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ALMOST AUTOMORPHIC AND BIJECTIVE FACTORS OF SUBSTITUTION SHIFTS

ALVARO BUSTOS-GAJARDO, JOHANNES KELLENDONK, AND REEM YASSAWI

ABSTRACT. In this article we completely characterise constant length substitution shifts which have a proper almost automorphic factor, or which have a bijective substitution factor such that the factor map is injective on at least one point. Our approach is algebraic: we characterise these dynamical properties in terms of a finite semigroup defined by the substitution. We characterise the existence of almost automorphic factors in terms of Green's \mathcal{R} -relation and the existence of bijective factors in terms of Green's \mathcal{L} -relation. Our results are constructive.

1. INTRODUCTION

We are interested in the existence of factors with specific properties for shifts defined by substitutions of constant length. Here, by dynamical system (X, T) we mean a homeomorphism T acting on a compact metric space X . If (X, T) and (Y, S) are two dynamical systems and $F : X \rightarrow Y$ is a continuous surjective map such that $F \circ T = S \circ F$ then (Y, S) is called a factor of (X, T) , (X, T) is called an extension of (Y, S) , and F is called the factor map. The factor map is *almost one-to-one* if Y has a dense orbit of points, each of which have a unique pre-image. If the factor map is almost one-to-one we say that (X, T) is an *almost one-to-one extension* of (Y, S) .¹ (X, T) is called *almost automorphic* if it is an almost one-to-one extension of its maximal equicontinuous factor. If an almost automorphic system is not equicontinuous we call it properly almost automorphic. We are interested in characterising two different scenarios regarding a shift (X_θ, σ) of a constant length substitution θ :

- (1) (X_θ, σ) has a proper almost automorphic factor.
- (2) (X_θ, σ) is *almost bijective*, by which we mean that it is an almost one-to-one extension of a *bijective* substitution shift.

In the first scenario, looking for almost automorphic factors which are shift factors already solves the problem. This is because the substitution systems we consider have odometers as maximal equicontinuous factors, and a proper almost one-to-one extension of an odometer is necessarily conjugate to a shift [8, Theorem 6.4]. By [17, Theorem 22], these shift factors must be substitution shifts. So the first scenario boils down to characterising aperiodic primitive substitutions of constant length which have a factor given by a primitive aperiodic constant length

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¹Here we follow the terminology of de Vries [7], which sometimes diverges from that of Veech, who calls any almost one-to-one extension an almost automorphic extension.

substitution shift with a coincidence (in the sense of Dekking). We also mention Martin’s Theorem [16, Thm.8.11] which implies that, if a dynamical system (X, T) has a proper almost automorphic factor then it also has a proper almost automorphic factor which is an almost one-to-one extension of the maximal equicontinuous factor of (X, T) .

Before we explain our methods let us indicate why the study of factors of the type above is interesting beyond the pure interest in them. One reason is simply that certain properties of dynamical systems—and here we think in particular of spectral properties—pass over to their extensions. For instance, it is commonly believed that bijective substitution shifts should have a singular component in their maximal spectral type, and this property would then be true also for substitutions which factor onto a bijective substitution. Another reason is Veech’s structure theorem for point distal systems which applies to constant length substitutions. This structure theorem says that any minimal system with a residual set of distal points is an \mathcal{AZ} -flow, that is, it has an almost one-to-one extension which is itself the top system of a tower of AI extensions of the trivial system (pt, id) containing only one point. Here an AI extension is a composition of an almost one-to-one extension with an isometric extension. In the light of that theorem one might ask what this tower looks like for constant length substitutions. Our findings give a partial answer to this question, namely, if the constant length substitution shift (X_θ, σ) is an almost one-to-one extension of a bijective substitution shift (X_η, σ) and this bijective substitution shift admits a proper almost automorphic factor (Y, σ) then we have the chain of factors

$$(1.1) \quad X_\theta \xrightarrow{F_1} X_\eta \xrightarrow{F_2} Y \xrightarrow{\pi_Y} \mathcal{G} \xrightarrow{O} pt$$

where F_1 and π_Y are almost one-to-one, and O isometric, and \mathcal{G} is the maximal equicontinuous factor of (X_θ, σ) . If F_2 is isometric, then (1.1) is a chain of AI-extensions (also called a Veech tower) already for (X_θ, σ) and the latter is even a *strict* \mathcal{AZ} -flow, as we do not have to go over to an almost one-to-one extension (X', T) to obtain a tower. Martin [16] provided a criterion for when this is the case, see also Theorem 3.29. We find below a necessary and sufficient criterion for when a general constant length substitution (X_θ, σ) admits a proper almost automorphic factor (Y, σ) . While the non-existence of a proper almost automorphic factor rules out that X_θ is a strict \mathcal{AZ} -flow, its existence does not imply that the factor map is isometric. We leave the question of when the factor map is isometric to future work.

A third reason to study in particular shift factors associated with bijective substitutions is that the latter constitute one of the few families for which a fairly explicit description of the Ellis semigroup exists [13] and since factor maps induce epimorphisms of Ellis semigroups, we obtain information on the Ellis semigroup of (X_θ, T) in this way. This will be presented in future work.

We study both scenarios above using a combination of two sets of tools. The first tool is algebraic, the semigroup S_θ of a substitution θ ; see Definition 2.8. This semigroup has been extensively used in the case when it is a group G_θ , i.e., when the substitution is bijective. For example, G_θ is used to characterise the automorphism group [14, 17], and, if it is commutative, then (X_θ, σ) has a singular component in its maximal spectral type [2, 19]. Also, it is a fundamental building block of the Ellis semigroup of a bijective substitution [13]. It is interesting that we use Green’s

\mathcal{R} relation to prove Theorem 1.1, and Green's \mathcal{L} relation to prove Theorems 1.2 and 1.3.

The second set of tools is classical and involves building topologically conjugate versions of (X_θ, σ) using *collaring* and *k-shifting* (Definitions 2.1 and 2.2). In particular, collaring allows us to control the radius of a putative factor map $F : X_\theta \rightarrow Y$, and *k-shifting* allows us to compose F with a “translation”. We use both these constructions to limit and manipulate possible factor maps, and this results in theorem statements that are constructive, i.e., given a length- ℓ substitution shift, one can explicitly determine whether or not it has a proper almost automorphic or bijective shift factor.

Let $\theta^{(-l,r)}$ denote the $(-l,r)$ -collaring of θ , which is the model of θ that we work with if $F : X_\theta \rightarrow Y$ has left and right radius l and r . To state our first result, we distinguish between different families of factor maps. The simplest factor maps are those induced by *inner encodings*. An inner encoding of θ is roughly speaking a substitution η obtained by a code from θ ; see Definition 2.3. The factor map induced by the inner encoding is a factor map between the substitution shifts $F : X_\theta \rightarrow X_\eta$ which can be characterised by the fact that it has radius zero and sends fixed points of the substitution θ to fixed points of η . Inner encodings arise whenever there is an equivalence relation on the alphabet \mathcal{A}_θ of the substitution θ such that if $a \sim b$, then as words, $\theta(a) \sim \theta(b)$. Thus inner encodings define partitions \mathcal{P} of \mathcal{A}_θ . Given a substitution θ , there is a natural way to define a partition \mathcal{P}_θ of its alphabet, which we call *the coincidence partition of θ* (Definition 3.2), and which yields an inner encoding of θ , called the *canonical inner encoding of θ* ; its definition involves Green's \mathcal{R} -relation on the kernel of S_θ . The shift of this inner encoding is almost automorphic. However one cannot guarantee that the inner encoding is aperiodic so that its shift space is infinite. Our first result, consisting of Theorem 3.21 and Corollary 3.26, is

Theorem 1.1. *Let θ be a primitive, constant length, aperiodic substitution, with pure base $\tilde{\theta}$. Then θ has a proper almost automorphic factor if and only if the canonical inner encoding of $\tilde{\theta}^{(-1,1)}$ is aperiodic.*

The beauty of this result is that it is quite simple to verify its conditions for a fixed substitution. Both Martin [16] and later Herning [11] have worked on this question, but only in the case where θ is bijective; see below for a discussion of their results. In particular, Martin was interested in finding almost automorphic factors of which the original substitution shift is an isometric extension. This is not always the case: in Example 3.30 we present a proper almost automorphic factor which is not an isometric factor.

Note that as a corollary, we can show that there exist substitution shifts for which the maximal *tame* factor, [9], equals the maximal equicontinuous factor. For, a tame factor must be almost automorphic [12], and with Theorem 1.1, we can give many examples of substitution shifts with no proper almost automorphic factor.

Our second result characterises, in terms of the semigroup S_θ , when the substitution shift (X_θ, σ) is an almost one-to-one extension of a bijective substitution shift. We make use of the fact that S_θ admits a kernel, i.e., a minimal bilateral ideal, which is a union of minimal left ideals. The *column rank* of θ is the rank of any element in the kernel of S_θ . We first show the following, which is Theorems 4.1 and 4.2 of this article.

Theorem 1.2. *Let θ be a constant length substitution with column rank c and height h . Suppose that $c > h$. The following are equivalent.*

- (1) *(X_θ, σ) is an almost one-to-one extension of a bijective substitution shift (X_η, σ) where η is an inner encoding of θ with an alphabet of c letters.*
- (2) *The semigroup S_θ of θ has a unique minimal left ideal.*

Note that if (X_θ, σ) is an almost one-to-one extension of any constant length substitution shift (X_η, σ) then η has to have the same column rank. So the requirement that the alphabet of η has c letters in (1) comes for free. Therefore, if θ is aperiodic and $c = 1$, there is no such factor as η must be aperiodic for the factor map to be almost one-to-one. We also show that in the case that $c = h$, η is periodic.

To drop the condition that the bijective factor comes from an inner encoding, let $\theta^{(+k)}$ be the k -shifted extension of θ . This version of θ is especially useful when one considers factor maps F which do not map fixed points of θ to fixed points. We show

Theorem 1.3. *Let θ be an aperiodic primitive constant length- ℓ substitution with column rank c and height h . Suppose that $c > h$. The following are equivalent:*

- (1) *(X_θ, σ) is almost bijective.*
- (2) *There exist $0 \leq n, k \leq C$ such that the semigroup $S_{(\theta^n)^{(+k)}}$ contains a unique minimal left ideal.*

Moreover, C can be explicitly obtained. However in general it is doubly exponential in ℓ ; see the statement of Theorem 4.19.

We discuss prior work concerning Theorem 1.1. In his work [16], Martin gives a necessary and sufficient condition for a bijective substitution shift to be an isometric extension of a proper almost automorphic factor in [16, Theorem 8.08], and his condition (A) in fact translates to our condition that the minimal sets for the collared $\theta^{(0,1)}$ of the substitution form a partition; see Theorem 3.29. This then guarantees that the factor map F_2 in (1.1) is isometric. For more general constant length substitutions, even apparently general bijective substitutions, factor maps to proper almost automorphic factors need not be isometric, see Example 3.30, and an interesting question that remains is what conditions guarantee that (X_θ, σ) is an isometric extension of a proper almost automorphic factor. Later, in his thesis, Herning [11] re-approaches the question of existence of a proper almost automorphic factor; it seems he was unaware of Martin's work. He answers this question in the negative by finding bijective substitutions that do not have a proper almost automorphic factor. In [11, Theorem 4.24], Herning characterises bijective length- p substitutions of prime length that have a proper almost automorphic factor. Once translated, his characterisation is very similar to ours and Martin's. Our Theorem 1.1 extends these results to characterise when any primitive constant length substitution, not just bijective, has a proper almost automorphic factor.

We summarise the contents of this paper. In Section 2, we set the background, fix notation, and define collared, shifted and inner encoded substitutions. We then limit the radii of factor maps for a length- ℓ substitution, extending techniques that exist in the literature for invertible factor maps. In Section 3, we set the stage to prove Theorem 1.1. To do this, we define *outer encodings* of a substitution. These

are substitutions whose semigroup is an epimorphic image of S_θ . As such, they do not necessarily give rise to factor maps between substitution shifts. However, we show in Proposition 3.18 that any inner encoding with column rank 1 must factor through the *canonical outer encoding*; see Definition 3.1. This is the key tool to prove Theorem 1.1, once we bound possible factor maps using the tools from Section 2. In Section 4, we prove Theorem 1.3 in three successive steps. First we prove Theorem 1.2, which characterises when a substitution has a bijective factor arising from an inner encoding, i.e., a radius zero factor map which sends fixed points to fixed points. Next, in Theorem 4.11, we give a characterisation while relaxing the condition that the factor map has radius 0. Finally, in Theorem 4.19, we eliminate the condition that the factor map send fixed points to fixed points. We strive to isolate the requirements on the substitution, noting that generally, purely algebraic results do not need the restriction to primitive aperiodic substitutions. We illustrate with examples throughout. In particular, we show in Examples 3.22 and 4.21 that the Rudin-Shapiro shift has a proper almost automorphic factor but is not almost bijective. In Examples 3.15 and 3.23 we find an example with no almost automorphic factors, and in Example 3.28 we illustrate the procedure for a substitution with height. In Example 3.30 we distinguish our work from Martin's, and also give an example where the canonical outer encoding factors nontrivially onto a nontrivial inner encoding. Example 4.9 consists of a substitution with uncountably many singular fibres which is almost bijective. Finally in Examples 4.14 and 4.22 we have a substitution which factors onto a bijective substitution, indirectly, i.e., via a factor map which sends fixed points to non-fixed points.

2. PRELIMINARIES

2.1. Constant length substitutions. A *length- ℓ substitution* θ is an ordered collection of ℓ maps, the so-called *column maps* $\theta_m : \mathcal{A} \rightarrow \mathcal{A}$, $m = 0, \dots, \ell - 1$, on a finite set \mathcal{A} (or \mathcal{A}_θ if more precision is needed), its alphabet. The substitution θ can be understood as a map which associates to a letter $a \in \mathcal{A}$ the word $\theta(a) := \theta_0(a) \cdots \theta_{\ell-1}(a)$ and to a word $a_1 \cdots a_k$ the word

$$(2.1) \quad \theta(a_1 \cdots a_k) = \theta(a_1) \cdots \theta(a_k),$$

of length $k\ell$, and to the bi-infinite sequences $\cdots u_{-2}u_{-1}u_0u_1 \cdots$ the bi-infinite sequence

$$\theta(\cdots u_{-2}u_{-1}u_0u_1 \cdots) := \cdots \theta(u_{-2})\theta(u_{-1})\theta(u_0) \cdot \theta(u_0)\theta(u_1) \cdots.$$

Here the \cdot indicates the position between the negative indices and the nonnegative indices.

A bi-infinite sequence u is *θ -periodic* if $\theta^k(u) = u$ for some $k \geq 1$. If $k = 1$ then we say that u is a *fixed point*. By taking a power of θ if necessary, we will assume that each θ -periodic point is θ -fixed. We say that a finite word is *allowed* for θ if it appears somewhere in $\theta^k(a)$ for some $a \in \mathcal{A}$ and some $k \in \mathbb{N}$. The *substitution shift* (X_θ, σ) is the dynamical system where the space X_θ consists of all bi-infinite sequences all of whose subwords are allowed for θ . We equip X_θ with the subspace topology of the product topology on $\mathcal{A}^{\mathbb{Z}}$, making the left shift map σ a continuous \mathbb{Z} -action.

In this article our techniques are a combination of algebraic arguments involving finite semigroups, and dynamical techniques applied to the dynamical system

(X_θ, σ) generated by θ . For the algebraic arguments, very few constraints are imposed on θ . For the dynamical arguments, we collect the various properties of substitutions which will play a role.

- *Primitivity.* We say that θ is *primitive* if there is some $k \in \mathbb{N}$ such that for any $a, a' \in \mathcal{A}$, the word $\theta^k(a)$ contains at least one occurrence of a' . For dynamical arguments, substitutions will mostly be assumed primitive. Primitivity of θ implies that X_θ is the shift-orbit closure of any θ -periodic point, and (X_θ, σ) is minimal. If θ is primitive, then $X_\theta = X_{\theta^n}$ for each $n \in \mathbb{N}$. Thus, by considering a power of θ if necessary, we will assume that all θ -periodic points are θ -fixed.
- *Aperiodicity.* We say that θ is *aperiodic* if X_θ does not contain any σ -periodic sequences. This is the case if and only if X_θ is an infinite space.
- *Bijectivity.* We say that θ is *bijective* if all column maps $\theta_m : \mathcal{A} \rightarrow \mathcal{A}$ are bijective.

2.2. The maximal equicontinuous factor of a length- ℓ substitution. Let \mathbb{Z}_ℓ denote the ℓ -adic integers, i.e., the inverse limit of cyclic groups $\varprojlim \mathbb{Z}/\ell^n \mathbb{Z}$. Let $\mathbb{Z}_{\bar{\ell}, h} := \varprojlim \mathbb{Z}/\ell^n h \mathbb{Z}$ and let $1 := (\cdots, 0, 0, 1)$; addition in $\mathbb{Z}_{\bar{\ell}, h}$ is performed with carry. If θ is primitive and aperiodic, then Dekking's theorem [6] tells us that $(\mathbb{Z}_\ell, +1)$ is an equicontinuous factor of (X_θ, σ) . Furthermore, there is an h , with $0 < h < \ell$, with h coprime to ℓ , such that $(\mathbb{Z}_{\bar{\ell}, h}, +1)$, is the maximal equicontinuous factor of (X_θ, σ) . The integer h is called the *height* of θ , and we say that θ has *trivial height* if $h = 1$.

We fix the factor map $\pi : X_\theta \rightarrow \mathbb{Z}_\ell$ from a primitive aperiodic length- ℓ substitution shift (X_θ, σ) to $(\mathbb{Z}_\ell, +1)$ with which we work in this article. We will specify it by requiring $\pi(u) = 0$ if and only if u is a θ -fixed point. We refer the reader to [6] for details.

Given the substitution θ , the substitution θ^n is a length- ℓ^n substitution. If $0 \leq j \leq \ell^n - 1$, we use θ_j^n to denote its j -th column map. The *column rank* of a substitution θ is defined as the minimal number of distinct letters in the image of a column map of θ^n , for some n . In other words,

$$(2.2) \quad c = c(\theta) := \inf_{j, n} \{ |\theta_j^n(\mathcal{A})| : 0 \leq j < \ell^n \}.$$

We say that θ has a *coincidence* if $c = h$. In this case, (X_θ, σ) is *almost automorphic*, i.e., an almost one-to-one extension of its maximal equicontinuous factor. For details, see [6]. We remark that our notion of column rank agrees with Dekking's original definition of column number when $h = 1$; however it is in this paper more useful to use the present definition, as it is also in [15], where Lemanczyk and Müllner show that h divides c , and that (X_θ, σ) is a somewhere c -to-one extension of $(\mathbb{Z}_\ell, +1)$.

2.3. Collared and shifted substitutions. It will be necessary to consider collared substitutions and k -shifted substitutions of the substitution θ , which yield shifts that are topologically conjugate to (X_θ, σ) . The notation we use in the following definition will be useful when we consider recasting factor maps of left radius $l \geq 0$ and right radius $r \geq 0$ as codings.

Definition 2.1. Let $n = (-l, r)$ where $l, r \geq 0$. The n -collared extension of θ is the substitution $\theta^{(n)}$ of the same length whose alphabet consists of the allowed

$r + 1 + l$ -letter words of θ and which is given as follows. Given an allowed word $a_{-l} \dots a_r$ compute $a'_{-l} \dots a'_{\ell(r+1)-1} := \theta(a_{-l} \dots a_r)$ and set

$$\theta^{(n)}_m(a_{-l} \dots a_r) := a'_{m-l} \dots a'_{m+r}.$$

If we take $l = r = 0$ then we obtain $\theta^{(n)} = \theta$. If $l = 0$ then $\theta^{(n)}$ is the so-called r -sliding block representation of θ ; see [19, Section 5.4]. We denote by $\mathcal{A}^{(2)} \subset \mathcal{A}^2$ the set of allowed 2-letter words.

Definition 2.2. Let $0 \leq k \leq \ell - 1$. The k -shifted extension of θ is the substitution $\theta^{(+k)}$ of the same length whose alphabet is $\mathcal{A}^{(2)}$ and which is given as follows. Given an allowed word $a_0 a_1$, write $\theta(a_0 a_1) = a'_0 \dots a'_{2\ell-1}$ and set

$$\theta^{(+k)}_m(a_0 a_1) := a'_{m+k} a'_{m+k+1}.$$

Note that $\theta^{(+0)} = \theta^{(0,1)}$.

2.4. Factors, codes and encoded substitutions. Recall that a factor map between two shifts, $X \subset \mathcal{A}^{\mathbb{Z}}$, $Y \subset \mathcal{B}^{\mathbb{Z}}$ is a continuous surjective map $F : X \rightarrow Y$ which intertwines the shifts, $F \circ \sigma = \sigma \circ F$. The Curtis-Hedlund-Lyndon theorem states that such a factor map is defined by a local rule, that is, there are integers $l \geq 0$ and $r \geq 0$ and a surjective map $\varphi : \mathcal{A}^{r+1+l} \rightarrow \mathcal{B}$, so that

$$(F(x))_n = \varphi(x_{n-l}, \dots, x_{n+r})$$

for each $n \in \mathbb{Z}$. The quantities l, r are called the left and right radius of F respectively. If F is radius zero, i.e., $l = r = 0$, then the local rule $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ of F is called a *code*. We use uppercase letters to denote factor maps, and lowercase greek letters to denote local rules. If we are given a local rule φ which defines a factor map, we will denote it by F_φ .

Definition 2.3. Let η and θ be length- ℓ substitutions. We say that η is an *inner encoding* of θ if there exists a surjective map $\beta : \mathcal{A}_\theta \rightarrow \mathcal{A}_\eta$ which intertwines the column maps of the substitutions, i.e.,

$$\begin{array}{ccc} \mathcal{A}_\theta & \xrightarrow{\theta_m} & \mathcal{A}_\theta \\ \beta \downarrow & & \downarrow \beta \\ \mathcal{A}_\eta & \xrightarrow{\eta_m} & \mathcal{A}_\eta \end{array}$$

commutes.

If we need more precision then we denote the inner encoding also by the pair (η, β) . A diagram chase shows that if θ is primitive then also η is primitive. If (η, β) is an inner encoding of θ , then β is the code of a factor map $F_\beta : X_\theta \rightarrow X_\eta$. However, given an arbitrary code $\beta : \mathcal{A}_\theta \rightarrow \mathcal{A}_\eta$ for a factor map $F_\beta : X_\theta \rightarrow X_\eta$, it will in general not intertwine the substitutions as above. We will see an example of this below.

Lemma 2.4. Consider a substitution θ with its $n = (-l, r)$ -collaring $\theta^{(n)}$. Let $\iota : \mathcal{A}_{\theta^{(n)}} \rightarrow \mathcal{A}_\theta$ be given by

$$(2.3) \quad \iota(a_{-l} \dots a_r) := a_0.$$

Then (θ, ι) is an inner encoding of $\theta^{(n)}$, and the factor map F_ι is injective.

Proof. Clearly $\iota \circ \theta^{(n)} = \theta \circ \iota$. To proceed we consider $n = (0, 1)$ to not overburden the notation. By definition, every 2-letter word of a sequence $x \in X_{\theta^{(0,1)}}$ is allowed for $\theta^{(0,1)}$ and hence of the form $\cdots (a_i a_{i+1})(a_{i+1} a_{i+2}) \cdots$. Hence

$$\cdots (a_i a_{i+1})(a_{i+1} a_{i+2}) \cdots \xrightarrow{F_i} \cdots a_i a_{i+1} \cdots$$

is injective. The argument for different n is similar. \square

Corollary 2.5. *The code $\mathcal{A}_{\theta^{(+k)}} \ni a_0 a_1 \xrightarrow{\iota} a_0 \in \mathcal{A}_\theta$ gives rise to an injective factor map $F_i : X_{\theta^{(+k)}} \rightarrow X_\theta$ (a conjugacy) which satisfies*

$$\sigma^k \circ \theta \circ F_i = F_i \circ \theta^{(+k)}.$$

Proof. All $\theta^{(+k)}$ have the same alphabet, namely $\mathcal{A}^{(2)}$, and the same shift space as $\theta^{(0,1)}$. Indeed, with the notation of Definition 2.2,

$$a_0 a_1 \xrightarrow{\theta^{(+k)}} (a'_k a'_{k+1})(a'_{k+1} a'_{k+2}) \cdots (a'_{\ell+k-1} a'_{\ell+k})$$

and

$$a_0 a_1 \xrightarrow{\theta^{(0,1)}} (a'_0 a'_1)(a'_1 a'_2) \cdots (a'_{\ell-1} a'_\ell)$$

so that as maps on that shift space we have

$$\theta^{(+k)} = \sigma^k \circ \theta^{(0,1)}.$$

Now the first statement is Lemma 2.4 for $l = 0$ and $r = 1$ and the equation follows from $F_i \circ \theta^{(0,1)} = \theta \circ F_i$. \square

We say that a factor map $F : X_\theta \rightarrow X_\eta$ between two substitution shifts of equal length *preserves the fixed point fibre* if it maps the fixed points of any power of θ to fixed points of any power of η .

Lemma 2.6. *Let η and θ be length- ℓ substitutions and let $\beta : \mathcal{A}_\theta \rightarrow \mathcal{A}_\eta$ be a code. If (η, β) is an inner encoding of θ then the factor map F_β preserves the fixed point fibre. Hence if (η, β) is an aperiodic inner encoding of θ , then the equicontinuous factor map $\pi : (X_\theta, \sigma) \rightarrow \mathbb{Z}_\ell$ factors through F_β . Conversely, if θ is primitive and $F_\beta : X_\theta \rightarrow X_\eta$ preserves the fixed point fibre, then (η, β) is an inner encoding.*

Proof. Recall that we can assume, by going over to a power of the substitution if needed, that all θ -periodic points of θ are fixed. Let $v.u$ be a fixed point of θ and hence also of θ^m for any $m \geq 1$. We denote by $u = u_0 \dots$ and $\dots v_{-1} = v$ the right and left infinite parts.

Suppose that η is inner encoded by β , that is, for each n and a , $\beta(\theta^n(a)) = \eta^n(\beta(a))$. Then

$$\begin{aligned} F_\beta(v.u) &= \lim_n \beta \theta^n(v_{-1}.u_0) = \lim_n \eta^n(\beta(v_{-1}).\beta(u_0)) \\ &= \eta(\lim_n \eta^n(\beta(v_{-1}).\beta(u_0))) = \eta(F_\beta(v.u)). \end{aligned}$$

The assumption that η is aperiodic implies by Dekking's theorem that both substitution shifts have $(\mathbb{Z}_\ell, +1)$ as an equicontinuous factor. Recall that we have fixed these equicontinuous factor maps by requiring that they map the fixed point fibre to $0 \in \mathbb{Z}_\ell$. As F maps the fixed point fibre of θ to that of η we see that the equicontinuous factor map $\pi : X_\theta \rightarrow \mathbb{Z}_\ell$ factors through F at least for all points in the orbit of a fixed point of θ , and then by minimality for all points of X_θ .

Finally we prove the converse. Suppose that F_β sends fixed points to fixed points, so that $F_\beta(v.u)$ is a fixed point of η^n for any $n \geq 1$. If $m < \ell^n$, then β maps the

m -th letter of u , which is $\theta_m^n(u_0)$, to the m th letter of $\beta(u)$, which is $\eta_m^n(\beta(u_0))$. In particular, β maps $\theta_m^n(\theta_k(u_0))$ to $\eta_m^n(\eta_k(\beta(u_0))) = \eta_m^n(\beta(\theta_k(u_0)))$. If θ is primitive all letters arise as $\theta_k^N(u_0)$ for some N and k showing that $\eta_m(\beta(a)) = \beta(\theta_m(a))$ for all $a \in \mathcal{A}_\theta$. \square

Note that, if θ is aperiodic, then, for $k > 0$, $\theta^{(+k)}$ cannot be an inner encoding. Indeed, let u be a fixed point of $\theta^{(+k)}$ and suppose that F_i maps u to a fixed point v of θ . Applying the equation of Corollary 2.5 we get

$$v = F_i \circ \theta^{(+k)}(u) = \sigma^k \circ \theta \circ F_i(u) = \sigma^k(v)$$

a contradiction.

Any map $\beta : \mathcal{A} \rightarrow \mathcal{B}$ between sets defines a partition $\mathcal{P}_\beta = \{\beta^{-1}(b) : b \in \mathcal{B}\}$. We call \mathcal{P}_β the *partition associated to β* .

Lemma 2.7. *Let θ be a length- ℓ substitution on \mathcal{A} .*

(1) *If (η, β) is an inner coding of θ then the partition \mathcal{P}_β associated to β satisfies*

$$\forall m \forall A \in \mathcal{P}_\beta \exists B \in \mathcal{P}_\beta \text{ such that } \theta_m(A) \subset B.$$

(2) *Conversely, if there is a partition \mathcal{P} of \mathcal{A} such that*

$$\forall m \forall A \in \mathcal{P} \exists B \in \mathcal{P} \text{ such that } \theta_m(A) \subset B,$$

then the canonical projection $\beta : \mathcal{A} \rightarrow \mathcal{P}$ defines an inner coding (η, β) of θ through $\eta_m := \beta \theta_m \beta^{-1}$.

Proof. Suppose that (η, β) is an inner coding of θ . Then for all m we have $\eta_m \circ \beta = \beta \circ \theta_m$. This implies that for all $b \in \mathcal{B}$, $\beta \theta_m$ has the same value on all elements of $\beta^{-1}(b)$. In other words for any $A \in \mathcal{P}_\beta$ we have that $\beta \theta_m(A)$ is a singleton, and this implies that $\theta_m(A)$ must be a subset of a member of \mathcal{P}_β .

As for the converse, if \mathcal{P} is a partition with the required property then we can define $\mathcal{B} := \mathcal{P}$, the code $\beta : \mathcal{A} \rightarrow \mathcal{B}$ to be the map that sends $a \in \mathcal{A}$ to the member of \mathcal{P} to which it belongs and for $b \in \mathcal{B}$, $\eta_m(b)$ is defined to be the member of \mathcal{P} which contains $\theta_m(b)$, i.e., $\eta_m := \beta \theta_m \beta^{-1}$. \square

If \mathcal{P} satisfies (2) of Lemma 2.7, we call the associated inner encoding the *inner encoding defined by the partition \mathcal{P}* .

Although not all codes give rise directly to inner encoded substitutions, they induce an inner encoded substitution in the following way: Let θ be a substitution on the alphabet \mathcal{A} and $\tau : \mathcal{A} \rightarrow \mathcal{B}$ be a code. If \mathcal{P}_τ does not satisfy Condition (2) of Lemma 2.7, we can define a finer partition $\tilde{\mathcal{P}}_\tau$ which has this property, notably through the equivalence relation $a \sim b$ if $\forall n \geq 0, 0 \leq m < \ell^n, \tau(\theta_m^n(a)) = \tau(\theta_m^n(b))$. We denote the associated inner encoded system by (η_τ, β_τ) and call it the *inner encoding defined by τ* . The alphabet of η_τ is usually smaller than \mathcal{A} and usually larger than \mathcal{B} . There is thus a code $\tau' : \mathcal{A}_{\eta_\tau} \rightarrow \mathcal{B}$ which comes with η and satisfies $\tau = \tau' \circ \beta_\tau$. This is summarised in the left hand side of Figure 1.

2.5. Factors of substitution shifts. Let $\mathcal{F}(X)$ denote the semigroup of maps from X to itself. In this article we will be working extensively with the following semigroup.

Definition 2.8. The semigroup S_θ of a length- ℓ substitution θ over the alphabet \mathcal{A} is the subsemigroup of $\mathcal{F}(\mathcal{A})$ generated by the column maps $\theta_i, i = 0, \dots, \ell - 1$.

$$\begin{array}{ccc}
\mathcal{A}_\theta & \xrightarrow{\beta_\tau} & \mathcal{A}_{\eta_\tau} \\
\tau \downarrow & \swarrow & \tau' \\
\mathcal{B} & &
\end{array}
\qquad
\begin{array}{ccc}
X_\theta & \xrightarrow{F_{\beta_\tau}} & X_{\eta_\tau} \\
F_\tau \downarrow & \swarrow & F_{\tau'} \\
Y & &
\end{array}$$

FIGURE 1. On the left: The commuting diagram of codes defined by a code τ and its inner encoding (η_τ, β_τ) . On the right: The commutative diagram of factor maps induced by the codes on the left. Under its assumptions Theorem 2.9 implies that $F_{\tau'}$ is a conjugacy.

In this section our aim is to show that, up to conjugacy, any dynamical factor of a primitive, aperiodic, constant length substitution θ of trivial height is an inner encoding of a collaring $\theta^{(n)}$ of θ , where $n = (-l, r)$ with $l, r \leq 1$. Later we will see using the suspension construction how the result transposes to the case of any height. As the proofs of many of the results in this section are gentle modifications of previously established results, we include them in Appendix A.

We first state in this context Theorem 2.9, which is a slight modification of [17, Theorem 22] when the substitution is assumed of trivial height, and which we use extensively. It focuses on shift factors which are given by a code. Such shift factors are also called ℓ -automatic. While the statement of [17, Theorem 22] does not mention inner encoded substitutions, a look at its proof shows that the substitution referred to in that theorem is exactly the inner encoded substitution defined by the code. In the original statement of Theorem 2.9 the substitution is assumed *pair aperiodic*. This was not a restriction as there is always a power of θ which is pair aperiodic. Here we assume that $S_\theta \subset S_{\theta^2}$, and show in Lemma A.1 that this is also not a restriction as there is always a power of θ which satisfies this condition. The proof of Theorem 2.9, that we provide in Appendix A, is new.

Theorem 2.9. *Let θ be a primitive aperiodic length- ℓ substitution. Suppose that $S_\theta \subset S_{\theta^2}$. Let $F_\tau : X_\theta \rightarrow Y \subset \mathcal{B}^{\mathbb{Z}}$ be a factor which is induced by a code $\tau : \mathcal{A}_\theta \rightarrow \mathcal{B}$. Then there exists (η, β) which is inner encoded by θ and surjective code $\tau' : \mathcal{A}_\eta \rightarrow \mathcal{B}$ such that $\tau = \tau' \circ \beta$ and the induced map $F_{\tau'} : X_\eta \rightarrow \mathcal{B}^{\mathbb{Z}}$ is injective and has image $Y = F_{\tau'}(X_\eta)$. In other words $F_{\tau'}$ is a conjugacy between X_η and Y .*

We have summarised the statement in the second commuting diagram of Figure 1, which follows from the first commuting diagram.

Proposition 2.10. *Let θ be a primitive, aperiodic length- ℓ substitution, and let $F : X_\theta \rightarrow Y \subset \mathcal{B}^{\mathbb{Z}}$ be a factor map. There exists an $n = (-l, r)$ -collaring $\theta^{(n)}$ and a code $\tau : \mathcal{A}_{\theta^{(n)}} \rightarrow \mathcal{B}$ such that $F = F_\tau \circ F_\tau^{-1}$. If, in addition, Y is an aperiodic length- ℓ substitution shift and F preserves the fixed point fibre, then $l, r \leq 1$.*

The statement of Proposition 2.10 is summarised in Figure 2. Its proof is based on that of [5, Proposition 3.19]; we provide it in Appendix A.

Corollary 2.11. *Let θ be an aperiodic, primitive, length- ℓ substitution on \mathcal{A}_θ .*

- (1) *If X_θ has a proper almost automorphic factor then it has a shift factor which is an almost one-to-one extension of the maximal equicontinuous factor of X_θ .*

$$\begin{array}{ccc}
X_\theta & \xleftarrow{F_l} & X_{\theta^{(n)}} \\
& \searrow F & \downarrow F_\tau \\
& & Y
\end{array}$$

FIGURE 2. The commutative diagram for Prop. 2.10.

- (2) If $F : X_\theta \rightarrow Y \subset \mathcal{B}^{\mathbb{Z}}$ is a shift factor then there exists an $n = (-l, r)$ -collaring $\theta^{(n)}$ with $l, r \leq 1$, a natural number $p \geq 1$, a length- l^p substitution η and a code $\beta : \mathcal{A}_{\theta^{(n)}^p} \rightarrow \mathcal{A}_\eta$ such that (η, β) is an inner encoding of $\theta^{(n)P}$ and (X_η, σ) is conjugate to (Y, σ) . Moreover, the conjugacy is given by a code $\tau' : \mathcal{A}_\eta \rightarrow \mathcal{B}$.

Proof. If X_θ has a proper almost automorphic factor then we may assume, by Theorem A.5, that this factor is a proper almost automorphic extension of the maximal equicontinuous factor of X_θ . By [8, Theorem 6.4], this factor is a shift factor.

As for the second statement, we apply Proposition 2.10 to obtain a code with its factor map $F_\tau : X_{\theta^{(n)}} \rightarrow Y$ for some collaring $\theta^{(n)}$ of θ such that the diagram in Figure 2 commutes. Let p be such that $S_{\theta^p} \subset S_{\theta^{2p}}$. We apply Theorem 2.9 to $F_\tau : X_{\theta^{(n)}^p} \rightarrow Y$ to obtain the inner encoding (η_τ, β_τ) of $\theta^{(n)P}$. This situation is summarised in the right half of the following diagram and its left half follows from Lemma 2.4 as $X_{\theta^{(n)}^p}$ is equal to $X_{\theta^{(n)}}$.

$$\begin{array}{ccccc}
X_\theta & \xrightarrow{F_l^{-1}} & X_{\theta^{(n)}^p} & \xrightarrow{F_{\beta_\tau}} & X_{\eta_\tau} \\
& \searrow F & \downarrow F_\tau & \swarrow F_{\tau'} & \\
& & Y & &
\end{array}$$

If $n = (-l, r)$ with $l, r \leq 1$ then we are done, the factor $X_\theta \rightarrow Y$ is conjugate to the factor $F_{\beta_\tau} : X_{\theta^{(n)}^p} \rightarrow X_{\eta_\tau}$ where (η_τ, β_τ) is an inner encoding.

If $n = (-l, r)$ with $l > 1$ or $r > 1$ then we need one more step. As F_l and F_{β_τ} are both obtained from inner encodings, they preserve the fixed point fibres. It follows that the composition $\tilde{F} := F_{\beta_\tau} \circ F_l^{-1} : X_\theta \rightarrow X_{\eta_\tau}$ preserves the fixed point fibre. We repeat the whole argument above but with X_{η_τ} in place of Y . We can apply Proposition 2.10, to obtain the commutative diagram

$$\begin{array}{ccccc}
X_\theta & \xleftarrow{F_l} & X_{\theta^{(\tilde{n})}^{\tilde{p}}} & \xrightarrow{F_{\beta_{\tilde{\tau}}}} & X_{\eta_{\tilde{\tau}}} \\
& \searrow \tilde{F} & \downarrow F_{\tilde{\tau}} & \swarrow F_{\tilde{\tau}'} & \\
& & X_{\eta_\tau} & &
\end{array}$$

however this time with $\tilde{n} = (\tilde{l}, \tilde{r})$ with $\tilde{l}, \tilde{r} \leq 1$, as X_{η_τ} is a substitution shift. This gives us a chain of conjugacies, namely between the factor $F : X_\theta \rightarrow Y$ and $\tilde{F} : X_\theta \rightarrow X_{\eta_\tau}$ as we saw above, and then between $\tilde{F} : X_\theta \rightarrow X_{\eta_\tau}$ and $F_{\beta_{\tilde{\tau}}} : X_{\theta^{(\tilde{n})}^{\tilde{p}}} \rightarrow X_{\eta_{\tilde{\tau}}}$. \square

3. ALMOST AUTOMORPHIC FACTORS

In this section we completely characterise when an aperiodic primitive substitution θ of constant length has a proper almost automorphic factor. Corollary 2.11 tells us that the existence of such a factor implies that there is an inner encoding of a power of a collared version of θ whose associated substitution shift is an almost one-to-one extension of the maximal equicontinuous factor of X_θ . Here we attack the converse. We will find that there are always inner encodings of a collared version of θ which have a coincidence, but the desired almost automorphic factor will exist only if the relevant inner encoding is aperiodic. We first restrict to the case of trivial height, leading to Theorem 3.21, and then treat the general case using the pure base of θ in Corollary 3.26.

3.1. The semigroup of a length- ℓ substitution. We defined the semigroup S_θ of a substitution of constant length θ to be the semigroup generated by the column maps, see Definition 2.8. In Appendix A we provide statements and proofs of some classical results from semigroup theory. We denote by $S_\theta^{(n)}$ the subset of maps of S_θ which have rank (size of the image) smaller or equal n . If $S_\theta^{(n)}$ is not empty then it is a two-sided ideal of S . Recall from (2.2) that the column rank of θ is the smallest rank of a product of column maps. From Corollary A.8 of the appendix we obtain the following.

Lemma 3.1. *Let θ be a constant length substitution. The kernel of S_θ is $S_\theta^{(c)}$, where c is the column rank of θ .*

Definition 3.2. The *minimal sets* of the substitution θ are the images of the maps of S_θ of minimal rank. We denote the family of minimal sets by U_θ , i.e.,

$$U_\theta := \{\text{im}f : f \in \ker S_\theta\}.$$

If U_θ is a cover of \mathcal{A} , that is, $\mathcal{A} = \bigcup_{f \in \ker S_\theta} \text{im}f$, then we call the substitution *essentially surjective*. Given U_θ we define a relation on the members by $A \sim B$ if $A \cap B \neq \emptyset$. The transitive closure of this relation is an equivalence relation on $\bigcup_{f \in \ker S_\theta} \text{im}f$ which defines a partition which we call the *coincidence partition* and denote by \mathcal{P}_θ .

Example 3.3. Consider the following substitution of length 4:

$$\theta : a \mapsto abcc \quad b \mapsto badd \quad c \mapsto cacd \quad d \mapsto dbdc.$$

We see that $\theta_0 = \mathbf{1}$, the identity map. One easily checks that any product of these maps which contains at least one θ_i with $0 < i < 4$ has rank 2 and hence the column rank is 2 and $S_\theta = \{\mathbf{1}\} \cup S_\theta^{(2)}$. $S_\theta^{(2)}$ is the kernel. It contains the following elements:

$$\begin{array}{lllll} \theta_1\theta_2 & : & a \mapsto a & b \mapsto b & c \mapsto a & d \mapsto b \\ \theta_1\theta_1 & : & a \mapsto a & b \mapsto b & c \mapsto b & d \mapsto a \\ \theta_2 & : & a \mapsto c & b \mapsto d & c \mapsto c & d \mapsto d \\ \theta_3\theta_3 & : & a \mapsto d & b \mapsto c & c \mapsto c & d \mapsto d \\ \theta_1 & : & a \mapsto b & b \mapsto a & c \mapsto a & d \mapsto b \\ \theta_1\theta_1\theta_2 & : & a \mapsto b & b \mapsto a & c \mapsto b & d \mapsto a \\ \theta_3 & : & a \mapsto c & b \mapsto d & c \mapsto d & d \mapsto c \\ \theta_3\theta_2 & : & a \mapsto d & b \mapsto c & c \mapsto d & d \mapsto c \end{array}$$

$S_\theta^{(2)}$ has two right ideals, namely one generated by θ_1 and the other by θ_3 . The elements of the right ideal generated by θ_1 have image $\{a, b\}$ while those of the right ideal generated by θ_3 have image $\{c, d\}$. Thus $U_\theta = \{\{a, b\}, \{c, d\}\}$ and since this is a partition we have $\mathcal{P}_\theta = U_\theta$. For later use we also note that $S_\theta^{(2)}$ has two left ideals, one generated by θ_2 and the other by θ_3 .

Lemma 3.4. *A primitive substitution is essentially surjective.*

Proof. Suppose that θ is primitive. Let $a \in \mathcal{A}$ and $b \in \text{img } f$ for some $f \in \ker S_\theta$. By primitivity a occurs in $\theta^N(b)$ for some N . Hence $a \in \text{img } f$ for some $g \in S_\theta$. Clearly $gf \in \ker S_\theta$. Hence $a \in \bigcup_{f \in \ker S_\theta} \text{img } f$. \square

The condition of primitivity is sufficient but not necessary: the length 1 substitution $\theta = \text{id}$ on any alphabet \mathcal{A} is not primitive, but $U_\theta = \{\mathcal{A}\}$ and therefore θ essentially surjective. Not all substitutions are essentially surjective: the length 2 substitution on $\{a, b\}$, $\theta(a) = aa$, $\theta(b) = ab$ is not primitive and b does not belong to a minimal set.

Lemma 3.5. *Let θ be an essentially surjective substitution. Then \mathcal{P}_θ is a partition of \mathcal{A} which satisfies Condition (2) of Lemma 2.7. Its associated inner encoding $(\eta_{\mathcal{P}_\theta}, \beta_{\mathcal{P}_\theta})$ has column rank $c = 1$.*

Proof. Let $A \in U_\theta$, that is, $A = \text{img } g$ for some $g \in \ker S_\theta$. Let $f \in S_\theta$. Then $f(A) \in U_\theta$ as $fg \in \ker S_\theta$. Now if $A \cap A' \neq \emptyset$ then also $f(A) \cap f(A') \neq \emptyset$. Thus if two members A, A' of the cover U_θ belong to the same member of \mathcal{P}_θ then also $f(A)$ and $f(A')$ belong to the same member of \mathcal{P}_θ . This implies Condition (2) of Lemma 2.7 for \mathcal{P}_θ . Furthermore, if f belongs to the kernel of S_θ then $f(A) = \text{img } fg = \text{img } f$ and so $f(A)$ is the same for all $A \in U_\theta$. Hence $\beta_{\mathcal{P}_\theta} f \beta_{\mathcal{P}_\theta}^{-1}$ has rank 1, and so $\eta_{\mathcal{P}_\theta}$ has column rank 1. \square

Definition 3.6. Let θ be an essentially surjective substitution with coincidence partition \mathcal{P}_θ . We call the inner encoding defined by \mathcal{P}_θ the *canonical inner encoding* of θ . We denote this inner encoding $(\eta_{\mathcal{P}_\theta}, \beta_{\mathcal{P}_\theta})$ by $(\eta_\theta, \beta_\theta)$.

The next lemma tells in particular us how the canonical inner codings of θ and its powers are related.

Lemma 3.7. *Let θ be a substitution and $N \geq 1$. Then $\ker S_{\theta^N} = \ker S_\theta$. In particular U_θ and U_{θ^N} coincide. Furthermore, if the substitution is essentially surjective then the inner encodings $(\eta_{\theta^N}, \beta_{\theta^N})$ and $(\eta_\theta, \beta_\theta)$ associated to θ^N and θ satisfy $\eta_{\theta^N} = \eta_\theta^N$ and $\beta_{\theta^N} = \beta_\theta$.*

Proof. By definition of the column rank, θ and θ^N have the same column rank. Clearly $S_{\theta^N}^{(c)} \subset S_\theta^{(c)}$. To see that the inclusion is surjective, recall that any element f of $\ker S_\theta$ is completely regular and hence we can factorise $f = f f^{0N-1}$, where we recall that f^0 is the idempotent generated by f . This shows that $\ker S_\theta = \ker S_{\theta^N}$ and immediately implies $U_\theta = U_{\theta^N}$ and, if U_θ is a cover, that $\beta_\theta = \beta_{\theta^N}$. Let $0 \leq j \leq \ell^N - 1$. There are j_1, \dots, j_N such that $\theta^N_{j_1} = \theta_{j_1} \cdots \theta_{j_N}$. Hence $\eta_{\theta^N_j} = \beta_\theta \theta_{j_1} \cdots \theta_{j_N} \beta_\theta^{-1} = \eta_{\theta_{j_1}} \cdots \eta_{\theta_{j_N}}$. \square

As η_θ has column rank $c = 1$ and h divides c , it has a coincidence and trivial height. Its associated dynamical system is thus almost automorphic. However, it need not be aperiodic, nor, if it is aperiodic, does it have to have the same maximal

equicontinuous factor as θ . The latter happens always if θ has non-trivial height, because then the maximal equicontinuous factor of η_θ is strictly smaller than that of θ . In the rest of this section we will show two things: under the assumption of trivial height, if $\eta_{\theta^{(n)}}$ is periodic for all the collared versions of θ (we only need $n = (-l, r)$ with $l, r \leq 1$) then X_θ does not admit a proper almost automorphic factor, and if the height of θ is non-trivial we can reduce the task to working with the pure base of θ .

We provide an algebraic property of S_θ characterising column rank 1. A semigroup is *left zero* if every element acts as a zero element when multiplying from the left, i.e. $xy = x$ for all $x, y \in S$. If S is completely simple, then it is left zero if and only if the \mathcal{R} -relation is trivial (equal to the diagonal relation).

Lemma 3.8. *A constant length substitution θ has column rank 1 if and only if $\ker S_\theta$ is a left zero semigroup. If this is the case and if θ is essentially surjective then $\ker S_\theta \ni x \mapsto \text{im}x \in \mathcal{A}$ is a bijection.*

Proof. θ has column rank 1 if and only if S_θ contains an element of rank 1 which is equivalent to saying that $\ker S_\theta$ contains exactly the maps of rank 1 of S_θ . Any collection of rank 1 maps from $\mathcal{F}(\mathcal{A})$ forms a left zero semigroup.

Now, suppose that the column rank is $c > 1$. If S_θ has more than one minimal left ideal, then $\ker S_\theta$ is not a left zero semigroup. If S_θ has a unique minimal left ideal, then, as we will see in Theorem 4.1, θ admits an inner encoding η such that S_η is a non-trivial group. In particular $\ker S_\eta = S_\eta$ and since an inner encoding induces an epimorphism from $\ker S_\theta$ to $\ker S_\eta$, $\ker S_\theta$ cannot be left zero.

Any rank 1 map can be identified with the unique element in its image. Hence $\ker S_\eta \ni x \mapsto \text{im}x \in \mathcal{A}$ is injective and, if U_θ is a cover, also surjective. \square

3.2. The canonical outer encoding.

Definition 3.9. Let θ and η be length- ℓ substitutions. We say that η is *outer encoded* by θ if there is an epimorphism $\Phi : S_\theta \rightarrow S_\eta$ such that $\eta_m = \Phi(\theta_m)$ for $0 \leq m \leq \ell - 1$.

We may also say that (η, Φ) is outer encoded by θ , that η is an outer encoded substitution of θ , or that η is an outer encoding of θ . Note that if (η, β) is an inner encoding of θ , then η is outer encoded with epimorphism

$$\Phi_\beta(f) = \beta f \beta^{-1}.$$

However an outer encoding does not necessarily define an inner encoding or induces a factor map; see Example 3.16.

Corollary 3.10. *If (η, Φ) is outer encoded by θ , then the restriction of Φ to $\ker S_\theta$ is an epimorphism onto $\ker S_\eta$.*

Proof. This follows from Lemma A.10 as S_θ is finite and hence admits a kernel. \square

Definition 3.11. Let θ be a length- ℓ substitution with column rank c . Let $\mathcal{B} = \ker(S_\theta)/\mathcal{R}$, and ${}^c\Phi : S_\theta \rightarrow \mathcal{F}(\mathcal{B})$ be the morphism defined by

$$(3.1) \quad {}^c\Phi(f)([x]_{\mathcal{R}}) := [fx]_{\mathcal{R}}.$$

The *canonical outer encoding* of θ is the substitution ${}^c\theta : \mathcal{B} \rightarrow \mathcal{B}^\ell$ defined by

$$(3.2) \quad {}^c\theta_m := {}^c\Phi(\theta_m).$$

The semigroup of a bijective substitution is a group. Therefore $S_\theta = \ker S_\theta$ and S_θ/\mathcal{R} consists of a single point. Hence the canonical outer encoding of θ is the one letter periodic substitution, which is, of course, a trivial inner encoding.

Example 3.12. The canonical outer encoded substitution of the collared Thue-Morse system $\theta^{(0,1)}$ is the period doubling substitution, and it is also an inner encoding, see Section 3.5. Note that we must take a collared version of θ , as the canonical outer encoding of any bijective substitution is trivial.

We are interested in finding almost automorphic factors. The following lemma explains our interest in canonical outer encodings. Note there is no need to assume that θ is primitive.

Lemma 3.13. *The canonical outer encoding $({}^c\theta, {}^c\Phi)$ of θ is primitive and has column rank 1, i.e., has trivial height and a coincidence.*

Proof. We will show that there is an element $f \in S_{c_\theta}$ whose image contains only one letter; this implies that ${}^c\theta$ has column rank 1 and therefore a coincidence and trivial height. By Lemma A.10, ${}^c\Phi$ restricts to an epimorphism from $\ker S_\theta$ to $\ker S_{c_\theta}$. Given $f \in \ker S_\theta$ and $[y]_{\mathcal{R}} \in \ker S_\theta/\mathcal{R}$ we have ${}^c\Phi(f)([y]_{\mathcal{R}}) = [fy]_{\mathcal{R}} = [f]_{\mathcal{R}}$. We thus see that the image of ${}^c\Phi(f)$ contains only the letter $[f]_{\mathcal{R}}$. Hence the column rank is 1.

To see that ${}^c\theta$ is primitive, we need to show that for any two letters $[x], [y] \in \ker S_{c_\theta}/\mathcal{R}$, there is an $f \in S_{c_\theta}$ such that $[fx] = [y]$. As any left ideal of $\ker S_\theta$ intersects any right ideal of $\ker S_\theta$, any two classes $[x], [y] \in \ker S_{c_\theta}/\mathcal{R}$ have representatives x, y which belong to the same \mathcal{L} -class. This means that there is $f \in S_{c_\theta}$ such that $y = fx$. \square

Proposition 3.14. *Let θ be an essentially surjective substitution and $({}^c\theta, {}^c\Phi)$ be the canonical outer encoding of θ . Then ${}^c\theta$ is an inner encoding of θ if and only if U_θ is a partition of \mathcal{A} . In this case, and upon identifying $\ker S_\theta/\mathcal{R}$ with U_θ , we have ${}^c\theta = \eta_\theta$, the canonical inner encoding of θ .*

Proof. Recall that the alphabet of ${}^c\theta$ is $\ker S_\theta/\mathcal{R}$. By Lemma A.7 the map $\ker S_\theta/\mathcal{R} \ni [x]_{\mathcal{R}} \mapsto \text{im}x \in U_\theta$ is a bijection.

If U_θ is a partition of \mathcal{A} then it coincides with the coincidence partition \mathcal{P}_θ and hence we may identify $\beta_\theta : \mathcal{A} \rightarrow \mathcal{P}_\theta$ with the code $a \mapsto [x]_{\mathcal{R}}$ where x is any function from $\ker S_\theta$ which contains a in its image. Say $a = x(b)$. We then have ${}^c\theta_m(\beta_\theta(a)) = [\theta_m x]_{\mathcal{R}}$ while $\beta_\theta(\theta_m(a)) = \beta_\theta(\theta_m x(b)) = [\theta_m x]_{\mathcal{R}}$. Hence ${}^c\theta = \eta_\theta$.

Suppose that ${}^c\theta$ is an inner encoding of θ , i.e. there is a code $\beta : \mathcal{A} \rightarrow \ker S_\theta/\mathcal{R}$ such that $\beta\theta_m = {}^c\theta_m\beta$. By Lemma 3.13, ${}^c\theta$ has column rank 1 and so $\ker S_{c_\theta}$ contains only rank 1 maps. Hence, for $g_1, g_2 \in \ker S_{c_\theta}$ the condition $g_1 \neq g_2$ is equivalent to $\text{im}g_1 \cap \text{im}g_2 = \emptyset$. Let $f_i \in ({}^c\Phi)^{-1}(g_i) \cap \ker S_\theta$. If $\text{im}f_1 \cap \text{im}f_2 \neq \emptyset$ then there are $a_1, a_2 \in \mathcal{A}$ such that $f_1(a_1) = f_2(a_2)$. It follows that $g_1(\beta(a_1)) = \beta f(a_1) = \beta f(a_2) = g_2(\beta(a_2))$, hence $\text{im}g_1 \cap \text{im}g_2 \neq \emptyset$, hence $g_1 = g_2$. Thus $\text{im}f_1 \cap \text{im}f_2 \neq \emptyset$ implies ${}^c\Phi(f_1) = {}^c\Phi(f_2)$ which means that f_1 and f_2 are \mathcal{R} -related. By Lemma A.6 they then have the same image. Thus the elements of U_θ either coincide or do not intersect. As we assumed that $U_{\mathcal{A}}$ covers \mathcal{A} it is a partition of \mathcal{A} . \square

Example 3.15. We return to Example 3.3 whose cover of minimal sets U_θ is a partition and thus the canonical outer encoding is inner encoded. Setting $A = \{a, b\}$

and $C = \{c, d\}$ the inner encoded substitution is given by

$$\eta_\theta : A \mapsto AACC \quad C \mapsto CACC.$$

Clearly it is aperiodic, and hence X_{η_θ} a proper, almost automorphic factor of X_θ .

Example 3.16. Consider the substitution

$$\begin{aligned} \theta : a &\mapsto acaef & e &\mapsto egaef \\ b &\mapsto bdbde & f &\mapsto dfbfg \\ c &\mapsto ceccg & g &\mapsto cecge. \\ d &\mapsto dfbde \end{aligned}$$

Its minimal sets are $U_\theta = \{A = \{a, b, c\}, B = \{c, d, e\}, C = \{e, f, g\}\}$. The canonical outer encoding is $A \mapsto ABABC, B \mapsto BCABC, C \mapsto BCACC$. The cover U_θ does not form a partition. It generates the partition $\mathcal{P}_\theta = \{\{a, b, c, d, e, f, g\}\}$ which leads to the periodic substitution

$$\eta_\theta : D \mapsto DDDDD$$

and thus we cannot conclude that X_θ has a proper almost automorphic factor.

Lemma 3.17. *Any inner encoding of an essentially surjective substitution is essentially surjective.*

Proof. Let (η, β) be inner encoded by $\theta, \beta : \mathcal{A} \rightarrow \mathcal{B}$. Let $b \in \mathcal{B}$. As U_θ covers \mathcal{A} there is $a \in \mathcal{A}$ and $f \in \ker S_\theta$ such that $\beta(f(a)) = b$. Hence $\beta f \beta^{-1}(\beta(a)) = b$. Moreover, $\beta f \beta^{-1} \in \ker S_\eta$. Hence $b \in U_\eta$. \square

Proposition 3.18. *Let θ be a substitution which is essentially surjective. Let ${}^c\theta$ be the canonical outer encoding of θ . If (η, Φ) is outer encoded by θ , and η is essentially surjective and has column rank 1, then (η, Φ) is outer encoded by ${}^c\theta$.*

Proof. Let $({}^c\eta, {}^c\Phi_\eta)$ be the canonical outer encoding of η . We claim that there is a morphism $\varphi : S_{{}^c\theta} \rightarrow S_{{}^c\eta}$ such that the diagram

$$\begin{array}{ccc} S_\theta & \xrightarrow{{}^c\Phi} & S_{{}^c\theta} \\ \Phi \downarrow & & \downarrow \varphi \\ S_\eta & \xrightarrow{{}^c\Phi_\eta} & S_{{}^c\eta} \end{array}$$

is commutative. By Lemma A.10 $\Phi(\ker S_\theta) = \ker S_\eta$ and Φ preserves the \mathcal{R} -relation. Therefore, given $t, s \in S_\theta$ and $x \in \ker S_\theta$, $[tx]_{\mathcal{R}_\theta} = [sx]_{\mathcal{R}_\theta}$ implies $[\Phi(t)\Phi(x)]_{\mathcal{R}_\eta} = [\Phi(s)\Phi(x)]_{\mathcal{R}_\eta}$ (here \mathcal{R}_θ is the \mathcal{R} -relation on S_θ and \mathcal{R}_η is the \mathcal{R} -relation on S_η). Also by Lemma A.10, any $y \in \ker S_\eta$ has a pre-image under Φ in $\ker S_\theta$, so we see that ${}^c\Phi(t) = {}^c\Phi(s)$, which is $[tx]_{\mathcal{R}_\theta} = [sx]_{\mathcal{R}_\theta}$ for all $x \in \ker S_\theta$, implies ${}^c\Phi_\eta(\Phi(t)) = {}^c\Phi_\eta(\Phi(s))$, which is $[\Phi(t)y]_{\mathcal{R}_\eta} = [\Phi(s)y]_{\mathcal{R}_\eta}$ for all $y \in S_\eta$. Thus φ is well defined through the formula $\varphi({}^c\Phi(t)) = {}^c\Phi_\eta(\Phi(t))$.

As η has column rank 1 and is essentially surjective, $\ker S_\eta$ can be identified with the alphabet of η and therefore ${}^c\Phi_\eta : S_\eta \rightarrow \mathcal{F}(\ker S_\eta)$ is injective. Hence $\Phi(t) = ({}^c\Phi_\eta)^{-1}(\varphi({}^c\Phi(t)))$ showing that it factors through ${}^c\Phi$. \square

Definition 3.19. Let θ be an essentially surjective substitution. Its *maximal inner encoding with column rank 1* is an inner encoded substitution η of θ , such that any other inner encoding of θ which has column rank 1 factors, via an inner encoding, through η .

We now show that maximal inner encodings with column rank 1 exist.

Theorem 3.20. *Let θ be an essentially surjective substitution. Its maximal inner encoding with column rank 1 is the canonical inner encoding η_θ of θ . In particular, θ admits an aperiodic inner encoding with column rank 1 if and only if its canonical inner encoding η_θ is aperiodic.*

Proof. By Lemma 3.5 η_θ is an inner encoding with column rank 1. Suppose that η is an inner encoding of θ with column rank 1. We saw in Proposition 3.18 that it must be an outer encoding of the canonical outer encoding ${}^c\theta$. Furthermore the epimorphism $\Phi : S_{{}^c\theta} \rightarrow S_\eta$ restricts to an epimorphism from $\ker S_{{}^c\theta}$ to $\ker S_\eta$. Since both have column rank 1, this restriction of Φ is an epimorphism between left zero semigroups, hence a surjective map from the alphabet of ${}^c\theta$, which is U_θ , to the alphabet of η . On the other hand, the code from \mathcal{A} to the alphabet of η is given by a partition \mathcal{P} . Any member of U_θ must therefore be a subset of an element of \mathcal{P} . Thus any element of \mathcal{P}_θ is a subset of an element of \mathcal{P} . It follows that η is an inner encoding of the inner encoding defined by \mathcal{P}_θ , which is the canonical inner encoding η_θ . \square

3.3. Trivial height. Recall that a substitution has a coincidence if and only if its height coincides with its column rank. In this subsection we consider the case in which θ has height 1. This allows us to exploit the above results about inner encoded substitutions with column rank 1 for the analysis of almost automorphic factors.

Theorem 3.21. *Let θ be a primitive substitution of trivial height. Then X_θ has a proper almost automorphic factor if and only if the canonical inner encoding associated to $\theta^{(n)}$, for some $n = (-l, r)$, $0 \leq l, r \leq 1$, is aperiodic.*

We remark that it is sufficient to check whether the canonical inner encoding of $\theta^{(-1,1)}$ is aperiodic, as all other $\theta^{(n)}$ are inner encodings of $\theta^{(-1,1)}$. However if one suspects that X_θ has a proper almost automorphic factor, in practice it is easier to check whether X_θ or $X_{\theta^{(0,1)}}$ give almost automorphic factors, especially if doing these computations by hand.

Proof. Suppose that X_θ has a proper almost automorphic factor. By Cor. 2.11, we may assume that this factor is a shift factor which is an almost one-to-one extension of the maximal equicontinuous factor of X_θ , and furthermore, that it is given by some inner encoding of $\theta^{(n)^p}$ for some $|n| \leq 1$ and $p \geq 1$. As the factor is properly almost automorphic, this inner encoding must be aperiodic and have column rank h , the height of θ . But by assumption, $h = 1$. It now follows from Theorem 3.20 that the canonical inner encoding $\eta_{(\theta^{(n)})^p}$ is aperiodic and has column rank 1. By Lemma 3.7 $\eta_{(\theta^{(n)})^p} = \eta_{\theta^{(n)}}^p$ and so also $\eta_{\theta^{(n)}}$ is aperiodic and has column rank 1.

Conversely, the substitution shift defined by the canonical inner encoding of a collaring $\theta^{(n)}$ with $|n| \leq 1$ is an almost automorphic factor of X_θ , as canonical inner encodings have column rank 1. This factor is equicontinuous if and only if the substitution is periodic. \square

Example 3.22. Consider the Rudin-Shapiro substitution, which is defined as

$$\begin{aligned} \theta : a &\mapsto ac & c &\mapsto ab \\ b &\mapsto dc & d &\mapsto db. \end{aligned}$$

It is known that this substitution has a proper almost automorphic factor, see for example [1]; we redo the calculation in our setting. Its minimal sets are $\{\{a, d\}, \{b, c\}\}$, which form a partition but which yield a periodic substitution. The substitution $\theta^{(0,1)}$ is

$$\begin{aligned} \theta^{(0,1)} : A &\mapsto BF & E &\mapsto AC \\ B &\mapsto BE & F &\mapsto AD \\ C &\mapsto HE & G &\mapsto GD \\ D &\mapsto HF & H &\mapsto GC, \end{aligned}$$

and the minimal sets are $\{\{A, H\}, \{B, G\}, \{C, F\}, \{E, D\}\} = \{p, q, r, s\}$, which again form a partition, and whose associated substitution is

$$\begin{aligned} \eta_{\theta^{(0,1)}} : p &\mapsto qr & r &\mapsto ps \\ q &\mapsto qs & s &\mapsto pr, \end{aligned}$$

which can be verified to be aperiodic.

Example 3.23. We return to Example 3.16 which has height 1. We saw that the coincidence partition of θ contains only one element and so θ has no aperiodic inner encoding with a coincidence. By the comment after Theorem 3.21, it is enough to check whether the canonical inner encoding of $\theta^{(-1,1)}$ is aperiodic. A more elaborate calculation shows that the coincidence partition of $\theta^{(-1,1)}$ has three elements $\mathcal{P}_{\theta^{(-1,1)}} = \{\alpha, \beta, \gamma\}$ and that the canonical inner encoded substitution is given by

$$\begin{aligned} \eta_{\theta^{(-1,1)}} : \alpha &\mapsto \beta\gamma\beta\alpha\alpha \\ \beta &\mapsto \beta\gamma\beta\alpha\alpha \\ \gamma &\mapsto \beta\gamma\beta\alpha\alpha \end{aligned}$$

which is periodic. We conclude that X_θ does not admit a proper almost automorphic factor.

3.4. Non-trivial height. One direction of Theorem 3.21 does not need the assumption of trivial height. Indeed, if the canonical inner encoding η_θ of θ is aperiodic then X_{η_θ} is a proper almost automorphic factor of X_θ , though one with trivial height. Using Theorem A.5 we can then even obtain a proper almost automorphic factor which has the same maximal equicontinuous factor as X_θ . But, as will show the next example, if the height is not trivial then the existence of a proper almost automorphic factor does not imply that it is given by an inner encoding of $\theta^{(-1,1)}$.

Example 3.24. The substitution

$$\theta : a \mapsto aba \quad b \mapsto bac \quad c \mapsto cab$$

has height 2 and column rank 2. So X_θ is almost automorphic. It thus has a proper almost automorphic factor (namely itself). The cover of minimal sets is

given by $U_\theta = \{\{a, b\}, \{a, c\}\}$ and so we see that the coincidence partition has only one element and the canonical inner encoding is periodic. The collared substitution

$$\theta^{(-1,1)} : A \mapsto ADB \quad B \mapsto BEA \quad C \mapsto EAC \quad D \mapsto EAD \quad E \mapsto CAC$$

has minimal sets

$$U_{\theta^{(-1,1)}} = \{\{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \{B, D\}, \{B, E\}\}.$$

Again the partition generated by $U_{\theta^{(-1,1)}}$ contains only one element so that the canonical inner encoding is periodic. This shows that the condition of trivial height in Theorem 3.21 cannot be dropped.

To overcome the problem outlined in the last example we will work with the pure base of θ . We recall from [6] the following results: If θ is primitive and has height h , then there exists a σ^h -periodic clopen partition of X_θ , $X_\theta = \bigsqcup_{k \in \mathbb{Z}/h\mathbb{Z}} X_\theta^k$, and $\sigma(X_\theta^k) = X_\theta^{k+1}$. Moreover, X_θ^0 is invariant under θ and there exists a subset $\mathcal{A}' \subset \mathcal{A}^h$ such that X_θ^0 consists precisely of the sequences $x \in X_\theta$ for which $x_0 \dots x_{h-1} \in \mathcal{A}'$. Define a substitution θ' on \mathcal{A}' as follows: Given $a_0 \dots a_{h-1} \in \mathcal{A}'$ compute $\theta(a_0 \dots a_{h-1}) = a'_0 \dots a'_{h\ell-1}$ and set

$$\theta'_k(a_0 \dots a_{h-1}) = a'_{kh} \dots a'_{(k+1)h-1}.$$

θ' is called the pure base of θ . It corresponds to the restriction of θ to X_θ^0 but expressed in the alphabet \mathcal{A}' .

The suspension of a \mathbb{Z} -action (X, φ) with $\mathbb{Z}/h\mathbb{Z}$ is the space $X \times \mathbb{Z}/h\mathbb{Z}$ equipped with the \mathbb{Z} -action

$$(3.3) \quad T_\varphi(x, i) := \begin{cases} (x, i+1) & \text{if } 0 \leq i < h-1 \\ (\varphi(x), 0) & \text{if } i = h-1 \end{cases}$$

The shift (X_θ, σ) is, for a substitution of height h , topologically conjugate to a suspension of the substitution shift $(X_{\theta'}, \sigma)$ with the finite group $\mathbb{Z}/h\mathbb{Z}$. Here we have denoted the shift action on $X_{\theta'}$ by σ' in order to distinguish it from the shift action σ on X_θ . The conjugacy is given by

$$X_{\theta'} \times \mathbb{Z}/h\mathbb{Z} \ni (x', i) \mapsto \sigma^i(x) \in X_\theta,$$

where on the left hand side x' is a sequence of letters from \mathcal{A}' , that is, a sequence of allowed (for θ) words of length h whereas on the right hand side x is the sequence of letters from \mathcal{A} that one obtains when one interprets x' as a sequence of letters in \mathcal{A} . Note that $(x', i) \mapsto (\sigma'(x'), i)$ on the left corresponds to $\sigma^i(x) \mapsto \sigma^{h+i}(x)$ on the right. The suspension construction is functorial and immediately implies:

- If $F : (X_{\theta'}, \sigma') \rightarrow (Y, \varphi)$ is a factor map then $F \times \mathbf{1} : (X_{\theta'} \times \mathbb{Z}/h\mathbb{Z}, T_{\sigma'}) \rightarrow (Y \times \mathbb{Z}/h\mathbb{Z}, T_\varphi)$ is a factor map and any factor map of $(X_{\theta'} \times \mathbb{Z}/h\mathbb{Z}, T_{\sigma'})$, up to a rotation, arises in this way. In particular the MEF of (X_θ, σ) is conjugate to $(\mathbb{Z}_\ell \times \mathbb{Z}/h\mathbb{Z}, T_{+1})$.
- (Y, φ) is almost automorphic if and only if $(Y \times \mathbb{Z}/h\mathbb{Z}, T_\varphi)$ is almost automorphic.

Recall that a topological dynamical system (X, T) is a minimal nontrivial almost automorphic extension of an odometer if and only if it is topologically conjugate to a shift [8, Theorem 6.4]. Combining this with the remarks above, we obtain the following.

Proposition 3.25. *Let θ be a primitive, aperiodic substitution of constant length, with pure base θ' . Then (X_θ, σ) has a proper almost automorphic factor if and only if $(X_{\theta'}, \sigma')$ has a proper almost automorphic factor.*

Corollary 3.26. *Let θ be a primitive, aperiodic substitution of constant length, with pure base θ' . Then X_θ has a proper almost automorphic factor if and only if the canonical inner encoding of $\theta'^{(-1,1)}$ is aperiodic.*

We describe how to construct the desired almost automorphic factor of X_θ when $X_{\theta'}$ has an aperiodic almost automorphic factor $X_{\eta'}$ via the map $F' : X_{\theta'} \rightarrow X_{\eta'}$. η' necessarily has height 1. Define a new length- ℓ substitution η with alphabet $\mathcal{A}_\eta = \{a_j : a \in \mathcal{A}_{\eta'}, 1 \leq j \leq h\}$ as follows. Define $i : \mathcal{A}_{\eta'} \rightarrow \mathcal{A}_\eta^h$ by $i(a) = a_1 \dots a_h$. Now let η be the unique length- ℓ substitution which satisfies $\eta \circ i = i \circ \eta'$. That is, we “split” each $a \in \mathcal{A}_{\eta'}$ into h different letters a_1, \dots, a_h in such a way that the concatenation of the length ℓ words $\eta(a_1) \dots \eta(a_h)$ is the word obtained from $\eta'(a)$ by splitting every letter. As defined, η has height h and pure base η' . Thus, X_η is also a suspension of $X_{\eta'}$ over $\mathbb{Z}/h\mathbb{Z}$. It can be seen that X_η is almost automorphic over $(\mathbb{Z}_{\bar{\ell}, h}, +1)$. See Example 3.28 for such a construction.

3.5. Two-letter collaring of bijective substitutions and their canonical outer encodings. In this section we apply our results above to study when a bijective substitution shift has a proper almost automorphic factor. This reproduces the results of Martin [16] and those of Herning [11] in our semigroup based approach. In the process we revisit the work in [13, Section 4]. We start with bijective substitutions of trivial height. Remember that for bijective substitutions, S_θ is a group and so ${}^c\theta$ is trivial, being defined on a one letter alphabet. This does not mean that a bijective substitution shift does not admit a proper almost automorphic substitutional factor, but only that these may only be seen when working with the collared versions of the substitution. We consider here only the collaring $\theta^{(0,1)}$.

For θ a bijective substitution on \mathcal{A} , let $\mathcal{A}^{(2)}$ be the set of allowed two-letter words for θ . Recall that the 2-collared substitution $\theta^{(0,1)}$ associated to θ is the substitution on $\mathcal{A}^{(2)}$ of the same length given by

$$\theta_m^{(0,1)}(a, b) = (\theta_m(a), \theta_{m+1}(a)), \quad 0 \leq m < \ell - 1, \quad \theta_{\ell-1}^{(0,1)}(a, b) = (\theta_{\ell-1}(a), \theta_0(b))$$

We assume that $\theta_0 = \theta_{\ell-1} = \mathbf{1}$ so that all θ -periodic points are fixed; thus all maps $\theta_m^{(0,1)}$ with $m < \ell - 1$ have rank $c = |\mathcal{A}|$ while $\theta_{\ell-1}^{(0,1)}$ is equal to the identity on $\mathcal{A}^{(2)}$ and hence has rank equal to $|\mathcal{A}^{(2)}|$. We know that the column rank of a substitution is a conjugacy invariant and so the column rank of $\theta^{(0,1)}$ must also be c . The kernel of $S_{\theta^{(0,1)}}$ is generated by $\theta_m \text{pr}_1 \times \theta_{m+1} \text{pr}_1$, $m < \ell - 1$, where $\text{pr}_1 : \mathcal{A}^{(2)} \rightarrow \mathcal{A}$ is the projection onto the first factor. Recall that the right ideals of $\ker S_{\theta^{(0,1)}}$ are in one-to-one correspondence to the images of these maps. As θ_m is bijective, the image of $\theta_m \text{pr}_1 \times \theta_{m+1} \text{pr}_1$ coincides with that of $\text{pr}_1 \times \theta_{m+1} \theta_m^{-1} \text{pr}_1$ and so uniquely is determined by the map $\theta_{m+1} \theta_m^{-1}$. We thus see that the set of right ideals of $\ker S_{\theta^{(0,1)}}$ is in one-to-one correspondence the set

$$I_\theta := \{\theta_{m+1} \theta_m^{-1} | m = 0, \dots, \ell - 2\}$$

which plays a prominent role in the description of the Ellis semigroup of the substitution shift (X_θ, σ) and is also called the R -set of the substitution [13]. This gives us the first part of

Corollary 3.27. *The alphabet of the canonical outer encoding ${}^c\theta^{(0,1)}$ of $\theta^{(0,1)}$ can be identified with I_θ . Under this identification it is given by*

$${}^c\theta^{(0,1)}_m(\theta_i\theta_{i-1}^{-1}) = \theta_{m+1}\theta_m^{-1}, \quad 0 \leq m < \ell - 1, \quad {}^c\theta^{(0,1)}_{\ell-1}(\theta_i\theta_{i-1}^{-1}) = \theta_i\theta_{i-1}^{-1}$$

It is an aperiodic inner encoding of $\theta^{(0,1)}$ if and only if for all $f, g \in I_\theta$ and $\forall a \in \mathcal{A}$: $f(a) \neq g(a)$. Thus the following is a necessary condition for ${}^c\theta^{(0,1)}$ to be an inner encoding:

$$(3.4) \quad |I_\theta| \times |\mathcal{A}| = |\mathcal{A}^{(2)}|$$

and so the code of the inner encoding is a $|\mathcal{A}|$ -to-1 map.

Proof. Recall that the canonical outer encoding is an inner encoding if the images of the maps $\theta_m \text{pr}_1 \times \theta_{m+1} \text{pr}_1$, $m < \ell - 1$, either coincide or do not overlap. As the images of $\theta_m \text{pr}_1 \times \theta_{m+1} \text{pr}_1$ and $\theta_{m'} \text{pr}_1 \times \theta_{m'+1} \text{pr}_1$ coincide if and only if $\theta_m \theta_{m-1}^{-1} = \theta_{m'} \theta_{m'-1}^{-1}$ this is exactly the condition stated. If this is the case, then the number of letters of $\theta^{(0,1)}$ is equal to the maximal choice of distinct maps $\theta_m \text{pr}_1 \times \theta_{m+1} \text{pr}_1$, $1 \leq m < \ell - 1$ (the size of I_θ) times the size of the image of one of them, which is $|\mathcal{A}|$. \square

We go through some of the examples in [13]. The canonical outer encoding of any aperiodic 2-letter bijective substitution is an inner encoded substitution, because for those $\mathcal{A}^{(2)} = \mathcal{A} \times \mathcal{A} = \{aa, ab, ba, bb\}$, $S_\theta = \mathbb{Z}/2\mathbb{Z} = \{\mathbf{1}, \mathbf{f}\}$ and $I_\theta = \{\mathbf{1}, \mathbf{f}\}$, where \mathbf{f} interchanges a with b . Indeed, this implies that the inner encoding is given by the partition $\{\{aa, bb\}, \{ab, ba\}\}$. The simplest example is the Thue-Morse substitution

$$\theta : a \mapsto abba \quad b \mapsto baab$$

and the inner encoded substitution associated to $\theta^{(0,1)}$ is the well known period doubling substitution

$${}^c\theta^{(0,1)} : A \mapsto ABAA \quad B \mapsto ABAB$$

which has a coincidence.

Our next example, from [13, Section 6.3], is

$$\begin{aligned} \theta : a &\mapsto abcca \\ &b \mapsto babab \\ &c \mapsto ccabc. \end{aligned}$$

It does not satisfy (3.4), as $\mathcal{A}^{(2)}$ has five letters and five is a prime number. Thus the substitution shift does not have an aperiodic almost automorphic factor.

As our last example, from [13, Section 6.3] has nontrivial height, we go through the required details carefully.

Example 3.28. The substitution

$$\begin{aligned} \theta : a &\mapsto abadcba & c &\mapsto cdcbadc \\ &b \mapsto badcbab & d &\mapsto dcbadcd \end{aligned}$$

has height 2. To find out whether the shift generated by this substitution has a proper almost automorphic factor, then, as it has height 2, we move to its pure base by Proposition 3.25. This is given by:

$$\begin{array}{ll} \tilde{\theta}: 0 \mapsto 3010102 & 2 \mapsto 2102102 \\ 1 \mapsto 2101013 & 3 \mapsto 3013013 \end{array}$$

where each of the four symbols in the new substitution represents a two-letter-word from the original (respectively, 0, 1, 2, 3 correspond to $[ad]$, $[cb]$, $[cd]$, $[ab]$). By inspection, we see that the minimal sets of this new substitution are disjoint, and thus its coincidence partition is given by $\mathcal{P}_{\tilde{\theta}} = \{\{0, 1\}, \{2, 3\}\}$, which means that the map $\tilde{F}: X_{\tilde{\theta}} \rightarrow X_{\tilde{\eta}}$, whose local rule is a code and sends 0, 1 to A and 2, 3 to B , is a factor map to the aperiodic, primitive, almost automorphic substitution shift given by:

$$\tilde{\eta}: A \mapsto BAAAAAB \quad B \mapsto BAABAAB.$$

As in the construction description after Proposition 3.25, because the original substitution θ has height 2, to find a proper almost automorphic factor of X_{θ} we introduce a height-2 suspension η of $\tilde{\eta}$ by “splitting” each symbol into two, moving from e.g. $A \mapsto BAAAAAB$ to $Aa \mapsto BbAaAaAaAaBb$, which is a concatenation of two length 7 words. The new substitution, almost automorphic by construction, is

$$\begin{array}{ll} \eta: A \mapsto BbAaAaA & a \mapsto aAaAaBb \\ B \mapsto BbAaAaB & b \mapsto bAaAaBb. \end{array}$$

The previously defined map \tilde{F} induces a factor map $F: X_{\theta} \rightarrow X_{\eta}$. To define it explicitly, we use the fact that each element of $\{0, 1, 2, 3\}$ corresponds to a two-letter word in X_{θ} and is mapped to either A or B , which also corresponds to the two-letter words Aa or Bb in X_{η} , so we expect F to map any instance of, say, ad in some $x \in X_{\theta}$ to the word Aa in the corresponding $F(x) \in X_{\eta}$. We can accomplish this by giving F left- and right-radius 1; accordingly, its local rule will be:

$$\begin{array}{ll} aba \mapsto b & adc \mapsto a \\ bab \mapsto B & bad \mapsto A \\ cba \mapsto a & cdc \mapsto b \\ dcb \mapsto A & dcd \mapsto B. \end{array}$$

3.6. Veech towers. Primitive constant length substitution shifts are point distal dynamical systems and hence are \mathcal{AI} -flows. The following questions arise naturally in this context: When are they strict \mathcal{AI} -flows? Can we exhibit in this case a tower of AI-extensions for them? And if they are not strict \mathcal{AI} -flows can we find an almost one-to-one extension with its tower of AI-extensions? We do not answer these questions completely, but explain how our results on the existence of certain factors sheds light on them.

A factor map $F: (X, T) \rightarrow (Y, S)$ is *isometric* if for each $\epsilon > 0$, there exists $\delta > 0$ such that whenever $x_1, x_2 \in X$ are such that $F(x_1) = F(x_2)$ and $d(x_1, x_2) \leq \delta$, then $d(T^n(x_1), T^n(x_2)) \leq \epsilon$ for each integer n (d is the metric on X). If Y is minimal, F isometric and finite-to-one, then F is k -to-one for some k .

We say that (X, T) is an almost isometric extension (AI-extension for short) of (Z, R) if it has a factor $F: (X, T) \rightarrow (Y, S)$ and (Y, S) has a factor $\pi: (Y, S) \rightarrow (Z, R)$ such that F is isometric and π is almost one-to-one.

We recall Martin's characterisation of when (X_θ, σ) is an AI-extension of its maximal equicontinuous factor. For ease of notation we state only the version for trivial height; for details of the general case, see [16, Theorem 8.08].

Theorem 3.29. *Let θ be a bijective, primitive aperiodic substitution of length ℓ and of trivial height. The following are equivalent:*

- (1) *The cover $\mathcal{U}_{\theta^{(0,1)}}$ is a partition.*
- (2) *X_θ is an almost isometric extension of its maximal equicontinuous factor.*

Note that by Proposition 3.14, the first condition of the theorem is equivalent to the statement that the canonical outer encoding ${}^c\theta^{(0,1)}$ is an aperiodic inner encoding.

Martin's theorem is concerned with isometric extensions of almost automorphic substitutions, and one must distinguish this from characterising when a substitution has a proper almost automorphic factor. We give next an example of a bijective substitution θ which has an aperiodic almost automorphic factor η , but where the factor map $F : X_\theta \rightarrow X_\eta$ is not an isometry.

Example 3.30. Consider the following bijective substitution

$$\begin{aligned} \theta : a &\mapsto abdb & c &\mapsto cddbc \\ b &\mapsto baaca & d &\mapsto dccad. \end{aligned}$$

The collaring $\theta^{(0,1)}$ is a substitution on a 14-letter alphabet whose minimal sets are

$$\begin{aligned} A &:= \{(aa), (bb), (cc), (dd)\} \\ B &:= \{(ab), (ba), (cd), (dc)\} \\ C &:= \{(ac), (bd), (ca), (db)\} \\ D &:= \{(ad), (bc), (ca), (db)\} \\ E &:= \{(ab), (ba), (cc), (dd)\} \\ F &:= \{(aa), (bb), (cd), (dc)\}, \end{aligned}$$

so that $\mathcal{U}_{\theta^{(0,1)}}$ is not a partition. We have

$$\mathcal{P}_{\theta^{(0,1)}} = \{ \{(aa), (ab), (ba), (bb), (cc), (cd), (dc), (dd)\}, \{(ac), (ad), (bc), (bd), (ca), (db)\} \}$$

consists of two sets, and the canonical inner encoding is

$$\eta : 0 \mapsto 00110 \quad 1 \mapsto 00111,$$

which is aperiodic. Furthermore the factor map from X_θ to X_η is not isometric, sending eight θ -fixed points to one fixed point for η , and six to the other. Finally, note that the canonical outer encoding

$$\begin{aligned} {}^c\theta^{(0,1)} : A &\mapsto BACDE & D &\mapsto BACDC \\ B &\mapsto BACDF & E &\mapsto BACDA \\ C &\mapsto BACDD & F &\mapsto BACDB \end{aligned}$$

is not an inner encoding, but projects to the nontrivial inner encoding η via the code $A, B, E, F \mapsto 0, C, D \mapsto 1$.

Martin's results implies that bijective substitutions of trivial height which satisfy the condition of Theorem 3.29 are strict \mathcal{AZ} -flows having the following Veech tower of AI extensions

$$X_\theta \xrightarrow{F_{\beta_\theta}} X_{c_\theta(0,1)} \xrightarrow{F} \mathbb{Z}_\ell \xrightarrow{O} pt$$

Here F_{β_θ} and O are isometric while F is almost one-to-one. We do not know of any result which is similar to Martin's theorem but holds for non-bijective constant length substitutions. Finding conditions which guarantee that F_{β_θ} is isometric for non-bijective substitutions remains an open question. We can however say that, if the inner encoded substitution associated to $\theta^{(0,1)}$ is periodic, then X_θ is not a strict \mathcal{AZ} -flow. This is a direct consequence of Martin's Theorem A.5 which implies that if X_θ is a strict \mathcal{AF} -flow then it must be an AI extension of its maximal equicontinuous factor.

Lemanczyk and Müllner [15] found an elegant construction of a substitution ζ whose dynamical system (X_ζ, σ) is an almost one-to-one extension of (X_θ, σ) and which always factors onto the system (X_{c_θ}, σ) defined by the outer encoded substitution ${}^c\theta$. This system has the tower of extensions

$$X_\zeta \xrightarrow{F_1} X_{c_\theta} \xrightarrow{F_2} \mathcal{G} \xrightarrow{O} pt$$

and is therefore a Veech tower if F_1 is isometric. As this is an interesting ansatz we explain some details from [15] using our notation. Given a primitive length ℓ substitution θ of trivial height, let U_θ be its collection of minimal sets. Define

$$\mathcal{C} := \{(a, M) : M \in U_\theta, a \in M\},$$

and recall the canonical outer encoding ${}^c\theta$ of θ . Define $\zeta : \mathcal{C} \rightarrow \mathcal{C}^\ell$ by

$$\zeta_i(a, M) = (\theta_i(a), {}^c\theta_i(M)).$$

In [15] the outer encoded substitution is directly defined (without reference to the semigroup of the substitution) and called the synchronising substitution. Furthermore, ζ is what is called there the joining of θ with ${}^c\theta$.

Proposition 3.31 ([15]). *(X_ζ, σ) is an almost one-to-one extension of (X_θ, σ) .*

Proof. The code $(a, M) \mapsto a$ defines a factor $F : X_\zeta \rightarrow X_\theta$. We claim that if θ has column rank c , so does ζ . For each i and each $M \in U_\theta$, $|\theta_i(M)| = c$. This implies that $|\{\zeta_i(a, M) : a \in M\}| = c$, so ζ has column rank at least c . Also, one can assume, by moving to a power of θ is necessary, that there are i and $M^* \in U_\theta$ such that $\theta_i(\mathcal{A}) = M^* \in U_\theta$. This implies that $\{\zeta_i(a, M) : a \in M, M \in U_\theta\} = \{(a, M^*) : a \in M^*\}$, so ζ has column rank at most c .

Furthermore if $z \in \mathbb{Z}_\ell$ and $\pi^{-1}(z) = \{x^{(1)}, \dots, x^{(c)}\}$ is a regular fibre in X_θ , then for each $x^{(i)}$, $(\phi^{-1}(x^{(i)}))_n = (x_n^{(i)}, \{x_n^{(1)}, \dots, x_n^{(c)}\})$. Thus X_ζ is an almost one-to-one extension of X_θ . \square

In Section 6 of [15] one finds an example with a factor map $F_1 : X_\zeta \rightarrow X_{c_\theta}$ which is not isometric. This raises the following question which we leave for future work: When is the factor map F_1 isometric?

4. FACTORING ONTO A BIJECTIVE SUBSTITUTION

In this section, we characterise, using the semigroup S_θ , when a substitution shift is almost bijective. First we restrict to the case where the factor map preserves the

fixed point fibre. We state the two main theorems and prove them in the next section.

Theorem 4.1. *Let θ be a constant length substitution with column rank c . The following are equivalent:*

- (1) S_θ has a unique minimal left ideal.
- (2) There is a bijective substitution η on a c -letter alphabet which is an inner encoding of θ .

If these conditions are satisfied then η is uniquely determined by θ .

As η is an inner encoding, it is always primitive. The following result shows that η has to be aperiodic and that the associated factor map $F_\beta : X_\theta \rightarrow X_\eta$ has to be almost one-to-one, except if θ has a coincidence.

Theorem 4.2. *Let θ be a primitive, aperiodic length- ℓ substitution with column rank c and height h . Suppose that there is a bijective substitution η on a c -letter alphabet which is inner encoded by θ . Then η is periodic if and only if $c = h$. Moreover, if $c > h$ then the factor map $X_\theta \rightarrow X_\eta$ induced by the inner encoding is almost one-to-one.*

In general, the existence of a bijective substitution factor for a substitution is not linked to the existence of a bijective substitution factor for its pure base; see Example 4.23. The issue here is that we characterise the existence of a bijective factor in terms of the semigroup \mathcal{S}_θ , and the relationship between this semigroup and that of the pure base of θ is not clear.

4.1. More preliminaries from semigroup theory. In order to prove the above theorems we need to further analyse sub-semigroups of the semigroup $\mathcal{F}(X)$ of maps from $X \rightarrow X$. Recall that the partition defined by a map $f : X \rightarrow Y$ is $\mathcal{P}_f = \{f^{-1}(y) | y \in Y\}$.

Definition 4.3. We say that a map $g : X \rightarrow X$ preserves a partition $\mathcal{P} \subset \mathcal{P}(X)$ if $g^{-1}(\mathcal{P}) \subset \mathcal{P}$.

Stated differently, let $\mathcal{P} = \{A_i | i \in I\}$, then g preserves \mathcal{P} if for all $i \in I$ there is a unique $j \in I$ such that $g^{-1}(A_i) = A_j$. Note that g does not necessarily preserve \mathcal{P}_g .

Lemma 4.4. *Let $g : X \rightarrow X$ preserve a partition \mathcal{P} . Then $g^{-1}|_{\mathcal{P}}$ is injective and hence bijective if \mathcal{P} is finite.*

Proof. Let $A, B \in \mathcal{P}$. Suppose $g^{-1}(A) = g^{-1}(B)$ which means $A \cap \text{img} = B \cap \text{img}$. Since A and B are either equal or have empty intersection, $A \cap \text{img} = B \cap \text{img}$ is the case if $A = B$ or $A \cap \text{img} = B \cap \text{img} = \emptyset$. But $A \cap \text{img} = \emptyset$ means $g^{-1}(A) = \emptyset$, a possibility which is excluded, as a partition does not contain the empty set. \square

Lemma 4.5. *Let $g : X \rightarrow X$ and $f : X \rightarrow Y$. If $\mathcal{P}_f = \mathcal{P}_{f \circ g}$ then g preserves \mathcal{P}_f .*

Proof. By assumption

$$\{f^{-1}(y) | y \in Y\} = \{g^{-1}(f^{-1}(y)) | y \in Y\}$$

which says exactly that $g^{-1}(\mathcal{P}_f) = \mathcal{P}_f$. \square

A map $p : X \rightarrow X$ is an idempotent if and only if it preserves its partition and maps each member $A \in \mathcal{P}_p$ to a single point in A .

Lemma 4.6. *Let $p : X \rightarrow X$ be an idempotent. If p preserves \mathcal{P} and $|\mathcal{P}| = |\mathcal{P}_p| < +\infty$ then $\mathcal{P}_p = \mathcal{P}$.*

Proof. As p maps each member $A \in \mathcal{P}_p$ to a single point in A , \mathcal{P}_p is the finest partition preserved by p . Hence if p preserves \mathcal{P} then it preserves the partition generated by \mathcal{P} and \mathcal{P}_p which can't be finer than \mathcal{P}_p . Therefore $|\mathcal{P}| = |\mathcal{P}_p| < +\infty$ implies $\mathcal{P}_p = \mathcal{P}$. \square

Lemma 4.7. *Let $p, q : X \rightarrow X$ two idempotents. If p preserves \mathcal{P}_q then $qp = q$.*

Proof. Suppose p preserves \mathcal{P}_q . Then it also preserves the partition generated by \mathcal{P}_p and \mathcal{P}_q . As \mathcal{P}_p is the finest partition preserved by p , \mathcal{P}_q must be coarser than \mathcal{P}_p or $\mathcal{P}_q = \mathcal{P}_p$. It follows that $p^{-1}|_{\mathcal{P}_q}$ is the identity map from \mathcal{P}_q to itself. This implies $q^{-1} = p^{-1}q^{-1}$ hence $qp = q$. \square

The condition of the following lemma is satisfied for all compact right topological semigroups and so in particular for any finite semigroup.

Lemma 4.8. *Let S be a sub-semigroup of $\mathcal{F}(X)$ which admits a kernel which contains an idempotent. The following are equivalent.*

- (1) *All elements of S preserve the partition defined by the idempotent.*
- (2) *S has a unique minimal left ideal, i.e., the kernel is left simple.*

Proof. $1 \Rightarrow 2$. Let $q \in S$ be a minimal idempotent such that all elements of S preserve \mathcal{P}_q . Let $p \in S$ be another minimal idempotent. By assumption p preserves \mathcal{P}_q . By Lemma 4.7 we have $qp = q$. Hence q lies in the minimal left ideal generated by p . Since all idempotents of a minimal left ideal generate the same left ideal, therefore S has a unique minimal left ideal.

$2 \Rightarrow 1$. Let p be an idempotent in the unique minimal left ideal L . Let $s \in S$. Then ps lies in the kernel of S which coincides with L . By Lemma A.6, $\mathcal{P}_{ps} = \mathcal{P}_p$. By Lemma 4.5, s preserves \mathcal{P}_p . \square

Proof of Theorem 4.1. Suppose first that \mathcal{S}_θ contains a unique minimal left ideal. By Lemma 4.8 all its elements preserve a partition $\mathcal{B} = \{A_1, \dots, A_k\}$ defined by any element of that ideal. Note that k must be the column rank c . Define the length- ℓ substitution η on the alphabet \mathcal{B} as that having column maps η_m such that

$$\eta_m^{-1} := \theta_m^{-1}|_{\mathcal{B}}.$$

In other words, $\eta_m(A_i)$ is the unique A_j which contains $\theta_m(a_i)$ for some $a_i \in A_i$. This is well-defined as θ_m preserves the partition \mathcal{B} . By Lemma 4.4 η is bijective. The code β is given by $a \mapsto A_i$ for all $a \in A_i$.

To prove the converse of the statement, suppose now there is a bijective substitution with column maps η_m on a c -letter alphabet \mathcal{B} which is inner encoded by θ , that is, there is a map $\beta : \mathcal{A} \rightarrow \mathcal{B}$ such that

$$\eta_m \circ \beta = \beta \circ \theta_m$$

for all $m = 0, \dots, \ell - 1$. Since η is bijective this implies that the partition \mathcal{P}_β defined by β must be the same as that defined by each $\beta \circ \theta_m$ and then also the same as that defined by any $\beta \circ f$ for any $f \in \mathcal{S}_\theta$. By Lemma 4.5 all elements of \mathcal{S}_θ must preserve \mathcal{P}_β . Let p be a minimal idempotent of \mathcal{S}_θ . By definition, its column rank is c , which is also the rank of β . Lemma 4.6 implies therefore that $\mathcal{P}_p = \mathcal{P}_\beta$. Now, we conclude with Lemma 4.8 that \mathcal{S}_θ has a unique minimal left ideal.

The uniqueness of the η_m follows from the fact that it is entirely determined by the θ_m and the partition induced from any idempotent of S_θ . \square

Proof of Theorem 4.2. Suppose that η is a bijective substitution on a c -letter alphabet which is inner encoded by θ . Clearly, if $c = 1$ then η is periodic. Now suppose $c > 1$.

It is one of the properties of the height h that there is a code $\beta : \mathcal{A}_\theta \rightarrow \mathbb{Z}/h\mathbb{Z}$ which induces a factor map $F_\beta : (X_\theta, \sigma) \rightarrow (\mathbb{Z}/h\mathbb{Z}, +1)$ such that $\beta(\theta(a)) = \beta(\theta(b))$ if $\beta(a) = \beta(b)$ [6]. In other words the partition induced by β satisfies the condition (2) of Lemma 2.7 and so defines a substitution η' which is inner encoded by θ . This substitution is periodic, as its substitution space is finite. If $c = h$ it satisfies the properties of Theorem 4.1 and hence must be (up to a renaming of its alphabet) equal to η .

Suppose now that η is h' -periodic. Hence $(X_\eta, \sigma) \cong (\mathbb{Z}/h'\mathbb{Z}, +1)$ and $\eta^{N}_{nh'} = \eta_0$ for all $n \in \mathbb{N}$ and N such that $\ell^N > nh'$. Taking a power we may assume that $\eta_0 = \mathbf{1}$. Clearly $1/h'$ is an eigenvalue of (X_η, σ) , hence also of (X_θ, σ) . If $1/h' \in \mathbb{Z}_\ell$ then there are n, N such that $nh' = \ell^N$. Hence, for all $0 \leq k < \ell$

$$\mathbf{1} = \eta^{N+1}_{k\ell^N} = \eta^N_0 \eta_k = \eta_k$$

which contradicts the primitivity of η , as $c > 1$. Hence h' must divide the height h of θ . As η is bijective we have $c = |\mathcal{A}_\eta| \leq |X_\eta| = h' \leq h$.

We show that the factor map $F : X_\theta \rightarrow X_\eta$ induced by the inner encoding is almost one-to-one provided η is aperiodic. $\pi_\theta : X_\theta \rightarrow \mathbb{Z}_\ell$ and $\pi_\eta : X_\eta \rightarrow \mathbb{Z}_\ell$ the factor maps onto the equicontinuous factor \mathbb{Z}_ℓ which map the fixed points of θ and η , resp., to 0. Then $\pi_\theta = \pi_\eta \circ F$. The regular points for θ , are those $z \in \mathbb{Z}_\ell$ which have a fibre of size c , similarly for η . For both substitutions they are known to form a residual set. Hence their intersection is not empty. Let z be regular for both θ and η . Then F must restrict to a bijection on the corresponding fibre. Hence F is almost one-to-one. \square

Remark. It follows from Lemma 3.7 that our characterisation of when a substitution allows for an inner encoded bijective substitution is stable under taking powers.

Example 4.9. Consider the primitive aperiodic substitution

$$\begin{array}{ll} \theta: 0 \mapsto 021 & 2 \mapsto 201 \\ 1 \mapsto 130 & 3 \mapsto 310 \end{array}$$

which is easily checked to have column rank 2. The partition defined by the map θ_2 is

$$\mathcal{P}_{\theta_2} = \{\{0, 2\}, \{1, 3\}\}.$$

It is easily seen to be preserved by all products of θ_i 's. Hence S_θ has a unique minimal left ideal. The code

$$\beta: 0, 2 \mapsto a \quad 1, 3 \mapsto b$$

gives rise to the inner encoded bijective aperiodic substitution

$$\begin{array}{l} \eta: a \mapsto aab \\ b \mapsto bba. \end{array}$$

As an aside we mention that the substitution shift defined by θ has uncountably many irregular fibers and hence the factor map $F_\beta : X_\theta \rightarrow X_\eta$ is not one-to-one on uncountably many points.

4.2. Factors which preserve the fixed point fibre. In Theorem 4.1, we characterised the substitution shifts which admit a bijective inner encoding. In this section, we extend this to characterising substitution shifts which admit a bijective substitution shift as a factor via a factor map which preserves the fixed point fibre.

Lemma 4.10. *Let $n = (-l, r)$. If $S_{\theta^{(n)}}$ has a unique minimal left ideal then S_θ has a unique minimal left ideal.*

Proof. As θ is an inner encoding of the collared substitution $\theta^{(n)}$, the semigroup S_θ is a homomorphic image of $S_{\theta^{(n)}}$. The result follows therefore from Lemma A.10. \square

Theorem 4.11. *Let θ be a primitive aperiodic constant length substitution with height h and column rank $c > h$. The following are equivalent.*

- (1) (X_θ, σ) is an almost one-to-one extension of a bijective substitution shift (X_η, σ) via a factor map F which preserves the fixed point fibre.
- (2) (X_θ, σ) factors onto a c -letter bijective substitution shift (X_η, σ) via a factor map F which preserves the fixed point fibre.
- (3) S_θ has a unique minimal left ideal.

Proof. To show that (1) implies (2), we only have to show that the bijective substitution η is on c letters. As F is almost one-to-one, X_θ and X_η must have the same maximal equicontinuous factor, hence the same height and therefore also the same column rank. For a bijective substitution the column rank equals the size of its alphabet.

To see that (2) implies (3), suppose that $F : X_\theta \rightarrow X_\eta$ is a factor map which preserves the fixed point fibre and where η is a bijective substitution on c letters. By Corollary 2.11(2) there exist $n = (-l, r)$, $l, r \leq 1$, $p \geq 1$, an inner encoding (η', β) of $\theta^{(n)^p}$, and a code $\tau' : \mathcal{A}_{\eta'} \rightarrow \mathcal{A}_\eta$ such that $F_{\tau'} : X_{\eta'} \rightarrow X_\eta$ is a conjugacy. As each of F and F_β preserve the fixed point fibres, $F_{\tau'}$ must also preserve the fixed point fibre. By Lemma 2.6 (η^p, τ') is an inner encoding of η' . Hence $(\eta^p, \tau' \circ \beta)$ is an inner encoding of $\theta^{(n)^p}$. By assumption η and also η^p is a substitution on c letters. Now Theorem 4.1 implies that $S_{\theta^{(n)^p}}$ has a unique minimal left ideal. By Lemmata 3.7 and 4.10, S_θ has a unique minimal left ideal.

Finally we come to (3) implies (1). Suppose that S_θ has a unique minimal left ideal. By Theorem 4.1 there exists a bijective substitution on a c letter alphabet which is an inner encoding of θ . By Lemma 2.6 the associated factor map preserves the fixed point fibre. By Theorem 4.2 the factor map is almost one-to-one. \square

We remark that Theorems 4.1 and 4.11 imply that all factor maps to a bijective substitution which fix the fixed point fibre must have radius zero.

Example 4.12. The substitution shift X_θ of Example 3.3 does not admit a factor map onto an aperiodic bijective substitution shift which fixes the fixed point fibre. Indeed, any aperiodic bijective substitution shift which is a factor of X_θ must have 2 letters, as the column rank of θ is 2, and this is excluded as the semigroup S_θ has two minimal left ideals.

4.3. Bijective factors which do not preserve the fixed point fibre. In this section we extend Theorem 4.11 to substitutions which have a bijective substitution factor via a factor map which does not send fixed points to fixed points. To do this we need to recall a little more information on the arithmetic properties of factor maps between substitutions.

4.4. Factors of substitution shifts and their κ -values. Let (X, σ) and (Y, σ) be infinite minimal shifts with a common equicontinuous factor (\mathcal{G}, R) and fixed equicontinuous factor maps $\pi_X : X \rightarrow \mathcal{G}$ and $\pi_Y : Y \rightarrow \mathcal{G}$. Let $\text{Fac}(X, Y)$ be the collection of factor maps from (X, σ) to (Y, σ) . Following [5] we define the map $\kappa : \text{Fac}(X, Y) \rightarrow \mathcal{G}$ through

$$\kappa(F) := \pi_Y(F(x)) - \pi_X(x).$$

$\kappa(F)$ is also called the κ -value of F . By minimality, it does not depend on x . But it depends on the choice of π_X and π_Y and in the framework of substitution shifts, which we consider here, we continue to choose them in such a way that the fixed points of the substitutions are mapped to 0. By [5, Theorem 3.3] the map κ satisfies

- (1) if (Z, σ) is another shift with equicontinuous factor \mathcal{G} and $F \in \text{Fac}(X, Y)$, $G \in \text{Fac}(Y, Z)$ then

$$\kappa(G \circ F) = \kappa(G) + \kappa(F),$$

- (2) if $\min_{g \in \mathcal{G}} |\pi_X^{-1}(g)| = \min_{g \in \mathcal{G}} |\pi_Y^{-1}(g)| = c < \infty$ then $\kappa : \text{Fac}(X, Y) \rightarrow \mathcal{G}$ is at most c -to-one and

$$\{g \in \mathcal{G} : |\pi_Y^{-1}(g)| > c\} \subset \{g \in \mathcal{G} : |\pi_X^{-1}(g)| > c\} + \kappa(F),$$

for all $F \in \text{Fac}(X, Y)$.

If $F : X_\theta \rightarrow X_\eta$ is a factor map between two length- ℓ substitution shifts, then $\kappa(F) = 0$ if and only if θ -fixed points are mapped to η -fixed points, i.e., if F preserves the fixed point fibre. In what follows, we will take $\mathcal{G} = \mathbb{Z}_\ell$, i.e.,

$$\kappa : \text{Fac}(X_\theta, X_\eta) \rightarrow \mathbb{Z}_\ell,$$

even if θ has nontrivial height, so that the c in (2) above is the column rank of θ and η .

The following adaptation of [5, Proposition 3.24] tells us that κ -values of factors between two constant length substitution shifts cannot take any value. We say that $z \in \mathbb{Z}_\ell$ is *rational* if it is eventually periodic. This naming is motivated by the fact that if $z \in \mathbb{Z}_\ell$ is eventually periodic, then it is the ℓ -adic expansion of a rational number.

Proposition 4.13. *Let θ and θ' be primitive, aperiodic length- ℓ substitutions and $F \in \text{Fac}(X_\theta, X_{\theta'})$. Then $\kappa(F)$ is rational.*

Proof. In the case where the factor map is a conjugacy, and the heights of θ and η are trivial, the statement is [5, Proposition 3.24]. That proposition can be readily modified to hold for the case in which the heights of θ and η are not necessarily trivial, namely by taking for \mathcal{G} not the maximal equicontinuous factor but \mathbb{Z}_ℓ .

If the factor map F is not a conjugacy we apply Proposition 2.10 to $Y = X_{\theta'}$ from which we deduce that $\kappa(F) = \kappa(F_\tau)$ as, by Lemma 2.6, F_τ preserves the fixed point fibre, c.f. Figure 2 with $Y = X_{\theta'}$. Going over to $\theta^{(n)}$ if necessary we may thus assume that $F = F_\tau$ for some code $\tau : \mathcal{A}_\theta \rightarrow \mathcal{A}_{\theta'}$.

By Theorem 2.9 applied to $Y = X_{\theta'}$ there is an inner encoding (η_τ, β_τ) of θ , and a code $\tau' : \mathcal{A}_{\eta_\tau} \rightarrow \mathcal{A}_{\theta'}$ such that $F_{\tau'} : X_\eta \rightarrow X_{\theta'}$ is a conjugacy, c.f. Figure 1 with $Y = X_{\theta'}$. As, by Lemma 2.6, F_{β_τ} preserves the fixed point fibre, we have $\kappa(F_\tau) = \kappa(F_{\tau'})$. Now we apply [5, Proposition 3.24] to the conjugacy $F_{\tau'}$ to obtain that $\kappa(F_{\tau'})$ and hence $\kappa(F_\tau)$ is rational. \square

If $F : X_\theta \rightarrow X_\eta$ is a factor map onto a bijective substitution shift with $\kappa(F) = m \in \mathbb{Z}$, then $\tilde{F} := \sigma^{-m} \circ F : X_\theta \rightarrow X_\eta$ is also a factor map onto a bijective substitution and, as $\kappa(\tilde{F}) = 0$, we can apply Theorem 4.11 to conclude that \mathcal{S}_θ has a unique left minimal ideal. In other words, Theorem 4.11 extends verbatim to the case in which the kappa-value of the factor map is an integer.

However, the following example shows that there exist factor maps $F : X_\theta \rightarrow X_\eta$ where η is bijective, where $\kappa(F) \notin \mathbb{Z}$, and where \mathcal{S}_θ has more than one minimal left ideal.

Example 4.14. Take the following two substitutions:

$$\begin{array}{lll} \eta: a \mapsto abcba & \theta: 0 \mapsto 35203 & 3 \mapsto 41534 \\ b \mapsto bcacb & 1 \mapsto 35214 & 4 \mapsto 02140 \\ c \mapsto cabac & 2 \mapsto 41520 & 5 \mapsto 02153. \end{array}$$

Clearly η is bijective. The words of length two in \mathcal{L}_η are

$$\{ab, ac, ba, bc, ca, cb\},$$

and by mapping $(ab), (ac), (ba), (bc), (ca), (cb)$ to $0, 1, 2, 3, 4, 5$, respectively, it can be verified that $\theta = \eta^{(+1)}$. Thus X_θ factors onto the bijective substitution shift X_η . However, \mathcal{S}_θ does *not* have a unique minimal left ideal. Indeed, it has two minimal left ideals, one associated to the partition $\{\{0, 1\}, \{2, 3\}, \{4, 5\}\}$, and the other to the partition $\{\{0, 5\}, \{1, 3\}, \{2, 4\}\}$; see Lemma A.6 which shows how a the \mathcal{L} -class constituting a minimal left ideal is associated to a partition. This does not contradict Theorem 4.11 because, as we will see below, the conjugacy $F_\tau : X_\theta \rightarrow X_\eta$ which is given by the code

$$\begin{array}{l} \tau : 0, 1 \mapsto a \\ 2, 3 \mapsto b \\ 4, 5 \mapsto c \end{array}$$

is a factor map with nonzero κ -value. In particular, Lemma 4.15 will tell us that $\kappa(F_\tau) = -1/4 = -1/(\ell - 1)$.

We now show how to “correct” such factor maps to obtain factor maps with κ -value 0. Recall the k -shifted extension of θ from Definition 2.2.

Lemma 4.15. *Let $\theta : \mathcal{A} \rightarrow \mathcal{A}^\ell$ be an aperiodic primitive length- ℓ substitution, and let $\zeta := \theta^{(+k)}$ be its k -shifted extension, with $0 \leq k < \ell$. Then there is a conjugacy $F : X_\zeta \rightarrow X_\theta$ with $\kappa(F) = \frac{k}{1-\ell}$.*

Proof. Recall from Corollary 2.5 that $F_i : X_\zeta \rightarrow X_\theta$ is a conjugacy. We consider its inverse F_i^{-1} . By iteration of the equation of Corollary 2.5 we obtain

$$\zeta^n \circ F^{-1} = F^{-1} \circ (\sigma^k \circ \theta)^n = F^{-1} \circ \sigma^{k \frac{1-\ell^n}{1-\ell}} \circ \theta^n.$$

Let u be a fixed point of θ . Then $v = \lim_{n \rightarrow \infty} \zeta^n(F_i^{-1}u)$ is a fixed point of ζ (we assume that the periodic points of ζ are fixed points). Then, by continuity of F_i^{-1} ,

$$v = F_i^{-1} \circ \lim_{n \rightarrow \infty} \sigma^{k \frac{1-\ell^n}{1-\ell}}(u),$$

so that $\kappa(F_i^{-1}) = \frac{-k}{1-\ell}$. The result follows. \square

Corollary 4.16. *Let $\theta : \mathcal{A} \rightarrow \mathcal{A}^\ell$ be an aperiodic primitive length- ℓ substitution, and let $\frac{p}{q} \in \mathbb{Z}_\ell$ with p, q coprime. Then there exists some substitution ζ and a conjugacy $F : X_\zeta \rightarrow X_\theta$ that satisfies $\kappa(F) = \frac{p}{q}$.*

Proof. As shown in Lemma 4.15, if $0 \leq k < \ell$ and $\zeta = \theta^{(+k)}$, then $\kappa(F_i) = \frac{k}{1-\ell}$. Similarly, if we replace θ by θ^m for some value of m and take $\zeta = (\theta^m)^{(+k)}$, the corresponding factor map has κ -value $\frac{k}{1-\ell^m}$, where we can take k to be any value between 0 and $\ell^m - 1$.

To prove the general statement, suppose first that $-q < p \leq 0$, and that p and q have no common divisors. As we assume that $p/q \in \mathbb{Z}_\ell$, then q and ℓ are coprime, so that $\ell^{\varphi(q)} \equiv 1 \pmod{q}$, where φ is Euler's totient function. Thus $\ell^{\varphi(q)} - 1 = hq$ for some $h > 0$. Since $-q < p \leq 0$, we have $0 \leq -hp < hq < \ell^{\varphi(q)}$, so we can take $\zeta = (\theta^{\varphi(q)})^{(+(-hp))}$, and the conjugacy $F_i : X_\zeta \rightarrow X_\theta$ will satisfy:

$$\kappa(F_i) = \frac{-hp}{1 - \ell^{\varphi(q)}} = \frac{-hp}{-hq} = \frac{p}{q},$$

as desired.

For the general case, it is enough to note that $\frac{p}{q} = M + \frac{p_0}{q}$ for some integer M and $-q < p_0 \leq 0$, where p_0 will also be coprime to q , so it suffices to take $\zeta = (\theta^{\varphi(q)})^{(+(-hp_0))}$, as above, and $\sigma^M \circ F_i$ as the desired conjugacy. \square

Define, for a substitution with column rank c ,

$$\mathcal{F}_\theta := \{w = w_k \dots w_1 \in (\mathbb{Z}/\ell\mathbb{Z})^+ : |\theta_{w_1} \circ \dots \circ \theta_{w_k}(\mathcal{A})| = c\}.$$

Note that if \mathcal{F}_θ contains a length k word, then θ^k has a column with c elements.

Lemma 4.17. *Let θ be a constant length substitution. Let (η, β) be an inner encoding of θ . The length $j(\eta)$ of the shortest word in \mathcal{F}_η is bounded by the length $j(\theta)$ of the shortest word in \mathcal{F}_θ .*

Proof. Let $U_\theta = \{\text{im } f : f \in \ker S_\theta\}$. We first show that $U_\eta = \beta(U_\theta)$. Indeed, by Lemma A.10 we have $\ker S_\eta = \beta(\ker S_\theta)\beta^{-1}$ and hence

$$U_\eta = \{\text{im } f : f \in \ker S_\eta\} = \{\text{im } \beta \circ g : g \in \ker S_\theta\}.$$

By definition, $w_k \dots w_1$ is a word in \mathcal{F}_θ if and only if $\theta_{w_1} \circ \dots \circ \theta_{w_k}(\mathcal{A}) \in U_\theta$. Now, by the above,

$$\eta_{w_1} \circ \dots \circ \eta_{w_k}(\mathcal{B}) = \beta \circ \theta_{w_1} \circ \dots \circ \theta_{w_k}(\mathcal{A}) \in \beta(U_\theta) = U_\eta.$$

This implies $\mathcal{F}_\theta \subset \mathcal{F}_\eta$ and hence $j(\eta) \leq j(\theta)$. \square

Lemma 4.18. *Let $\theta : \mathcal{A} \rightarrow \mathcal{A}^\ell$ be an aperiodic primitive length- ℓ substitution, and let $F : X_\theta \rightarrow X_\zeta$ be a factor map. If \mathcal{F}_θ contains a word of length j , then $n\kappa(F) \in \mathbb{Z}$ for some $0 \leq n \leq (\ell - 1)(\ell^j - 1)$.*

Proof. In the case where the factor map is a conjugacy, the statement is [5, Proposition 3.24], with the appropriate modifications to account for height, as discussed in the proof of Proposition 4.13. Otherwise we proceed as in the proof of Proposition 4.13 and apply Proposition 2.10 to $Y = X_{\theta'}$. We may thus assume that $F = F_\tau$ for some code $\tau : \mathcal{A}_\theta \rightarrow \mathcal{A}_\zeta$.

Theorem 2.9 tells us that there is a pair (η, β) which is inner encoded by θ , and a code $\tau' : \mathcal{A}_\eta \rightarrow \mathcal{A}_\zeta$ such that $F_{\tau'} : X_\eta \rightarrow X_\zeta$ is a conjugacy, and $\kappa(F_\tau) = \kappa(F_{\tau'})$. Then again by [5, Proposition 3.24], the denominator of $\kappa(F_{\tau'})$ (always assuming that it is coprime to the numerator) is at most $(\ell - 1)(\ell^j - 1)$, where $j = j(\eta)$ is the length of the shortest word in \mathcal{F}_η . Now by Lemma 4.17, we have that $j(\eta) \leq j(\theta)$. Thus the denominator of $\kappa(F_{\tau'})$ is at most $(\ell - 1)(\ell^{j(\theta)} - 1)$. Since $\kappa(F_{\tau'}) = \kappa(F_\tau)$, the result follows. \square

Theorem 4.19. *Let θ be an aperiodic primitive constant length- ℓ substitution with column rank $c > 1$. The following are equivalent:*

- (1) X_θ is an almost one-to-one extension of an aperiodic bijective length- ℓ substitution shift.
- (2) There exists $0 \leq n, k$ such that the semigroup $S_{(\theta^n)_{(+k)}}$ contains a unique minimal left ideal.

Moreover, if j is the length of the shortest word in \mathcal{F}_θ , then $n \leq (\ell - 1)(\ell^j - 1)$ and $0 \leq k \leq \ell^n - 1$.

Proof. Let $F : (X_\theta, \sigma) \rightarrow (X_\eta, \sigma)$ where η is bijective, defined on a c -letter alphabet. By Proposition 4.13, $\kappa(F)$ is rational, and by Lemma 4.18, $\kappa(F) = p/q$ where $1 \leq q \leq (\ell - 1)(\ell^j - 1)$, where j is the length of the shortest word in \mathcal{F}_η . Using $\varphi(q) \leq (\ell - 1)(\ell^j - 1)$, and taking a shift if necessary, Corollary 4.16 tells us that there is $n \leq (\ell - 1)(\ell^j - 1)$ and $0 \leq k \leq \ell^n$, and a conjugacy $\tilde{F} : (X_{(\theta^n)_{(+k)}}, \sigma) \rightarrow (X_\theta, \sigma)$ such that $\kappa(F) + \kappa(\tilde{F}) \in \mathbb{Z}$. As $\kappa(F \circ \tilde{F}) \in \mathbb{Z}$, then $S_{(\theta^n)_{(+k)}}$ must have a unique minimal left ideal by Theorem 4.11.

For the converse, if $S_{(\theta^n)_{(+k)}}$ has a unique minimal left ideal, then by Theorem 4.11, $(X_{(\theta^n)_{(+k)}}, \sigma)$ factors dynamically onto a bijective substitution shift. As $(X_{(\theta^n)_{(+k)}}, \sigma)$ is conjugate to (X_θ, σ) , the result follows. \square

The following corollary follows from a straightforward application of the comments in Section 3.5 and underlines the interest in finding bijective factors.

Corollary 4.20. *Suppose that X_θ is almost bijective with corresponding bijective substitution η such that $\mathcal{U}_{\eta(0,1)}$ is a partition of $\mathcal{A}_\eta^{(2)}$. Then X_η factors isometrically onto a proper almost automorphic substitution shift X_ζ and*

$$X_\theta \xrightarrow{F_1} X_\eta \xrightarrow{F_2} X_\zeta \xrightarrow{F_3} \mathbb{Z}_\ell \xrightarrow{O} \{\text{pt}\}$$

is a Veech tower for X_θ , F_1, F_3 being almost one-to-one and F_2, O isometric.

Example 4.21. We return to the Rudin Shapiro substitution θ , in Example 3.22. One verifies that the length j of the shortest forbidden word in \mathcal{F}_θ is $j = 1$, so by Theorem 4.19, X_θ factors onto a bijective substitution shift if and only if $S_{\theta^{(+k)}}$ contains a unique minimal left ideal, where $0 \leq k \leq 1$. One can verify that each of $S_{\theta^{(+)}}$ and $S_{\theta^{(+)}}$ have two minimal left ideals. Therefore θ does not factor onto a bijective substitution shift.

We remark that if $(\ell - 1)(\ell^j - 1) > 1$, then by inspecting the proof of the second part of Theorem 4.19, namely using $\varphi(q) \leq q - 1$ if $q > 1$, we can improve the bound to $n \leq (\ell - 1)(\ell^j - 1) - 1$ and $0 \leq k \leq \ell^n - 1$. We will use this in the examples below.

Example 4.22. We return to Example 4.14, with θ and η defined there. We take $\zeta := \theta^{(+3)}$, and we claim that ζ has a unique minimal ideal. The reason why we make this choice $\theta^{(+3)}$ is that, by Lemma 4.15, the natural conjugacy $G : X_\zeta \rightarrow X_\theta$ satisfies $\kappa(G) = -3/4$. Then $\kappa(\sigma \circ F) = 3/4$, and $F \circ \sigma \circ G : X_\zeta \rightarrow X_\eta$ has κ -value zero. Note though that $F \circ \sigma \circ G$ has right radius one, so we will need to work with $X_{\zeta^{(2)}}$. Theorem 4.11 guarantees that $\mathcal{S}_{\zeta^{(2)}}$ has a unique minimal left ideal, and Lemma 4.10 guarantees that so also does \mathcal{S}_ζ . Indeed, to define ζ , we simultaneously list and code the words of length two in \mathcal{L}_θ :

$$\begin{array}{lll} A = 02, & B = 03, & C = 14, \\ D = 15, & E = 20, & F = 21, \\ G = 34, & H = 35, & I = 40, \\ J = 41, & K = 52, & L = 53; \end{array}$$

it then can be verified that the 3-shifted extension $\zeta := \theta^{(+3)}$ of θ has a unique minimal left ideal, generated by the partition $\{\{A, B, K, L\}, \{C, D, G, H\}\{E, F, I, J\}\}$.

We end with an example that shows that if a substitution shift has a bijective substitution factor, this does not imply that its pure base has a bijective substitution factor, illustrating that we cannot reduce to work with the pure base.

Example 4.23. Consider the following substitution on $\{a, \bar{a}, b, c, d, e, f\}$:

$$\begin{aligned} \theta: a &\mapsto adc \\ \bar{a} &\mapsto \bar{a}dc \\ b &\mapsto bea \\ c &\mapsto cfb \\ d &\mapsto d\bar{a}e \\ e &\mapsto ebf \\ f &\mapsto fcd. \end{aligned}$$

It is routine to check that this substitution is primitive and has height 2. We can define a factor map $F_\tau : X_\theta \rightarrow X_\eta$, which has local rule $\tau : \{a, \bar{a}, b, c, d, e, f\} \rightarrow \{a, b, c, d, e, f\}$, with $\tau(a) = \tau(\bar{a})$, and τ is the identity otherwise; η is the resulting inner encoding. It can be verified that η is bijective. As such, we can check that S_θ has indeed a unique minimal left ideal, satisfying the conditions of Theorem 4.1.

Given that both θ and η have height 2, one may proceed as in the proof of Proposition 3.25 and define a factor map between the pure bases $\hat{\theta}$ and $\hat{\eta}$ of both substitutions, but, in general, the property of being bijective is *not* a conjugacy invariant. In fact, $\hat{\eta}$ is not a bijective substitution. Furthermore, it cannot be taken to be conjugate to one: it can be verified that the maximal equicontinuous factor map of $X_{\hat{\eta}}$ has two irregular fibres modulo \mathbb{Z} , namely 0 and $\dots 1, 1, 1 = -1/2$, but any subshift that is conjugate to one arising from a bijective substitution can only have one irregular fibre modulo \mathbb{Z} .

Theorem 4.19 tells us that to decide whether $\tilde{\theta}$ is almost bijective, we need to compute the minimal left ideals of $(\tilde{\theta}^n)^{(+k)}$ with $1 \leq n \leq 3$ and $0 \leq k \leq 27$. Via automated computation, we have verified that $\tilde{\theta}$ has no bijective substitution shift factors.

Appendices

A. APPENDIX R

We first prove Theorem 2.9. The following lemma tells us that the assumption in Theorem 2.9, that $S_\theta \subset S_{\theta^2}$, is not a deal breaker.

Lemma A.1. *Let S_θ be a sub-semigroup of $\mathcal{F}(\mathcal{A})$ which is generated by a family of maps $\theta \in \mathcal{F}(\mathcal{A})$. If \mathcal{A} is finite then there exists $n \in \mathbb{N}$ such that $S_{\theta^n} = S_{\theta^{2n}}$ where θ^n is the set of maps which are n -fold products of elements of θ .*

Proof. Clearly, $S_{\theta^2} \subset S_\theta$, and if $S_\theta = S_{\theta^2}$ then also $S_{\theta^2} = S_{\theta^4}$. Hence, the infinite chain

$$S_\theta \supset S_{\theta^2} \supset S_{\theta^4} \supset \cdots$$

must stabilise, as S_θ is finite. \square

We use the following results.

Theorem A.2. [17, Theorem 5] *Let θ be a primitive aperiodic length- ℓ substitution and $\pi_\theta : X_\theta \rightarrow \mathbb{Z}_\ell$ the equicontinuous factor map which maps all fixed points (or θ -periodic points) to 0. Let $\tau : \mathcal{A}_\theta \rightarrow \mathcal{B}$ be a code and $Y = F_\tau(X_\theta)$. Then there is a factor map $\pi_Y \rightarrow \mathbb{Z}_\ell$ such that $\pi_\theta = \pi_Y \circ F_\tau$.*

The following proposition follows from the recognizability of constant length substitutions, which was essentially shown by Dekking in [6].

Proposition A.3. *Let θ be a primitive aperiodic length- ℓ substitution and $\pi_\theta : X_\theta \rightarrow \mathbb{Z}_\ell$ the equicontinuous factor map which maps all fixed points (or θ -periodic points) to 0. If $x, x' \in X_\theta$ satisfy $\pi_\theta(x) = \pi_\theta(x')$ then, for all $N \in \mathbb{N}$ there exists $n \in \mathbb{Z}$ such that $\sigma^n(x)$ and $\sigma^n(x')$ belong to $\theta^N(X_\theta)$. Here θ is viewed as a map on X_θ where it is injective, but not surjective.*

Recall that S_θ is the sub-semigroup of $\mathcal{F}(\mathcal{A})$ generated by the column maps of the substitution. Recall that a partition \mathcal{P} of \mathcal{A} defines a code $\tau : \mathcal{A} \rightarrow \mathcal{B} = \mathcal{P}$, $\tau(a)$ is the member $A \in \mathcal{P}$ to which a belongs. If for each $f \in S_\theta$ and for each $A \in \mathcal{P}$ there is $A' \in \mathcal{P}$ with $f(A) \subset A'$, then the code defines an inner encoded substitution (η, τ) . If not, we can define a finer partition $\tilde{\mathcal{P}}$ which has this property, notably through the equivalence relation $a \sim b$ if for each $f \in S_\theta$, $\tau(f(a)) = \tau(f(b))$. We denote the associated inner encoded system by (η, β) . The alphabet $\mathcal{A}_\eta = \beta(\mathcal{A})$ of η sits in between \mathcal{A} and \mathcal{B} , because $a \sim b$ means that in particular $\tau(a) = \tau(b)$. Hence τ factorises $\tau = \tau' \circ \beta$ where $\tau' : \beta(\mathcal{A}) \rightarrow \mathcal{B}$ is a code. In what follows we use the fact that $|\mathcal{A}|(|\mathcal{A}| - 1)$ is the number of unequal ordered pairs in \mathcal{A} .

Lemma A.4. *Let θ be a primitive aperiodic length- ℓ substitution over \mathcal{A} . Let $N = |\mathcal{A}|(|\mathcal{A}| - 1)$. Suppose that $S_\theta \subset S_{\theta^2}$. Let $a, b \in \mathcal{A}$. If $\tau(\theta^N(a)) = \tau(\theta^N(b))$ then $\forall f \in S_\theta$ we have $\tau(f(a)) = \tau(f(b))$.*

Proof. Any $f \in S_\theta$ has the form $\theta_{i_n} \cdots \theta_{i_1}$ for some n . Let, for $j \leq n$, $f^{(j)} = \theta_{i_j} \cdots \theta_{i_1}$ and $f^{(0)} = \mathbf{1}$. If $f(a) \neq f(b)$ then $f^{(j)}(a) \neq f^{(j)}(b)$ for all $j \leq n$ and so, if $n \geq N$ then the sequence of unequal pairs $(f^{(j)}(a), f^{(j)}(b))_{0 \leq j \leq n} \subset \mathcal{A}^2$ hits at least one point twice, say $(f^{(j_1)}(a), f^{(j_1)}(b)) = (f^{(j_2)}(a), f^{(j_2)}(b))$. Let

$$\tilde{f} = \theta_{i_n} \cdots \theta_{j_2} \theta_{j_1-1} \cdots \theta_{i_1}$$

that is, leave out θ_i for $i = j_1, \dots, j_2 - 1$ in the product for f). Iterating if necessary $\tilde{f} \in S_\theta$ has at most N factors θ_i . Furthermore, $\tilde{f}(a) = f(a)$ and $\tilde{f}(b) = f(b)$. Now $S_\theta \subset S_{\theta^2}$ implies that we can write \tilde{f} as a product of N factors. Therefore there is m such that $\tilde{f} = \theta^N_m$. It follows that $\tau(\theta^N(a)) = \tau(\theta^N(b))$ implies $\tau(\tilde{f}(a)) = \tau(\tilde{f}(b))$ implies $\tau(f(a)) = \tau(f(b))$. \square

Proof of Theorem 2.9. Given the code $\tau : \mathcal{A} \rightarrow \mathcal{B}$ we have seen above that we can construct an inner encoding (η, β) of θ and $\tau = \tau' \circ \beta$. It remains to show that $F_{\tau'} : X_\eta \rightarrow Y \subset \mathcal{B}^{\mathbb{Z}}$ is injective. Clearly $X_\eta = F_\beta(X_\theta)$. Let $\tilde{x}, \tilde{y} \in X_\eta$, so $\tilde{x} = F_\beta(x)$ and $\tilde{y} = F_\beta(y)$ for some $x, y \in X_\theta$, and suppose that $F_{\tau'}(\tilde{x}) = F_{\tau'}(\tilde{y})$. Hence $F_\tau(x) = F_\tau(y)$. By Theorem A.2, $\pi_\theta(x) = \pi_Y(F_\tau(x)) = \pi_Y(F_\tau(y)) = \pi_\theta(y)$. Lemma A.3 implies that there is n such that $\sigma^n(x), \sigma^n(y) \in \theta^N(X_\theta)$. Let x', y' be pre-images, $\theta^N(x') = \sigma^n(x)$ and $\theta^N(y') = \sigma^n(y)$. We thus have $F_\tau(\theta^N(x')) = F_\tau(\theta^N(y'))$ which means, for all k , $\tau(\theta^N(x'_k)) = \tau(\theta^N(y'_k))$. Lemma A.4 implies that $\tau(f(x'_k)) = \tau(f(y'_k))$, for all k . Hence $\beta(x'_k) = \beta(y'_k)$, for all k . As θ maps members of the partition defined by β into members of that partition we have $\beta(\theta(x'_k)) = \beta(\theta(y'_k))$ hence, by iteration, $\beta(x_k) = \beta(y_k)$ for all k . Hence $\tilde{x} = \tilde{y}$. \square

Next, we give a proof of Proposition 2.10. It is based on the proof of [5, Proposition 3.19], where there is a similar result bounding the radius of an automorphism of a constant length substitution shift. Here, we bound the radius of a factor map of a constant length substitution shift.

Proof of Proposition 2.10. Let l, r be the left, right radius of F respectively, and let $n = (-l, r)$. Then there is a code $\tau : \mathcal{A}_{\theta^{(n)}} \rightarrow \mathcal{B}$ such that $F_\tau : X_{\theta^{(n)}} \rightarrow Y$ is a factor map and such that $F = F_\tau \circ F_i^{-1}$. Note that by Lemmas 2.4 and 2.6, F_i^{-1} maps fixed points to fixed points.

To prove the second statement, suppose that $Y = X_\eta$ for some length- ℓ substitution η and that F preserves the fibre of fixed points. Let $R = \max\{l, r\}$. We know that $(F(x))_i$ is determined by $x_{[i-R, i+R]}$, and thus $(F(x))_{[0, \ell^n]}$ is determined by $x_{[-R, \ell^n + R]}$. Thus, if we choose $n \geq \lceil \log_\ell R \rceil$, this ensures $R \leq \ell^n$, hence $(F(x))_{[0, \ell^n]}$ is entirely determined by $x_{[-\ell^n, 2\ell^n]}$.

Fix $n \geq \lceil \log_\ell R \rceil$. Since F sends fixed points to fixed points, then $F(\theta^n(X_\theta)) \subseteq \eta^n(X_\eta)$. Define $G : X_\theta \rightarrow X_\eta$ as $G := \eta^{-n} \circ F \circ \theta^n$. Note that $F \circ \theta^n$ maps X_θ to $\eta^n(X_\eta)$, and the map $\eta^{-n} : \eta^n(X_\eta) \rightarrow X_\eta$ is well-defined by recognisability of η , so G is also well-defined. It is not hard to check that G is continuous and $G \circ \sigma = \sigma \circ G$, so G is also a factor map.

Given knowledge of $x_{[-1, 1]}$, we know $(\theta^n(x))_{[-\ell^n, 2\ell^n]}$ and hence $(F \circ \theta^n(x))_{[0, \ell^n]}$ is also determined, as discussed above.

Recall that a substitution η is *injective* if the map $\eta : \mathcal{A} \rightarrow \mathcal{A}^\ell$ is injective. If η is injective, then $(F \circ \theta^n(x))_{[0, \ell^n]}$ determines $(\eta^{-n} \circ F \circ \theta^n(x))_0$, and thus G has left and right radius at most 1. Also, G must send fixed points to fixed points. If η is non-injective on letters, we can replace it with an injectivisation $\tilde{\eta}$ of η , which is the standard example of an inner encoding of η , with two letters identified if and

only if their images under η are equal. As the associated conjugacy $X_\eta \rightarrow X_{\bar{\eta}}$ has radius 0, this will not change the desired result, that G has left and right radius at most 1.

It remains to show that the radius restriction for G implies the same restriction for F . Minimality implies that any factor map $X_\theta \rightarrow X_\eta$ is entirely determined by the image of a single point. As the fixed point fibres of substitutions are finite, there can only be finitely many factor maps $F_i : X_\theta \rightarrow X_\eta$ which preserve the fixed point fibre. Let $\mathcal{F} := \{F_1, \dots, F_k\}$ be the collection of those. Let l_i, r_i be the left and right radius of each F_i . If we define $R = \max\{l_1, r_1, \dots, l_k, r_k\}$ and take $n \geq \lceil \log_\ell R \rceil$, then the above argument shows that $\eta^{-n} \circ F_i \circ \theta^n$ preserves the fixed point fibre of θ and hence the map $\mathcal{F} \rightarrow \mathcal{F}$ given by $F_i \mapsto \eta^{-n} \circ F_i \circ \theta^n$ is well-defined. This map is a bijection. For, if $\eta^{-n} \circ F_i \circ \theta^n = \eta^{-n} \circ F_j \circ \theta^n$, then $F_i \circ \theta^n(u) = F_j \circ \theta^n(u)$ for any point $u \in X_\theta$, and in particular for any fixed point u of θ , so that $F_i(u) = F_j(u)$ for u a fixed point. Now minimality implies that the map is injective and since \mathcal{F} is finite it is also surjective.

To conclude, we have seen above that every factor map of the form $\eta^{-n} \circ F_i \circ \theta^n$ has left and right radius at most 1 and so the same must hold for all F_i . \square

APPENDIX M

We formulate a theorem of Martin in a slightly more general way and provide the proof for the convenience of the reader [16, Thm.8.11].

Theorem A.5. *Let (Y, S) be an almost automorphic factor of (X, T) . Then there exists a factor (\tilde{Y}, \tilde{S}) of (X, T) which is both an extension of (Y, S) and an almost one-to-one extension of the maximal equicontinuous factor of (X, T) . If (Y, S) is a proper almost automorphic factor of (X, T) then (\tilde{Y}, \tilde{S}) is also a proper almost automorphic factor of (X, T) . If the factor map $F : X \rightarrow Y$ is isometric then the factor map $\tilde{F} : X \rightarrow \tilde{Y}$ is also isometric.*

Proof. See Figure 3. Let $\pi_X : X \rightarrow X_{max}$ be the maximal equicontinuous factor map. Define the equivalence relation on X by $x_1 \sim x_2$ if $F(x_1) = F(x_2)$ and $\pi_X(x_1) = \pi_X(x_2)$. Let $\tilde{Y} = X / \sim$ and let $\tilde{F} : X \rightarrow \tilde{Y}$ denote the canonical surjection. Clearly π_X factors $\pi_X = \tilde{\pi} \circ \tilde{F}$, and $F : X \rightarrow Y$ factors $F = G \circ \tilde{F}$. In particular \tilde{Y} is an extension of Y . Let Y_{max} be the MEF of Y with factor map π_Y . By maximality, X_{max} factors onto Y_{max} and we can arrange the factor map $\rho : X_{max} \rightarrow Y_{max}$ in such a way that $\pi_Y \circ F = \rho \circ \pi_X$.

We claim that $\tilde{\pi} : \tilde{Y} \rightarrow X_{max}$ is almost one-to-one: Let $\eta \in Y_{max}$ such that $\pi_Y^{-1}(\eta)$ contains a single element, say y . Let $\xi \in \rho^{-1}(\eta)$. Let $x \in \pi_X^{-1}(\xi)$. Then $\pi_Y \circ F(x) = \eta$, hence $F(x) = y$. Hence any two elements $x_1, x_2 \in \pi_X^{-1}(\xi)$ satisfy $F(x_1) = F(x_2)$ and $\pi_X(x_1) = \pi_X(x_2)$ so that $\tilde{F}(x_1) = \tilde{F}(x_2)$. It follows that $\tilde{\pi}^{-1}(\xi)$ contains only one element.

We claim that (\tilde{Y}, \tilde{S}) is not equicontinuous. Indeed, if it were equicontinuous then Y would also be equicontinuous, as it is a factor of \tilde{Y} .

Finally, if $F = G \circ \tilde{F}$ is isometric then \tilde{F} must also be isometric. \square

APPENDIX J

We recall some background material on subsemigroups of the semigroup $\mathcal{F}(X)$ of maps from X to itself; see also [4, 10, 18]. Here X is just a set and the semigroup

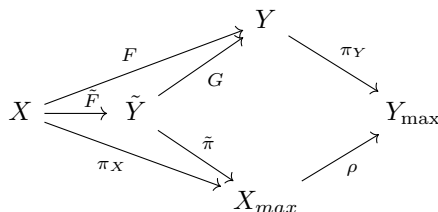


FIGURE 3. The commuting diagram for Theorem A.5.

product is composition of functions. For our purposes X will be a finite set. We denote by $\mathcal{P}(X)$ the set of subsets of X .

Let $f : X \rightarrow Y$. We denote by $f^{-1} : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ the pre-image map but simply write $f^{-1}(y)$ for $f^{-1}(\{y\})$. The map f defines an equivalence relation as $x \sim x'$ if $f(x) = f(x')$. We denote the associated partition by \mathcal{P}_f , that is,

$$\mathcal{P}_f = \{f^{-1}(y) : y \in Y\}.$$

The cardinality of \mathcal{P}_f equals the rank of f , that is the cardinality of its image $\text{im}f$.

Recall two of Green's equivalence relations \mathcal{L} , \mathcal{R} . They give a first way to approach and organise a semigroup S . We say that $a, b \in S$ are \mathcal{L} -related, or \mathcal{R} -related, if they generate the same left, or right ideal, respectively. If S is a group, then these relations coincide with the full relation. Each of the above relations partition the semigroup. \mathcal{R} is a left congruence and therefore S/\mathcal{R} a left- S -module. \mathcal{L} is a right congruence and therefore S/\mathcal{L} a right- S -module.

Lemma A.6. *Let S be a semigroup of $\mathcal{F}(X)$. If $f, g \in S$ are \mathcal{R} -related then $\text{im}f = \text{im}g$. If $f, g \in S$ are \mathcal{L} -related then $\mathcal{P}_f = \mathcal{P}_g$.*

Proof. Let f and g be \mathcal{R} -related, that is, $f = g$ or there is f', g' such that $f = gg'$, $g = ff'$. Then clearly $\text{im}f \subset \text{im}g$ and $\text{im}g \subset \text{im}f$.

Let f and g be \mathcal{L} -related, that is, $f = g$ or there is f', g' such that $f = g'g$, $g = f'f$. Then $f^{-1}(x) = g^{-1}(g'^{-1}(x))$ showing that the members of the partition \mathcal{P}_f are unions of members of the partition of \mathcal{P}_g , i.e. the partition \mathcal{P}_g is finer than \mathcal{P}_f . A symmetric argument shows that $\mathcal{P}_f = \mathcal{P}_g$. \square

An element $f \in S \subset \mathcal{F}(X)$ is *completely regular* if there exists $g \in S$ such that $fgf = f$ and $fg = gf$. This implies that fg is an idempotent. As normal inverses are unique we call fg the idempotent associated to f and denote it f^0 .

A *completely simple* semigroup is a semigroup which has no proper bilateral ideals and which contains an idempotent. The *kernel* of a semigroup, if it exists, is its smallest bilateral ideal. If X is finite then any sub-semigroup of $\mathcal{F}(X)$ admits a kernel. We denote the kernel of S by $\ker S$; if it contains an idempotent it is completely simple, and any element is completely regular.

Lemma A.7. *Let X be a finite set and $S \subset \mathcal{F}(X)$. Then S is completely simple if and only if all its functions have the same rank. Moreover, if $f, g \in S$ belong to the same right ideal, then $g^0f = f$, while if f, g belong to the same left ideal, then $f = fg^0$.*

Proof. Given $n \in \mathbb{N}$, the subset of functions of rank $\leq n$ form a bilateral ideal in S . Hence the condition that all functions have the same rank is necessary for simplicity.

Now suppose that all functions of S have the same rank. Let $f \in S$. Then $\text{im}f = \text{im}f^k$ for all $k \geq 1$. As X is finite there must be a $k > 0$ such that $f = f^{k+1}$. Hence $f^k = f^0 \in S$. The restriction of f^0 to $\text{im}f$ is the identity. Let also $g \in S$. As $fg \in S$ the argument above shows that there is $k > 0$ such that the restriction of $(gf)^k$ to $\text{im}gf = \text{im}g$ is the identity. It follows that $(gf)^k g = g$. Hence g belongs to the bilateral ideal generated by f . Choosing $f \in \ker S$ we see that $S = \ker S$ and so is simple.

Suppose that $f, g \in S$ belong to the same right ideal. Then $f = gg'$ for some $g' \in S$. Hence $g^0 f = g^0 gg' = gg' = f$. Similarly, if $f, g \in S$ belong to the same left ideal. Then $f = g'g$ for some $g' \in S$. Hence $fg^0 = g'gg^0 = f$. \square

Corollary A.8. *Let X be a finite set and $S \subset \mathcal{F}(X)$. The kernel of S is given by its functions of minimal rank. It is completely simple.*

The Rees structure theorem ([4, Theorem 3.5] or [13, Theorem 2.1]) tells us that a completely simple semigroup (without zero) is isomorphic to a matrix semigroup $S \cong I \times G \times \Lambda$ where I indexes the right ideals of S , Λ indexes the left ideals of S , G is a group and there is an $I \times \Lambda$ matrix M such that multiplication in $I \times G \times \Lambda$ is defined as

$$(i, g, \lambda)(i', g', \lambda') = (i, gM_{\lambda, i'}g', \lambda')$$

The right ideals are given by $\{i\} \times G \times \Lambda$, $i \in I$, and the left ideals by $I \times G \times \{\lambda\}$, $\lambda \in \Lambda$. In particular one sees that any right ideal intersects any left ideal non-trivially.

Proposition A.9. *Let $S \subset \mathcal{F}(X)$ be a completely simple semigroup and $f, g \in S$. The following are equivalent.*

- R1 f and g are \mathcal{R} -related.
- R2 f and g belong to the same right ideal.
- R3 f and g have the same image.

Moreover, the following are equivalent.

- L1 f and g are \mathcal{L} -related.
- L2 f and g belong to the same left ideal.
- L3 f and g define the same partition.

Proof. A completely simple semigroup (without zero) is the disjoint union of its right ideals and all the right ideals are simple. As the \mathcal{R} -class of any element of a simple right ideal is that right ideal, we have equivalence between R1 and R2. It remains to show that R3 implies R2: Suppose that f and g belong to the same left ideal. Then $fg^0 = f$. If also $\text{im}f = \text{im}g$ then $g^0 f = f$, hence f and g belong to the same right ideal. Now suppose that f and g are arbitrary elements satisfying $\text{im}f = \text{im}g$. Then there is $h \in S$ which is in the same right ideal as g and the same left ideal as f . The first property implies $\text{im}h = \text{im}g$ and, by the above, h is also in the same right ideal as f . Hence f and g belong to the same right ideal.

The equivalence between L1 and L2 is shown as for right ideals. We show that L3 implies L2: Suppose that f and g belong to the same right ideal. Then $g^0 f = f$. If also $\mathcal{P}_f = \mathcal{P}_g$ then $fg^0 = f$, hence f and g belong to the same left ideal. The rest of the argument is as for right ideals. \square

Lemma A.10. *Let S be a semigroup which admits a kernel. Let $\Phi : S \rightarrow T$ be an epimorphism onto another semigroup T . The restriction of Φ to the kernel of S*

is an epimorphism onto the kernel of T . Moreover, if S has a unique minimal left ideal then also T has a unique minimal left ideal.

Proof. Any epimorphism preserves ideals: We show this for left ideals. Let I be a left ideal of S and $t \in T$; we ought to show that $t\Phi(I) \subset \Phi(I)$. Since Φ is onto there is $s \in S$ such that $\Phi(s) = t$. Then $t\Phi(I) = \Phi(sI) \subset \Phi(I)$.

Furthermore, the preimage of an ideal is an ideal: We show this for left ideals. Let I be a left ideal of T and $s \in S$; we ought to show that $s\Phi^{-1}(I) \subset \Phi^{-1}(I)$. Indeed, $\Phi(s\Phi^{-1}(I)) = tI \subset I$.

Now the above implies that $I := \Phi(\ker S)$ is the kernel of T : Indeed, by the first paragraph it is a bilateral ideal. Let $J \subset I$ be a bilateral ideal of T . Then $\Phi^{-1}(J)$ is a bilateral ideal in S by the second paragraph. $\Phi^{-1}(J)$ thus contains $\ker S$. Hence $\Phi(\ker S) \subset J$ showing that $J = I$.

Suppose now that $\ker S$ is left simple and that L_1, L_2 are left ideals of $\ker T$. Then $\Phi^{-1}(L_1) \cap \ker S$ and $\Phi^{-1}(L_2) \cap \ker S$ belong to the same left ideal. As epimorphisms preserve ideals their image under Φ belongs to the same left ideal. Hence L_1 and L_2 belong to the same left ideal. Hence $L_1 = L_2$. \square

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