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Abstract. We introduce a class of unbounded quasiconvex integrands for which we prove a lower semicontinuity theorem of the associated integrals.

1. Introduction and main result

Let $m, N \in \mathbb{N}^*$ be two positive integers. Let $W : \mathbb{M} \to [0, \infty]$ be a lower semicontinuous function where $\mathbb{M}$ is the space of m rows N columns matrices. Let $p \in [1, \infty[$. We consider the integral functional $I : W^{1,p}(\Omega; \mathbb{R}^m) \to [0, \infty]$ defined by

$$I(u) := \int_\Omega W(\nabla u(x))dx.$$

In vectorial calculus of variations, i.e. when $\min(N, m) > 1$, Morrey [25] introduced the concept of quasiconvexity as a necessary and sufficient condition for weak lower semicontinuity on $W^{1,\infty}(\Omega; \mathbb{R}^m)$. This condition is distinct from convexity and is very difficult to grasp due to its non-local character [21]. Quasiconvexity is a necessary and sufficient condition in the case of integrands with p-polynomial growth conditions [1]; for the case of finite integrands, see [22, 24, 31, 30, 27, 28, 18].

Very little is known about the problem of lower semicontinuity for integrals I when the quasiconvex integrands take the value ∞. Of course, there is the well-known case where W is convex and lower semicontinuous; in this case, the associated integral is lower semicontinuous. On the other hand, the concept of polyconvexity [14], which allows W to take the value ∞ and which is not necessarily convex, is suitable in the context of hyperelasticity as it allows for considering the following two basic conditions: the impossibility of reducing a finite volume to zero volume and the non-interpenetration of matter, i.e.

$$\lim_{\det F \to 0} W(F) = \infty \quad \text{and} \quad (\forall F \in \mathbb{M} \quad W(F) < \infty \iff \det F > 0). \tag{1}$$

We should mention other classes of integrands which can take infinite values, and which have been studied for their connections to weak lower semicontinuity of I. One of such class is the closed $W^{1,p}$-quasiconvex integrands introduced by Pedregal [26]. These integrands have been caracterized by Kristensen [20] as being supremum of quasiconvex functions with p-growth conditions (this also discussed in [22, 15]), thus allowing them to take the value ∞. Additionally, various classes of integrands satisfying conditions related to the blow-up method of proof (see [19]) have emerged in recent years, as highlighted in [24, 23, 32]. It is worth noting that this blow-up method has been used in problems of relaxation, homogenization, and Γ-convergence problems since the mid-1990s.

Our aim here is to propose a new class of integrands which can take the value ∞, and which contains naturally the class of lower semicontinuous convex integrands.

Denote by $\mathrm{dom}W := \{\xi \in \mathbb{M} : W(\xi) < \infty\}$ the effective domain of W. We consider the following assertions on W:

(A_1) $\mathrm{int}(\mathrm{dom}W) \neq \varnothing$ and there exists $C > 0$ such that for every $\xi, \zeta \in \mathbb{M}$ and every $t \in]0, 1[$

$$W(t\xi + (1-t)\zeta) \leqslant C(1 + W(\xi) + W(\zeta));$$

(A$_2$) W is ru-usc, i.e. there exists $\xi_0 \in \mathrm{int}(\mathrm{dom}W)$ such that $\overline{\lim}_{t\to 1^-} \Delta_W^{a,\xi_0}(t) \leqslant 0$ where for some $a > 0$

$$\Delta_W^{a,\xi_0}(t) := \sup_{\xi\in \mathrm{dom}W} \frac{W(t\xi + (1-t)\xi_0) - W(\xi)}{a + W(\xi)}.$$

The condition (A$_1$) means that there is self-control by the function W itself over potential bumps it may have. This hypothesis was initially introduced in [5] following works in relaxation and homogenization [29, 3]. Subsequently, it has been used in problems of Γ-convergence, homogenization, and relaxation for unbounded functionals in [11, 7, 8, 10]. Note that this condition implies that dom$W$ the effective domain of W is convex.

The concept of a ru-usc function (A$_2$) was introduced in [2] and further studied in [6, 9]. It has been used in works concerning relaxation, homogenization, and Γ-convergence of unbounded integrals, see [3, 4, 11, 17, 7, 8, 10].

When W satisfies (A$_1$) and (A$_2$), we say that W is a ru-usc function with self-control over its (own potential) bumps. The following proposition provides conditions under which (A$_1$) is satisfied.

Proposition 1. *We have*

(i) *if W is convex with* $\mathrm{int}(\mathrm{dom}W) \neq \varnothing$ *then* (A$_1$) *and* (A$_2$) *hold;*

(ii) *if W has G-growth from above and below with $G : \mathbb{M} \to [0,\infty]$ satisfying* (A$_1$)*, i.e.* $\mathrm{int}(\mathrm{dom}G) \neq \varnothing$*, and there exist $\alpha, \beta > 0$ such that for all $\xi \in \mathbb{M}$*

$$\alpha G(\xi) \leqslant W(\xi) \leqslant \beta(1 + G(\xi)), \tag{2}$$

then (A$_1$) *holds.*

A function $W : \mathbb{M} \to [0,\infty]$ is $W^{1,p}$-quasiconvex if

(A$_3$) for every $\xi \in \mathbb{M}$

$$W(\xi) = \inf\left\{\int_Y W(\xi + \nabla\varphi(x))dx : \varphi \in W_0^{1,p}(Y;\mathbb{R}^m)\right\}$$

where $Y =]0,1[^N$. When $p = \infty$, we say that W is quasiconvex.

We say that $I : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,\infty]$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$, if for every $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and every $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ satisfying $u_n \rightharpoonup u$ we have

$$\underline{\lim}_{n\to\infty} \int_\Omega W(\nabla u_n(x))dx \geqslant \int_\Omega W(\nabla u(x))dx.$$

Here the main result of the paper.

Theorem 1. *Assume that $p > N$ and that* (A$_1$) *and* (A$_2$) *hold. Then I is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$ if and only if W is $W^{1,p}$-quasiconvex, i.e. satisfies* (A$_3$)*.*

Our result is related to the questions raised by the Ball-Murat Conjecture [13, Conjecture 3.7, p. 232] which states that if W is continuous, then I is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$ (sequentially weakly* lower semicontinuous if $p = \infty$) if and only if W is $W^{1,p}$-quasiconvex. While we are clearly far from providing some answers to these questions, our result may be viewed as a step in this direction.

The plan of the paper is as follows. Section 2 is devoted to the proof of Theorem 1. In Section 3, we give some examples. In Subsection 3.1 we provide a class of ru-usc integrands under differentiability assumption. In Subsection 3.2, we give an example of unbounded ru-usc quasiconvex integrands satisfying the two basic conditions in hyperelasticity (1) in the two dimensional case. Then, we prove Proposition 1 in Section 4.

2. Proof of Theorem 1

The necessary condition of the sequentially weakly lower semicontinuity on $W^{1,p}(\Omega;\mathbb{R}^m)$ of I is given by [13, Corollary 3.2, p. 231].

Consider $\xi_0 \in \text{int}(\text{dom}W)$ given by (A_2). We can assume that $\xi_0 = 0$, indeed, we can make the following change of integrand $\widetilde{W}(A-\xi_0) := W(A)$ for all $A \in \mathbb{M}$. The new integrand $\widetilde{W}$ satisfies (A_1) and (A_2) with $\Delta^{a,0}_{\widetilde{W}}$.

Let us show that if W is $W^{1,p}$-quasiconvex, i.e. (A_3) holds, then $I : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,\infty]$ defined by

$$I(u) := \int_\Omega W(\nabla u(x))dx.$$

is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$.

Let $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and a sequence $\{u_\varepsilon\}_{\varepsilon>0} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ satisfy

$$u_\varepsilon \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^m) \quad \text{and} \quad \varliminf_{\varepsilon\to 0} I(u_\varepsilon) = \lim_{\varepsilon\to 0} I(u_\varepsilon) < \infty.$$

It follows that $\nabla u_\varepsilon(\cdot) \in \text{dom}W$ a.e. in Ω, and since $\text{dom}W$ is convex, it follows that $\nabla u(\cdot) \in \overline{\text{dom}W}$ a.e. in Ω. By the line segment principle, we have $t\nabla u(\cdot) \in \text{int}(\text{dom}W)$ a.e. in Ω for all $t \in [0,1[$.

Since the compact imbedding of Sobolev spaces into $L^\infty(\Omega;\mathbb{R}^m)$ because $p > N$, we can assume, up to a subsequence, that

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_{L^\infty(\Omega;\mathbb{R}^m)} = 0. \tag{3}$$

We set $\mu_\varepsilon := W(\nabla u_\varepsilon(\cdot))\mathcal{L}_N\lfloor_\Omega$ for all $\varepsilon > 0$. The sequence of positive Borel measures $\{\mu_\varepsilon\}_{\varepsilon>0}$ is bounded, i.e.

$$\sup_{\varepsilon>0} \mu_\varepsilon(\Omega) < \infty,$$

there exists then a subsequence (not relabelled) and a Borel measure μ such that $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ weakly in the sense of measures. To prove the lower semicontinuity result, it is sufficient to show that for $\mathcal{L}_N$-a.e. $x_0 \in \Omega$

$$\frac{d\mu}{d\mathcal{L}_N}(x_0) \geqslant W(\nabla u(x_0)), \tag{4}$$

indeed, by Alexandrov theorem

$$\varliminf_{\varepsilon\to 0} I(u_\varepsilon) = \lim_{\varepsilon\to 0} I(u_\varepsilon) \geqslant \varliminf_{\varepsilon\to 0} \mu_\varepsilon(\Omega) \geqslant \mu(\Omega) \geqslant \int_\Omega \frac{d\mu}{d\mathcal{L}_N}(x)dx \geqslant \int_\Omega W(\nabla u(x))dx.$$

We have for $\mathcal{L}_N$-a.e. $x_0 \in \Omega$ and for every $s \in]0,1[$

$$\frac{d\mu}{d\mathcal{L}_N}(x_0) = \lim_{\rho\to 0} \varliminf_{\varepsilon\to 0} \frac{\mu_\varepsilon(B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))} = \lim_{\rho\to 0} \overline{\lim_{\varepsilon\to 0}} \frac{\mu_\varepsilon(B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))}, \tag{5}$$

indeed, by Alexandrov theorem we can write that for every $x_0 \in \Omega$ and every $(s,\rho) \in]0,1[^2$

$$\frac{\mu(B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))} \leqslant \varliminf_{\varepsilon\to 0} \frac{\mu_\varepsilon(B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))} \leqslant \overline{\lim_{\varepsilon\to 0}} \frac{\mu_\varepsilon(B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))} \leqslant \frac{\mu(\overline{B}_{s\rho}(x_0))}{\mathcal{L}_N(\overline{B}_{s\rho}(x_0))}$$

now, by passing to the limits $\rho \to 0$ we get (5).

Fix $t \in]0,1[$. Let $\tau \in]t,1[$ be such that $\Delta^{a,0}_W(\tau) < \infty$. Let $x_0 \in \Omega$ satisfy:

$$W(\tau\nabla u(x_0)) < \infty; \tag{6}$$

$$\nabla u(x_0) \in \overline{\text{dom}W}; \tag{7}$$

$$\frac{d\mu}{d\mathcal{L}_N}(x_0) = \lim_{s\to 1^-} \overline{\lim_{\rho\to 0}} \varliminf_{\varepsilon\to 0} \frac{\mu_\varepsilon(B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))} = \lim_{s\to 1^-} \overline{\lim_{\rho\to 0}} \overline{\lim_{\varepsilon\to 0}} \frac{\mu_\varepsilon(B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))} < \infty; \tag{8}$$

$$0 = \lim_{\rho\to 0} \frac{1}{\rho} \|u - u_{x_0}\|_{L^\infty(B_{s\rho}(x_0);\mathbb{R}^m)}. \tag{9}$$

The last equation (9) comes from the fact that every Sobolev function in $W^{1,p}(\Omega;\mathbb{R}^m)$ with $p > N$ is almost everywhere differentiable; $u_x(\cdot) := u(x) + \nabla u(x)(\cdot - x)$ is the affine tangent map to u at $x \in \Omega$.

To simplify the notation, let us denote $\Delta_W^{a,0} = \Delta_W^a$. We also set for every $(u, O) \in W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega)$

$$I(u, O) = \int_O W(\nabla u(x))dx$$

where $\mathcal{O}(\Omega)$ is the set of all open subsets of Ω.

By using (A_2) the ru-usc assumption on W, we have for every $t, s, \rho \in]0, 1[$ and every $\varepsilon > 0$

$$\begin{aligned}\frac{I(tu_\varepsilon, B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))} &\leqslant \frac{I(u_\varepsilon, B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))} + \Delta_W^a(t)a + \Delta_W^a(t)\frac{I(u_\varepsilon, B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))}\\ &\leqslant \frac{I(u_\varepsilon, B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))} + \Delta_W^a(t)a + \Delta_W^a(t)\frac{I(u_\varepsilon, B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))}\\ &= \frac{\mu_\varepsilon(B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))} + \Delta_W^a(t)a + \Delta_W^a(t)\frac{\mu_\varepsilon(B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))}\end{aligned}$$

passing to the limits $\varepsilon \to 0, \rho \to 0$ and $s \to 1$ we have

$$\begin{aligned}\overline{\lim_{s\to 1}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\frac{I(tu_\varepsilon, B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))} &\leqslant \lim_{s\to 1}\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\frac{\mu_\varepsilon(B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))} + \Delta_W^a(t)a\\ &\quad + \Delta_W^a(t)\lim_{s\to 1^-}\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\frac{\mu_\varepsilon(B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))}\\ &= \frac{d\mu}{d\mathcal{L}_N}(x_0) + \Delta_W^a(t)a + \Delta_W^a(t)\frac{d\mu}{d\mathcal{L}_N}(x_0)\\ &= \frac{d\mu}{d\mathcal{L}_N}(x_0) + \Delta_W^a(t)\left(a + \frac{d\mu}{d\mathcal{L}_N}(x_0)\right),\end{aligned}$$

letting $t \to 1$ and using (A_2) we obtain

$$\overline{\lim_{t\to 1}}\,\overline{\lim_{s\to 1}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\frac{I(tu_\varepsilon, B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))} \leqslant \frac{d\mu}{d\mathcal{L}_N}(x_0).$$

So, a sufficient condition to have (4) is to show

$$W(\nabla u(x_0)) \leqslant \overline{\lim_{t\to 1}}\,\overline{\lim_{s\to 1}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\frac{I(tu_\varepsilon, B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))}. \tag{10}$$

Let us prove (10).

Let $\kappa \in]0, 1[$ and let $(s, \rho) \in]0, 1[^2$ satisfy $B_{s\rho}(x_0)$. Let $\chi_\kappa \in W_0^{1,\infty}(B_{s\rho}(x_0); [0, 1])$ satisfy

$$\chi_\kappa \equiv 1 \text{ on } \overline{B}_{\kappa s\rho}(x_0), \quad \chi_\kappa \equiv 0 \text{ on } \Omega\backslash B_{s\rho}(x_0) \quad \text{and} \quad \|\nabla\chi_\kappa\|_{L^\infty(B_{s\rho}(x_0);\mathbb{R}^m)} \leqslant \frac{4}{s\rho(1-\kappa)}.$$

We define $u_{x_0}(x) := u(x_0) + \nabla u(x_0)(x - x_0)$ for all $x \in \Omega$.

Let $\varepsilon > 0$. We set

$$v_\varepsilon^\kappa := \chi_\kappa u_\varepsilon + (1 - \chi_\kappa)u_{x_0}$$

We see that $v_\varepsilon^\kappa \in u_{x_0} + W_0^{1,p}(B_{s\rho}(x_0);\mathbb{R}^m)$, implying the existence of $\varphi \in W_0^{1,p}(B_{s\rho}(x_0);\mathbb{R}^m)$ such that

$$v_\varepsilon^\kappa = u_{x_0} + \varphi.$$

Let $t' \in]0, 1[$ be such that $t = \tau t'$. We have

$$t\nabla v_\varepsilon^\kappa(x) = \begin{cases} t\nabla u_\varepsilon(x) & \text{if } x \in B_{\kappa s\rho}(x_0)\\ t'\left(\tau\chi_\kappa(x)\nabla u_\varepsilon(x) + \tau(1 - \chi_\kappa(x))\nabla u(x_0)\right) & \\ \quad +(1 - t')\dfrac{t}{1 - t'}\nabla\chi_\kappa(x) \otimes (u_\varepsilon(x) - u_{x_0}(x)) & \text{if } x \in B_{s\rho}(x_0)\backslash\overline{B}_{\kappa s\rho}(x_0).\end{cases}$$

and by (A_3) the $W^{1,p}$-quasiconvexity of W, we use [13, Proposition 2.3, p. 228] to see that the inequality in the definition of $W^{1,p}$-quasiconvexity holds over any nonempty bounded open set with zero measure of their boundary

$$W(t\nabla u(x_0)) = \inf_{\varphi\in W_0^{1,p}(B_{s\rho}(x_0);\mathbb{R}^m)} \fint_{B_{s\rho}(x_0)} W(t\nabla u(x_0)+\nabla\varphi(x))dx \leqslant \fint_{B_{s\rho}(x_0)} W(t\nabla v_\varepsilon^\kappa(x))dx.$$

Letting $\varepsilon\to 0, \rho\to 0$ and $s\to 1$ we get

$$\begin{aligned}\overline{\lim_{s\to 1}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}} \inf_{\varphi\in W_0^{1,p}(B_{s\rho}(x_0);\mathbb{R}^m)} \fint_{B_{s\rho}(x_0)} W(t\nabla u(x_0)+\nabla\varphi(x))dx \\ \leqslant \overline{\lim_{s\to 1}}\,\overline{\lim_{\kappa\to 1}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}} \fint_{B_{s\rho}(x_0)} W(t\nabla v_\varepsilon^\kappa(x))dx\end{aligned}$$

and when $t\to 1$ we have by lower semicontinuity of W

$$W(\nabla u(x_0)) \leqslant \overline{\lim_{t\to 1}}\,\overline{\lim_{s\to 1}}\,\overline{\lim_{\kappa\to 1}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}} \fint_{B_{s\rho}(x_0)} W(t\nabla v_\varepsilon^\kappa(x))dx.$$

Therefore (10) is a consequence of

$$\overline{\lim_{t\to 1}}\,\overline{\lim_{s\to 1}}\,\overline{\lim_{\kappa\to 1}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}} \fint_{B_{s\rho}(x_0)} W(t\nabla v_\varepsilon^\kappa(x))dx \leqslant \overline{\lim_{t\to 1}}\,\overline{\lim_{s\to 1}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}} \fint_{B_{s\rho}(x_0)} W(t\nabla u_\varepsilon(x))dx. \tag{11}$$

Let us prove (11). We set $U_{s,\kappa}^\rho := B_{s\rho}(x_0)\backslash\overline{B}_{\kappa s\rho}(x_0)$. By using the definition of $t\nabla v_\varepsilon^\kappa$ we have

$$\begin{aligned}&\fint_{B_{s\rho}(x_0)} W(t\nabla v_\varepsilon^\kappa(x))dx \\ \leqslant&\frac{1}{\mathcal{L}_N(B_{s\rho}(x_0))}\int_{B_{\kappa s\rho}(x_0)} W(t\nabla u_\varepsilon(x))dx + \frac{1}{\mathcal{L}_N(B_{s\rho}(x_0))}\int_{U_{s,\kappa}^\rho} W(t\nabla v_\varepsilon^\kappa(x))dx \\ \leqslant&\fint_{B_{s\rho}(x_0)} W(t\nabla u_\varepsilon(x))dx + \frac{1}{\mathcal{L}_N(B_{s\rho}(x_0))}\int_{U_{s,\kappa}^\rho} W(t\nabla v_\varepsilon^\kappa(x))dx,\end{aligned}$$

so, (11) holds if

$$\overline{\lim_{t\to 1}}\,\overline{\lim_{s\to 1}}\,\overline{\lim_{\kappa\to 1}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}} \frac{1}{\mathcal{L}_N(B_{s\rho}(x_0))}\int_{U_{s,\kappa}^\rho} W(t\nabla v_\varepsilon^\kappa(x))dx = 0. \tag{12}$$

We are then reduced to prove (12). By using (A_1) we have

$$\begin{aligned}&\frac{1}{\mathcal{L}_N(B_{s\rho}(x_0))}\int_{U_{s,\kappa}^\rho} W(t\nabla v_\varepsilon^\kappa(x))dx \\ \leqslant& C\left(\frac{\mathcal{L}_N(U_{s,\kappa}^\rho)}{\mathcal{L}_N(B_{s\rho}(x_0))} + \frac{1}{\mathcal{L}_N(B_{s\rho}(x_0))}\int_{U_{s,\kappa}^\rho} W(\tau\chi_\kappa\nabla u_\varepsilon + \tau(1-\chi_\kappa)(\nabla u(x_0)))dx\right. \\ &\left. + \frac{1}{\mathcal{L}_N(B_{s\rho}(x_0))}\int_{U_{s,\kappa}^\rho} W(\Psi_{\varepsilon,\rho}(x))dx\right) \\ \leqslant& C\left(\left(1-\kappa^N\right)\left(1+W(\tau\nabla u(x_0))\right) + \frac{1}{\mathcal{L}_N(B_{s\rho}(x_0))}\int_{U_{s,\kappa}^\rho} W(\tau\nabla u_\varepsilon)dx\right. \\ &\left. + \frac{1}{\mathcal{L}_N(B_{s\rho}(x_0))}\int_{U_{s,\kappa}^\rho} W(\Psi_{\varepsilon,\rho}(x))dx\right)\end{aligned} \tag{13}$$

where $\Psi_{\varepsilon,\rho}(\cdot) := \frac{t}{1-t'}\nabla\chi_\kappa(\cdot)\otimes(u_\varepsilon(\cdot)-u_{x_0}(\cdot))$ satisfies

$$\|\Psi_{\varepsilon,\rho}\|_{L^\infty(B_{s\rho}(x_0);\mathbb{R}^m)} \leqslant \frac{t}{1-t'}\frac{4}{s\rho(1-\kappa)}\left(\|u_\varepsilon-u\|_{L^\infty(B_{s\rho}(x_0);\mathbb{R}^m)} + \|u-u_{x_0}\|_{L^\infty(B_{s\rho}(x_0);\mathbb{R}^m)}\right).$$

Since $0 \in \mathrm{int}(\mathrm{dom}W)$, there exists $R > 0$ such that $W(F) < \infty$ for all $F \in \mathbb{M}$ satisfying $|F| \leqslant R$. Using Lemma 5, we can deduce

$$M_R := \sup_{|\xi| \leqslant R} W(\xi) < \infty. \tag{14}$$

By (9) there exists $\rho_R \in]0,1[$ such that for every $\rho \in]0, \rho_R[$ one has

$$\frac{t}{1-t'} \frac{4}{s(1-\kappa)} \frac{1}{\rho} \|u - u_{x_0}\|_{L^\infty(B_{s\rho}(x_0);\mathbb{R}^m)} \leqslant \frac{R}{2}.$$

Fix $\rho \in]0, \rho_R[$. Since (3) there exists $\varepsilon_\rho \in]0,1[$ such that for every $\varepsilon \in]0, \varepsilon_\rho[$

$$\frac{t}{1-t'} \frac{4}{s\rho(1-\kappa)} \left(\|u_\varepsilon - u\|_{L^\infty(B_{s\rho}(x_0);\mathbb{R}^m)} \right) \leqslant \frac{R}{2}.$$

It follows that $\|\Psi_{\varepsilon,\rho}\|_{L^\infty(B_{s\rho}(x_0);\mathbb{R}^m)} \leqslant R$ for every $\varepsilon \in]0, \varepsilon_\rho[$ and consequently by (14)

$$\overline{\lim_{\varepsilon \to 0}} \frac{1}{\mathcal{L}_N(B_{s\rho}(x_0))} \int_{U^\rho_{s,\kappa}} W(\Psi_{\varepsilon,\rho}(x))dx \leqslant \overline{\lim_{\varepsilon \to 0}} \frac{\mathcal{L}_N(U^\rho_{s,\kappa})}{\mathcal{L}_N(B_{s\rho}(x_0))} \fint_{U^\rho_{s,\kappa}} \sup_{|\xi| \leqslant R} W(\xi)dx \leqslant (1-\kappa^N) M_R.$$

Now, by letting $\rho \to 0$ and $\kappa \to 1$ we obtain

$$\overline{\lim_{\kappa \to 1}} \, \overline{\lim_{\rho \to 0}} \, \overline{\lim_{\varepsilon \to 0}} \frac{1}{\mathcal{L}_N(B_{s\rho}(x_0))} \int_{U^\rho_{s,\kappa}} W(\Psi_{\varepsilon,\rho}(x))dx = 0. \tag{15}$$

Using (A_1) and (14) we have with $C' := C + CM_R + 1$

$$\begin{aligned}
\frac{1}{\mathcal{L}_N(B_{s\rho}(x_0))} \int_{U^\rho_{s,\kappa}} W(\tau \nabla u_\varepsilon)dx &\leqslant \frac{1}{\mathcal{L}_N(B_{s\rho}(x_0))} \int_{U^\rho_{s,\kappa}} C(1 + W(\nabla u_\varepsilon) + W(0))dx \\
&\leqslant C\left((1-\kappa^N) + \frac{\mu_\varepsilon(U^\rho_{s,\kappa})}{\mathcal{L}_N(B_{s\rho}(x_0))} \right) + C(1-\kappa^N)W(0) \\
&\leqslant C'\left((1-\kappa^N) + \frac{\mu_\varepsilon(U^\rho_{s,\kappa})}{\mathcal{L}_N(B_{s\rho}(x_0))} \right)
\end{aligned}$$

and

$$\begin{aligned}
\overline{\lim_{\rho \to 0}} \, \overline{\lim_{\varepsilon \to 0}} \frac{\mu_\varepsilon(U^\rho_{s,\kappa})}{\mathcal{L}_N(B_{s\rho}(x_0))} &= \overline{\lim_{\rho \to 0}} \, \overline{\lim_{\varepsilon \to 0}} \left(\frac{\mu_\varepsilon(B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))} - \frac{\mu_\varepsilon(B_{s\kappa\rho}(x_0))}{\mathcal{L}_N(B_{s\kappa\rho}(x_0))} \kappa^N \right) \\
&\leqslant \overline{\lim_{\rho \to 0}} \, \overline{\lim_{\varepsilon \to 0}} \frac{\mu_\varepsilon(B_{s\rho}(x_0))}{\mathcal{L}_N(B_{s\rho}(x_0))} - \underline{\lim_{\rho \to 0}} \, \underline{\lim_{\varepsilon \to 0}} \frac{\mu_\varepsilon(B_{s\kappa\rho}(x_0))}{\mathcal{L}_N(B_{s\kappa\rho}(x_0))} \kappa^N \\
&= \frac{d\mu}{d\mathcal{L}_N}(x_0)(1-\kappa^N)
\end{aligned}$$

thus, we have

$$\overline{\lim_{\rho \to 0}} \, \overline{\lim_{\varepsilon \to 0}} \frac{1}{\mathcal{L}_N(B_{s\rho}(x_0))} \int_{U^\rho_{s,\kappa}} W(\tau \nabla u_\varepsilon)dx \leqslant C' \left(1-\kappa^N \right) \left(1 + \frac{d\mu}{d\mathcal{L}_N}(x_0) \right)$$

so, taking (8) into account and letting $\kappa \to 1$ we obtain

$$\overline{\lim_{\kappa \to 1}} \, \overline{\lim_{\rho \to 0}} \, \overline{\lim_{\varepsilon \to 0}} \frac{1}{\mathcal{L}_N(B_{s\rho}(x_0))} \int_{U^\rho_{s,\kappa}} W(\tau \nabla u_\varepsilon)dx = 0. \tag{16}$$

Now, passing to the limits in (13) by using (15), (16), we obtain (12). ■

3. Examples

3.1. **Example of a class of differentiable ru-usc integrands.** Let $W : \mathbb{M} \to [0, \infty]$ be a function. We consider the following conditions:

(E_1) dom$W$ is a nonempty open and convex set satisfying $0 \in \text{dom}W$;
(E_2) W is of class C^1 on dom$W$ and there exists $K > 0$ such that for every $\xi \in \text{dom}W$

$$|D_\xi W(\xi) \cdot \xi| \leqslant K(1 + W(\xi)).$$

The class of integrands satisfying (E_2) was introduced in [15] (see also [12]) to deriving the Euler-Lagrange equation for minimization problems under the two basic conditions (1) of hyperelasticity.

Proposition 2. *Assume that* (E_1)*, and* (E_2) *hold. Assume that W has G-growth, i.e.* (2) *holds with $G : \mathbb{M} \to [0, \infty]$ satisfying* (A_1)*. Then W satisfies* (A_1) *and* (A_2).

Proof of Proposition 2. By Proposition 1 (ii) the condition (A_1) holds. Let $\xi \in \text{dom}W$. Then for every $t \in]0, 1[$

$$W(t\xi) - W(\xi) = \int_1^t D_\xi W(s\xi) \cdot \xi ds = \int_1^t \frac{1}{s} D_\xi W(s\xi) \cdot s\xi ds.$$

It follows that

$$\begin{aligned} W(t\xi) - W(\xi) \leqslant \int_t^1 \frac{1}{|s|} |D_\xi W(s\xi) \cdot s\xi| \, ds &\leqslant K \int_t^1 \frac{1}{s}(1 + W(s\xi)) ds \\ &\leqslant KC(1 + W(\xi) + W(0)) \int_t^1 \frac{1}{s} ds \\ &\leqslant \ln\left(\frac{1}{t}\right) C''(1 + W(\xi)), \end{aligned}$$

where $C'' = KC(1 + W(0))$. Thus $\Delta_W^{1,0}(t) \leqslant -\ln(t)$ for all $t \in]0, 1[$. ■

Remark 1. It is easy to see that the condition (A_1) is equivalent to

$$\forall (t, \xi, \zeta) \in [0, 1] \times \mathbb{M} \times \mathbb{M} \qquad W(t\xi + (1 - t)\zeta) \leqslant C \max(W(\xi), W(\zeta))$$

for some $C > 0$. In case $C = 1$ the condition (A_1) means that W is quasiconvex in the sense of convex analysis, i.e.

(E_3) $W(t\xi + (1 - t)\zeta) \leqslant \max(W(\xi), W(\zeta))$ for all $(t, \xi, \zeta) \in [0, 1] \times \mathbb{M} \times \mathbb{M}$.

Consequently, if (E_1), (E_2) and (E_3) hold, then W satisfies (A_1) and (A_2).

3.2. **Example of unbounded ru-usc quasiconvex function.** We assume here that $m = N = 2$. In this section we show how to construct an unbounded ru-usc quasiconvex function $W : \mathbb{M} \to [0, \infty]$ which is not convex and satisfying the two basic conditions of hyperelasticity: the non-interpenetration of the matter and the non possibility to compress a finite volume of matter into zero volume

$$\forall \xi \in \text{dom}W \quad \det(I + \xi) > 0 \quad \text{and} \quad \lim_{\det(I+\xi) \to 0} W(\xi) = \infty. \tag{17}$$

Every $\xi \in \mathbb{M}$ is denoted by

$$\xi := \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}.$$

The set

$$C := \left\{ \xi \in \mathbb{M} : 1 + \xi_{11} > |\xi_{12}| \text{ and } 1 + \xi_{22} > |\xi_{21}| \right\}$$

satisfies

(C_1) $0 \in C$;
(C_2) C is convex and open;
(C_3) $\det(I + \xi) > 0$ for all $\xi \in C$;
(C_4) $\text{SO}(3) \not\subset I + C$;

(C_5) $\operatorname{tr}(\operatorname{cof}(I+\xi)^{\intercal}(I+\zeta)) > 0$ for all $\xi, \zeta \in C$ where $\operatorname{cof}(F)$ denotes the matrix of cofactors of a 2 rows and 2 columns matrix F.

Let $g : \mathbb{M} \to [0, \infty]$ be defined by

$$g(\xi) := \begin{cases} h(\det(I+\xi)) & \text{if } \xi \in C \\ \\ \infty & \text{otherwise} \end{cases}$$

where $h :]0, \infty[\to [0, \infty]$ is a function satisying:

(H_1) h is convex and nonincreasing;

(H_2) there exists $r \leqslant 1$ such that for every $\lambda \in]0,1[$ and every $x \in]0, \infty[$

$$h(\lambda x) \leqslant \frac{1}{\lambda^r} h(x); \tag{18}$$

(H_3) there exists $K > 0$ such that for every $(x, y) \in]0, \infty[^2$

$$h(\lambda x + (1-\lambda)y) \leqslant K(1 + h(x) + h(y)).$$

Note that the function h can be chosen to satisfy $\lim_{x \to 0} h(x) = \infty$ (for instance, $h(x) := \frac{1}{x}$ for all $x>0$ satisfying (18) with $r = 1$.)

Proposition 3. *We have*

(i) *$\widetilde{g}$ is polyconvex where $\widetilde{g}(\cdot) := g(\cdot - I)$;*

(ii) *there exists $\gamma > 0$ such that for every $\xi, \zeta \in C$ and every $\lambda \in]0,1[$ it holds*

$$g(\lambda\xi + (1-\lambda)\zeta) \leqslant \gamma(1 + g(\xi) + g(\zeta));$$

(iii) *g is ru-usc.*

Proof. We have (i) because we can write $\widetilde{g}(F) = \varphi(F, \det(F))$ with $\varphi : \mathbb{M} \times \mathbb{R} \to [0, \infty]$ is the convex function defined by

$$\varphi(F, s) := \begin{cases} h(s) & \text{if } F \in I + C \\ \\ \infty & \text{otherwise.} \end{cases}$$

We show (ii). Fix $\xi, \zeta \in C$ and $\lambda \in]0,1[$. We set $F := I + \xi$ and $F' := I + \zeta$. Then for every $(\xi, \zeta) \in C \times C$

$$\det(\lambda(I+\xi) + (1-\lambda)(I+\zeta)) = \lambda^2 \det F + (1-\lambda)^2 \det F' + \lambda(1-\lambda)\operatorname{tr}\left(\operatorname{cof}(F)^{\intercal}\left(F'\right)\right).$$

Since (C_5), we have

$$\det(\lambda F + (1-\lambda)F') \geqslant \lambda^2 \det F + (1-\lambda)^2 \det F'.$$

Using properties (H_1), (H_2) and (H_3) of h, we have

$$\begin{aligned} g(\lambda\xi + (1-\lambda)\zeta)) = h\left(\det(\lambda F + (1-\lambda)F')\right) &\leqslant h(\lambda^2 \det F + (1-\lambda)^2 \det F') \\ &\leqslant K\left(1 + h(\lambda \det F) + h((1-\lambda)\det F')\right) \\ &\leqslant K\left(1 + \lambda^{1-r} h(\det F) + \lambda^{1-r} h(\det F')\right) \\ &\leqslant K\left(1 + g(\xi) + g(\zeta)\right). \end{aligned} \tag{19}$$

Applying inequality (19) with $\zeta = 0$, and using properties of h, we have for every $\xi \in C$ and every $t \in]0,1[$

$$\begin{aligned} g(t\xi) = h\left(t^2 \det F + (1-t)^2\right) \leqslant h\left(t^2 \det F\right) &\leqslant \frac{1}{t^{2r}} h(\det F) \\ &= \frac{1}{t^{2r}} h(\det F) - h(\det F) + g(\xi) \\ &\leqslant \frac{1 - t^{2r}}{t^{2r}}(1 + g(\xi)) + g(\xi) \end{aligned}$$

which shows that g is ru-usc with $\Delta_g^{1,0}(t) \leqslant \frac{1-t^{2r}}{t^{2r}}$. ■

Let $W : \mathbb{M} \to [0,\infty]$ be defined by

$$W(\xi) := \begin{cases} f(\xi) + g(\xi) & \text{if } \xi \in C \\ \infty & \text{otherwise,} \end{cases} \tag{20}$$

where $f : \mathbb{M} \to [0,\infty]$ is a quasiconvex function with p-polynomial growth, i.e. there exist $c, C > 0$ such that for every $\xi \in \mathbb{M}$

$$c|\xi|^p \leqslant f(\xi) \leqslant C(1 + |\xi|^p). \tag{21}$$

The following proposition shows that such a W is consistent with the assumptions of Theorem 1 as well as with the two basic conditions of hyperelasticity.

Proposition 4. *Assume that $p > 2$. If h satisfies $\lim_{x\to 0} h(x) = \infty$ then W is a quasiconvex function satisfying* (A_1), (A_2) *and* (17).

The only point to verify is that a quasiconvex function with p-polynomial growth is ru-usc, we refer to [11, Sect. 3].

4. Auxiliary results

Proof of Proposition 1. Let us prove (i). Let $W : \mathbb{M} \to [0,\infty]$ be a convex function with $\mathrm{intdom}W \neq \varnothing$. It easy to see that (A_1) holds. Let $t \in [0,1[$ and $\xi \in \mathrm{dom}W$. Since $\mathrm{intdom}W \neq \varnothing$, take $\xi_0 \in \mathrm{intdom}W$. We have

$$W((1-t)\xi + t\xi_0) - W(\xi) \leqslant (1-t)(W(\xi_0) + W(\xi))$$

which shows that $\Delta_W^{a,0}(t) \leqslant 1 - t$ with $a = W(\xi_0)$.

Let us prove (ii). We start by proving (A_1). Consider $W : \mathbb{M} \to [0,\infty]$ having G-growth with $G : \mathbb{M} \to [0,\infty]$ satisfying (A_1), i.e. there exist $\gamma, \alpha, \beta > 0$ such that for every $\xi, \zeta \in \mathbb{M}$ and every $t \in [0,1]$

$$G(t\xi + (1-t)\zeta) \leqslant \gamma(1 + G(\xi) + G(\zeta)) \quad \text{and} \quad \alpha G(\xi) \leqslant W(\xi) \leqslant \beta(1 + G(\xi)).$$

First, it is direct to see that $\mathrm{dom}W = \mathrm{dom}G$, thus $\mathrm{int}(\mathrm{dom}W) = \mathrm{int}(\mathrm{dom}G) \neq \varnothing$. Now, fix $\xi, \zeta \in \mathbb{M}$ and $t \in [0,1]$, we have

$$\begin{aligned} W(t\xi + (1-t)\zeta) \leqslant \beta(1 + G(t\xi + (1-t)\zeta)) &\leqslant \beta(1 + \gamma(1 + G(\xi) + G(\zeta))) \\ &\leqslant \beta \max(1,\gamma)(1 + G(\xi) + G(\zeta)) \\ &\leqslant \beta \max(1,\gamma)\left(1 + \frac{1}{\alpha}W(\xi) + \frac{1}{\alpha}W(\zeta)\right) \\ &\leqslant C\,(1 + W(\xi) + W(\zeta)) \end{aligned}$$

where $C := \beta \max(1,\gamma) \max(1, \frac{1}{\alpha})$. ■

The following proposition was proved for the first time in [5] (see also [11, lemma 4.1]).

Proposition 5. *Let V be a finite dimensional vector space. Let $f : V \to [0,\infty]$ be a Borel measurable function. Assume that*

(i) *there exists $r > 0$ such that $f(v) < \infty$ whenever $|v| \leqslant r$ for all $v \in V$;*
(ii) *there exists $C > 0$ such that for every $(v,w) \in V \times V$ we have*

$$\sup_{t\in]0,1[} f(tv + (1-t)w) \leqslant C(1 + f(v) + f(w)).$$

Then

$$\sup_{|v|\leqslant r} f(v) < \infty.$$

The proof of Proposition 5 is a consequence of the following lemma:

Lemma 1. *Let $L : \mathbb{R}^d \to [0,\infty]$ be a function. Assume that*

(i) *there exists $r > 0$ such that $L(v) < \infty$ whenever $|v|_\infty \leqslant r$ for all $v \in \mathbb{R}^d$;*

(ii) *there exists a nondecreasing concave function* $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for every* $(v, w) \in \mathrm{dom}L \times \mathrm{dom}L$ *we have*

$$\sup_{t\in]0,1[} L(tv + (1-t)w) \leqslant \Phi(\max(L(v), L(w))).$$

Then

$$\sup_{|v|_\infty \leqslant r} L(v) < \infty.$$

Proof of Lemma 1. Let $\mathcal{S} := \left\{v = (v_1, \dots, v_d) \in \mathbb{R}^d : \forall k \in \{0, \dots, d\} \quad v_k \in \{-r, 0, r\}\right\}$. Since (i) we have $L^* := \sup_{v\in\mathcal{S}} L(v) < \infty$.

We define d equivalence relations on $\mathbb{R}^d$. Let $(v, w) \in \mathbb{R}^d \times \mathbb{R}^d$. For each $i \in \{1, \dots, d-1\}$, we say that $v \sim_i w$ if there exists $u = (u_1, \dots, u_k, \dots, u_d) \in \mathcal{S}$ such that

$$v - w = \sum_{k=i+1}^{d} u_k e_k$$

where $\{e_k\}_{k=1}^d$ is the standard basis of $\mathbb{R}^d$. When $i = d$, we say that $v \sim_d w$ if $v = w$. We denote by $[v]_i$ the equivalence class of v for the relation $\sim_i$ for all $i \in \{0, \dots, d\}$. Let $v \in \mathbb{R}^d$ satisfy $|v|_\infty \leqslant r$. We claim that is enough to show that

$$\forall i \in \{1, \dots, d\} \quad \forall w \in \left[\sum_{k=1}^{i} v_k e_k\right]_i \qquad L(w) \leqslant \Phi^i(L^*). \tag{22}$$

where we set for every $t \in \mathbb{R}_+$

$$\Phi^1(t) := \Phi(t) \quad \text{and} \quad \Phi^i(t) := \Phi(\Phi^{i-1}(t)) \quad \text{for all } i \geqslant 2.$$

Indeed, when $i = d$ we have $\left[\sum_{k=1}^d v_k e_k\right]_d = \{v\}$, so by using (22) we get

$$L(v) \leqslant \Phi^d(L^*),$$

and then taking the supremum over all $v \in \mathbb{R}^d$ satisfying $|v|_\infty \leqslant r$, we obtain

$$\sup_{|v|_\infty \leqslant r} L(v) \leqslant \Phi^d(L^*).$$

Let us prove (22) by induction. Let $w \in [v_1 e_1]_1$. There exists $u = (u_1, \dots, u_k, \dots, u_d) \in \mathcal{S}$ such that $w = v_1 e_1 + \sum_{k=1}^d u_k e_k$, which can be rewritten as

$$w = \frac{|v_1|}{r}\left(\mathrm{sign}(v_1) r e_1 + \sum_{k=2}^{d} u_k e_k\right) + \left(1 - \frac{|v_1|}{r}\right)\left(0 + \sum_{k=2}^{d} u_k e_k\right).$$

By (ii) and using the fact that Φ is nondecreasing, we have

$$L(w) \leqslant \Phi\left(\max\left(L\left(x, \mathrm{sign}(v_1) r e_1 + \sum_{k=2}^{d} u_k e_k\right), L\left(x, \left(\sum_{k=2}^{d} u_k e_k\right)\right)\right)\right) \leqslant \Phi(L^*).$$

So, we proved that $L(w) \leqslant \Phi(L^*)$ for all $w \in [v_1 e_1]_1$.

Let $i \in \{1, \dots, d\}$. Assume that

$$\forall j \in \{1, \dots, i\} \quad \forall w \in \left[\sum_{k=1}^{j} v_k e_k\right]_j \qquad L(w) \leqslant \Phi^j(L^*). \tag{23}$$

Let $w \in \left[\sum_{k=1}^{i+1} v_k e_k\right]_{i+1}$. We are going to show that $L(w) \leqslant \Phi^{i+1}(L^*)$. There exists $u = (u_1, \dots, u_k, \dots, u_d) \in \mathcal{S}$ such that $w = \sum_{k=1}^{i+1} v_k e_k + \sum_{k=i+2}^{N} u_k e_k$, which can be rewritten as

$$w = \frac{|v_{i+1}|}{r}\left(\sum_{k=1}^{i} v_k e_k + \mathrm{sign}(v_{i+1}) r + \sum_{k=i+2}^{d} u_k e_k\right) + \left(1 - \frac{|v_{i+1}|}{r}\right)\left(\sum_{k=1}^{i} v_k e_k + \sum_{k=i+2}^{d} u_k e_k\right).$$

By (ii), it follows that

$$L(w) \leqslant \Phi\Bigg(\max\left(L\left(x, \sum_{k=1}^{i} v_k e_k + \operatorname{sign}(v_{i+1})r + \sum_{k=i+2}^{d} u_k e_k\right), L\left(x, \sum_{k=1}^{i} v_k e_k + \sum_{k=i+2}^{d} u_k e_k\right)\right)\Bigg).$$

We see that

$$\sum_{k=1}^{i} v_k e_k + \operatorname{sign}(v_{i+1})r + \sum_{k=i+2}^{d} u_k e_k \in \left[\sum_{k=1}^{i} v_k e_k\right]_i \text{ and } \sum_{k=1}^{i} v_k e_k + \sum_{k=i+2}^{d} u_k e_k \in \left[\sum_{k=1}^{i} v_k e_k\right]_i,$$

so, by (ii) and using the fact that Φ is nondecreasing, we have by using (23)

$$L(w) \leqslant \Phi\Big(\Phi^i(L^*)\Big) = \Phi^{i+1}(L^*). \blacksquare$$

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