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Hale–Silkowski criterion for exact controllability of difference delay systems and 1-D hyperbolic PDEs

Sébastien Fueyo^{1*}

November 14, 2025

Abstract

This paper addresses the control problem for linear difference delay equations (LDDEs). It is shown that the L^1 exact controllability criterion derived in [10] can be reduced to a condition that requires verification only on a compact subset of the complex plane, thereby simplifying the analysis of controllability for LDDEs. This result can be seen as an analogue of the Hale–Silkowski criterion for exponential stability. Furthermore, a continuity property of exact controllability with respect to the delays is established. As an application, an exact controllability criterion is derived for one-dimensional hyperbolic partial differential equations.

1 Introduction

The present paper deals with the exact controllability of linear difference delay equations (LDDEs) of the form

$$x(t) = \sum_{j=1}^N A_j x(t - \Lambda_j) + Bu(t), \quad t \geq 0, \quad (1)$$

where, given three strictly positive integers d , m , and N , A_1, \dots, A_N are fixed $d \times d$ matrices with real entries, the state $x(t)$ and the control $u(t)$ belong to \mathbb{R}^d and \mathbb{R}^m respectively, and B is a fixed $d \times m$ matrix with real entries. Without loss of generality, the delays $\Lambda_1, \dots, \Lambda_N$ are positive real numbers so that $0 < \Lambda_1 < \dots < \Lambda_N$.

Beyond the fact that LDDE systems are useful in themselves and can model physical phenomena, we have to emphasize that they are also crucial for deriving interesting results in the study of transport systems modeled by one-dimensional (1-D) hyperbolic partial differential equations (PDEs). Indeed, the method of characteristics reduces the 1-D hyperbolic PDEs to a subclass of LDDEs, see [4, 6, 12, 15].

Surprisingly, most controllability results have been established for neutral differential equations (see [25, 33, 37, 44]), even though the methodology—particularly that of [44]—can be applied to LDDEs as well. The aforementioned results let the question concerning the controllability criteria for LDDEs open. Due to the form of Equation (1), both the control space and the state space can be

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chosen of the same nature, such as, for instance, L^q for some $q \in [1, +\infty)$. Since LDDEs are infinite-dimensional systems, there exist multiple notions of controllability, such as approximate, exact, and relative controllability, each capturing different aspects of control properties in this setting.

To the limit of our knowledge, the first work [29] to study the controllability of system (1) provides a characterization of relative controllability in terms of algebraic conditions. Regarding approximate and exact controllability, the first algebraic conditions were established in [13] for cases where the delays are rationally dependent. The approach involved reducing the original LDDEs to systems with a single delay and performing elementary algebraic computations on matrices. This study also provided the first nontrivial algebraic conditions for approximate and exact controllability in systems of dimension two ($N = 2$) with a single input ($m = 1$). The derived controllability criteria were based on an observability inequality; however, this method is inherently limited to LDDEs with low dimensions and single-input configurations.

Finally, approximate and exact controllability results were established in [10, 18] using algebraic techniques based on convolution algebras; see [26, 35] for foundational works on the topic. More specifically, System (1) fits within the realization theory framework developed by Y. Yamamoto [42, 43]. As an initial result, approximate controllability criteria can be directly inferred from [44]. The realization theory approach interprets the control system (1) as an input-output system, where the input $u(\cdot)$ is applied for negative times, and the output is the system's state trajectory $x(\cdot)$. It can then be shown that exact controllability is equivalent to solving a Bézout identity over an algebra of Radon measures [10], transforming the problem into an interpolation task known as the *corona problem*. The solvability of the corona problem led to an L^1 exact controllability criterion, as detailed in [18].

The frequency exact controllability criterion for LDDEs is analogous to the criterion for strong stabilization of neutral differential equations provided in [22]. It can also be seen as the counterpart of the frequency stability criterion, which characterizes the exponential stability of the origin of System (1) without control (i.e., $B \equiv 0$), as described in [21]. In the current state of the literature, stability results are more comprehensive, as they include an easier criterion for cases where the delays are rationally independent. Specifically, the stability condition simplifies to the Hale–Silkowski criterion, as detailed in the PhD dissertation of [38] and the paper [2]. Furthermore, it has been shown that a system is exponentially stable for all sets of delays if it is exponentially stable for at least one set of rationally independent delays. Regarding exact controllability criteria for cases with rationally independent delays, the topic remains largely unexplored. The objective of this paper is to address this gap in the literature by investigating the controllability of LDDEs while leveraging the rational independence structure of the delays.

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set. Regarding exact controllability criteria for cases with rationally independent delays, the topic remains largely unexplored. The objective of this paper is to address this gap in the literature by investigating the controllability of LDDEs while leveraging the rational independence structure of the delays.

The contributions of this paper are twofold. First, Theorem 4.2 establishes an exact controllability criterion that leverages the rational dependence structure of the delays, thereby simplifying the result presented in [18]. The main improvement here is that this paper reduces the L^1 controllability criterion obtained in [18] to a criterion that needs to be checked only on a compact set, thanks to the use of the rational dependence of the delays. Moreover, this work goes further by providing sufficient controllability conditions in the L^q spaces for all $q \in [1, +\infty)$. As a direct consequence of Theorem 4.2, a continuity property for exact controllability is obtained and stated in Corollary 4.7.

Although the frequency controllability criterion shares formal similarities with the Hale–Silkowski (HS) criterion for stability, their implications differ significantly. This distinction can be summarized as follows:

- The HS criterion for *stability* is remarkably powerful: it implies that if a system is exponentially stable for one set of rationally independent delays, it remains exponentially stable for all delay configurations (i.e., it is robust to variations in the delays). This criterion ultimately reduces to verifying that the spectral radius of a family of matrices (defined on a compact set) is strictly less than 1.

- In contrast, the situation for *controllability* is more delicate. At this stage, we are only able to prove that controllability at a given set of rationally independent delays implies controllability in a neighborhood of that point. Extending this result to the entire parameter space appears much more difficult, due to the nature of the rank condition involved—unlike the invertibility condition used for stability, it is less robust and more complex to analyze.

Finally, these results are applied to derive exact controllability criteria for one-dimensional hyperbolic partial differential equations, as presented in Theorem 5.4. The remaining of the paper is organized as follows. Section 2 introduces the notations used in this article. In Section 3, the definition of the exact controllability in finite time for LDDEs is given, while the end of the section states some controllability criteria which can be deduced by combining the ideas developed in the two papers [10] and [18]. The main result is stated in Section 4 and a controllability criterion, using the rational independence structure of the delays, is obtained. Finally, as an application, an exact controllability criterion is given for one dimensional hyperbolic partial differential equations in Section 5.

2 Notations

This paper adopts the following notation:

- \mathbb{N} and \mathbb{N}^* are the sets of nonnegative and positive integers, respectively.
- The set $\{1, \dots, N\}$ is denoted by $\llbracket 1, N \rrbracket$ for any $N \in \mathbb{N}^*$.
- \mathbb{Z} , \mathbb{R} , \mathbb{R}_- , \mathbb{R}_+ , \mathbb{R}_+^* and \mathbb{C} are used to denote the sets of integers, real numbers, nonpositive, non-negative, strictly positive real numbers and complex numbers respectively.
- For $s \in \mathbb{C}$, $\Re(s)$ and $\Im(s)$ denote the real and imaginary part of s , respectively.

Let $\mathcal{M}(\mathbb{R}_-)$ denote the space of Radon measures on \mathbb{R} with compact support contained in \mathbb{R}_- . By the Riesz–Markov representation theorem, distributions of order zero identify with Radon measures.

In particular, the Dirac distribution δ_t at $t \in \mathbb{R}_-$ belongs to $M(\mathbb{R}_-)$. Endowed with the convolution $*$, the space $M(\mathbb{R}_-)$ becomes a commutative unital algebra where the unit is δ_0 . For further details on the Radon measures algebras, see [10].

Let I a closed interval of \mathbb{R} and d an integer. For $q \in [1, +\infty]$, the space $L^q(I, \mathbb{R}^d)$ denotes the space of functions defined on I with values in \mathbb{R}^d for which the p -th power of their norm on \mathbb{R}^d is Lebesgue integrable. The space of the continuous maps from $[0, +\infty)$ into $L^q([0, 1], \mathbb{R}^n)$ is noted by $C^0([0, +\infty), L^q([0, 1], \mathbb{R}^n))$.

Given two positive integers N and m , elements $x \in \mathbb{C}^N$ are considered as column vectors, hence the transposition x^T of x is a row vector. For $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in \mathbb{C}^N$ and $n = (n_1, \dots, n_N) \in \mathbb{Z}^N$, we write $\Lambda \cdot n := n_1 \Lambda_1 + \dots + n_N \Lambda_N$. The set of $N \times m$ matrices with coefficients in \mathbb{C} is denoted by $\mathcal{M}_{N,m}(\mathbb{C})$. For $A \in \mathcal{M}_{N,m}(\mathbb{C})$, A^* represents its conjugate transpose matrix. We use $\|\cdot\|$ to denote a norm for every finite-dimensional space (over \mathbb{C}). The identity matrix of $\mathcal{M}_{N,N}(\mathbb{C})$ is written I_d .

Let $F(\cdot)$ be a holomorphic matrix-valued map. The closure of the image of $F(\cdot)$ in the space of complex matrices $\mathcal{M}_{N,m}(\mathbb{C})$ is denoted by $\overline{F(\mathbb{C})}$, that is,

$$\overline{F(\mathbb{C})} := \left\{ A \in \mathcal{M}_{N,m}(\mathbb{C}) \mid \exists (p_n)_{n \in \mathbb{N}} \subset \mathbb{C}, \lim_{n \rightarrow +\infty} \|F(p_n) - A\| = 0 \right\}. \quad (2)$$

Given M and B two complex matrices of size $d \times d$ and $d \times m$ respectively, with $d, m \in \mathbb{N}^*$. The $d \times (d + m)$ matrix $[M, B]$ denotes the concatenation of the two matrices M and B . Furthermore, $\text{rank } [M, B]$ represents the dimension of the range of the matrix $[M, B]$.

3 Definitions, prerequisites and first results

This work presents a complete analysis of exact controllability for System (1), expanding the results from [10, 18] through detailed derivations and new interpretations. The foundation rests on the following proposition, provable via standard inductive arguments, which establishes the well-posedness of the control problem:

Proposition 3.1. *Let $T > 0$, $u \in L^q([0, T], \mathbb{R}^m)$, and $x_0 \in L^q([-\Lambda_N, 0], \mathbb{R}^d)$. There exists a unique solution $x \in L^q([-\Lambda_N, T], \mathbb{R}^d)$ such that $x(\theta) = x_0(\theta)$ for all $\theta \in [-\Lambda_N, 0]$ and $x(\cdot)$ satisfies Equation (1) for almost all $t \in [0, T]$.*

Given the infinite-dimensional nature of LDDEs, the notion of finite-time L^q exact controllability is introduced for System (1), which serves as the central focus of this work.

Definition 3.2. System (1) is said to be L^q exactly controllable in time $T > 0$ if for every $x_0, \phi \in L^q([-\Lambda_N, 0], \mathbb{R}^d)$, there exists $u \in L^q([0, T], \mathbb{R}^m)$ such that the trajectory $x \in L^q([-\Lambda_N, T], \mathbb{R}^d)$ satisfies

$$x(T + \theta) = \phi(\theta), \quad \theta \in [-\Lambda_N, 0]. \quad (3)$$

The next necessary and sufficient criteria for the L^1 exact controllability in time $d\Lambda_N$ of System (1) are given. Since the criteria are of Hautus-type, i.e. given in the frequency domain, the matrix-valued holomorphic map $H : \mathbb{C} \rightarrow \mathcal{M}_{d,d}(\mathbb{C})$ is defined by

$$H(p) := I_d - \sum_{j=1}^N e^{-p\Lambda_j} A_j, \quad \forall p \in \mathbb{C}. \quad (4)$$

Note that, it is easy to see that the matrix-valued map $\mathbf{H} : \mathbb{C} \rightarrow \mathcal{M}_{d,d}(\mathbb{C})$, with $\mathbf{H}(p) = H^{-1}(p)$ for $p \in \mathbb{C}$, is the transfer function associated with System (1), where the input is the control $u(\cdot) \in L^2([0, +\infty), \mathbb{R}^m)$ and the output is the state trajectory $x(\cdot)$ with initial data $x_0 \equiv 0$. It is worth noting that the transfer function \mathbf{H} exists at least in a complex right half plane because linear difference delay systems are exponentially bounded, see [21].

Theorem 3.3. *System (1) is L^1 exactly controllable in time $d\Lambda_N$ if and only if $\text{rank}[A_N, B] = d$ and one of the following conditions hold true:*

(i) $\text{rank}[M, B] = d$ for every $M \in \overline{H(\mathbb{C})}$.

(ii) There exists $\alpha > 0$ such that, for every $p \in \mathbb{C}$,

$$\inf \{ \|g^T H(p)\| + \|g^T B\| \mid g \in \mathbb{C}^d, \|g\| = 1 \} \geq \alpha.$$

(iii) There exists $\alpha > 0$ such that, for every $p \in \mathbb{C}$,

$$\det(H(p)H^*(p) + BB^*) \geq \alpha.$$

Proof. By Theorem 5.2 in [18], we have that System (1) is L^1 exactly controllable in time $d\Lambda_N$ if and only if $\text{rank}[A_N, B] = d$ and $\text{rank}[M, B] = d$ for every $M \in \overline{H(\mathbb{C})}$. Thanks to [10, Proposition 3.12], Items (i), (ii) and (iii) are equivalent under the assumption $\text{rank}[A_N, B] = d$. \square

Remark 3.4. The controllability result stated in Theorem 3.3 relies on the set $\overline{H(\mathbb{C})}$. However, a precise characterization of this set remains challenging. According to Equation (2), one can only assert that $I_d \in \overline{H(\mathbb{C})}$. Nonetheless, this information is not particularly helpful for controllability purposes, since $\text{rank}[I_d, B] = d$ holds trivially.

An open question is whether the exact controllability criteria stated in Theorem 3.3 also serve as criteria for the L^q exact controllability of System (1), for all $q \in [1, +\infty)$. Currently, the following partial answer can be provided, which is a straightforward consequence of [10, Corollary 5.12] and [18, Theorem 4.2]. The proof is outlined to ensure the article remains self-contained.

Theorem 3.5. *If $\text{rank}[A_N, B] = d$ and one of the Items (i), (ii) and (iii) stated in Theorem 3.3 hold true then System (1) is L^q exactly controllable for all $q \in [1, +\infty)$.*

Proof. Since Items (i), (ii), and (iii) are equivalent under the assumption $\text{rank}[A_N, B] = d$ by [10, Proposition 3.12], it suffices to show that $\text{rank}[A_N, B] = d$ and Item (i) imply the L^q exact controllability for all $q \in [1, +\infty)$.

We begin with the case $q = 1$, for which a characterization in terms of an identity involving Radon measures can be established. Define two elements Q and P of $M(\mathbb{R}_-)$ by

$$Q := \delta_{-\Lambda_N} I_d - \sum_{j=1}^N \delta_{-\Lambda_N + \Lambda_j} A_j, \quad P := B\delta_0. \quad (5)$$

According to [10, Proposition 5.10 and Corollary 5.12], the system is L^q exactly controllable for all $q \in [1, +\infty)$ if there exist matrices R and S with entries in $M(\mathbb{R}_-)$ such that

$$Q * R + P * S = \delta_0 I_d. \quad (6)$$

Under the assumptions $\text{rank}[A_N, B] = d$ and Item (i), the existence of such R and S follows from a reduction to a one-dimensional setting using the method introduced by P. Fuhrmann [19], and the application of [18, Theorem 4.2], which provides sufficient conditions for the solvability of such identities. This completes the proof. \square

Remark 3.6. The identity (6) is referred to as a *Corona problem*. While different functional algebras can be considered, the most well-known result in this direction is Carleson’s theorem, which addresses the solvability of the Corona problem for bounded holomorphic functions in the unit disk; see [9]. In our setting, a partial answer for matrices with entries in $M(\mathbb{R}_-)$ has been provided in [18], for a certain subclass of Radon measures.

Remark 3.7. For readers interested in the relationship between the solvability of a Corona problem and controllability, let us briefly outline the conceptual link, which has its roots in classical input-output system theory. This connection, while perhaps less emphasized in recent decades, is seeing a resurgence — notably in [10], where it is made fully explicit in the context of linear difference delay equations (LDDEs).

This methodology originates in the pioneering works of Kalman and was further developed by Rouchaleau [35], Willems [41], and Kamen and Yamamoto [27, 43]. The key idea is to reinterpret the control problem — in our case, System (1) — as an input-output system, where the control u acts as an input signal and system state y serves as the output.

In this framework, the entire system is characterized by two matrix-valued Radon measures, Q and P , defined in (5), and the system’s behavior can be described via the convolution of these measures with input signals. The core insight is that exact controllability (often referred to as reachability in this context) is equivalent to the solvability of the Bézout identity (6). This identity expresses a *left-coprimeness condition* in the convolution algebra $M(\mathbb{R}_-)$, and its solvability guarantees that any desired output (i.e., state) can be reached from zero by a suitable input u , which can be explicitly constructed using the convolution with S , Q , and the target.

Such an approach allows one to reformulate the controllability problem as an algebraic one, opening the door to techniques from functional analysis and operator theory — particularly the theory of generalized inverse systems and factorization problems in measure algebras. Foundational references on this perspective include [43, Theorem 4.4], and for the specific case of LDDEs, rigorous derivations and proofs can be found in [10, Corollary 5.12] and [17, Theorem 4.1].

Remark 3.8. The controllability criteria stated in Theorem 3.3 are relatively recent, and their analytical or numerical verification remains a challenging task. The main difficulty arises from the involvement of the closure $\overline{H(\mathbb{C})}$, which requires analyzing limiting behaviors in the complex plane—potentially at infinity. Such result can be also reduced as the intersection of curves in some particular cases, but the computations of the intersection of curves is not trivial, see [10, Section 6].

In the general case, one of the primary objectives of this paper is precisely to address this issue by identifying conditions under which controllability can be assessed without relying on such asymptotic arguments, thereby enabling more tractable—albeit still nontrivial—verification criteria.

One may conjecture that, if System (1) is L^q exactly controllable for some $q \in [1, +\infty)$, then $\text{rank}[A_N, B] = d$, and one of Items (i), (ii), or (iii) stated in Theorem 3.3 holds. Unfortunately, it

seems complicated to confirm this conjecture. Section 4 presents a simplification of the controllability criteria in Theorem 3.3 and Theorem 3.5, achieved through a decomposition of the delays into rationally independent elements.

4 Hale–Silkowski criterion for exact controllability of LDDEs

Thanks to the definition of $H(\cdot)$ given in Equation (4), and under the assumption that $\text{rank}[A_N, B] = d$, it is sufficient to verify either criterion (ii) or (iii) from Theorem 3.3 within a bounded vertical strip of the complex plane in order to determine the L^1 -controllability of System (1). To outline the argument for Item (ii) (the reasoning for Item (iii) is analogous), observe that $H(p_n) \rightarrow I_d$ as $n \rightarrow +\infty$ with $\Re(p_n) \rightarrow +\infty$. Hence, there exists $\alpha > 0$ such that criterion (ii) is automatically satisfied for all $p \in \mathbb{C}$ with $\Re(p) > \alpha$. On the other hand, as $n \rightarrow +\infty$ with $\Re(p_n) \rightarrow -\infty$, the function $H(p_n)$ behaves asymptotically like $e^{-p_n \Lambda_N} A_N$. In view of the assumption $\text{rank}[A_N, B] = d$, this guarantees that criterion (ii) is also satisfied for all $p \in \mathbb{C}$ with $\Re(p) < -\alpha$, for some sufficiently large $\alpha > 0$. Therefore, it suffices to verify the criterion only for values of p such that $|\Re(p)| \leq \alpha$.

It is well known that the rational dependence between the delays can be exploited to obtain a simpler stability criterion for System (1) in the case where $B \equiv 0$. This criterion, known as the Hale–Silkowski criterion, only needs to be verified on a compact subset of the complex plane. The result is based on the PhD thesis of Silkowski [38] and is detailed in [2]. Some extensions to non-autonomous systems are available; see [12].

This section establishes an exact controllability counterpart to the Hale–Silkowski criterion (previously applied to stability). Theorems 3.3 and 3.5 are strengthened, culminating in a new equivalent condition: the positivity of a function on a compact set. As a consequence, continuity properties of exact controllability are stated when the delays are rationally independent.

The analysis requires fundamental definitions about the delay rational structure. For $\tilde{\Lambda} \in \mathbb{R}^N$, define

$$Z(\tilde{\Lambda}) := \{n \in \mathbb{Z}^N \mid \tilde{\Lambda} \cdot n = 0\},$$

encoding all integer-linear relations between delay components [12, 29]. For $\tilde{\Lambda} \in \mathbb{R}^N$, the components of $\tilde{\Lambda}$ are said to be *rationally independent* when $Z(\tilde{\Lambda})$ is reduced to the singleton $\{0\}$. Observe that, thanks to [12, Proposition 3.9], there exist a positive integer $K \leq N$, a rationally independent family (r_1, \dots, r_K) of K positive real numbers, and nonnegative integers $m_{j,k}$ for $j \in \llbracket 1, N \rrbracket$ and $k \in \llbracket 1, K \rrbracket$ such that

$$\Lambda_j = \sum_{k=1}^K m_{j,k} r_k. \quad (7)$$

Let $M = (m_{j,k})_{j \in \llbracket 1, N \rrbracket, k \in \llbracket 1, K \rrbracket}$ denote the coefficient matrix, and define $\mu := \max_{1 \leq j \leq N} \sum_{k=1}^K |m_{j,k}|$ as the maximal row sum. The sets of rational dependence relations $Z(\cdot)$ can be used to define a partial order \succeq between vectors, reflecting the fact that one is “more rationally dependent than the other”: given $\tilde{\Lambda} \in \mathbb{R}^N$, we write

$$\tilde{\Lambda} \succeq \Lambda \quad \text{if} \quad Z(\tilde{\Lambda}) \supseteq Z(\Lambda).$$

Let $V(\Lambda)$ denote the set of vectors dominating Λ with respect to this partial order, i.e.,

$$V(\Lambda) := \{\Lambda' \in \mathbb{R}^N \mid \Lambda' \succeq \Lambda\},$$

where \succeq represents the given partial ordering relation. By [12, Proposition 3.9], it follows that

$$V(\Lambda) = \text{Ran } M, \quad (8)$$

showing that $\text{Ran } M$ does not depend on the choice of the decomposition of Λ in Equation (7) and $V(\Lambda)$ is a closed subspace of \mathbb{R}^N .

In the following, the rational independence of a set of real numbers is a key assumption that enables the use of Kronecker's approximation theorem, which can be stated as follows (see, for instance, [23, Theorem 2, Chapter 2]).

Theorem 4.1 (Kronecker). *Let a_1, \dots, a_L be real numbers that are rationally independent, and let β_1, \dots, β_L be arbitrary real numbers. Given any $\varepsilon > 0$ and $Q > 0$, there exist integers n, p_1, \dots, p_L such that*

$$|na_\ell - \beta_\ell - p_\ell| < \varepsilon \quad \text{for all } \ell = 1, \dots, L,$$

with $|n| \geq Q$, and the sign of n can be chosen arbitrarily.

It turns out that the following controllability criterion can be established.

Theorem 4.2. *Under the assumption that $\text{rank } [A_N, B] = d$, Item (ii) of Theorem 3.3 is equivalent to*

(iv) *For every $p_1 \in \mathbb{R}$ and $(\theta_1, \dots, \theta_N) \in V(\Lambda) \cap ([-\mu\pi, \mu\pi])^N$, we have*

$$\det \left(R(p_1, \theta_1, \dots, \theta_N) R^*(p_1, \theta_1, \dots, \theta_N) + BB^* \right) > 0, \quad (9)$$

with $R(p_1, \theta_1, \dots, \theta_N) := I_d - \sum_{j=1}^N e^{-p_1 \Lambda_j} e^{i\theta_j} A_j$. Moreover, there exist real numbers $\underline{p} \leq \bar{p}$ such that condition (9) automatically holds for every $p_1 < \underline{p}$ and for every $p_1 > \bar{p}$.

In particular, System (1) is L^1 (respectively, L^q for any $q \in [1, +\infty)$) exactly controllable in time $d\Lambda_N$ if and only if (respectively, if) $\text{rank } [A_N, B] = d$ and condition (iv) holds true.

Remark 4.3. Since $([-\mu\pi, \mu\pi])^N$ is compact and $V(\Lambda)$ is closed, their intersection $V(\Lambda) \cap ([-\mu\pi, \mu\pi])^N$ is also compact. Consequently, Item (iv) of Theorem 4.2 yields a controllability criterion that needs to be verified only on the compact set defined by $p_1 \in [\underline{p}, \bar{p}]$ and $(\theta_1, \dots, \theta_N) \in V(\Lambda) \cap ([-\mu\pi, \mu\pi])^N$, where the real numbers $\underline{p} \leq \bar{p}$ are explicitly defined in the proof, as given in equations (17) and (15).

Before the proof of Theorem 4.2, we start by a technical lemma.

Lemma 4.4. *The two following items hold true.*

- 1) *Item (iv) of Theorem 4.2 is not satisfied if and only if there exist $\tilde{p}_1 \in \mathbb{R}$, $(\theta_1, \dots, \theta_N) \in \text{Ran } M \cap ([-\mu\pi, \mu\pi])^N$ and $g \in \mathbb{C}^d$ such that $\|g\| = 1$ and $g^T B = 0$ and $g^T R(\tilde{p}_1, \theta_1, \dots, \theta_N) = 0$.*
- 2) *Under the assumption that $\text{rank } [A_N, B] = d$, we have that Condition (ii) in Theorem 3.3 is not satisfied if and only if there exist $g \in \mathbb{C}^d$ and $(p_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ with bounded real part such that $\|g\| = 1$, $g^T B = 0$, and $\lim_{n \rightarrow \infty} g^T H(p_n) = 0$.*

Proof. Let us prove Item 1). We recall that $V(\Lambda) = \text{Ran } M$.

If Item (iv) of Theorem 4.2 is not satisfied, then, since the matrix $RR^* + BB^*$ is Hermitian and positive semi-definite, it implies the existence of $\tilde{p}_1 \in \mathbb{R}$ and $(\theta_1, \dots, \theta_N) \in \text{Ran } M \cap ([-\mu\pi, \mu\pi])^N$ such that

$$\det(R(\tilde{p}_1, \theta_1, \dots, \theta_N)R^*(\tilde{p}_1, \theta_1, \dots, \theta_N) + BB^*) = 0. \quad (10)$$

We deduce from Equation (10), the existence of some $g \in \mathbb{C}^d$ such that $\|g\| = 1$ and

$$g^T R(\tilde{p}_1, \theta_1, \dots, \theta_N)R^*(\tilde{p}_1, \theta_1, \dots, \theta_N) (g^T)^* + g^T BB^* (g^T)^* = 0. \quad (11)$$

Equation (11) yields $g^T B = 0$ and $g^T R(\tilde{p}_1, \theta_1, \dots, \theta_N) = 0$, achieving to prove one of the implication. Since the proof of the converse is trivial, we reach the result of Item 1).

Let us prove Item 2). If Condition (ii) in Theorem 3.3 is not satisfied, there exist a sequence $(p_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ and a sequence of vectors $(g_n)_{n \in \mathbb{N}} \in (\mathbb{C}^d)^{\mathbb{N}}$ such that $\|g_n\| = 1$ for all $n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} g_n^T B = 0$, and $\lim_{n \rightarrow \infty} g_n^T H(p_n) = 0$. We remark that $H(p)$ behaves like I_d and $-e^{-p\Lambda_N} A_N$ when the real part of p tends to $+\infty$ and $-\infty$ respectively. Then the assumption $\text{rank } [A_N, B] = d$ implies that the real part of $(p_n)_{n \in \mathbb{N}}$ is bounded, and thus $(H(p_n))_{n \in \mathbb{N}}$ is also bounded. Without loss of generality, $\lim_{n \rightarrow \infty} g_n = g$ for some vector $g \in \mathbb{C}^d$ and we deduce that $\|g\| = 1$, $g^T B = 0$, and $\lim_{n \rightarrow \infty} g^T H(p_n) = 0$, which proves one of the two implications. The proof of the converse is obvious. \square

We are ready to prove the main result of the paper.

Proof Theorem 4.2. We begin by proving that the first part of Item (iv) is equivalent to Item (ii) of Theorem 3.3, namely that for every $p_1 \in \mathbb{R}$ and every $(\theta_1, \dots, \theta_N) \in V(\Lambda) \cap ([-\mu\pi, \mu\pi])^N$, the relation (9) holds.

By Equation (7), we have

$$H(p) = I_d - \sum_{j=1}^N A_j e^{-\Re(p)\Lambda_j} \prod_{k=1}^K (e^{-i\Im(p)r_k})^{m_{j,k}}, \quad p \in \mathbb{C}. \quad (12)$$

Let us prove by contraposition that the first part of (iv) defined above implies (ii) following the argument in [2, Theorem 3.1]. By Item 2) of Lemma 4.4, there exist $g \in \mathbb{C}^d$ and $(p_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ with bounded real part such that $\|g\| = 1$, $g^T B = 0$, and

$$\lim_{n \rightarrow \infty} g^T H(p_n) = 0. \quad (13)$$

We can extract a subsequence (still denoted $(p_n)_{n \in \mathbb{N}}$) such that there exists $\tilde{p}_1 \in \mathbb{R}$ so that $\Re(p_n) \xrightarrow{n \rightarrow +\infty} \tilde{p}_1$ and, for every $k \in \llbracket 1, K \rrbracket$, there exists $\tilde{\theta}_k \in \mathbb{R}$ such that $e^{-i\Im(p_n)r_k} \xrightarrow{n \rightarrow +\infty} e^{i\tilde{\theta}_k}$. Without loss of generality, we can assume that $\tilde{\theta}_k \in \mathbb{R}$ belongs to $[-\pi, \pi]$ for all $k \in \llbracket 1, K \rrbracket$. Thus, by Equations (12) and (13) we have

$$g^T R(\tilde{p}_1, \theta_1, \dots, \theta_N) = 0,$$

where $\theta_j := \sum_{k=1}^K \tilde{\theta}_k m_{j,k}$ for $j \in \llbracket 1, N \rrbracket$. We remark that $(\theta_1, \dots, \theta_N) \in V(\Lambda) \cap ([-\mu\pi, \mu\pi])^N$, where the last equality follows from Equation (8) and the definition of μ . This completes the proof that the first part of (iv) implies (ii) with the use of Item 1) of Lemma 4.4.

Conversely, let us prove by contraposition that (ii) implies the first part of (iv). Thanks to Item 1) of Lemma 4.4, there exist $\tilde{p}_1 \in \mathbb{R}$, $(\theta_1, \dots, \theta_N) \in \text{Ran } M \cap ([-\mu\pi, \mu\pi])^N$ and $g \in \mathbb{C}^d$ such that $\|g\| = 1$ and

$$g^T B = 0 \text{ and } g^T R(\tilde{p}_1, \theta_1, \dots, \theta_N) = 0.$$

Let $\tilde{\theta}_1, \dots, \tilde{\theta}_K$ be such that $\theta_j = \sum_{k=1}^K \tilde{\theta}_k m_{j,k}$ for $j \in \llbracket 1, N \rrbracket$. By applying the Kronecker approximation theorem (see Theorem 4.1) to the rationally independent set

$$a_i = -\frac{r_i}{2\pi}, \quad \beta_i = -\frac{\tilde{\theta}_i}{2\pi}, \quad \text{for } i = 1, \dots, K,$$

we deduce the existence of a sequence $(\alpha_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that

$$e^{i(\tilde{\theta}_k - \alpha_n r_k)} \longrightarrow 1 \quad \text{as } n \rightarrow +\infty, \quad \text{for all } k = 1, \dots, K.$$

We deduce by using Equation (12) that

$$g^T H(\tilde{p}_1 + i\alpha^n) \xrightarrow{n \rightarrow +\infty} g^T R(\tilde{p}_1, \theta_1, \dots, \theta_N) = 0.$$

Thus, (ii) implies the first part of (iv) by Item 2) of Lemma 4.4.

We now prove the second part of Item (iv), namely that there exist real numbers $\underline{p} \leq \bar{p}$ such that condition (9) automatically holds for every $p_1 < \underline{p}$ and every $p_1 \geq \bar{p}$. First notice that for every p_1 nonnegative real number and $(\theta_1, \dots, \theta_N) \in ([-2\pi, 2\pi])^N$, we have

$$RR^* \geq \left(1 - 2e^{-p_1 \Lambda_1} \sum_{j=1}^N \|A_j\|_f \right) I_d, \quad (14)$$

where $R := R(p_1, \theta_1, \dots, \theta_N) = I_d - \sum_{j=1}^N e^{-p_1 \Lambda_j} e^{i\theta_j} A_j$, the order is that of Hermitian matrices, and

$\|\cdot\|_f$ denotes the Frobenius norm. To establish the inequality in (14), we used the fact that the spectral radius of $(A + A^*)/2$ is always bounded by $\|A\|_f$ for any complex matrix A . Setting

$$\bar{p} = \max \left(0, \frac{\ln \left(4 \sum_{j=1}^N \|A_j\|_f \right)}{\Lambda_1} \right), \quad (15)$$

one immediately deduces that $RR^* \geq \frac{1}{2} I_d$ as soon as $p_1 > \bar{p}$ and hence (9) is satisfied for $p_1 > \bar{p}$.

To get an estimate of \underline{p} , first recall that the rank condition on $[A_N, B]$ is equivalent to the fact that the real-symmetric matrix $A_N A_N^T + B B^T$ is positive definite and consider λ_1 its smallest eigenvalue, which is positive. We also have that for every nonpositive real number p_1 and $(\theta_1, \dots, \theta_N) \in [-2\pi, 2\pi]^N$,

$$\begin{aligned} RR^* &\geq e^{-2p_1 \Lambda_N} \left(A_N A_N^T - e^{p_1 \Lambda_N} (A_N + A_N^T) + \sum_{j=2}^N e^{p_1 (\Lambda_N - \Lambda_j)} (e^{-i\theta_j} A_N A_j^T + e^{i\theta_j} A_j A_N^T) \right) \\ &\geq e^{-2p_1 \Lambda_N} \left(A_N A_N^T - 2e^{p_1 (\Lambda_N - \Lambda_{N-1})} \|A_N\|_f (1 + \sum_{j=2}^N \|A_j\|_f) I_d \right). \end{aligned} \quad (16)$$

Set

$$\underline{p} = \min \left(0, \frac{1}{\Lambda_N - \Lambda_{N-1}} \ln \left(\frac{\lambda_1}{8 \|A_N\|_f (1 + \sum_{j=2}^N \|A_j\|_f)} \right) \right). \quad (17)$$

Let $z \in \mathbb{C}^N$, $\|z\| = 1$ and $p_1 < \underline{p}$. Assume first that $z^* A_N A_N^T z \geq \frac{\lambda_1}{2}$. Then, by using (16), one gets that $z^* R R^* z \geq e^{-2p_1 \Lambda_N} \frac{\lambda_1}{4}$. On the other hand, if $z^* A_N A_N^T z < \frac{\lambda_1}{2}$, then

$$z^* B B^T z = z^* (A_N A_N^T + B B^T) z - z^* A_N A_N^T z > \frac{\lambda_1}{2}.$$

We thus have proved that for every $z \in \mathbb{C}^N$ and $\|z\| = 1$, one has that $z^* (R R^* + B B^T) z \geq \frac{\lambda_1}{4}$, i.e., $R R^* + B B^T$ is a positive definite matrix as soon as $p_1 < \underline{p}$.

The last part of the theorem is a straightforward application of Theorem 3.3 and Theorem 3.5. \square

Remark 4.5. Condition (iv) is reminiscent of the Hale–Silkowsi criterion for strong stability, see for instance [5, 12, 21, 30].

Remark 4.6. In the single-delay case $N = 1$, Condition (iv) of Theorem 4.2 is nothing else but Condition (iii) of Theorem 3.3.

As an immediate consequence of Theorem 4.2, the exact controllability property exhibits openness in an appropriate topological sense when delays are rationally independent. In another words, it is showed that, if System (1) is L^1 exactly controllable in time $d\Lambda_N$ for a set of rationally independent delays Λ then System (1) is L^1 exactly controllable in time $d\tilde{\Lambda}_N$ for all set of rationally independent delays $\tilde{\Lambda} = (\tilde{\Lambda}_1, \dots, \tilde{\Lambda}_N) \in (\mathbb{R}_+^*)^N$ contained in an open neighborhood of Λ .

Corollary 4.7. *Assume that System (1) is L^1 exactly controllable in time $d\Lambda_N$ and the delays are rationally independent. Then there exists $\epsilon > 0$ such that, for every $\tilde{\Lambda} \in (\mathbb{R}_+^*)^N$ satisfying $\|\tilde{\Lambda} - \Lambda\| < \epsilon$, the system obtained from (1) by replacing the vector of delays Λ by $\tilde{\Lambda}$ is L^1 exactly controllable in time $d\tilde{\Lambda}_N$.*

Proof. When the components of $\Lambda \in (\mathbb{R}_+^*)^N$ are rationally independent, we have that $V(\Lambda) \cap [-\mu\pi, \mu\pi]^N = [-\pi, \pi]^N$.

Let $K \subset \mathbb{R}$ be a compact set to be specified later. For every $p_1 \in K$ and every $(\theta_1, \dots, \theta_N) \in [-\pi, \pi]^N$, Equation (9) holds. In particular, by continuity of the determinant, there exists $\epsilon > 0$ such that, for any $\tilde{\Lambda} \in (\mathbb{R}_+^*)^N$ satisfying $\|\tilde{\Lambda} - \Lambda\| < \epsilon$, we have, for all $p_1 \in K$ and $(\theta_1, \dots, \theta_N) \in [-\pi, \pi]^N$,

$$\det (\tilde{R}(p_1, \theta_1, \dots, \theta_N) \tilde{R}^*(p_1, \theta_1, \dots, \theta_N) + B B^*) > 0, \quad (18)$$

where

$$\tilde{R}(p_1, \theta_1, \dots, \theta_N) := I_d - \sum_{j=1}^N e^{-p_1 \tilde{\Lambda}_j} e^{i\theta_j} A_j.$$

Moreover, as shown in the proof of Theorem 4.2, it is possible to choose the compact set K independently of $\tilde{\Lambda}$ so that (18) also holds for all $p_1 \notin K$. Therefore, by applying Theorem 3.3, we conclude that the system obtained from (1) by replacing the delay vector Λ with $\tilde{\Lambda}$ is exactly controllable in L^1 in time $d\tilde{\Lambda}_N$. \square

5 Application to the exact controllability of 1-D hyperbolic PDEs

We now illustrate the applicability of the algebraic controllability criterion developed in Section 4 to a class of boundary-controlled partial differential equations, namely one-dimensional linear hyperbolic systems. These systems naturally arise in the modeling of various physical and engineering processes, including gas pipelines [20], information transport over networks [8], or thermal flows in distributed infrastructures [32]. Their properties—such as stability [6, 7], stabilizability [1, 28], and controllability [15]—have been the subject of extensive research in recent years. In this section, we rely on the equivalence established in [11, Section 3.1] between one-dimensional hyperbolic PDEs and linear time-delay systems. This equivalence allows us to apply the algebraic tools developed in previous sections to derive a controllability criterion. We start by recalling the general form and modeling significance of these hyperbolic PDEs, as well as a brief overview of related works.

5.1 Context and Related Works

In this paper, the linear one-dimensional hyperbolic system under consideration are of the form

$$\begin{cases} \partial_t R(t, x) + \Lambda(x)\partial_x R(t, x) + D(x)R(t, x) = 0, \\ \begin{pmatrix} R^+(t, 0) \\ R^-(t, 1) \end{pmatrix} = M \begin{pmatrix} R^+(t, 1) \\ R^-(t, 0) \end{pmatrix} + Bu(t), \end{cases} \quad (19)$$

where

- t and x belong to $[0, +\infty)$ and $(0, 1)$ respectively;
- $\Lambda(x)$ is a diagonal matrix denoted $\text{diag}\{\lambda_1(x), \dots, \lambda_n(x)\}$. It is assumed that there exists $\tilde{n} \in \{0, \dots, n\}$ such that $\lambda_i(x) > 0 > \lambda_j(x)$ for every $i \in \{1, \dots, \tilde{n}\}$, $j \in \{\tilde{n}+1, \dots, n\}$, $x \in [0, 1]$. Furthermore, one has:

$$\lambda_i, \frac{1}{\lambda_i} \in L^\infty((0, 1), \mathbb{R}), \text{ for every } i \in \{1, \dots, n\};$$

- the components of the solution R are split into those corresponding to positive and negative velocities, i.e.,

$$R = \begin{pmatrix} R^+ \\ R^- \end{pmatrix} \quad \text{with} \quad \begin{cases} R^+ = (R_1, \dots, R_{\tilde{n}})^T, \\ R^- = (R_{\tilde{n}+1}, \dots, R_n)^T; \end{cases}$$

- $D(x)$ is a diagonal matrix denoted by $\text{diag}\{d_1(x), \dots, d_n(x)\}$ where $d_i \in L^1((0, 1), \mathbb{R})$ for every $i \in \{1, \dots, n\}$;
- $u: \mathbb{R}_+ \rightarrow \mathbb{R}^m$ denotes the control law;
- B and M are $n \times m$ and $n \times n$ real matrices respectively.

The study of controllability properties for PDEs, particularly for hyperbolic systems, has a long and rich history. One of the pioneering contributions is due to Russell [36], who introduced the use of observability inequalities to derive controllability results. System (19) falls into the class of

boundary control systems, since the control acts through the boundary. For a general introduction to this framework, we refer the reader to the textbooks [14, 39].

The method of observability inequalities has been extensively developed for systems of the form (19), particularly when the system matrices exhibit specific structures; see, for instance, [31, 34, 40].

A different line of research is the backstepping method, which has led to powerful tools for the stabilization and control of hyperbolic systems such as (19); see, e.g., [1, 15, 16, 24, 45]. This framework underlies a wide range of applications, including traffic flow control [45].

However, both observability-based methods and backstepping, in their current forms, generally rely on restrictive assumptions on the structure of the matrices M and B . They may often accommodate more general transport or diffusion terms (e.g., $D(x)$ not necessarily diagonal), but they are typically not designed to address arbitrary matrices M and B . Yet such general configurations arise naturally in certain engineering applications—for example, in electronic circuits involving high-frequency amplifiers and transmission lines, which are modeled by Telegrapher’s equations; see [3].

To address such cases, a more recent and complementary approach has emerged: the algebraic method. This framework focuses on general matrices M and B , but typically assumes $D(x)$ to be diagonal. This line of work builds on the equivalence between hyperbolic PDEs and time-delay systems from [11, Section 3.1], enabling the use of tools from delay systems theory to study boundary controllability. It was revived and formalized in [11], relying on earlier algebraic methods developed by Yamamoto [44].

In the present work, we adopt this latter approach. We assume that $D(x)$ is diagonal, as previously stated, to remain consistent with the framework of [11]. Nevertheless, we emphasize that it is well known that such systems can be transformed—via an appropriate change of variables—into an equivalent form where $D(x) \equiv 0$.

5.2 Controllability results

The aim of this section is to give a L^q controllability criterion for hyperbolic systems given in Section 4 by using of time delay systems.

We begin by addressing the well-posedness of the control system (19). More specifically, we first provide a definition of a solution to System (19), which is directly inspired by the method of characteristics and does not require any a priori regularity assumptions on the solution.

The definition adopted here is taken from [11, Definition 3.1] and builds upon the framework introduced in [12, Definition 4.1]. A related concept, known as a broad solution, was later introduced in [15, Definition 3.1] for a more general class of hyperbolic systems extending beyond the particular form of (19).

Definition 5.1. Let $T > 0$, $u : [0, T] \rightarrow \mathbb{R}^m$, and $\bar{R} : [0, 1] \rightarrow \mathbb{R}^n$. A function $R : [0, T] \times [0, 1] \rightarrow \mathbb{R}^n$ is called a *solution* of system (19) on the interval $[0, T]$ with initial condition \bar{R} and control u , if the following conditions hold:

1. **Initial condition:** $R(0, x) = \bar{R}(x)$ for every $x \in [0, 1]$;
2. **Boundary condition:** the second Equation (19) is satisfied for every $t \geq 0$;

3. **Characteristic propagation:** for every $i \in \llbracket 1, n \rrbracket$, every $t \in [0, T]$, and every $x \in [0, 1]$, the identity

$$R_i \left(t + \int_x^{x+h} \frac{d\xi}{\lambda_i(\xi)}, x + h \right) = e^{-\int_x^{x+h} \frac{d_i(\xi)}{\lambda_i(\xi)} d\xi} R_i(t, x) \quad (20)$$

holds for every $h \in \mathbb{R}$ such that $x + h \in [0, 1]$ and $t + \int_x^{x+h} \frac{d\xi}{\lambda_i(\xi)} \in [0, T]$.

Note that, in Definition 5.1, we consider as solutions functions defined at every point and with no regularity assumption (not even measurability), so we require them to be defined at all points and we do not identify functions equal almost everywhere. To give a sense to the solution in L^p , we make the usual abuse of notation of considering elements of L^q as functions instead of classes of equivalence of functions with respect to almost everywhere equality. For more details, the reader is referred to [11, Section 3.1]. From [11, Proposition 3.6], we obtain the following existence and uniqueness result for solutions.

Proposition 5.2. *Let $q \in [1, +\infty]$, $T > 0$, $\bar{R} \in L^q((0, 1), \mathbb{R}^n)$, and $u \in L^q((0, T), \mathbb{R}^m)$. Then system (19) admits a unique solution $R : [0, T] \times [0, 1] \rightarrow \mathbb{R}^n$, with initial condition \bar{R} and control u , such that for every $t \in [0, T]$, we have $R(t, \cdot) \in L^q((0, 1), \mathbb{R}^n)$.*

We are now in a position to introduce the notion of exact controllability in the L^q -framework.

Definition 5.3. Let $T > 0$ and $q \in [1, +\infty]$. System (19) is said to be L^q -exactly controllable in time T if, for every $\phi, \psi \in L^q([0, 1], \mathbb{R}^n)$, there exists a control $u \in L^q([0, T], \mathbb{R}^m)$ such that the solution R of (19) with initial condition ϕ and control u satisfies $R(T, \cdot) = \psi$.

Before establishing the controllability criteria, we introduce some preliminary definitions:

– The vector $T = (\tau_1, \dots, \tau_n)$ consists of positive real numbers defined by

$$\tau_i = \int_0^1 \frac{dx}{|\lambda_i(x)|}, \quad i \in \llbracket 1, n \rrbracket; \quad (21)$$

– The matrix K is given by

$$K = M \cdot \text{diag} \left(e^{-\int_0^1 \frac{d_1(x)}{|\lambda_1(x)|} dx}, \dots, e^{-\int_0^1 \frac{d_n(x)}{|\lambda_n(x)|} dx} \right).$$

The components of the vector $T = (\tau_1, \dots, \tau_n)$ are called the *delays of the hyperbolic system* (19). Similarly as in Section 4, there exist a positive integer $\tilde{K} \leq n$, a rationally independent family $(\tilde{r}_1, \dots, \tilde{r}_{\tilde{K}})$ of \tilde{K} positive real numbers, and nonnegative integers $\tilde{m}_{j,k}$ for $j \in \llbracket 1, n \rrbracket$ and $k \in \llbracket 1, \tilde{K} \rrbracket$ such that

$$\tau_j = \sum_{k=1}^{\tilde{K}} \tilde{m}_{j,k} \tilde{r}_k \quad (22)$$

and $\mu_{\text{hyp}} = \max_j \sum_{k=1}^{\tilde{K}} |\tilde{m}_{j,k}|$. The following theorem provides exact controllability criterion of System (19).

Theorem 5.4. *The hyperbolic system (19) is L^1 -exactly (respectively, L^q for every $q \in [0, +\infty)$) controllable in time $\tau_1 + \dots + \tau_n$ if and only if (respectively, if) $\text{rank}[K, B] = n$ and the following condition holds:*

For every $p_1 \in [\underline{p}, \bar{p}]$ and $(\theta_1, \dots, \theta_n) \in V(\mathbb{T}) \cap ([-\mu_{\text{hyp}}\pi, \mu_{\text{hyp}}\pi])^n$, we have

$$\det \left(R_{\text{hyp}}(p_1, \theta_1, \dots, \theta_n) R_{\text{hyp}}^*(p_1, \theta_1, \dots, \theta_n) + BB^* \right) > 0, \quad (23)$$

where the matrix $R_{\text{hyp}}(p_1, \theta_1, \dots, \theta_n)$ satisfies

$$R_{\text{hyp}}(p_1, \theta_1, \dots, \theta_n) := I_d - K \text{diag}\{e^{-p_1\tau_1}e^{i\theta_1}, \dots, e^{-p_1\tau_n}e^{i\theta_n}\}.$$

Moreover, there exist real numbers $\underline{p} \leq \bar{p}$ such that condition (23) automatically holds for every $p_1 < \underline{p}$ and for every $p_1 > \bar{p}$.

Remark 5.5. The controllability condition outlined in Theorem 5.4 must be validated within the compact domain defined by $p_1 \in [\underline{p}, \bar{p}]$ and $(\theta_1, \dots, \theta_n) \in V(\mathbb{T}) \cap ([-\mu_{\text{hyp}}\pi, \mu_{\text{hyp}}\pi])^n$. The bounds \underline{p} and \bar{p} are real numbers explicitly derived in the proof of the theorem, as detailed in equations (26) and (30).

Proof. Let us consider the class of linear delay differential equations (LDDEs) of the form

$$\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = K \begin{pmatrix} y_1(t - \tau_1) \\ \vdots \\ y_n(t - \tau_n) \end{pmatrix} + Bu(t), \quad t \geq 0. \quad (24)$$

System (24) constitutes a subclass of the more general delay systems described in (1), but evolves on the smaller state space $\prod_{i=1}^n L^1([- \tau_i, 0], \mathbb{R})$, as opposed to the larger space $L^1([- \tau_n, 0], \mathbb{R}^n)$. Moreover, as established in [11, Proposition 3.7], there exists a bijective correspondence between the solutions of system (24) and those of the hyperbolic system (19). Consequently, with minor adaptations, the proof of Theorem 4.2 extends directly to establish the desired result.

In particular, for hyperbolic systems, we are able to obtain finer estimates for $\underline{p} \leq \bar{p}$. Let $D = \text{diag}(e^{-p_1\tau_1}e^{i\theta_1}, \dots, e^{-p_1\tau_n}e^{i\theta_n})$. We get, for every p_1 nonnegative real number and $(\theta_1, \dots, \theta_n) \in ([-2\pi, 2\pi])^n$, that

$$\begin{aligned} R_{\text{hyp}} R_{\text{hyp}}^* &\geq (1 - 2\|KD\|_f) I_d, \\ &\geq (1 - 2\|K\|_f \|D\|_f) I_d, \\ &\geq \left(1 - 2e^{-p_1\tau_1} \sqrt{n} \|K\|_f\right) I_d, \end{aligned} \quad (25)$$

thus \bar{p} can be chosen as

$$\bar{p} = \max \left(0, \frac{\ln \left(4\sqrt{n} \|K\|_f \right)}{\tau_1} \right). \quad (26)$$

It follows from (25) that $R_{\text{hyp}} R_{\text{hyp}}^* \geq \frac{1}{2} I_d$ whenever $p_1 > \bar{p}$.

Let us determine \underline{p} . We have

$$R_{\text{hyp}} R_{\text{hyp}}^* = I_d - KD - D^* K^* + KDD^* K^*. \quad (27)$$

Consider a unit vector $x \in \mathbb{C}^n$ and we use $\|\cdot\|_2$ as the euclidean norm on \mathbb{C}^n . We deduce from (27) that

$$\begin{aligned} x^* R_{\text{hyp}} R_{\text{hyp}}^* x &\geq 1 - 2\|D^* K^* x\|_2 + \|D^* K^* x\|_2^2, \\ &= (\|D^* K^* x\|_2 - 1)^2, \\ &= (\|K^* x\|_2 \|D^* v\|_2 - 1)^2, \end{aligned} \quad (28)$$

with $v := K^* x / \|K^* x\|_2$ so that v is a unit vector. We get, for every p_1 nonpositive real number and $(\theta_1, \dots, \theta_N) \in ([-2\pi, 2\pi])^N$, that

$$\|D^* v\|_2^2 \geq e^{-2p_1 \tau_1}. \quad (29)$$

The rank condition on $[K, B]$ guarantees that the matrix $K^T K + B^T B$ is real symmetric and positive definite. Denote by λ_1 its smallest, and necessarily positive, eigenvalue. Set

$$\underline{p} = \min \left(0, -\frac{1}{\tau_1} \ln \left(\frac{\sqrt{2}}{\sqrt{\lambda_1}} \right) \right) \quad (30)$$

and assume that $p_1 < \underline{p}$. If $\|B^* x\|_2^2 \geq \lambda_1/2$, we have the result. If $\|B^* x\|_2^2 < \lambda_1/2$, we have $\|K^* x\|_2^2 \geq \lambda_1/2$ so that

$$\|K^* x\|_2 \|D^* v\|_2 \geq e^{-p_1 \tau_1} \frac{\sqrt{\lambda_1}}{\sqrt{2}}. \quad (31)$$

From (28), (29) and (31), we get

$$x^* R_{\text{hyp}} R_{\text{hyp}}^* x > 0, \quad (32)$$

thus, in all cases, $R_{\text{hyp}} R_{\text{hyp}}^* + BB^T$ is a positive definite matrix as soon as $p_1 < \underline{p}$ achieving the obtention of the estimates \underline{p} and \bar{p} . \square

We now establish a continuity property of the exact controllability criterion with respect to the characteristic speeds. Specifically, we show that small perturbations of the velocity profiles λ_i , in the L^∞ -norm, preserve the exact controllability property—provided the original system is controllable and the delays are rationally independent. This result follows by adapting the arguments developed in the proof of Corollary 4.7.

Corollary 5.6. *Assume that System (19) is L^1 exactly controllable in time $\tau_1 + \dots + \tau_n$, and that the delays τ_1, \dots, τ_n are rationally independent.*

Then there exists $\epsilon > 0$ such that, for every $i \in \{1, \dots, n\}$ and every $\tilde{\lambda}_i \in L^\infty((0, 1), \mathbb{R})$ satisfying

$$\|\tilde{\lambda}_i - \lambda_i\|_{L^\infty((0,1))} < \epsilon,$$

the system obtained from (19) by replacing each λ_i with $\tilde{\lambda}_i$ is L^1 exactly controllable in time $\tilde{\tau}_1 + \dots + \tilde{\tau}_n$.

6 Conclusion

This paper presents a simplified exact controllability criterion for linear difference delay equations (LDDEs), by explicitly exploiting the rational dependence structure of the delays. While the general L^1 -based controllability criterion derived in [18] relies on solving a Bézout identity over a noncompact frequency domain, our approach reduces the verification to a compact subset of the complex plane. This simplification is made possible by the arithmetic properties of the delays and the use of Kronecker-type approximation results.

In addition, we establish a continuity property for exact controllability with respect to the delays. Specifically, we show that if a system is exactly controllable for a given set of rationally independent delays, then this property holds for all delays in a neighborhood of that set. This yields a new structural understanding of the robustness of exact controllability under small perturbations of the delay vector.

As an application, the developed framework is extended to 1-D hyperbolic partial differential equations via the method of characteristics. The resulting criterion inherits the advantages of the LDDE case: it is explicit and verified on a compact set. This opens up new perspectives for both theoretical analysis and control design in distributed parameter systems governed by transport dynamics.

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