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# WEIGHTED CSCK METRICS ON KÄHLER VARIETIES

by

Chung-Ming Pan & Tat Dat Tô

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*Abstract.* — We study the weighted constant scalar curvature Kähler equations on mildly singular Kähler varieties. Under an assumption on the existence of certain resolution of singularities, we prove the existence of singular weighted constant scalar curvature Kähler metrics when the weighted Mabuchi functional is coercive. This extends the work of Chen and Cheng to the singular weighted setting.

## Contents

Introduction.....	1
1. Preliminaries.....	4
2. A priori estimates.....	10
3. Existence of singular wscK metrics.....	15
Appendix A. Weighted Aubin–Yau inequality.....	22
Appendix B. Weighted local Chen–Cheng’s $\mathcal{C}^2$ -estimate.....	24
References.....	28

## Introduction

The constant scalar curvature Kähler (cscK) metric problem has become one of the central focus in Kähler geometry during the last decades. The Yau–Tian–Donaldson conjecture asserts that given a compact Kähler manifold with a fixed Kähler class, the existence of cscK metrics in the Kähler class is equivalent to an algebro-geometric notion called “K-stability”.

Several progresses in the literature [DR17a, BDL20, CC21a, CC21b] have shown that the existence of a unique cscK metric in a Kähler class is equivalent to the coercivity of the Mabuchi functional, whose minimizers are cscK metrics. When the Kähler class  $\alpha = c_1(L)$  for some ample line bundle  $L$ , Boucksom–Hisamoto–Jonsson [BHJ19] demonstrated that the coercivity of Mabuchi functional implies uniform K-stability (see [DR17b, Der18, SD18, SD20] for transcendental setup). Conversely, C. Li [Li22b] (and the recent transcendental version by Mesquita-Piccione [MP24]) showed that the strong uniform K-stability implies the coercivity of Mabuchi functional. The remaining challenge in proving the uniform Yau–Tian–Donaldson conjecture lies in establishing strong uniform K-stability from uniform K-stability.

Typical examples of cscK metrics are Kähler–Einstein metrics. Motivated by Minimal Model Program and moduli theory, Kähler–Einstein metrics have been well studied on smooth and mildly singular Kähler varieties [Aub78, Yau78, EGZ09, CDS15, BBE<sup>+</sup>19, BBJ21, LTW21, LTW22, Li22a] and their families [Koi83, RZ11a, RZ11b, SSY16, LWX19, DGG23, PT24] etc.

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However, there are very few results regarding cscK metrics in the singular setting. We shall focus on the analytic part of the Yau–Tian–Donaldson conjecture on mildly singular varieties, particularly the equivalence between the existence of singular cscK metrics and the coercivity of the Mabuchi functional. In a recent joint work with Trusiani [PTT23], when the Mabuchi functional is coercive, we establish the existence of singular cscK metrics on  $\mathbb{Q}$ -Gorenstein smoothable Kähler varieties with log terminal singularities. One of the key ingredients is the stability of the coercivity of the Mabuchi functional [PTT23, Thm. A]. A similar strategy for establishing openness of coercivity has been applied to the resolution setting by Boucksom–Jonsson–Trusiani [BJT24] under an appropriate condition on the resolution.

This article aims to remove the additional smoothable assumption and to investigate existence results in a more general “weighted” setting. The weighted framework introduced by Lahdili [Lah19] contains various notions of canonical Kähler metrics.

In the sequel, we shall consider  $X$  to be an  $n$ -dimensional compact Kähler variety with log terminal singularities,  $T \subset \text{Aut}_{\text{red}}(X)$  a maximal real torus in the reduced automorphism group, and a  $T$ -invariant Kähler metric  $\omega$  on  $X$ . Before presenting our results, we introduce the following additional condition:

**Condition (A).** — *There exists a  $T$ -equivariant resolution of singularities  $\pi : Y \rightarrow X$  such that  $Y$  is Kähler and  $\pi$  is an isomorphism over  $\pi^{-1}(X^{\text{reg}})$ . Also, there exist a Kähler metric  $\omega_Y$  on  $Y$ , a positive constant  $K_1 > 0$ , and a function  $\rho_1 \in \text{QPSH}(Y)$  such that*

$$\text{Ric}(\omega_Y) \geq -K_1(\pi^*\omega + dd^c\rho_1) \quad \text{and} \quad \int_Y e^{-K_1\rho_1}\omega_Y^n < +\infty.$$

In the above notation,  $\text{QPSH}(Y)$  is the set of all quasi-plurisubharmonic functions on  $Y$ . A resolution of singularities described in Condition (A) is referred to as a *resolution of Fano type* in [BJT24] (see [BJT24, Sec. 4.1] for further discussions and examples).

Under Condition (A), we establish the following existence theorem:

**Theorem A.** — *Let  $(X, \omega)$  be a compact Kähler variety with log terminal singularities and satisfy Condition (A). If the Mabuchi functional  $\mathbf{M}_\omega$  on  $X$  is  $T_{\mathbb{C}}$ -coercive, then  $X$  admits a singular cscK metric in  $\{\omega\}$  which also minimizes  $\mathbf{M}_\omega$ , where  $T_{\mathbb{C}}$  is the complexification of  $T$ .*

*In particular, if  $\text{Aut}(X)$  is discrete, the coercivity of Mabuchi functional implies the existence of singular cscK metrics in  $\{\omega\}$ .*

The main idea for proving Theorem A is to establish uniform a priori estimates for the solutions to a family of cscK equations on the resolution of singularities. This result generalizes the work of Chen and Cheng [CC21a] to the degenerate setting. Moreover, our method can be extended to a weighted setup, which will be discussed later.

We briefly explain the scheme of proving Theorem A here when  $\text{Aut}(X)$  is discrete. Consider a family of Kähler metrics  $\omega_\varepsilon := \pi^*\omega + \varepsilon\omega_Y$  on  $Y$  for  $\varepsilon \in (0, 1]$  and denote by  $E$  the exceptional set of  $\pi$ . Under Condition (A), [BJT24, Thm. A] have shown the openness of uniform coercivity on  $Y$ ; namely, if  $\mathbf{M}_\omega$  is coercive on  $X$ , then  $(\mathbf{M}_{\omega_\varepsilon})_\varepsilon$  are uniformly coercive for all sufficiently small  $\varepsilon > 0$ . By [CC21b, Thm. 1.2], there exists a smooth pair  $(\varphi_\varepsilon, F_\varepsilon)$  solving the following system of equations:

$$\begin{cases} (\omega_\varepsilon + dd^c\varphi_\varepsilon)^n = e^{F_\varepsilon}\omega_Y^n, & \sup_Y \varphi_\varepsilon = 0, \\ \Delta_{\omega_\varepsilon, \varphi_\varepsilon} F_\varepsilon = -\bar{s}_{\omega_\varepsilon} + \text{tr}_{\omega_\varepsilon, \varphi_\varepsilon}(\text{Ric}(\omega_Y)), \end{cases} \quad (\text{cscK}_\varepsilon)$$

where  $\omega_{\varepsilon, \varphi_\varepsilon} := \omega_\varepsilon + dd^c\varphi_\varepsilon$  and  $\bar{s}_{\omega_\varepsilon} := \frac{nc_1(Y) \cdot [\omega_\varepsilon]^{n-1}}{[\omega_\varepsilon]^n}$ . Then under Condition (A), the uniform coercivity of  $\mathbf{M}_{\omega_\varepsilon}$  yields a uniform control on the entropy  $\mathbf{H}_\varepsilon(\varphi_\varepsilon)$ . We establish uniform a priori estimates based on the entropy bound (see Theorem 2.4 for a general version):

**Theorem B.** — *Let  $(\varphi_\varepsilon, F_\varepsilon)$  be the solution to  $(\text{cscK}_\varepsilon)$ . Assume that Condition (A) holds and for all  $\varepsilon > 0$  sufficiently small,*

$$\mathbf{H}_\varepsilon(\varphi_\varepsilon) := \int_Y \log \left( \frac{\omega_{\varepsilon, \varphi_\varepsilon}^n}{\omega_Y^n} \right) \omega_{\varepsilon, \varphi_\varepsilon}^n \leq C$$

for some constant  $C > 0$  independent of  $\varepsilon$ . Then there exists a uniform constant  $C_0 > 0$  such that

$$\|\varphi_\varepsilon\|_{L^\infty} \leq C_0, \quad F_\varepsilon \leq C_0 \quad \text{and} \quad F_\varepsilon \geq K_2\rho - C_0,$$

where  $\rho \in \text{QPSH}(Y) \cap C^\infty(Y \setminus \text{Exc}(\pi))$  such that  $\{\rho = -\infty\} = \text{Exc}(\pi)$ , and  $\pi^*\omega + dd^c\rho$  is a Kähler current with  $\text{Ric}(\omega_Y) \leq K_2(\omega + dd^c\rho)$  for some  $K_2 > 0$ . Moreover, for any compact set  $K \subset Y \setminus E$  and  $\ell \in \mathbb{N}$ , we have  $\|\varphi_\varepsilon\|_{C^\ell(K, \omega_Y)} \leq C_{K, \ell}$  for some uniform constant  $C_{K, \ell} > 0$ .

As a result, one can extract a subsequence  $\varphi_\varepsilon$  converging in  $C_{\text{loc}}^\infty(Y \setminus E)$  to  $\varphi_0 \in \text{PSH}(Y, \pi^*\omega) \cap L^\infty(Y)$  which is smooth on  $Y \setminus E$  and solves the cscK equation there.

We make a more detailed comment on our proof here. For the  $L^\infty$ -estimate, we follow the method of Guo and Phong [GP22] utilizing auxiliary complex Monge–Ampère equations (see also [GPT23]). We stress that Condition (A) is essentially used for the  $L^\infty$ -estimate in this approach. Then we provide an integral estimate for Laplacian adapting [CC21a, Thm 3.1] and [DD22, Thm. 4.1] and higher order estimates in the regular part following from Chen–Cheng’s local estimates [CC21a, Prop. 6.1, arXiv version].

In the singular Kähler–Einstein setting, it is well-established that the coercivity of the Mabuchi functional is equivalent to the existence of a unique singular Kähler–Einstein metric (cf. [Dar17, DG18a, BBE<sup>+</sup>19]). Theorem A and Theorem B in particular show that singular Kähler–Einstein metrics on a Kähler varieties with discrete automorphism group can be approximated by cscK metrics, provided Condition (A) holds. This recovers a recent theorem by Székelyhidi [Szé24, Thm. 3].

In Section 3.3, we present a method for constructing singular cscK metrics inspired by the result of Arezzo and Pacard [AP06] under Condition (A). Additionally, we propose a mixed construction that combines our previous result on the smoothable setting [PTT23].

**Weighted setting.** — Our approach above, in fact, applies to dealing with a singular version of weighted constant scalar curvature Kähler (wcscK) metrics. The weighted setting was originally introduced by Lahdili [Lah19] (see also [Ino22]), including various notions of canonical Kähler metrics, for example, extremal metrics and Kähler–Ricci solitons. For further results on the existence of wcscK metrics, we refer to [AJL23, Lah23, DJL24, DJL25, HL24] and the references therein.

We quickly review the basic setup and notations for the weighted cscK metrics here. The reader is referred to Section 1 for more details.

Denote by  $\mathfrak{t}$  the Lie algebra of  $T$ . Since  $\omega$  is  $T$ -invariant, it admits a moment map  $m_\omega : X \rightarrow \mathfrak{t}^\vee$ . Consider functions  $v, w \in C^\infty(\mathfrak{t}^\vee, \mathbb{R})$  with  $v > 0$  on  $m_\omega(X)$ . A metric  $\omega'$  is a  $(v, w)$ -cscK metric if it has bounded potential near  $X^{\text{sing}}$  and it solves the following weighted scalar curvature equation  $S_v(\omega') = w(m_{\omega'})$  on  $X^{\text{reg}}$ , where  $S_v$  is the weighted scalar curvature.

In the smooth setting, Lahdili [Lah19] has proved that wcscK equation is the Euler–Lagrange equation of a  $(v, w)$ -weighted Mabuchi functional  $\mathbf{M}_{v, w}$ . This also works in the singular context (cf. [BJT24, Sec. 4]). With the approach we mentioned, Theorem A can be extended to the weighted situation:

**Theorem C.** — *Let  $(X, \omega)$  be a compact Kähler variety with log terminal singularities that satisfies the above setting. Assume that  $(X, \omega)$  satisfies Condition (A) and  $\log v$  is concave on  $\mathfrak{t}^\vee$ . If the weighted Mabuchi functional  $\mathbf{M}_{v, w}$  is  $T_{\mathbb{C}}$ -coercive, then  $X$  admits a singular  $(v, w)$ -cscK metric in  $\{\omega\}$ , which also minimizes  $\mathbf{M}_{v, w}$ .*

We remark that the “concavity of  $\log v$ ” is used in the  $C^2$ -estimate for extending Theorem B to the weighted setting (cf. Theorem 2.4). This is explained in detail in Appendix A.

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## 1. Preliminaries

In this section, we recall several basic notions of pluripotential theory on singular spaces and weighted cscK metrics. We define  $d^c := \frac{i}{2}(\bar{\partial} - \partial)$  and then we have  $dd^c = i\partial\bar{\partial}$ .

**1.1. Pluripotential theory on normal Kähler varieties.** — Let  $(X, \omega)$  be an  $n$ -dimensional normal compact Kähler variety. By Kähler variety, we mean an irreducible reduced complex analytic space equipped with a Kähler form. A Kähler metric  $\omega$  on  $X$  is locally a restriction of a Kähler metric defined near the image of a local embedding  $j : X \xrightarrow[\text{loc.}]{} \mathbb{C}^N$ . More generally, a smooth form  $\alpha$  on  $X$  is a smooth form on  $X^{\text{reg}}$  such that  $\alpha$  extends smoothly under any local embedding  $X \xrightarrow[\text{loc.}]{} \mathbb{C}^N$ .

*Definition 1.1.* — A function  $\phi : X \rightarrow \mathbb{R} \cup \{-\infty\}$  is  $\omega$ -plurisubharmonic ( $\omega$ -psh for short) if  $\phi + u$  is plurisubharmonic where  $u$  is a local potential of  $\omega$ ; i.e.  $\phi + u$  is the restriction of an psh function defined near an open neighborhood of  $\text{im}(j)$ . Denote by  $\text{PSH}(X, \omega)$  the set of all integrable  $\omega$ -psh functions.

By Bedford–Taylor’s theory [BT82], the complex Monge–Ampère operator can be extended to bounded  $\omega$ -psh functions on smooth complex manifolds. In the singular setting, the complex Monge–Ampère operator of locally bounded psh functions can also be defined by taking zero through singular locus (cf. [Dem85] for more details).

**1.1.1. Finite energy class.** — Set  $V := \int_X \omega^n$ . For all  $\varphi \in \text{PSH}(X, \omega) \cap L^\infty(X)$ , the Monge–Ampère energy is defined by

$$\mathbf{E}(\varphi) := \frac{1}{(n+1)} \sum_{j=0}^n \int_X \varphi \omega_\varphi^j \wedge \omega^{n-j},$$

where  $\omega_\varphi := \omega + dd^c \varphi$ . The energy satisfies  $\mathbf{E}(\varphi + c) = \mathbf{E}(\varphi) + cV$  for all  $c \in \mathbb{R}$  and for  $\varphi, \psi \in \text{PSH}(X, \omega) \cap L^\infty(X)$ , if  $\varphi \leq \psi$  then  $\mathbf{E}(\varphi) \leq \mathbf{E}(\psi)$  with equality iff  $\varphi = \psi$ . By the later property,  $\mathbf{E}$  extends uniquely to  $\text{PSH}(X, \omega)$  by

$$\mathbf{E}(\varphi) := \inf \{ \mathbf{E}(\psi) \mid \varphi \leq \psi \in \text{PSH}(X, \omega) \cap L^\infty(X) \}.$$

The finite energy class is then given as

$$\mathcal{E}^1(X, \omega) = \{ \varphi \in \text{PSH}(X, \omega) \mid \mathbf{E}(\varphi) > -\infty \}.$$

The space of finite energy potentials  $\mathcal{E}^1(X, \omega)$  admits a metric topology induced by the so-called  $d_1$ -distance, which can be expressed as follows (cf. [Dar17, Thm. 2.1], [DG18a, Thm. B])

$$d_1(u, v) = \mathbf{E}(u) + \mathbf{E}(v) - 2\mathbf{E}(P_\omega(u, v)),$$

where  $P_\omega(u, v) := (\sup \{ w \in \text{PSH}(X, \omega) \mid w \leq \min(u, v) \})^*$ . From the definition, one can verify that  $d_1(u, 0) = -\mathbf{E}(u)$  for any  $u \leq 0$ .

**1.2. Weighted cscK metrics.** — In this section, we review some basic concepts related to weighted cscK metrics.

**1.2.1. Automorphism group and holomorphic vector fields.** — We recall here certain well-known properties of the Lie algebra of  $\text{Aut}(X)$  and some of its subgroups (cf. [LS94, Gau20]).

Let  $(X, \omega)$  be a compact Kähler manifold and denote  $J$  to be its complex structure. The automorphism group  $\text{Aut}(X)$  is a complex Lie group, whose Lie algebra  $\mathfrak{h} = \{\xi \in \Gamma(TX) \mid \mathcal{L}_\xi J = 0\}$  consists of real holomorphic vector fields on  $X$ . A vector field  $\xi$  is real holomorphic if and only if  $\zeta := \frac{1}{2}(\xi - iJ\xi)$  is a holomorphic, so  $\mathfrak{h}$  can be identified with  $H^0(X, T_X)$  the space of holomorphic vector fields. Denote by  $\text{Aut}_0(X)$  the identity component of  $\text{Aut}(X)$ .

The Albanese torus is defined by (cf. [Uen75, Thm. 9.7])

$$\text{Alb}(X) = H^0(X, \Omega_X^1)^\vee / H_1(X, \mathbb{Z}),$$

where the inclusion  $H_1(X, \mathbb{Z}) \hookrightarrow H^0(X, \Omega_X^1)^\vee$  is induced via the map  $\alpha \mapsto \int_c \alpha$  for any loop  $c$  and holomorphic 1-form  $\alpha$ . Then one can define a Lie group homomorphism  $\tau : \text{Aut}_0(X) \rightarrow \text{Alb}(X)$  as follows. Fix a  $x_0 \in X$ , for any  $\sigma \in \text{Aut}_0(X)$ , we define  $\tau(\sigma) : \alpha \mapsto \int_{x_0}^{\sigma \cdot x_0} \alpha$ , which does not depend on  $x_0$  as  $\alpha$  is harmonic. The homomorphism  $\tau$  induces an action of  $\text{Aut}_0(X)$  on  $\text{Alb}(X)$  by translation. Moreover, the derivative of  $\tau$  at the identity is  $\tau' : \mathfrak{h} \rightarrow H^0(X, \Omega_X^1)^\vee$ ,  $\xi \mapsto (\tau'(\xi) : \alpha \mapsto \alpha(\xi))$ . Define  $\mathfrak{h}_{\text{red}} := \ker \tau'$ , which consists of all holomorphic vector field  $\xi \in H^0(X, T_X)$  such that  $\alpha(\xi) = 0$  for all  $\alpha \in H^0(X, \Omega_X^1)$ . By [LS94, Thm. 1], we have

$$\mathfrak{h}_{\text{red}} = \{\xi \in H^0(X, T_X) \mid \xi = \nabla^{1,0} f = g^{j\bar{k}} \partial_{\bar{k}} f \frac{\partial}{\partial z^j}, f \in \mathcal{C}^\infty(X, \mathbb{C})\}.$$

Then  $\mathfrak{h}_{\text{red}}$  generates a Lie subgroup  $\text{Aut}_{\text{red}}(X) \subset \text{Aut}_0(X)$  which acts trivially on the Albanese torus. We also identify  $\mathfrak{h}_{\text{red}} \subset \mathfrak{h}$  with the Lie algebra of real holomorphic vector fields with zeros (cf. [LS94, Thm. 1]).

**1.2.2. Weighted setting.** — Fix a real torus  $T \subset \text{Aut}_{\text{red}}(X)$  with Lie algebra  $\mathfrak{t} \subset \mathfrak{h}_{\text{red}}$ . Any closed, real,  $T$ -invariant  $(1, 1)$ -form  $\omega$  admits a moment map  $m_\omega : X \rightarrow \mathfrak{t}^\vee$ , that is a unique (up to additive constant)  $T$ -invariant smooth map such that for each  $\xi \in \mathfrak{t}$ ,  $m_\omega^\xi := \langle m_\omega, \xi \rangle : X \rightarrow \mathbb{R}$  satisfies

$$-dm_\omega^\xi = i_\xi \omega = \omega(\xi, \cdot);$$

in other words,  $m_\omega^\xi$  is a Hamiltonian function of  $\xi$  with respect to  $\omega$ .

Denote by  $P_\omega := \text{im}(m_\omega)$  a compact set in  $\mathfrak{t}^\vee$  and  $\mathcal{H}_\omega^T := \mathcal{H}^T(X, \omega)$  the set of  $T$ -invariant, strictly positive, smooth,  $\omega$ -psh functions. For any  $\phi \in \mathcal{H}_\omega^T$ , we normalize  $m_\phi := m_{\omega_\phi}$  by  $m_\phi^\xi = m_\omega^\xi + d^c \phi(\xi)$  for all  $\xi \in \mathfrak{t}$ . Then from [Lah19, Lem. 1], under this normalization, for any  $\phi \in \mathcal{H}_\omega^T$ , one has  $P_{\omega_\phi} = P_\omega$  and

$$\int_X m_\phi^\xi \omega_\phi^n = \int_X m_\omega^\xi \omega^n, \quad \forall \xi \in \mathfrak{t}. \quad (1.1)$$

Let  $v, w \in \mathcal{C}^\infty(\mathfrak{t}^\vee, \mathbb{R})$  with  $v > 0$  on  $P := P_\omega$ . We recall the following notations and definitions from [BJT24, Sec. 3],

**Definition 1.2.** — Let  $(\theta, m_\theta)$  be a  $T$ -invariant pair and  $f$  be a  $T$  distribution.

(i) A moment map  $m_{dd^c f}$  for the  $T$ -invariant  $(1, 1)$ -current  $dd^c f$  is defined by

$$m_{dd^c f}^\xi := d^c f(\xi) = i_\xi d^c f;$$

(ii) For a smooth function  $g : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ , and another  $T$ -invariant pair  $(\omega, m_\omega)$ ,

$$\langle (g(v))'(m_\omega), m_\theta \rangle := g'(v(m_\omega)) \sum_\alpha v_\alpha(m_\omega) m_\theta^{\xi_\alpha}$$

where  $(\xi_\alpha)_\alpha$  is a basis of  $\mathfrak{t}$ , and  $(v_\alpha)_\alpha$  are the partial derivatives of  $v$  with respect to the dual basis  $(\xi^\alpha)_\alpha$  in  $\mathfrak{t}^\vee$ ;

(iii) The  $v$ -weighted trace is given by

$$\text{tr}_{\omega, v} \theta := \text{tr}_\omega \theta + \langle (\log v)'(m_\omega), m_\theta \rangle;$$

(iv) The  $v$ -weighted Laplacian is defined as

$$\Delta_{\omega, v} f := \text{tr}_{\omega, v}(dd^c f) = \Delta_\omega f + \langle (\log v)'(m_\omega), m_{dd^c f} \rangle;$$

(v) The  $v$ -weighted Ricci curvature is

$$\mathrm{Ric}_v(\omega) := \mathrm{Ric}(v(m_\omega)\omega^n) = -dd^c \log(v(m_\omega)\omega^n) = \mathrm{Ric}(\omega) - dd^c \log v(m_\omega);$$

(vi) The  $v$ -weighted scalar curvature is

$$S_v(\omega) := \mathrm{tr}_{\omega,v} \mathrm{Ric}_v(\omega) = S(\omega) - \frac{\langle v'(m_\omega), \Delta_\omega m_\omega \rangle + \Delta_\omega v(m_\omega)}{v(m_\omega)}.$$

As [BJT24, (3.4)], by applying the interior product  $i_{\bar{\zeta}}$  to the trivial relation  $d^c f \wedge \omega^n = 0$ , one can obtain

$$m_{dd^c f}^{\bar{\zeta}} \omega^n = nd^c f \wedge \omega^{n-1} \wedge i_{\bar{\zeta}} \omega.$$

Combining the above formula with

$$dv(m_\omega) = \sum_{\alpha} v_{\alpha}(m_\omega) dm_{\omega}^{\bar{\zeta}_{\alpha}} = - \sum_{\alpha} v_{\alpha}(m_\omega) i_{\bar{\zeta}_{\alpha}} \omega, \quad \text{and} \quad \langle v'(m_\omega), m_{dd^c f} \rangle := \sum_{\alpha} v_{\alpha}(m_\omega) m_{dd^c f}^{\bar{\zeta}_{\alpha}},$$

one can derive that

$$\langle v'(m_\omega), m_{dd^c f} \rangle = n \frac{dv(m_\omega) \wedge d^c f \wedge \omega^{n-1}}{\omega^n} = \mathrm{tr}_{\omega}(dv(m_\omega) \wedge d^c f).$$

Therefore, we have the following formula for the weighted Laplacian

$$\Delta_{\omega,v} f = \Delta_{\omega} f + \mathrm{tr}_{\omega}(d \log v(m_\omega) \wedge d^c f). \quad (1.2)$$

The  $v$ -weighted Monge–Ampère operator is defined as

$$\mathrm{MA}_v(\phi) := \mathrm{MA}_{\omega,v}(\phi) := v(m_\phi)(\omega + dd^c \phi)^n$$

for any  $\phi \in \mathcal{H}_{\omega}^T$ . From [BJT24, (3.32)], one has the following integration-by-parts formula

$$\int_X g(\Delta_{\omega_\phi,v} f) \mathrm{MA}_v(\phi) = \int_X f(\Delta_{\omega_\phi,v} g) \mathrm{MA}_v(\phi) = - \int_X \langle df, dg \rangle_{\omega_\phi} \mathrm{MA}_v(\phi), \quad (1.3)$$

for all  $T$ -invariant distributions  $f, g$  such that at least one of which is smooth, where

$$\langle df, dg \rangle_{\omega} := \mathrm{tr}_{\omega}(df \wedge d^c g).$$

**1.2.3. Weighted cscK equations.** — Consider the weighted cscK problem

$$S_v(\omega_\phi) = w(m_\phi).$$

The above equation can be rewritten as

$$\begin{cases} v(m_\phi)(\omega + dd^c \phi)^n = e^F \omega^n, & \sup_X \phi = 0, \\ \Delta_{\phi,v} F = -w(m_\phi) + \mathrm{tr}_{\phi,v}(\mathrm{Ric}(\omega)), \end{cases}$$

where  $\mathrm{tr}_{\phi,v} := \mathrm{tr}_{\omega_\phi,v}$  and  $\Delta_{\phi,v} := \Delta_{\omega_\phi,v}$ . For any volume form  $\mu_X$  on  $X$ , taking  $\tilde{F} := F + \log \frac{\omega^n}{\mu_X}$ , the original coupled equations are equivalent to the following coupled equations

$$\begin{cases} v(m_\phi)(\omega + dd^c \phi)^n = e^{\tilde{F}} \mu_X, & \sup_X \phi = 0, \\ \Delta_{\phi,v} \tilde{F} = -w(m_\phi) + \mathrm{tr}_{\phi,v}(\mathrm{Ric}(\mu_X)). \end{cases}$$

Since  $P$  is compact, there are positive constants  $C_v$  and  $C_w$  such that, on  $P$ ,

$$C_v^{-1} \leq v + \sum_{\alpha} |v_{\alpha}| + \sum_{\alpha,\beta} |v_{\alpha\beta}| \leq C_v, \quad \text{and} \quad C_w^{-1} \leq |w| + \sum_{\alpha} |w_{\alpha}| + \sum_{\alpha,\beta} |w_{\alpha\beta}| \leq C_w.$$

This yields that

$$|\mathrm{tr}_{\omega_\phi,v}(\eta) - \mathrm{tr}_{\omega_\phi}(\eta)| = |\langle (\log v)'(m_\phi), m_\eta \rangle| \leq C_{v,\eta},$$

where  $C_{v,\eta}$  depending only on  $C_v$  and  $\eta$ . In particular, for any  $\phi, \psi \in \mathcal{H}^T(X, \omega)$  and  $\eta := \omega_\psi$ ,

$$|\mathrm{tr}_{\omega_\phi,v}(\omega_\psi) - \mathrm{tr}_{\omega_\phi}(\omega_\psi)| = |\langle (\log v)'(m_\phi), m_\psi \rangle| \leq C_v, \quad (1.4)$$

since  $m_\phi(X) = m_\omega(X) = P$  is compact.

**1.2.4. Weighted Mabuchi functional.** — Let  $\mu_X$  be a fixed volume form on  $X$ . The  $v$ -weighted relative entropy is defined by

$$\mathbf{H}_v(\phi) := \mathbf{H}_{v,\mu_X}(\phi) := \int_X \log \left( \frac{\mathbf{MA}_v(\phi)}{\mu_X} \right) \mathbf{MA}_v(\phi).$$

The  $w$ -weighted Monge–Ampère energy  $\mathbf{E}_w := \mathbf{E}_{\omega,w} : \mathcal{H}_\omega^T \rightarrow \mathbb{R}$  is given by

$$(d\mathbf{E}_w)_\phi(f) := \int_X f w(m_\omega) \omega_\phi^n, \quad \mathbf{E}_w(0) = 0$$

for any  $f \in C^\infty(X)^T$  (cf. [Lah19, Lem. 3]).

Fix  $\theta$  a  $T$ -invariant closed  $(1,1)$ -form and  $m_\theta$  is its moment map. Then the twisted weighted Monge–Ampère energy  $\mathbf{E}_v^\theta := \mathbf{E}_{\omega,v}^\theta : \mathcal{H}_\omega^T \rightarrow \mathbb{R}$  is the primitive of  $\left( v(m_\phi) n\theta \wedge \omega_\phi^n + \langle v'(m_\phi), m_\theta \rangle \omega_\phi^n \right)$  with  $\mathbf{E}_{\omega,v}^\theta(0) = 0$ ; in other words,

$$(d\mathbf{E}_{\omega,v}^\theta)_\phi(f) = \int_X f \left( v(m_\phi) n\theta \wedge \omega_\phi^n + \langle v'(m_\phi), m_\theta \rangle \omega_\phi^n \right), \quad \mathbf{E}_{\omega,v}^\theta(0) = 0.$$

for any  $f \in C^\infty(X)^T$  (see [Lah19, Lem. 4] for the well-definedness).

**Definition 1.3.** — The weighted Mabuchi energy  $\mathbf{M}_{v,w}$  is an Euler–Lagrange functional of  $\mathcal{H}_\omega^T \ni \phi \mapsto \mathbf{M}'_{v,w}(\phi) := (w(m_\phi) - S_v(\omega_\phi)) \mathbf{MA}_v(\phi)$ .

From [Lah19, BJT24], we have the following lemma.

**Lemma 1.4.** — The following formula holds on  $\mathcal{H}_\omega^T$

$$\mathbf{M}_{v,w} := \mathbf{M}_{v,w,\mu_X} = \mathbf{H}_{v,\mu_X} + \mathbf{E}_v^{-\text{Ric}(\mu_X)} + \mathbf{E}_{vw}.$$

The above lemma follows from the fact that the functional  $\mathcal{H}_\omega^T \ni \phi \mapsto \mathbf{H}_v + \mathbf{E}_v^{-\text{Ric}(\mu_X)}$  is an Euler–Lagrange functional for  $\phi \mapsto -S_v(\omega_\phi) \mathbf{MA}_v(\phi)$  (cf. [BJT24, Lem. 3.29]).

**Lemma 1.5.** — Any  $(v,w)$ -cscK metric in  $\{\omega\}$  is a minimizer of  $\mathbf{M}_{v,w,\mu_X}$  for any choice of  $\mu_X$ .

*Proof.* — It follows from [Lah23, Cor. 1] that any  $(v,w)$ -cscK metric in  $\{\omega\}$  is a minimizer of  $\mathbf{M}_{v,w,\omega^n}$ . On the other hand, by [BJT24, Lem. 3.48] there is a constant  $C$  such that  $\mathbf{M}_{v,w,\mu_X} = \mathbf{M}_{v,w,\omega^n} + C$ , hence any  $(v,w)$ -cscK metric in  $\{\omega\}$  is also a minimizer of  $\mathbf{M}_{v,w,\mu_X}$ .  $\square$

**1.2.5. Weighted extremal metrics.** — We recall here the definition of weighted extremal metrics and relative weighted Mabuchi energy, which is a modification of Mabuchi energy such that it is  $T_{\mathbb{C}}$ -invariant. We refer to [BJT24, Sec. 3.6] for more details.

**Lemma 1.6** ([BJT24, Lem. 3.34]). — The weighted Mabuchi functional  $\mathbf{M}_{v,w}$  is translation invariant and  $T_{\mathbb{C}}$ -invariant iff for all affine function  $\ell = \zeta + c \in \mathfrak{t} \oplus \mathbb{R}$  on  $\mathfrak{t}^\vee$  we have  $\int_X \ell(m_\omega) \mathbf{M}'_{v,w}(0) = 0$ , i.e.

$$\int_X \ell(m_\omega) w(m_\omega) \mathbf{MA}_v(0) = \int_X \ell(m_\omega) S_v(\omega) \mathbf{MA}_v(0).$$

Now, let  $w_0 \in C^\infty(P, \mathbb{R}_{>0})$ . The weighted Futaki–Mabuchi pairing on the space  $\mathfrak{t} \oplus \mathbb{R}$  of affine functions on  $\mathfrak{t}^\vee$  is defined by

$$\langle \ell, \ell' \rangle := \int_X \ell(m_\omega) \ell'(m_\omega) w_0(m_\omega) \mathbf{MA}_v(0) = \int_X \ell(m_\omega) \ell'(m_\omega) \mathbf{MA}_{vw_0}(0).$$

Then this is positive definite, and there exists the unique affine function  $\ell^{\text{ext}} = \ell_{\omega,v,w_0}^{\text{ext}}$  on  $\mathfrak{t}^\vee$  such that

$$\langle \ell, \ell^{\text{ext}} \rangle = \int_X \ell(m_\omega) S_v(\omega) \mathbf{MA}_v(0). \quad (1.5)$$

Then the weighted Mabuchi energy  $\mathbf{M}_{v,w}$  with  $w = w_0 \ell^{\text{ext}}$  is  $T_{\mathbb{C}}$ -invariant. We define  $\mathbf{M}_{v,w_0}^{\text{rel}} := \mathbf{M}_{v,w_0 \ell^{\text{ext}}}$  is the relative weighted Mabuchi energy.

**Definition 1.7.** — Let  $v, w_0 \in C^\infty(P, \mathbb{R}_{>0})$ . A metric  $\omega$  is called a  $(v, w_0)$ -extremal metric if it is  $(v, w)$ -cscK with  $w = w_0 \ell^{\text{ext}}$ .



**1.2.6. Extremal Kähler metrics.** — We explain here how the problem of finding extremal Kähler metrics is a special case of the one for weighted setting.

Let  $(X, \omega)$  be a compact Kähler manifold, and let  $g$  be the Riemannian metric defined by  $\omega$ . The metric  $\omega$  is said to be extremal if the Hamiltonian vector field  $\xi_\omega = J\nabla S(\omega)$  is a Killing vector field for  $g$ , i.e.  $\mathcal{L}_{\xi_\omega} g = 0$ .

Let  $T$  be a maximal compact torus of  $\text{Aut}_{\text{red}}(X)$  and  $\alpha$  be a Kähler class. From [FM95] (see also [Lah19, Sec. 3.1]), the projection of  $S(\omega)$  with respect to  $L^2(\omega^n)$ -inner product, to the sub-space  $\{m_\omega^\xi + c : \xi \in \mathfrak{t}, c \in \mathbb{R}\}$  is written as  $\Pi_\omega^T(S(\omega)) = m_\omega^{\xi_{\text{ext}}} + c_{\text{ext}}$  for some  $\xi_{\text{ext}} \in \mathfrak{t}$  where  $\xi_{\text{ext}}$  only depends on  $T$  and  $\{\omega\}$ . In particular, it implies that  $c_{\text{ext}}$  also depend only on  $T$  and  $\{\omega\}$ , since

$$nc_1(X) \cdot \{\omega\}^{n-1} = \int_X S(\omega)\omega^n = \int_X \Pi_\omega^T(S(\omega))\omega^n = \int_X m_\omega^{\xi_{\text{ext}}}\omega^n + c_{\text{ext}}\{\omega\}^n,$$

where the last integral only depends  $\{\omega\}$  and  $\xi_{\text{ext}}$  by (1.1). One can also normalize  $\int_X m_\omega^{\xi_{\text{ext}}}\omega^n = 0$  to get  $c_{\text{ext}} = \bar{s} := n \frac{nc_1(X) \cdot \{\omega\}^{n-1}}{\{\omega\}^n}$ . Therefore, we obtain the unique affine function  $\ell^{\text{ext}}(p) = \langle \xi_{\text{ext}}, p \rangle + c_{\text{ext}}$  defined in (1.5) and the problem of finding extremal metric is equivalent to the one for  $(1, \ell^{\text{ext}})$ -cscK metric.

**1.2.7. Extension on  $\mathcal{E}^{1,T}$  and coercivity.** — Denote by  $\mathcal{E}_\omega^{1,T} := \mathcal{E}^1(X, \omega)^T \subset \mathcal{E}^1(X, \omega)$  the space of  $T$ -invariant finite energy potentials and  $\mathcal{E}_{\text{norm}}^{1,T}(X, \omega) := \{u \in \mathcal{E}_\omega^{1,T} \mid \sup_X u = 0\}$ . From [BJT24, Prop. 3.41], one can extend all functionals above on  $\mathcal{E}_\omega^{1,T}$ . On  $\mathcal{E}_{\text{norm}}^{1,T}(X, \omega)$ , we have  $d_1(u, 0) = -\mathbf{E}_\omega(u)$ ; hence,

$$d_T(u, 0) := \inf_{\sigma \in T_C} d_1(\sigma^*u, 0) = \inf_{\sigma \in T_C} -\mathbf{E}_\omega(\sigma^*u) =: -\mathbf{E}_{\omega, T}(u).$$

**Definition 1.8.** — The weighted Mabuchi functional  $\mathbf{M}_{v, w}$  is  $T_C$ -coercive if there exist constants  $A > 0$  and  $B > 0$  such that

$$\mathbf{M}_{v, w}(\phi) \geq Ad_T(\phi, 0) - B$$

for any  $\phi \in \mathcal{E}_{\text{norm}}^{1,T}(X, \omega)$ .

**1.3. Weighted variational formalism in the singular setting.** — We recall here the weighted formalism for the variational problem of singular wscK metrics on Kähler varieties with log terminal singularities as introduced in [BJT24, Sec. 3.8, 4.1].

**1.3.1. Reduced automorphism group and moment maps.** — Let  $(X, \omega)$  be a normal compact Kähler variety. The automorphism group  $\text{Aut}(X)$  is a complex Lie group, whose Lie algebra is  $H^0(X, T_X) \simeq H^0(X^{\text{reg}}, T_{X^{\text{reg}}})$  the space of holomorphic vector fields, that are global section of the tangent sheaf  $T_X := \text{Hom}(\Omega_X^1, \mathcal{O}_X)$ . There exists an  $\text{Aut}_0(X)$ -equivariant resolution of singularities  $\pi : \tilde{X} \rightarrow X$ , i.e.  $\pi$  is an isomorphism over  $X^{\text{reg}}$  and any  $\sigma \in \text{Aut}_0(X)$  can be extended to a unique  $\sigma' \in \text{Aut}_0(\tilde{X})$ . Hence, we get the inclusion  $\text{Aut}_0(X) \subset \text{Aut}_0(\tilde{X})$ . Moreover, any holomorphic vector field on  $\tilde{X}$  descends to an element in  $H^0(X^{\text{reg}}, T_{X^{\text{reg}}})$ , so we get  $\text{Aut}_0(X) \simeq \text{Aut}_0(\tilde{X})$ .

**Definition 1.9.** — We define the *reduced automorphism group*  $\text{Aut}_{\text{red}}(X) \subset \text{Aut}_0(X)$  to be the identity component of subgroup of  $\text{Aut}_0(X) \simeq \text{Aut}_0(\tilde{X})$  acting trivially on the Albanese torus of  $\tilde{X}$ .

This definition is independent of the choice of  $\text{Aut}_0(X)$ -equivariant resolution of singularities  $\pi : \tilde{X} \rightarrow X$ , by the bimeromorphic invariance of Albanese torus (cf. [Uen75, Prop. 9.12]). Since  $\text{Aut}_0(X) \simeq \text{Aut}_0(\tilde{X})$ , we get  $\text{Aut}_{\text{red}}(X) \simeq \text{Aut}_{\text{red}}(\tilde{X})$ .

Denote  $\mathcal{Z}$  (reps.  $\mathcal{Z}'$ ) the space of locally  $dd^c$ -exact real  $(1, 1)$ -form (resp. currents) on  $X$ . In particular, any current  $\theta \in \mathcal{Z}'$  can be written as  $\theta = \omega + dd^c u$  for some  $\omega \in \mathcal{Z}$  and  $u$  is a distribution. Then the group  $\text{Aut}(X)$  acts on  $\mathcal{Z}$  and  $\mathcal{Z}'$ . In particular,  $\text{Aut}_0(X)$  acts trivially on the the classes of  $\mathcal{Z}$  (cf. [BJT24, Lem. 3.54]): for any  $\sigma \in \text{Aut}_0(X)$ , there exist  $f \in C^\infty(X)$  such that  $\sigma^*\omega = \omega + dd^c f$ .

Fix a compact torus  $T \subset \text{Aut}_{\text{red}}(X)$  with Lie algebra  $\mathfrak{t}$ . Any  $T$ -invariant  $(1, 1)$ -form (resp. current)  $\omega \in \mathcal{Z}$  admits a moment map  $m = m_\omega : X \rightarrow \mathfrak{t}^v$  which is a  $T$ -invariant smooth map (resp.

a distribution), unique up to an additive constant, such that for each  $\xi \in \mathfrak{t}$ ,  $m^\xi := \langle m, \xi \rangle : X \rightarrow \mathbb{R}$  satisfies

$$-dm^\xi = i_{\xi} \omega = \omega(\xi, \cdot).$$

Denote by  $P := \text{im}(m_\omega)$  which is a compact subset in  $\mathfrak{t}^\vee$  and normalize such that  $P = \text{im}(m_{\omega_\phi})$  for all  $\phi \in \mathcal{C}^\infty(X)^T$ . Take  $v, w \in \mathcal{C}^\infty(\mathfrak{t}^\vee, \mathbb{R})$  with  $v > 0$  on  $P$ .

**1.3.2. Singular weighted cscK metrics.** — Suppose that  $X$  is  $\mathbb{Q}$ -Gorenstein, meaning that  $X$  is normal and  $K_X$  is a  $m$ -Cartier for some  $m \in \mathbb{N}^*$ . Below, we recall the definitions of adapted measures and log terminal singularities from [EGZ09, Sec. 5-6].

**Definition 1.10.** — Let  $h^m$  be a smooth hermitian metric on  $mK_X$ . Taking  $\Omega$  a local generator of  $mK_X$ , the adapted measure associated with  $h^m$  is defined by

$$\mu_h := i^{n^2} \left( \frac{\Omega \wedge \bar{\Omega}}{|\Omega|_{h^m}^2} \right)^{1/m}.$$

This definition does not depend on the choice of  $\Omega$ , and two adapted measures differ by a smooth positive density. The Ricci form of the adapted measure  $\mu_h$  is given as

$$\text{Ric}(\mu_h) := dd^c \log |\Omega|_{h^m}^{2/m}.$$

The  $\mathbb{Q}$ -Gorenstein variety  $X$  has log terminal singularities if the measure  $\mu_h$  has finite masses near  $X^{\text{sing}}$ .

We now consider  $X$  to be log terminal. As in [EGZ09, Sec. 5],  $\text{Ric}(\mu_h)$  is canonically attached to an element in  $H^0(X, \mathcal{C}_X^\infty / \text{PH}_X)$  where  $\mathcal{C}_X^\infty$  (resp.  $\text{PH}_X$ ) is the subsheaf of continuous functions on  $X$  that are local restrictions of smooth functions (resp. pluriharmonic functions) under local embeddings. The first Chern class of  $\text{Ric}(\mu_h)$ , denoted  $\{\text{Ric}(\mu_h)\}$ , is the image of  $\text{Ric}(\mu_h)$  in  $H^1(X, \text{PH}_X)$  via the connecting homomorphism  $\{\bullet\}$  in the following exact sequence

$$H^0(X, \mathcal{C}_X^\infty) \rightarrow H^0(X, \mathcal{C}_X^\infty / \text{PH}_X) \xrightarrow{\{\bullet\}} H^1(X, \text{PH}_X) \rightarrow 0$$

which is induced by the short exact sequence  $0 \rightarrow \text{PH}_X \rightarrow \mathcal{C}_X^\infty \rightarrow \mathcal{C}_X^\infty / \text{PH}_X \rightarrow 0$ .

Assume that  $h^m$  is a  $T$ -invariant metric on  $mK_X$  so that  $\text{Ric}(\mu_h)$  is an equivariant curvature form. Fix a compact torus  $T \subset \text{Aut}_{\text{red}}(X)$  with Lie algebra  $\mathfrak{t}$ . Set  $P := P_\omega := m_\omega(X) \subset \mathfrak{t}^\vee$  and take  $v \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$ . For any  $\phi \in \mathcal{H}_\omega^T$ , the weighted Ricci current of  $\omega_\phi := \omega + dd^c \phi$  is defined as

$$\text{Ric}_v^T(\omega_\phi) = \text{Ric}^T(\text{MA}_v(\phi)) := -dd^c \log \left( \frac{v(m_\phi) \omega_\phi^n}{\mu_h} \right) + \text{Ric}(\mu_h).$$

and the  $v$ -weighted scalar curvature is the distribution expressed by

$$S_v(\omega_\phi) := \text{tr}_{\omega_\phi}(\text{Ric}^T(\omega_\phi)).$$

One can extend the weighted energy and the weighted Ricci energy on  $\mathcal{E}_\omega^{1,T}$  (cf. [PTT23, Sec. 4.1.2]). Fix an adapted measure  $\mu_X$ , for any  $w \in \mathcal{C}^\infty(P, \mathbb{R})$ , the weighted Mabuchi energy  $\mathbf{M}_{v,w} : \mathcal{E}_\omega^{1,T} \rightarrow \mathbb{R} \cup \{+\infty\}$  can be expressed as

$$\mathbf{M}_{v,w} := \mathbf{M}_{v,w,\mu_X} := \mathbf{H}_v + \mathbf{E}_v^{-\text{Ric}(\mu_X)} + \mathbf{E}_{vw}.$$

**Definition 1.11.** — Let  $v \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$  and  $w \in \mathcal{C}^\infty(P, \mathbb{R})$ . Then  $\omega_\phi \in \{\omega\}$  is a singular  $(v, w)$ -cscK metric if  $\phi \in \text{PSH}(X, \omega) \cap L^\infty(X)$  which is also smooth on  $X^{\text{reg}}$ , and

$$S_v(\omega_\phi) = w(m_\phi) \quad \text{on } X^{\text{reg}}.$$

Let  $w_0 \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$  and  $\ell^{\text{ext}}$  be the unique affine function  $\ell^{\text{ext}} = \ell_{\omega,v,w_0}^{\text{ext}}$  on  $\mathfrak{t}^\vee$  such that

$$\langle \ell, \ell^{\text{ext}} \rangle = \int_X \ell(m_\omega) S_v(\omega) \text{MA}_v(0).$$

Then  $\omega_\phi$  is called a singular  $(v, w_0)$ -extremal metric if it is a singular  $(v, w)$ -cscK metric with  $w = w_0 \ell^{\text{ext}}$ .

Let  $\pi : Y \rightarrow X$  be an  $T$ -equivariant resolution of singularities. We can also define  $\mathbf{M}_{\pi^*\omega}^{\text{rel}} : \pi^*\mathcal{H}_\omega^T \rightarrow \mathbb{R}$  by

$$\mathbf{M}_{\pi^*\omega}^{\text{rel}}(\pi^*u) := \mathbf{H}_{\omega_Y^n}(\pi^*u) + \mathbf{E}_{\pi^*\omega}^{-\text{Ric}(\omega_Y)}(\pi^*u) + \mathbf{E}_{\pi^*\omega, v\omega^{\text{ext}}}(\pi^*u).$$

Under Condition (A), following [BJT24, Lem. 4.21], one can extend the weighted energy, the weighted Ricci energy, and  $\mathbf{M}_{\pi^*\omega}^{\text{rel}}$  to  $\mathcal{E}_{\pi^*\omega}^{1,T}$ .

## 2. A priori estimates

In this section, we shall establish a priori estimates for wscK equations on compact Kähler manifolds when the reference metrics are degenerating. These estimates are crucial for obtaining the existence of singular wscK metrics in the next section.

Once and for all, we fix  $X$  to be an  $n$ -dimensional compact Kähler manifold,  $T \subset \text{Aut}_{\text{red}}(X)$  a compact torus with Lie algebra  $\mathfrak{t}$ , and  $\omega$  a  $T$ -invariant Kähler metric on  $X$  with the moment polytope  $P$  of  $\omega$  and total volume  $V$ . Take  $v \in \mathcal{C}^\infty(\mathfrak{t}^\vee, \mathbb{R})$  such that  $v > 0$  on  $P$ .

**2.1.  $L^\infty$ -estimates.** — In the following, we shall establish a uniform  $L^\infty$ -estimate following the strategy of Guo and Phong [GP22, Thm. 3] with certain modifications (cf. [PTT23, Thm. 5.4]).

**Theorem 2.1.** — *Let  $\mu$  be a  $T$ -invariant smooth volume form such that  $\omega^n = g\mu$  with  $\|g^{1/n}\|_{L^\infty} \leq K_0$  for some constant  $K_0 > 0$ . Suppose that  $(\varphi, F) \in \mathcal{H}^T(X, \omega) \times \mathcal{C}^\infty(X)^T$  is a solution to the coupled equations*

$$\begin{cases} v(m_\varphi)(\omega + dd^c\varphi)^n = e^F\mu, & \sup_X \varphi = 0, \\ \Delta_{\varphi, v}F = -S + \text{tr}_{\varphi, v}(\text{Ric}(\mu)), \end{cases}$$

for some  $S \in \mathcal{C}^\infty(X)^T$ . In addition, assume that there are positive constants  $K_1, K_3, K_4$  such that

- (1)  $-K_1(\omega + dd^c\rho_1) \leq \text{Ric}(\mu)$ , where  $\rho_1$  is a  $T$ -invariant quasi-psh function,  $\sup_X \rho_1 = 0$  and  $\int_X e^{-K_1\rho_1} d\mu < +\infty$ ;
- (2)  $\mathbf{H}_\mu(\varphi) = \int_X \log\left(\frac{\omega_\varphi^n}{\mu}\right) \omega_\varphi^n \leq K_3$ ;
- (3) there exists  $\alpha > 0$  such that  $\int_X e^{-\alpha(\varphi - \sup_X \varphi)} \mu \leq K_4$  for all  $\varphi \in \text{PSH}(X, \omega)$ .

Then there is a uniform constant  $C > 0$  depending only on  $n, \max_X S, V, \alpha, K_0, K_1, K_3, K_4, C_v, \mu, \rho_1$  such that

$$\|\varphi\|_{L^\infty} \leq C \text{ and } F \leq C.$$

Furthermore, if we have that

$$\text{Ric}(\mu) \leq K_2(\omega + dd^c\rho_2),$$

for a  $T$ -invariant  $\omega$ -psh function  $\rho_2 \in \mathcal{C}^\infty(X \setminus Z)$ , with  $Z := \{\rho_2 = -\infty\}$  and  $\sup_X \rho_2 = 0$ , there is a constant  $C_2 > 0$  depending only on  $n, \max_X |S|, V, \alpha, K_0, K_1, K_2, K_3, K_4, C_v, \mu, \rho_1$  such that

$$F \geq K_2\rho_2 - C_2.$$

*Proof.* — Consider  $\tau_k : \mathbb{R} \rightarrow \mathbb{R}_{>0}$  a sequence of positive smooth functions decreasing towards the function  $x \mapsto x \cdot \mathbb{1}_{\mathbb{R}_{>0}}(x)$ . Let  $\phi_k$  be a solution to the following auxiliary complex Monge–Ampère equation

$$V^{-1}(\omega + dd^c\phi_k)^n = \frac{\tau_k(-\varphi + \lambda F) + 1}{A_k} e^F \mu, \quad \sup_X \phi_k = 0, \quad (2.1)$$

where

$$A_k = \int_X (\tau_k(-\varphi + \lambda F) + 1) e^F \mu \xrightarrow{k \rightarrow +\infty} \int_X (-\varphi + \lambda F)_+ e^F \mu + \int e^F \mu = A_\infty.$$

We remark that  $\phi_k$  is also  $T$ -invariant since the RHS of (2.1) is  $T$ -invariant. Applying Young's inequality with  $\chi(s) = (s+1)\log(s+1) - s$  and  $\chi^*(s) = e^s - s - 1$ ,

$$\int_X (-\varphi) e^F \omega^n \leq \int_X \chi(\alpha^{-1} e^F) \omega^n + \int_X \chi^*(-\alpha\varphi) \omega^n$$

where  $\alpha > 0$  is the constant in (3). It follows from (2) and (3) that  $C_v^{-1}V \leq A_\infty \leq C(K_3, K_4, V)$ . Thus  $C_v^{-1}V \leq A_k \leq C_1 = C(K_3, K_4, V)$  for  $k$  sufficiently large.

By strong openness [Ber13, GZ15], we have a constant  $\delta > 0$  such that  $I := \int_X e^{-(K_1+\delta)\rho_1} < +\infty$ . Using Demailly's approximation theorem [Dem92] (see also [Dem12, Sec. 14B]) and Demailly-Kollár's convergence result [DK01, Main Thm. 0.2 (2)], there exists a sequence quasi-plurisubharmonic functions  $(\rho_{1,k})_k$  with analytic singularities and smooth away from their singular locus, such that  $-K_1(\omega + dd^c \rho_{1,k}) - \frac{1}{k}\omega \leq \text{Ric}(\mu)$ ,  $\rho_{1,k} \rightarrow \rho_1$  in  $L^1$  as  $k \rightarrow +\infty$ , and  $\int_X e^{-(K_1+\delta)\rho_{1,k}} \leq 2I$  for all  $k \gg 1$ . Hence,  $-K'_1(\omega + dd^c \rho'_{1,k}) \leq \text{Ric}(\mu)$  where  $K'_1 = K_1 + \frac{1}{k}$  with  $k > 1/\delta$  and  $\rho'_{1,k} = \frac{K_1}{K_1+1/k}\rho_{1,k}$ . Replacing  $\rho_1$  by  $\rho'_{1,k}$  and  $K_1$  by  $K'_1$ , one can assume  $\rho_1$  has analytic singularities.

Consider the function

$$\Phi = -\epsilon(-\phi_k + \Lambda)^{\frac{n}{n+1}} - \varphi + \lambda(F + K_1\rho_1)$$

with  $\Lambda = \left(\frac{2n}{n+1}\epsilon\right)^{n+1}$  and  $\epsilon = \left(\frac{(n+1)(n+\lambda\bar{s}+L)}{n^2}\right)^{\frac{1}{n+1}} A_k^{\frac{1}{n+1}}$ , where  $\bar{s} := \max_X S$ ,  $L := \lambda(C_{v,\mu} + K_1 C_v) + 2C_v$  with  $C_{v,\mu} > 0$  a constant independent of  $\varphi$  so that  $\langle (\log v)'(m_\varphi), m_{\text{Ric}(\mu)} \rangle \geq -C_{v,\mu}$ , and  $\lambda > 0$  is a constant such that  $n + \lambda\bar{s} > 0$  and  $\lambda K_1 < 1/2$ . Since  $\rho_1(x) \rightarrow -\infty$  as  $x \rightarrow Z_1 := \{\rho_1 = -\infty\}$ , the maximal points of  $\Phi$  only occur in  $X \setminus Z_1$ . Fix  $x_0 \in X \setminus Z_1$  a maximal point of  $\Phi$ . At  $x_0$ , we have

$$\begin{aligned} 0 &\geq \Delta_{\varphi,v}\Phi = \Delta_{\omega_\varphi}\Phi + \langle (\log v)'(m_\varphi), m_{dd^c\Phi} \rangle \\ &\geq \frac{\epsilon n}{n+1}(-\phi_k + \Lambda)^{-\frac{1}{n+1}} \Delta_{\omega_\varphi}\phi_k - \Delta_{\omega_\varphi}\varphi + \lambda\Delta_{\varphi,v}(F + K_1\rho_1) \\ &\quad + \frac{\epsilon n}{n+1}(-\phi_k + \Lambda)^{-\frac{1}{n+1}} \langle (\log v)'(m_\varphi), m_{dd^c\phi_k} \rangle - \langle (\log v)'(m_\varphi), m_{dd^c\Phi} \rangle. \end{aligned}$$

Since  $-C_v \leq \langle (\log v)'(m_\varphi), m_{dd^c\psi} \rangle \leq C_v$  for any  $\phi, \psi \in \text{PSH}(X, \omega)^T$  (see (1.4)), we infer that

$$\begin{aligned} 0 &\geq \frac{\epsilon n}{n+1}(-\phi_k + \Lambda)^{-\frac{1}{n+1}} \Delta_{\omega_\varphi}\phi_k - \Delta_{\omega_\varphi}\varphi + \lambda\Delta_{\varphi,v}(F + K_1\rho_1) - C_v \left( \frac{\epsilon n}{n+1} \Lambda^{-\frac{1}{n+1}} + 1 \right) \\ &= \frac{\epsilon n}{n+1}(-\phi_k + \Lambda)^{-\frac{1}{n+1}} (\text{tr}_{\omega_\varphi} \omega_{\phi_k} - \text{tr}_{\omega_\varphi} \omega) - \text{tr}_{\omega_\varphi}(\omega_\varphi - \omega) + \lambda(-S + \text{tr}_{\omega_\varphi}(\text{Ric}(\mu) + K_1 dd^c \rho_1)) \\ &\quad + \lambda \left( \langle (\log v)'(m_\varphi), m_{\text{Ric}(\mu)} \rangle + K_1 \langle (\log v)'(m_\varphi), m_{dd^c\rho_1} \rangle \right) - 2C_v. \end{aligned}$$

Then

$$\begin{aligned} 0 &\geq \frac{n^2\epsilon}{n+1}(-\phi_k + \Lambda)^{-\frac{1}{n+1}} \left( \frac{\tau_k(-\varphi + \lambda F) + 1}{A_k} \right)^{1/n} - n - \lambda\bar{s} - L + \left( 1 - \frac{n\epsilon}{n+1} \Lambda^{-\frac{1}{n+1}} - \lambda K_1 \right) \text{tr}_{\omega_\varphi} \omega \\ &\geq \frac{n^2\epsilon}{n+1}(-\phi_k + \Lambda)^{-\frac{1}{n+1}} \left( \frac{\tau_k(-\varphi + \lambda F) + 1}{A_k} \right)^{1/n} - n - \lambda\bar{s} - L. \end{aligned}$$

Since  $n + \lambda\bar{s} > 0$  and  $\lambda K_1 < 1/2$ , at  $x_0$ , we obtain

$$-\varphi + \lambda(F + K_1\rho_1) \leq -\varphi + \lambda F \leq \left( \frac{(n + \lambda\bar{s} + L)(n+1)}{n^2\epsilon} \right)^n A_k (-\phi_k + \Lambda)^{\frac{n}{n+1}}; \quad (2.2)$$

therefore,  $\Phi(x_0) \leq 0$  and  $\Phi \leq 0$  on  $X$ . By the choice of  $\epsilon, \Lambda$  and  $V \leq A_k \leq C(K_3, K_4, V)$ , and Young's inequality, we derive that for any  $\delta > 0$

$$\lambda(F + K_1\rho_1) \leq -\varphi + \lambda(F + K_1\rho_1) \leq C(V, K_1, K_3, K_4)(-\phi_k + \Lambda)^{\frac{n}{n+1}} \leq -\delta\phi_k + C_2, \quad (2.3)$$

with  $C_2 = C(\delta, V, K_2, K_3, K_4)$ .

From Condition (A), we have  $\int_Y e^{-K_1\rho_1}\mu < +\infty$ . The strong openness [Ber13, GZ15] yields a constant  $0 < a \ll 1$  such that  $\int_X e^{-(1+a)K_1\rho_1}\mu \leq C_a$  for some constant  $C_a > 0$ . By (2.3), Hölder inequality and (3), for  $\beta = \frac{1+a/2}{\lambda}$  and  $\delta > 0$  such that  $\delta\beta < \alpha/\gamma^*$ , with  $\gamma = \frac{1+a}{1+a/2}$  and  $\frac{1}{\gamma} + \frac{1}{\gamma^*} = 1$ , we obtain

$$\int_X e^{\beta\lambda F} \mu \leq e^{\beta C_2} \int_X e^{-\frac{\alpha}{\gamma^*}\phi_k - (1+\frac{a}{2})K_1\rho_1} \mu = e^{\beta C_2} \int_X e^{-\frac{\alpha}{\gamma^*}\phi_k - (1+a)K_1\rho_1/\gamma} \mu \leq C(a, C_a, \alpha). \quad (2.4)$$

By a refined version of Kołodziej's  $L^\infty$ -estimate [Kol98] (see [DGG23, Thm. A] for the version we referred), a uniform control  $C_v^{-1} \leq v(m_\varphi) \leq C_v$  and (2.4), we obtain  $\|\varphi\|_{L^\infty} \leq C(n, V, \alpha, K_1, K_3, K_4, C_v, \mu, \rho_1)$ . Also, combining the  $L^\infty$ -estimate of  $\varphi$  with (2.4), we infer that

$$\|(\tau_k(-\varphi + \lambda F) + 1)e^F\|_{L^{p'}(X, \omega^n)} \leq C(n, \alpha, V, K_1, K_3, K_4, C_v, \mu, \rho_1)$$

for some  $p' > 1$  and for all  $k > 0$  sufficiently large. Again, Kołodziej's  $L^\infty$ -estimate yields a uniform bound  $\|\phi_k\|_{L^\infty} \leq C(n, V, \alpha, K_1, K_3, K_4, C_v, \rho_1)$ . Then the inequality (2.2) provides a uniform upper bound for  $F$ .

In the second part, we consider the function  $H := F + (K_2 + 1)\varphi - K_2\rho_2$ . Since  $\rho_2 = -\infty$  along  $Z$ , one can assume that  $H$  admits a minimum at  $x_0 \in X \setminus Z$ . At  $x_0$ , we have

$$\begin{aligned} 0 &\leq \Delta_{\varphi, v} H = -S + \operatorname{tr}_{\omega_\varphi}(\operatorname{Ric}(\mu)) + (K_2 + 1)n - (K_2 + 1) \operatorname{tr}_{\omega_\varphi} \omega - K_2 \operatorname{tr}_{\omega_\varphi}(\omega + dd^c \rho_2) + K_2 \operatorname{tr}_{\omega_\varphi} \omega \\ &\quad + \langle (\log v)'(m_\varphi), m_{\operatorname{Ric}(\mu)} \rangle + (K_2 + 1) \langle (\log v)'(m_\varphi), m_{dd^c \varphi} \rangle - K_2 \langle (\log v)'(m_\varphi), m_{dd^c \rho_2} \rangle \\ &\leq -S + \operatorname{tr}_{\omega_\varphi}(\operatorname{Ric}(\mu) - K_2(\omega + dd^c \rho_2)) + (K_2 + 1)n - \operatorname{tr}_{\omega_\varphi} \omega + C_{v, \mu} + 2(K_2 + 1)C_v \\ &\leq (K_2 + 1)n - S - \operatorname{tr}_{\omega_\varphi} \omega + C_{v, \mu} + 2(K_2 + 1)C_v \\ &\leq (K_2 + 1)n - S - n \left( \frac{\omega^n}{\omega_\varphi^n} \right)^{1/n} + C_{v, \mu} + 2(K_2 + 1)C_v \\ &\leq (K_2 + 1)n - S - n \left( \frac{g v(m_\varphi)}{e^F} \right)^{1/n} + C_{v, \mu} + 2(K_2 + 1)C_v \\ &\leq (K_2 + 1)n - \min_X S - nK_0 C_v^{-1/n} e^{-F/n} + C_{v, \mu} + 2(K_2 + 1)C_v. \end{aligned}$$

Therefore, at  $x_0$ ,  $F(x_0) \geq -C(n, K_0, K_2, \min_X S, C_{v, \mu}, C_v)$ . We obtain

$$\begin{aligned} F &\geq K_2(\rho_2 - \rho_2(x_0)) - C(n, K_0, K_2, \min_X S, C_{v, \mu}, C_v) - (K_2 + 1)\|\varphi\|_{L^\infty} \\ &\geq K_2\rho_2 - C(n, K_0, \dots, K_4, \max_X |S|, C_v, \mu, \rho_1) \end{aligned}$$

as required.  $\square$

**2.2. Local  $L^p$ -estimate for Laplacian.** — Let  $\omega_X$  be another  $T$ -invariant Kähler metric on  $X$ . Since  $P_{\omega_X} = \operatorname{im}(m_{\omega_X})$  is compact, one can further assume that

$$C_v^{-1} \leq |v| + \sum_\alpha |v_\alpha| + \sum_{\alpha, \beta} |v_{\alpha\beta}| \leq C_v \quad \text{on } P_{\omega_X}.$$

Consider the following wscK equations

$$\begin{cases} v(m_\varphi)(\omega + dd^c \varphi)^n = e^F \omega_X^n, & \sup_X \varphi = 0, \\ \Delta_{\varphi, v} F = -S + \operatorname{tr}_{\varphi, v}(\operatorname{Ric}(\omega_X)). \end{cases} \quad (2.5)$$

In this section, we shall further assume  $t^\vee \ni \zeta \mapsto \log v(\zeta) \in \mathbb{R}$  to be *concave*. With the concavity condition on  $\log v$ , from Lemma A.1, we have the following weighted Aubin–Yau type inequality:

$$\Delta_{\varphi, v} \log \operatorname{tr}_{\omega_X} \omega_\varphi \geq \frac{\Delta_{\omega_X, v} F}{\operatorname{tr}_{\omega_X} \omega_\varphi} - \mathcal{B} \operatorname{tr}_{\omega_\varphi} \omega_X - \mathcal{C}$$

for some uniform constant  $\mathcal{B}, \mathcal{C} > 0$ .

**Proposition 2.2.** — Suppose that  $(\varphi, F) \in \mathcal{H}^T(X, \omega) \times C^\infty(X)^T$  is a solution to (2.5). Fix  $p > 1$ . Assume that  $\omega \leq C_\omega \omega_X$ ,  $\operatorname{Ric}(\omega_X) \leq A\omega_X$ ,  $\operatorname{Bisec}(\omega_X) \geq -B$ ,  $\omega + dd^c \rho \geq C_\rho \omega_X$  with  $C_\rho > 0$ , and

$$\|\varphi\|_{L^\infty} \leq C_0, \quad F \leq C_0, \quad \text{and} \quad F \geq K_2\rho - C_0,$$

where  $\rho \in C^\infty(X \setminus \{\rho = -\infty\})$  is a  $T$ -invariant  $\omega$ -psh function. Then for any  $\mathcal{K}$  compact set of  $X \setminus \{\rho = -\infty\}$ , one has the following estimate

$$\|\operatorname{tr}_{\omega_X} \omega_\varphi\|_{L^{2p+2}(\mathcal{K}, \omega_X^n)} \leq C_1,$$

where  $C_1$  only depends on  $\mathcal{K}, n, p, A, B, C_0, C_\rho, \min_X S$ .

*Proof.* — We shall adapt the approach in the smooth setting of Chen and Cheng [CC21a] for cscK metrics and also its generalization by [DJL24] and [HL24] for wcscK metrics. We highlight some differences in the following:

- We shall work with the trace taken with respect to the reference metric  $\omega_X$  instead of  $\omega$  as  $\omega$  is moving when we are going to apply the result. However, the metric  $\omega$  still plays a role and must be carefully merged during the computations, especially since we only have the upper bound  $\omega \leq C_\omega \omega_X$ .
- In the last step, we need to use bounds on  $\varphi$  and  $F$ . Special attention is required for  $F$  since its lower bound is uniform only up to a term involving  $K_2\rho$ , which is not bounded from below.

Take  $u := e^{-a(F+b\varphi-b\rho)} \operatorname{tr}_{\omega_X} \omega_\varphi \geq 0$ , where  $a, b$  are constants to be determined later. Recall that from (1.2),

$$\Delta_{\omega, v} f = \Delta_\omega f + \operatorname{tr}_\omega (d \log v(m_\omega) \wedge d^c f);$$

hence, we have

$$\begin{aligned} \Delta_{\varphi, v} u &= \Delta_{\varphi, v} e^{\log u} = \frac{|du|_{\omega_\varphi}^2}{u} + u \Delta_{\omega_\varphi} \log u + \operatorname{tr}_{\omega_\varphi} (d \log v(m_{\omega_\varphi}) \wedge d^c u) \\ &= \frac{|du|_{\omega_\varphi}^2}{u} + u \Delta_{\omega_\varphi} \log u + u \operatorname{tr}_{\omega_\varphi} (d \log v(m_{\omega_\varphi}) \wedge d^c \log u) = \frac{|du|_{\omega_\varphi}^2}{u} + u \Delta_{\varphi, v} \log u \\ &\geq u \Delta_{\varphi, v} \log u = -au \Delta_{\varphi, v} (F + b\varphi - c\rho) + u \Delta_{\varphi, v} \log(\operatorname{tr}_{\omega_X} \omega_\varphi). \end{aligned} \quad (2.6)$$

Using  $\operatorname{Ric}(\omega_X) \leq A\omega_X$ ,  $|\langle (\log v)'(m_\varphi), m_{\operatorname{Ric}(\omega_X)} \rangle| \leq C_{v, \operatorname{Ric}(\omega_X)}$ ,  $\operatorname{tr}_{\varphi, v} \omega_\varphi \leq n + C_v$ , and  $\operatorname{tr}_{\omega_\varphi} \omega_\rho \geq C_\rho \operatorname{tr}_{\varphi, v} \omega_X - C_v$ , we derive

$$\begin{aligned} \Delta_{\varphi, v} (F + b\varphi - b\rho) &= -S + \operatorname{tr}_{\varphi, v} (\operatorname{Ric}(\omega_X)) + b \operatorname{tr}_{\varphi, v} (dd^c \varphi - dd^c \rho) \\ &\leq -\min_X S + A \operatorname{tr}_{\omega_\varphi} \omega_X + \langle (\log v)'(m_\varphi), m_{\operatorname{Ric}(\omega_X)} \rangle - b \operatorname{tr}_{\varphi, v} \omega_\rho + b \operatorname{tr}_{\varphi, v} \omega_\varphi \\ &\leq -\min_X S + (A - bC_\rho) \operatorname{tr}_{\omega_\varphi} \omega_X + b(2C_v + n) + C_{v, \operatorname{Ric}(\omega_X)} \\ &= C_1 + (A - bC_\rho) \operatorname{tr}_{\omega_\varphi} \omega_X \end{aligned} \quad (2.7)$$

where  $C_1 > 0$  only depends on  $-\min_X S, b, C_v, C_{v, \operatorname{Ric}(\omega_X)}$ .

By a weighted Aubin–Yau’s inequality for weighted Monge–Ampère equation obtained in [DJL24] (see also Lemma A.1 for the version we apply), we have

$$\Delta_{\varphi, v} \log \operatorname{tr}_{\omega_X} \omega_\varphi \geq \frac{1}{\operatorname{tr}_{\omega_X} \omega_\varphi} \Delta_{\omega_X, v} F - \mathcal{B} \operatorname{tr}_{\omega_\varphi} \omega_X - \mathcal{C} \quad (2.8)$$

where  $\mathcal{B}, \mathcal{C}$  depend on  $\omega_X, C_v$ . Combining (2.6), (2.7), and (2.8), we get

$$\Delta_{\varphi, v} u \geq e^{-a(F+b\varphi-b\rho)} \left\{ -(aC_1 + \mathcal{C}) \operatorname{tr}_{\omega_X} \omega_\varphi + \Delta_{\omega_X, v} F + (abC_\rho - Aa - \mathcal{B}) \operatorname{tr}_{\omega_X} \omega_\varphi \operatorname{tr}_{\omega_\varphi} \omega_X \right\}.$$

Using the fact that

$$\operatorname{tr}_{\omega_X} \omega_\varphi \operatorname{tr}_{\omega_\varphi} \omega_X \geq (\operatorname{tr}_{\omega_X} \omega_\varphi)^{\frac{n}{n-1}} \left( \frac{\omega_X^n}{\omega_\varphi^n} \right)^{\frac{1}{n-1}} = (\operatorname{tr}_{\omega_X} \omega_\varphi)^{\frac{n}{n-1}} (ve^{-F})^{\frac{1}{n-1}} \geq (\operatorname{tr}_{\omega_X} \omega_\varphi)^{\frac{n}{n-1}} (C_v e^F)^{\frac{-1}{n-1}},$$

and choosing  $b$  large enough such that  $abC_\rho - Aa - \mathcal{B} \geq abC_\rho/2$ , we get

$$\Delta_{\varphi, v} u \geq -(aC_1 + \mathcal{C})u + \frac{abC_\rho}{2} C^{-\frac{1}{n-1}} e^{\frac{-F}{n-1}} (\operatorname{tr}_{\omega_X} \omega_\varphi)^{\frac{1}{n-1}} u + e^{-a(F+b\varphi-b\rho)} \Delta_{\omega_X, v} F. \quad (2.9)$$

Since  $|\nabla u|_{\omega_\varphi}^2 \operatorname{tr}_{\omega_X} \omega_\varphi \geq |\nabla u|_{\omega_X}^2$ , we infer that

$$\frac{1}{2p+1} \Delta_{\varphi, v} u^{2p+1} = u^{2p} \Delta_{\varphi, v} u + 2pu^{2p-1} |\nabla u|_{\omega_\varphi}^2 \geq u^{2p} \Delta_{\varphi, v} u + 2pu^{2p-2} e^{-a(F+b\varphi-b\rho)} |\nabla u|_{\omega_X}^2 \quad (2.10)$$

By (2.9), (2.10), and the weighted integration-by-parts formula (1.3), we obtain

$$\begin{aligned}
0 &= \int_X \frac{1}{2p+1} \Delta_{\varphi, v} u^{2p+1} v(m_\varphi) \omega_\varphi^n \\
&\geq 2p \int_X 2pu^{2p-2} e^{-a(F+b\varphi-b\rho)+F} |\nabla u|_{\omega_X}^2 \omega_X^n - (aC_1 + C) \int_X u^{2p+1} e^F \omega_X^n \\
&\quad + \frac{abC_\rho}{2} C_v^{-\frac{1}{n-1}} \int_X u^{2p+1} e^{\frac{n-2}{n-1}F} (\operatorname{tr}_{\omega_X} \omega_\varphi)^{\frac{1}{n-1}} \omega_X^n + \underbrace{\int_X u^{2p} e^{-a(F+b\varphi-b\rho)+F} \Delta_{\omega_X, v} F \omega_X^n}_{=: I}.
\end{aligned} \tag{2.11}$$

Set  $G = (1-a)F - ab(\varphi - \rho)$  and consider  $a > 1$ . The last term  $I$  can be written as the following two parts

$$I = \int_X u^{2p} \Delta_{\omega_X, v} F e^G \omega_X^n = \underbrace{\frac{1}{1-a} \int_X u^{2p} \Delta_{\omega_X, v}(G) e^G \omega_X^n}_{=: I_1} + \underbrace{\frac{ab}{1-a} \int_X u^{2p} \Delta_{\omega_X, v}(\varphi - \rho) e^G \omega_X^n}_{=: I_2}.$$

For  $I_1$ , the weighted integration-by-parts formula (1.3) implies

$$\begin{aligned}
I_1 &= \frac{1}{1-a} \int_X u^{2p} \Delta_{\omega_X, v}(G) e^G v(m_{\omega_X})^{-1} \mu_{\omega_X, v} \quad \text{where } \mu_{\omega_X, v} = v(m_{\omega_X}) \omega_X^n \\
&= \frac{1}{a-1} \int_X u^{2p} |\nabla G|_{\omega_X}^2 e^G \omega_X^n + \frac{1}{a-1} \int_X n2pu^{2p-1} e^G du \wedge d^c G \wedge \omega_X^{n-1} \\
&\quad - \frac{1}{a-1} \int_X nu^{2p} e^G v(m_{\omega_X})^{-1} d(v(m_{\omega_X})) \wedge d^c G \wedge \omega_X^{n-1}.
\end{aligned}$$

By Cauchy–Schwarz inequality,

$$\left| 2pu^{2p-1} \frac{du \wedge d^c G \wedge \omega_X^{n-1}}{\omega_X^n} \right| \leq \frac{4p^2}{2} u^{2p-2} |\nabla u|_{\omega_X}^2 + \frac{1}{2} u^{2p} |\nabla G|_{\omega_X}^2$$

and

$$\begin{aligned}
\left| u^{2p} v(m_\omega)^{-1} \frac{d(v(m_\omega)) \wedge d^c G \wedge \omega_X^{n-1}}{\omega_X^n} \right| &\leq \frac{v(m_{\omega_X})^{-2}}{2} u^{2p} |\nabla v(m_{\omega_X})|_{\omega_X}^2 + \frac{1}{2} u^{2p} |\nabla G|_{\omega_X}^2 \\
&\leq C_2 u^{2p} + \frac{1}{2} u^{2p} |\nabla G|_{\omega_X}^2,
\end{aligned}$$

where  $C_2$  only depends on  $C_v$  and  $|\nabla(v(m_{\omega_X}))|_{\omega_X}^2$ . The above two inequalities yield

$$I_1 \geq -\frac{2p^2}{a-1} \int_X u^{2p-2} |\nabla u|_{\omega_X}^2 e^G \omega_X^n - \frac{C_2}{a-1} \int_X u^{2p} e^G \omega_X^n. \tag{2.12}$$

We next control the term  $I_2$ . We compute

$$\begin{aligned}
\Delta_{\omega_X, v}(\varphi - \rho) &= \operatorname{tr}_{\omega_X, v}(\omega_\varphi - \omega_\rho) = \operatorname{tr}_{\omega_X}(\omega_\varphi - \omega_\rho) + \langle (\log v)'(m_{\omega_X}), (m_\varphi - m_\rho) \rangle \\
&\leq \operatorname{tr}_{\omega_X} \omega_\varphi + C_3
\end{aligned}$$

where  $C_3$  only depends on  $P, C_v$ . Since  $a > 1$ , we get

$$I_2 \geq \frac{ab}{1-a} \int_X u^{2p} \operatorname{tr}_{\omega_X} \omega_\varphi e^G \omega_X^n + \frac{abC_3}{1-a} \int_X u^{2p} e^G \omega_X^n. \tag{2.13}$$

Combining (2.11), (2.12) and (2.13), one can derive

$$\begin{aligned}
0 &\geq 2 \left( p - \frac{p^2}{a-1} \right) \int_X u^{2p-2} e^G |\nabla u|_{\omega_X}^2 \omega_X^n - (aC_1 + C) \int_X u^{2p+1} e^F \omega_X^n \\
&\quad + \frac{abC_\rho}{2} C_v^{-\frac{1}{n-1}} \int_X u^{2p+1} e^{\frac{n-2}{n-1}F} (\operatorname{tr}_{\omega_X} \omega_\varphi)^{\frac{1}{n-1}} \omega_X^n + \frac{ab}{1-a} \int_X u^{2p} \operatorname{tr}_{\omega_X} \omega_\varphi e^G \omega_X^n + \frac{abC_3 + C_2}{1-a} \int_X u^{2p} e^G \omega_X^n.
\end{aligned}$$

Taking  $a > 1$  sufficiently large so that  $p > p^2/(a-1)$ , and using the upper bound  $F \leq C_0$ , one gains  $\frac{n-2}{n-1}F \geq F - C_0/(n-1)$  and thus,

$$0 \geq -C_4 \int_X (\operatorname{tr}_{\omega_X} \omega_\varphi)^{2p+1} e^{(2p+1)(G-F)+F} \omega_X^n - C_5 \int_X u^{2p} e^G \omega_X^n + C_6 \int_X (\operatorname{tr}_{\omega_X} \omega_\varphi)^{2p+1+\frac{1}{n-1}} e^{(2p+1)(G-F)+F} \omega_X^n,$$

where  $C_4, C_5, C_6$  only depend on  $a, b, p, C_0, C_\rho, C_v, n$  and  $|\nabla(v(m_{\omega_X}))|_{\omega_X}^2$ . Set  $\mu := e^{(2p+1)(G-F)+F}\omega_X^n = e^{-[a(2p+1)-1]F-ab(2p+1)\varphi+ab(2p+1)\rho}\omega_X^n$ . Choose  $b \gg 1$  such that  $ab(2p+1) > K_2(a(2p+1) - 1)$ . By  $F \geq K_2\rho - C_0$  in the assumption,

$$C_7^{-1}e^{ab(2p+1)\rho} \leq \frac{\mu}{\omega_X^n} \leq C_7,$$

and it implies that  $C_8^{-1} \leq \int_X \mu \leq C_8$ . Therefore, we obtain

$$0 \geq -C_4 \int_X (\mathrm{tr}_{\omega_X} \omega_\varphi)^{2p+1} d\mu - C_5 \int_X (\mathrm{tr}_{\omega_X} \omega_\varphi)^{2p} d\mu + C_6 \int_X (\mathrm{tr}_{\omega_X} \omega_\varphi)^{2p+1+\frac{1}{n-1}} d\mu,$$

and Hölder's inequality shows  $\|\mathrm{tr}_{\omega_X} \omega_\varphi\|_{L^{2p+1}(X, \mu)} \leq C$ . This concludes the proof.  $\square$

**2.3. Local higher-order estimates.** — We next provide higher-order estimates that are away from the degenerating locus. We shall use the following local estimate, which generalizes [CC21a] for cscK equations (see also [DJL24, HL24]). Its proof follows a similar argument in [CC21a] with further analysis of the weighted terms. For full details, the reader is referred to Appendix B.

**Theorem 2.3.** — *Assume that  $\mathfrak{t}^\vee \ni \zeta \mapsto \log v(\zeta) \in \mathbb{R}$  is concave. Let  $\phi$  be a smooth solution of*

$$\begin{cases} v(m_{dd^c\phi}) \det(\phi_{k\bar{j}}) = e^G \\ \Delta_{\phi, v} G = -S = -w(m_{dd^c\phi}) \end{cases}$$

in  $B_1(0) \subset \mathbb{C}^n$ , where  $\Delta_{\phi, v} f := \phi^{k\bar{j}} \partial_k \bar{\partial}_{\bar{j}} f + \mathrm{tr}_{dd^c\phi}(d \log v(m_{dd^c\phi}) \wedge d^c f)$ , such that  $\Delta\phi \in L^p(B_1(0))$  and  $\mathrm{tr}_\phi \omega_{\mathrm{eucl}} \in L^p(B_1(0))$  for some  $p > 3n$ . Then there exists a constant  $A > 0$ , depending only on  $C_v, C_w, p, \|S\|_{L^\infty}, \|\Delta\phi\|_{L^p(B_1(0))}, \|\mathrm{tr}_\phi \omega_{\mathrm{eucl}}\|_{L^p(B_1(0))}$ , such that  $|\phi_{k\bar{j}}| \leq A, |\nabla G| \leq A$  in  $B_{\frac{1}{2}}(0)$  and for any  $k \geq 2$

$$\|D^k \phi\|_{L^\infty(B_{1/2}(0))} \leq C(k, A, \|\phi\|_{L^\infty(B_1(0))}).$$

Combining Theorem 2.1, Proposition 2.2, and Theorem 2.3, we obtain the following result:

**Theorem 2.4.** — *Under the same assumption of Theorem 2.1 and assuming that  $\log v$  is concave, there is a uniform constant  $C > 0$  depending only on  $n, \max_X S, V, \alpha, K_0, K_1, K_2, K_3, K_4, C_v, \mu, \rho_1$  such that*

$$\|\phi\|_{L^\infty} \leq C \text{ and } F \leq C \text{ and } F \geq K_2\rho_2 - C,$$

where  $\rho_2$  is a quasi-plurisubharmonic function with analytic singularities, such that  $\omega + dd^c \rho_2$  is a Kähler current with  $\mathrm{Ric}(\mu) \leq K_2(\omega + dd^c \rho_2)$ .

Moreover, for any compact set  $K \subset X \setminus \{\rho_2 = -\infty\}$  and  $\ell \in \mathbb{N}$ , we have  $\|\varphi_\varepsilon\|_{C^\ell(K, \omega_X)} \leq C_{K, \ell}$  for some uniform constant  $C_{K, \ell} > 0$ , where  $\omega_X$  is a fixed Kähler metric on  $X$ .

### 3. Existence of singular wscck metrics

We shall combine the a priori estimates obtained in the previous section and the uniform coercivity established in [BJT24] to construct singular weighted cscK metrics.

**3.1. Setup and Theorem B.** — In the sequel, we always assume the following setting:

**Setting 3.1.** — Let  $X$  be an  $n$ -dimensional compact Kähler variety with log terminal singularities. Fix a compact torus  $T \subset \mathrm{Aut}_{\mathrm{red}}(X)$  and denote by  $\mathfrak{t} = \mathrm{Lie}(T)$ . Assume that  $X$  admits a  $T$ -equivariant resolution of singularities  $\pi : Y \rightarrow X$  with  $K_Y = \pi^* K_X + \sum_i a_i E_i$ ,  $a_i > -1$ ,  $Y$  is Kähler, and  $\pi$  is an isomorphism over  $X^{\mathrm{reg}}$ . Given a  $T$ -invariant Kähler metric  $\omega$  on  $X$ , by [Bou04, Thm. 3.17], there exists a  $T$ -invariant quasi-psh function  $\rho$  that is smooth on  $Y \setminus E$  and has analytic singularities along  $E$  such that  $\pi^* \omega + dd^c \rho$  is a Kähler current.



Let  $\omega_Y$  be a  $T$ -invariant Kähler metric on  $Y$ . For  $\varepsilon \in (0, 1]$ , we denote by  $\omega_\varepsilon := \pi^*\omega + \varepsilon\omega_Y$  and  $P_\varepsilon := \text{im}(m_{\omega_\varepsilon}) = \text{im}(m_{\pi^*\omega}) + \varepsilon \text{im}(m_{\omega_Y})$  which contained in a compact set of  $\mathfrak{t}^\vee$  for all  $\varepsilon$  small. Let  $v, w \in \mathcal{C}^\infty(\mathfrak{t}^\vee, \mathbb{R})$  with  $v > 0$  on  $P_\omega$ . Since  $P_\omega$  is compact, there are constants  $C_v, C_w \geq 1$  such that for any  $\varepsilon$  sufficiently small, on  $P_\varepsilon$ ,

$$C_v^{-1} \leq v + \sum_\alpha |v_\alpha| + \sum_{\alpha, \beta} |v_{\alpha\beta}| \leq C_v, \quad \text{and} \quad C_w^{-1} \leq |w| + \sum_\alpha |w_\alpha| + \sum_{\alpha, \beta} |w_{\alpha\beta}| \leq C_w.$$

Denote by  $\ell_\varepsilon^{\text{ext}}$  (reps.  $\ell^{\text{ext}}$ ) the extremal affine function on  $\mathfrak{t}^\vee$  associated to  $\omega_\varepsilon$  (resp.  $\omega$ ). Since  $\omega_\varepsilon$  converges smoothly to  $\pi^*\omega$ , the moment map  $m_{\omega_\varepsilon}$  converges smoothly to  $m_{\pi^*\omega}$  and  $\ell_\varepsilon^{\text{ext}} \rightarrow \ell^{\text{ext}}$  in  $\mathfrak{t} \oplus \mathbb{R}$  (cf. [BJT24, Lem. 4.18]).

Assume that a smooth pair  $(\varphi_\varepsilon, F_\varepsilon)$  solves

$$\begin{cases} v(m_{\varphi_\varepsilon})(\omega_\varepsilon + dd^c \varphi_\varepsilon)^n = e^{F_\varepsilon} \omega_X^n, & \sup_Y \varphi_\varepsilon = 0, \\ \Delta_{\varphi_\varepsilon, v} F_\varepsilon = -(w \ell_\varepsilon^{\text{ext}})(m_{\varphi_\varepsilon}) + \text{tr}_{\varphi_\varepsilon, v}(\text{Ric}(\omega_Y)). \end{cases} \quad (3.1)$$

Under Setting 3.1, one can conclude Theorem B by applying Theorem 2.4.

**3.2. Proof of Theorem C.** — We now prove the following result on the existence of singular weighted cscK metrics.

**Theorem 3.2.** — *Under Setting 3.1, suppose that Condition (A) is fulfilled and  $\log v$  is concave. If the weighted Mabuchi functional  $\mathbf{M}_{\omega, v, w \ell^{\text{ext}}}$  is coercive, then  $X$  admits a singular  $(v, w)$ -extremal metric (i.e.  $(v, w \ell^{\text{ext}})$ -cscK metric) in  $\{\omega\}$ , and it is also a minimizer of  $\mathbf{M}_{\omega, v, w \ell^{\text{ext}}}$ .*

*Proof.* — **Part 1: existence.** Since  $\mathbf{M}_X^{\text{rel}} := \mathbf{M}_{\omega, v, w \ell^{\text{ext}}}$  is coercive, there are positive constants  $A_0$  and  $B_0$  such that

$$\mathbf{M}_X^{\text{rel}} \geq A_0(-\mathbf{E}_{\omega, T}) - B_0 \quad \text{on } \mathcal{E}_{\text{norm}}^1(X, \omega)^T.$$

By [BJT24, Thm. A], for any  $A \in (0, A_0)$ , there exists  $B > 0$  such that for all sufficiently small  $\varepsilon > 0$ ,

$$\mathbf{M}_\varepsilon^{\text{rel}} \geq A(-\mathbf{E}_{\omega_\varepsilon, T}) - B \quad \text{on } \mathcal{E}_{\text{norm}}^1(Y, \omega_\varepsilon)^T, \quad (3.2)$$

where

$$\mathbf{M}_\varepsilon^{\text{rel}} := \mathbf{M}_{\omega_\varepsilon, v, w \ell_\varepsilon^{\text{ext}}} = \mathbf{H}_{\varepsilon, v, \omega_Y^n} + \mathbf{E}_{\varepsilon, v}^{-\text{Ric}(\omega_Y)} + \mathbf{E}_{\varepsilon, v, w \ell_\varepsilon^{\text{ext}}}$$

the weighted Mabuchi functional on  $(Y, \omega_\varepsilon)$ . By the existence result obtained in [DJL25, HL24], there is a smooth pair  $(\varphi_\varepsilon, F_\varepsilon)$  solving

$$\begin{cases} v(m_{\varphi_\varepsilon})(\omega_\varepsilon + dd^c \varphi_\varepsilon)^n = e^{F_\varepsilon} \omega_X^n, & \sup_Y \varphi_\varepsilon = 0 \\ \Delta_{\varphi_\varepsilon, v} F_\varepsilon = -(w \ell_\varepsilon^{\text{ext}})(m_{\varphi_\varepsilon}) + \text{tr}_{\varphi_\varepsilon, v}(\text{Ric}(\omega_X)) \end{cases}$$

Namely,  $\omega_{\varepsilon, \varphi_\varepsilon}$  is a  $(v, w \ell_\varepsilon^{\text{ext}})$ -cscK metric and  $\varphi_\varepsilon$  minimizes  $\mathbf{M}_\varepsilon^{\text{rel}}$ . One can also assume that  $\mathbf{E}_{\omega_\varepsilon, T}(\varphi_\varepsilon) = \mathbf{E}_{\omega_\varepsilon}(\varphi_\varepsilon)$ .

Recall that  $\pi^*\omega + dd^c \rho \geq C_\rho \omega_Y$ . We shall verify the assumptions in Theorem 2.1 with  $\rho_2 = \rho$  and  $\mu = \omega_Y^n$ . It is obvious that the condition (3) holds. We now show that the uniform control on the entropies  $\mathbf{H}_\varepsilon(\varphi_\varepsilon)$ , which will confirm the condition (2).

If a closed  $(1, 1)$ -current  $\Theta \geq -A\pi^*\omega$  for some constant  $A > 0$ , then

$$\mathbf{E}_{\varepsilon, v}^\Theta(\psi) \leq C(A, v, \Theta)(-\mathbf{E}_\varepsilon(\psi) + 1). \quad (3.3)$$

for any  $\varepsilon \in (0, 1]$  and  $0 \geq \psi \in \text{PSH}(Y, \omega_\varepsilon) \cap L^\infty(Y)$  (c.f. [BJT24, Lem. 4.22]). By Condition (A) and the strong openness [Ber13, GZ15], there is a constant  $a > 1$  such that  $\int_Y e^{-aK_1 \rho_1} \omega_Y^n < +\infty$ . Set  $\mu_Y := e^{-aK_1 \rho_1} \omega_Y^n$ . Similar to [BJT24, Sec. 4.4], we consider

$$\widehat{\mathbf{H}}_{\varepsilon, v}(\psi) := \mathbf{H}_{\mu_Y}(\text{MA}_{\varepsilon, v}(\psi)) \quad \text{and} \quad \widehat{\text{Ric}} := \text{Ric}(\omega_Y) + K_1 dd^c \rho_1,$$

then  $\widehat{\text{Ric}} \geq -K_1 \pi^*\omega \geq -K_1 \omega_\varepsilon$  by assumption. It follows from [BJT24, Lem. A.2] and [BJT24, Prop. 3.44] that

$$\frac{1}{a} \widehat{\mathbf{H}}_{\varepsilon, v}(\psi) = \frac{1}{a} \mathbf{H}_{\varepsilon, v}(\psi) + \int_Y K_1 \rho_1 \text{MA}_{\varepsilon, v}(\psi) \quad (3.4)$$

and

$$\mathbf{E}_{\varepsilon, v}^{-\widehat{\text{Ric}}}(\psi) = \mathbf{E}_{\varepsilon, v}^{-\text{Ric}(\omega_Y)}(\psi) - \int_Y K_1 \rho_1 \text{MA}_{\varepsilon, v}(\psi) + \int_Y K_1 \rho_1 \text{MA}_{\varepsilon, v}(0). \quad (3.5)$$

Therefore, we get

$$\frac{1}{a} \mathbf{H}_{\varepsilon, v, \omega_Y^n}(\psi) + \mathbf{E}_{\varepsilon, v}^{-\text{Ric}(\omega_Y)}(\psi) = \frac{1}{a} \widehat{\mathbf{H}}_{\varepsilon, v}(\psi) + \mathbf{E}_{\varepsilon, v}^{-\widehat{\text{Ric}}}(\psi) - \int_Y K_1 \rho_1 \text{MA}_{\varepsilon, v}(0), \quad (3.6)$$

with  $\int_Y K_1 \rho_1 \text{MA}_{\varepsilon, v}(0) \rightarrow \int_Y K_1 \rho_1 v(m_{\pi^* \omega})(\pi^* \omega)^n$  as  $\varepsilon \rightarrow 0$ . Here without loss of generality, we assume that  $\int_Y \rho_1 v(m_{\pi^* \omega})(\pi^* \omega)^n = 0$ . Then one can derive

$$\begin{aligned} \mathbf{M}_{\varepsilon}^{\text{rel}}(\psi) &= \mathbf{H}_{\varepsilon, v, \omega_Y^n}(\psi) + \mathbf{E}_{\varepsilon, v}^{-\text{Ric}(\omega_Y)}(\psi) + \mathbf{E}_{\varepsilon, v}^{w\ell^{\text{ext}}}(\psi) \\ &\geq \left(1 - \frac{1}{a}\right) \mathbf{H}_{\varepsilon, v, \omega_Y^n}(\psi) + \mathbf{E}_{\varepsilon, v}^{-\widehat{\text{Ric}}}(\psi) + \mathbf{E}_{\varepsilon, v}^{w\ell^{\text{ext}}}(\psi) - C. \end{aligned} \quad (3.7)$$

From Lemma 1.5,  $\varphi_{\varepsilon}$  is a minimizer of the Mabuchi functional  $\mathbf{M}_{\varepsilon}^{\text{rel}}$ ; hence,  $\mathbf{M}_{\varepsilon}^{\text{rel}}(\varphi_{\varepsilon}) \leq \mathbf{M}_{\varepsilon}^{\text{rel}}(0) \leq C_1$ . By (3.2),

$$-\mathbf{E}_{\varepsilon}(\varphi_{\varepsilon}) \leq \frac{C_1 + B}{A} =: D \quad \text{and} \quad -\mathbf{E}_{\varepsilon, v}^{w\ell^{\text{ext}}}(\varphi_{\varepsilon}) \leq D',$$

for all  $\varepsilon$  small, and then (3.3) and (3.7) imply that

$$\left(1 - \frac{1}{a}\right) \mathbf{H}_{\varepsilon, v, \omega_Y^n}(\varphi_{\varepsilon}) \leq C - \mathbf{E}_{\varepsilon, v}^{w\ell^{\text{ext}}}(\varphi_{\varepsilon}) + \mathbf{E}_{\varepsilon, v}^{-\widehat{\text{Ric}}}(\varphi_{\varepsilon}) \leq C_1 + D' + C(K_1, v, \widehat{\text{Ric}})D.$$

Therefore,  $\mathbf{H}_{\varepsilon, v, \omega_Y^n}(\varphi_{\varepsilon}) \leq C'$  for all sufficiently small  $\varepsilon > 0$  and for some uniform constant  $C' > 0$ . We have

$$\begin{aligned} |\mathbf{H}_{\varepsilon, v, \omega_Y^n}(\varphi_{\varepsilon}) - \mathbf{H}_{\varepsilon, \omega_Y^n}(\varphi_{\varepsilon})| &= \left| \int_Y \log \left( \frac{v(m_{\varphi_{\varepsilon}}) \omega_{\varepsilon, \varphi_{\varepsilon}}^n}{\omega_Y^n} \right) v(m_{\varphi_{\varepsilon}}) \omega_{\varepsilon, \varphi_{\varepsilon}}^n - \int_Y \log \left( \frac{\omega_{\varepsilon, \varphi_{\varepsilon}}^n}{\omega_Y^n} \right) \omega_{\varepsilon, \varphi_{\varepsilon}}^n \right| \\ &= \left| \int_Y \left(1 - \frac{1}{v(m_{\varphi_{\varepsilon}})}\right) \log \left( \frac{v(m_{\varphi_{\varepsilon}}) \omega_{\varepsilon, \varphi_{\varepsilon}}^n}{\omega_Y^n} \right) v(m_{\varphi_{\varepsilon}}) \omega_{\varepsilon, \varphi_{\varepsilon}}^n + \int_Y \log(v(m_{\varphi_{\varepsilon}})) \omega_{\varepsilon, \varphi_{\varepsilon}}^n \right| \leq C'_v C' + C'_v, \end{aligned}$$

where  $C'_v > 0$  depend only on  $C_v$ ; hence, we obtain a uniform upper bound for  $\mathbf{H}_{\varepsilon, \omega_Y^n}(\varphi_{\varepsilon})$  as required.

By Theorem 2.4, we obtained uniform  $L^{\infty}$  and local  $\mathcal{C}^{\ell}$ -estimates for  $\varphi_{\varepsilon}$ . The Arzelà–Ascoli theorem shows that there is a subsequence  $\varphi_{\varepsilon_j}$  converging in  $\mathcal{C}_{\text{loc}}^{\infty}(Y \setminus E)$  to  $\varphi_0 \in \text{PSH}(Y, \pi^* \omega) \cap L^{\infty}(Y) \cap \mathcal{C}^{\infty}(Y \setminus E)$  which satisfies  $(v, w\ell^{\text{ext}})$ -cscK equations on  $Y \setminus E$ . This deduces the existence of a singular weighted cscK metric  $\omega + dd^c \psi_0$  in  $\{\omega\}$  on  $X$ , where  $\psi_0 \in \text{PSH}(X, \omega) \cap L^{\infty}(X) \cap \mathcal{C}^{\infty}(X^{\text{reg}})$  and  $\varphi_0 = \pi^* \psi_0$ .

**Part 2: Minimizer.** It remains to show that  $\psi_0$  is a minimizer of  $\mathbf{M}_X^{\text{rel}}$  on  $\mathcal{E}_{\omega}^{1, T}$ . Note that  $\varphi_{\varepsilon}$  are uniformly bounded and converging locally smoothly to  $\pi^* \psi_0$  on  $Y \setminus E$ . One can derive  $\mathbf{E}_{\varepsilon, v}^{w\ell^{\text{ext}}}(\varphi_{\varepsilon}) \rightarrow \mathbf{E}_{v}^{w\ell^{\text{ext}}}(\pi^* \psi_0)$  as  $\varepsilon \rightarrow 0$ . Thus, by (3.6) with  $a = 1$  and by the semi-continuity with respect to strong convergence of  $\widehat{\mathbf{H}}_{\varepsilon, v}$  and  $\mathbf{E}_{\varepsilon, v}^{-\widehat{\text{Ric}}}$  (cf. [BJT24, Lem. 4.22]), one gets

$$\liminf_{\varepsilon \rightarrow 0} \mathbf{M}_{\varepsilon}^{\text{rel}}(\varphi_{\varepsilon}) \geq \widehat{\mathbf{H}}_v(\pi^* \psi_0) + \mathbf{E}_v^{-\widehat{\text{Ric}}}(\pi^* \psi_0) + \mathbf{E}_v^{w\ell^{\text{ext}}}(\pi^* \psi_0),$$

where we used the fact that  $\int_Y K_1 \rho_1 \text{MA}_{\varepsilon, v}(0) \rightarrow \int_Y K_1 \rho_1 v(m_{\pi^* \omega})(\pi^* \omega)^n = 0$ . It follows from [BJT24, Lem. 4.26] that

$$\widehat{\mathbf{H}}_v(\pi^* \psi_0) + \mathbf{E}_v^{-\widehat{\text{Ric}}}(\pi^* \psi_0) \geq \mathbf{H}_{\mu_X}(\psi_0) + \mathbf{E}_v^{-\text{Ric}(\mu_X)}(\psi_0).$$

This shows that

$$\liminf_{\varepsilon \rightarrow 0} \mathbf{M}_{\varepsilon}^{\text{rel}}(\varphi_{\varepsilon}) \geq \mathbf{M}_X^{\text{rel}}(\psi_0). \quad (3.8)$$

Fix an arbitrary  $u \in \mathcal{E}_{\omega}^{1, T}$ . Without loss of generality, assume that  $\mathbf{H}_{v, \mu_X}(u) < +\infty$  and denote by  $f = \omega_u^n / \omega^n$  which is  $T$ -invariant. By [PTT23, Lem. 3.4], there exists  $0 \leq f^j \in \mathcal{C}^{\infty}(X)$  converging to  $f$  in  $L^X$  as  $j \rightarrow 0$  with  $\chi(s) := (s+1) \log(s+1) - s$ . Consider  $f^{T, j}(x) := \int_T f^j(\sigma \cdot x) d\mu(\sigma)$ ,

where  $\mu$  is the Haar measure on  $T$  normalized by  $\int_T d\mu(\sigma) = 1$ . We claim that  $f^{T,j} \in \mathcal{C}^\infty(X)$  also converges to  $f$  in  $L^\chi$ . Indeed, since  $\chi$  is convex,

$$\chi(|f^{T,j} - f|(x)) = \chi\left(\left|\int_T (f^{T,j} - f)(\sigma \cdot x) d\mu(\sigma)\right|\right) \leq \int_T \chi(|f^j - f|(\sigma \cdot x)) d\mu(\sigma).$$

By Fubini's theorem and  $T$ -invariance of  $\omega$ , we obtain that

$$\begin{aligned} \int_{x \in X} \chi(|f^{j,T} - f|(x)) \omega^n(x) &\leq \int_{x \in X} \left( \int_{\sigma \in T} \chi(|f^j - f|(\sigma \cdot x)) d\mu(\sigma) \right) \omega^n(x) \\ &= \int_{\sigma \in T} \left( \int_{x \in X} \chi(|f^j - f|(\sigma \cdot x)) \omega^n(x) \right) d\mu(\sigma) = \int_{\sigma \in T} \left( \int_{x \in X} \chi(|f^j - f|(\sigma \cdot x)) \omega^n(\sigma \cdot x) \right) d\mu(\sigma). \end{aligned}$$

Hence,  $f^{T,j}$  converges to  $f$  in  $L^\chi$ . Consider  $u_j \in \mathcal{C}^\infty(X^{\text{reg}}) \cap L^\infty(X)$  solving

$$(\omega + dd^c u_j)^n = c_j f^{T,j} \omega^n \quad \text{with} \quad \sup_X u_j = 0$$

where  $c_j > 0$  is a normalizing constant. Moreover, it follows from [CC24, Thm. 1.3] that  $u_j$  is continuous on  $X$ . By [PTT23, Lem. 3.4],  $u_j$  converges strongly to  $u$ , and  $\mathbf{H}_{\mu_X}(u_j) \rightarrow \mathbf{H}_{\mu_X}(u)$ ; thus,  $\mathbf{M}_{v,\omega^{\text{ext}}}(u_j) \rightarrow \mathbf{M}_{v,\omega^{\text{ext}}}(u)$ .

Then for each  $j$  fixed, one can find a family of functions  $u_{j,\varepsilon} \in \text{PSH}(Y, \omega_\varepsilon)$  such that

- $(u_{j,\varepsilon})_{\varepsilon \in (0,1)}$  are uniformly bounded and continuous;
- $u_{j,\varepsilon}$  converges locally smoothly on  $Y \setminus E$  as  $\varepsilon \rightarrow 0$ ;
- $u_{j,\varepsilon}$  decreases to  $\pi^* u_j$  as  $\varepsilon \rightarrow 0$ .

Since  $\pi^* u_j$  is continuous on  $Y$ , Dini's theorem implies  $u_{j,\varepsilon}$  converges uniformly to  $\pi^* u_j$  as  $\varepsilon \rightarrow 0$ . Indeed, for a fixed  $j$ , consider  $u_{j,\varepsilon}$  the unique solution to the following equation

$$(\omega_\varepsilon + dd^c u_{j,\varepsilon})^n = e^{u_{j,\varepsilon} - \pi^* u_j} \pi^* (c_j f^{T,j} \omega^n).$$

Then by [EGZ09], one has a uniform  $\mathcal{C}^0$ -estimate for  $u_{j,\varepsilon}$  on  $Y$ , local  $\mathcal{C}^2$ -estimate for  $u_{j,\varepsilon}$  away from  $E$ , and  $u_{j,\varepsilon}$  converges locally smoothly to  $\pi^* u_j$  in  $Y \setminus E$ . We now check that  $u_{j,\varepsilon}$  is decreasing as  $\varepsilon \rightarrow 0$ . For  $0 < \varepsilon_1 < \varepsilon_2$ , we have

$$(\omega_{\varepsilon_2} + dd^c u_{j,\varepsilon_1})^n \geq (\omega_{\varepsilon_1} + dd^c u_{j,\varepsilon_1})^n = e^{u_{j,\varepsilon_1} - \pi^* u_j} \pi^* (c_j f^{T,j} \omega^n),$$

so  $u_{j,\varepsilon_1}$  is a subsolution to the equation

$$(\omega_{\varepsilon_2} + dd^c \varphi) = e^{\varphi - \pi^* u_j} \pi^* (c_j f^{T,j} \omega^n).$$

Thus, we obtain that  $u_{j,\varepsilon_1} \leq u_{j,\varepsilon_2}$  for any  $0 < \varepsilon_1 < \varepsilon_2$ .

Combining with (3.8), we get

$$\mathbf{M}_X^{\text{rel}}(\psi_0) \leq \liminf_{\varepsilon \rightarrow 0} \mathbf{M}_\varepsilon^{\text{rel}}(\varphi_\varepsilon) \leq \liminf_{\varepsilon \rightarrow 0} \mathbf{M}_\varepsilon^{\text{rel}}(u_{\varepsilon,j}) = \mathbf{M}_X^{\text{rel}}(u_j),$$

where the second inequality follows from the fact that  $\varphi_\varepsilon$  is a minimizer for  $\mathbf{M}_{\varepsilon,v,\omega^{\text{ext}}}$  and the last equality follows from Lemma 3.3 below. Letting  $j \rightarrow +\infty$ , we obtain that  $\mathbf{M}_{v,\omega^{\text{ext}}}(\psi_0) \leq \mathbf{M}_{v,\omega^{\text{ext}}}(u)$  for arbitrary  $u \in \mathcal{E}_\omega^{1,T}$  and this finishes the proof.  $\square$

**Lemma 3.3.** — Consider  $u \in \text{PSH}(X, \omega)^T \cap \mathcal{C}^0(X)$  such that  $u$  is smooth on  $X^{\text{reg}}$  and  $\mathbf{H}_{\mu_X}(u) < +\infty$ . Assume that  $u_\varepsilon \in \text{PSH}(Y, \omega_\varepsilon)^T \cap \mathcal{C}^0(Y) \cap \mathcal{C}^\infty(Y \setminus E)$  converges smoothly to  $\pi^* u$  in  $Y \setminus E$ , uniformly on  $Y$  and  $\mathbf{H}_{\varepsilon,\omega_Y^n}(u_\varepsilon) \rightarrow \mathbf{H}_{\omega_Y^n}(\pi^* u)$  as  $\varepsilon \rightarrow 0$ . Then  $\mathbf{M}_\varepsilon^{\text{rel}}(u_\varepsilon) \rightarrow \mathbf{M}_X^{\text{rel}}(u)$  as  $\varepsilon \rightarrow 0$ .

*Proof.* — The proof follows a similar approach to that of [BJT24, Prop. 4.16] where the authors assume  $u$  to be smooth on  $X$  instead of  $X^{\text{reg}}$ . Here, we only provide a brief outline and indicate the necessary modifications.

Set  $D = K_Y - \pi^* K_X = \sum_i a_i E_i$  and define  $\rho_{Y/X} = \log \frac{\pi^* \mu_X}{\omega_Y^n}$ . Then we have

$$[D] - dd^c \rho_{Y/X} = \pi^* \text{Ric}(\mu_X) - \text{Ric}(\omega_Y^n).$$

For  $u \in \text{PSH}(X, \omega) \cap L^\infty(X)$  and any smooth  $(1, 1)$ -form  $\theta$  (or positive closed  $(1, 1)$ -current),

$$\mathbf{E}_{\pi^*\omega}^\theta(\pi^*u) := \sum_{j=0}^{n-1} \int_Y (\pi^*u)\theta \wedge (\pi^*\omega + dd^c \pi^*u)^j \wedge (\pi^*\omega)^{n-1-j}.$$

Note that  $\mathbf{E}_{\pi^*\omega}^{[D]}(\pi^*u) = 0$  for any  $u \in \text{PSH}(X, \omega) \cap L^\infty(X)$ . Following the same argument in [BJT24, Lem. 4.15], one can infer that for any  $u \in \text{PSH}(X, \omega) \cap L^\infty(X)$  and smooth on  $X^{\text{reg}}$  we have

$$(\mathbf{H}_{\omega_Y^n, v}(\pi^*u) + \mathbf{E}_{\pi^*\omega}^{-\text{Ric}(\omega_Y)}(\pi^*u)) - (\mathbf{H}_{\mu_X, v}(u) + \mathbf{E}_\omega^{-\text{Ric}(\mu_X)}(u)) = \mathbf{E}_{\pi^*\omega}^{[D]}(\pi^*u) = 0;$$

therefore,

$$\mathbf{H}_{\omega_Y^n, v}(\pi^*u) + \mathbf{E}_{\pi^*\omega}^{-\text{Ric}(\omega_Y)}(\pi^*u) = \mathbf{H}_{\mu_X, v}(u) + \mathbf{E}_\omega^{-\text{Ric}(\mu_X)}(u). \quad (3.9)$$

Since  $u_\varepsilon$  converges uniformly to  $\pi^*u$ , we have  $\mathbf{E}_{\omega_\varepsilon}^{-\text{Ric}(\omega_Y)}(u_\varepsilon) \rightarrow \mathbf{E}_{\pi^*\omega}^{-\text{Ric}(\omega_Y)}(\pi^*u)$  as  $\varepsilon \rightarrow 0$ . Moreover, since  $\ell_\varepsilon^{\text{ext}} \rightarrow \ell^{\text{ext}}$  in  $\mathfrak{t} \oplus \mathbb{R}$ , we have  $\mathbf{E}_{\varepsilon, v\omega \ell^{\text{ext}}}(u_\varepsilon) \rightarrow \mathbf{E}_{\omega, v\omega \ell^{\text{ext}}}(u)$ . The hypothesis  $\mathbf{H}_{\varepsilon, \omega_Y^n}(u_\varepsilon) \rightarrow \mathbf{H}_{\omega_Y^n}(\pi^*u)$  implies  $\mathbf{H}_{\varepsilon, \omega_Y^n, v}(u_\varepsilon) \rightarrow \mathbf{H}_{\omega_Y^n, v}(\pi^*u)$  by generalized dominated convergent theorem and  $u_\varepsilon \rightarrow \pi^*u$  locally smoothly on  $Y \setminus E$ . All in all, these yield

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \mathbf{M}_\varepsilon^{\text{rel}}(u_\varepsilon) &= \mathbf{H}_{\omega_Y^n, v}(\pi^*u) + \mathbf{E}_{\pi^*\omega}^{-\text{Ric}(\omega_Y)}(\pi^*u) + \mathbf{E}_{\omega, v\omega \ell^{\text{ext}}}(u) \\ &= \mathbf{H}_{\mu_X, v}(u) + \mathbf{E}_\omega^{-\text{Ric}(\mu_X)}(u) + \mathbf{E}_{\omega, v\omega \ell^{\text{ext}}}(u) = \mathbf{M}_X^{\text{rel}}(u), \end{aligned}$$

as required, where the second equality follows from (3.9).  $\square$

**3.2.1. Singular cscK and extremal metrics.** — We consider the problem of finding a singular cscK metric. In this case  $v = 1$  and  $w_0 = \bar{s}_\omega$ , where  $\bar{s}_\omega := \frac{n}{V_\omega} c_1(X) \cdot [\omega]^{n-1}$ .

*Proof of Theorem A.* — **Case 1:**  $\text{Aut}_0(X) = \{\text{Id}\}$ . In this case, we have  $\ell^{\text{ext}} = 1$ , and the result follows straightforwardly from Theorem 3.2.

**Case 2:**  $\text{Aut}_0(X) \neq \{\text{Id}\}$ . Take  $T \subset \text{Aut}_{\text{red}}(X)$  to be a maximal torus and  $\pi : Y \rightarrow X$  to be the  $T$ -equivariant resolution of singularities in Condition (A). Since in our case  $\mathbf{M}_\omega = \mathbf{M}_{v, w_0}$  is  $T_{\mathbb{C}}$ -coercive, it is bounded from below on  $\mathcal{E}^{1, T}$ , so  $\mathbf{M}_\omega$  is  $T_{\mathbb{C}}$ -invariant (cf. [CC21b, Lem. 4.11]). Therefore, we get  $\ell^{\text{ext}} = 1$ . The existence of a singular cscK metric on  $X$  now follows from Theorem 3.2.  $\square$

Let  $X$  be a normal compact Kähler variety with log terminal singularities. Fix a Kähler class  $\alpha$  and take a maximal torus  $T \subset \text{Aut}_{\text{red}}(X)$ . Let  $\pi : Y \rightarrow X$  be a  $T$ -equivariant resolution of singularities. We now consider the problem of the existence of singular extremal metrics in the Kähler class  $\alpha$ .

Let  $\xi_{\text{ext}}$  be the extremal vector field defined by  $T$  and  $\alpha$  (cf. Section 1.2.6) and let  $\omega \in \alpha$  be a  $T$ -equivariant Kähler metric. A singular extremal metric in this setting is defined as a positive current of the form  $\pi^*\omega + dd^c \varphi$  where  $\varphi \in \text{PSH}(Y, \pi^*\omega) \cap L^\infty(Y)$  and  $\varphi$  is smooth away from  $\text{Exc}(\pi)$ . Additionally,  $\pi^*\omega + dd^c \varphi$  is a genuine  $(1, w_{\text{ext}})$ -cscK metric on  $Y \setminus \text{Exc}(\pi)$  where  $w_{\text{ext}}(p) = \langle \xi_{\text{ext}}, p \rangle + c_{\text{ext}}$  and  $c_{\text{ext}}$  is a constant depending only on  $\alpha$  and  $T$ .

Under Condition (A), Theorem C implies the following existence result of singular extremal metrics:

**Theorem 3.4.** — *Under the above setting, moreover, assume that  $X$  satisfies Condition (A). If the weighted Mabuchi functional  $\mathbf{M}_{\omega, v, w_{\text{ext}}}$  on  $X$  is  $T_{\mathbb{C}}$ -coercive, then  $X$  admits a singular extremal metric in  $\alpha$ .*

**3.3. Constructing examples of singular cscK metrics.** — We shall give a way to construct examples of singular cscK metric in the spirit of Arezzo–Pacard [AP06] and use variational argument and our existence result. Before illustrating the process, we need the following lemma:

**Lemma 3.5.** — *Let  $(X, \omega)$  be a compact Kähler variety with log terminal singularities and let  $f : Y \rightarrow X$  be a blowup along a compact submanifold  $S \subset X^{\text{reg}}$  of codimension  $\geq 2$ . Consider  $\omega_Y$  a Kähler metric on  $Y$  and  $\omega_\varepsilon = f^*\omega + \varepsilon\omega_Y$  for  $\varepsilon \in [0, 1]$ . If  $\mathbf{M}_\omega$  is coercive, then  $\mathbf{M}_{\omega_\varepsilon}$  is coercive for all sufficiently small  $\varepsilon > 0$ .*

*Proof.* — The proof follows the same strategy in [PTT23, Thm. 4.11] and [BJT24, Thm. A]. Since  $f$  is a blowup of  $S$ , for  $m \in \mathbb{N}^*$  so that  $mK_Y$  and  $mK_X$  are both  $\mathbb{Q}$ -Cartier, there is a smooth hermitain metric  $h_Y$  of  $mK_Y$  such that

$$-\frac{1}{m}\Theta(mK_Y, h_Y) \geq -Df^*\omega \quad (3.10)$$

for some constant  $D \geq 0$ . Denote by  $\Theta_Y = \frac{1}{m}\Theta(mK_Y, h_Y)$  and  $\Theta_X = \frac{1}{m}\Theta(mK_X, h_X)$  for some hermitian metric  $h_X$  on  $mK_X$ . We also set  $\mu_Y$  (resp.  $\mu_X$ ) to be the corresponding probability measure of  $h_Y$  (resp.  $h_X$ ).

Recall that from [PTT23, Sec. 4],

$$\mathbf{M}_{\omega_\varepsilon} = \mathbf{H}_{\mu_Y, \omega_\varepsilon} + \bar{s}_\varepsilon \mathbf{E}_{\omega_\varepsilon} - n\mathbf{E}_{\Theta_Y, \omega_\varepsilon} - C_\varepsilon \quad \text{on } \mathcal{E}^1(Y, \omega_\varepsilon)$$

and

$$\mathbf{M}_\omega = \mathbf{H}_{\mu_X, \omega} + \bar{s}\mathbf{E}_\omega - n\mathbf{E}_{\Theta_X, \omega} - C \quad \text{on } \mathcal{E}^1(X, \omega)$$

where  $\bar{s}_\varepsilon = \frac{nc_1(Y) \cdot [\omega_\varepsilon]^{n-1}}{[\omega_\varepsilon]^n}$ ,  $\bar{s} = \frac{nc_1(X) \cdot [\omega]^{n-1}}{[\omega]^n}$ ,  $C_\varepsilon = \mathbf{H}_{\mu_Y, \omega_\varepsilon}(0) = \int_Y \log\left(\frac{\omega_\varepsilon^n}{\mu_Y}\right) \mu_Y$  and  $C = \mathbf{H}_{\mu_X, \omega}(0) = \int_X \log\left(\frac{\omega^n}{\mu_X}\right) \mu_X$ . Let  $A_0, B_0 > 0$  be two constant such that  $\mathbf{M}_\omega \geq A_0(-\mathbf{E}_\omega) - B_0$  on  $\mathcal{E}_{\text{norm}}^1(X, \omega)$ . We claim that for all  $0 < A < A_0$  there are  $\varepsilon_0 > 0$  and  $B > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$ ,

$$\mathbf{M}_{\omega_\varepsilon} \geq A(-\mathbf{E}_{\omega_\varepsilon}) - B$$

s on  $\mathcal{E}_{\text{norm}}^1(Y, \omega_\varepsilon)$ .

Suppose otherwise, for an  $A \in (0, A_0)$ , there are  $\varepsilon_k \rightarrow 0$ ,  $B_k \rightarrow +\infty$ , as  $k \rightarrow +\infty$ , and  $u_k \in \mathcal{E}_{\text{norm}}^1(Y, \omega_{\varepsilon_k})$  such that

$$\mathbf{M}_{\omega_{\varepsilon_k}}(u_k) < A(-\mathbf{E}_{\omega_{\varepsilon_k}}(u_k)) - B_k.$$

Without loss of generality, one can assume that  $u_k$  is bounded. Otherwise, by [PTT23, Lem. 3.4], one can find a sequence bounded  $\omega_k$ -psh functions  $(v_{k,j})_j$  converging strongly to  $u_k$  and their entropies also converge to  $\mathbf{H}_{\mu_Y, \omega_{\varepsilon_k}}(u_k)$ . Then for sufficiently large  $j$ , we have  $\mathbf{M}_{\omega_{\varepsilon_k}}(v_{k,j}) < A(-\mathbf{E}_{\omega_{\varepsilon_k}}(v_{k,j})) - B_k$ .

Note that from (3.10), for all  $\psi \in \mathcal{E}_{\text{norm}}^1(Y, \omega_\varepsilon)$ ,

$$\begin{aligned} -n\mathbf{E}_{\Theta_Y, \omega_\varepsilon}(\psi) &= -\frac{1}{V_\varepsilon} \sum_{j=0}^{n-1} \int_Y \psi \Theta_Y \wedge \omega_{\varepsilon, \psi}^j \wedge \omega_\varepsilon^{n-1-j} \\ &\geq \frac{1}{V_\varepsilon} \sum_{j=0}^{n-1} \int_Y \psi Df^*\omega \wedge \omega_{\varepsilon, \psi}^j \wedge \omega_\varepsilon^{n-1-j} \geq (n+1)D\mathbf{E}_{\omega_\varepsilon}(\psi). \end{aligned}$$

Hence,

$$\mathbf{H}_{\mu_Y, \omega_\varepsilon}(u_k) + (\bar{s}_{\varepsilon_k} + (n+1)D)\mathbf{E}_{\omega_{\varepsilon_k}}(u_k) \leq \mathbf{M}_{\omega_{\varepsilon_k}}(u_k) < A(-\mathbf{E}_{\omega_{\varepsilon_k}}(u_k)) - B_k.$$

After enlarging  $D$ , one may assume that for all  $\varepsilon \in [0, 1]$ ,  $(n+1)D + A - \bar{s}_\varepsilon > \delta$  for a uniform  $\delta > 0$ . Then we have  $B_k < ((n+1)D + A - \bar{s}_{\varepsilon_k})(-\mathbf{E}_{\omega_{\varepsilon_k}}(u_k))$  and this implies that  $d_k := -\mathbf{E}_{\omega_{\varepsilon_k}}(u_k) \rightarrow +\infty$  as  $k \rightarrow +\infty$ . Let  $g_k(s)$  be the  $d_1$ -geodesic connecting 0 and  $u_k$  in  $\mathcal{E}_{\text{norm}}^1(Y, \omega_{\varepsilon_k})$ . Fix a constant  $R > 0$  and define  $v_k := g_k(R)$ . By the convexity of Mabuchi functional [PTT23, Prop. 4.7],

$$\mathbf{M}_{\omega_{\varepsilon_k}}(v_k) \leq \frac{d_k - R}{d_k} \mathbf{M}_{\omega_{\varepsilon_k}}(0) + \frac{R}{d_k} \mathbf{M}_{\omega_{\varepsilon_k}}(u_k) \leq \frac{R}{d_k} (Ad_k - B_k) \leq AR.$$

From the expression of Mabuchi functional, we have  $\mathbf{H}_{\mu_Y, \omega_{\varepsilon_k}}(v_k) \leq ((n+1)D + A - \bar{s}_{\varepsilon_k})R$ .

In the argument below, although  $Y$  is singular, corresponding proofs in [BJT24] proceed exactly the same. By the strong compactness [BJT24, Thm. 2.10], up to a subsequence,  $v_k$  converges in  $v_0 \in \mathcal{E}_{\text{norm}}^1(Y, f^*\omega)$  and  $\mathbf{E}_{\omega_{\varepsilon_k}}(v_k) \rightarrow \mathbf{E}_{f^*\omega}(v_0)$  as  $k \rightarrow +\infty$ . Note that  $v$  can descend to a function in  $\mathcal{E}_{\text{norm}}^1(X, \omega)$ , which we still denote by  $v$ . By [BJT24, Lem. 4.16],  $\mathbf{H}_{\mu_X, \omega}(v) - n\mathbf{E}_{\Theta_X, \omega}(v) \leq \mathbf{H}_{\mu_Y, f^*\omega}(v) - n\mathbf{E}_{\Theta_Y, f^*\omega}(v)$  and [BJT24, Lem. 4.6 and Lem. 4.9] shows that  $\mathbf{M}_{\omega_0}(v_0) \leq \liminf_{k \rightarrow +\infty} \mathbf{M}_{\omega_{\varepsilon_k}}(v_k)$ . All in all, we obtain

$$A_0R - B_0 \leq \mathbf{M}_\omega(v_0) \leq \liminf_{k \rightarrow +\infty} \mathbf{M}_{\omega_{\varepsilon_k}}(v_k) \leq AR.$$

Letting  $R = \frac{B_0}{A_0 - A} + 1$ , this yields a contradiction.  $\square$

**3.3.1. Construct singular cscK on blowups of singular KEs.** — Let  $(X, \omega)$  be a compact Kähler variety with log-terminal singularity. Suppose that  $K_X$  is  $m$ -Cartier for some  $m \in \mathbb{N}^*$ , and pick a hermitian metric  $h$  on  $mK_X$ . Assume that either

- $K_X$  is ample and  $\omega = \frac{i}{m} \Theta(mK_X, h)$ ; or
- $K_X$  numerically trivial and  $\frac{i}{m} \Theta(mK_X, h) = 0$ ; or else
- $K_X$  is anti-ample,  $\omega = -\frac{i}{m} \Theta(mK_X, h)$ , and  $(X, K_X)$  is K-stable.

Note that in the above cases,  $\{\omega\}$  contains a unique singular Kähler–Einstein metric and the Mabuchi function with respect to  $\omega$  is coercive. We further assume that  $X$  admits a resolution of Fano type  $\pi : \widehat{X} \rightarrow X$ .

Let  $f : Y \rightarrow X$  be a blowup of  $N$  distinct points in the smooth locus of  $X$ . Denote the irreducible components of the exceptional divisor of  $f$  by  $(D_i)_i$ . Since  $f$  is an isomorphism near the singularities of  $Y$  and  $X$ , one can verify that  $Y$  also admits a resolution of Fano type.

For any  $a := (a_1, \dots, a_N) \in \mathbb{R}_{>0}^N$ , there exists a constant  $\delta_a > 0$ , such that for all  $\delta \in (0, \delta_a)$ , the class  $\{f^*\omega\} - \delta \sum_{i=1}^N a_i c_1(\mathcal{O}(D_i))$  contains a Kähler metric  $\omega_{Y,a,\delta}$ . By Lemma 3.5, the Mabuchi functional with respect to  $f^*\omega + \varepsilon \omega_{Y,a,\delta}$  is coercive for all sufficiently small  $\varepsilon$ . Theorem A then ensures the existence of a singular cscK metric in the class  $(1 + \varepsilon)\{f^*\omega\} - \varepsilon \delta \sum_{i=1}^N a_i c_1(\mathcal{O}(D_i))$  on  $Y$ .

If  $X$  is a Kähler variety with log terminal singularities,  $\text{Aut}(X)$  is discrete and it admits a crepant resolution  $\pi : Y \rightarrow X$ , it is not difficult to check that  $\pi : Y \rightarrow X$  satisfies Condition (A). In the case of Kähler–Einstein varieties with log terminal singularities and discrete automorphism groups that admit crepant resolutions, the above construction provides a method to produce numerous singular cscK metrics on their blowups at points within the smooth locus.

In dimension two, all surfaces with canonical singularities admit crepant resolutions. In dimension three, the famous result of [BKR01, Thm. 1.2] establishes that singular varieties locally modeled on  $\mathbb{C}^3/G$ , where  $G < \text{SL}(3, \mathbb{C})$  is finite, also admit crepant resolutions. Three-dimensional ODP singularity also admits a crepant resolution, and it is not a quotient singularity.

We also extract the following example from [Szé24, Rmk. 34] and [BJT24, Example 4.10] that is not a crepant resolution.

**Example 3.6 (Isolated cone singularities).** — Let  $V$  be a smooth projective variety and let  $L$  be an ample line bundle on  $V$ . Set  $C_a(V, L) := \text{Spec} \sum_{m \geq 0} H^0(V, L^m)$  the corresponding affine cone. Assuming  $K_V \sim_{\mathbb{Q}} r \cdot L$  for some  $r \in \mathbb{Q}$ , by [Kol13, Lem. 3.1],  $C_a(V, L)$  is klt if and only if  $r < 0$ . Moreover,  $C_a(V, L)$  is canonical if and only if  $r \leq -1$ . Therefore, one can choose  $r \in (-1, 0)$  to get a klt isolated singularity which is not canonical.

Assume that  $X$  has klt isolated singularities, and each singular point is locally isomorphic to an affine cone  $C_a(V, L)$  where  $V$  is a Fano manifold. Then blowing up the singularities yields a resolution of singularities such that  $\pi : Y \rightarrow X$  is projective. In this case,  $-K_Y = -\pi^*K_X - \sum_i a_i E_i$  is  $\pi$ -nef if and only if  $X$  has canonical singularities, i.e.  $a_i \geq 0$ . When  $-K_Y$  is  $\pi$ -nef, Condition (A) holds by choosing  $\rho_1 = 0$ .

We now consider a slightly more general situation that  $-K_Y$  is not  $\pi$ -nef and explain Condition (A) in this setting. Let  $\mu_X$  and  $\mu_Y$  be two smooth  $T$ -invariant volume forms on  $X$  and  $Y$ , respectively. Then define a  $T$ -invariant current  $\text{Ric}(\mu_Y) + dd^c \psi := \pi^* \text{Ric}(\mu_X) - \sum_i a_i [E_i]$ , where  $\psi = -\sum_i a_i \log |s_i|_{h_i}^2$  for some smooth Hermitian metric  $h_i$  on  $\mathcal{O}_X(E_i)$ . Since  $E_i$  are disjoint in our construction, one has  $\{\pi^*\omega\} - \sum_{a_i > 0} a_i c_1(\mathcal{O}_X(E_i)) > 0$ . Therefore, there exists a constant  $K > 0$  such that  $-\sum_{a_i > 0} a_i [E_i] \geq -K\{\pi^*\omega\} - \sum_{a_i > 0} dd^c a_i \log |s_i|_{h_i}^2$  for some smooth hermitian metrics  $h'_i$  on  $\mathcal{O}_X(E_i)$ . We then have  $\text{Ric}(\mu_Y) + dd^c(\psi + \sum_{a_i > 0} a_i \log |s_i|_{h'_i}^2) \geq -K\pi^*\omega$ . Set  $\phi := \psi + \sum_{a_i > 0} a_i \log |s_i|_{h'_i}^2 = u - \sum_{a_j < 0} a_j \log |s_j|_{h_j}^2$  for some  $u \in \mathcal{C}^\infty(Y)$ . One can obtain  $\int_X e^{-\phi} \mu_Y^n < +\infty$  as  $a_j > -1$ . Hence, Condition (A) holds by taking  $\rho_1 = \phi/K$ .

**3.3.2. Mixing construction with smoothing.** — Let us stress that this construction also works on the Q-Gorenstein smoothable setting. Consider a Q-Gorenstein smoothing  $f : (\mathcal{X}, \omega_{\mathcal{X}}) \rightarrow \mathbb{D}$  of  $(X_0, \omega_{\mathcal{X}|_{X_0}})$  where  $X_0$  is a Kähler–Einstein variety with log terminal singularities and  $\text{Aut}(X)$  is discrete, and  $\{\omega_{\mathcal{X}|_{X_0}}\}$  is a Kähler–Einstein class. Denote by  $\mathcal{Z}$  the singular set of  $\mathcal{X}$ . Take a finite set of points  $\{p_1, \dots, p_N\}$  in  $X_0^{\text{reg}}$ . There exists smooth curves  $C_1, \dots, C_N$  in  $\mathcal{X}$  such that these curves are disjoint, each  $C_i$  intersects  $X_0$  transversely at the single point  $p_i$ ,  $C_i \subset \mathcal{X} \setminus \mathcal{Z}$ , and the restriction  $f|_{C_i} : C_i \rightarrow \mathbb{D}$  is an isomorphism. Now consider the blowup map  $\mu : \mathcal{Y} \rightarrow \mathcal{X}$  along all the curves  $(C_i)_i$  and let  $Y_0 = \text{Bl}_{\{p_1, \dots, p_N\}} X$ . Then  $\pi = f \circ \mu : \mathcal{Y} \rightarrow \mathbb{D}$  is a Q-Gorenstein smoothing of  $(Y_0, \omega_{\mathcal{Y}, \varepsilon|_{Y_0}})$  where  $\omega_{\mathcal{Y}, \varepsilon} = \mu^* \omega_{\mathcal{X}} + \varepsilon \omega_{\mathcal{Y}}$ . Here  $\omega_{\mathcal{Y}}$  is a hermitian metric on  $\mathcal{Y}$  and relatively Kähler that defined by

$$\omega_{\mathcal{Y}} = \mu^* \omega_{\mathcal{X}} - \sum_{i=1}^N a_i \Theta_{h_i}(\mathcal{O}(E_i))$$

where  $E_i$  is the exceptional divisor over  $C_i$ ,  $a_i > 0$  and  $h_i$  is some hermitian metric on  $\mathcal{O}(E_i)$  such that  $\omega$ . Since the Mabuchi functional  $\mathbf{M}_{\omega_{\mathcal{X}|_{X_0}}}$  on  $X_0$  is coercive, it follows from Lemma 3.5  $\mathbf{M}_{\omega_{\mathcal{Y}, \varepsilon|_{Y_0}}}$  on  $Y_0$  as well. By [PTT23, Thm. C], this implies the existence of a cscK metric in the class  $\{\omega_{\mathcal{Y}, \varepsilon|_{Y_0}}\}$ .

For examples of smoothable Calabi–Yau varieties, we refer the reader to [DG18b, Sec. 8]. For the smoothable Fano case, [LX19, Liu22] prove that mildly singular cubic varieties in dimensions three and four are K-stable. Additionally, explicit examples of K-stable singular cubic threefolds can be found, for instance, in [CTZ24, Sec. 3, 4] and [CMTZ24, Sec. 5].

### Appendix A. Weighted Aubin–Yau inequality

In this section, for the reader’s convenience, we provide detailed proof of a Laplacian inequality (see also [DJL24, Lem. 5.6]), which generalizes [Siu87, p. 98–99] to the weighted setting.

**Lemma A.1.** — *Let  $\omega$  and  $\omega_X$  be two T-invariant Kähler metrics. Assume that  $\varphi \in \mathcal{H}_{\omega}^T$  satisfies*

$$v(m_{\varphi})(\omega + dd^c \varphi)^n = e^F \omega_X^n.$$

*Then there exist positive constants  $\mathcal{B}_0, \mathcal{C}_0$  such that*

$$\begin{aligned} \Delta_{\omega_{\varphi}, v} \log \text{tr}_{\omega_X} \omega_{\varphi} &\geq \frac{\Delta_{\omega_X, v} F}{\text{tr}_{\omega_X} \omega_{\varphi}} - \mathcal{B}_0 \text{tr}_{\omega_{\varphi}} \omega_X - \mathcal{C}_0 \\ &+ \frac{1}{\text{tr}_{\omega_X} \omega_{\varphi}} \sum_{\alpha, \beta} \left( \frac{v_{\alpha}(m_{\varphi}) v_{\beta}(m_{\varphi})}{v(m_{\varphi})^2} - \frac{v_{\alpha\beta}(m_{\varphi})}{v(m_{\varphi})} \right) \langle dm_{\varphi}^{\xi_{\alpha}}, dm_{\varphi}^{\xi_{\beta}} \rangle_{\omega_X}. \end{aligned}$$

*In particular, if  $\mathfrak{t}^{\vee} \ni p \mapsto \log v(p) \in \mathbb{R}$  is concave, then*

$$\Delta_{\omega_{\varphi}, v} \log \text{tr}_{\omega_X} \omega_{\varphi} \geq \frac{\Delta_{\omega_X, v} F}{\text{tr}_{\omega_X} \omega_{\varphi}} - \mathcal{B}_0 \text{tr}_{\omega_{\varphi}} \omega_X - \mathcal{C}_0.$$

*We note that the constants  $\mathcal{B}_0 = B + 2C_v^2 C_{\text{Ric}}$  and  $\mathcal{C}_0 = C_v^2 (C_2 + C_v^2 C_{\xi})$  where*

- $\text{Bisec}(\omega_X) \geq -B$  is a negative lower bound for the bisectional curvature of  $\omega_X$ ;
- $C_v > 0$  is a constant such that

$$\begin{aligned} C_v^{-1} &\leq v + \sum_{\alpha} |v_{\alpha}| + \sum_{\alpha, \beta} |v_{\alpha\beta}| \leq C_v \quad \text{on } P = \text{im}(m_{\omega}), \\ C_v^{-1} &\leq |v| + \sum_{\alpha} |v_{\alpha}| + \sum_{\alpha, \beta} |v_{\alpha\beta}| \leq C_v \quad \text{on } P_{\omega_X} = \text{im}(m_{\omega_X}); \end{aligned} \tag{A.1}$$

- $C_2$  depending only on  $n$  and  $C_1$  where  $C_1 > 0$  is a constant so that for any  $\alpha$ ,

$$-C_1 \omega_X \leq \mathcal{L}_{J_{\xi_{\alpha}}} \omega_X \leq C_1 \omega_X; \tag{A.2}$$

- $C_{\text{Ric}} > 0$  and  $C_{\xi} > 0$  are constants so that  $|m_{\text{Ric}(\omega_X)}^{\xi_{\alpha}}| \leq C_{\text{Ric}}$  and  $|\xi_{\alpha}|_{\omega_X}^2 \leq C_{\xi}$ , for all  $\alpha$ , respectively.

**Remark A.2.** — The concavity condition on  $\log v(p)$  holds in many interested cases. Here we extract some examples from [Lah19, Sec. 3]:

- cscK and extremal metrics:  $v(p) = 1$ ,
- Kähler–Ricci solitons:  $v(p) = e^{\langle \xi, p \rangle}$  for some fixed  $\xi \in \mathfrak{t}$ ,
- Kähler metrics given by the generalized Calabi construction:  $v(p) = \prod_j (\langle \xi_j, p \rangle + a_j)^{d_j}$  with  $d_j > 0$ .

*Proof.* — Before entering the proof, we recall a basic equality of Lie derivative about the quotient of volume forms. Suppose that  $\alpha$  is an  $(n, n)$ -form and  $\beta$  is a volume form on  $X$ . For all vector fields  $V$ , one can derive the following

$$\mathcal{L}_V \left( \frac{\alpha}{\beta} \right) = \frac{\mathcal{L}_V \alpha}{\beta} - \frac{\alpha}{\beta} \cdot \frac{\mathcal{L}_V \beta}{\beta}.$$

As a consequence, we have

$$\mathcal{L}_V \left( \frac{\omega_\varphi^n}{\omega_X^n} \right) = \frac{n(\mathcal{L}_V \omega_\varphi) \wedge \omega_\varphi^{n-1}}{\omega_\varphi^n} - \frac{n(\mathcal{L}_V \omega_X) \wedge \omega_X^{n-1}}{\omega_X^n} \quad (\text{A.3})$$

and

$$\mathcal{L}_V (\text{tr}_{\omega_X} \omega_\varphi) = \frac{n(\mathcal{L}_V \omega_\varphi) \wedge \omega_X^{n-1} + n(n-1)(\mathcal{L}_V \omega_X) \wedge \omega_\varphi \wedge \omega_X^{n-1}}{\omega_X^n} - (\text{tr}_{\omega_X} \omega_\varphi) \frac{n(\mathcal{L}_V \omega_X) \wedge \omega_X^{n-1}}{\omega_X^n} \quad (\text{A.4})$$

By the standard Aubin–Yau’s inequality, we have

$$\Delta_{\omega_\varphi} \log \text{tr}_{\omega_X} \omega_\varphi \geq \frac{\Delta_{\omega_X} (F - \log v(m_\varphi))}{\text{tr}_{\omega_X} \omega_\varphi} - B \text{tr}_{\omega_\varphi} \omega_X,$$

and thus,

$$\begin{aligned} & \Delta_{\omega_\varphi, v} \log \text{tr}_{\omega_X} \omega_\varphi \\ & \geq \frac{\Delta_{\omega_X, v} (F - \log v(m_\varphi))}{\text{tr}_{\omega_X} \omega_\varphi} - B \text{tr}_{\omega_\varphi} \omega_X + \text{tr}_{\omega_\varphi} [d \log v(m_\varphi) \wedge d^c \log \text{tr}_{\omega_X} \omega_\varphi] \\ & \quad - \frac{1}{\text{tr}_{\omega_X} \omega_\varphi} \text{tr}_{\omega_X} \left[ d \log v(m_{\omega_X}) \wedge d^c \log \left( \frac{\omega_\varphi^n}{\omega_X^n} \right) \right] \\ & = \frac{\Delta_{\omega_X, v} F}{\text{tr}_{\omega_X} \omega_\varphi} - B \text{tr}_{\omega_\varphi} \omega_X + \frac{1}{\text{tr}_{\omega_X} \omega_\varphi} \left\{ \text{tr}_{\omega_\varphi} [d \log v(m_\varphi) \wedge d^c \text{tr}_{\omega_X} \omega_\varphi] - \Delta_{\omega_X} \log v(m_\varphi) \right. \\ & \quad \left. - \text{tr}_{\omega_X} [d \log v(m_{\omega_X}) \wedge d^c \log v(m_{\omega_\varphi})] - \text{tr}_{\omega_X} \left[ d \log v(m_{\omega_X}) \wedge d^c \log \left( \frac{\omega_\varphi^n}{\omega_X^n} \right) \right] \right\}. \end{aligned} \quad (\text{A.5})$$

It suffices to establish a suitable lower bound for the terms in  $\{\dots\}$  in (A.5). Note that  $P$  does not depend on  $\varphi$  and  $d^c = J^{-1} \circ d \circ J$ . By Cartan’s formula,  $i_V d^c f = -i_{JV} df = -\mathcal{L}_{JV} f$ . Using (A.4), (A.2) and (A.1), we obtain the following:

$$\begin{aligned} \text{tr}_{\omega_\varphi} (d \log v(m_\varphi) \wedge d^c \text{tr}_{\omega_X} \omega_\varphi) &= \frac{1}{v(m_\varphi)} \sum_\alpha v_\alpha(m_\varphi) i_{\xi_\alpha} d^c \text{tr}_{\omega_X} \omega_\varphi = -\frac{1}{v(m_\varphi)} \sum_\alpha v_\alpha(m_\varphi) (\mathcal{L}_{J\xi_\alpha} \text{tr}_{\omega_X} \omega_\varphi) \\ &= -\frac{1}{v(m_\varphi)} \sum_\alpha v_\alpha(m_\varphi) \left( \frac{n(\mathcal{L}_{J\xi_\alpha} \omega_\varphi) \wedge \omega_X^{n-1} + n(n-1)(\mathcal{L}_{J\xi_\alpha} \omega_X) \wedge \omega_\varphi \wedge \omega_X^{n-2}}{\omega_X^n} - \text{tr}_{\omega_X} \omega_\varphi \cdot \frac{n(\mathcal{L}_{J\xi_\alpha} \omega_X) \wedge \omega_X^{n-1}}{\omega_X^n} \right) \\ &\geq -\frac{1}{v(m_\varphi)} \sum_\alpha v_\alpha(m_\varphi) \left( \frac{n(\mathcal{L}_{J\xi_\alpha} \omega_\varphi) \wedge \omega_X^{n-1}}{\omega_X^n} \right) - C_2 C_v^2 \text{tr}_{\omega_X} \omega_\varphi \\ &= \frac{1}{v(m_\varphi)} \sum_\alpha v_\alpha(m_\varphi) \Delta_{\omega_X} m_\varphi^{\xi_\alpha} - C_2 C_v^2 \text{tr}_{\omega_X} \omega_\varphi \\ &= \frac{1}{v(m_\varphi)} \left( \Delta_{\omega_X} v(m_\varphi) - \sum_{\alpha, \beta} v_{\alpha\beta}(m_\varphi) \langle dm_\varphi^{\xi_\alpha}, dm_\varphi^{\xi_\beta} \rangle_{\omega_X} \right) - C_2 C_v^2 \text{tr}_{\omega_X} \omega_\varphi \\ &= \Delta_{\omega_X} \log v(m_\varphi) + \sum_{\alpha, \beta} \left( \frac{v_\alpha(m_\varphi) v_\beta(m_\varphi)}{v(m_\varphi)} - \frac{v_{\alpha\beta}(m_\varphi)}{v(m_\varphi)} \right) \langle dm_\varphi^{\xi_\alpha}, dm_\varphi^{\xi_\beta} \rangle_{\omega_X} - C_2 C_v^2 \text{tr}_{\omega_X} \omega_\varphi \end{aligned} \quad (\text{A.6})$$



where  $C_2 > 0$  is a constant depending only on  $C_1$  and  $n$ . Recall that  $C_{\text{Ric}} > 0$  is a constant such that  $|m_{\text{Ric}(\omega_X)}^{\zeta_\alpha}| \leq C_{\text{Ric}}$  for all  $\alpha$ . Since  $P_{\text{Ric}} := \text{im}(m_{\text{Ric}(\omega_X)})$  does not depend on the representative  $\text{Ric}(\omega_X)$  in  $c_1(X)$ , we also have  $|m_{\text{Ric}(\omega_\varphi)}^{\zeta_\alpha}| \leq C_{\text{Ric}}$  for all  $\alpha$ , and thus,

$$\begin{aligned} \text{tr}_{\omega_X} \left[ d \log v(m_{\omega_X}) \wedge d^c \log \left( \frac{\omega_\varphi^n}{\omega_X^n} \right) \right] &= \frac{1}{v(m_{\omega_X})} \sum_\alpha v_\alpha(m_{\omega_X}) m_{dd^c \log \frac{\omega_\varphi^n}{\omega_X^n}}^{\zeta_\alpha} \\ &= \frac{1}{v(m_{\omega_X})} \sum_\alpha v_\alpha(m_{\omega_X}) (m_{\text{Ric}(\omega_X)}^{\zeta_\alpha} - m_{\text{Ric}(\omega_\varphi)}^{\zeta_\alpha}) \leq 2C_v^2 C_{\text{Ric}}. \end{aligned} \quad (\text{A.7})$$

Since  $-dm_\varphi^{\zeta_\alpha} = i_{\zeta_\alpha} \omega_\varphi$ , we remark that from (A.1) and  $|\zeta_\alpha|_\omega^2 \leq C_\zeta$ , one has

$$\begin{aligned} |\text{tr}_{\omega_X} [d \log v(m_{\omega_X}) \wedge d^c \log v(m_\varphi)]| &= \left| \frac{1}{v(m_{\omega_X})v(m_\varphi)} \sum_\alpha v_\alpha(m_{\omega_X}) m_{dd^c v(m_\varphi)}^{\zeta_\alpha} \right| \\ &= \left| \frac{1}{v(m_{\omega_X})v(m_\varphi)} \sum_\alpha v_\alpha(m_{\omega_X}) (i_{\zeta_\alpha} d^c v(m_\varphi)) \right| = \left| \frac{1}{v(m_{\omega_X})v(m_\varphi)} \sum_\alpha v_\alpha(m_{\omega_X}) (-i_{J\zeta_\alpha} dv(m_\varphi)) \right| \\ &= \left| \frac{1}{v(m_{\omega_X})v(m_\varphi)} \sum_\alpha v_\alpha(m_{\omega_X}) \left( \sum_\beta v_\beta(m_\varphi) i_{J\zeta_\alpha} i_{\zeta_\beta} \omega_\varphi \right) \right| \leq C_v^4 \left| 2\Re \sum_{i,j} (g_\varphi)_{ij} (\zeta'_\alpha)^i (\zeta''_\beta)^j \right| \\ &\leq C_v^4 C_\zeta \text{tr}_{\omega_X} \omega_\varphi \end{aligned} \quad (\text{A.8})$$

where  $\zeta_\alpha = \zeta'_\alpha + \zeta''_\alpha$  so that  $\zeta'_\alpha \in \Gamma(X, T_X^{1,0})$  is the holomorphic part of  $\zeta_\alpha$ ,  $\zeta''_\alpha = \overline{\zeta'_\alpha}$ , and under local coordinates  $\zeta'_\alpha = \sum_i (\zeta'_\alpha)^i \partial_i$  and  $C_\zeta$  is a constant satisfying  $|\zeta_\alpha|_{\omega_X}^2 \leq C_\zeta$  for every  $\alpha$ .

To conclude, combining (A.5), (A.6), (A.7) and (A.8), we finally obtain

$$\begin{aligned} &\Delta_{\omega_\varphi, v} \log \text{tr}_{\omega_X} \omega_\varphi \\ &\geq \frac{\Delta_{\omega_X, v} F}{\text{tr}_{\omega_X} \omega_\varphi} - B \text{tr}_{\omega_\varphi} \omega_X - \frac{2C_v^2 C_{\text{Ric}}}{\text{tr}_{\omega_X} \omega_\varphi} - C_v^2 (C_2 + C_v^2 C_\zeta) + \frac{1}{\text{tr}_{\omega_X} \omega_\varphi} \sum_{\alpha, \beta} \left( \frac{v_\alpha(m_\varphi) v_\beta(m_\varphi)}{v(m_\varphi)^2} - \frac{v_{\alpha\beta}(m_\varphi)}{v(m_\varphi)} \right) \langle dm_\varphi^{\zeta_\alpha}, dm_\varphi^{\zeta_\beta} \rangle_{\omega_X} \\ &\geq \frac{\Delta_{\omega_X, v} F}{\text{tr}_{\omega_X} \omega_\varphi} - (B + 2C_v^2 C_{\text{Ric}}) \text{tr}_{\omega_\varphi} \omega_X - C_v^2 (C_2 + C_v^2 C_\zeta) + \frac{1}{\text{tr}_{\omega_X} \omega_\varphi} \sum_{\alpha, \beta} \left( \frac{v_\alpha(m_\varphi) v_\beta(m_\varphi)}{v(m_\varphi)^2} - \frac{v_{\alpha\beta}(m_\varphi)}{v(m_\varphi)} \right) \langle dm_\varphi^{\zeta_\alpha}, dm_\varphi^{\zeta_\beta} \rangle_{\omega_X} \end{aligned}$$

as required.  $\square$

## Appendix B. Weighted local Chen–Cheng’s $C^2$ -estimate

This section aims to prove Theorem 2.3. Recall that we have two local equations on  $B_1(0) \subset \mathbb{C}^n$ ,

$$v(m_{dd^c \phi}) \det(\phi_{i\bar{j}}) = e^G \quad \text{and} \quad \Delta_{\phi, v} G = -S = -w(m_{dd^c \phi}).$$

We first prove the following lemma:

**Lemma B.1.** — *There exists positive constants  $K, C$  depending on  $v, w, \|G\|_{L^\infty(B_1(0))}$  and  $\|\phi\|_{L^\infty(B_1(0))}$  such that*

$$\Delta_{\phi, v} u \geq -C \cdot (\Delta \phi) \cdot u \quad \text{on } B_1(0)$$

where  $u = e^{\frac{G}{2}} |dG|_\phi^2 + K \Delta \phi$ .

The proof of Lemma B.1 follows similarly the global version in [DJL24, Sec. 6]. Given the local contexts and for the sake of completeness, we present detailed proof here.

*Proof.* — We first remark that by arithmetic and geometric means inequality, one has

$$\Delta \phi \geq n \det(\phi_{i\bar{j}})^{1/n} = n e^{G/n} v(m_{dd^c \phi})^{-1/n} \geq c \quad \text{and} \quad \text{tr}_\phi \omega \geq n \det(\phi_{i\bar{j}})^{-1/n} = n e^{-G/n} v(m_{dd^c \phi})^{1/n} \geq c.$$

where  $\omega$  is the Euclidean metric and  $c > 0$  is a constant depending only on  $n, \|G\|_{L^\infty(B_1(0))}$  and  $C_v$ . We next review the following estimate by Chen and Cheng [CC21a, p. 16, (4.3)]. Under the

normal coordinates with respect to  $\phi_{i\bar{j}}$ , we may assume that  $\omega_{i\bar{j}}(x_0) = \delta_{ij}$ ,  $\phi_{i\bar{j}}(x_0) = \lambda_i \delta_{ij}$  and  $\partial_k \phi_{i\bar{j}}(x_0) = 0$  at a fixed point  $x_0$ , then, at  $x_0$ , we have

$$\begin{aligned}
e^{-\frac{\zeta}{2}} \Delta_\phi (e^{\frac{\zeta}{2}} |dG|_\phi^2) &= \Delta_\phi |\nabla^\phi G|_\phi^2 + \frac{1}{2} \phi^{i\bar{i}} (\partial_i G \cdot \partial_{\bar{i}} |\nabla^\phi G|_\phi^2 + \partial_{\bar{i}} G \cdot \partial_i |\nabla^\phi G|_\phi^2) + \frac{1}{4} |\nabla^\phi G|_\phi^4 + \frac{1}{2} \Delta_\phi G \cdot |\nabla^\phi G|_\phi^2 \\
&= \phi^{i\bar{i}} (\partial_i \Delta_\phi G \cdot \partial_{\bar{i}} G + \partial_{\bar{i}} \Delta_\phi G \cdot \partial_i G) + \text{Ric}_\phi (\nabla^\phi G, \nabla^\phi G) + |\nabla^\phi \nabla^\phi G|_\phi^2 + |\nabla^\phi \bar{\nabla}^\phi G|_\phi^2 \\
&\quad + \frac{1}{2} \phi^{i\bar{i}} \phi^{j\bar{j}} (G_i G_j G_{i\bar{j}} + G_i G_{\bar{j}} G_{j\bar{i}} + G_{\bar{i}} G_j G_{ij} + G_{\bar{i}} G_j G_{i\bar{j}}) + \frac{1}{4} |\nabla^\phi G|_\phi^4 + \frac{1}{2} \Delta_\phi G \cdot |\nabla^\phi G|_\phi^2 \\
&\geq \phi^{i\bar{i}} (\partial_i \Delta_\phi G \cdot \partial_{\bar{i}} G + \partial_{\bar{i}} \Delta_\phi G \cdot \partial_i G) + \text{Ric}_\phi (\nabla^\phi G, \nabla^\phi G) + |\nabla^\phi \bar{\nabla}^\phi G|_\phi^2 \\
&\quad + \frac{1}{2} \phi^{i\bar{i}} \phi^{j\bar{j}} (G_i G_j G_{j\bar{i}} + G_{\bar{i}} G_j G_{i\bar{j}}) + \frac{1}{2} \Delta_\phi G \cdot |\nabla^\phi G|_\phi^2
\end{aligned} \tag{B.1}$$

where  $\text{Ric}_\phi := \text{Ric}(dd^c \phi) = -dd^c \log \det(\phi_{i\bar{j}})$ . The last inequality comes from the fact that

$$\frac{1}{4} |\nabla^\phi G|_\phi^4 + \frac{1}{2} \phi^{i\bar{i}} \phi^{j\bar{j}} (G_i G_j G_{i\bar{j}} + G_{\bar{i}} G_j G_{ij}) + |\nabla^\phi \nabla^\phi G|_\phi^2 = \left| G_{ij} + \frac{1}{2} \sqrt{\lambda_i^{-1} \lambda_j^{-1}} G_i G_j \right|_\phi^2 \geq 0.$$

From the first equation, we have  $\text{Ric}(dd^c \phi) = -dd^c (G - \log v(m_{dd^c \phi}))$ . Rewrite (B.1) as follows

$$\begin{aligned}
e^{-\frac{\zeta}{2}} \Delta_\phi (e^{\frac{\zeta}{2}} |dG|_\phi^2) &\geq \phi^{i\bar{i}} (\partial_i \Delta_\phi G \cdot \partial_{\bar{i}} G + \partial_{\bar{i}} \Delta_\phi G \cdot \partial_i G) - \phi^{i\bar{i}} \phi^{j\bar{j}} G_{i\bar{j}} G_j G_{\bar{j}} \\
&\quad + \phi^{i\bar{i}} \phi^{j\bar{j}} (\log v(m_{dd^c \phi}))_{i\bar{j}} G_i G_{\bar{j}} + \phi^{i\bar{i}} \phi^{j\bar{j}} G_{i\bar{j}} G_i G_{\bar{j}} + |\nabla^\phi \bar{\nabla}^\phi G|_\phi^2 + \frac{1}{2} \Delta_\phi G \cdot |dG|_\phi^2 \\
&= 2 \text{tr}_\phi (d\Delta_\phi G \wedge d^c G) + \phi^{i\bar{i}} \phi^{j\bar{j}} (\log v(m_{dd^c \phi}))_{i\bar{j}} G_i G_{\bar{j}} + |\nabla^\phi \bar{\nabla}^\phi G|_\phi^2 + \frac{1}{2} \Delta_\phi G \cdot |dG|_\phi^2.
\end{aligned}$$

Considering the weighted version of the above inequality, one can infer

$$\begin{aligned}
e^{-\frac{\zeta}{2}} \Delta_{\phi, v} (e^{\frac{\zeta}{2}} |dG|_\phi^2) &\geq 2 \text{tr}_\phi (d\Delta_\phi G \wedge d^c G) + \phi^{i\bar{i}} \phi^{j\bar{j}} (\log v(m_{dd^c \phi}))_{i\bar{j}} G_i G_{\bar{j}} + |\nabla^\phi \bar{\nabla}^\phi G|_\phi^2 + \frac{1}{2} \Delta_\phi G \cdot |dG|_\phi^2 \\
&\quad + \frac{1}{2} \text{tr}_\phi (d \log v(m_{dd^c \phi}) \wedge d^c G) |dG|_\phi^2 + \text{tr}_\phi (d \log v(m_{dd^c \phi}) \wedge d^c |dG|_\phi^2) \\
&= 2 \text{tr}_\phi (d\Delta_\phi G \wedge d^c G) + \phi^{i\bar{i}} \phi^{j\bar{j}} (\log v(m_{dd^c \phi}))_{i\bar{j}} G_i G_{\bar{j}} + |\nabla^\phi \bar{\nabla}^\phi G|_\phi^2 + \frac{1}{2} \Delta_{\phi, v} G \cdot |dG|_\phi^2 \\
&\quad + \text{tr}_\phi (d \log v(m_{dd^c \phi}) \wedge d^c |dG|_\phi^2)
\end{aligned} \tag{B.2}$$

Under normal coordinates with respect to  $\phi_{i\bar{j}}$  at  $x_0$ , we obtain

$$\begin{aligned}
\text{tr}_\phi (d \log v(m_{dd^c \phi}) \wedge d^c |dG|_\phi^2) &= \frac{1}{2} \phi^{i\bar{i}} \left( (\log v(m_{dd^c \phi}))_i [\phi^{j\bar{j}} G_j G_{\bar{j}}]_{\bar{i}} + (\log v(m_{dd^c \phi}))_{\bar{i}} [\phi^{j\bar{j}} G_j G_{\bar{j}}]_i \right) \\
&= \frac{1}{2} \phi^{j\bar{j}} \left( G_{\bar{j}} \phi^{i\bar{i}} (\log v(m_{dd^c \phi}))_i G_{j\bar{i}} + G_j \phi^{i\bar{i}} (\log v(m_{dd^c \phi}))_i G_{\bar{j}\bar{i}} + G_{\bar{j}} \phi^{i\bar{i}} (\log v(m_{dd^c \phi}))_{\bar{i}} G_{ji} + G_j \phi^{i\bar{i}} (\log v(m_{dd^c \phi}))_{\bar{i}} G_{\bar{j}\bar{i}} \right) \\
&= 2 \text{tr}_\phi (dG \wedge d^c [\text{tr}_\phi (d \log v(m_{dd^c \phi}) \wedge d^c G)]) - \Re \left( \phi^{i\bar{i}} \phi^{j\bar{j}} (\log v(m_{dd^c \phi}))_{ij} G_i G_{\bar{j}} \right) - \phi^{i\bar{i}} \phi^{j\bar{j}} (\log v(m_{dd^c \phi}))_{i\bar{j}} G_{\bar{i}} G_j
\end{aligned}$$

and the inequality (B.2) can be expressed as

$$\begin{aligned}
e^{-\frac{\zeta}{2}} \Delta_{\phi, v} (e^{\frac{\zeta}{2}} |dG|_\phi^2) &\geq 2 \text{tr}_\phi (d\Delta_{\phi, v} G \wedge d^c G) + |\nabla^\phi \bar{\nabla}^\phi G|_\phi^2 + \frac{1}{2} \Delta_{\phi, v} G \cdot |\nabla^\phi G|_\phi^2 \\
&\quad - \Re \left( \phi^{i\bar{i}} \phi^{j\bar{j}} (\log v(m_{dd^c \phi}))_{ij} G_i G_{\bar{j}} \right).
\end{aligned} \tag{B.3}$$

Recall that

$$dm_{dd^c \phi}^{\zeta_\alpha} = -i \zeta_\alpha dd^c \phi = -i \zeta_\alpha (\sqrt{-1} \phi_{p\bar{q}} dz^p \wedge d\bar{z}^q) = -\sqrt{-1} \phi_{p\bar{q}} (\zeta'_\alpha)^p d\bar{z}^q + \sqrt{-1} \phi_{p\bar{q}} (\zeta''_\alpha)^{\bar{q}} dz^p.$$

Hence,

$$\begin{aligned}
\partial_{i\bar{j}}^2 \log v(m_{dd^c \phi}) &= \partial_i \left\{ \sum_\alpha (\partial_\alpha \log v)(m_{dd^c \phi}) \cdot \sqrt{-1} \phi_{j\bar{\ell}} (\zeta''_\alpha)^{\bar{\ell}} \right\} \\
&= \sum_{\alpha, \beta} (\partial_{\alpha\beta}^2 \log v)(m_{dd^c \phi}) \cdot [-\phi_{i\bar{q}} \phi_{j\bar{\ell}} (\zeta''_\beta)^{\bar{q}} (\zeta''_\alpha)^{\bar{\ell}}] + \sum_\alpha (\partial_\alpha \log v)(m_{dd^c \phi}) \cdot \sqrt{-1} [\phi_{j\bar{\ell}} (\zeta''_\alpha)^{\bar{\ell}} + \phi_{j\bar{\ell}} (\zeta''_\alpha)^{\bar{\ell}}]_i.
\end{aligned} \tag{B.4}$$

Note that  $\zeta'_\alpha$  is a holomorphic vector field for any  $\alpha$ . Therefore,  $(\zeta'_\alpha)_i^j = 0 = (\zeta''_\alpha)_i^{\bar{j}}$  for any  $i, j \in \{1, \dots, n\}$ .

Combining (B.3), (B.4), and the fact that  $(\zeta''_\alpha)_{\bar{i}}^{\bar{\ell}} = 0$ , under normal coordinates with respect to  $\phi$  at  $x_0$ , we get

$$\begin{aligned} e^{-\frac{G}{2}} \Delta_{\phi,v} (e^{\frac{G}{2}} |dG|_\phi^2) &\geq 2 \operatorname{tr}_\phi (d\Delta_{\phi,v} G \wedge d^c G) + |\nabla^\phi \bar{\nabla}^\phi G|_\phi^2 + \frac{1}{2} \Delta_{\phi,v} G \cdot |\nabla^\phi G|_\phi^2 \\ &\quad + \Re \sum_{\alpha,\beta} (\partial_{\alpha\beta}^2 \log v) (m_{dd^c\phi}) (\zeta''_\beta)^{\bar{i}} (\zeta''_\alpha)^{\bar{j}} G_{\bar{i}} G_{\bar{j}} \\ &\geq 2 \operatorname{tr}_\phi (d\Delta_{\phi,v} G \wedge d^c G) + |\nabla^\phi \bar{\nabla}^\phi G|_\phi^2 + \frac{1}{2} \Delta_{\phi,v} G \cdot |\nabla^\phi G|_\phi^2 - 4C_v^4 C_\xi |dG|_\omega^2 \end{aligned}$$

where  $C_\xi > 0$  is a constant such that  $|\zeta|_\omega^2 \leq C_\xi$ . Using the second equation  $\Delta_{\phi,v} G = -w(m_{dd^c\phi})$ , we then derive

$$\begin{aligned} \operatorname{tr}_\phi (d\Delta_{\phi,v} G \wedge d^c G) &= -\operatorname{tr}_\phi (dw(m_{dd^c\phi}) \wedge d^c G) = -\sum_\alpha w_\alpha(m_{dd^c\phi}) \operatorname{tr}_\phi (dm_{dd^c\phi}^{\xi_\alpha} \wedge d^c G) \\ &= \sum_\alpha w_\alpha(m_{dd^c\phi}) \Re(\zeta'_\alpha)^i G_i \geq -C_\xi C_w |dG|_\omega \geq -C_1 - (\Delta\phi) \cdot |dG|_\phi^2 \end{aligned}$$

where  $C_1 > 0$  depending only on  $C_\xi, C_v$ . Then

$$e^{-\frac{G}{2}} \Delta_{\phi,v} (e^{\frac{G}{2}} |dG|_\phi^2) \geq -2C_1 - C_2(\Delta\phi) |dG|_\phi^2 + |\nabla^\phi \bar{\nabla}^\phi G|_\phi^2 \quad (\text{B.5})$$

where  $C_2 > 0$  depend only on  $C_\xi, C_v, C_w$ .

Next, we compute, under normal coordinates with respect to  $\phi_{i\bar{j}}$ ,

$$\begin{aligned} \Delta_{\phi,v}(\Delta\phi) &= \Delta_\phi(\Delta\phi) + \operatorname{tr}_\phi (d \log v(m_{dd^c\phi}) \wedge d^c \Delta\phi) \\ &= -\operatorname{tr}_\omega \operatorname{Ric}(dd^c\phi) + \operatorname{tr}_\phi (d \log v(m_{dd^c\phi}) \wedge d^c \Delta\phi) \\ &= -\Delta G + \Delta \log v(m_{dd^c\phi}) + \operatorname{tr}_\phi (d \log v(m_{dd^c\phi}) \wedge d^c \Delta\phi) \\ &\geq -\Delta G + \sum_{\alpha,\beta} (\partial_{\alpha\beta}^2 \log v) (m_{dd^c\phi}) \langle dm_{dd^c\phi}^{\xi_\alpha}, dm_{dd^c\phi}^{\xi_\beta} \rangle - C_3 \Delta\phi \\ &\geq -\Delta G - 2C_v^2 (\Delta\phi)^2 - C_3 \Delta\phi, \end{aligned}$$

where  $C_3$  depend only on  $n$  and a constant  $C > 0$  such that  $-C\omega \leq \mathcal{L}_{\xi_\alpha} \omega \leq C\omega$ . The fourth line follows from the same estimate (A.6).

Then we obtain

$$\Delta_{\phi,v} (e^{\frac{G}{2}} |dG|_\phi^2 + K\Delta\phi) \geq -C_1 e^{\frac{G}{2}} - C_2 (\Delta\phi) e^{\frac{G}{2}} |dG|_\phi^2 + e^{\frac{G}{2}} |\nabla^\phi \bar{\nabla}^\phi G|_\phi^2 - K\Delta G - 2C_v^2 (\Delta\phi)^2 - C_3 \Delta\phi.$$

Recall that  $|\nabla^\phi \bar{\nabla}^\phi G|_\phi^2 = \sum_{i,j} \phi^{i\bar{i}} \phi^{j\bar{j}} |G_{i\bar{j}}|^2$ . By Cauchy–Schwarz inequality, we have the following estimate

$$|\nabla^\phi \bar{\nabla}^\phi G|_\phi^2 + \frac{K^2}{4} (\Delta\phi)^2 = \sum_{i,j} \phi^{i\bar{i}} \phi^{j\bar{j}} |G_{i\bar{j}}|^2 + \frac{K^2}{4} \sum_{i,j} \phi_{i\bar{i}} \phi_{j\bar{j}} \geq \sum_{i,j} K |G_{i\bar{j}}| \geq K \sum_i K |G_{i\bar{i}}| \geq K |\Delta G|.$$

Moreover, since  $\Delta\phi \geq c$ ; hence, we derive the following estimate

$$\Delta_{\phi,v} u = \Delta_{\phi,v} (e^{\frac{G}{2}} |dG|_\phi^2 + K\Delta\phi) \geq -C_4 \cdot K \cdot (\Delta\phi) \cdot u,$$

for a constant  $C_4 > 0$  depending only on  $n, C_v, C_w$  and some sufficient large  $K > 0$  depending on  $C_v, C_1, C_3, c$  and  $\|G\|_{L^\infty(B_1(0))}$ .  $\square$

Using Lemma B.1, one can obtain Theorem 2.3 following an argument in Chen–Cheng’s article [CC21a]:

*Proof of Theorem 2.3.* — Let  $\eta$  be a positive smooth function with compact support in  $B_1(0)$ ,  $\eta \equiv 1$  on  $B_{3/4}(0)$  and  $0 \leq \eta \leq 1$ . We first claim that there is a positive constant

$$D_1 = D_1(n, \|G\|_{L^\infty(B_1(0))}, v, w, \|\Delta\phi\|_{L^{n+1}(B_1(0))}, C_\xi, C_\eta)$$

such that  $\| |dG|_\phi^2 \|_{L^1(B_{3/4}(0))} \leq D_1$ , where  $C_\eta > 0$  is a constant such that  $|d\eta|_\omega, |dd^c\eta|_\omega \leq C_\eta$  and  $C_\xi > 0$  is a constant such that  $|\zeta_\alpha|_\omega \leq C_\xi$ .

To see the claim, set  $C_G > 0$  a constant such that  $|G| \leq C_G$  and  $e^{-G} \leq C_G$  on  $B_1(0)$ . Then we have

$$\begin{aligned}
& \int_{B_{3/4}(0)} |dG|_\phi^2 \omega^n \leq \int_{B_1(0)} \eta |dG|_\phi^2 e^{-G} v(m_{dd^c \phi}) (dd^c \phi)^n \leq C_G C_v \int_{B_1(0)} \eta dG \wedge d^c G \wedge (dd^c \phi)^{n-1} \\
& = C_G C_v \int_{B_1(0)} -\eta G dd^c G \wedge (dd^c \phi)^{n-1} - G d\eta \wedge d^c G \wedge (dd^c \phi)^{n-1} \\
& = C_G C_v \int_{B_1(0)} \eta G n [w(m_{dd^c \phi}) (dd^c \phi)^n + d \log v(m_{dd^c \phi}) \wedge d^c G \wedge (dd^c \phi)^{n-1}] \\
& \quad + \frac{1}{2} C_G C_v \int_{B_1(0)} G^2 dd^c \eta \wedge (dd^c \phi)^{n-1} \\
& \leq n e^{C_G} C_G^2 C_v^2 C_w \int_{B_1(0)} \omega^n + \frac{1}{2} C_G^3 C_v C_\eta \int_{B_1(0)} (\Delta \phi)^{n-1} \omega^n \\
& \quad - \frac{1}{2} n C_G C_v \left[ \int_{B_1(0)} G^2 \eta dd^c \log v(m_{dd^c \phi}) \wedge (dd^c \phi)^{n-1} + G^2 d \log v(m_{dd^c \phi}) \wedge d^c \eta \wedge (dd^c \phi)^{n-1} \right] \\
& \leq n e^{C_G} C_G^2 C_v^2 C_w \int_{B_1(0)} \omega^n + \frac{1}{2} C_G^3 C_v C_\eta \int_{B_1(0)} (\Delta \phi)^{n-1} \omega^n + \frac{1}{2} n C_G C_v \left[ \int_{B_1(0)} e^{C_G} C_G^2 C_v^3 C_\xi^2 (\Delta \phi) \omega^n + e^{C_G} C_G^2 C_v^3 C_\xi C_\eta \omega^n \right] \\
& \leq D_1.
\end{aligned}$$

Hence, the claim follows. We provide more details about the inequality next to the last one. Since

$$\partial_{\bar{j}} \log v(m_{dd^c \phi}) = \frac{1}{v(m_{dd^c \phi})} \sum_{\alpha} v_{\alpha} (m_{dd^c \phi}) (-\sqrt{-1} \phi_{k\bar{j}}(\zeta'_{\alpha})^k),$$

we have  $\text{tr}_{\phi}(d \log v(m_{dd^c \phi}) \wedge d^c \eta) \leq C_v^2 C_{\xi} C_{\eta}$ ; thus,

$$d \log v(m_{dd^c \phi}) \wedge d^c \eta \wedge (dd^c \phi)^n \leq C_v^2 C_{\xi} C_{\eta} (dd^c \phi)^n = C_v^2 C_{\xi} C_{\eta} \frac{e^G}{v(m_{dd^c \phi})} \omega^n \leq e^{C_G} C_v^3 C_{\xi} C_{\eta} \omega^n.$$

On the other hand, for the term involving  $dd^c \log v(m_{dd^c \phi})$ ,

$$\begin{aligned}
\partial_{\bar{j}}^2 \log v(m_{dd^c \phi}) &= \partial_i \left\{ \sum_{\alpha} (\partial_{\alpha} \log v)(m_{dd^c \phi}) (-\sqrt{-1} \phi_{k\bar{j}}(\zeta'_{\alpha})^k) \right\} \\
&= \sum_{\alpha, \beta} (\partial_{\alpha\beta}^2 \log v)(m_{dd^c \phi}) \phi_{i\bar{q}} \phi_{k\bar{j}}(\zeta'_{\beta})^{\bar{q}} (\zeta'_{\alpha})^k + \sum_{\alpha} (\partial_{\alpha} \log v)(m_{dd^c \phi}) (-\sqrt{-1}) [\phi_{k\bar{j}}(\zeta'_{\alpha})^k + \phi_{k\bar{j}}(\zeta'_{\alpha})^k]_i.
\end{aligned}$$

Note that  $\mathcal{L}_{\zeta_{\alpha}} dd^c \phi = 0$  from the  $G$ -invariant assumption. Therefore,

$$\begin{aligned}
0 &= \mathcal{L}_{\zeta_{\alpha}} dd^c \phi = d(i_{\zeta_{\alpha}} dd^c \phi) = d\left(\sqrt{-1} \phi_{k\bar{\ell}}(\zeta'_{\alpha})^k dz^{\ell} - \sqrt{-1} \phi_{k\bar{\ell}}(\zeta''_{\alpha})^{\bar{\ell}} dz^k\right) \\
&= \left(\sqrt{-1} \phi_{k\bar{\ell}}(\zeta'_{\alpha})^k dz^i \wedge dz^{\ell} - \sqrt{-1} \phi_{k\bar{\ell}}(\zeta''_{\alpha})^{\bar{\ell}} dz^j \wedge dz^k\right) - \sqrt{-1} \phi_{k\bar{\ell}}(\zeta''_{\alpha})^{\bar{\ell}} dz^i \wedge dz^k + \sqrt{-1} \phi_{k\bar{\ell}}(\zeta'_{\alpha})^k dz^j \wedge dz^{\ell} \\
&= \sqrt{-1} \left(\phi_{\gamma\bar{\beta}}(\zeta'_{\alpha})^{\gamma} + \phi_{\alpha\bar{\delta}}(\zeta''_{\alpha})^{\bar{\delta}}\right) dz^{\alpha} \wedge dz^{\beta} - \sqrt{-1} \phi_{k\bar{\ell}}(\zeta''_{\alpha})^{\bar{\ell}} dz^i \wedge dz^k + \sqrt{-1} \phi_{k\bar{\ell}}(\zeta'_{\alpha})^k dz^j \wedge dz^{\ell}
\end{aligned}$$

and this implies  $\phi_{\gamma\bar{\beta}}(\zeta'_{\alpha})^{\gamma} + \phi_{\alpha\bar{\delta}}(\zeta''_{\alpha})^{\bar{\delta}} = 0$ . In the normal coordinates of  $dd^c \phi$ , we get

$$\phi^{i\bar{j}} \phi_{k\bar{j}}(\zeta'_{\alpha})^k = \frac{1}{2} \left( \phi^{i\bar{j}} \phi_{k\bar{j}}(\zeta'_{\alpha})^k + \overline{\phi^{i\bar{j}} \phi_{\ell\bar{j}}(\zeta'_{\alpha})^{\ell}} \right) = \frac{1}{2} \phi^{i\bar{j}} \left( \phi_{k\bar{j}}(\zeta'_{\alpha})^k + \phi_{i\bar{\ell}}(\zeta''_{\alpha})^{\bar{\ell}} \right) = 0.$$

This yields that  $\text{tr}_{\phi} dd^c \log v(m_{dd^c \phi}) \leq C_v^2 C_{\xi}^2 (\Delta \phi)$  and

$$dd^c \log v(m_{dd^c \phi}) \wedge (dd^c \phi)^{n-1} \leq C_v^2 C_{\xi}^2 (\Delta \phi) (dd^c \phi)^n \leq e^{C_G} C_v^3 C_{\xi}^2 (\Delta \phi) \omega^n.$$

We also remark that  $\Delta_{\phi, v} f = \frac{1}{2v(m_{dd^c \phi})} \partial_{\alpha} (v(m_{dd^c \phi}) g_{\phi}^{\alpha\bar{\beta}} \partial_{\bar{\beta}} f)$  where  $\alpha, \beta$  are real coordinates and  $[(g_{\phi})_{\alpha\bar{\beta}}]_{1 \leq \alpha, \beta \leq 2n}$  is the Riemannian metric associate to  $dd^c \phi$ . Therefore, it follows from Lemma B.1, that

$$\partial_{\alpha} (v(m_{dd^c \phi}) g_{\phi}^{\alpha\bar{\beta}} \partial_{\bar{\beta}} u) \geq -2Cv(m_{dd^c \phi}) (\Delta \phi) u.$$

Using [CC21a, Lem. 6.3, arXiv version]), we derive that

$$\|u\|_{L^{\infty}(B_{1/2}(0))} \leq D_2 (\|u\|_{L^1(B_{3/4}(0))} + 1) \leq C, \quad (\text{B.6})$$

where  $D_2$  depends on  $C_v, C_w, p, \|S\|_{L^\infty}, \|\Delta\phi\|_{L^p(B_1(0))}, \|\mathrm{tr}_\phi \omega\|_{L^p(B_1(0))}$ . Recall that  $u = e^{\frac{G}{2}} |dG|_\phi^2 + K\Delta\phi$ ; hence we get

$$\|u\|_{L^1(B_{3/4}(0))} \leq e^{C_G/2} \| |dG|_\phi^2 \|_{L^1(B_{3/4}(0))} + K \|\Delta\phi\|_{L^1(B_{3/4}(0))} \leq e^{C_G/2} D_1 + K \|\Delta\phi\|_{L^1(B_{3/4}(0))}.$$

Then combining this with (B.6) implies the uniform  $L^\infty$ -estimates of  $|dG|_\phi$  and  $\Delta\phi$  on  $B_{1/2}(0)$  which only depend on  $C_v, C_w, p, \|S\|_{L^\infty}, \|\Delta\phi\|_{L^p(B_1(0))}, \|\mathrm{tr}_\phi \omega\|_{L^p(B_1(0))}$ . Then the standard Evans–Krylov estimate and bootstrapping argument imply higher order estimates on  $B_{1/2}(0)$  for  $\phi, G$  which depend on  $C_v, C_w, p, \|S\|_{L^\infty}, \|\Delta\phi\|_{L^p(B_1(0))}, \|\mathrm{tr}_\phi \omega\|_{L^p(B_1(0))}$ .  $\square$

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