle is not limited to the direct possession of something by someone but is broader in scope, allowing it to introduce entire sentences attributed to a noun. The use of 的 is as follow: Modifier + 的 + Modified. For instance, noun + 的 + noun: “我的书” (I + 的 + book) which means “my book”; adjective + 的 + noun: “新的书” (new + 的 + book), meaning “new book”; sentence + 的 + noun: “我姐姐买的书” (my sister bought + 的 + book), meaning “the book my sister bought”.

Research Method

In this paper we reflect on a specific classroom event which shows some aspects of the typical daily practice in multilingual classes, both from the students' and teacher's viewpoint. Our study regards an experience in an 8th-grade classroom in the most multicultural province of Italy, Prato (centre of Italy). In the class, more than three quarter of the students were of Chinese culture, of first or second generation, and their language proficiency in both Chinese and Italian was varied. The first author (researcher) acted as an observer. During a workshop activity conducted by the teacher, the students were asked to "draw the flattened cylinder of base radius = 3cm, with height = 10cm, then calculate the area of the flattened figure". Further to the calculation, the students were asked to create a short explanation of the whole process. We focus on this second step based on field notes and a student protocol. Due to the multilingual setting, many students faced huge obstacles in describing the steps followed. The activity took two hours, in which students worked at their desks, frequently asking the teacher about their resolution and argumentation.

A Chinese student protocol: analysis of Aly's argumentation

In this paper, we focus on the work of a Chinese student (Aly) who was still learning the Italian language. Our interest in Aly's protocol is two-fold: first, it allows a linguistic reflection on the transition from L1 to L2 syntactic structure during a mathematical argumentation; second, it highlights how Aly, following the teacher's feedback on her argumentation, *replaced* concretely the first argumentation with a new argumentative text that was glued *onto* the previous one. After talking individually with the teacher, Aly didn't show the previous argumentation again to the researcher, who was curious to compare the former and the latter versions. In Figure 2 we see the first version elaborated by Aly. Even if both researcher and teacher knew a little about the Chinese structure of a sentence, they initially found it difficult to understand the text.

RECTANGLE OF BASE OF IS EQUAL OF CIRCLE CIRCUMFERENCE
 AND RECTANGLE OF AREA OF THE EQUAL BASE TIMES HEIGHT, CIRCUMFERENCE CIRCUMFERENCE
 I CALCULATE
 $C = \pi \cdot d = 3,14 \cdot 3 \cdot 2 =$ SO BASE BASE OF THE RECTANGLE AND
 AREA OF RECTANGLE $= A = b \cdot h = 6\pi \cdot 10$ SO THAT I FIND AREA OF RECTANGLE
 AREA OF CIRCLE $= A = \pi r^2 = \pi \cdot 3^2$ THE I HAVE TWO CIRCLES, THEN I HAVE TO MULTIPLY
 $\pi \cdot 3^2 \cdot 2$ THE TOTAL SURFACE SUM UP THE TOTAL SURFACE IS THE SUM OF
 ALL THE AREAS.

Figure 2: English transcription of Aly's first argumentation.

With the aim to highlight the reasons behind Aly's writing and how these are rooted in her familiar language (Standard Chinese), we will de-construct the linguistic structure of her argumentation. We will find the emergence of the syntax of the native language of the student, even if the argumentation appears to be in Italian. Moreover, we should consider that this case study is representative of a student in the process of L2 learning, particularly during the transition from her/his native language to the syntactic structure of the host language, and who is proficient enough to develop a full L2-like argumentation. For this reason, we believe that this event represents a link in the long chain of second language learning, and a key moment in the learning of mathematics in multilingual classrooms, given

the role of argumentation and communication in mathematics. In the analysis, we focus on the first sentence; from the second sentence onward, Aly’s work changes: her writing becomes linear, she starts to speak in the first person, describing her actions (I calculate, I find, etc.). The first sentence of the first version of Aly’s argumentation (Figure 2) can be analysed, from a syntactic point of view, in four steps (Figure 3), which are useful to *deconstruct* the process which led Aly to her final argumentation (Figure 4). In *italics* we highlight the exact words (and signs) used by Aly in her protocol and corresponding to the step under analysis. In Figure 3, we use *bold* for the same purpose. The middle column shows how the syntactic structure of the step would be used systematically in the sentence. The last column shows how Aly actually used it.

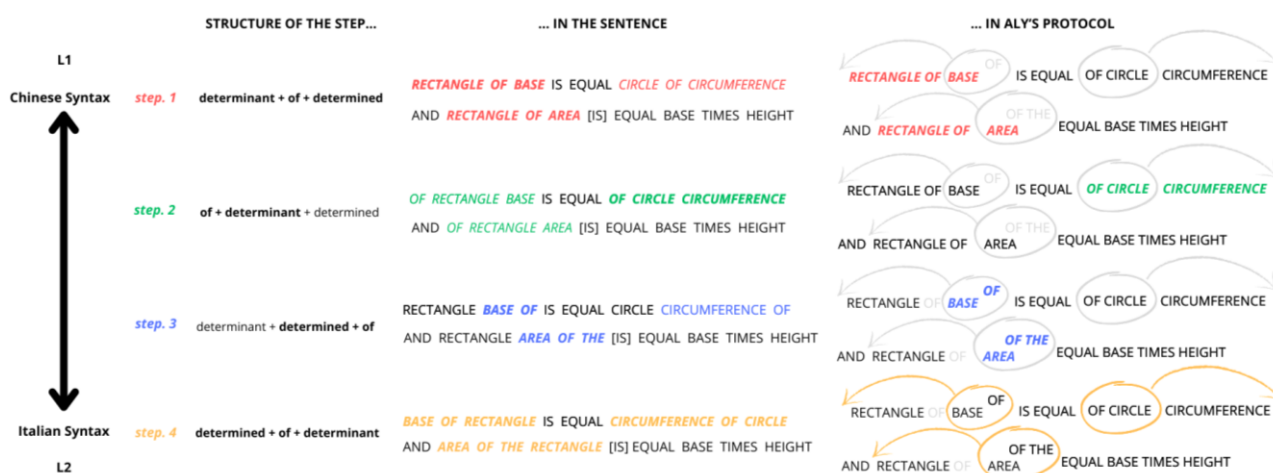


Figure 3: The four-level analysis of the first sentence.

1 - Naïve arrangement, close to Chinese Syntax (L1), (step 1, Figure 3): *rectangle of base* is equal circle of circumference and *rectangle of area* [is] equal base times height.

Here the syntactic structure is intimately rooted in the Chinese syntax, even if no Chinese word occurs. The key aspect is the structure “determinant + of + determined” (*rectangle of base*’ and *rectangle of area*’). This aspect emerges from Aly’s cultural and linguistic heritage where, as discussed above, the 的 particle follows this rule. We can see that *of circle circumference*’ by Aly doesn’t follow this pattern; this led us to the second step, where the “determinant + of + determined” structure starts to move toward the Italian one, “determined + of + determinant”.

2 - Hybrid syntax, between Chinese (L1) and Italian (L2), (step 2, Figure 3): *of rectangle base* is equal *of circle circumference* and *of rectangle area* [is] equal base times height.

Now the structure “of + determinant” is clear, even if embedded in a structure uncommon for western syntax (the determined comes last). The structure *of circle*’ instead of the Chinese *circle of*’ matches exactly the typical Italian order. This means that in this case Aly is already working toward the “right” – shared and accepted, by an Italian speaker – direction even if the determined follows the determinant in a “of + determinant + determined” structure.

3 - Hybrid syntax, between Chinese (L1) and Italian (L2), (step 3, Figure 3): *rectangle base of* is equal of circle circumference and *rectangle area of the* [is] equal base times height.

Again, Aly shifted toward Italian syntax by adding the words ‘of’ and ‘of the’ after the determined, respectively, ‘base’ and ‘area’. This led to a hybrid syntax in which the Italian structure “determined + of” is embedded in a Chinese syntax “determinant + determined”. Thus, we observe the pattern “determinant + determined + of” (‘rectangle base of’ and ‘rectangle area of the’).

4 - Reconstruction in the making, close to Italian Syntax (L2), (step 4, Figure 3): rectangle (base of) [backward arrow] is equal (of circle) [forward arrow] circumference and rectangle (area of the) [backward arrow] [is] equal base times height.

In the last step, circles and arrows were used to change the order of the words. The fourth version of the first sentence and the following sentences are all coherent and correct for the Italian syntax: the “determined + of + determinant” structure emerged explicitly. In this fourth step it’s clear how the structure is deeply changed, compared to the first step (closer to Aly’s familiar language), through the second and third (a melted syntax) to finally conform to the expectations of the Italian native speakers. In Figure 2, we see that the difference between the first and the second steps is not “clear”, even for Aly. Before adding the ‘of’ (step 3) or circles and arrows (step 4), Aly is in the between of steps 1 and 2 (see middle column, Figure 3). Indeed, her first sentence is not coherent. She worked in the same exact way for the first and third occurrence of the structure “determinant + determined” (step 1), and in a different way for the second occurrence (step 2). In the former case, an additional ‘of’ was needed after the determined (‘base of’, ‘area of’), so the pattern “determined + of” became clear since step 3, and put in front of the determinant (‘rectangle’) using a backward arrow, in step 4; in the latter case, the ‘of’ stays and the circle encloses all the pattern “of + determinant” (‘of circle’), clear since step 2, and it is postponed after the determined (‘circumference’) using a forward arrow, in step 4. Aly is pulled back and forth by her (familiar) L1 and the new (school) L2. The real switch occurs during step 4, where the three previous perspectives are merged, changing the syntactic order using arrows. The final reconstruction of the sentence occurs in step 5, after the teacher suggested to Aly to re-write the argumentation.

5 - Revision (Figure 4): base of the rectangle is equal circumference of circle, area of the rectangle = $b \cdot h$.

BASE OF THE RECTANGLE IS EQUAL CIRCUMFERENCE OF CIRCLE, AREA OF THE

RECTANGLE = $b \cdot h$, CIRCUMFERENCE = $\pi \cdot d = 3,14 \cdot 6$, SO I CALCULATE BASE OF THE RECTANGLE

AND AREA OF RECTANGLE = $b \cdot h = 6\pi \cdot 10$ SO THAT I FIND AREA OF RECTANGLE,

AREA OF CIRCLE = $\pi r^2 = \pi \cdot 3^2$ (SINCE I HAVE TWO CIRCLES, THEN I NEED TO MULTIPLY BY TWO = $\pi \cdot 3^2 \cdot 2$)

THE TOTAL SURFACE IS THE SUM OF ALL THE AREAS

Figure 4: The final argumentation by Aly and its translation.

Except for the conclusion where “equal base times height” becomes symbolic, in the fifth step we see a re-construction, coherent with the fourth step, toward an entirely Italian syntax. Looking at the translation in Figure 4, the first sentence links circumference and rectangle and recalls the area of the rectangle. The next sentence shows the length of the circumference, so Aly determines the length of

the base of the rectangle, and explicitly calculates the area of the rectangle. Then Aly finds the area of a circle, and then multiplies it by two, since (*poi chè*) there are two circles in the cylinder. Finally, the total surface is the sum of the three areas.

Discussion and Conclusions: Language diversity as/is an obstacle or a resource?

To address the research question, we should add some details regarding the experience discussed. After the teacher's intervention and the composition of her second argumentation, Aly, discussing with the researcher, looked ashamed of her first argumentation. The fact that the teacher didn't value her first attempt but asked Aly to re-write her argumentation can be read in different ways. The teacher appeared to underestimate the role the first four steps had for Aly. The teacher wanted a complete Italian-style argumentation, imposing the student's L2 as the dominant language of the class. The teacher believed that Aly could achieve step 5, as she did (also thanks to her good proficiency in L2). These perspectives indeed coexist: Aly created a good argumentation (product, Figure 4), but covered concretely her process (four first steps, Figure 3), gluing the product (Figure 4) *on* the process (Figure 2). Therefore, the researcher asked the teacher why she made Aly write the argumentation again. The teacher explained that the first argumentation (Figure 2) was tricky to understand. She recalled the fact that, in the Chinese language, the structure is "determinant + determined". Although the teacher knew this fact about the Chinese language, she didn't *accept* Aly's argumentation. However, she was aware that Aly was able to develop a full L2-like argumentation: in Figure 2, from the second sentence onward, the argumentation was much more mature. Thus, the teacher believed that Aly's L2 proficiency was enough to make her manage the last, fifth, step, even in the first sentence. Nevertheless, we believe that the teacher didn't value Aly's effort (the process), and just encouraged her to improve her product, overly focusing on the transition between step 4 and 5. If the second aim can be useful to try to bring the student up to the "standard level" of L2, the first aspect cannot be ignored. Aly didn't value the first steps as necessary steps (*her* personal path) to success. She deleted the previous steps and hid (from others, even from the researcher who asked to see her first argumentation) her (meaningful) processes.

In conclusion we could argue that the case study addresses the research question posed above focusing on an argumentation *in the making* and highlighting the strategic role of the teacher in helping the students with weak L2 proficiency on this path. From both the student and teacher point of view, we can say that the diversity of languages can be, at the first sight, an obstacle. The student can conform her/his product to the teacher expectations and hide her/his deep processes, because she/he is afraid of being misunderstood and misjudged by the teacher. In this case, the language diversity of the student doesn't become a resource because her/his argumentation is not accepted by the teacher, since it is not rooted in the dominant language (Italian). Here the focus of the student and the teacher is on the product, which should conform to the L2 syntax, so the process of the student (involving cultural and affective aspects) might be neglected. On the other hand, the native language (Chinese) – and especially its syntactic structure – can be a strategic resource for the non-native speaking student (a deep core to draw on, according to the iceberg metaphor of Cummins, 2000) and an opportunity for the teacher to open and explore the hidden learning processes by non-native speakers. This process of syntactic construction and re-construction appears in a very clear way in the Aly protocol. In particular, the first sentence on which we focused (since it shows the process in the making) is the

compression of a non-linear and multilevel process (Figure 3), due the passage from L1 to L2. According to our analysis, the co-presence of many languages can be used by the teacher to make the student work on her/his linguistic proficiency and to make her/his argumentations clearer. Thus, the obstacle for the student becomes a resource *thanks* to the teacher.

The transition between the L1 and L2 is rarely caught *in the making* of a mathematical argumentation but can enlighten how (that is, through which processes) non-native speaking students become proficient in L2. It is significant to point out that the case study has limitations due the language proficiency of the particular student, which is highly individual. Aly is proficient enough in L2 to develop an autonomous and completely L2 discourse. However, this is just one of the many paths that a non-native speaker could take to reach to a L2 proficiency, during a mathematical argumentation, in a specific socio-cultural setting. Here the language of the teacher (the host country), a L2 for the non-native speaking student, is the dominant language, while the student's L1 is the non-dominant language. Since each teaching practice is, according to Bishop (1991), an enculturation process in which culture matters, the same can be said for accepted and encouraged languages. The teacher therefore not only acts as an *enculturator*, but also as an *acculturator*. The teacher, through her/his power, is able to include and accept in her/his culture (class, school, system) the distant student's culture (usually extraneous to the institution). To switch, as suggested by Cummins (2000, p. 28), from an *either/or* perspective to a *both/and* with regard to L1 and L2, we should involve each language in the teaching/learning practice. Giving up to the "dominant versus non-dominant" dichotomy, leads to liberation, *empowering* each student.

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Two interactional practices used by early years practitioners to encourage children to “try again” in preschool mathematics learning

Natalie Flint¹, Elizabeth Stokoe², Katie Seabridge¹, Tim Jay³, Ann Dowker⁴, Colin Foster¹, Gaia Scerif⁴ and Victoria Simms⁵

¹Loughborough University, Department of Mathematics Education, Loughborough, England, United Kingdom; n.d.flint@lboro.ac.uk, k.seabridge@lboro.ac.uk, c.foster@lboro.ac.uk

²The London School of Economics and Political Science, London, England, United Kingdom; e.stokoe@lse.ac.uk

³University of Nottingham, Nottingham, England, United Kingdom; t.jay@nottingham.ac.uk

⁴University of Oxford, Oxford, England, United Kingdom; ann.dowker@psy.ox.ac.uk, gaia.scerif@psy.ox.ac.uk

⁵Ulster University, Coleraine, Northern Ireland, United Kingdom; v.simms@ulster.ac.uk

Children’s mathematics learning outcomes depend on the quality of education that they experience before starting formal schooling. This research aims to understand how practitioners working with 3–4-year-olds enable children to ‘try again’ in mathematics learning interactions. We used conversation analysis to analyse a corpus of over 115 hours of video data collected in English preschool education settings with 3–4-year-old children. We identify and describe sequences in which the child displayed at least one mathematical error or misunderstanding and in which practitioners scaffold children’s learning to enable them to try again. While previous work has discussed scaffolding in secondary education, this paper examines the practice in an early years context. We identified two communicative practices through which this occurs: (1) reformulating the question, and (2) proposing a joint activity. These two practices exemplify scaffolding in preschool interactions, and how they enable children to try again. These preliminary findings are foundational for future work on practitioner responses to incorrect answers in mathematics learning.

Keywords: Early years mathematics, classroom interaction, scaffolding, conversation analysis

Introduction

In this study we explore instances of children persevering, or not giving up, in mathematics learning. Using video recordings of naturally occurring preschool interactions in England with children aged 3–4, we explore what practitioners do to encourage children to try again. We draw on literature from early years education and conversation analysis to explore the practices used to encourage children to try again.

Theoretical Underpinnings

Children’s experiences in early years settings are associated with mathematics achievement (Melhuish et al., 2013; Sylva et al., 2004; von Spreckelsen et al., 2019). But to understand how these environments influence children’s mathematical learning, and provide information needed by early years practitioners and families, we need to identify the nuances of children’s interactions with adults.

This research shares the view that Edwards and Mercer (1987, p. 1) describe in their work on classroom interaction; “all education is essentially about the development of some shared understanding, some mutuality of perspectives”. Edwards and Mercer suggest that “there is a real possibility that by pooling their experiences they [will] achieve a new level of understanding beyond that which either had before” (Edwards & Mercer, 1987, p. 187). In other words, by interacting with other people, a “reciprocal examination of logical statements” occurs (Rogoff, 1990, p. 192). People share knowledge and do not just learn what the other person knows.

The concept of scaffolding in children’s learning was introduced by Vygotsky (published in English, 1978) and has subsequently been applied to education by many researchers. van de Pol, Volman & Beishuizen (2010), offer a framework of scaffolding at all levels of education and highlight different ways in which practitioners can scaffold, including (1) *feeding-back* information about the student’s performance; (2) *giving hints* in the form of clues or suggestions; (3) *instruction* on what to do or how to do something; (4) *explaining* by providing further information or clarification; (5) *modelling* by offering behaviour for imitation, and (6) *questioning*. Here, we use Conversation Analysis to examine how these strategies are used by practitioners in naturally occurring early years mathematics learning interactions.

Classroom Interactions and Conversation Analysis

One of the most documented concrete interactional practices in the classroom is the IRF/IRE sequence through which, across three turns, teachers *initiate*, children *respond*, and the teacher takes a third turn that, in IRF, constitutes *feedback* and in IRE is *evaluation* (Mehan, 1979; Sinclair & Coulthard, 1975). Ingram (2021) outlines some of the research done on IRF sequences in mathematics learning, which is primarily in secondary education. For example, Ingram, Pitt and Baldry (2015) explored how mathematics teachers handled mathematical errors in whole-class interactions in secondary school education in England. They found that, despite pedagogical recommendations, errors were predominantly treated as something to be avoided. In many cases of errors in relation to mathematical vocabulary, the teacher immediately corrected the error.

Conversation analytic literature on education has examined the third turn with great interest. This typically refers to the turn which comes after an adjacency pair, such as Question-Response. The subsequent third turn is a space which allows us to identify the procedural elements of learning interactions. Lee (2007, p. 180) notes that the third turn position is important but resists the “blanket terms” “evaluation”, “follow-up” and “feedback” and instead focuses on the actions that are achieved by the teachers in these third turns, including parsing, steering, and class management. Koole and Elbers (2014) used conversation analysis to examine mathematics classrooms in Dutch secondary schools. They used empirical data to examine the notion of ‘responsiveness’ (contingency) that is treated as a fundamental characteristic of scaffolding. Our research on scaffolding in early years mathematics learning interactions addresses the gap at the intersection of mathematics education, early years education, and conversation analysis in relation to scaffolding.

Data and Method

The data we analyse here was collected in British preschool settings for children aged 3–4 years (however, one setting had all children from age 2 years to age 4 years in one room – rather than a

separate ‘toddler room’ and ‘preschool room’). For this study, three preschool settings were visited across five weeks (totaling 15 days, or over 115 hours of video). Interactions were recorded throughout the day to capture the range of activities and interactions happening, including more structured learning interactions, such as ‘circle time’, and other activities including mealtimes. All interactions and activities were initiated by the practitioners or the children without direction or intervention from the research team. On some occasions, the interactions recorded were dictated by other constraints, such as recording a group of children where all children had consent to be recorded, or not recording outside on particularly windy days, as the audio quality would be poor. Ethical approval for the study was provided by the Loughborough University Ethics Committee and full written consent was gained from all parents/guardians and staff before recording began. Video data were transcribed both verbatim and for conversation analysis, using Jefferson’s (2004) system for capturing the pace, intonation, and placement of turns at talk augmented by Mondada’s (2019) notations that encode embodied actions (e.g., gestures, body torque, gaze) into the transcript.

Conversation analysis is a systematic method used for analysing ordinary, naturally occurring interactions. It addresses what one of its founders terms the set of “generic orders of organization in talk-in-interaction” (see Schegloff, 2007, p.xiii) on which many of the foundational advances in conversation analytic research over the past six decades have focused, including turn-taking, action formation, sequence organisation, repair, word selection, and overall structural organisation. For example, talk and embodied conduct accomplishes action (e.g., greetings, questions, requests, complaints, invitations, etc.) organised across *sequences* (e.g., basic two-turn pairs of actions to longer and more complex sequences) (also see Clift, 2016). In the current study, we aimed to identify and describe the core communicative practices comprising children in the early years and practitioners, family members, and peers. We collected all instances whereby the topic was mathematical and in which children responded with an incorrect answer to a practitioner’s initiating action. We then examined what it is that practitioners do to encourage children to try again.

Analysis

We analyse two instances from our collection and illustrate our evolving analysis with one example each of the practices through which practitioners encourage children to try again. The patterns we have recognised are as follows: (1) reformulating the question, and (2) proposing a joint activity. Multimodal resources are drawn on in both extracts and are described using Mondadian multimodal transcription conventions (Mondada, 2019).

Reformulating the Question

In Extract 1, two practitioners are sat with a group of children at the start of the day for circle time and have just talked about the day of the week and the date. Immediately prior to the start of Extract 1, the children have been showing the number eight on their fingers, which Practitioner-1 (P-1) praises at line 1. P-1¹ initiates with a question about what month they are in. Practitioner-2 (P-2) is present and sat with the group but is taking a supportive role.

¹ P-1 is out of the view of the camera; therefore, her gestures and gaze are not transcribed.

Extract 1 – What month are we in? – 20230918_Circle-Time (03:10)

01 P-1: Yeah, well done.=And what's the- (.) what month are we in.
 02 (.)
 03 P-1: °What month are we in.°=
 04 MAISIE: =Ju:ne.
 05 (.)
 06 P-1: Not qu[i:te.]
 07 P-2: [Not quite] good guess.
 08 (.)
 09 P-1: Not quite.
 10 (.)
 11 MAISIE: February?
 12 P-1: So we ha:d.
 13 (.)
 14 MAISIE: °Ma[:rch?°]
 15 P-1: [We just had] we've had August so what month comes
 16 <after> August.
 17 (.)
 18 MAISIE: [September.]
 19 (OTHER): [September.]
 20 P-2: £°mmm°£
 p-2 £nods-£
 21 P-1: Well done. September. and what sound does September start with.

After completing a previous sequence where the children have been showing the number eight on their fingers, the practitioner begins a new sequence (what we might call the I in IRE), though this is designed as connecting to, or continuing, the prior with a preface. Practitioner-1 initiates repair on her first attempt at the question “and what's the” before reformulating and rapidly repeating the repaired question on line 03 “what month are we in” (lines 01-03). One child, Maisie, responds quickly with the name of a month, but not the month they are currently in. Here, the practitioners give a dispreferred response “not quite” rather than a direct correction or challenge. The use of “not quite” in the same Turn Construction Unit (TCU) nevertheless mitigates the dispreferred response further by adding praise. Their initial feedback to Maisie’s answer acknowledges that Maisie’s attempt is of the right category of answer (i.e., Maisie answers “June” rather than “Monday”). The practitioners do not attempt to correct Maisie, which often occurs when an answer is treated as a typical repair (i.e., some issue with speaking, hearing or understanding that participants aim to deal with as swiftly as possible – which we see in all types of talk, not just in interactional contexts or with children). In these data, the practitioners acknowledge the attempt and assess it.

Following this initial evaluation of her first answer, Maisie attempts again. After another attempt from Maisie (line 11), Practitioner-1 begins to give feedback which scaffolds the child to getting the answer (although this is initially cut off as Maisie attempts a third guess). After more than one attempt, Practitioner-1 provides a prompt for Maisie to draw on a particular area of knowledge. Here, Practitioner-1 reminds Maisie of the name of the previous month, thus encouraging her to orient to her knowledge of the sequence of the months of the year (also referred to as “funneling” (Bauersfeld, 1988)). This is followed by a reformulation of the question, as Practitioner-1 asks, “so what month comes after August”. The question here is simplified, as the practitioner funnels her scaffolding. There is emphasis on “after” as the word is said more slowly than the surrounding talk. This draws attention to the relationship between the hint, that the last month was “August”, and the question

being asked. This prompt, together with the reformulated question, requires the children to do more than guess by naming a month, but instead gives a hint (van de Pol et al., 2010) to help them to remember sequences and patterns that they might know. This prompt encourages Maisie to draw on this area of knowledge, thus leading to her (and another child) to getting the answer at lines 17-18. Following this answer, Practitioner-1 praises the child for finding the answer and repeats the answer. Here, the practitioners offer encouraging support in the form of assessments following the first attempt. After more than one attempt, Practitioner-1 scaffolds the child, enabling the child to make further attempts, and it is scaffolding that ultimately encourages the child to find the answer.

Proposing a Joint Activity

In Extract 2, Practitioner-3 (P-3), a different practitioner in the same setting, is sat with a group of children who are playing with the sand tray. In the sand tray there are eight plastic shapes and tools (such as brushes) to be used in the sand. Surrounding the sand tray, there are laminated cards with images of the 2D shapes with their names printed underneath. The sequence of interest here is an interaction between P-3 and Izzi (beginning at line 7). Lines 1-6 are provided for context.

Extract 2 – How many sides does a square have? – 20230925_Indoor-Sand (07:56)

01 P-3: who can think of another shape we don't have
 02 (.)
 03 JADA: ☞↑this ↑shape↑
 jada ☞holds regular hexagon-->
 04 P-3: °you'v-° you've just *drawn °around that shape°*
 p-3 *points to shape in the sand*
 05 (.)
 06 P-3: (°wh[at do you-°]
 07 IZZI: [I MADE A SQ]UA:RE (.) I MADE A ↓SQUA:RE:.
 08 P-3: oh good girl: how many sides does a square have
 09 IZZI: err:: (.) three?
 10 P-3: ooh. not quite how many sides does a square have☞☞
 jada -->☞☞
 11 P-3: shall we count them
 12 IZZI: THREE::
 13 P-3: o[oh. not *qui]:te.
 p-3 *picks up square-->
 14 JADA [£seve::n£]
 15 P-3: you ↓girls are tricking me.=how many sides does
 16 a square have.*
 p-3 *points using brush at top side-->
 17 (.)
 18 IZZI: ♡o *:ne, ♡
 izzi ♡points at top left corner of square♡
 p-3 -->*
 19 IZZI: ♡two:, ♡
 izzi ♡points at bottom side of square♡
 20 IZZI ♡three:, ♡
 izzi ♡points at right side of square♡
 21 IZZI ♡↑four. ♡=
 izzi ♡points at top side of square♡
 22 P-3: =amazin. well done. *
 p-3 puts square down-->*

At line 1, the practitioner asks the children for another shape that they have not drawn and talked about yet. Jada proposes a shape; however, they have already drawn this. At line 7, Izzi announces that she has made a square. Practitioner-3 acknowledges and praises Izzi before asking her a question about the characteristics of a square, namely, how many sides a square has (line 8). Izzi responds “three” hesitantly following “err” and with rising intonation. The practitioner responds by acknowledging that Izzi’s answer is close, “not quite” (line 10), much like the responses to the first attempt in Extract 1. The practitioner then asks the same question again but follows this up with a proposal to count as a joint activity (line 11). Research on children’s play has identified common ways in which children propose joint activities, typically in the form of “Let’s X” or “How about X” (Stivers & Sidnell, 2016). Clayman and Heritage (2014) identify proposals in mundane conversation (between adults) that adopt the stance of both self and other as beneficiaries take the form of “Shall we X”, whereas in noncongruent cases, offers or requests are treated as proposals. Here, the proposal of the joint activity takes the form of “Shall we count them”.

Following this initial proposal, Izzi does not participate in the counting, but instead offers the same answer again, which receives a similar “not quite” response. Jada gives a mock answer of “seven”, which she elongates and says with smiley voice. The practitioner treats Jada’s response as non-serious. She then re-asks the question, again without reformulating. However, she begins the counting activity by pointing at the sides of the shape with a brush (after picking up the shape after Izzi has repeated her incorrect answer). Izzi then participates in the counting activity, pointing and counting out loud. She counts the numbers, one, two and three with rising intonation and elongation, and “four” without any elongation and falling (turn-final) intonation, marking this as the cardinal number. Practitioner-3 then praises Izzi, and the conversation moves on to discuss other shapes.

In this case, the proposal of a joint activity is not successful in the first instance. It is only upon the practitioner initiating the activity herself that the children join in. However, by turning to a joint activity which includes the practitioner (but could also invite the other children to participate in the interaction), the child tries again and reaches the answer. Here, the practitioner does not model the behaviour in full, but instead she models the activity that needs to be done, and once the child has begun participating, she steps back and allows the child to find the answer. Here, we see an example of how ‘modelling’ (van de Pol et al., 2010) can be done by an early years practitioner.

Discussion and Conclusion

The aim of this paper was to examine how practitioners support children to try again in mathematics learning interactions. We have presented two sequential structures that we have observed from these preschool interactional data. We have discussed (1) reformulating the question, and (2) proposing a joint activity. These practices have demonstrated how early years practitioners successfully get children to try again after a mathematical error.

The findings presented here are preliminary and lay the foundation for future work on practitioner responses to incorrect answers in mathematics learning. The two practices identified here were selected as they have been prevalent in our initial analyses – but we do not suggest that they represent the full range of practices. In future, we will explore further scaffolding practices.

In both cases presented here, the initial response in the third turn to the child's response is an initial assessment that acknowledges the attempt as being a good attempt such as "not quite" or "very close" or "good guess". In Extract 1, this is all that is offered following one attempt, allowing the children to attempt again. As Extract 1 comes from a whole-class interaction, this could be to allow another child an attempt prior to doing any scaffolding. In Extract 2, the scaffolding comes immediately after the assessment, however here, the practitioner is working with a small group of children. In both cases, the assessment acknowledges the answer as being a good attempt, but implies that this is not the correct answer, which is typically responded to with "well done" or "good boy/girl" or a confirmation of the answer. We have explored how two types of scaffolding, set out by van de Pol et al. (2010), are achieved interactionally, namely, *giving hints* and *modelling*.

In the cases examined, the third part is rarely confined to a third turn. At first glance, the IRE and IRF sequence suggests that each element represents one turn at talk. There is an implication from this that the third part of these structures (i.e., the evaluation, follow-up, or feedback) should be confined to one turn (or *minimal post-expansion*). However, in the cases we have examined, the third part consists of more than one turn and is instead an extended sequence (or *non-minimal post-expansion*) (see Schegloff, 2007). This study contributes to our understanding of the third turn in these sequences.

Acknowledgment

We would like to thank the nurseries, schools, practitioners, and children for participating. This work was supported by UKRI Economic and Social Research Council [grant number ES/W002914/1].

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Intersubjectivity in peer interactions – the emergence of responsive moves in block play situations

Esther Henschen¹, Anna-Marietha Vogler² and Martina Teschner¹

¹University of Education Ludwigsburg, Germany; henschen@ph-ludwigsburg.de

²Martin Luther University Halle-Wittenberg, Germany

While in current studies on interaction quality and learning support in interactions in preschool, adaptivity and micro-adaptive learning support provide the theoretical basis for research, the following article will focus on the emergence of situational intersubjectivity, which is expressed through a shared understanding of the situation and the interrelatedness of interactive contributions that can be seen as a precondition for scaffolding processes. To reconstruct this interrelated interactive interplay of actions, the empirical part of the article refers to the construct of responsiveness. In children's block play interactions in which a common goal is pursued, it can be seen that responsiveness is a guiding principle and thus enables scope for participation in mathematical discourse.

Keywords: Intersubjectivity, responsiveness, play situations, preschool, peer interactions.

Introduction

Interaction (OECD, 2004) and the quality of interaction (Wullschleger et al., 2023) is seen as a key element for subject-related learning in kindergartens: A high interaction quality provides potential for subject-related learning because changes in the quality of micro-adaptive learning support go hand in hand with the changes in children's mathematical competency (Dunekacke et al., 2024). Simultaneously, empirical findings from a large number of studies on (preschool) teacher-child discourse indicate that empirically effective forms of learning support (long dialogues, focussed feedback, argumentative depth) occur rarely in practice (cf. Anders et al., 2012; König, 2009; Siraj-Blatchford & Manni, 2008; Vogler, 2020). However, besides the elaboration of (further) professional development concepts for preschool teachers to support effective forms of learning support (Bruns et al., 2021), it also seems relevant to investigate further discourses in kindergarten with regard to their interaction quality. Discourses among peers are an obvious choice in this context, as peer interactions make up a large part of the interaction time in preschools, and subject-related learning also happens through collective mathematical argumentation (Henschen et al., 2022).

Adaptivity and "micro-adaptive learning support" (Dunekacke et al., 2024) often provide the theoretical basis for current research on interaction quality and learning support in interactions in preschool. In contrast, this article will focus on the emergence of situational intersubjectivity, which is expressed through a shared understanding of the situation and the interrelatedness of interactive contributions that can be seen as a precondition for scaffolding processes. To reconstruct this interrelated interactive interplay of actions, the empirical part of this article will refer to the construct of responsiveness based on the work of Beck and Vogler (2023). Based on theoretical explanations and empirical analyses, the significance of intersubjectivity for professional learning support is presented using peer interactions in preschool settings as an example. The analyses show that high

interaction quality, which is rather lacking in teacher-child interactions, is emergent in peer interactions.

Adaptivity, Scaffolding and Sustained Shared Thinking

In the current discussion on mathematics education, the quality of interaction in early mathematics education is measured quantitatively based on the degree of adaptivity (Dunekacke et al., 2024). Adaptivity or micro-adaptive learning support is described in these quantitative studies as "the fit between the actions of the kindergarten teacher and the mathematical activities of the children" (Dunekacke et al., 2024, p. 13). The fact that these studies focus on the match between the teacher's instruction and the child's reaction in the operationalisation (Dunekacke et al., 2024) poses a challenge because theoretical explanations of adaptivity or adaptive support in pedagogical studies, which are referenced here (van de Pol et al., 2022), understand adaptivity in a different sense: "Adaptivity is the extent to which teachers adapt their regulation of the learning process to the student's understanding" (Dunekacke et al., 2024). In the context of interaction quality, the concept of scaffolding is at the centre of these educational studies: adaptive support is understood as a "key feature of scaffolding" (van de Pol et al., 2023, p. 766). In early (mathematical) education, the concept of scaffolding is also made relevant by various studies on learning support (for a summary see: Krammer, 2017). For further theoretical considerations, the concept of scaffolding is a key point for the exploration of interaction quality in early mathematics education.

Therefore, "the first common characteristic in the various definitions of scaffolding is contingency" (van de Pol et al. 2010, pp. 274-275). Contingency is often described as "responsiveness, tailored, adjusted, differentiated, titrated, or calibrated support". It is characterised on the one hand by the support being appropriate to the child's current learning level. On the other hand, "a teacher [...] acts contingently when he/she adapts the support in one way or another to a (group of) student(s)" (van de Pol et al. 2010, pp. 274-275). This description of contingency implies a hierarchical structure of the interaction process and assumes that there is an asymmetry in the knowledge among the participants. As a consequence, the knowledgeable adult is required to realise the support.

Intersubjectivity as an important feature of (subject-related) support

What is central for the scaffolding process or support, however, is the condition of intersubjectivity, which is characterised in the interactions by mutual reference between the participants. Intersubjectivity is a key element in the scaffolding process because it is the condition for the adaptation of support by others (Elstrodt et al., 2019). Negotiating a shared topic can be identified in this context as a constitutive element of helpful support (Krammer, 2014). In addition to educational and didactical studies, research in the field of language acquisition also emphasises situational intersubjectivity as a key element for the quality of interaction. In this context, Tomasello (2008, 156) cites Quine: "In acquiring [language], we must rely entirely on cues that are intersubjectively accessible and allow us to recognise what we need to say and when". This means that the shared attention that arises from collective interactions with shared goals is relevant for children's (language) learning (Tomasello, 2008). Intersubjectivity or an intersubjective understanding of the situation can therefore be recognised as a relevant feature of subject-related support. However, support and subject-related learning at an early age does not only take place with adults in interactions: Peers also play an

important role in learning and socialisation (Henschen et al., 2022). While recent work in mathematics education has shown that there is often no intersubjective understanding of the situation in preschool discourses (Krammer, 2017; Vogler, 2024), it seems appropriate to investigate the complementary discourses of the preschool–the peer discourses–for such intersubjectivity and subject-related learning. In the following, the focus is therefore on how the intersubjectivity that is emergent in peer interactions can be reconstructed.

Theoretical considerations on the reconstruction of intersubjectivity

Based on these theoretical considerations and the resulting question, it is therefore necessary to reconstruct how intersubjectivity - and thus the conditions for a high level of interaction quality - is established reciprocally between those involved in the interaction.

In the analyses, responsiveness is used in this context as a sensitising concept (Blumer, 1954) in order to reconstructively capture this reciprocal referencing (Beck & Vogler, 2023). Interactions are described as responsive if they emphasise the content of learners' ideas, adopt and pursue these ideas and make connections to the subject content (Robertson et al., 2015). In order to be able to integrate the content of learners' ideas into the process of negotiation of mathematical meaning, it is a necessary condition to understand or interpret the children's contributions in relation to their mathematical content. This understanding can be seen as a responsive act: One has to engage with someone else's idea and react to it in order to understand it. From this interactionist perspective, responsiveness can therefore be characterised as a coordination process between two people. In contrast to the work on responsiveness in classroom discourse, however, responsiveness is less to be understood here as a one-sided interaction in which a teacher adapts the actions. Rather, responsiveness is a phenomenon of equal coordination in interaction in which all interactants orientate themselves towards the needs of others in order to react to them in a prompt and coherent manner (Beck & Vogler, 2023). Responsiveness is therefore understood in this article as a discursive characteristic. In this context, Beck and Vogler (2023) propose a distinction between two different modes of responsiveness in the interactive interplay of the classroom based on analyses of empirical data from primary school mathematics classrooms: The adaptation of action in interactive interplay in relation to (1) mathematical content (measure of content-related responsiveness) and the adaptation in relation to (2) interactive discursive skills of the learners. The latter enable learners to participate in the discourse but is not primarily associated with a negotiation process of mathematically rich topics (measure of linguistic-discursive responsiveness). The way (technical vs. linguistic) and the quality of orientation towards the ideas of the other's contributions can differ greatly (Bishop et al., 2022; Beck & Vogler, 2023).

Research desiderata and questions

While previous studies in mathematics education have tended to focus on the responsiveness between the teacher and the learner or between the teacher and the child, this article looks at the discourse between children in kindergarten during block play. This appears to be a worthwhile approach for two reasons. Firstly, there are no clear roles between peers in kindergarten, meaning none of the interactants has the task of acting responsively due to a role attributed to them from outside. Responsive action then occurs because it is necessary for those involved in the interaction and around

a certain topic. On the other hand, a large part of the interaction time in kindergarten takes place between children and learning also occurs during these times (Henschen et al., 2022). In addition to studies focusing on asymmetrical teaching-learning processes, it therefore seems important to also understand how children construct knowledge with each other. Establishing an intersubjective understanding of the situation and thus the children's responsive actions towards each other could be a key to this. From this perspective, it may be possible to develop approaches for learning processes in professional-child discourses. The article therefore examines the following question: To what extent are interactions in block play situations characterized by content-related and linguistic-discursive responsiveness?

Data and methodology

Data corpus – block play with peers in everyday kindergarten life

Ethnographical video data was created as part of Henschen's dissertation (2020) has been used to answer this question. The interactions analysed for this article originate from a free play phase (completely videotaped) in a kindergarten. The observed children were 5-6 years old, and they played with the Sonos construction material with which various structures (buildings, vehicles, etc.) can be built using plug-in connections and which is always available to the children in the institution (see figures in the transcripts below).

Methodological (pre)considerations and methods of analysis

Henschen (2020) used qualitative content analysis to show that the children addressed a wide range of mathematical content in the block play situations. With the help of interaction analyses, it was possible to build on this microanalytically and the thematic negotiation in the scenes could be traced (Vogler et al., 2023); in addition, functional argumentation analyses allowed the reconstruction of collective and multifaceted argumentations (Vogler et al., 2022). All of these analyses have already shown that children work collaboratively during block play and content and methods relevant to mathematical learning can be found in negotiation processes. The assumption is that responsive moves is a precondition for this, which is why it seems worthwhile to analyse this data with this focus.

Beck and Vogler (2023) also use interaction analysis to reconstruct responsiveness in interactions. In this qualitative analysis method, the utterances of teachers and learners are first interpreted individually in the order in which they occur so that their relationships with each other can then be traced based on conversation analysis. This makes it possible to work out both the negotiated mathematical meaning as well as the characteristics of the turn-taking system and thus the discursive referencing. The analysis method is therefore also used here.

Analyses of an empirical example of block play situations in kindergarten

Before the analysed situation begins, Ron and Max have already been working with a pre-existing construction made of Sonos material. In minute 26, after Max has attached a construction with wheels (the children refer to it as a "forklift") to the top of the structure. Ron remarks, "Why all the way to the top, then they can't get it down at all". Max then says that "they" have a "ladder", and Ron suggests that they build a ladder. After the two of them have made a short piece of the ladder (4 rungs), they turn to the structure with this piece.

004 Ron: und wie kommen die dann da hoch/
(And then how do they get up there?)

In connection with Ron's previous statement that the forklift cannot be brought down from the very top, the question can be understood as an initiation. However, this impulse is not disjunctive in the interaction process; rather, it can be understood as a common starting point for the play situation. In the sense of Tomasello's (2008) remarks on intentionality, this can be understood as an invitation to think together about the construction problem. As the play progresses, the children initially agree to build a ladder and start to do so. This leads to disagreement. Ron disagrees with Max's suggestion to fix the (too) short piece of the ladder and therefore asks for clarification, which does not directly follow.

026 Max: nein\ wir stapeln des in dem Stockwerk\
(No, we'll stack it on this floor.)
027 *lays the ladder through the building (figure 1, left)*
028 wir stapeln des so (We can stack it like this)
029 *makes a stacking motion with his hand*
030 Ron: *takes out the ladder*
031 nein\ wir stapeln des so hoch
(No, we will stack it this high.)
032 *places the ladder diagonally on the "floor".*
033 weil sonst kommen die doch gar nicht da hoch
(Because otherwise they wouldn't be able to get up there.)



Figure 1: pictures of the building activities

Max's utterance can be understood as a counterproposal to the fixing. He suggests integrating the ladder into the building that is highly flexible and offers the possible utility for the shared play context. In this context, Max uses the expression "stacking", as does Ron in the situation. It becomes obvious how both are interwoven with each other and can mutually stimulate each other.

The linguistic-discursive responsiveness seems to be emergent in the interaction between the two children: Ron connects his statements to Max's previous remarks by adapting his syntax and his choice of words to Max's. Ron does, however, distance himself from Max's idea of stacking the ladder on the floor but refers to this idea in terms of content by marking an alternative suggestion (otherwise) and argues why his solution is more viable against the background of the originally agreed upon goal (to get up). This interpretation is supported by the following negotiation.

052 Ron: aber sonst können sie dort gar nicht hochkommen
(otherwise they can't get up there at all)
053 Max: aber die bauen die Leiter zusammen \ du Fatz#
(But they build the ladder, you goof#)
054 *looks at Ron*
055 Ron: alle Leitern/ (all ladders?)
056 Max: Ja (Yes)
057 *looks towards Ron and continues building the ladder in his hand*
058 Ron: Okay\ Ich leg die mal so hin gell dann können die da hoch

059

(Okay. I'll put them there like this, then they can go up there)
puts the ladder down, leans it against the building at an angle

In line 52, Ron picks up again on the outline of the aim of the construction, which mathematically shows a link to spatial orientation and the topographical idea of connection (Henschen, 2020, p. 318) from line 033: "and how do they get up there?" where Max refers to this initial question in line 053. Ron calls Max a "goof" (German: Fatz). On the one hand, this symbolizes his involvement in the action and, on the other, he rejects Ron's solution as possibly too complicated for him. Instead, Max shifts the solution to the narrative play action. Ron responds to Max's objection by making a simple, linguistically adapted request, which signals to Max his concession; in this context, "all" may also stand for "your and my ladder pieces". After Max agrees with Ron, Ron initially confirms his concession with "okay". The responsiveness in Ron's further statement "I'll put them down like this" is shown in the openness, simplicity, and concreteness of what is said. It is interesting to note that he nevertheless keeps the initial common goal of "coming up" in mind.

After some time, the two children work together to build a large ladder from existing pieces and continually extend it. In the following phases of the play activity, the focus is on the further handling of the longer ladder that has now been created and on attaching the ladder to the building as well as a possible anchoring (anchoring problem). A linguistic-discursive responsiveness becomes apparent again and again, as the same words and similar syntactic structures occur in successive utterances.

In contrast to this, the following section shows more content-related responsiveness when Ron then also succeeds in convincing Max to accept a solution to the "anchoring problem".

- 140 Ron: Aber man kann es doch so befestigen und das ist gut\
dann hebt sie besser\
(But you can fix it like that and that's good, then it will hold better.)
141 *points to the upper end of the ladder and the upper edge of the structure*
- 142 Max: Aber wir brauchen erst wieder so ein langes (...)
(But we need another long one like that first.)
- 157 Ron: warte des müssen wir# (wait, we have to#)
158 *together with Max he pulls the ladder upwards out of the structure*
- 159 Max: ausbauen (remove)
160 *tries to fix the rod to the corner again*
- 161 Ron: *together with Max, he grabs the ladder*
162 des müssen wir da richtig rein die Leiter#
(we have to really get the ladder in there#)
- 163 Max: richtig einbauen (installing correctly)
164 *together they connect the ladder to the top corner of the building,
the ladder now hangs parallel in front of the building (figure 1, right)*

Ron's absolute will to fix the ladder, which he had repeatedly articulated before this scene, now meets with Max' response for the first time. This is not expressed through a linguistic-responsive action by Max through the expression of approval, but rather through a reference to the mathematical content-related task that arises, namely finding a suitable part to connect and close the gap. Content-related responsiveness is evident here. The discourse soon reaches its peak when the children alternately continue the sentence with the appropriate content and realise the imagined solution by acting together. The children interact not only responsively in a linguistic-discursive way, but also in terms of content in the negotiation process, which could be due to an intersubjectivity that is inherent in collaborative play and can be a prerequisite for learning through joint play.

Conclusion

Based on the importance of intersubjectivity for micro-adaptive and subject-related learning support, this study has investigated how the negotiation of shared meaning is mutually established between the participants of the interaction. The construct of responsiveness provided valuable insights into the children's negotiation processes. The analysed collaborative block play situation is characterised by a high degree of linguistic-discursive and content-related responsiveness. While a high degree of linguistic-discursive responsiveness often emerges in interactions with preschool teachers in everyday kindergarten life (Beck & Vogler, 2022), the example of children in free play shows that this context also fosters content-related responsiveness. While a high degree of linguistic-discursive responsiveness primarily keeps the interaction going, it can be assumed that content-related responsiveness is crucial for mathematical learning. The responsiveness seems to be a manifestation of the efforts to develop intersubjectivity in collaborative play. This also explains the children's long-lasting negotiations which hold great potential for mathematical learning.

Responsiveness can be seen as a guiding principle in children's block play interactions in which a common goal is pursued. This responsiveness opens up a participatory space in mathematical discourse. Cooperative play (Tomasello, 2008) therefore also appears to be of particular importance for the creation of learning environments. The current focus on researching learning environments with rule-based games (Wullschleger, 2023) uses competing game mechanisms. This article has further differentiated both theoretically and empirically that intersubjectivity can be seen as a requirement for a successful subject-specific learning process in cooperative play situations. These findings have shown that it is valuable to continue researching cooperative peer play situations.

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Observations on naming and labelling practices in angle work with teachers: how might small language choices support mathematical activity?

Jenni Ingram, Robert Ward-Penny, Elizabeth Kimber and Núria Planas

University of Oxford, UK;

Jenni.Ingram@education.ox.ac.uk, Robert.Ward-Penny@education.ox.ac.uk,

Elizabeth.Kimber@education.ox.ac.uk, Nuria.Planas@education.ox.ac.uk

There is a growing interest in how word choice in mathematics classroom interactions can influence the meanings and perspectives that students develop. In this paper we contribute to this by offering examples to highlight the variation in how mathematical objects and relationships can be described and the implications these differences may have on the learning of some students. These examples are drawn from a discussion between a group of researchers and teachers working together on a geometric task as part of a project where they are collaborating to develop resources to support language-responsive teaching. We focus in particular on the naming and labelling of angles and the relationships between angles, and the different roles these names and labels may have for the fluency, precision and meaning in mathematical communication.

Keywords: Noticing, naming and labelling, fluency and precision.

Introduction

Language plays a central role in teaching and learning mathematics. It enables students to express and explore both abstract concepts and the semantic relationships between them, as well as being constituent to practices such as arguing, reasoning, and justifying. The attendant importance of language to pedagogy is reflected in a growing body of international research exploring the potential for language-responsive approaches to teaching mathematics (Erath et al., 2021). Recent research, in particular, has highlighted the importance of the choice of specific words, phrases or representations to communicate about meaning but also to support the development of that meaning (Götze & Baiker, 2023).

In this paper and in our work with teachers, we emphasise smaller units of mathematical talk and language and the relationship between these and the mathematical meanings and concepts they relate to. This builds on a growing body of research focusing on the intertwining of these smaller units with mathematical practices (Götze & Baiker, 2023; Planas et al., 2024; Ingram, 2021). These smaller units include individual words, phrases or sentences that focus on a mathematical object, relationship or action and may or may not be specific to the mathematics register.

One example where this level of granularity has been helpful concerns the connected actions of naming and labelling. With specific regard to naming, much of the existing research has been on teachers' noticing of naming in their own talk (e.g. Adler, 2021; Planas et al., 2024), with contrasts made between mathematically correct and precise names and more ambiguous names or pronouns such as 'it' or 'that'. In classroom interactions, mathematical correctness and precision may be in

tension with developing a shared meaning and the efficiency and fluency of communication (Adler, 1998; Planas et al., 2024).

The account below draws on a small section of one episode from a design-based collaborative research project focused on language-responsive mathematics teaching to illuminate the importance of small words and short phrases and sentences in mathematical communication, and the value of paying attention to their various possible uses and meanings, and the consequences these may have on learners. While we offer data from the project in this paper, these are to illustrate the arguments we are making which are tentative and evolving as we work alongside teachers and students. Our aim is not to present findings, but rather to provoke a conversation around these small units and their role in mathematics teaching and learning.

The professional development (methods)

The authors are part of a research team conducting a wider design-based research project (Cobb et al., 2003) which involves working alongside a group of ten secondary mathematics teachers in a series of professional development (PD) sessions. A central part of each session is a discussion built around mathematically rich tasks that some of the teachers already use in their classrooms. These discussions are intended to foreground topic-specific language demands and discourse practices, in line with the recommendations of Erath et al. (2021), with separate sessions focusing on linear equations, probability and angle properties of parallel lines.

Participation in the project is intended to promote teacher awareness of the use and import of language, with the goal of supporting pedagogy. It is hoped that drawing attention to challenges and variation relating to language use, within and across topic-specific domains, will support teachers in recognising, working with and, where appropriate, facilitating flexibility in learners' own mathematical thinking and use of language. This awareness can arise from the activities and discussions led by the research team, through their discussions with each other about their experiences using the tasks, as well as reflection on what their students say in their lessons that follow the PD. In this way the PD is based on the Discipline of Noticing (Mason, 2002).

The PD workshop from which the data for the discussion below is taken focused on identifying the linguistic demands and opportunities for student reasoning in a task involving a sequence of (connected) geometrical diagrams focused on using properties of parallel lines to identify a missing angle. The interaction we focus on involves the teachers and researchers/teacher educators reporting back to each other how they identified the missing angle, with each teacher focusing on a different image. All participants had access to all the diagrams, but the discussion was deliberately managed so that the diagrams could not be annotated. This forced the participants to communicate their reasoning verbally.

Theoretical stance

The members of the research team have experience of different theoretical frames, including ethnomethodology, sociocultural theory and systemic functional linguistics (SFL), and we consider this diversity to have enriched our engagement with the data. What is often central to each of these

perspectives is the focus on communication in interaction, and an understanding of learning as developing shared meanings and ways of communicating.

From a SFL perspective, in mathematics education research the focus is on how linguistic communication represents ideas and interpersonal relationships (Halliday, 1994). From an ethnomethodological perspective, both what is said and how it is said make a difference to how people act in interaction (Heritage, 1984). From a sociocultural perspective, communication carries with it the social and cultural meanings that have been established over time (Werstsch, 1988). Beyond mathematics education, there is also a wealth of research on social interaction that has demonstrated how word choice can impact behaviour (e.g., Heritage & Robinson, 2011) and the perspectives we take.

For the purposes of this paper, language is language use, that is, the use of linguistic forms, such as words, phrases and sentences, in articulation with other modes of communication, such as visual diagrams. Moreover, we view language use as indicative of choice (words, phrases, gestures or images), though not necessarily conscious choice, and that the meanings which may be construed depend on the interactional context. Within mathematics education research, word choice is an under-researched aspect of linguistic communication. Language use at the level of choosing single words, phrases or relatively short sentences has important functions in mathematical communication, including the function of naming concepts, processes or practices in particular ways, each of which encodes meanings.

Following Mason (2010), we treat learning mathematics as involving students perceiving similarities, relationships and properties and discerning differences or features in a new way. Learning about angle properties of parallel lines involves recognising the similarities in the relative positions of the lines and angles that result in angles having the same size, which we might name as alternate or corresponding angles. Distinctions need to be made between lines that have the properties of being parallel and those that do not, so that the properties of parallel lines can be used either dynamically by transforming one angle into another through rotation or translation, or statically by deducing sequentially the relationships between different angles.

Results and discussion

In this paper, we draw on two aspects of the interaction in the PD session to highlight the variation in language used and the implications this has for ambiguity in communication that is often indicative of fluency in communication and the tensions between mathematical clarity and precision and the communication of reasoning. We show how word choice conveys different meanings and perspectives (i.e. where attention is being focused or directed), as well as how the sequencing of these words can affect the communication of reasoning.

In Table 1, we have collected the different names used to refer to angles or relationships between angles during the discussion by the participants around the sequence of tasks. The names have been partly organised in rows to reflect some immediate connections, but in general the structure of the table does not necessarily indicate related use or adjacent cases.

This table demonstrates a substantial variation in how angles may be named. Some of the names for an angle are simple labels (e.g. A, angle A), whilst some reflect properties of the angle such as its size (e.g., the seventy degree angle). Other names locate the angle in relation to another angle (e.g. the alternate angle) whilst some follow an agreed convention, relating the angle to other geometric elements (e.g. BDC^2). In switching between these, students have to navigate this variation, often in contexts where the name and the relationship or property being named is a relatively new idea.

A single angle	A collection of two or more angles	Relationships between angles
the seventy degree angle	the angles	angles in a triangle add up to one hundred and eighty degrees
the alternate angle	the alternate angles	are alternate
A	lots of 40° angles	equal
it		equivalent
angle A		angles at a point equal three hundred and sixty
angle forty degrees		vertically opposite angles are equal
a plane angle		opposite angles
co-interior angle	co-interior angles	co-interior angles
third angle		the corresponding angles
the one hundred and twenty		angles in a triangle
BDC		angles around a point

Table 1: Ways that angles or relationships between angles were named

The names used to indicate relationships also vary. Some of these reflect permanent or generalised relationships or rules (e.g., angles on a straight line add to one hundred and eighty degrees), while others are local to the specific angles being talked about (e.g., the alternate angles). Sometimes, the clarity of a label can shift over time; for instance, as one of our participating teachers noted, naming angles with their size can be effective until more than one angle of the same size becomes relevant in an argument, as is often the case when teaching angles in parallel lines.

² Note: This is what was said so the angle notation usually used in formal mathematics writing has not been added

We notice here that there are only subtle differences in the ways that some of these properties are used to describe angles and their properties or relationships. Such differences might be difficult to notice, particularly for students facing wider linguistic challenges. For example, *the alternate angle* refers to an angle with a specific relationship to the angle being previously talked about. *The alternate angles* refers to two angles that have the relationship that they are equal (in size) because they are alternate angles, and this relationship can be perceived as the result of a transformation (rotation around a point) or as a static rule for equal angles (so-called Z angles). Finally, *are alternate* focuses on the relationship as a property that these angles have.

Mason (2010, p. 23) offers that “each technical term in mathematics signals the fact that those who developed the term needed a label”. However, we note here that although these needs may be consequent from a task, they originate with individuals; students will have different needs as they are cast in different roles as speakers and listeners within the classroom, as they develop their mathematical knowledge, and as they engage in different mathematical activities. The subtle semantic differences between names may fade when viewed through the lens of experience and with fluency, but each difference holds some potential to confuse or frustrate a learner and is thus worthy of attention. Even whilst working as experienced mathematicians, our participating teachers showed differences in how they engaged with the process of labelling. After a discussion of the final problem, one teacher ventured “I think the labelling definitely helped that one”, but a second teacher reflected how the “labelling was helpful, but it took me some time to actually match the labelling to the picture.”

Fluency and precision

We also offer here that fluency or precision is not an absolute, but rather inclusive of sensitivity to what is important and what is sufficient, both of which can be influenced by and inferred from context. This resonates with Grice’s maxims of communication and Adler’s dilemma of transparency (1998). When we interpret the situation as being one in which what is unsaid is understood, efficiency can be emphasised over precision (e.g., angles around a point as an abbreviation for angles around a point sum to three hundred and sixty degrees), and this assumption of a shared understanding of what is unsaid suggests a fluency in the ideas, rules or properties being communicated.

Precision has two roles in mathematics classroom interaction – precision in the use of technical terms, but also precision in terms of the shared attention on what is being talked about. These are not necessarily the same in the teaching and learning of mathematics, with mathematical precision sometimes obscuring or distracting from the focus (Adler, 1998). Precision in the shared attention can also be achieved through the use of meaning-related words and phrases and not solely through technical terms from a mathematics register.

Meaning is dependent upon the interactional context and the choice of words. Communication about relationships, in particular, needs students to pay attention not only to the relationship, but also to the objects between which the relationship exists. Describing an angle as “the seventy degree angle” or as “the alternate angle” emphasises different features of the angle – is it the size that is important or the relationship to another angle? We also note that naming by size draws attention to a local property of the problem whereas naming by relationship may allow for variation in the problem that maintains

this relationship, drawing attention to a more global property of the angle. Where the focus is on relationships between angles, a lack of precision in the naming of the angles or even incorrect use of names or non-mathematical names may not impede communication and could also support the focus on the relationship, e.g. this angle here is alternate to that one there.

Adler (1998) stresses the need for teachers to make decisions about whether to emphasise mathematical precision or emphasise meaning by using words familiar to the students. What we stress here is that this is not a dichotomous choice, and this choice needs to be influenced by focusing on what we want students to notice and the distinctions we want students to make. Mathematical ideas, objects or relationships that are in focus can be foregrounded by using the technical term, while more informal language can support the backgrounding of other features.

Language in service of intersubjectivity

We now offer more detail of two incidents from the same PD session which we consider to be interesting.

The first instance arose when a teacher described verbally their method of solving a problem that involved a diagram that included two pairs of parallel lines and three angles labelled with their size and a fourth angle whose size needed to be found (Figure 1a). They began with the following step:

“I added in three parallel lines that cut through the seventy degree angle, the hundred and fifty degree angle and the hundred degree angle, that were all parallel to the brown lines...”

There was some confusion in the group after the end of this description, with a second teacher sharing that “...because I’m not seeing it, it’s hard for me to visualise what’s happening” and indicating some confusion about the parallel lines. The first teacher then reframed the key step:

“So I inserted three parallel lines that were all parallel to the browns, at the seventy, at the one hundred and fifty and at the hundred...”

Even though this contained no new information, this reformulation proved to be effective, and the second teacher said that they could now place the parallel lines. One reading of this situation is that in the new version of their statement, the first teacher shared the information in an order that was more aligned with the (unseen) visualisation of the second teacher. In this case, a small shift in language use would have had a meaningful consequence for the learner. The placement of the “three parallel lines” requires the listener to know parallel to what and where to draw these lines (through the three labelled angles).

The second event concerned a diagram with four line segments in two separated parts (Figure 1b).

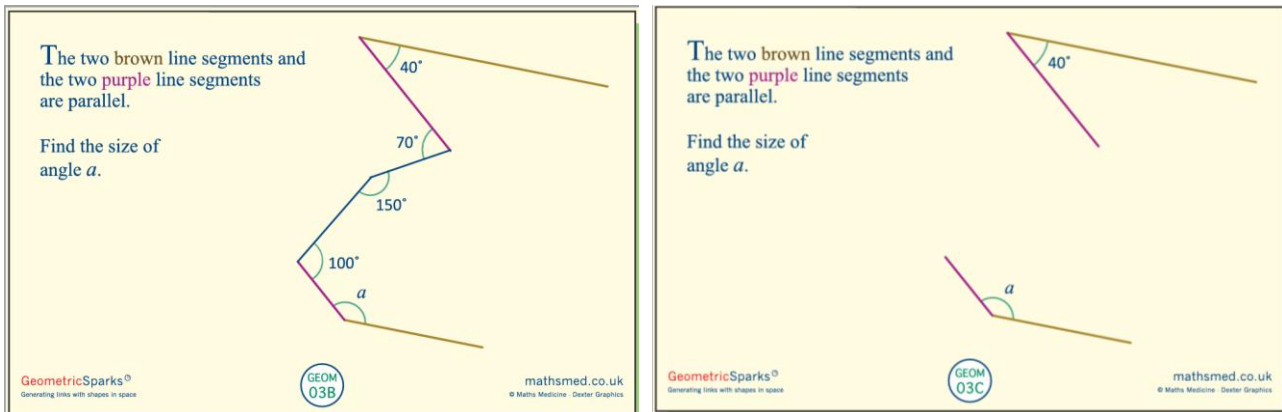
A solution to this problem was shared by one of the researchers:

“I’m not extending or adding lines... I’m not thinking about the static geometry I’m thinking about dynamic geometry... if I move it, I translate angle a up or to the left or to the right... it remains angle a... I change the position, but I don’t change the measure.”

The researcher then offered the step of overlapping the brown line segments, so that the two angles labelled in the diagram were placed next to each other, and the two purple line segments would form one longer line segment.

In the fruitful discussion that followed, the same teacher who had expressed confusion in the first incident several minutes later indicated that sufficient intersubjectivity had been achieved. Moreover, it had prompted further mathematical thinking:

“I really like that approach... to be truthful, that would not be my first option but no, I’ve just learned something because even while you were explaining I’ve just seen another way of doing that question...”



Figures 1a and 1b: Two angles tasks (reprinted with permission from Küchemann, 2023)

In a contrasting way to that described above, it is possible that this success may have been supported by a closer match between the researcher’s verbal account and the teacher’s subsequent internal visualisation. This invites the question of whether this alignment may hint at the difficulties students and teachers may encounter when following and making sense of a different perspective (e.g. making sense of a dynamic account when your own method was based on a static account), potentially supported by specific linguistic features.

Conclusion

Our work to date on this project has repeatedly pointed towards the importance of paying attention to small words and phrases within mathematical language, and being mindful of the multiple interpretations which can hide behind them. That is, there is a need to make sure we are naming what we are paying or drawing attention to. There is worth in becoming more aware of, and beginning to unpick, the complexity of what we are asking learners to do.

Angles, and the language associated with naming angles, can invoke diverse intuitions and understandings: of a static object, of a measure, of a vertex, and of a dynamic turn. Relationships between angles become even more complex, and incautious use of language risks consolidating or compounding students’ difficulties. This concern is particularly salient in the cases of students with other linguistic disadvantages.

We do not propose any specific measures here to manage this concern or suggest what good practice should look like (or even that any unqualified set of standard practices could exist). Instead, we offer that by supporting the noticing skills of teachers, we are both developing their awareness of this issue and upholding their agency to make informed and deliberate decisions.

Acknowledgements

This project has been partly funded by the Nuffield Foundation, but the views expressed are those of the authors and not necessarily the Foundation. Visit www.nuffieldfoundation.org

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Students explain their ways of solving division tasks in the context of measurement

Jessica Mähnert¹ and Kirstin Erath¹

¹Martin Luther University Halle-Wittenberg, Germany; jessica.maehmert@mathematik.uni-halle.de
kirstin.erath@mathematik.uni-halle.de

When learning fraction division, students are expected to develop conceptual as well as procedural understanding. In this paper, we give attention to how students explain when solving tasks involving fraction division. The aim is to explore language means and mathematical ways employed by students when solving these tasks. The qualitative study identifies (topic-specific) language means that students use to talk about their ideas. Our findings reveal how important meaning-related language means and formal-related language means are for building a conceptual understanding of dividing fractions. The analyses also show how students use multiple ways rather than relying on the reciprocal algorithm to solve measurement tasks. Thus, conceptual understanding of fraction division should not be isolated but rather interconnected with other areas such as inverse operation, part-whole concept, equivalence of fractions, and repeated addition.

Keywords: Fraction division, measurement model, language means, conceptual understanding, design research.

Introduction

Fraction division is an important concept that students need to master as a basis for advanced algebraic topics. In Germany, the *invert and multiply* algorithm for dividing fractions appears easy to be taught and to be learned, since this method involves only inverting the divisor fraction and then multiplying it by the dividend fraction. However, the reality shows that many sixth graders in Germany have difficulties in understanding and solving tasks for fraction division (Padberg, 1982). A study conducted by Padberg (1982) found that only 32% of division exercises were answered correctly by 243 students in year 7. The study highlights that the primary cause of the high error rate is the use of incorrect solution strategies. This is an important finding, as it implies that the concept of dividing fractions can be challenging to grasp and retain. Therefore, it is crucial to ensure that learners retain the division rule by integrating it into their conceptual understanding. By including conceptual understanding in the learning process of the division algorithm, learners can recognize incorrect results and comprehend the significance of performing the reciprocal. Wherever conceptual understanding is to be developed, language plays a crucial role. As students' language resources are diverse, it is important for equitable mathematics education to design instruction that enhances language in mathematics classrooms for all students (Erath et al., 2021). Engaging students in rich discourse practices seems to be particularly important for their conceptual understanding of mathematics (Erath et al., 2021). Moreover, designing such materials and instruction requires identifying the demands of topic-specific language and meaning-related language means (Prediger & Zindel, 2017). While the conceptual interpretations of dividing fractions have already been formulated, the language used to explain the concept has not yet been empirically identified. This phenomenon was the starting point for engaging in a design research project that was intended to

foster students' conceptual understanding of fraction division in a content- and language-integrated teaching-learning arrangement.

Theoretical background: Language means for explaining the meaning of dividing fractions

When learning fraction division, students are expected to develop conceptual understanding as well as procedural understanding. The idea of conceptual understanding, which refers to the meaning behind mathematical operations or terms, has been seen as separate from procedural skills, which involve algorithms used to perform those operations, however, research shows that these two types of knowledge are closely intertwined (Kilpatrick et al., 2001). To define understanding, we refer to the definition provided by Hiebert and Carpenter (1992), which states that understanding is a network of internal representations. A concept or procedure is considered understood if its mental representation is integrated into a network of representations, and the strength and number of connections determine the depth of understanding. To work with this definition, we conceptualize mathematical knowledge as a network of conceptual and procedural knowledge elements that need to be developed and connected in a network. Therefore, fraction division will be examined from both a procedural and a conceptual point of view.

The division can be interpreted as partitive division or measurement division (Gregg & Gregg, 2007). In the partitive division, the total and the number of groups are known, and one must find the size of each group (e.g. John has $\frac{3}{4}$ of a litre of orange juice. He pours the juice equally into 4 glasses. How much juice is in each glass?). The present study is based on the latter conception of measurement, in which the total and the size of each group are well known which leads to finding the number of groups. An example of measurement from the study (see Figure 1, Task 1) is: How many times does $\frac{1}{2}$ fit into 1? In this case, the solution requires dividing 1 by $\frac{1}{2}$.

In Germany, dividing fractions involves procedural knowledge such as forming the reciprocal of the divisor, multiplying the numerator, and multiplying the denominator. However, to develop a conceptual understanding of division, it is important to go beyond simply memorizing arithmetic rules and to comprehend the connections between numbers and the relationships between division and other arithmetic operations. For instance, division and multiplication are inversely related. Additionally, division can also be seen as the repeated addition of equal subtractions. In the context of creating a content- and language-integrated teaching-learning arrangement, tasks should be designed to enable various ways of explaining the solution of the measurement task.

Language plays an important role in the successful development of the conceptual understanding described above. In this context, two functions of language can be distinguished; the communicative and the cognitive function: Language is an important means of organizing one's own thinking (communicative function) and at the same time language opens up new possibilities of thinking and knowing (cognitive function) (Maier & Schweiger, 1999). In the field of fraction division, language as a tool for thinking is particularly important for providing meaning to the mathematical concept. To build conceptual understanding, learners need not only to write down their mathematical calculation method but also to explain the meaning behind the calculations, so the discourse practice of explaining proves to be central in this context (Erath et al., 2021).

Teachers seem to focus too often on formal related language means like divide, divisor, and reciprocal. However, while formal-related language means are important for explaining arithmetic operations, it is equally important to use language means that convey the meaning behind these concepts. These so-called meaning-related language means (Wessel, 2015) refer to the meaning of the concepts behind the formal operation of division and are key to helping learners explain their mathematical calculation method and understand the reasoning behind the calculations. Research has shown that meaning-related language means are particularly important for building conceptual understanding, but still for each mathematical topic, these meaning-related phrases need to be identified by epistemological analysis (Prediger & Zindel, 2017).

Thus, this paper examines students' language means that are used to explain the division in the measurement context. Additionally, we present conceptual ideas in which students explain the measurement task. So, the present study aims to explore the conceptual depth to which learners explain the measurement problem and the language means (formal-related and meaning-related) they use by answering the research question *What language means do learners use and to what extent do different language means fit different conceptualisations?*

Methods: Design Research

As the focus is on learning as a process in which students are actively engaged in making sense of mathematical ideas, design research was chosen as a methodology (Gravemeijer & Cobb, 2006). Design research relies on an iterative interaction between designing teaching-learning materials, conducting design experiments, and analysing the teaching and/ or learning processes. In the presented project, one design experiment cycle in a laboratory setting was conducted so far.

Research context


Based on the findings from existing literature (see theoretical background), a content- and language-integrated teaching-learning arrangement was developed, aiming first at the concept of fractional division before engaging in automating calculation skills. During the first design cycle, the emphasis was on comprehending the idea of measurement using fraction strips. With regard to helping students understand the division of fractions, the challenge was to develop sequences of activities that help students to interpret situations that could involve division of fractions. Thus, the idea of measurement was chosen due to the potential to build an understanding through fraction strips, and to encourage learners to utilize intuitive methods rather than relying on memorized methods of calculation.

In this teaching-learning arrangement, students are introduced to a method for determining how many times a fraction can be contained within a whole, which they must comprehend and then apply on their own. To illustrate this, students have various green fraction strips at their disposal. By placing the green fraction strips next to each other, they determine how many times these strips fit into a whole, represented by the blue strip. To illustrate this, an example of how many times $\frac{1}{3}$ fit into a whole is given in Figure 1, Task 1a. In the teaching-learning arrangement, the students begin with common fractions that fit into a whole (Figure 1, Task 1) and progress to proper fractions that fit into any whole number. Subsequently, students solve the task of determining how many times $\frac{1}{365}$ fit into a whole. This task cannot be solved using fraction strips. However, the learners were then asked to explain their way of solving the task, which is central for building conceptual understanding and

requires rather meaning-related language means. At this point, we have not discussed with students that this task may be viewed as a division problem. The aim is to examine how students articulate their explanations and what conceptual understanding they utilize in their explanations. Both the meaning and formal-related language means are of interest in this task, as well as the conceptual ideas presented by the learners.

Task 1: Now determine how many times a size fits into a whole using the fraction strips. Draw a picture for each task and write an answer sentence.

a) How many times does $\frac{1}{3}$ fit into a whole? b) How many times does $\frac{1}{4}$ fit into a whole?



c) How many times does $\frac{1}{5}$ fit into a whole?

Task 2: How many times does $\frac{1}{365}$ fit into a whole? How can you quickly find out how many times $\frac{1}{365}$ fit into a whole? Write down your calculation method.

Figure 1: Insight into the teaching-learning arrangement with selected learner responses

Methods of data collection

A total of 15 learners from two different schools' sixth-grade classes in a city in central Germany participated in the first cycle of design experiments. The students had previous experience with arithmetic operations with fractions. Students worked in groups of two to three learners for around 90 minutes on the provided learning materials. The first author attended all sessions and intervened in the learning process to a minimum. The sessions were videotaped, and students' written materials were collected.

Methods of data analysis

The analysis of the initiated teaching-learning processes was conducted with regard to the way in which the learners solved the measurement task. Subsequently, the language means used by the learners were analyzed. In the first step, we identified all sequences in which students worked on Task 2 (Figure 1). Then, each of these sequences was transcribed by the first author, starting from the moment students attended to Task 2 and ending when is marked in interaction that working on this task is finished. The analysis is based on these transcripts and follows the process of inductive category formation of how learners solve the measurement task, as described by Mayring (2015).

Empirical insights

Table 1 shows a summary of the analysis. It links mathematics and language by listing mathematical ideas students use for solving divisional tasks involving measurement in the first column and providing sample learner utterances that illustrate the discourse practice of explaining listed in the second column. The learners' utterances in the second column are from various learners. In total, we identified four different ways (first column of Table 1) with which learners solve the measurement task (Figure 1, Task 2).

Table 1: Four ways to solve Task 2 and excerpts from students' explanations

Conceptual idea used in the calculation method	Examples of learners' utterances
Inverse operation	I would now say that it fits in 365 times. Because it fits in 365 times and therefore 365 times. Then I get a whole if I multiply $\frac{1}{365}$ by 365.
part-whole concept	Well, $\frac{1}{365}$ is also 1 out of 365. $\frac{1}{365}$ must then fit 365 times so that we make a whole.
equivalence of fractions	We could write that we have a whole. A whole is the same as $\frac{365}{365}$. We would then have to make a lot of strokes in this whole and always at the same distance. Now we just have to work out how many times $\frac{1}{365}$ fits into a whole. If you now calculate $\frac{1}{365}$ times 365, you get $\frac{365}{365}$. So, it has to fit 365 times.
repeated addition	You place the $\frac{1}{365}$ -strip as often as necessary until you get a whole. Then you only have to multiply by 365, because you have to add 365 times to get a whole.

To gain a better understanding of the language means that learners use and of how different language means fit different conceptualizations, we investigate the examples from Table 1 in detail.

First, we will refer to the learner utterance in line 1, which was categorized as inverse operation because the measurement task was solved by using the inverse operation multiplication. This means that, given a division task, the learner calculated the number by which the divisor must be multiplied to obtain the dividend, the total amount. By saying “Then I get a whole if I multiply $\frac{1}{365}$ by 365.” (Table 1, line 1) the close connection between multiplication and division becomes clear. The learner employed an inverse method, figuring out the number by which $\frac{1}{365}$ must be multiplied to make a whole. We only identified *multiply* as formal-related language means, while we identified *fit in*, *times*, and *whole* as meaning-related language means. They were categorized as meaning-related because *times* refers to the meaning of multiplication, *whole* refers to the mathematical concept of fractions, and *fit in* refers to the meaning of division. The language means *whole* and *fits in* result from the context offered (see Figure 1). This example shows that the operation of division and its relationship to multiplication is central.

In line 2 of the table, the student uses only meaning-related language means to explain the result of the measurement task. By saying “ $\frac{1}{365}$ is also 1 out of 365” (Table 1, line 2) the learner means that it represents one part out of a total of 365 equal parts that make up the whole. So, the meaning-related language means *1 out of 365* is used here. This understanding is foundational to the concept of fractions and their relationships to wholes. Through explaining, the learner recognizes that $\frac{1}{365}$ fit 365 times into a whole. The student only uses meaning-related language means such as *fit*, *times*, and *whole*, as seen in the example in line 1 of the table. However, this approach may not be suitable for solving all measurement tasks, e.g. how many times $\frac{3}{365}$ fit into a whole?

The utterance in line 3 shows that learners combine meaning-related and formal-related language means when explaining their result of the measurement task. Since the learner refers to the

equivalence of fractions, it is categorized as an explanation involving equivalence. The idea behind this is as follows: Since the result of the measurement task is not immediately obvious, the learner chooses a way to find a common division for the two numbers. By saying “A whole is the same as $\frac{365}{365}$,” (Table 1, line 3), the student extends 1 whole to $\frac{365}{365}$ to solve the measurement task at hand. Furthermore, the student demonstrates an understanding of the concept of refining by referring to it in Table 1, line 3. The student states “We would then have to *make a lot of strikes in this whole and always at the same distance.*”, which suggests that he recognizes to refine the whole into smaller bars. The italicized part was identified as meaning-related because it refers to the concept of equivalence. The student then considers how often $\frac{1}{365}$ fit into $\frac{365}{365}$. To arrive at his answer, he multiplies the numerator of $\frac{1}{365}$ by 365, which helps him determine how many times $\frac{1}{365}$ can fit into $\frac{365}{365}$. He then uses the formal-related language means to express the subsequent comparison that $\frac{1}{365}$ fit a total of 365 times, by saying, “If you now calculate $\frac{1}{365}$ times 365, you get $\frac{365}{365}$.” (Table 1, line 3). The term *calculate* was identified as formal-related. By finding a common denominator between two fractions, the learner extends one whole to match the other fraction, making it easier to compare and determine how many times one quantity fits into the other. At this point, measurement can also be understood as an indirect comparison of two quantities using a base unit. To explain the solution of the measurement task, the learner uses a combination of meaning-related and formal-related language means. This method of calculation involves measuring using conceptual knowledge of fractions and language: If the distance cannot be measured accurately with one unit of measurement, the unit of measurement is subdivided. The representation of fraction strips combined with measurement can be used to promote understanding. In sum, this demonstrates the necessity of not introducing the concept of expanding and reducing formally but rather building up conceptual knowledge for the equivalence of fractions. Otherwise, the learner might not be able to solve the measurement task by referring to the equivalence of fractions. Therefore, the equivalence of fractions can be used as a model for explaining. According to Kilpatrick et. al (2001), linking conceptual and procedural understanding is essential for a deeper understanding of the mathematical concept of dividing fractions. All division tasks including measurement can be solved by finding a common division of two fractions and comparing the numerators to determine how often one fraction fits into the other.

The learners’ explanation in line 4 pertains to the mathematical idea of repeated addition involving adding $\frac{1}{365}$ repeatedly until a whole is reached. Since this requires 365 repetitions, it can be simplified by multiplying $\frac{1}{365}$ by 365. By saying “You place the $\frac{1}{365}$ -strip as often as necessary until you get a whole.” (Table 1, line 4) the focus is on the mathematical meaning and the idea of measurement is articulated. The phrase *place as often* illustrates the concept of adding $\frac{1}{365}$ fraction-strip, while the second sentence demonstrates the learner’s use of formal-related language means, including multiplication, and addition. The learner not only connects the meaning of the concept to formal-related language means, but also demonstrates a connection to the context in which the idea of measurement is expressed. In the case of measurement, the learners must connect meaning-related language means to formal-related ones to build a strong foundation of mathematical understanding. Both phrases *to place as often* and *to add* emphasize the notion of addition and emphasize that combining meaning-related and formal-related language means can aid in developing mathematical comprehension. Whereas *to add* was identified as a formal language means, *to place as often* was

identified as a meaning-related language means because it refers to the concept of addition. While meaning-related language supports the development of conceptual understanding, formal-related language is utilized to express mathematical accuracy. The learners' statement highlights the significance of combining both types of language to enhance mathematical understanding.

It is important to note that not all measurement tasks can be solved using repeated addition or part-whole concept because these ways are very time-consuming and cannot be continued for more complex tasks. These ways can be helpful when the concept is still being developed. The idea of inverse operation is viable, but there are cases where the result of a division task cannot be immediately recalled, as there are no memorized number sequences in the range of fractions, unlike with the division of whole numbers, for example. In such cases, the concept of equivalence, as explained above, can be applied to solve such tasks. To utilize this concept, learners must also employ other meaning-related language means that are related to the concept of equivalence of fractions. This means that learners must be able to interpret the language means and the conceptual ideas provided.

Conclusion

The focus is on language means and the different ways in which students solve the measurement task. The findings indicate that students use various ways to solve these tasks. Division therefore is in contrast to the other arithmetic operations, because a division task can be solved by applying different relationships, rather than just using the reciprocal algorithm. Therefore, it is important to build a conceptual understanding of division of fractions, including the inverse operation, part-whole concept, equivalence of fractions, and repeated addition.

In answering the research question, we could identify different language means used by the students. They use meaning-related and formal-related language means to solve measurement tasks, demonstrating a combination of conceptual and procedural knowledge. This is evident in their use of mathematical equivalency and repeated addition. Here we have found that learners use different language means, indicating that the tasks in the context of measurement seem to activate a diverse repertoire of language means among learners. This is particularly interesting in terms of knowledge construction (see theoretical background), as Kilpatrick et. al (2001) has already emphasized that both types of knowledge (conceptual and procedural) complement each other and are important for learning mathematics successfully. This suggests that the context of measurement can be a good starting point for discussing fraction division, but further research is needed.

In summary, our findings match previous research (Erath et al., 2021) in that we also observe that learning mathematics includes thinking and speaking mathematically. Language is crucial in developing conceptual understanding, as it allows students to explore connections between the division of fractions and other mathematical topics. In this context, the cognitive function of language is central to the development of conceptual understanding. When language is used as a tool of thinking, students can explore the various relationships that the division of fractions has with other mathematical topics. To support this development, language means, both meaning-related and formal-related, should be used to enhance students' understanding of mathematical concepts and their ability to communicate their ideas effectively. Students need to understand and justify calculation methods, rather than just memorizing them because Padberg (1982) shows that learning the invert and multiply algorithm can be challenging for students. Overall, students' learning processes need to

cover both conceptual understanding and procedural skills, for both the current topics and foundations from previous years, with language aiding in this development.

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The role of the debating reviewer to promote communication and dialogic learning within and between thinking groups

Annamaria Miranda

Dipartimento di Matematica, University of Salerno, Italy; amiranda@unisa.it

Research argues that engaging students in cooperative problem-solving activities in which every student plays a role within a thinking group is a promising way to consolidate mathematical knowledge and metaknowledge through communication and social interaction. To make this strategy more powerful, a new performer, the Debating Reviewer, who stimulates the debate within a group during the solving process as well as between groups in the final reflective phase, is introduced. Then, the issue of promoting students' dialogic learning through interactions, discourses, and reflections, is faced. In this paper, a preliminary investigation of students' perceptions of the new role is given.

Keywords: Language, communication, discourse, dialogic learning.

Introduction

Dialogic learning methodologies have been widely celebrated in recent years as pedagogical processes that engage students in activities to excite cognitive abilities and promote deep learning. Dialogic learning is learning that takes place through a written or spoken conversation between one or more people. The notion of dialogic learning is not new; it has origins in several theories (Bakhtin, 1981; Freire, 1970; Habermas, 1981/1984; Sfard, 2008; Wells, 1999). Meyers and Jones (1993) describe active learning approaches as those that provide students with the opportunity to discuss, interact with, and reflect on the content, ideas and issues of a subject, thereby promoting dialogic learning. Active learning and dialogic learning, apart from being significant in themselves, are also crucial capabilities in facilitating the acquisition of many other skills and capabilities such as problem-solving skills, collaborative learning, critical thinking, creative thinking, reflection thinking, abstraction, communication skills, and so on. Researchers in education are creating and assessing methods for helping students develop these capabilities, but in practice this is not so easy. In a recent book (Manalo, 2019), some strategies to bridge the gap between the work of researchers and what instructors and teachers are taught as practical are provided and discussed. In the context of mathematics education, these strategies are strictly connected to commognition theory (Sfard, 2008) and to education for the use of natural language in approaching theoretical aspects of mathematics (Boero et al., 2008). The term 'commognition' is born from the union of communication and cognition and refers to the approach to learning based on the idea that thinking can be conceptualised as communication with oneself (Sfard, 2008). In accordance with the resulting vision of non-dualistic human cognition, mathematics is a historically established discourse, and learning math means becoming part of this form of communication. Yet,

mastery of natural language in its logical, reflective, exploratory, and command functions emerges as one of the crucial conditions in approaching more-or-less elementary theoretical aspects of mathematics. Indeed, only if students reach a sufficient level of familiarity with the use of natural language in the proposed mathematical activities they perform in a satisfactory way and fully profit from these activities. (Boero et al., 2008, p. 262)

My preliminary studies fit into this panorama. The aim is to find strategies for stimulating dialogic learning and effective communication, highlighting the necessity of natural language. A methodology that guarantees a dialectical approach to the study and critical use of thought is the debate. A characteristic element of the debate is the reflection that intervenes to consolidate the connection between experience and learning, as well as to inaugurate cooperation and comparison. Collaborative reflection enlarges the horizons of thought because during the dialogue with others, when they perceive things differently, they propose new points of view, raise unprecedented questions, and question the assumptions made. In collective reflection, “we interpret what we do by involving ourselves and others in conversations, debates, and reflections on what we have understood individually or collectively. We appreciate the importance and relationships of all the parties involved” (Chang, 2019, pp. 95–97). I face the issue of promoting students’ learning through interactions, discourses, and reflections that engage them in cooperative problem-solving activities. Inspired by some studies confirming university students’ meaningful learning outcomes working in role-structured groups, the Thinking Group (Miranda, 2023), my experimental design foresees students playing specific individual and collective roles in a learning path developing into four activities. To strengthen reflection, communication, and dialogic learning within and between groups, I define a new individual role, the Debating Reviewer, encouraging the construction of knowledge and metaknowledge through thinking (discourses within groups) and metathinking (reflections on different strategies between groups), and discuss some perceptions of the debating reviewer role by those who played as well as by those who didn’t play it, during the learning path and at the end of it.

Theoretical framework

The theoretical underpinning for the design, execution, and analysis of the learning path is the idea that thinking can be conceptualised as communication with oneself (Sfard, 2008), and stimulated by collaborative reflection and debate (Chang, 2019), both principles support dialogic learning. A good candidate to stimulate these theories’ tenets within the activities seems to be the Thinking Group methodology (Miranda, 2023). The strategy originates by intertwining the Thinking Classroom (TC) construct and role-playing methodology (RP), playing some cognitive roles. A TC is “a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together, and constructing knowledge and understanding through activity and discussion” (Liljedahl, 2016, p. 362). In a TC, students can move freely to solve appropriately chosen problems on non-permanent boards arranged along the walls, working in randomly organised groups. RP is a form of experiential learning (Russell & Shepherd, 2010). Students take on assigned roles and act out those roles through a scripted play. The role play can be conducted one-to-one (individual role) or as a group role play, with each member of the group taking on a role or character (collective role). Playing a role facilitates learning by fostering learners’ cognitive, behavioural, motivational, affective, and sociocultural engagement with the subject matter. Specific individual roles (*Boss, Promoter, Critical Mind, Blogger*), named *cognitive roles*, each corresponding to a cognitive function being activated when a mathematician faces a problem (e.g., looking for paths, questioning themselves, organising themselves, systematising the findings, etc.), were successfully experienced in cooperative working activities at the university level (Albano et al., 2022; Miranda, 2023). The Thinking Group model (TG) foresees structuring a classroom into groups, each playing a collective role, in which each student performs

an individual cognitive role. Specifically, a “TG is a TC in which each student performs an assigned cognitive role, as a solver or as an observer” (Miranda, 2023, p. 722).

Research confirms meaningful cognitive, metacognitive, and affective outcomes in problem-solving activities experimenting with TG at the university level. However, for the valuation of heterogeneities and reflections within each thinking group (in solving the mathematical problem) and between thinking groups (in comparing different solution strategies), it was found that the model needed to be integrated by moments of collaborative reflection and debate. So, the introduction of a new role, the debating reviewer, stimulating these dialogic characteristics, was necessary.

The definition of debating reviewer

To magnify participation and social interaction within and between thinking groups and debate and reflection between groups, I define a new role to play and call it the *Debating Reviewer (DR)*.

The DR is a supervisor of the problem-solving process, a double reviewer, of the roles played and of the solving process and product, who stimulates the debate in two phases: first in the solving phase, between members of a Thinking Group, then, in the final reflective phase, between TGs in which the entire "class group" has been divided. His or her task within the TG is not only to monitor the roles played and to discuss the mathematical problem to be solved, but also to stimulate TGs to produce well-written collective documents narrating the experience lived during the activity to be reviewed by him or her. He or she is also, in the phase of collective reflection, both the spokesperson of the work produced by the TGs, of which he or she was supervisor and reviewer, and the moderator of the discussion, mainly concerning the debate about comparing the different strategies students applied.

Reflection conducted with collective discussion at the end of a problem-solving activity is considered an important phase for the consolidation of students' learning. It is therefore relevant to ask how the reflective debate conducted by the DR in the dual role can affect learning for the construction of knowledge and the development of cognitive, metacognitive, and affective skills.

RQ: How was the role of the Debating Reviewer perceived by both those who played it and those who didn't? How did the dialogue and debate help students reflect, consolidate their knowledge, and feel improved in their strategic problem-solving approaches?

Methodology

The experience involved 36 students attending the second year of a bachelor's degree in mathematics within a topology course held at a southern Italian university during the academic year 2021–2022. Besides acquiring content knowledge and dealing with the fundamental concepts of general topology, the main educational goal was to construct students' mathematical reasoning and communicative skills. The experimental design provides a path engaging students in four cooperative working activities (CW1, CW2, CW3, and CW4) each of which is devoted to the understanding of a specific topological concept (closure, continuity, compactness, and connectedness). The tasks were designed to promote students' construction of knowledge by themselves, going from example generation to acquiring concepts to the free production of conjectures and proofs. In every CW activity, the task consisted of three problems concerning the understanding of a specific concept: the first two dealt with the construction of examples satisfying the topological property defining that concept, under given constraints; the third one asked to provide some characterization related to the investigated

property. At the beginning of the path, the participants have been split into four Thinking Groups named TG1, TG2, TG3, and TG4, each consisting of a Debating Reviewer and two subgroups of four students supervised by her or him, a Solver group, and an Onlooker group. Each student in a subgroup, both in the Solver and Onlooker group, was assigned a specific role in relation to specific actions/processes to be performed according to the cognitive function they personified: *Boss, Promoter, Critical mind, Blogger, Debating Reviewer*. The design foresaw that any cognitive role, except DR, was played on two levels of engagement in the problem-solving process: the solving level and the reflecting level. At the solving level, the Solver group was devoted to collectively solving the problem by acting according to the cognitive function to perform. At the reflecting level, an Onlooker group observed how the Solver group was working by reflecting on how students acted with respect to both the mathematical problem and their individual role. To foster awareness of all the cognitive functions, the students changed both individual and collective roles with each new activity. However, the DRs preserved their role as DR along the activities but changed the TG they belonged to.

Data collection and data analysis

All the data concerning the learning activity has been digitally stored via the Moodle platform. The collected data consists of three Google Docs to produce at the end of each CW activity:

- 1) a personal logbook reporting the reflections concerning the individual roles played
- 2) a collective logbook narrating the mathematical solving process shared with the subgroup
- 3) the DR's review to draw up after receiving the collective written documents

and a final feedback questionnaire related to the personal experience students lived during the path.

I focus my qualitative analysis on how students perceived the DR role, looking at the DR's personal document as well as all students' answers to two questions selected from the feedback questionnaire:

9. Do you believe the Thinking Groups' and the class's overall learning process has improved because of the participation of the DR role? 10. Did the debate sections help you to consolidate your knowledge at the end of each activity?

More in detail, I looked for whether significant 'dialogic' learning processes emerged from the interaction with the DR. To accomplish this, through thematic analysis (Braun & Clarke, 2006), identified some indicators of dialogic learning, and then went on to observe the occurrence of these components in the discourses provided by students: *specific and natural language, public speaking, debating, comparison between groups, and internalising concepts*. Specifically, I grouped DRs' self-perceptions and other students' perceptions of DRs (perceptions of DR felt by students who are not personifying the DR role), and for each of these, I identified significant students' excerpts, within data 1) and 3) for the first group and from the feedback questionnaire for both groups, denoting students by S1,...,S36, to understand if and how they were harbingers of dialogic learning.

Results and discussion

Perception of the role of the debating reviewer by the students who played it

The first DRs' responses as reviewers (from data document 3) that caught my attention highlighted outcomes in one's 'perceived dialectical mastery,' both specific and natural. Students who have played the role of DR have had a way to recognise and develop dialectical capabilities. Indeed, in the review sheet related to activity CW2, the DR S10 commented on the work of the solver group, stating

that “*the entire document has been drafted in a clear and comprehensive manner. The language used is rigorous, but at the same time very intuitive*”. Knowing how to distinguish an artificial exposure from a fluid one is essential in any working context, especially where human interaction or group management are required. In our setting, knowing how to expose mathematics in a comprehensive way, using a language suitable for the audience, would ensure that each student learns and that the work proceeds smoothly. The importance of choosing a suitable language is also highlighted by DR S30, who states that “*the proofs carried out are correct and conducted in an appropriate language, even if they have not been fully included in the collective report. The document should be improved for publication with more references to definitions and a better organisation of the text in the narrative of the various stages, so that the reasoning can be more understandable to the reader.*” The DR highlights the importance of presenting unambiguously what he or she has learned to allow the entire audience an easy understanding, since learning as a form of communication and participation is conceived as an intrinsically collective, or social, rather than individual phenomenon (Sfard, 2008). Furthermore, the details in the proofs, examples, and definitions recalled significantly improve the quality of the work produced, making it tangible and structured. This maturity in the use of the language was not immediate but rather developed by participating in one activity after another, as the DR S30 notes in the individual sheet CW3: “*A further observation that I feel to share is that, as activities pass, groups are gaining greater mastery of language, improving their ability to solve a problem, compare, and collaborate in a group.*” DR S10’s comments on the CW2 activity show that the solver group “*has made good use of the topic tools available*”. There is, therefore, from the DR a look at the problem of optimising the resources available to make them expendable effectively. Using the study material with maturity allows you to tackle a mathematical problem and produce a solution strategy that combines simplicity and convenience (Schoenfeld, 1992). The individual sheet created by the DR S16 for the CW4 activity reports: “*I and another member of the observer group were inclined towards a simpler topology, [...] comparing ourselves with the rest of the group, but we realised that our proposal would complicate the remainder of the exercises*”. From the DR’s analysis of this comment, it emerges that, in general, simplicity does not imply convenience and that some choices, apparently convenient, should be considered from a perspective reflecting on benefits not necessarily immediate. For a DR, working with different TGs can complicate the process of adaptation as it requires interaction with different minds and working strategies. Every time, it is necessary to set up and build a relationship of trust with the members of the new TG. At the same time, changing TGs often has great training potential because it exposes the DR to varying development environments. This is what the DR S34, who, in the individual activity sheet CW4 (from data document 1), reports: “*by moving into another TG, I was able to observe and listen to other ideas and thoughts from my colleagues.*” The DR has, therefore, had the opportunity to grasp different aspects of reasoning and to meet many different perspectives, thus developing fluidity in thinking and adaptability. One of the tasks of the DR is to manage the discussions related to the solution process to ensure that all the members of the TG make known their ideas and proposals and that cooperation is always accompanied by harmony and fluidity. Students who have worked as DRs have heard and done this role from the beginning of the path. In fact, in the individual activity sheet of CW2, S30 claims that his role is to “*manage the discussions with respect to the assigned roles.*” The DR played a key role in managing the debate during and at the end of each of the cooperative working activities, which served to highlight the critical points of the problem-solving process and the

contribution offered by every assigned role. He or she also highlights the ways in which the team deals with difficulties, thus allowing us to understand whether the working process is complete and functional, whether the work is distributed in a balanced way among the members. DRs appreciated the potential of their role (from questions 9 and 10 of the feedback questionnaire):

S30: Afterwards, I can consider myself *lucky* to have played this role, and the biggest contribution has been made in the *debate* with the whole class. I would have liked to *compare* my comments on the group with those of the other DRs before the discussion between groups and then bring back similarities and differences.

Comparison with other groups to negotiate the best strategy to solve the problem emerges strongly:

S16: Having played the role of DR, I think that the role was useful, especially in two phases: during the revision phase through the *comparison* between the documents produced by the solver and onlooker subgroups, as well as during the *comparison between the various groups*, and this was fun.

S10: The *debate* was particularly useful in *comparing the various solutions* made by the groups. Seeing multiple solutions to the same problem, especially the most complex ones, stimulated new ideas and new solution strategies. Many times, it has been surprising that different approaches have produced the same conclusions.

Also, the theme of *public explaining well the groups' solution process* comes out:

S16: My task was to describe the activity carried out; this helped me to face the fear of *speaking in front of people*, and above all, it "forced" me to deepen the topics to expose them adequately, trying to make as few mistakes as possible.

S34: The final debate served to improve *the exposition of the topic*, which will be very useful in the future, especially for those who want to teach.

Perception of the role of the DR by students who did not play it

The current paragraph is dedicated to the opinions collected from students who did not play the role of DR during the activities (excerpts from answers to questions 9 and 10 of the questionnaire). The final debate made it possible to compare the different strategies adopted by the groups to deal with the proposed problems. It is interesting to observe how, although adopting different strategies and techniques, sometimes the same solution is reached, thus having a view of the same problem from many distinct perspectives (Chang, 2019). This aspect of the debate was grasped and appreciated not only by the DRs who made it possible but also by the students who held other roles. Most students believe that the role of the DR has had a great positive impact on the entire class. Communicative, participative, and dialogic aspects of learning (Sfard, 2008) emerge. Also, the *public exposition of the solution process* and the *moderation of the debate* dealing with the reflection about the strategies are highly appreciated by students interacting with the DR:

S4: I believe that DR role has been important throughout the entire activity; has taken care of the part of the *exposition of the group's work*, and consequently, since to present something adequately, you need to have excellent knowledge of the subject, DR has ensured that every step in *the solution process was clear* and was explained in an exhaustive way; she has also taken care of the organisation of the debate among the members of the group and of giving support in the *drafting of the final document*. I think, however, that the advantages of the presence of DR are that she/he checks that the roles are respected and that she/he *supervises the whole process and product* from an external rather than an internal point of view.

S7: I believe that the presence of the DR has benefited all of us a lot, even to people who were less confident and who often did not intervene during the lessons as, in *presenting the project to everyone*, it has allowed us to have debates, which are

important for consolidating definitions, for the creation of conjectures and observations, such as the link between a Hasse diagram and being a topology.

Students interacting with the DR recognize his or her utility for the *comparison between groups*:

S8: I believe that the debate at the end of the activity *is an important moment* of the experience, the one in which you encounter the *reasoning of the other groups* and learn new and different ways to solve the same problem.

S18: The DR allowed the *comparison between the solvers and the observers*, enforced the opposing opinions by grasping the points of disagreement and smoothing them out in a critical-rational way, and selected the most appropriate arguments.

The final debate made students more confident that they had *internalized the contents of the subject* and allowed them to have a broader view of the problem and stimulate new ideas:

S6: The debates made at the end of the various activities were very important for me, as they made me certain that I had *internalized the concepts* and allowed me to go beyond my own ideas and those of my group mates. Thanks to these debates, I realized that the same problem can be solved in many ways, and so I began to have a broader vision in such a way as to accept and understand the thoughts of others. [...] Knowing approaches different from your own allows you to enrich yourself with the experiences of others.

S17: The final debate about CW's activities has served to broaden my horizons not only in solving the problem but also, above all, in terms of my knowledge of the subject. This is because, in my opinion, comparison is always the father of constructive criticism, without which it is never possible to grow professionally.

S22: Other students' strategies honestly almost challenged me in a positive competition with the sole purpose of raising the bar. It was a 'test': understanding or not the paths taken by other groups means having *internalized or not the topic*.

The role of the DR revealed a great dialogic learning opportunity for all the students. It was a complex role activating different aspects of learning and allowing them to simultaneously deploy many skills ranging from theoretical and intuitive knowledge to the property of language, up to diplomacy.

Conclusions

The effective contribution that the DR has made to the experience of cooperative working activities lies in the role of mediator and moderator, and thanks to his/her intervention it has been possible to create a dialogue first within any working group, during the development of the proposed problems, and secondly between the DRs, during the final phase comparing the different solution strategies used by the working groups. Thinking 'I wouldn't have taken that path' or "that road was simpler" allows students to develop critical thinking. The DR role encourages not only the construction of students' mathematics knowledge, but also the developing of their meta-knowledge. Further investigations are required. New directions on how the role of DR impact on the collective identity and if the interaction with ChatGPT would empower it, especially in the debating phase, are coming.

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Precision of mathematical descriptions in German Sign Language

Swetlana Nordheimer¹, Gabriele Unterhitzberger¹ and Christian Rathmann¹

¹Humboldt-Universität zu Berlin, Germany; swetlana.nordheimer@hu-berlin.de

The aim of this paper is to show how geometric problems can be formulated in sign languages. Based on the theoretical framework of conventionalised and productive mathematical signs (Langer, 2005; Papaspyrou et. al., 2008; Barth et. al., 2022; Rathmann 2022) applied to geometrical contexts, we present a case study with a non-signing mathematician who is given the task of solving geometrical problems formulated in German Sign Language. Theoretical considerations are exemplified by mathematical signs used in geometry and the concepts behind them. The case study serves as the empirical part of a larger research project focusing on Signed Geometry. The data includes drawings made by the mathematician as part of his problem-solving process. Analyses of these drawings provide empirical evidence for the expressive power of Signed Geometry when it comes to precise mathematical descriptions of spatial problems.

Keywords: geometry, sign languages, mathematical education, problem-solving, discourse modes.

Introduction

The goal of this paper is to introduce the possibilities of achieving enhanced precision of mathematical descriptions through the use of sign languages. In our case study, we will explore this topic and find empirical evidence for the idea that the interplay of productive and conventionalised signs not only provides a powerful tool for representing phenomena of spatial geometry, but also offers new ways of increasing the precision of the description of geometrical objects, their properties, and their transformations in three-dimensional space. To this end, we will first introduce the reader to the theoretical aspects of conventionalised and productive signs as applied to geometric concepts and problems. Although these investigations include linguistic aspects, they are of a didactic nature since the main aim is to provide teaching materials in German Sign Language (DGS) for deaf learners with high mathematical abilities. In this context, the formulation of mathematical problems and tasks in precise, perceivable, and understandable terms for deaf problem solvers, sign languages are instruments of scientific research.

Our results can be directly applied in schools and universities, or at least serve as a source of inspiration for teachers and researchers. In particular, (hearing) teachers and researchers who are still in the process of learning sign languages as a second language in the new visuo-gestural modality can benefit from the mathematical examples given and use them not only as teaching materials in the classroom or as empirical research data, but also as learning resources to develop their own sign language and didactic skills. In order to explore the proposition of increased precision in the formulation of mathematical problems through the use of sign languages, we decided to conduct several case studies with undergraduate mathematics students who were exposed to geometric problems in German Sign Language. In this short paper we focus on one of these studies. In particular, the participant was asked to draw sketches of the signed problems in order to understand and solve them. His problem solving activity was accompanied by an interview. Here we analyse one of the problems he solved and its solution to illustrate the power of using sign language to formulate precise mathematical statements. First, some theoretical aspects of conventionalised and productive signs

will be presented. For a more detailed presentation of the theoretical background and connections to other work in the field, see Nordheimer et. al. (2024).

Theoretical Background

Mathematical signs can be categorised into conventionalised signs, which can be collected in lexical, and productive signs, which are created dynamically on the flow using depicting constructions, classifier constructions and other morphosyntactic constructions of sign languages. Examples of conventionalised signs can be found in lexicons such as SIGN2MINT (Max-Planck-Gesellschaft zur Förderung der Wissenschaften e.V. [MPG], 2021). Conventionalised signs often have iconic roots, motivated by the similarity of forms or actions used by the referent. This similarity is not always obvious to sign language users, so signs of an originally iconic properties may be used as symbols. For example, a conventionalised sign for "WÜRFEL", which means cube, found by SIGN2MINT is motivated by the action of throwing a dice. To describe a cube and its geometric properties, conventionalised signs can be combined with productive signs that represent/depict the shape of the object or parts of it, its properties, its position in space, specific actions to form the object and its possible transformations (see Figure 1).

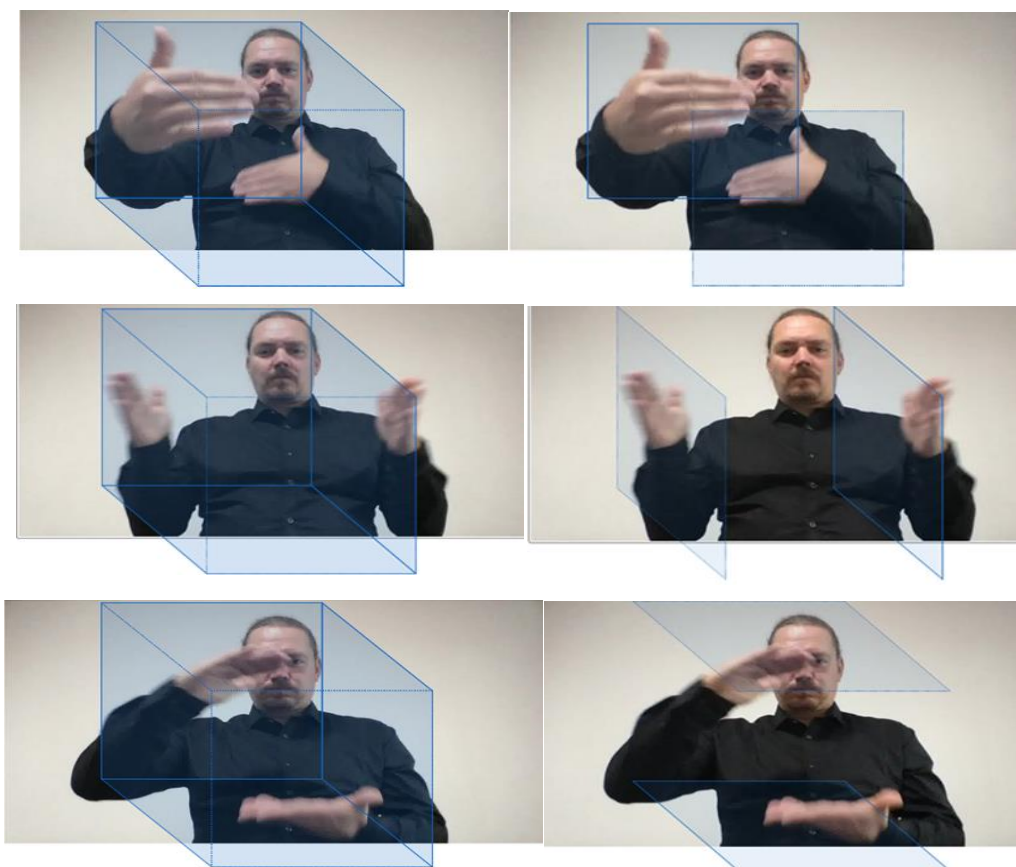


Figure 1: Three dimensional description of a cube

To help readers who are not familiar with sign languages, transparent sketches produced by GeoGebra are used here. This can help to see the images represented by a particular sign. Images represented by signs are dynamic, only valid for a short time and only partially visible. However, people who are sign language users can create mental images without the assistance of GeoGebra. Here we can only present some excerpts from the signed and dynamic version, which is available at:

<https://stemsil.eu/project/mathe-adventkalender-7-2/?lang=de>. It can help to see the richness and dynamics of the problem including the movements, directions, changes in speed and velocity.

Traditional geometric word problems often contain drawings, diagrams, sketches, texts or oral explanations. Let's take the geometric problem in Figure 2 as an example. First of all, we would like to give a translation of the text for non-German speaking readers:

"Cut off the corners of a clay cube through the centre of each edge. (One such corner is cut off in the picture to illustrate this). How many corners does the remaining solid have?"

The problem contains a drawing of a cube with one of its corners cut off and a descriptive text. Although the drawing is shown next to the text, it is not an integral part of the text. The participant still has to switch between the written text and the drawing to solve the problem.

19. Von einem Knetwürfel werden die Ecken abgeschnitten, und zwar durch die Mitte einer *jeden* Kante (in der Abb. ist zur Veranschaulichung eine solche Ecke abgeschnitten). Wie viele Ecken hat der verbleibende Körper?

- (A) 6 (B) 8 (C) 12 (D) 18 (E) 24

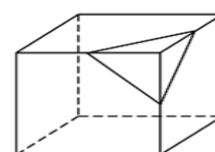


Figure 2: Original problem (CR: Känguru der Mathematik Deutschland)

In a signed version of the same problem, drawings are created in the air with hands and fingers as an integral part of the explanation in sign language. It would not be easy for problem solvers without sign language skills to distinguish between the drawing (which is more iconic) and the description (which is more symbolic). This distinction is not always necessary for deaf sign language users to understand and solve a problem. However, it can be assumed that problem solvers with a high level of spatial geometry skills can compensate for the lack of sign language skills by using mathematical skills to interpret the depicting constructions. Productive signs as depicting constructions can be compared to geometric drawings, while mental images can be associated with visible or invisible geometric objects. Our main conjecture in this study is that without knowledge of the meaning of the conventionalised signs used to describe the problem, problem solvers without any sign language proficiency cannot understand the task sufficiently to solve it successfully. To illustrate the interplay between conventionalised and productive signs, we refer to the signed version of the original problem shown in Figure 2. The signed version in DGS (German Sign Language) was created by Tino Sell, who studied mathematics and German Sign Language Education.

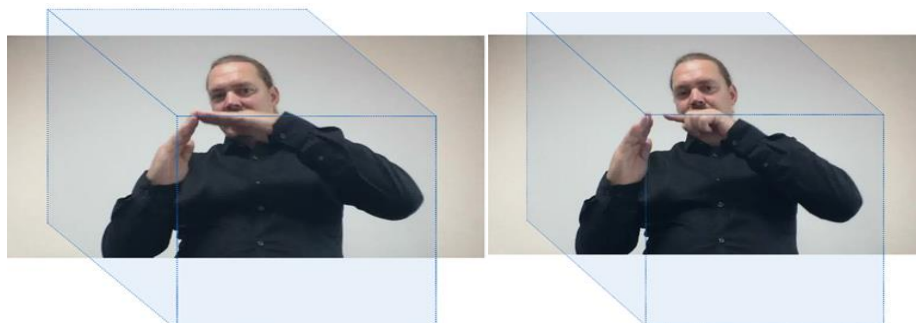


Figure 3: Marking the upper right vertices and starting to draw the edge

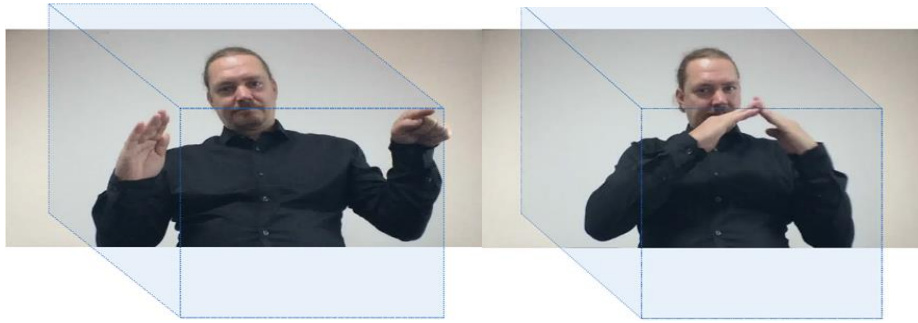


Figure 4: Drawing the edge and marking the upper left vertice



Figure 5: Drawing the edge again and marking its midpoint

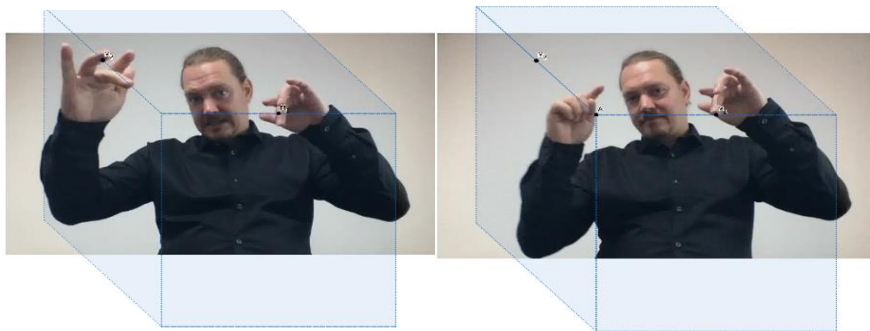


Figure 6: Using the first the midpoint as reference point, marking the second midpoint and vertice

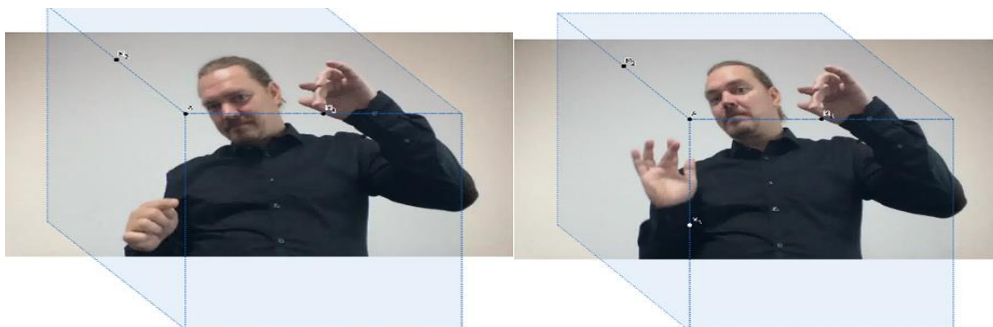


Figure 7: Using the first midpoint as reference point, marking the third midpoint



Figure 8: Using the first midpoint as reference point, connecting it to the two other midpoints

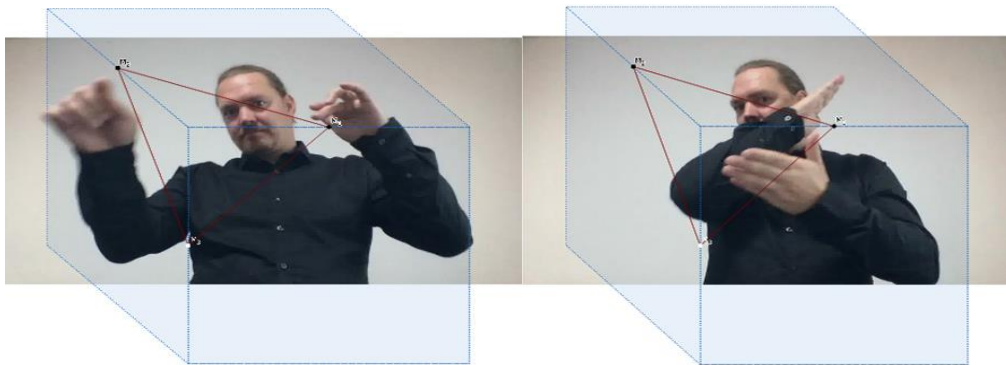


Figure 9: Connecting all midpoints to a triangle and cutting the cube

Problems like the one just described were used by the *Bonn Matheclub* as starting points for learning environments for deaf learners. In order to create a learning environment where deaf learners and tutors could feel comfortable and free to express their ideas without being observed as case study participants, we avoided involving them in our case studies. In the rather small sign language Deaf community in Germany, it is challenging to anonymise data, especially when learners taking part in national and international mathematical competitions are part of the project. Therefore, we preferred to conduct our study with a non-signing learner who belongs to the larger population of hearing learners.

Methodology

In order to investigate the expressiveness and precision of the language of signs in mathematics, we have chosen the method of the case study. Our only participant, whose results we would like to present here, is a highly proficient mathematics student who was educated in a specialised mathematics school. He works as a tutor for younger students at the university and, according to his results in International Mathematical Olympiads, is one of the best young mathematicians in the world. As he wishes to remain anonymous, we will call him simply Alan. Alan is not proficient in German Sign Language, but he agreed to try to solve problems in sign language and to explain his solution in German. The study took place in the summer of 2023 and consisted of two individual sessions of one hour each. He was given a notebook with ten videos of geometric problems. The videos varied in length from one to two minutes. Alan was free to choose the speed at which the videos played, to repeat them as often as he liked, to draw sketches and to ask the researcher questions in spoken German. In two hours, Alan solved six out of ten problems. The data from our study includes Alan's sketches and notes of conversations between Alan and the researchers. We decided to avoid video and audio recording in order to create a more natural setting for the problem-solving process. The

problem-solving process started with looking at the problem, followed by drawing diagrams, and led to a problem-based dialogue with elements of teaching. At the end of the problem-solving process and before the final viewing, Alan was shown conventional signs needed to understand the problem. In contrast to the signs presented in this paper, Alan was given videos without transparent GeoGebra sketches and could only see natural signs. He had to use his imagination to recognise the figures, solids, their properties, and possible transformations. Allan's drawings allow us to draw some conclusions about what he could imagine, at least in part. In this paper we will focus only on the problem as described in Figure 2 as an example.

Case Study

Alan decided to slow down the video in DGS (German Sign Language) to 0.25 of the normal speed. In total, he had to watch the videos four times. He drew sketches after each viewing. All the drawings are numbered according to the steps of the solution and are presented in Tables 1 and 2.

Table 1: Alan's drawings after the first and second step

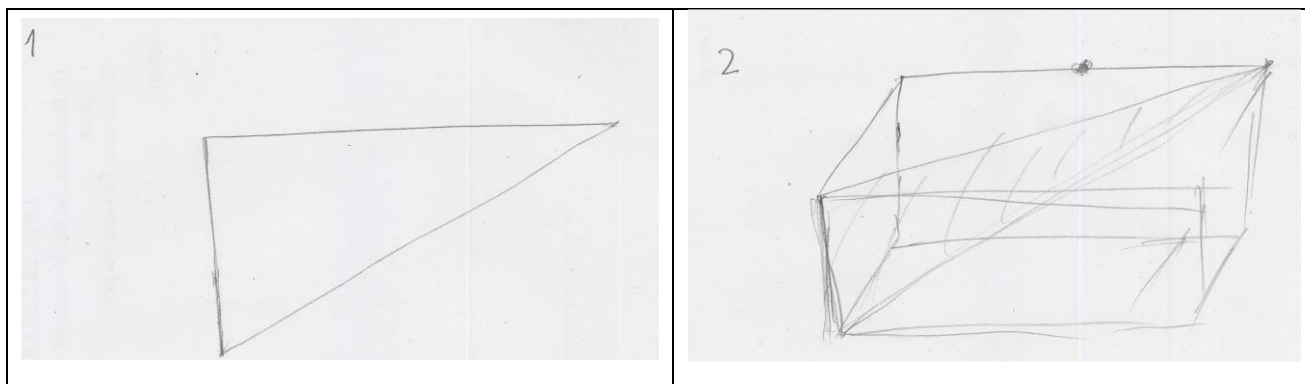
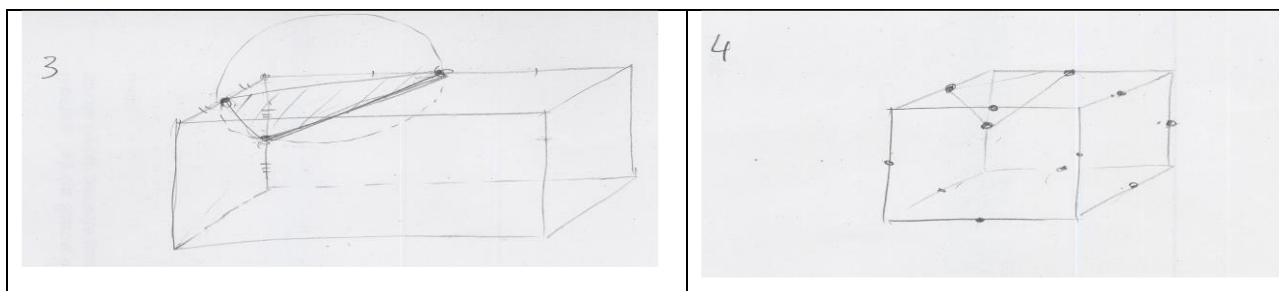


Table 2: Alan's drawings after the first and second step



First viewing: After the first viewing, Alan was able to recognise the triangle. Unlike the signed problem where the triangle was equilateral, Alan's triangle was right-angled. The cuboid was not there.

Second viewing: In the second viewing, Alan recognised that the triangle was inside the cuboid. However, Alan could not recognise the cube. He marked a point on one of the edges, roughly in the middle. The position of the triangle was still not correct.

Third viewing: After the third viewing, Alan adjusted the position of the triangle and obtained a drawing that was very close to the two-dimensional projection of the three-dimensional signed drawing. He had also drawn a circle around the triangle. This led us to believe that he interpreted the conventionalised sign as "sum" or "total", which he visualised by the circle-like sign. In this case, a symbol was interpreted as part of the depicting construction.

Fourth viewing: After the third viewing, Allan was told the meaning of the conventional signs "cube", "midpoint", "how much" and "total". With this information he looked at the signed problem again, drew an oblique image of a cube and formulated the following solution to the problem: "A cube has 8 vertices. 8 vertices plus 2 times 8. That gives a total of 24 vertices". After Alan had written down his solution on a piece of paper, the dialog continued as follows:

Student: 24 edges.
Researcher: Are you sure?
Student: ... (marking every midpoint of the edge in the drawing 4 (table 2))
We have 12 edges and 12 midpoints, the right answer is 12.

To solve the problem, the student had to go back to the sketches and draw the midpoints of each edge to find a general solution where the set of vertices of the new solid is derived from the set of midpoints of the cube's edges.

Conclusion

As a start, the case study presented above provides empirical evidence of the expressive power of mathematical signs as an interplay of conventionalised terms and productive constructions. More in-depth theoretical research, conducted with deaf experts who can combine the fields of mathematical education and sign language linguistics, is needed to specify and generalise the mechanisms of achieving mathematical precision in different domains and levels of mathematical thinking, and to provide a broader theoretical and empirical base.

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Language aspects related to the angle concept in different countries

Frode Rønning¹, Arindam Bose² and Kirstin Erath³

¹Norwegian University of Science and Technology, Trondheim, Norway; frode.ronning@ntnu.no

²Tata Institute of Social Sciences, Mumbai, India; arindam.bose@tiss.edu

³Martin Luther University Halle-Wittenberg, Germany; kirstin.erath@mathematik.uni-halle.de

This study is based on curricula and textbooks aimed at pupils of age 10-12 from three different countries, and on interviews with teachers of pupils for the same age group and from the same countries. The aim is to identify language aspects related to the angle concept that are emphasised in different settings to uncover differences between the countries both in the learning material and in teachers' practices, and to discuss whether such differences could lead to specific challenges that may be linked to linguistic differences. Our study is set in a sociocultural paradigm, seeing knowledge as situated. Furthermore, we rely on semiotic theory and on different aspects of the angle concept as identified in previous research.

Keywords: Angles, semiotic theory, everyday and mathematical language (register)

Introduction

Elementary geometric concepts have a natural place in everyday language, and children encounter such concepts long before being exposed to schooling. This may be an advantage but also a challenge because the meaning of a word in the everyday language (register) may not be the same as in the mathematical language (register). Geometric concepts are often connected to images that can influence the mathematical thinking in a positive way, but images can also be inadequate and disturb the learning if they “become fixed as unconsciously effective ‘tacit models’” (vom Hofe, 1998, p. 317). Therefore, the close connection to everyday life and concrete everyday objects may be an obstacle to learning the concept related to an (abstract) mathematical object (Rønning, 2023; Rønning & Strømskag, 2019). In this paper, we will investigate how the concept of *angle* is presented in curricula and textbooks in three countries, and we will identify some challenges arising in teachers' practices when working with this concept in different languages and different cultures.

The study is part of the project EsCo³ and is based on written material (curriculum texts and textbooks), and interviews with two to three teachers from each of the countries Germany, India and Norway. Each interview focused on one particular topic, and each teacher was asked to elaborate on key ideas within the topic, approaches used to teach it, and linguistic challenges that students encounter when learning the topic. One of the chosen topics was *angle properties*.

We aim to identify similarities and differences between the approaches in the three countries, and to inquire into whether challenges in the teachers' practices may be influenced by the culture and the language of instruction. Recognising that the angle concept has many different aspects (Dohrmann & Kuzle, 2015; Etzold, 2017; Mitchelmore & White, 2000), we will aim to identify the aspects that are

³Establishing a collaborative relationship for research on school mathematics teaching with linguistically disadvantaged learners (<https://www.education.ox.ac.uk/research/esco-establishing-a-collaborative-relationship-for-research-on-school-mathematics-teaching-with-linguistically-disadvantaged-learners/>). This project was funded by John Fell Fund.

present in the data for the study. A preliminary analysis of the interview data has led us to identify three prevailing topics: measuring/calculations, embodied learning/everyday activities, and linguistic aspects/words used. In this paper, we will concentrate on the topic *linguistics aspects/words used*.

The angle concept

The angle concept is central in school mathematics from the early years, and it forms an important basis for several other topics. Dohrmann and Kuzle (2015) describe two essentially different approaches to angles, having different roots: the *rotation approach* and the *congruence approach*. The rotation (dynamic) approach can be traced back to Klein and the angle as a measure for a rotation. The congruence (static) approach can be traced back to Euclid, and in more modern times to Hilbert (Dohrmann & Kuzle, 2015). In the congruence approach the concept of angle is necessary to describe relations between geometrical objects within an axiomatic system. Etzold gives examples of situations, apparently having nothing in common, that all can be described by the angle concept. He claims that there seems to be some mathematical commonalities to all situations where angles occur, but that these are not sufficient to completely describe every situation, e.g., a set of points in an angular domain is not suitable to describe rotations (Etzold, 2017, pp. 35–36).

The descriptions above largely fit with the observations of Mitchelmore and White (2000) who found that “[t]hree particular classes of angle definition occur repeatedly: an amount of turning about a point between two lines; a pair of rays with a common end-point; and the region formed by the intersection of two half-planes” (p. 209). Mitchelmore and White claim that common to all the aspects is that any definition of angle involves “two lines meeting at a point and some relation between them” (1998, p. 5). This, they refer to as the *standard general angle concept*. Dohrmann and Kuzle (2015, p. 37), in their empirical research, observed misconceptions such as angles being compared by comparing the length of the legs, and that the rotation aspect of an angle was largely missing.

Language and learning mathematics

Words used to denote geometric concepts often have a different meaning in the everyday register compared to the mathematics register, and such discrepancies may be language specific, as shown by Rønning and Strømskag (2019), in a different context from geometry. In semiotic literature, the relation between a sign and a reference context is described in various ways, e.g., by the epistemological triangle (Steinbring, 2005). The sign (in our case, the word *angle*) does not have a meaning of its own. This meaning must be produced in a mediation between the sign and a reference context. However, this mediation is not entirely subjective and arbitrary but determined by epistemological conditions of mathematical knowledge (Steinbring, p. 22). In this mediation, examples and activities are important, but “mathematical concepts cannot be sufficiently explained only within the frame of cultural behavior, usage and interpretation” (Steinbring, p. 19). Steinbring emphasises that the system illustrated in the epistemological triangle is not static, but developing, based on actions by the learner, and interaction between teacher and learner (pp. 22–23).

Methods of the study

Three kinds of data were used for this paper. First, the Norwegian National Curriculum, the curriculum for secondary schools (Sekundarschulen) in Saxony-Anhalt in Germany and the National

Curriculum of India (known as NCF-2005)⁴ (NCERT, 2005), limited to two states in India. The curricula were analysed with a focus on how the word and the concept *angle* appears, and to what extent the curricula give information on respective learning goals or concepts. Second, two to three common textbooks from each country were selected, and the chapters on angles were analysed with respect to their focus, definition of angle, and static or dynamic perspectives on angles. Third, audio data from teacher interviews were analysed. The interview data is part of the data collected during the EsCo project. Semi-structured interviews were conducted with experienced teachers working with linguistically disadvantaged pupils and took place in seven country contexts including Germany, India, and Norway. All teachers participated in at least three interviews focusing on one of the following topics: linear equations, proportional reasoning, angle properties, and probability. In each interview, teachers were asked similar questions about what they perceived to be the linguistic challenges both they and their learners faced in their mathematics classrooms and the strategies the teachers used to address these challenges. This paper only refers to the interviews on angle properties. In our analysis, we included all teacher utterances related to the topic *linguistics aspects/words used*.

Presentation in curriculum texts and textbooks

In the Norwegian National Curriculum⁵ for grades 1-10, the word *angle* occurs only once, in a learning goal for Grade 4 (age 9-10), with no indication of what the concept means or how it is expected to be worked with. It is therefore necessary to consult textbooks to get an impression of how the concept of angle is presented to learners. Two book series were investigated, and they both introduce the classification of angles into acute, right and obtuse in Grade 4, showing (typical) examples of the three categories. One book (Dahl & Nohr, 2022) presents three pictures of angles with the words right, acute and obtuse, accompanied by the question “why do you think the angles are called right, acute and obtuse?” (p. 108). The other book (Fritzen et al., 2022) states that “where two straight lines meet, we get an angle”, that “an acute angle is smaller than a right angle” and that “an obtuse angle is larger than a right angle” (p. 12). The words “smaller” and “larger” point to some kind of measurement, but there is no explanation of what exactly is measured. In the same book series for Grade 6 (Kongsnes et al., 2021), it is stated that “[t]wo lines meeting at a point, form an angle. The two lines are called the legs of the angle. The point where the lines meet, is called the vertex” (p. 10). It is not clear what exactly the angle is, it is just said that the lines *form* an angle. This book also states that angles are measured in degrees and that an acute angle is smaller than 90 degrees, a right angle is 90 degrees, and an obtuse angle is between 90 and 180 degrees (p. 10), leaving it open whether an angle could be larger than 180°. Also here, it is not clear what is being measured. Gulbrandsen et al. (2020) make a connection with circles by stating as a fact that “a circle is 360°” (p. 84) and angles are seen as fractions of a circle. In this context, angles larger than 360° will not make sense. The theoretical presentation in the books points to a static angle concept and that angles cannot exceed 360°, but still there are activities and exercises going beyond this.

⁴ The school curriculum in India was revised in 2023 and new textbooks for schools are under preparation. The current study is based on the 2005 curriculum (NCF-2005) and the textbooks based on this curriculum.

⁵ <https://www.udir.no/lk20/mat01-05/kompetansemaal-og-vurdering/kv18>

In one of the books for Grade 4 (Dahl & Nohr, 2022), an exercise is set in the context of a skateboard with a question about how many degrees are turned when doing one and a half rounds with the skateboard, and a second question about how many degrees are turned when doing three rounds. This clearly points to a dynamic angle concept and also requires the interpretation that an angle could be larger than 360° , although the word *angle* is not used. In one of the books for Grade 6 (Kongsnes et al., 2021), an exercise is formulated with the question “Does the angle have a fixed size, or can it change size?”, accompanied by pictures where one picture shows an open car door. This points to something in between a dynamic and a static angle concept – for any position of the door, the angle is fixed, but since the door can stay open in many positions, there is a certain dynamic aspect involved.

Compared to the Norwegian curriculum, the word angle occurs in the curriculum for secondary schools in Saxony-Anhalt⁶ not only more often but also more detailed and comprehensive. The angle concept is in the focus of two key competence areas, “Basic geometric concepts and symmetry” and “Angle relations” for Grades 5 and 6 (age 10 to 12). There, not only topic specific mathematical competences such as identify and sketch different types of angles are listed but also basic knowledge aspects such as the technical vocabulary acute, right, obtuse, straight, reflex, and full angles. Even though it is more detailed than the Norwegian curriculum, also the German curriculum does not indicate what the concept behind the word angle entails. The subsequent textbook analysis showed that both examined textbooks (Bernhard et al., 2017; Hecht et al., 2012), similarly to the Norwegian textbooks, do not explicitly define the concept angle, but state that an angle is *bounded* (German: *begrenzt*) by two legs or rays with the same starting point called vertex. Furthermore, there is a huge emphasis in both books on the classification of angles. Interestingly, and also analogous to the observations in the Norwegian textbooks, both German textbooks include tasks supporting a dynamic perspective by referring to clocks or to so called “angle discs”, even though the definitions focus on a static perspective of angles. In Hecht et al. (2012), for example, “angle discs” are introduced right at the beginning of the new topic. In one of the tasks, the children are asked to systematically vary the angles (dynamic perspective) in order to differentiate between the different types of angles in a static perspective. Thus, similar to the Norwegian task with the car door, the static perspective on angles is addressed here as soon as an angle is set. Before that, however, the dynamic perspective is in the foreground. In addition, some tasks in both textbooks refer to everyday contexts such as finding angles in the classroom or favourable shooting angle in front of the goal on a football field.

The Indian textbooks, published by NCERT⁷, are based on NCF-2005. In the textbook for Grade 5, *angle* is presented so that a learner “gets the feel of an angle through observation and paper folding” (NCERT, 2006, Grades 1–5, p. 4), and the following specific mathematical competencies are listed: a) Identify right angles in the environment, b) Classify angles into right, acute and obtuse angles, c) Represent right angle, acute angle and obtuse angle by drawing and tracing. Like in Germany, the NCERT books for Grades 5 and 6 offer mathematical competences related to angles. For example,

⁶ Fachlehrplan Sekundarschule: https://lisa.sachsen-anhalt.de/fileadmin/Bibliothek/Politik_und_Verwaltung/MK/LISA/Unterricht/Lehrplaene/Sek/Anpassung/lp_sks_mathe_01_08_2019.pdf

⁷ NCERT (National Council of Educational Research and Training) is the federal body for framing the national curriculum for school education in India and for preparing textbooks. Different states in India can follow either the NCERT curriculum and textbooks or prepare their own state curricula and textbooks aligned with NCERT.

the Grade 5 textbook *Math Magic*, has a chapter on *Shapes and Angles* which introduces pupils to fundamental properties of angles and their classification. Angles are illustrated through relatable examples, such as the “angles formed by the hands of a clock” or the “corners of a square”. Through interactive exercises, the aim is to make pupils learn to identify and measure angles and also explore angle relationships within shapes, leading to developing geometric reasoning. In the Grade 7 mathematics textbook (NCERT, 2006), angles are further explored in depth and also connected to geometric shapes such as different types of triangles, quadrilaterals and circles. The chapter on *Lines and Angles* introduces the concepts of complementary and supplementary angles, angles formed by transversals, and angle properties within polygons. There are different activities and problem-solving tasks to build a deeper understanding of angle relationships and their applications in various geometric contexts.

Akin to the Norwegian and German textbooks, the NCERT curriculum does not present any explicit definition of angle but focuses on practical applications and problem-solving strategies (NCERT, 2005). The focus is on building meaning of the angle concept through everyday contexts and the emphasis is on real-world applications and contextual examples. Exercises in the NCERT textbooks often incorporate everyday scenarios, such as measuring angles in architectural designs or identifying angles in natural phenomena, aiming at building procedural fluency.

Linguistic aspects and challenges

This part is based on data from the interviews. N3⁸ explains that “it is difficult to explain what an angle is without using the word angle”. N1 points out that “the word angle is an integrated part of our language, and most people have used it or heard it used”. The term “angle”, like many other geometrical terms, are well known from everyday life, which may be helpful, but also carries extra challenges for pupils not having Norwegian as their first language. This is emphasised by N1, when she says that “they also have to learn the word”. For pupils with Norwegian as their first language, there are challenges because words may have different meanings in the everyday register compared to the mathematical register. This was pointed out by N2 in connection with words characterising types of angles, e.g., *acute* and *obtuse* (Rønning, 2023). An acute angle is in Norwegian called *spiss*, which means *sharp* (like the tip of a pencil). This can be confusing because an angle which is almost 90° does not look *spiss* (sharp). Regarding the word *stump* (in English *blunt*), the challenge is different because the word *stump* is an old-fashioned Norwegian word which has no meaning for young people. A right angle is called *rett vinkel* in Norwegian, and the word *rett* also means right, as in correct. N2 remarks in connection with *right angle* that pupils have asked “what does a wrong angle look like?” Hence, the question “why do you think the angles are called right, acute and obtuse?” in the textbook by Dahl and Nohr (2022) may not make any sense to pupils.

Both German teachers report that their students experience linguistic challenges during lessons on angles. Teacher G1 states that her students often struggle with talking about different types of angles because phrases like “smaller than” or “greater than” need to be filled with mathematical meaning. This challenge might arise since there is a lack of definition of the concept angle (at least in textbooks)

⁸ The teachers are referred to with a letter indicating the country (G=Germany, I=India, and N=Norway) and a number.

and therefore, similar to the phenomenon described from the Norwegian data, it is not clear what is being measured. Referring to the definition of acute angles G1 reports: “They [the learners] always say from 0 to 90 degrees. [...] Yes, that's where it starts, that 0 and 90 degrees don't belong. How do you write it down? Do you write it down with a relationship equation or “between 0 and 90 degrees”, so you have a lot of linguistic means, how can you express it differently? How do I express in my language that I don't bring in 0 and 90? So you have to work intensively on this when it comes to angles.” And she adds that “with the linguistic formulations—next to each other, opposite each other—these are the difficulties that keep coming up, these verbal abbreviations”. Thus, G1 highlights the students' challenges with talking about relations. This is extended by teacher G2 towards typical mathematical phrasing: “But there are always formulations such as “the bigger... the smaller...” some students do not understand what is going on. You need to practice a lot.” G2 is very aware of language varieties, and it is important for her that pupils can explain in their own words: “I have accepted many formulations that have hit the maths, let's put it that way. I didn't always pull myself up by the textbook.” Furthermore, she states that she has different expectations for different children's explanations, particularly for those with special needs.

Thus, we can see interesting differences between the reports of the teachers in our study. While the Norwegian and Indian teachers talk about the everyday meaning of words that are also part of the technical mathematical vocabulary, the German teachers put more emphasis on phrases and students' explanations of concepts. As the insights from the Indian data show, in this context, the transition from everyday to technical language is even more in focus.

All the three Indian teachers reported that *angle* is introduced in the mathematics textbooks by referring to a “corner”. I2 referred to the Grade 6 NCERT textbook that it mentions, “जब कोने (corner) बनते हैं तो कोण (angle) भी बनते हैं (when there is a corner, there is an angle) (NCERT Grade 6, 2006, p. 61). कोन (*kon*, corner) is a commonly used word in many Indian languages to refer to a corner, while the technical register for “angle” is कोण (*konn*) which is pronounced similar to *kon*. Hence, the semantic use of the words *konn* and *kon* are similar, implicitly referring to the angle embedded in a corner. The above sentence is the starting sentence of the section on *angles* in Grade 6. I2 also reported that textbooks mention that जब रेखा या रेखाखण्ड मिलते हैं तो कोण बनते हैं (angle is formed when lines or line segments meet), consistent with the congruence approach (Dohrmann & Kuzle, 2015). The word for a *right angle* in Hindi is समकोण (*samkonn*) where the word सम is often used to denote “equal” in other mathematical contexts. The implicit meaning which emerges here is that when the angle of a straight line (180°) is split equally in two parts, we get a 90° angle in each part, and that is how the word “equal” is connected with the meaning of a right angle. The words for acute and obtuse angles are न्यून कोण (*nyun konn*) and अधिक कोण (*adhik konn*) respectively where semantically these words refer to *minimum* and *more* respectively. These implicitly refer to “less than” and “more than” (ostensibly keeping 90° in mind) and therefore connect with acuteness and obtuseness. Interestingly, these mathematical registers are not used in the everyday register. All the three teachers reported that they use examples from everyday contexts such as household objects and their shapes to explain the meanings of these mathematical registers, e.g., wall corners, lamp sheds carved inside walls, shape of huts and so on. Examples of curved shapes and surfaces are used as non-examples. I2 specifically mentioned that the mathematical registers prescribed in the textbooks are often not used in the

everyday language and therefore are unfamiliar to the students, such as the words for supplementary and complementary angles, acute and obtuse angles, transversals and so on. In many cases, these words are not used in everyday conversation and create language challenges for the learners. Therefore, I2 uses different common words and objects to connect to the meanings of these registers for better meaning making.

Discussion and outlook

This paper focuses on *angle*, one of the most fundamental concepts in geometry. This concept can be defined in different ways (Mitchelmore & White, 2000) and can be introduced and thought of both from a static (congruence) and a dynamic (rotation) perspective (Dohrmann & Kuzle, 2015).

The analysis of curricula and textbooks from the three countries shows that the angle concept is described and realised differently with respect to detail and with respect to the set focus in the different country contexts. However, in all three countries, there is no explicit definition of angle offered in the written documents. The concept is implicitly defined using examples, which means that the sign *angle* has mainly a semiotic function by referring to an empirical reference context (an image), whereas the epistemological function is missing (Steinbring, 2005). There is also a certain vagueness about what it means to measure an angle, which may be connected to the vagueness about what an angle is.

In the interview data, teachers from Norway and India set a focus on the everyday meaning of words and on the transition from everyday language to a more technical mathematical language. However, the German teachers put more emphasis on students' explanations and a precise phrasing of relations. This might be traced back to the curriculum documents and textbooks. In comparison, the German curriculum includes the most technical vocabulary, and this is reflected in the textbooks. In contrast, the Indian curriculum and textbooks put larger emphasis on meaning-making through everyday contexts.

Altogether, we were able to observe that teachers in all three country contexts are aware of different linguistic challenges connected to the mathematical topic angles and that they all support their learners by preparing links to everyday language and contexts and by being aware of language varieties. In the future, we plan to deepen our analysis on linguistic aspects/words used and enrich it with the interwoven topics of *measuring/calculations* and *embodied learning/everyday activities*.

Acknowledgement

The work reported was supported by a grant from the John Fell Oxford University Press Research (OUP) Fund to Jenni Ingram. We thank the teachers who participated in this study, the wider research team from the project for their support in the data collection and analysis, and Mrinal Jyoti Baruah for help with the analysis of the data from India.

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Heuristics for actualising epistemological potentials in multilingual mathematics activities

Ulrika Ryan and Petra Svensson Källberg

Malmö University, Faculty of Education and Society, Malmö, Sweden; ulrika.ryan@mau.se,
petra.svensson@mau.se

This paper contributes to the discussions about how students' multiple languages and experiences with multiple mathematical practices can be recognised as epistemological resources for learning in mathematics classroom activities. Drawing on theoretical and empirical findings we suggest some heuristics for actualising epistemological potentials in multilingual mathematics activities.

Keywords: Multilingualism, epistemological potentials, heuristics, conceptual understanding.

This paper contributes to the discussions about how students' multiple languages and experiences with multiple mathematical practices can be recognised as epistemological resources for learning in mathematics classroom activities (Prediger & Uribe, 2021; Schüler-Meyer et al., 2023). The aim is to suggest heuristics for arrangements of mathematics tasks as mathematics activities that can actualise different epistemological potentials. Previous studies have indicated that attention to epistemological dimensions of multiple language use and experiences of multiple mathematical practices can support students' understanding of mathematical concepts (Prediger et al., 2019) and provide recognition of immigrant students' resources for mathematics learning (Chronaki et al., 2022). Despite that, few studies have focused on heuristics for arranging classroom activities that actualise different epistemological potentials for mathematics learning (Schüler-Meyer et al., 2023). We respond to this by addressing the following questions:

- What pedagogical arrangements can actualise variations of epistemological potentials of multiple language use and experiences with multiple mathematical practices?
- What heuristics can be formulated for actualising variations of epistemological potentials of multiple language use and experiences with multiple mathematical practices?

Theoretical framing

To suggest some heuristics for actualising different epistemological potentials in multilingual mathematics activities we use a framework by Ryan et al. (2021) for analysing epistemological potentials in multilingual mathematics activities. The framework brings together multiple language use and multiple mathematical practices to consider epistemological potentials in multilingual school mathematics activities (see Figure 1). We refer to multilingual school mathematics activities as all kinds of attitudes, experiences, and practical doings of mathematics that somehow are connected to multilingualism. The framework describes four different potentials positioned on an interface between two axes. In Ryan et al. (2021), the axes range from separating to synthesising aspects of language use and mathematical practices. Instead, here, we use a continuum from non-connecting to connecting. We make this change drawing on findings from Prediger et al. (2019) about bilingual complementary and connection modes, which highlight language nuances as multilingual resources that can support conceptual understanding (Prediger et al., 2019). Drawing on pragmatic linguistic

perspectives, we consider the meaning of a concept as its relations to other concepts (Brandom, 2000). Consequently, understanding a concept is to master its conceptual relations with other concepts. For example, the meaning of the concept of a square is in its relation to, for instance, four sides, four corners, and right angles. From this, it follows that a deeper understanding of a concept means mastering a larger number of connections with other concepts. In the bilingual complementary mode nuances of a conceptualisation are combined across multilingual resources but they are not being conceptually connected. An example of this may be when speakers translate between two named languages in terms of substituting one phrase (or word) for another phrase. Hence, the complementary mode focuses on substitution rather than on making language-embedded conceptual nuances explicit by highlighting conceptual relations. In the bilingual connection mode, the full repertoire of the multilingual speaker is used in functional and dynamic translanguaging practices (Prediger et al., 2019). Multilingual resources move across language use, which can support making connections between nuances in language-embedded conceptualisations of mathematical concepts. Since the connection mode focuses on connections, it can provide opportunities for conceptual understanding in terms of increased numbers of conceptual relations.

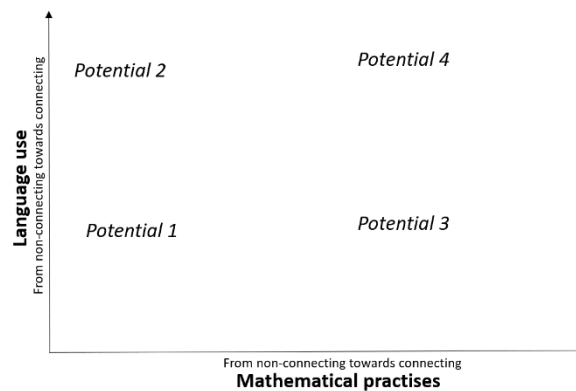


Figure 1: Framework for analysing epistemological potentials in multilingual mathematics activities

The x-axis of the framework displays a continuum that moves from epistemological potentials that do not connect multiple mathematical practices to potentials that connect multiple mathematical practices as resources for conceptual understanding. The y-axis displays a continuum that moves from epistemological potentials that do not connect multiple languages to potentials that connect multiple languages as resources for conceptual understanding. In practice, language use constitutes (mathematical) practices and (mathematical) practices constitute language use; therefore, we use the notion of an interface between two axes. Consequently, it is only in theory (i.e. on the axes) that the continuums along the axes (connections/non-connections) relate exclusively to languages or to practices. We have located 4 different epistemological potentials on the interface.

In *potential 1*, multiple languages and multiple mathematical practices are not connected as resources for conceptual understanding. Students’ multiple language resources are recognised and function as a lever to move students from informal school mathematics talk in students’ home language(s) to formal school mathematics talk in the language of instruction. Connections between multiple mathematical practices are not valued as resources for conceptual understanding. The focus is on

abandoning informal mathematical practices for the benefit of formal school mathematics practices. Mathematics activities that can actualise this potential activate the linguistic complementary mode.

In *potential 2*, multiple languages are connected, but different mathematical practices are not connected as resources for conceptual understanding. This means that students use their multiple language resources in dynamic, interconnected and functional ways (i.e., translanguaging) primarily for communication about school mathematics. Multiple mathematical practices are not connected in this potential. Consequently, mathematics activities do not specifically activate the bilingual connection mode to enhance conceptual understanding.

In *potential 3*, multiple languages are not connected but different mathematical practices are connected as resources for conceptual understanding. Here multiple language use relates to the complementary mode. The focus is on connecting multilingual students' experiences with mathematics to school mathematics as resources for conceptual understanding. To actualise this potential, mathematics classroom activities can, for example, be arranged so that they make connections between students' experiences with everyday household and community practices with school mathematics explicit. School mathematics could also be connected with school mathematics practices from places that differ from the official school institution, e.g., students' or family members' school mathematics experiences prior to migration.

In *potential 4*, multiple language use and multiple mathematical practices are connected as resources for conceptual understanding. This means that multilingual resources move across language use to make connections between nuances in conceptualisations of mathematical concepts (Prediger et al., 2019), and translanguaging practices are employed. This is integrated with connecting multilingual students' experiences with mathematics to school mathematics as resources for conceptual understanding as outlined in potential 3.

Mathematical tasks are usually at the heart of mathematics instruction that aims at enhancing conceptual understanding (Sullivan et al., 2015). Some scholars include the arrangement and activities associated with a mathematics task as part of the task (e.g., Sullivan et al., 2015) while others make a distinction between the task and the activities (e.g., Chua & Toh, 2018), which is what we do here. Hence, mathematical tasks may include textbook items, materials developed by teachers or found in curricula etcetera. Typically, mathematical tasks "are a series of easy-to-follow steps and guiding questions that prompt students to explore and discover the mathematical concepts embodied in the task" (Chua & Toh, 2018, p. 2). We assume that mathematical tasks can be arranged to elicit activities that provide opportunities for students to develop conceptual understanding. However, the same task could be arranged as a learning activity in different ways. Hence, variation of the pedagogical organisation of the same task can lead to different arranged classroom situations, which actualise different epistemological potentials.

When teachers choose tasks and arrange them as learning activities, they make pedagogical decisions which means that they handle dilemmas (Sullivan et al., 2015). Sullivan et al. (2015) identified five dilemmas that teachers need to handle when they choose a task and arrange it as a learning activity. The first dilemma concerns *context* - should the task and the arranged activity be context-free, or should the context relate to semi-reality or to reality? If so, what and whose reality? The second

dilemma concerns *language* - what social (informal-formal) languages, here we add named languages, should be accepted and what should they be used for? The third dilemma concerns *structure* - what kind of and how much scaffolding should be used? The fourth dilemma is about *distribution* - how and in what ways should the task and activity be cognitively demanding? The fifth dilemma concerns *levels of interaction* - how should students interact with each other and how should the teacher interact with students?

We assume that by exploring how the five dilemmas unfold when the same mathematics task is arranged to actualise the four different potentials described in the framework by Ryan et al. (2021), it is possible to suggest some heuristics for arranging mathematics activities that can actualise the different epistemological potentials. Ferrari et al. (2023) also used Sullivan et al.'s (2015) five dilemmas in relation to epistemological potentials of multiple language use.

Method

The method we used to formulate heuristics for arranging mathematics activities that actualise different potentials is theoretically and empirically informed. Theoretically we use the framework about epistemological potentials (Ryan et al., 2021) and the five dilemmas identified by Sullivan et al. (2015). We draw on design principles that recognise students' multiple languages (Erath et al., 2021; Schüler-Meyer et al., 2023; Prediger & Uribe, 2021). We draw on experiences from an ongoing research project (e.g. Källberg, et al., 2024), and from the 30 years of experience as mathematics teachers in compulsory school that we, the authors, share between us. As outlined below, we used five steps iteratively to develop the heuristics.

Step 1 We developed a tentative matrix (for the final version, see Figure 2). By means of the framework for epistemological potentials (Ryan et al., 2021) and the five dilemmas (Sullivan et al., 2015) we explored different general hypothetical task arrangement scenarios.

Step 2 We selected a mathematics task that we have experiences of arranging in activities in multilingual mathematics classrooms in an ongoing research project. The task concerns the development of conceptual understanding of multiplication by means of the area model. See the description below.

Rectangles with 12 squares

Materials: squared paper, pencil and 12 square tiles

How many different rectangles can you make with 12 squares? What do they look like? Build them and draw them on the squared paper. Describe the rectangles with a multiplication (for example, $4 \cdot 3$) (that shows how it looks). Use numbers and the multiplication symbol.

Step 3 Drawing on the tentative matrix, on findings from previous research (e.g., Erath et al., 2021), and from our experiences, we constructed four different arrangements of the multiplication task into classroom activities that could actualise the four different epistemological potentials respectively.

Step 4 We reflected on dilemmas in relation to the arrangements and refined the tentative matrix based on how the dilemmas were handled in the four different arrangements.

Step 5 Drawing on the refined matrix and the descriptions of the arrangements, we extracted and formulated heuristics.

Findings

Below we describe four different arrangements of the selected task. They are not complete or comprehensive descriptions. Since our research context is Swedish, the language of instruction is Swedish. Consequently, when we write Swedish below, we refer to the language of instruction.

An arrangement to actualise potential 1

Students who share home languages are put in language homogenous groups so that they can use their full linguistic repertoires when they engage in peer-talk to solve the task. They are encouraged to use all their languages for communication and to use digital translating tools when needed to understand the text in the task. Students receive concept lists that comprise naming of the mathematical concepts multiplication, rectangles, and squares in Swedish and in their home languages, complemented with the mathematical symbol for multiplication and visualisations of rectangles and squares. The group work is followed by whole classroom discussions where students show solutions on the board and explain them in Swedish. Teacher moves include facilitating the discussion and revoicing and rephrasing students' explanations using formal and informal mathematical language and mathematical symbols. The teacher makes sure that the students' translation of, for example, the multiplication 4 times 3 into an array model with four rows of three squares each is correctly notated as a multiplication and that it is verbally explained (as "four threes" or "four sets of three") to support the conceptual understanding of multiplication.

An arrangement to actualise potential 2

This arrangement is arranged in similar ways as the former but differs since language use connections are facilitated. It means that the students work in language-homogenous groups, receive concept lists and can use digital translating tools. But they are also asked to complement the concept list with explanations of the concepts written in their home language and in Swedish. When the students present their solutions to the task on the board, they get to choose what language(s) to use. The teacher moves are similar to the arrangement that actualises potential one. In addition, they include noticing and highlighting similarities and differences of the mathematical concepts in the students' different home languages. The teacher shows a linguistic interest and asks the students questions such as *How do you say multiplication in your home language?* Based on the students' responses and their solutions, the teacher empathises with how the word for multiplication is similar and different in the different languages to raise students' meta-linguistic awareness.

An arrangement to actualise potential 3

Before the lesson with the selected task, the students are assigned a homework task to interview parents or other family members about how they learned multiplication, how they do multiplications like 4 times 6 (if not knowing it by heart) and in what situations they use multiplication. Students are put in language homogenous groups and are encouraged to use their home languages. They are encouraged to present their homework and to notice similarities and differences between their family members' responses. Then the groups share their discussions with the whole class. The teacher

summarises the students' presentations on the board and highlights how different practices of multiplication are connected. Thereafter, the groups are encouraged to solve the selected task by engaging in peer-talk and to use concepts lists and digital translating tools, like in the first arrangement. The group work is followed by whole classroom discussions where students show solutions on the board and explain them in Swedish. Teacher moves are similar to the ones in the first arrangement. In addition, they include highlighting how students' different ways of solving the multiplication task can be connected to each other to support the conceptual understanding of multiplication.

An arrangement to actualise potential 4

Before the lesson with the selected task, the students were assigned a homework task to interview parents or other family members, asking the following questions: *What is multiplication called in your home language(s) or another language that you know? Is there more than one word for it? For example, in Swedish, we can say both multiplication and times. What does the word(s) for multiplication in your home language or another language that you know mean? How would you translate that into Swedish? How do you write the symbol for multiplication in your home language? In Swedish, it is usually written as a dot •, and How do you say five multiplied by five (or five times five) in your home language? Are there several ways to say it?* The students also ask questions about how they have been taught multiplication, and how they solve multiplications like 4 times 6 (if not knowing it by heart). The homework task is followed up in the same way as in the arrangement that actualise potential 3. But they do not only orally present their discussions to the whole group; in this arrangement, they also write the words and symbols for multiplication in their home languages on the board. The teacher summarises verbally and highlights different practices of multiplication and different meanings of the word multiplication. For example, in Swedish and English, the words multiplication and times mean an increase, with the *multi* meaning several, diverse or multiple, whereas, for example, the Arabic word for multiplication *darb* means “to hit” and/or “striking”. Then, the groups are encouraged to solve the selected task and use concepts lists and digital translating tools like in the other arrangements. Here, the group work is followed by whole class discussions where students are encouraged to say the concepts and operations in Swedish and in their home languages. In addition to the teacher moves above, teacher moves from the previous arrangements are enacted.

Heuristics

In this section, we present heuristics for arrangements of mathematics tasks as activities that can actualise different epistemological potentials in the matrix below (see Figure 2 below). The matrix combines the four epistemological potentials of Ryan et al.'s (2021) framework with the five dilemmas (numbered D1, D2 and so forth in the matrix) proposed by Sullivan et al. (2015).

Discussion

By exploring how the five dilemmas unfold when the same mathematics task is arranged differently to actualise the four different potentials described in the framework by Ryan et al. (2021) we have suggested some heuristics for arranging mathematics activities that can actualise the different epistemological potentials. It is important to note that each one of the potentials offers affordances and constraints. Hence, there is not one ideal potential and one ideal arrangement. Consider, for

example, how teachers need to balance the use of students' home languages with the demand for knowing mathematics in the dominant language for progression in the educational system. From this perspective, the arrangement for actualising potential 1 may be the optimal one. However, the arrangement for actualising potential 1 may be the one arrangement that recognises epistemological resources of multiple languages and mathematical practices the least. Adler pointed out this teacher dilemma already in 2001. Extracting and formulating the heuristics in the matrix was not a straightforward task. Some of the heuristics overlap and can be placed in more than one of the cells. One reason for that is that the framework consists of an interface between two axes. This means that the potentials that are located horizontally, for instance, potential one and three, and two and four, are similar with respect to the y-axis, while they differ with respect to the x-axis. This also applies vertically. For readability of the matrix, we decided to include some overlaps to reduce the number of flips between columns. Also, we do not claim that this matrix is complete. Rather, it needs to be put into practice to investigate its relevance as heuristics for actualising different epistemological potentials in multilingual mathematics activities. Hence, we invite teachers and researchers to use the heuristics and make improvements to them.

	Potential 1	Potential 2	Potential 3	Potential 4
D1	Consider a balance between "pure mathematics" and semi-reality in school mathematics discourse.	Consider a balance between "pure mathematics" and semi-reality in school mathematics discourse.	Consider how to integrate students' experiences with the concept by making connections to their homes and communities for example through homework activities that focus on mathematical practices.	Consider how to integrate students' experiences with the concept by making connections to their homes and communities for example through homework activities that focus on mathematical practices.
D2	Consider how translating tools, concept lists in multiple languages, peer talk in multiple languages can be integrated in individual and group activities.	Consider how to activate students' full language repertoires in all parts of the activity including whole class discussions.	Consider how translating tools, concept lists in multiple languages, peer talk in multiple languages can be integrated in individual and group activities.	See potential 2 and Consider how to activate students' full language repertoires to enhance meta-linguistic awareness in the activity. Ask questions about differences and similarities in mathematical concepts between languages and make connections.
D3	Consider revoicing using formal language in the language of instruction. Consider using multiple representations.	Consider how to make safe spaces for students to use their full language repertoires. Consider using multiple representations.	Consider how to make connections between multiple mathematical practices. For example, compare mathematical notation in different cultures, compare experiences of concept use in different school mathematical cultures to scaffold students' connection-making.	See potential 2 and 3 and Consider how to make connections between multiple languages. For example, compare the meaning of mathematical concepts in multiple languages to scaffold students' connection-making.
D4	Consider how students' different solutions can be used in whole class discussions. Consider how the task could be altered to make it more or less cognitively demanding.	Consider how students' different solutions can be used in whole class discussions. Consider how the task could be altered to make it more or less cognitively demanding.	Consider how to give time and space for students' connection-making activities since they are cognitively demanding.	Consider how to give time and space for students' connection-making activities since they are cognitively demanding.
D5	Consider individual and/or group work in language homogenous groups. Consider how to encourage students to use their available language resources. Consider giving students different roles in group work. Consider how to integrate explaining and exemplifying questions in the activity.	Consider individual and/or group work in language homogenous groups. Consider how to instruct students to use their full language repertoires. Consider giving students different roles in group work. Consider how to integrate explaining and exemplifying questions in the activity.	Consider individual and/or group work in language homogenous (or not) groups based on their different experiences with the concept at stake. Consider how to encourage students to use their available language resources. Consider giving students different roles in group work. Consider how to integrate explaining and exemplifying questions in the activity.	See potential 2 and 3 and Consider how to position yourself and your students as co-learners. For example, consider how to show curiosity about students' language repertoires and their experiences with multiple mathematical practices.

Figure 2: Matrix of heuristics

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Exploring parent-child interactions by learning mathematics: Repertoires-in-Use within a kindergarten-family learning environment

Ángela Uribe¹, Priska Sprenger² and Barbara Ott¹

¹St.Gallen University of Teacher Education, CH; angela.uribe@phsg.ch, barbara.ott@phsg.ch

²University of Education Freiburg, DE; priska.sprenger@ph-freiburg.de

This paper explores the interactions between Lami, a child fluent in Spanish and learning German and Swiss German, and his father, who moved to Switzerland three years ago. They engage in mathematical activities at home, introduced through videos. These activities are part of a broader learning environment that includes similar activities at kindergarten, conducted in the language of instruction. The paper examines how Lami and his father use their linguistic repertoires in the learning process. It provides a detailed analysis of the ‘sources of meaning’ used during their interactions, highlighting the diversity of repertoires, even within a single language. It reveals that the richness of language in mathematical interactions extends beyond the specific languages spoken, encompassing the full spectrum of an individual’s linguistic and experiential repertoire.

Keywords: Multilingualism, kindergarten, learning at home, ‘repertoires-in-use’

Introduction

Family engagement plays a crucial role in developing a multilingual environment that supports cognitive and language development in children (Cummins, 2000). Understanding the learning processes of multilingual children can be significantly enhanced by examining the interactions between parents and children during mathematical activities. These interactions often uncover a variety of cultural experiences and alternative mathematical approaches, thereby broadening the learning spectrum (Civil & Quintos, 2022).

In Switzerland, where this study is conducted, a considerable number of children are exposed to multiple languages at home: 33% of children under 15 years old encounter two languages, and 10% are exposed to three or more languages (BFS, 2021). Considering that kindergarten is part of the educational trajectory in Switzerland, where children typically attend from the age of four for two years, understanding how to coordinate family and school efforts becomes essential. Particularly, this coordination is crucial for effectively integrating aspects of multilingualism and parental involvement to support the learning process and to optimize the potential benefits of multilingualism from an early age (Baker, 2014).

In this design research study, we design a learning environment that bridges family and school contexts (Ott et al., 2024). We develop tasks for both settings that focus on developing an understanding of perceiving and using structures to determine the cardinality of sets through discussions about structured arrangements, using materials such as plastic eggs and egg cartons. Our aim is to engage the full linguistic repertoire of students through mathematically rich, playful activities. In this paper, the focus is placed on the case of Lami, a multilingual child whose linguistic repertoire includes Spanish, German, and Swiss German. We reconstruct parent-child interactions

with focus on naturally occurring argumentative situations during joint mathematical activities with materials. Furthermore, we use the approach of ‘repertoires-in-use’ to disentangle the resources used for meaning-making in these interactions.

Theoretical background

Argumentation in family interactions

Argumentation plays a crucial role not only in fostering multilingualism by promoting active, diverse language use (Moschkovich, 2002) but also in developing mathematical understanding. Beyond formal educational settings, argumentation is also deeply embedded in family dynamics, serving as a medium through which members negotiate and navigate daily challenges and differing viewpoints (Arcidiacono et al., 2022). Arguments are a natural part of family life, shaped by the relationships and dynamics within the family, and serve as both a ‘vehicle and site of socialization’ (Heller, 2023). Of our particular interest is to reconstruct how these naturally occurring argumentations unfold within the context of learning mathematics together at home.

‘Repertoires-in-Use’ for meaning-making

Different languages, dialects, and even language-related experiences can serve as ‘sources of meaning’ (Barwell, 2018) that can be activated to support the process of meaning-making. The sources of meaning that an individual has at their disposal may be difficult to reconstruct due to their complexity. This paper resonates with the concept of ‘repertoires-in-use’, which refers to the situational reconstruction of the use of one’s own repertoire for a specific purpose, specifically for making meaning by learning mathematics (Uribe & Prediger, 2021). The term ‘repertoire’, as defined by Lüdi (2016), refers to the full set of linguistic resources – encompassing languages, dialects, styles, and registers – that an individual has access to and employs to navigate different communicative contexts effectively. The conceptualization of ‘repertoires-in-use’ focuses on analyzing language use in multilingual mathematics learning, distinguishing between formal-related, meaning-related, and everyday language across activated languages. It also assesses graphical, symbolic, and contextual representations and examines how these resources are interconnected in the learning process.

To better understand the ‘repertoires-in-use’ for meaning-making, it is crucial to examine specific learning contents. Numerous studies have shown that the ability to perceiving and using structures to determine the cardinality of a set is the basis for successful arithmetical learning (e.g. Lüken, 2012; Mulligan & Mitchelmore, 2013). The study presented here focuses on such structuring processes as a fundamental aspect of mathematical learning (Sprenger & Benz, 2020).

Methodology of the design research study

The data presented in this paper are part of the project “MATHESprechen in kindergarten: Using language repertoires in mathematics productively right from the start”, which investigates the overarching question of the extent to which the use of multilingual resources is stimulated in kindergarten children through mathematical activities with materials in a learning environment. This paper investigates the following sub-question: Which sources from their repertoire do a multilingual child and their father use during playful family interactions focused on structuring processes, and how are these sources interconnected by the child and father during the learning process?

In Design Research methodology, we developed an initial design and investigated the research question, focusing on iterative refinement through two cycles of design, implementation, analysis, and redesign, aiming to both understand and improve the learning environment and theories.

Design approach

The designed learning environment consists of activities for learning the mathematical concept of perceiving and using structures to determine cardinality at kindergarten and in the family. Although the kindergarten setting is monolingual in the language of instruction and the family setting is monolingual or multilingual in the home language(s), depending on typical family communication patterns, the study aims to bridge these contexts to enhance learning and to identify which multilingual resources activate children as ‘sources of meaning’ by learning mathematics. For the family setting, videos were developed to guide the activities. For each activity one video provides parents with informational content, including suggestions such as potential questions to ask their children to stimulate discussion about structuring and other video is for parents to use with the children, explaining the activities through a narrator and demonstrating with two puppets acting as language models.

Methods of data gathering and data analysis

The learning environment was tested in a design experiment in September 2023 in two kindergarten groups in Switzerland with children in the last year of kindergarten. The first group consisted of 10 children (average age: 5 years, 7 months), the second of six children (average age: 5 years, 3 months). Two Albanian-speaking families, one Spanish-speaking family and one Bosnian-speaking family took part in the family part. The whole design experiment was videotaped.

Individual scenes, chosen for their direct relevance to the research question due to the meaningful interactions occurring between children and parents, were analyzed using the interpretative methodology of interaction analysis (Krummheuer & Naujok, 1999).

Empirical insights into Lami’s process

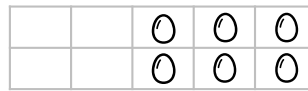
The analysis provides insights into the ‘repertoires-in-use’ of Lami and his father while learning mathematics oriented by the designed videos within the family setting. Lami, who is 5 years and 3 months old, moved to Switzerland three years ago. Both of Lami’s parents come from Spain. His father, with whom he conducts the activities, speaks Spanish and has limited understanding of German. The home language is Spanish. The transcript displays the original utterances along with their translations in parentheses. In the analysis, only the translations are shown. The use of gray coloring for these translations indicates that the original utterances were made in a different language.

Learning mathematics together at home with videos and materials

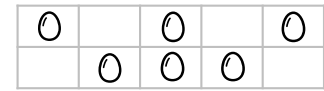
At home, Lami and his father engage in one activity in which one of them rolls the dice. Each of them then places the number of plastic eggs, as determined by the dice roll, into their own egg carton in any arrangement they choose. Afterwards, they compare their different arrangements and discuss these, including possible structures. Figure 1 provides a guideline for referring to the position of each egg in the text, as well as the arrangement made by Lami and his father for a dice roll of six.

P1	P2	P3	P4	P5
P6	P7	P8	P9	P10

Guideline for describing the positions (P) of the eggs in the arrangement



Lami's first arrangement of six eggs



Father's first arrangement of six eggs

Figure 1: Guideline for describing positions and arrangements

Repertoires-in-use by describing structures: Navigating additive and multiplicative language means

The analyzed scene starts when Lami rolls the dice again after he and his father have already played the game once. Lami initiates the scene by rolling a six, and he and his father then place the corresponding number of eggs into their respective egg cartons (see Figure 1). Upon seeing how his father has arranged the eggs, Lami exclaims: *“that’s not six.”* From this moment, the scene centers on arguing why both arrangements consist of six eggs. In response to Lami's objection, the father encourages counting rather than arguing: *“isn’t six? count, count.”* However, Lami does not follow this instruction. Instead, he attempts to engage in an argument by stating, *“look, you placed them differently.”* The father again requests to count. Lami counts the eggs one by one in the carton, but after completing the count and within the same turn, Lami reiterates his perspective, *“...but I placed them the same, because I won.”* The father seems unconvinced but does not ask Lami to clarify his point; instead, he rephrases it. The following analysis concentrates on the moment after this when Lami self-initiates another argument to support his reasoning: *“And look here, look.”* (turn 101). It appears that Lami suddenly notices something else in the arrangement that supports his argumentation.

- 101 Lami: Y mira aquí, mira. (And look here, look.) (*puts the dice on the ground and turns it so that the dice face 6 is oriented as it might lie in the father’s carton*) Es así. (It’s like this.) (*points to the dice on the dice*) Tú los has puesto en (You placed them in) (*turns the dice so that the dice face 6 is oriented as it lies in his carton, touches the eggs in the father’s carton in the order: P5, P3, P1*) uno, dos, tres. (one, two, three.) Y tres. (And three.) (*touches P7 – P9*) Aquí hay tres (Here are three) (*points to the top row P5, P3, P1*) y aquí (and here) (*points to squares P7-P9*) hay tres y es seis, pero (there are three and it’s six, but) (*takes the egg from P5 and places it in P4*) si los pones (if you place them) (*takes the egg from P1 and places it in P2*) así, es igual (like this, it’s the same) < (*places the eggs back as the father had placed them*)
- 102 Vater: Vale, vale, ¿y si los pongo así? (Okay, okay, and if I place them like this?) (*takes eggs from the bottom row and places them in squares P6, P8, P10, looks at the child*)
- 103 Lami: (*scratches his forehead, looks at the dice*) Queda dos, dos y dos. Tres, (It becomes two, two and two. Three,) (*shows three fingers*) tres dos. (three two.) (*looks at the father*)

Lami’s utterance in turn 101 is remarkable, as it is highly condensed and much information has to be unfolded: He articulates *“You place them in one, two, three. And three...”*. In the utterance, Lami counts the eggs on the top row independently, he then bundles the eggs in the lower row, by subitizing (seeing and naming the set at one glance) not by counting each egg individually, and by collectively referring to them as *“... And three ...”* (see Figure 2). He subsequently rephrases and summarizes, *“Here are three and here there are three”*, and then addresses his father’s earlier remark with, *“and it’s six...”*. This clarification underlines that Lami’s point is a different one: while the actual arrangement indeed consists of six eggs, it does not align with “his” game’s rules, because it does not

resemble a dice showing six. However, Lami points out that the eggs in the carton can indeed be rearranged to visually represent the six on a dice and execute the rearrangement: “*but if you place them like this, it’s the same.*”. Lami’s argumentation, referring to the arrangement of the points on the dice, remains primarily implicit and is supported mainly by gestures (see Figure 3).

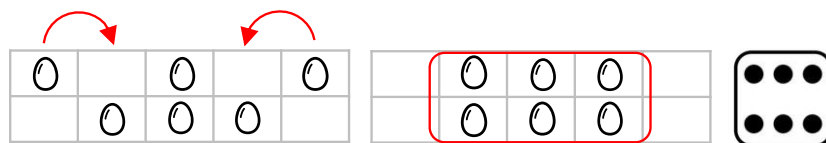
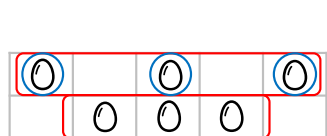


Figure 2: Lami's diagram in his father's arrangement

Figure 3: Lami's modification in his father's arrangement

The father's next move is a very productive one: “*Alright, alright, and if I put them like this?*” (turn 102). He shifts both eggs on the outside of the lower row one field outward (see Figure 4). Lami responds by describing the structure by unitizing the eggs with the phrase: “*It becomes two, two and two.*” (see Figure 5).

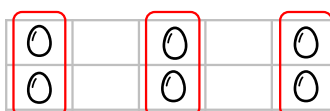
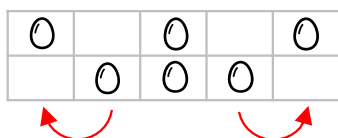


Figure 4: Father's changes in his own arrangement **Figure 5: Lami unitizing in groups of two**

The lexical choice Lami makes in Spanish, “quedar” is particularly interesting. It can be interpreted as “it remains”, “it leaves” or “it becomes”. In this specific context, the use of the singular form “queda” rather than the plural “quedan” suggests a reference to the structure as a whole rather than to the individual eggs. This leads us to believe that Lami is verbalizing the structure as a complete whole, meaning not just subsets, three subsets of two, but as a whole of six that can be structured into groups of twos. This interpretation is emphasized by Lami’s use of language in the second part of his turn: “tres dos” (*three twos*). Here, Lami verbally stresses the number “three” not the size of each group, but the number of groups. His expression “*three two*” is condensed. In Spanish, the distinction between the size of the groups and the number of groups remains very implicit. To make it explicit, one would need to unfold the expression into “three groups of two”. In this case, German, another language in Lami’s repertoire, could be used productively to also highlight the bundle as “drei 2er” (three twos) German provides a very condensed means to express this concept, effectively emphasizing the groupings in a condensed manner (Prediger, 2019).

- 104 Vater: Dos, dos .. ¿Tres por dos cuánto es? ¿Tres veces dos cuánto es? (Two, two ..Three (multiplied) by two how much is it? Three times two how much is it?)
- 105 Lami: No lo sé. (I don't know.) (*briefly looks back*)
- 106 Vater: ¿Tres veces dos cuánto es? ¿Dos, (Three times two, how much is it? Two, (*points to P5 and P10*) dos (two) (*points to P3 and P8*) y dos, (and two) (*points to P1 and P6*) cuánto es? how much is it?
- 107 Lami: (*leans forward to see the father's carton*) seis (six) (*looks at him*)
- 108 Vater: #Pah, (Pha,) (*laughs*) vamos, (come on,) (*takes eggs out of the carton*) tíralo otra vez. (throw it again.)

Lami’s way of structuring and verbally addressing the arrangement as “*three twos*” might have correctly led the father to think about multiplication. The father automatically switches to a formal-related register, asking, “*three (multiplied) by two, how much is it?...*” (turn 104). However, he does not use the extended expression “*multiplicado por*” (*multiplied by*) but the abbreviated “*por*”. This reduction to the preposition “*by*” alone, a typical way to denote multiplication in Spanish, fails to convey the conceptual core embedded in the highly technical expression of one multiplicand being multiplied by a multiplier. The use of the preposition “*por*” without a verb could lead to other interpretations, such as the everyday phrase “*three eggs for the price of two.*”. The father seems quickly to identify that this formal-related register doesn’t mean much to Lami and shifts the choice of his language means to something more meaning-related: “*...three times two, how much is it?*” (turn 104). However, this approach to expressing multiplication with the idea of “*times*”, also seems not to resonate much with Lami, who responds with “*I don’t know*” (turn 105). The father then in turn 106 unfolds “*three times two*” to “*two, two, and two*”, which seems to work, but in doing so, he only addresses an additive structure, while Lami, in his use of language, is actually addressing a more sophisticated multiplicative structure by referring to the groups as “*three twos*”. Lami addresses the multiplicative structure but is not yet able to formally name it. Lami and his father are navigating through different language registers. The expression “*two two and two*” leads Lami to say the correct answer. The father seems pleased that Lami gives the right result to the operation: “*six*”. Saying it so they conclude this sequence.

Figure 6 serves as summary and illustrates that while discussing the structure of the arrangement, Lami and his father primarily used Spanish, but employed different language registers within this language. Lami’s father navigated between formal-related and everyday language, primarily switching between these registers without explicitly connecting them to each other or the material. Lami, on the other hand, did not use formal-related language as he has not yet learned it, but is developing conceptual understanding by handling the materials and discussing them. He articulates the structure using language that is more related to meaning, even addressing more complex structures. These serve as a foundation for understanding the construction of other concepts, such as multiplication.

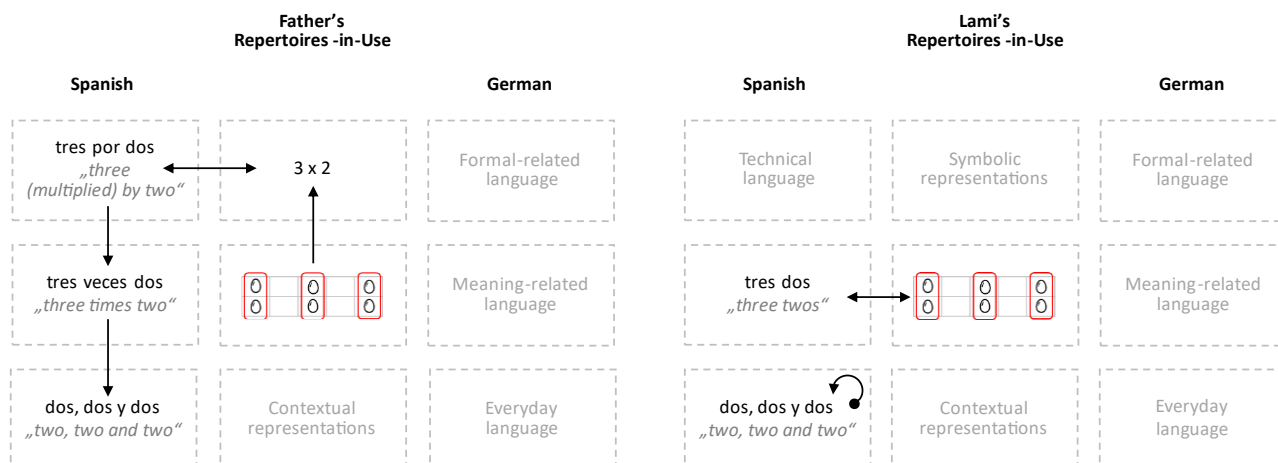


Figure 6: Summary of the repertoires-in-use of Lami and his father.

Discussion and Outlook

The empirical analysis of the presented case provides insights into the dynamics of early mathematics learning within a family setting and explores the diversity and complexity of linguistic repertoires-in-use. Lami and his father engage in an argumentation process, negotiating a common understanding of the meaning of “six” in a playful situation. The process reveals that the richness of language in mathematical interactions is not limited to the specific language spoken but encompasses the entire spectrum of an individual’s linguistic and experiential repertoire. Additionally, it shows that within a single language, in this case Spanish, different uses of language can be reconstructed. The analysis highlights how the father’s efforts to elicit counting from his son, possibly influenced by personal experiences in learning mathematics or cultural views, contrast with Lami’s approach to playing the game and arguing in line with his own game rules. This underscores the importance of providing parents with guidance on how to facilitate the learning situation at home to shape mathematical understanding effectively, without discarding but rather integrating their own resources in a meaningful way.

The reconstruction of the repertoires-in-use shows that while the father uses a more formal-related register, the child uses language that is more related to meaning. The language used by both illustrates how the father, triggered by Lami’s language use, identifies a multiplication in the arrangement but remains within an additive view of the situation. Meanwhile, Lami, although unable to formally name the multiplication, addresses the multiplicative structure by speaking about the arrangement. This situation clearly demonstrates how early mathematical experiences related to structural use are crucial for developing a foundation for the later understanding of concepts like multiplication.

Moving forward, further analysis of the project data should examine how these family dynamics and linguistic strategies might vary in other multilingual settings and across different levels of language proficiency.

Acknowledgment

The study is conducted within the project MATHEsprechen. We thank the foundation Movetia for their financial support and the teachers and families for their participation.

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Establishing sociomathematical norms in situations with early childhood teachers

Anna-Marietha Vogler

Martin-Luther-University Halle-Wittenberg, Germany;

The article assumes that young learners are already socialised into microcultures of doing mathematics in interactions with peers and teachers in kindergarten. Certain norms shape these cultures and are significant for further learning biography. While the microcultures of mathematics classrooms and the norms of classroom discourses have been described in mathematics education, thus far, there are fewer studies concerning the establishment of norms for early mathematics learning. Therefore, this article aims to analyse social and sociomathematical norms in teacher-child discourses. A focus is placed on the relevance of early childhood teachers in their role as representatives of mathematical culture. The analyses show that the emergent sociomathematical norms on mathematical concepts partly contradict viable mathematical concepts provided in the school discourse and are thus not adaptable.

Keywords: sociomathematical norms, teacher-child-interactions, early childhood education

Introduction

Children learn rules at an early age by participating in everyday interactions with caregivers such as parents, siblings, and peers. In kindergartens, the focus is often on social rules like eating at the table or following the rules of the game (Köymen et al., 2014). In addition to these social rules, however, subject-specific mathematical rules can also emerge in various situations in kindergarten, according to the assumption of the article. In this way, the first microcultures of mathematics emerge into which the children are socialised. Each of these microcultures (of doing mathematics) has its own rules, which are constructed by the participants in processes of negotiation (of mathematical meaning). What is considered mathematically acceptable can, therefore, vary depending on the respective context or discourse. What is learned also depends on this culture (Sierpiska & Lerman, 1996). In this context, mathematical discourses in kindergarten are the first institutionally shaped discourses (Vogler, 2024). Participation in these discourses is not only constitutive for the development of mathematical thinking, the mathematical and discursive competencies acquired in these discourses are also of great importance for the further learning biography in the subject (Benz et al., 2015). This can also apply to the acquired and internalised rules, which act as the cornerstone of interpretation processes in later discourses and can be consolidated or modified (Köymen et al., 2014). Early childhood teachers have a specific role in these early institutional discourses: they negotiate the various social and subject-related rules with the children (Vogler, 2024). While classroom discourses and the processes of constructing and interpreting established social and subject-specific rules in the form of social and sociomathematical norms have received considerable attention in research on mathematics education (Voigt, 1995; 1998; Yackel & Cobb, 1996), there has been less research on (discursive) practices in kindergarten and the norms inherent in them. Especially norms that are characteristic of mathematics-related discourses with early childhood teachers appear to be particularly important in terms of the perspective of continuous ('subject-related') learning at the

transition from the culture of doing mathematics in kindergarten to the mathematical discourse of school (Henschen et al., 2023). Not least because in both 'cultures', shared norms and the interpretation of these norms provide the basis for participation in (subject-specific) interactions and thus learning processes. An extension of the construct of sociomathematical norms to the kindergarten context therefore seems to be appropriate in order to close the research gap in the area of reconstructing social and also subject-related norms in early discourses. This concern is illustrated in the article by analysing a teacher-child situation in kindergarten. A particular focus is placed on the norms that are relevantly set by early childhood teachers – in their role as representatives of mathematical culture, which they assume from a normative perspective.

Mathematical learning from an interactionist perspective

The further explanations in this article are based on an interactionist perspective on (early) mathematical learning, as conceptualised in the anthology by Cobb and Bauersfeld (1995). From this interactionist perspective on mathematical learning, mathematics is understood as the product of social processes. Cultural and social aspects are consequently not peripheral conditions of mathematics learning but essential characteristics (Voigt, 1998; 1995). In this context, mathematical meanings or the rules of their use are constructed through processes of negotiation of (mathematical) meanings between participants in interactions. Thus, the interaction between learners and (early childhood) teachers becomes the focus of research interest. As a result, the product "mathematics" can also vary depending on the social context in which it is negotiated among the participants. Mathematics no longer appears as unchanged transferable or reproducible objective knowledge (Voigt, 1998; 1995).

Emergence of mathematical meaning in kindergarten discourses

Not only at school but also at an early age, children experience math-related discourse and negotiate rules together with different interaction partners (Vogler, 2024). The kindergarten is an important place for these negotiation processes, as this is where the first contact with institutional discourses takes place (Benz et al., 2015; Vogler, 2024). However, in contrast to the classroom, subject-related discourses in kindergarten rarely take place in the context of planned tasks but rather in spontaneous everyday settings and play situations. Despite the documented potential of these for early mathematical learning, there is a tendency towards school-based concepts that take little or no account of the individual cognitive development of young children (Benz et al., 2015). The institutional discourses of kindergartens are framed by the curricula of the various federal states – although not yet a mandatory requirement in Germany. Kindergarten is where the first mathematical experiences are made and the first mathematical concepts developed. The attendance of informed and responsible adults seems essential in order to realise this 'educational mission' in early mathematical learning situations (Benz et al., 2015). From a mathematics education perspective, the way in which the discipline of mathematics is represented and which interpretations and rules may prevail and be internalised in the mathematical process of negotiation depends on the early childhood teachers. The teachers therefore play a decisive role in establishing a culture of doing mathematics. However, there are currently no comprehensive subject-related training programmes for early childhood teachers in Germany (Wullenschleger et al., 2023). This seems especially critical, as there is not only evidence

of a positive correlation between the time spent with early childhood teachers and the development of children's academic and social skills (Goble et al., 2016), but some studies also show tendencies that a high quality of mathematical support in teacher-child discourse has a positive effect on children's mathematical skills at the end of their time in kindergarten and vice versa (Linberg & Kluczniok, 2020). With regard to this special role of early childhood teachers in supporting learning processes, how the microcultures of mathematics learning are organised in the context of early childhood teachers explicitly shaping mathematical situations is of importance.

Social and sociomathematical norms in kindergarten discourses

From the perspective of mathematics education, the construct of norms is appropriate for analysing these microcultures of mathematics learning, as the established norms in each of these microcultures define what is considered subject-related and socially acceptable as negotiated rules (Yackel & Cobb, 1996). Depending on the context in which the rules are (re-)constructed, they may differ. As Yackel and colleagues (2000) describe, it is not the presence or absence of norms that distinguish microcultures from one another. Rather, it is the nature of the emergent norms that differ from discourse to discourse of the various microcultures. Following previous findings concerning classroom microcultures and norms (Yackel et al., 2000), a distinction can be made between two types of norms: social and sociomathematical norms. Social norms are rules whose implementation does not initially require reference to mathematical content. They could therefore apply to any discourse. In (classroom) negotiation processes, the norms (including those of these authorities) are often only made explicit at moments when they are violated by members of a microculture (Sfard, 2008). The ability to interpret these norms of a culture in such moments of divergence can be understood as an important factor for the participation of learners in mathematical negotiation processes of the discourses of the respective microcultures (Vogler, 2024; Vogler et al., 2018). It seems necessary to investigate these interactions of early learning in kindergarten and to reconstruct which social and sociomathematical norms are negotiated there, since shared social and sociomathematical norms and the interpretation of these norms not only provide the basis for participation in (subject-specific) interactions and learning processes in school discourses, but the first (subject-specific) experiences of norms are already established at the kindergarten. Due to the special role of early childhood teachers in these early institutional discourses, the article will therefore analyse which norms emerge in interactions with these teachers. In particular, the article will concentrate on norms that concern mathematical concepts, because an early understanding of concepts also plays an important role in further discourses, and a sustainable development of conceptual knowledge is essential for mathematical learning. It therefore seems necessary to provide competent and coherent support with a view to further mathematical learning contexts.

Data and methodology

In order to interpretatively elaborate the construction of social and socio-mathematical norms in the teacher-child discourses, video data of early childhood teacher situations from the erStMaL (early Steps in Mathematics Learning) project were analysed, which were developed and realised by the teachers themselves in various mathematical domains (as specified in the project based on the content domains of the educational standards of the primary level) (Brandt & Vogel, 2017). It is assumed that

teachers realise topics that are particularly relevant to them in self-developed, mathematics-containing situations with planned tasks and thus significantly influence the establishment processes of sociomathematical norms (Vogler, 2024). A situation was selected with an early childhood teacher realising a game together with two children. Details on the selection of this situation can be found in Vogler (2024).

Methodology

The analysis of the data is carried out with the help of interaction analysis (Krummheuer, 2018). This allows the reconstruction of the negotiation of mathematical meaning. The presented scenes were selected with a focus on the reconstruction of norms: Efforts to establish social as well as sociomathematical norms are often only explicated in moments in which these norms are violated by members of a microculture (Sfard, 2008). This includes both social and sociomathematical norms. However, if a mathematical topic is being addressed, such as the terms for the different n-corners in the example, sociomathematical norms and their negotiation take centre stage. Such situations were selected on the basis of the interaction analyses. But, in kindergarten contexts, indicating a violation of the norm in interactions can not only be done by explaining the violation but also by ignoring it or by correcting it, for example, by asking questions (Vogler, 2024). The analyses show such situations.

Analyses of empirical examples

In the following scene, a sociomathematical norm can be reconstructed to describe tokens of a game in the negotiation process between a teacher and two children. In the process, terms are formed for various n-corners in the dialogue with the material. The early childhood teacher (ECT) sits at the table with the two children Karl and Konstantin (both approx. 4 years old). In the middle is a boardgame with 5x8 indentations for various flat geometric (hollow) solids with different top surfaces (circles, squares, hexagons, triangles, trapezia) in different colours: blue, red, green, and yellow. The scene begins when a previously uncommented placement of a solid on the board is interrupted, and the children are asked to name the different solids. Konstantin starts by naming two cylinders with the word circles. The teacher then asks Karl about his token. He is holding a hexagonal prism in his hand.



Figure 1: Illustration of situation with the children Karl and Konstantin

Analysis of scene I – Hexagon

040	ECT	Like this\ Do you know what it's called/
041	Karl	Noo\
042		<i>places the solid on the game board</i>
043	ECT	Let's count, there are corners on it\
044		<i>points to the red prism with hexagonal top surface</i>
045	<	<i>touches a corner of the hexagonal cover surface with the index finger</i>
046	<	One\
047		<i>touches another, neighbouring corner of the top surface</i>

048	Karl		Two\ touches neighbouring corners in a clockwise direction and says number words
049			
050		>	Three (.) Four (.) Five\ (...)
051	ECT	>	touches the last corner of the top surface that has not yet been touched
052	Karl		Six\ Exactly\ This is a hexagon (German: Sechseck)\
053	ECT		
054	Karl		A hexagon \

After Karl answers the teacher's question in line #041 with a clear "Noo\!", the teacher initiates a counting process in line #043 – possibly as a reaction to the boy's lack of conceptual knowledge – which refers to the corners of the base or top surface of the prism with a hexagonal top surface. She touches the first corner and says the number word "one". After a short break, Karl continues this process independently by touching each corner of the geometric figure in a clockwise direction, advancing one number word in the number word sequence and counting on. He continues the process according to established rules. The young boy's action obviously meets with the approval of the teacher. She does not interrupt the boy in the counting process. There already seems to be a sociomathematical norm for the counting process, which Karl uses to orientate his behaviour here as a matter of course. Karl's counting process probably stops because, when counting in a circle, the first element counted has to be memorised or marked in order to determine the last element. He is then supported in the process by the teacher, who taps on the sixth remaining corner with her finger (line #051). However, Karl's reaction confirms the interpretation that a norm of counting processes has already been established here: He adds the number word "six" in line #052 and is evaluated positively by the teacher. Finally, she names the "solid" in line #53 ("hexagon"). While in line #036 the teacher echoes the boy's term, it is now the boy who echoes the teacher's term. In German, the counting process to establish the concept is more obvious, as the term hexagon in German is made up of the number word and the suffix “-eck” (German: Sechseck). “Eck“ in German means angular solid. This action can be interpreted as an indication that a social norm has already been established in the group or at least between Karl and the teacher. Previously unknown and meaningful concepts are echoed. This is a pattern of interaction that is common to many teaching-learning discourses and serves to consolidate meanings and concepts through repeated presentation. In addition to this social norm from the field of teaching-learning contexts, however, efforts to establish another sociomathematical norm can also be traced in this scene. For example, a norm regarding the designation of special "shapes" with corners can be assumed here: In the case of "shapes" with corners, the designation of the shape obviously results from the numbers (n) of corners counted in conjunction with the suffix "-eck". From a mathematics education perspective, the problem is that the "shapes" in the situation are geometric solids from a mathematical perspective, where only the corners of the covering surfaces are taken into account in the counting process in the situation. In this way, the solids are labelled with terms of plane shapes. This emerging sociomathematical (concept) norm that emerges here can be described as incorrect from a mathematical perspective and is "consolidated" in the next scene.

Analysis of scene II – length of the roof rods

181	ECT	<	And the shape/ Can you tell Konstantin what it's called/
182		<	touches a cuboid with a square top surface
183	Karl		Yes\ There's a square\ A square has four sides of equal length\
184	ECT	>	

185 > *moves the index finger over all four sides of the square top surface*
 186 And four corners\ Would you like to count/
 187 *turns to Konstantin*
 187 Konst *Shakes his head*
 188 ECT Nono/
 189 Karl < one two three four
 190 < *touches a corner of the red square clockwise for each number word*
 191 > one two three four
 192 > *touches a corner of the top surface of the cuboid anti-clockwise for each number word*
 193 ECT Exactly\ quadrangle (.) or square \

Even this scene can be interpreted in the context of establishing the sociomathematical norm described above for labelling plane shapes with n corners. While the symmetrical prism with a hexagonal top surface was conceptualised in scene 1, in the analysed scene here it is a cuboid with a square base. In line #183, Karl refers to the "shape" as a square when asked by the teacher. In the following turn, the teacher then explains some of the characteristic elements of the square (four sides of equal length and four corners) and illustrates the sides of equal length gesturally by tracing each side on the top surface of the cuboid. After emphasising the characteristic of squareness in line #186, she immediately asks Constantine to count the corners. Parallels to the plot from scene 1 can be recognised here, and also a certain routine can be reconstructed in relation to the interaction pattern as defined by Voigt (1995). This routine can be interpreted as an indicator of the process of establishing the norm. Konstantin, however, refuses to comply with the request to count. Karl takes over the turn and taps clockwise on the respective corners of the square surface. He simultaneously says all the numbers in the number word series up to the number four. Immediately after this action, he touches the edge of the cover area again in an anti-clockwise direction and also simultaneously says the number words. The touching and naming of the number word series can be interpreted as a demonstration of knowledge. His action is positively evaluated by the teacher in line #193 with an "Exactly". The subsequent term "quadrangle or square" (line #193) can be understood as a reference to the sociomathematical norm in the overall view of the scene. As in scene 1, the term is also derived here by determining the number of corners of the n-cornered surface. The fact that the term rectangle is initially preferred to the mathematically more precise term of the square can again be seen as an indicator that this is a normative process. From a mathematical and didactic perspective, the two terms are by no means synonymous.

Concluding Remarks

The aim of the article was to reconstruct social and sociomathematical norms in a situation in which the focus is on the negotiation of mathematical concepts. One of the reconstructed norms involves the designation of the flat, angular surfaces of the tokens in the game. It is a norm for the concept formation that is very concise in German: the term for an angular shape is derived from the number word for the number of corners with the suffix -eck. While Karl possibly uses this norm to describe the prism with a hexagonal top surface, in the case of the cuboid he uses the more precise term square (instead of quadrangle). He only uses the counting routine to determine the corners after the dissent between Konstantin and the teacher. Konstantin's actions do not allow any conclusions to be drawn as to whether he interprets the established norm or the expectations that emerge situationally in the discourse. Overall, from a mathematical perspective, it seems to be problematic that a norm focusing

on mathematical terms is established in the process of negotiation in which geometric solids are designated by the terms for 2-dimensional shapes. If this norm is internalized in the long term, this can lead to moments of divergence for the children in the school discourse. As the first representatives of the institutional mathematical culture, the teacher thus conveys an image of mathematics that is not compatible. However, from the perspective of early mathematical education, it can also be considered whether the "incorrect" use of mathematical terms in the context of the solids at this point does not rather support the conceptual development of mathematical ideas in the children in the sense of the idea of "good enough for the moment". Nevertheless, an explicit focus on the covering surface would be beneficial. Without further processes of negotiation, which limit the framework of the established (subject-specific) norms to the cover, base, or top surfaces, the actions of the teachers will not meet the formulated demand for high-quality subject-related support in interactions and the coherence of early mathematical learning in kindergarten to the mathematical culture of the school. The reconstructions of the incorrect sociomathematical norms in the situation underline the great importance of subject-related knowledge for the design of mathematically correct and coherent educational processes by teachers as described in the current didactic literature (Dunekacke & Barenthien, 2021). Considering that, in general, sociomathematical norms are already established at an early age, it is also essential for primary school teachers to engage with the children's already established norms and to build on these at the beginning of elementary school. It will be the responsibility of future research to identify possible consistencies and divergences between the cultures of mathematics education in the institutions and to evaluate the consequences for facilitating a transition without 'discursive breaks'. In order to do this, it is important to clarify more in detail than is possible in this article which types of norms are emergent in other (less framed) discourses with teachers or peers in everyday daycare settings (Henschen et al., 2023).

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Addressing language diversity in early years mathematics: proposed classroom practices through a live brief assessment

Sinem Hizli Alkan¹ and Derya Sahin Ipek²

¹Anglia Ruskin University, United Kingdom; sinem.hizli-alkan@aru.ac.uk

²Eötvös Loránd University, Hungary; derya@student.elte.hu

Keywords: Language diversity, mathematics, early years, Live Brief assessment.

Introduction

Language diversity has gained increasing attention in teaching and learning mathematics in our diverse societies (Barwell et al., 2016; Planas et al., 2018). Teacher education programs have a critical role in challenging and preparing prospective teachers to address language diversity through critical, purposeful, and socio-cultural lenses in teaching mathematics.

In our study, year 1 primary education students were assigned to address a development idea from a local school titled "How to close the language gap in early years?". We examined those proposed practices addressing the following research question: *What are teaching practices that prospective teachers propose through Live Brief presentations to address language diversity in early years mathematics?*

Theoretical Framework

Drawing from Planas et al. (2018, p. 201), we understand language diversity as ‘the languages of the learners as they interact with mathematics but also to the languages for communication: official languages of instruction, languages of teaching, and languages of thinking and learning’. Language diversity’ goes beyond what ‘multilingualism’ or having ‘English as an additional language (EAL)’ refers to in the context of mathematics classrooms. Language diversity captures a broad range of communication mediums such as gestures, dialects, sign languages, diagrams, symbols, number systems, etc.

In this study, we used Moschkovich’s (2002) three perspectives –lexicon, register, and situated-sociocultural– as a framework to explore the connection between language and learning mathematics. The lexicon perspective highlights the importance of explicit vocabulary instruction, while the register perspective addresses the multiple meanings of mathematical terms. The situated-sociocultural perspective, as described by Moschkovich, views mathematics learning as involving more than just vocabulary acquisition; it focuses on how students use resources from various language registers to communicate mathematically.

Live Brief Assessments

This paper draws on Author 1’s involvement as a tutor for a mathematics module for year 1 primary education students in England. 118 year 1 undergraduate students participated in the Live Brief group presentation. The students were required to explain the role of language in teaching and learning mathematics, create an action plan for a school in a socio-culturally deprived area, and propose activities for whole-class, small group, and 1-1 settings.

We analysed 19 presentations, using reflexive thematic analysis across six stages (Braun & Clarke, 2021). After familiarising ourselves with the content, we generated codes linked to Moschkovich's perspectives. We constructed themes by summarising each perspective's presence, then reviewed and refined themes as explicit vocabulary teaching, scaffolding, and multi-sensory approaches. Finally, we structured the writing process by numbering presentations and coordinating tasks.

Findings and Conclusion

While a lexicon perspective was frequently observed in the activities, the presence of a situated socio-cultural perspective primarily in one-on-one activities is significant due to its social and discursive nature. The findings encompass three key themes: explicit vocabulary teaching, various scaffolding strategies, and the use of multi-sensory approaches.

Language Diversity: Barrier or Resource? Prospective teachers often held superficial perspectives on language diversity, lacking criticality. It is essential to explore their attitudes towards language diversity as these will shape their future practices. The traditional 'language gap' approach is limited; mathematics teaching should be language-rich, meaningfully involve students' socio-cultural and language backgrounds, and focus on their mathematical reasoning, problem-solving, and thinking processes (Lucas et al., 2008; Wright, 2020).

Mathematics as a communication tool? Prospective teachers addressed language diversity more in 1-1 interactions. This is likely due to a limited pedagogical repertoire for inclusive differentiation in whole-class settings and a lack of awareness on facilitating meaningful mathematical communication between peers. Viewing language diversity from a deficit perspective leads to individualized interventions rather than using language as a resource in small group and whole-class discussions.

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The relevance of language to a multilingual mathematics teacher's identity

Danai Dafnopoulou

Linnaeus University, Faculty of Technology, Växjö, Sweden; danai.dafnopoulou@lnu.com

Keywords: Multilingual mathematics teachers, teacher's identity, language diversity.

Research has reported that multilingual mathematics teachers lack the utilisation of teaching practices for embracing multilingualism (Ní Ríordáin, 2018). Even though research on teachers focused more on teachers' strategies when teaching in multilingual classrooms, the theoretical lens of identity has become a prominent approach when highlighting the relationship between social and individual aspects of teaching (Sfard, 2019). Identity, as a lens on language diversity matters, has mostly been used to explore students' mathematical identity (e.g. Schüler-Meyer, 2017), while according to Delacour (2020), multilingual mathematics teachers' identity is under-researched. In this regard, this poster aims to illustrate the multiplicity of experiences of multilingual mathematics teachers concerning teaching in multilingual mathematics classrooms in primary schools in Sweden. A multilingual mathematics teacher is used as an example to illustrate such experiences.

Theoretical background

Following the theoretical framing of Patterns of Participation (Skott, 2018), teachers' experiences are interpreted through the theoretical construction of multiple patterns of participation. A pattern of participation is a presentation and description of the significant practices and figured worlds related to an interaction. The teacher draws from different practices and figured worlds to interpret students' actions while contributing to their interaction (Skott, 2018). Social practices are a social phenomenon created as the individual teacher participates in an endeavour while interpreting the situation in multiple ways (Skott, 2018). Figured worlds are “socially and culturally constructed realms of interpretation in which specific actors and characters are recognised, and certain acts and particular outcomes are seen as more valuable than others” (Holland et al., 1998, p. 52). The distinction between the concepts is that the latter may exist in the school context for the teacher. However, the teacher does not renegotiate its broader meaning outside school (Skott, 2018), like in educational or research contexts. Therefore, a multilingual teacher's identity is viewed as shifting experiences of being, becoming and belonging in classroom interactions with multilingual students.

Research method

The teacher anonymised as Neda was followed over two academic years. She was a multilingual primary teacher⁹ and a leader teacher in mathematics in a multilingual primary school in Sweden. Neda was a teacher of 20 multilingual students, most of whom spoke Swedish, the official language of instruction. Eight more languages were part of the students' linguistic repertoires, and students used two languages during lessons. 25 observations of lessons of the same multilingual class were conducted, constituting background information for interviews. Primary data sources consisted of

⁹ Neda speaks Persian, Swedish and English. Neda used Swedish and occasionally English during lessons.

stimuli-recall interviews where Neda reflected on instances from the classroom. Those reflections allowed insights into the practices and figured worlds that contributed to interactions with students from her perspective. Moreover, semi-structured interviews related to her background and reflections on teachers' meetings or discussions with colleagues observed in the school added to the relevant practices and figured worlds. An inductive approach to data analysis was followed, focusing on different actors, objects, actions, and valued actions or outcomes.

Findings

This poster presented an example of a pattern of participation related to Neda's one-to-one interaction with a multilingual student working on a task on perimeter, units of area, and the concept of square. Findings revealed that language and language diversity matter and are relevant to how Neda re-engaged in interactions with multilingual students. Particularly, when Neda interacted with multilingual students, she did not view them as unable to work with mathematics; instead, she drew on multilingualism as a language barrier that limited communication between them because students did not have a Swedish language repertoire. She was responsible for overcoming those barriers to support students' understanding of mathematics content. She drew from and re-engaged with different professional development programs related to translanguaging and Singapore mathematics, which view language according to multimodality and the use of different representations, such as manipulatives. She needed to pick and adjust those representations accordingly to allow students to access the precise meaning of the mathematical concept.

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Using Maltese for the teaching and learning of equivalent fractions: an introductory outline of a design-based research study

Marie Therese Farrugia

University of Malta, Malta; marie.t.farrugia@um.edu.mt

Keywords: Design-based research, bilingual classrooms, equivalent fractions.

Introduction

Design-based research frequently has a two-pronged aim namely, focusing on both a specific mathematical topic and a wider instructional issue (Prediger et al., 2015). In my study, I developed language responsive lessons for the curricular topic *Equivalent Fractions* for a class of Grade 6 students (aged 10-11 years). According to Van de Walle et al. (2020), equivalence, and in particular, fraction equivalence, is often poorly understood by students. The wider focus of my research was to explore how the Maltese language can support students' meaning-making. The research can be described as dual design-based research (Gravemeijer & van Eerde, 2009), since I considered the learning of both the teacher and the students.

Context

In Malta, a former British colony, school mathematics is often accessed bilingually. The home language of the teacher and students is often Maltese, while English is the language of written mathematics (including imported textbooks, on-screen texts, students' note-taking and examinations). This situation frequently prompts an interweaving of both languages during verbal interaction. With respect to the mathematical terminology in Maltese, teachers and students draw on everyday mathematical terminology, or they might use idiosyncratic expressions. More commonly, they simply retain the mathematical term in English. A bilingual glossary, (Farrugia et al., 2024), is intended to provide teachers and learners with Maltese versions of mathematics terminology, should these be required as part of the educational process, for both verbal and written texts. For equivalent fractions, examples of key English expressions are: the names of fractions (*one third*, etc.), *equivalent fractions*, *same part of whole*, *multiply*, *divide*, *numerator*, *denominator*, *simplest form*. However, everyday words may also play an important role in the articulation of meaning (Prediger, 2019).

Principles of design-based research

My study aligns with Cobb et al.'s (2003) key design principles as follows:

- Generation of theory: My aim was to explore how the Maltese language can be used with purpose, verbally and in writing, to support meaning making.
- Highly interventionist: I designed a series of lessons that were delivered by the class teacher. The lessons formed part of the usual school curriculum.
- Prospective and reflective: Reflective discussions were held with the teacher *prior* to my drawing up the lessons, on a daily basis as the lessons were being delivered and at the end of the series of lessons.

- Iterative: Six consecutive lessons on the same curricular topic were given, with researcher-teacher discussion taking place after each lesson; the approach as a whole was later repeated with another teacher, grade and topic.
- Domain specific, yet with potential to inform prospective practice: The lesson design and reflections are specific to a particular teacher, class and topic. However, the exploration of the implementation and analysis of the realisation of the study (Stephan, 2015), will inform prospective practice on language responsive classrooms in Malta.

Purpose of the conference poster

The purpose of the poster was to provide a brief outline of the study. In my display, I outlined the key features of the design-based research with regard to the use of Maltese. I gave annotated illustrations of how oral Maltese was promoted for paired discussions, and how written Maltese was integrated into word problems. In my presentation, I highlighted that the researcher-teacher collaboration proved to be an interesting and reflective experience for both. I also reported two notable realisations that transpired from the study, namely, that the students took the use of written Maltese for mathematics very much in their stride and that the teacher *herself* also found the use of written Maltese helpful.

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Language issues in refugee mathematics education in Iceland

Ingólfur Gíslason and Freyja Hreinsdóttir

University of Iceland; ingolfug@hi.is

Keywords: Refugees, language hyperdiversity, translanguaging.

The mathematical education of refugee students

Refugees are highly heterogeneous in terms of language, educational background, and legal status. They have insecurity about their futures, a ruptured or impoverished schooling in their past, and traumatic experiences of hostility, violence or poverty. Refugees also bring with them their diverse epistemologies, cultures and criticality. Thus, teaching them presents a complex challenge. This research aims to understand language issues perceived by those who teach and make decisions about teaching mathematics to asylum seekers (waiting to receive a decision on their asylum claim). A common misconception is that mathematics transcends cultural and linguistic boundaries (Khilji & Xenofontos, 2023). Limited proficiency in the school language is often associated with limited mathematical abilities, leading teachers to use simplified mathematical instruction and tasks for refugees, risking limiting opportunities for mathematics learning (de Araujo, 2017). Refugee students may perform significantly better on mathematics assessments in their first language (L1) than in the school language (Attar et al., 2022). Teachers often use visual supports and explain vocabulary, while more linguistically oriented supports are usually lacking (Alisaari et al., 2024). In schools, *monolingualism* (the use of only the dominant language of society or school in education) is common; code-switching between L1 and their second language (L2) is often seen as a sign of linguistic and cognitive deficiency (García & Wei, 2014). In contrast, *translanguaging* (2021) “is a practice aimed at gaining communication and understanding, but it more widely refers to the fluid use of language as a resource by participants as they make sense of their worlds, identities, and cultures of mathematics” (Planas & Chronaki, 2021). Different linguistic decisions about educating refugees are being made (and reconsidered) worldwide, and more research is needed on their underpinning arguments and effects. This research aims to uncover and analyse these decisions and their underpinnings in Iceland.

Research methods

We have conducted one interview with a senior person involved with teaching and management in a school division for students seeking asylum. The interview was semi-structured, emphasising the construction of a preliminary picture of the decisions made about language issues in the mathematics education of asylum seekers in Iceland, including specific teaching methods and policies. Another aim of the interview was to create contact with the practitioners, making it possible to recruit additional participants and gain access to the classrooms as the research progresses.

Findings

Teaching asylum seekers in Iceland is confined to one site, *Birta*, located within a regular school. Birta provides Icelandic, mathematics, English and life skills classes. There are 40 students in this division who stay until they are either granted protection or deported, but only up to one school year,

after which they enter the general school system. Many students have had little schooling; some cannot read in any language and are unfamiliar with number symbols. All students' academic competencies are assessed upon arrival in an interview with an interpreter. According to our informant, the curriculum is adapted to accommodate students' varying competencies, emphasising "learning mathematics through learning to speak Icelandic and vice versa." Teachers speak only in Icelandic, while students use all their linguistic resources, including computer apps. The reasons for the teacher only to talk in Icelandic are that this "puts all students on equal grounding", it has proved unproductive for teachers to code-switch, and it is considered essential for students to learn Icelandic. Group work is arranged to mix students from different linguistic backgrounds, while board games and discussions offer opportunities to practice counting, arithmetic, and other mathematical concepts.

Conclusion

Insisting that the teacher only speaks Icelandic might seem to fall under the ideology of monolingualism. However, the idea of "learning mathematics through learning to speak Icelandic and vice versa" while discussing the world in terms of numbers and arithmetic (and other mathematical concepts) and students being free to use any linguistic means of communication resonates with translanguaging. Further interviews and classroom data will help refine this theoretical approach, give a deeper understanding of language issues in refugee mathematics education and identify opportunities for improvement or transformation.

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A language-responsive approach to understanding the scalar product: design of a content- and language-integrated teaching-learning arrangement

Olga Lomas

Paderborn University, Germany; olga.lomas@uni-paderborn.de

Keywords: Scalar product, design research, design principles, conceptual understanding.

Introduction

This poster provides insights into the dissertation project *A Language-responsive teaching-learning arrangement for conceptual understanding of the scalar product*. When it comes to teaching and learning the scalar product in German schools, learners often struggle with the interpretation of the scalar product when it is not equal to zero. This is because the scalar product is seen as a tool in analytic geometry, with predominant focus on applying the algebraic formula $\vec{a} * \vec{b} = a_1b_1 + a_2b_2$ for two vectors $\vec{a}, \vec{b} \in \mathbb{R}^2$, and the relationship between the scalar product and orthogonality of two vectors (Frohn, 2020, p. 35). Thus, there is a gap concerning the interpretation of the scalar product in relation to different types of angles, the equality of different scalar products and the geometric interpretation as an area of a rectangle.

Furthermore, the integration of different representations is necessary, as learning and teaching the scalar product in school activate previously learned subject-specific language means, e. g., language means of elementary geometry such as angle relationships and orthogonality of lines, but also new subject-specific language structures need to be established. In this context, language and other representations play an important role. Empirical studies on language in mathematics education in Germany predominantly focus mainly on topics in primary and lower secondary education, with some exceptions also for higher secondary education (Armbrust, 2022; Prediger & Sahin-Gür, 2020).

Design of the learning environment

For the reasons mentioned above, there is a need for a content- and language-integrated teaching design on the scalar product that combines conceptual understanding with subject- and language-integrated support for learners. The corresponding research questions is: *What should a language-responsive learning environment for conceptual understanding of the scalar product look like, and which design principles should be implemented?*

The dissertation project is embedded in the research paradigm of *Didactical Design Research* (Prediger & Zwetschler, 2013) and the poster displays two activities from the intended learning path. The first activity focuses on the impact of angles and lengths on the scalar product (Figure 1). Building on this, the second activity aims to consolidate the newly acquired knowledge by having the learners reinforce the relationships between the scalar product, angles, and lengths. In this context, the poster demonstrates how the design principles *relating different representations* (Wessel, 2015) and *engaging learners in rich discourse practices (especially describing and explaining)* (Erath, 2017) were considered during the conception of the learning environment.

What does an unequal to zero scalar product mean geometrically?

Examine how the scalar product $\vec{a} * \vec{b}$ changes:

- a) Use the sliders and describe, what changes for the arrow of \vec{b} .
- b) Describe, what happens to the scalar product when you move the arrow of \vec{b} along...
 - i. ...the ray.
 - ii. ...the cycle.
- c) Explain on which quantities the scalar product $\vec{a} * \vec{b}$ depends.

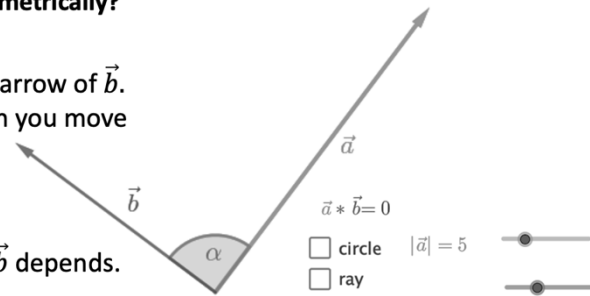


Figure 1: Activity from the teaching-learning environment

The design experiments have been conducted in a laboratory setting with German high school students in March and April 2024. In the next step, the video graphed data will be transcribed and analyzed using an inductive-deductive approach to develop local theories on typical learning pathways of high school students and on the effects of the design elements on supporting conceptual understanding and language production.

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Digital videos as an occasion for interaction?

Larissa Millich and Charlotte Rechtsteiner

PH Ludwigsburg, University of Education, Institute of Mathematics and Computer Science,
Ludwigsburg, Germany; larissa.millich@prom.ph-ludwigsburg.de; rechtsteiner@ph-ludwigsburg.de

Keywords: Social interaction, flexible mental calculation, digital videos.

Theoretical Framework

Researchers widely agree on the importance of social interaction for the generation of new knowledge (e.g., Boaler, 2002). Similarly, interaction between children during mathematics lessons is essential for developing their flexible mental calculation skills (Rathgeb-Schnierer, 2010). When children articulate their cognitive processes and actively exchange diverse viewpoints and problem-solving strategies, it facilitates the development of flexible mental calculation skills (ibid.).

In mathematics education research, it is recognized that social interaction alone does not generate new knowledge. Fruitful interaction in class relies on dedicated learning settings and the structure/timing of interactions, in addition to independent in-depth work (e.g., Boaler, 2002). Previous research focused on the creation of digital videos by primary school students, wherein they elucidated concepts, solutions, skills, or personal discovery processes. This approach is believed to offer significant potential for fostering interaction among students (e.g., Huhmann & Rechtsteiner, 2022).

In light of these findings, it is important to explore further learning opportunities that combine the use of digital videos and the development of flexible mental calculation skills in the context of interactions related to mathematics.

Overview of the Project

Previous research focused on the creation of digital videos by children and its benefits for learning processes. However, the potential of subsequent in-depth work with these videos has not yet been explored. In this study, digital videos created by children provide an opportunity for exchange and further engagement. The research question is twofold, aiming to provide insights into two dimensions:

The developmental level: What theoretical and empirical teaching-learning arrangements can be designed to stimulate flexible mental calculation skills?

The research level: Which learning opportunities regarding flexible mental calculation skills arise when a group interacts with a digital video created by other learners?

Design

To answer both dimensions of the research question, a didactical designed-based research approach was chosen, based on the FUNKEN model (Prediger et al., 2015). This approach involves iterative cycles of intervention and reflection during an experiment and/or between different experiments, which is typical for design-based research.

Procedure and data analyses

In July 2023, the initial research cycle was conducted at an elementary school in Baden-Württemberg, Germany. Five groups, each comprising four to five students, participated in learning activities focused on "Kombi-Gleichungen." After these learning activities the children watched videos produced by other children and were asked to discuss the content based on questions. Their interactions were recorded using video.

The collected data will be analysed using a dual-method approach which incorporates both interactionistic and epistemological perspectives. The analysis of the data from the first research cycle will be conducted from the interactionist perspective. The data will be examined in terms of how the students are involved in their interactive production (Krummheuer, 2007). Furthermore, we will examine how arguments are structured in the course of interaction (ibid.). Based on the results, adjustments will be made to the learning environment for the subsequent cycle.

Outlook

As we are currently analysing the data from the first cycle, at this point only preliminary observations can be made, providing the basis for the questions in the next phase of the research.

The learners appear to predominantly explain their actions but rarely justify or prove their statements. A possible explanation for this is that the instruction before the social interaction was too short and/or not intense enough. Additionally, although mathematical findings are formulated by individual learners, they are often not discussed by the group. Hence, interaction should be fostered instead of monologues. Furthermore, some children behave passively in the mathematical discussions and distract other children with their behaviour. For the second research cycle, it is therefore necessary to encourage more in-depth mathematical exchange among the children to render their participation more beneficial.

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Language challenges in learning mathematics for Ukrainian refugee students in UK schools

Volodymyr Proshkin

Loughborough University, Department of Mathematics Education, Loughborough, UK;

v.proshkin@lboro.ac.uk

Keywords: Language challenges, mathematics education, refugee students.

Introduction

Since 2022, a significant number of Ukrainian children have sought refuge in the UK due to the ongoing war in Ukraine. It is known that around 40,000 Ukrainian children were studying in UK schools in December 2023. Many schools have experience serving refugee students, but many teachers do not. I seek to answer the overarching research question, What challenges do Ukrainian students face while learning mathematics in a UK school?

Literature

Prentice (2022) provided examples of positive student support practices in schools in the UK, including supporting English language learning. Madziva and Thondhlana (2017) developed a model to conceptualise the quality of education for refugees in the UK based on the crucial role of English language proficiency in its implementation. Many previous studies have reported students' difficulties in learning mathematics due to insufficient understanding of the language of instruction (Ingram et al., 2024; etc.). However, identifying language challenges in mathematics learning among Ukrainian students in UK schools has not yet been the subject of scientific research.

Main results

From November 2023 to February 2024, I conducted interviews with 23 students and 18 parents, and surveys of 21 mathematics teachers. I followed a 6-step thematic analysis process: 1) familiarization, 2) coding, 3) generating themes, 4) reviewing themes, 5) defining and naming themes, and 6) reporting (Braun & Clarke, 2006). Out of the total number of students interviewed, 18 stated that they speak Ukrainian at home, while the remaining students communicate in Russian. According to feedback from students, parents, and teachers, the English language is the biggest obstacle in learning mathematics. All 23 students reported that they have or had difficulties with language while studying mathematics. They highlighted challenges in understanding mathematical terminologies, conditions of tasks, and teachers' questions. Students also mentioned that they could only partially discuss solutions with their peers or teachers. Some students refrained from speaking up due to feeling embarrassed or unintelligent, while others found it easier to write solutions on the board instead of explaining them verbally due to language barriers.

According to a survey conducted among mathematics teachers and parents, the problem of language has been identified as a major obstacle. 18 teachers and 16 parents pointed out that language has been a barrier to learning for students. For instance, one teacher reported that students often complete the mathematics but do not understand what the question is asking them to do because of the language barrier. Another teacher mentioned that some students arrive with little English, making it difficult

for them to communicate or make friends. Similarly, a parent said that their child is often quiet in class due to the fear of making mistakes in English.

Ukrainian students faced language barriers in mathematics during the initial months of learning in the UK. To address this, teachers employed various techniques, which were reported by the students and their parents. Some students were seated near the front of the classroom or with other Ukrainian students better at speaking English. However, this led to some students communicating with each other in their native language during class, which did not help to improve their understanding of English. Teachers explained the mathematical concepts in more detail and at a slower pace to ensure that the mathematical concepts were understood. Teaching assistants were also brought in to provide additional support. Students were allowed to use online translators and English-Ukrainian dictionaries, and tasks were translated into Ukrainian. Teachers' presentations were also made with Ukrainian translations. Moreover, students were encouraged to use alternative online resources. It is important to note that approximately half of the students reported that their teachers did not seem to take any particular action to mitigate language barriers.

Discussion and Conclusion

The language barriers gradually decrease as students integrate into local schools, which typically takes one to one and a half years. Many students continue to face language challenges even after this period. Some students provided feedback, stating that they still struggle to explain themselves freely due to the lack of language proficiency. Others mentioned that they tend to forget certain mathematical terms while speaking and still pronounce some words incorrectly. The issue is unique because most Ukrainian students study online at a Ukrainian school, which interferes with their studies in the UK. In further research, it is important to investigate which teaching strategies can effectively minimise the language barriers experienced by Ukrainian refugee students in mathematics.

Acknowledgement

Volodymyr Proshkin wishes to thank the INI and LMS for the financial support, Dr Colin Foster and the Department of Mathematics Education at Loughborough University for their hospitality.

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Interactive classroom activities with connected classroom technology in mathematics lessons

Johanna Zöchbauer, Markus Hohenwarter and Zsolt Lavicza

Johannes Kepler University Linz, Austria; johanna.zoechbauer@partner.jku.at

Keywords: Classroom interaction, connected classroom technology, educational resources, formative assessment.

Introduction

The power of connected classroom technology (CCT) or a classroom response system (CRS) is that it enables teachers not only to present tasks, but also to collect student responses (identifiable or anonymous) in real time and to display this data on a screen (Gustafsson, 2024; Wright et al., 2018). This can then form the basis for whole-class discussions (Gustafsson, 2024). According to Gustafsson (2024), however, integrating a CRS into mathematics instruction is not sufficient to guarantee success.

Our research focuses on a specific CCT: GeoGebra Classroom (GGC) – a virtual platform on the GeoGebra website. Initially, we collected ideas to develop this CCT to better meet the requirements of teachers. One aim that emerged was to give students a voice and to use their responses in various follow-up activities, such as whole-class discussions (Zöchbauer & Hohenwarter, 2020). In their summary of literature reviews of case studies, Quabeck et al. (2023) also identified “space to talk” as a main quality domain. This means that students need to talk more in mathematics lessons to enhance their mathematical learning opportunities (Quabeck et al., 2023).

Within a design science research process (Peppers et al., 2007), we are now conducting a Delphi study that seeks to elaborate how and to what extent GGC is utilized by teachers and in what ways it offers them opportunities to enhance their teaching practice. In the first step of the Delphi study, we developed categories and key points for a questionnaire based on expert interviews and literature findings. A central concern that emerged was classroom interaction.

Method

For the interviews, we chose 16 experts based on their experience with GGC and usage of it in the classroom at school, university, or in teacher education. To obtain broad international insights, we chose experts from various countries in Europe, America, and Asia. We conducted exploratory semi-structured interviews. The interviews were transcribed verbatim, and analysis of the qualitative data was based on qualitative content analysis (Mayring, 2015). Our overall research question was “How and to what extent can GGC support teachers and offer them new opportunities to enhance their teaching practice?” Hence, we asked interviewees to identify typical lesson scenarios that used GGC, including what teachers particularly like about it and how – if at all – student responses were used.

Findings

Our data analysis showed that GGC was used to teach a variety of topics in settings from elementary school up to university and teacher education. The examples of teaching scenarios reported mentioned

some of the innovative CCT features summarized by Wright et al. (2018), including monitoring students' processes, displaying student results, and assessing the learning progress of a whole class. These features make CCTs effective tools for developing formative assessment, provide students with immediate feedback and support, and encourage them to monitor and reflect on their progress (Wright et al., 2018). They also create opportunities for independent and collaborative learning, foster whole-class discussions and enable students to contribute to activities (Wright et al., 2018). Further, we found that GGC was used to underpin various methods, such as flipped classroom, think-pair-share, and games. Students used GGC to explore or rediscover concepts and complete exercises. Teachers also used it in lessons to collect students' work, to carry out research, and in remote teaching. One expert used it to conduct icebreaker activities in virtual learning settings, thus increasing social interaction.

Some of the teaching moves, scenarios, and students' activities we identified could not be assigned to one category only, but fitted into several, depending on what was considered. Some categories fell under a super-category called classroom interaction, which includes whole-class discussions, learning to give feedback, comparing strategies, group activities, and mathematical reasoning. We found that GGC was utilized by teachers in various ways and in a wide range of settings. In general, by using GGC, teachers were shifting away from a teacher-centered classroom to a student-centered classroom. GGC thus has great potential to activate students and provides ample scope for further investigation. Our poster will present the findings from the data analysis illustrated by examples reported by the interviewees.

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