Optimal Transport Model of Optical Flow Estimation: Constant and Varying Illumination Cases

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ABSTRACT

In this article, we consider a new modeling of optical flow estimation with constant or varying illumination based on mass optimal transport theory. While PDE classical approaches in motion analysis assume pixel-wise conservation of the intensity map along the flow of characteristics driven by the flow vector, the new formulation assumes a conservation of corresponding densities and yields a conservative transport equation encoding also the varying illumination. Namely, such variations are expressed as those of small volumes (of pixels) expressing compressibility constraint. When no variations of the illumination occur, we obtain an optimal transport generalized incompressible flow model more "rich" than classical ones. The model in this optimal transport approach offer a complete similarity to standard L^2 Monge-Kantorovich problem on mass transportation and generalized incompressible flows. We use the Benamou-Brenier algorithm for the optical flow computation. We present some numerical results and make some comparisons with more classical approaches to show the reliability of the optimal transport model, its perspectives and its limitations.

***K*eywords** optical flow estimation · illumination variations · PDEs · mass optimal transport · generalized incompressible flow · Benamou-Brenier formula

Introduction

The *optical flow* estimation has became a central problem in computer vision fields in the four last decades [3, 8, 17, 9]. It consists of determining a vector field, the optical flow, that governs the motion in a sequence of successive images. Several methods where considered and implemented in this field among which PDE ones appear as a reliable way to determine the vector and to give satisfactory answers to some related issues (see [31] and references therein). Number of applications, as video compression, movies restoration or autonomous driving technology, to name but few, show the importance of optical flow estimation and emphasize that it is still an active area of research to improve the models, the computations, and to include a variety of constraints such as varying illumination in a scene. In classical PDE approaches, the estimation is based on a pixel-wise constancy of the light intensity along flow characteristics which gives a rise to a non-conservative transport partial differential equation where the vector of the transport is precisely the optical flow. The models here turn out to be inverse problems and as such ill-posed. There is a huge literature related to various ways of solving this optical flow problem and a large number of numerical methods [21, 18, 31, 30, 25, 16]. The ill-posedness is usually treated within regularization procedures which enforce stable determination and yield dense optical flow. As it is well known, the regularization allows to ensure existence, uniqueness and stability, but at the same time, it is based on implicit assumptions (a priori) on the geometry of solutions and is highly dependent on the choice of regularization parameters. Regarding the varying illumination which induces "virtual motion", some PDE approaches were considered such as adding a constraint on the constancy of the norm of the gradient of the image intensity, assumed to be less sensitive to such variations, which leads to nonlinear system of PDE [8]. Sticking to the linear model of Horn and Schunck, Gennert and Negahdaripour proposed to relax the constancy assumption on the intensity, by adding linear variations of the illumination to the optical flow equation [14]. This approach, which might be well suited to some changes of illumination in certain scene, has the advantage of keeping linear the mathematical model, leading to simple

implementation and reasonable cost for solving the PDE while the variation of the illumination is encoded as a new component of the optical flow vector [15]. However, it suffers some limitations such as a "physical" interpretation and a rigorous justification, which should have a scene-independent meaning. Moreover, adding a new unknown in already ill-posed problem, without new "observation" makes the method strongly dependent on the regularization strategy, provides a dense variations of illumination, and has the same shortcomings inherited from the classical approach.

In this article, we introduce a new optimal transport based model for this optical flow estimation with varying illumination. Contrary to the classical ones which assume a transport of particles (pixels) with constant intensities a long trajectories (characteristics of the flow), we rather consider a transport of density map. The constancy assumption transforms into a "mass conservation" principle for a varying density "fluid" motion and the varying illumination appears as a "compressibility" constraint (variations of volumes). As such, the variations are directly linked to the real optical flow vector and encoded in its divergence operator. Therefore, the optical flow vector is scene independent in the sense that it does not require any particular assumption on the motion, nor requiring any regularization. When no such variations occurs, the model is simply an optimal transport generalized incompressible flow, where the intensities are conserved along the characteristics like in classical approaches, however with the supplementary incompressibility constraint, that is to say a well-posed PDE system. The models fall into optimal mass transportation theory and framework, and benefits from the huge mathematical and computational knowledge obtained in this field in the last two decades. In particular, this makes the approach easy to implement and having a great potential of generalizations to handle more constraints coming from the applications.

The optimal mass transportation problem seeks the most efficient way of transforming one distribution of mass to another, relative to a given cost function. The first optimal mass transport problem is due to Monge [22]. It was later reformulated by Kantorovich [20], and found applications in several scientific areas, e.g. operational research, economics, mathematical analysis and geometry. In particular, significant progress, both in theoretical and numerical sides, in optimal transport theory has led to its large use in several applied fields and engineering including image analysis problems such as image retrieval, registration and morphing, color and texture analysis, image denoising and restoration, morphometry, super-resolution, and lastly machine learning [24, 5, 27, 28]. We refer interested readers in optimal transport to the books of Villani [29], Santambrogio [27], and for more numerical aspects to Peyré and Cuturi [26] and references therein. Loosely speaking shifting from the classical approach in optical flow estimation, based on pixels transport under a Liouville flow, to the transport of densities (small volumes containing a certain number of pixels) can be viewed as moving a mass (from one location to another) by minimizing Monge-Kantorovich (cost) functional, that is a Wasserstein distance. The motion follows geodesics that minimize the kinetic energy, which in optimal transport words is expressed as minimizing the 2-Wasserstein metric among all curves connecting the two mass of respective densities (in our case): the initial and final images. The well known theorem of Benamou-Brenier characterizing such transport in term of PDEs and the algorithm they provide make the complete link between the optimal transport and the optical flow estimation particularly in the case of varying illumination. Under this view, the variation of the illumination is elegantly linked to the vector flow, highlighting that such perturbation (in densities) is a violation of volume conservation constraint. The equations (PDE) of video motion appear as the Euler-Lagrange system governing fluid motion in fluid mechanics. The results of optical flow computations and the determination of the variations of illumination outperform those from classical approach, such as the Gennert and Negahdaripour one. approach at the price of superior computations cost due to the use of Brenier-Benamou algorithm. However, this cost may be strongly reduced by using more recent algorithms in optimal transportation problems [27, 26] while preserving the advantages of this new formulation.

We notice that viewing the optical flow estimation as a denoising problem [1], leads the classical approach to perform such denoising in the framework of heat-like (parabolic) setting while in the new approach it amounts to use incompressible Euler system (hyperbolic), solution of which are more rich (including singularities -edges-, oscillations -textures, patterns- and sparsity).

The point of view developed in this article is a very promising way to address some related issues in computer vision problems thanks to the strong connections between analysis and geometry (and probability) within the optimal transport theory. Moreover, the models are valid for optical flow of evolving surfaces (manifolds) and not restricted to the Euclidean setting. With great efforts in modeling via regularization procedures and computations the classical approach based on Horn-Schunck seminal work have reached a certain maturity for technological use with some accuracy and acceptable cost of computations, the optimal transport approach comes to complete the arsenal of PDE methods in this field with deep insight in understanding complex motions, obtaining optical flow with its true regularity, that is, with edges and fine geometric patterns, in Euclidean and non-Euclidean framework.

Organization of the article

In Section 1, we recall the classical method to compute the optical based on the widely used *global* method of Horn and Schunck (see also Bruhn, Weickert and Schnörr). The varying illumination is introduced following the modification introduced by Gennert and Negahdaripour [14, 8, 18]. In Section 2 we recall briefly the optimal transport theory framework and the basic results we use in this article. In Section 3, we introduce the new optical flow model based on the optimal transport and establish its main properties. In Section 4, we recall the adapted Benamou-Brenier algorithm we use to solve the optical flow problem. Finally, in Section 5, we present some numerical simulations to show the accuracy and quality of solutions obtained with the new model and we give some comparisons with the classical one. We also underline both the advantages and some limitations to our approach.

1 Classical Optic Flow Formulation

In this section, we recall the most classical optical flow problem formulation. We define a function f from an image domain $\Omega \subset \mathbb{R}^2$ into $\mathbb{R}$ representing the intensity of a pixel $\mathbf{x} \in \Omega$ at an instant t by

$$\begin{array}{rccc} f: & [0,1]\times\Omega & \to & \mathbb{R} \\ & (t,\mathbf{x}) & \mapsto & f(t,\mathbf{x}). \end{array}$$

The estimation of the optical flow consists in finding the vector field $\mathbf{u} = (u_1, u_2)$, called the optic flow (or optical flow), describing the motion of each pixel between two frames of a given sequence. Most methods for the determination of $\mathbf{u}$ are based on the assumption that the intensity of a given pixel is constant between two successive frames. More precisely, the optic flow defines trajectories $\chi(t,\mathbf{x})$ along which this constancy assumption holds

$$\begin{cases} \dot{\chi} = \mathbf{u}, \\ \dot{\mathbf{u}} = 0, \\ \chi(0) = \mathbf{x}, \end{cases}$$

and the constancy assumption reads

$$\frac{d(f\circ\chi)}{dt} = 0,$$

which is familiarly written in the computer vision community

$$f(x+u_1, y+u_2, t+1) = f(x,y,t). \tag{1}$$

In the small displacements case (the frames are very close), this equation is linearized with a Taylor expansion and reads

$$\begin{cases} f_t + \mathbf{u}\cdot\nabla f = 0, & (t,\mathbf{x}) \in (0,1)\times\Omega, \\ f(0,\cdot) = f_0,\ f(1,\cdot) = f_1, & \mathbf{x}\in\Omega, \end{cases} \tag{2}$$

where the unknown is the vector field (u_1, u_2) and ϕ_t, ϕ_x and ϕ_y denotes the partial derivatives with respect to t, x and y respectively, for a function ϕ. With only one equation to determine two unknowns, this problem is ill-posed. It is called the aperture problem. Several methods have been considered among which the seminal approach of Lucas and Kanade [21] and its variational form due to Horn and Schunck [18], which consists in minimizing, over the Sobolev space $H^1(\Omega)^2$, the energy functional

$$E(\mathbf{u}) = \int_\Omega (f_t + \mathbf{u}\cdot\nabla f)^2 + \alpha(|\nabla u_1|^2 + |\nabla u_2|^2)\, d\mathbf{x},$$

where α is a nonnegative constant.

The above model assumes constant illumination and fails to capture accurate motion when this constraint is incorrect. In this last case, many authors relax (or enforce) the constancy assumption for example by adding a specific constraint (conservation of the modulus of the gradient of f) [7], or, with colored images, by using variables which are less sensitive to such illumination changes. Another approach proposed by Gennert and Negahdaripour [14] aims to modelize the illumination change with a new unknown acting as a modulation factor on the intensity. More precisely, the constancy assumption on the intensities (1) is modified with a linear transform as follows

$$f(x+u_1, y+u_2, t+1) = M(x,y,t)f(x,y,t) + T(x,y,t).$$

Then, setting $T = 0$ and M close to the identity, that is

$$M(x,y,t) = 1 + \delta m, \quad \text{and} \quad m_t = \lim_{\delta t\to 0}\frac{\delta m}{\delta t},$$

they derive the new transport equation

$$\partial_t f + \mathbf{u} \cdot \nabla f - m_t f = 0,$$

or equivalently

$$\frac{d\,(f \circ \chi)}{dt} = m_t f.$$

One of the advantages of this approach is to remain close to the variational, and linear, classical model above and to minimize the same energy for the new vector field $(\mathbf{u}, m_t)$

$$\int_\Omega \Big(\nabla f \cdot \mathbf{u} + \partial_t f - f\, m_t\Big)^2 \, d\mathbf{x} + \alpha \int_\Omega |\nabla u_1|^2 \, d\mathbf{x} + \alpha \int_\Omega |\nabla u_2|^2 \, d\mathbf{x} + \lambda \int_\Omega |\nabla m_t|^2 \, d\mathbf{x}.$$

We refer to [16] for details on this approach and its numerical implementation. Besides multiple advantages of this method such as linearity of the model, easy and cheap computations and accuracy for slow and smooth varying illuminations, it suffers some shortcomings as yielding a dense flow and depending on the regularization, to name but a few. In what follows, we consider an alternative modeling approach based on the shift from individual pixels transport to a transport of mass (small volumes of pixels) from a location to another to describe the motion. The approach fall into optimal transportation theory which gives us an interpretation of the variations of the illumination (and thus virtual motion in a scene) linking the movement and the illumination variations in terms of volume variations like in fluid motions. The optical flow problem in this case does not require any regularization. Moreover, the huge and rapid development of the optimal transport theory and associated numerical methods open the way to handle current issues not completely addressed in the majority of optical flow models.

To give a deep insight on the main difference between pixel-wise transport and mass optimal transport point of view, we recall the following reformulation of the classical approach to the optical flow estimation as a denoising problem: we denote by $A_0 = (\nabla f)^T(\nabla f)$, and A a positive definite regularization of A_0. We set $F = -f_t \nabla f$, then the optical flow appears as a minimizer of

$$\frac{1}{\lambda}\|\mathbf{u} - \widetilde{F}\|_A + \mathcal{R}(\mathbf{u})\,, \tag{3}$$

where $\lambda > 0$, $\widetilde{F} = A^{-\frac{1}{2}}F$, and $\|\mathbf{v}\|_A = (\mathbf{v}, A\mathbf{v})^{\frac{1}{2}}$, and R a convex regularizer [1]. This formulation contains almost all classical models and says that optical flow estimation is a denoising problem and that it amounts to determine $\mathbf{u}$ by solving a "heat-like" (parabolic) PDE with initial (noisy datum $\widetilde{F}$). The seminal model of Horn and Schunck [18] is the one corresponding to $\mathcal{R}$ as a Tikhonov regularization. We shall see that the optimal transport models amounts to perform such a denoising with incompressible Euler system (hyperbolic) when no variations of the illumination occur. Thus, without presuming a priori the geometry of the flow, the new approach encodes such a geometry.

2 Brief review of optimal transport

We give here a brief and short review for the basic principles of optimal transport theory and the material we need in this article. We refer interested readers to [29, 27, 26] and the references therein for details. The Monge optimal mass transport problem is to find the best way to transport one mass distribution into another while minimizing a cost function. Given two probability measures μ and ν defined on measure spaces X and Y (usually subsets of $\mathbb{R}^d$) and having densities, that is $\mu = \rho_0\, d\mathbf{x}$ and $\nu = \rho_1\, d\mathbf{x}$, the aim is to find a measurable map $T : X \to Y$ that transport (pushes) μ onto ν and minimizes the following objective function

$$C(\mu, \nu) := \inf_{T \in \mathrm{Tr}\,(\mu,\nu)} \int_X c(\mathbf{x}, T(\mathbf{x}))\rho_0(\mathbf{x})\, d\mathbf{x},$$

where $c : X \times Y \to \mathbb{R}_+$ is the cost function (the Euclidean distance in the original Monge formulation) and

$$\mathrm{Tr}\,(\mu, \nu) = \{T : X \to Y \mid T_\sharp \mu = \nu\}\,,$$

where $T_\sharp \mu$ represents the pushforward of μ and is characterized as

$$\nu(A) = \mu\big(T^{-1}(A)\big), \text{ for any measurable set } A \subset Y.$$

When the transport map T, if it exists, is smooth and the measures μ and ν have densities, it may be determined by solving the Monge-Ampère equation

$$\big|\mathrm{Det}\,(DT)\rho_1\big(T(\mathbf{x})\big)\big| = \rho_0(\mathbf{x}),$$

where DT denotes the Jacobian matrix of T (the matrix formed by taking the gradient of each coordinate of T). Notice that Monge problem is strongly nonlinear (in both the objective function and the constraints set) and does not admit

solutions for certain measures. The Kantorovich formulation of the transportation problem allows to circumvent these obstacles by relaxation which consists of optimizing over transportation plans, instead of transport maps. A transport plan is a probability measure $\gamma \in \mathcal{P}(X \times Y)$ with marginals μ and ν. One can think of γ as the joint distribution of ρ_0 and ρ_1 describing how much mass is being moved to different locations. Let γ be a plan with marginals μ and ν, i.e.

$$(\pi_X)_\sharp \gamma = \mu, \qquad (\pi_Y)_\sharp \gamma = \nu,$$

where $\pi_X : X \times Y \to X$, resp $\pi_Y : X \times Y \to Y$ are the canonical projections. Let $\Gamma(\rho_0, \rho_1)$ be the set of all such plans, then the Kantorovich's formulation can then be written as

$$C(\mu, \nu) = \min_{\gamma \in \Gamma(\mu,\nu)} \int_{X \times Y} c(\mathbf{x}, \mathbf{y}) \, d\gamma(\mathbf{x}, \mathbf{y}).$$

Unlike the Monge problem, in Kantorovich's formulation the constraints are linear with respect to γ. Therefore, Kantorovich's formulation is in the form of a convex optimization problem which admits a solution (may be not unique), that might be numerically solved by linear programming. Moreover, This formulation in terms of transport plans can deal with arbitrary measurable sets and has the ability to distribute mass from the one location in one density to multiple locations in another. For any transport map $T : X \to Y$, there is an associated transport plan, given by

$$\gamma = (I_d \times T)_\sharp \mu, \tag{4}$$

that is $\gamma(A, B) = \mu(\{\mathbf{x} \in A \mid T(\mathbf{x}) \in B\})$, The minimizer of the optimization problem of Kantorovich formulation is called the optimal transport plan γ^* and when a transport map T^* exists, it satisfies (4).

There are several advantages of the Kantorovich formulation such as a dual formulation of the problem and number of geometric properties in (metric) Wasserstein spaces, which make the optimal transport theory very attractive in many fields of application. In particular, the existence of solutions relies to classical direct method of calculus of variations. Namely, for a lower semi-continuous and bounded from below cost function c, there exists a minimizer to the Kantorovich problem. Moreover, in most applications, c is strictly convex ensuring the uniqueness of the optimal transport plan and there exists unique optimal transport map satisfying (4).

The important question in many applications is regarding the existence of an optimal transport map instead of a plan. It follows from Brenier's theorem in the case $c(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2$, extended by Gangbo and McCann to general cases [13], an explicit characterization of T^* (via Euler-Lagrange equation). More precisely, consider the so-called L^p-Kantorovich (or Wasserstein) distance between measures μ and ν which admit the densities ρ_0 and ρ_1, defined by: for $p \geq 1$

$$W_p(\mu, \nu)^p = \min_{\gamma \in \Gamma(\mu,\nu)} \int_{X \times Y} |\mathbf{x} - \mathbf{y}|^p \, d\gamma(\mathbf{x}, \mathbf{y}) = \inf_{T \in \mathrm{Tr}(\mu,\nu)} \int_{X \times Y} |\mathbf{x} - T(\mathbf{x})|^p \rho_0(\mathbf{x}) \, d\mathbf{x}.$$

W_p is a metric on $\mathcal{P}_p(\Omega)$ (Ω), $\Omega \subset \mathbb{R}^d$ where $\mathcal{P}_p(\Omega)$ is the set of Borel probability measures on Ω, with finite p-th moment. The metric space $(\mathcal{P}_p(\Omega), W_p)$ is referred to as the p-Wasserstein space. If Ω is bounded then for any $p \geq 1$, W_p metrizes the weak convergence of measures on $\mathcal{P}_p(\Omega)$. That is the convergence with respect to W_p is equivalent to weak convergence of measures. Note that the p-Wasserstein metric can equivalently be defined using the dual Kantorovich problem

$$W_p(\mu, \nu)^p = \sup_{\phi} \left\{ \int_\Omega \phi(\mathbf{x}) d\mu(\mathbf{x}) - \int_\Omega \phi^c(\mathbf{y}) d\nu(\mathbf{y}) \right\},$$

where the (c-)conjugate $\phi^c(\mathbf{y}) = \inf_{\mathbf{x}} \{\phi(\mathbf{x}) - |\mathbf{x} - \mathbf{y}|^p\}$. The following general theorem is proved in [29]

Theorem 2.1. *Let μ and ν be two Borel probability measures on compact measurable supports X and Y, respectively. When $c(\mathbf{x}, \mathbf{y}) = h(\mathbf{x} - \mathbf{y})$ for some strictly convex function h and μ is absolutely continuous with respect to the Lebesgue measure, then there exists a unique optimal transportation map $T^* : X \longrightarrow Y$ such that $T^*_\# \mu = \nu$ and*

$$\int_X h\big(\mathbf{x} - T^*(\mathbf{x})\big) d\mu(\mathbf{x}) = \min_{\gamma \in \Gamma(\mu,\nu)} \int_{X \times Y} h(\mathbf{x} - \mathbf{y}) d\gamma(\mathbf{x}, \mathbf{y}).$$

In addition, the optimal transport plan is unique and thus T^* is characterized by equation (4)*.*

We emphasis that for our very specific application to optical flow problem, we shall focus in what follows on the case $p = 2$, However, taking W_p for any $p \geq 1$ allows us to generalize the entire setting and formulations below to various optical flow problems (e.g. the Wasserstein metric with $p = 1$ is of great interest in computer vision).

In the case of the 2-Wasserstein distance the following characterization is obtained [4]:

Proposition 2.1. *(Brenier). The square of the* 2*-Wasserstein distance is equal to the infimum*

$$\inf_{(\rho,v)} \int_0^1 \int_\Omega \rho(t,\mathbf{x})|v(t,\mathbf{x})|^2 \, d\mathbf{x}\, dt, \tag{5}$$

among all pairs (ρ, v) *satisfying the constraints*

$$\partial_t \rho + div\,(\rho v) = 0, \tag{6}$$

and

$$\rho(0,\mathbf{x}) = \rho_0(\mathbf{x}), \qquad \rho(1,\mathbf{x}) = \rho_1(\mathbf{x}). \tag{7}$$

Moreover, formal optimality conditions give:

$$v = \nabla\psi \quad \text{and} \quad \psi_t + \frac{1}{2}|\nabla\psi|^2 = 0, \tag{8}$$

where ψ is the Kantorovich potential, that is the Lagrange multiplier associated to the constraints (6) and (7).
Note. The original result of Brenier is given for $\Omega = \mathbb{R}^d$, other versions in bounded domains and with boundary conditions may be found for example in Villani [29, 10].

3 Optimal Transport Based Model

Let us consider a general flow equation with a velocity $\mathbf{u}$, eventually depending on time, given by

$$\begin{cases} \dot{\chi}(t,\mathbf{x}) = \mathbf{u}(t,\chi(t,\mathbf{x})), \\ \chi(0,\mathbf{x}) = \mathbf{x} \end{cases}$$

Instead of taking a constancy assumption on the intensity values of the pixels as in the classical methods, let us consider a small volume $\mathcal{U}_t := \chi(t,\mathcal{U}_0)$ containing a number of particles and associate a continuous density function $f(t,\mathbf{x}(t))$ representing the intensity map per unit volume at the point $\mathbf{x}(t)$. We set

$$\mathcal{F}(t) := \int_{\mathcal{U}_t} f(t,\mathbf{x})\, d\mathbf{x} = \int_{\mathcal{U}_0} f\big(t,\chi(t,\mathbf{x})\big)|\det J(t,\mathbf{x})|\, d\mathbf{x},$$

where $J(t,\mathbf{x}) := \frac{\partial\chi(t,\mathbf{x})}{\partial\mathbf{x}}$ is the Jacobian matrix of χ. We now reformulate the constancy assumption as a conservation of the intensity in the small volumes which amounts to have a null material derivative

$$\frac{d\mathcal{F}(t)}{dt} = 0,$$

that is, with standard computations, the intensity conservation equation

$$\partial_t f + \text{div}\,(f\,\mathbf{u}) = 0.$$

with the "boundary" conditions

$$f(0,\cdot) = f_0 \quad \text{and} \quad f(1,\cdot) = f_1. \tag{9}$$

We get

$$\partial_t f + \mathbf{u}\,\nabla f = -\text{div}\,(\mathbf{u})\, f. \tag{10}$$

Recall that a volume transported by the flow is expressed via the ordinary differential equation

$$\begin{cases} \partial_t|\det J(t,\mathbf{x})| = \text{div}\left(\mathbf{u}\big(t,\chi(t,\mathbf{x})\big)\right)|\det J(t,\mathbf{x})|, \\ |\det J(0,\mathbf{x})| = 1. \end{cases}$$

That is,

$$|\det J(t,\mathbf{x})| = e^{\int_0^t \text{div}\,(\mathbf{u}(s,\chi(s,\mathbf{x})))\, ds}.$$

From the physics of light propagation and optical laws, it is common to model the losses in the intensity due to absorption and scattering by the attenuation rule:

$$\frac{d}{dt} f\big(t,\chi(t,\mathbf{x})\big) = m_t\big(\chi(t,\mathbf{x})\big)\, f\big(t,\chi(t,\mathbf{x})\big),$$

where $m_t(\mathbf{x})$ denotes the spatially varying attenuation coefficient. That is

$$f\big(t,\mathbf{x}(t)\big) = f(0,\mathbf{x}) e^{\int_0^t m_t(\mathbf{x}(s))\, ds}.$$

Comparing to (10), we obtain

$$m_t = -\text{div}(\mathbf{u}). \tag{11}$$

It follows that the varying illumination expresses a variation of the transported volumes along the flow.

Remark 3.1.

1. We emphasize that (11) states that m_t is not a supplementary, and somehow approximation, variable but is linked to the vector field $\mathbf{u}$ giving a "physical sense" to the variations of the illumination in a scene.
2. Unlike in physics, where the thermodynamics laws allow to decide whether we have a gas or a liquid, in computer vision there is no such knowledge and we have to decide between the pixel-wise or mass transportation description of the motion. We believe that the optimal mass transportation offers a more comprehensive and general approach to motion analysis and even more when varying illumination occurs.
3. The optimal transport makes consistent, and justify, the intuition of Gennert and Negahdaripour [14] as we will show in the case of discrete masses transport, but instead of adding a new variable to an inverse problem, it is a part of a well-posed one and the whole system does not require any regularization.

In the case of no variations of the illumination, we have, once again from optical laws

$$m_t = 0,$$

and

$$\partial_t f + \mathbf{u}\,\nabla f = 0.$$

We retrieve the relationship that the material derivative $\frac{d(f\circ\chi)}{dt} = 0$, that is the intensity is constant a long the characteristics. However, unlike the classical methods, we have the additional equation on the optical flow which expresses the incompressibility constraint

$$\operatorname{div}\mathbf{u} = 0.$$

This is the optimal transport generalized incompressible flow system of equations [6]. In all cases, the optimal transport formulation of the optical flow amount to compute the 2-Wasserstein distance between the measures $\mu = f_0(\mathbf{x})d\mathbf{x}$, and $\nu = f_1(\mathbf{x})d\mathbf{x}$, we assume here that

$$\int_\Omega f_0\, d\mathbf{x} = \int_\Omega f_1\, d\mathbf{x} = 1.$$

That is, to compute

$$W_2(\mu,\nu) = \inf_{(f,\mathbf{v})} \int_0^1 \int_\Omega f(t,\mathbf{x})|\mathbf{v}(t,\mathbf{x})|^2 d\mathbf{x}\, dt, \tag{12}$$

under the constraints

$$\partial_t f + \operatorname{div}(\mathbf{v}\, f) = 0, \tag{13}$$

and

$$f(0,\mathbf{x}) = f_0(\mathbf{x}), \qquad f(1,\mathbf{x}) = f_1(\mathbf{x}), \tag{14}$$

which is nothing else than the principle of the least action principle in mechanics. It follows from Brenier's theorem above that this optical flow problem is well posed, that is admits a unique solution $(f,\mathbf{u})$. Moreover, in addition to the characterization given in the theorem, we have as by product, the variations of the illumination

$$m_t(\mathbf{x}) = -\operatorname{div}\mathbf{u}(\mathbf{x}),$$

or the incompressibility constraint when there is no varying illumination in the scene.

Remark 3.2. As it was noted by several authors, there is a deep geometric interpretation behind the optimal transport problem for the 2-Wasserstein metric. Indeed, an important fact regarding this distance is Otto's presentation of a formal Riemannian metric for the space $\mathcal{P}_2(\Omega)$ [23]. Loosely speaking, consider a path $\mu(t)$ in $\mathcal{P}_2(\Omega)$ with smooth densities $f(t,\mathbf{x})$, then $s(t,\mathbf{x}) = \frac{\partial f(t,\mathbf{x})}{\partial t}$ can be seen as a tangent vector. This (infinitesimal) variation can be viewed as resulting form moving the mass by a vector field $\mathbf{u}$ such that

$$s = -\nabla\cdot(f\mathbf{u}).$$

There are many such vector fields. Otto defined the size of $s(t,\cdot)$ as (the square root of) the minimal kinetic energy of the vector field that produces the perturbation to density s. That is

$$(s,s) = \min_{\mathbf{u}\text{ satisfies (13)}} \int f\,|\mathbf{u}|^2. \tag{15}$$

Thanks to the Riemannian manifold structure of $\mathcal{P}_2(\Omega)$ and the inner product presented in equation (15) the 2-Wasserstein metric can be reformulated into finding the minimizer of the action (12) under the constraints (13) and (14), among all curves in $\mathcal{P}_2(\Omega)$ connecting μ and ν.

This interpretation shows that the optimal transport based model amount to determine constant speed geodesics in the probability space $(\mathcal{P}_2, W_2)$. Besides this nice geometric interpretation, we have a very general formulation of the optical problem not restricted to Euclidean spaces. A case example is the optical flow estimation between manifolds which is useful for some imaging problems such as evolving surfaces (with applications as in the analysis of 4D microscopy data).

It is tempting at a theoretical level to compare the classical method of optical flow with our optimal mass transportation approach. As mentioned in the beginning of the paper, the optical flow estimation might be reformulated as a "denoising" problem (3) with a heat-like equation in the classical methods as the regularizer yields a parabolic system (e.g. heat kernel, ...). The resulting model is strongly dependent on regularization parameters and the operators chosen and require implicit assumption on the geometry of the sought vector field. In the optimal transport approach the "denoising" is performed, for example in the constant illumination case, with the incompressible Euler system, for $T > 0$

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0, & \text{in } (0,T)\times\Omega, \\ \operatorname{div}\mathbf{u} = 0, & \text{in } (0,T)\times\Omega, \\ \mathbf{u}(0,x) = \mathbf{u}^\delta, & \text{in } \Omega, \\ \text{b.c.} \end{cases}$$

A bit more difficult to solve numerically but solutions have more striking features as it is known from the regularity results, e.g. solutions may have singularities -edges- or oscillations -textures- which is more realistic modeling for motion analysis (think of traffic flow). In this article, we do not solve such a system but in future work we will do as a hyperbolic approach to denoising seems an interesting alternative to parabolic scale spaces methods.

We notice for this formal comparison that it is possible to take the intensity as a scalar quantity to be conserved along the flow, and combine the previous considerations with mass conservation of pixels. This amount to state the conservation of f along the characteristics and might be obtained by choosing as variables the density of the pixels ρ (i.e. the number of pixels in a given volume) and the "momentum" ρf. Straightforward computations give then $\frac{df}{dt} = 0$ as in classical models but with continuity equation on ρ (mass conservation). When the density ρ is constant we retrieve the incompressibility constraint, otherwise, a production (reaction) term $\frac{df}{dt} = -\frac{f}{\rho}\frac{d\rho}{dt} = f \operatorname{div}\mathbf{u}$. This means that in the classical models the pixels travel along the flow like free noninteracting particles but in bounded domains one should add the continuity equation.

Another element in this comparison follows from a discrete representation of the problem. Actually, with no variations of the illumination, the optimal transport and the classical models give similar constancy of the intensity along the flow trajectories, though there are some important differences and consequences:

- Writing the two images as vectors with entries $f_\ell(\mathbf{x}_i)$, $\ell = 0, 1$ and the associate discrete measures $\mu = \sum_{i=1}^N (f_0)_i \delta_{\mathbf{x}_i}$ and $\nu = \sum_{j=1}^M (f_1)_j \delta_{\mathbf{y}_j}$, where δ_{x_i} and $\delta_{\mathbf{y}_j}$ are Dirac measures centered at $\mathbf{x}_i$ and $\mathbf{y}_j$ respectively, the Kantorovich problem can be written as,

$$\begin{aligned} &\min_\gamma \sum_i \sum_j c_{ij}\gamma_{ij} \quad \text{such that} \\ &\sum_j \gamma_{ij} = (f_0)_i, \quad \sum_i \gamma_{ij} = (f_1)_j, \\ &\gamma_{ij} \geq 0, i = 1,\dots,N \quad j = 1,\dots,M, \end{aligned}$$

  where $c_{ij} = c(\mathbf{x}_i, \mathbf{y}_j)$. Thus, we have an optimization problem with a linear objective function and linear constraints, therefore it is a linear programming problem. We note that the problem is convex, but not strictly, and the constraint provides a polyhedral set of $N \times M$ matrices, $\Gamma(\mu, \nu)$. When $M = N$ (here is the continuity equation) it turns out that this Monge-Kantorovich problem is an assignment problem. The transport plan γ^* may be represented by an interpolation function (McCann interpolation) [13]. Formally, this solution in term of geometric transport theory, is viewed as follows: if there exists a unique transportation map $T^*_\# \mu = \nu$ which minimizes the transportation cost, the geodesic (in the metric geodesic space $\mathcal{P}_2(\Omega)$ [29]) is obtained by moving the mass at constant speed from $\mathbf{x}$ to $T^*(\mathbf{x})$. Let $t \in [0,1]$ and $\mathbf{x} \in \Omega$ and let

$$T^*_t(\mathbf{x}) = (1-t)\mathbf{x} + tT^*(\mathbf{x}),$$

  be the position at time t of the mass initially at $\mathbf{x}$. Note that T^*_0 is the identity mapping and $T^*_1 = T^*$. Pushing forward the mass by T^*_t:

$$\mu^*(t) = T^*_t \mu,$$

provides the desired geodesic from μ to ν. We remark that the velocity of each particle

$$\partial_t T^* = T^*(\mathbf{x}) - \mathbf{x} = \partial_{\mathbf{x}} T^*,$$

which is nothing but the displacement of the optimal transportation map and the transport equation (2). Thus a very particular instance of the Monge optimal transportation problem gives the classical formulation of the optic flow problem of Lucas-Kanade (or Horn-Schunck) except that in this last model the mass conservation of the pixels is omitted. This suggests strongly that the optimal transport point of view admits as special instance the classical pixel-wise case completed by the mass conservation. Whereas, the classical approach resort to (somehow arbitrary) regularization which does not ensures a geodesic transport of measures, the optimal transport provides a mathematically closed system.

- The discrete case of mass transport where $N \neq M$ solved under the Kantorovich formulation allows us also to consider optical flow between images of different resolutions.
- For varying illumination this general approach based on optimal transportation leads to the constraint defining m_t in term of non conservation of the volume which seems to us a comprehensive explanation to the impact of such variations in motion analysis.

4 Numerical Method

There exists a plenty of rather different approaches to finding optimal transportation maps and plans [26]. This methods and solvers differ by their respective advantages and shortcomings among which the convergence rates or the computational cost which may scales from $O(N^3)$ (the most expensive ones like classical linear programming) to $O(N)$ (for the cheapest ones), N being the size of the spatial domain. We refer to [26] (and references therein) for more details on the numerical methods in optimal transport theory. In what follows we restrict ourselves to the use of the approach based on the augmented Lagrangian considered by Benamou and Brenier, even if its cost is higher compared to the classical optical flow methods based on Horn-Schunck method with Tikhonov regularization. We emphasize that our main goal in this article is to show the quality of the obtained optical flow and variations of the illumination rather than solving engineering problems that we will consider in a forthcoming work as well as alternative algorithms at hands. We also notice that this algorithm of augmented Lagrangian can be adapted to generalizations of the quadratic cost, based on action along paths to enforce some additional constraints encountered in real-life applications.

4.1 Augmented Lagrangian Method (Benamou-Brenier)

In their paper, Benamou and Brenier proposed to use the augmented Lagrangian method in order to solve (12) under constraints (13) and (14) [4, 11]. A brief overview of the method and its derivation is as follows. Let $\phi(t, \mathbf{x})$ be the space-time dependent Lagrange multiplier for constraints in equation (13) and (14). The Lagrangian is given by

$$\begin{aligned}\mathcal{L}(\phi, f, \mathbf{v}) &= \int_0^1 \int_\Omega f|\mathbf{v}|^2 + \phi\big(\partial_t f + r\mathrm{div}\,(f\mathbf{v})\big)\, d\mathbf{x}\, dt \\ &= \int_0^1 \int_\Omega \frac{|s|^2}{2f} - \partial_t\phi\, f - r\nabla\phi\, s\, d\mathbf{x}\, dt - \int_\Omega \big(\phi(0,\cdot)f_0 - \phi(1,\cdot)f_1\big)\, d\mathbf{x},\end{aligned}$$

where $s = f\mathbf{v}$ and we used integration by parts together with the equality constraints in equation (13) to obtain the second line. Note that

$$W_2^2(\mu, \nu) = \inf_{(f,s)} \sup_{\phi} \mathcal{L}(\phi, f, s). \tag{16}$$

Using the Legendre transform for $\frac{|s|^2}{2f}$ one can write

$$\frac{|s|^2}{2f} = \sup a(t,\mathbf{x})f(t,\mathbf{x}) + b(t,\mathbf{x})s(t,\mathbf{x}),\ \text{such that}\, a + \frac{|b|^2}{2} \leq 0,\ \forall t, \mathbf{x}.$$

Define $\psi = (f, s)$, and $q = (a, b)$, and their corresponding inner product to be

$$(\psi, q) = \int_0^1 \int_\Omega f(t,\mathbf{x})a(t,\mathbf{x}) + s(t,\mathbf{x}) \cdot b(t,\mathbf{x})\, d\mathbf{x}\, dt.$$

Next, we set

$$F(q) = \begin{cases} 0 & \text{if } a + \frac{|b|^2}{2} \leq 0, \forall t, \mathbf{x}, \\ +\infty & \text{else,} \end{cases}$$

and

$$G(\phi) = \int_\Omega \big(\phi(0,\cdot)f_0 - \phi(1,\cdot)f_1\big)\, d\mathbf{x},$$

then it is straightforward to show that equation (16) can be rewritten as

$$W_2^2(\mu,\nu) = \sup_{\psi}\inf_{\phi,q} F(q) + G(\phi) + \langle \psi, \nabla_{t,\mathbf{x}}\phi - q\rangle,$$

where $\nabla_{t,\mathbf{x}}$ denotes the space time gradient $(\partial_t, \nabla_{\mathbf{x}})$ and ψ the Lagrange multiplier for a new constraint, namely $\nabla_{t,\mathbf{x}}\phi = q$. Thus the augmented Lagrangian can be written as, for $r > 0$

$$\mathcal{L}_r(\phi,\psi,q) = F(q) + G(\phi) + \langle \psi, \nabla_{t,\mathbf{x}}\phi - q\rangle + \frac{r}{2}\langle \nabla_{t,\mathbf{x}}\phi - q, \nabla_{t,\mathbf{x}}\phi - q\rangle.$$

The properties of convergence and rates of convergence for such saddle point problems are well established [12]. Benamou and Brenier [4] used a variation of the Uzawa algorithm to solve the problem above. We note that recent methods based on Bregman iterations could also be used for solving such saddle point problem. The Uzawa algorithm in this case reads: given $(\phi^{n-1}, q^{n-1}, \psi^n)$, $n \geq 1$

- Find ϕ^n, solution of

$$\begin{cases} -r\Delta_{t,\mathbf{x}}\phi^n = \nabla_{t,\mathbf{x}}\cdot(\psi^n - rq^{n-1}), & \text{in } [0,1]\times\Omega, \\ r\partial_t\phi^n(0,\cdot) = f_0 - f^n(0,\cdot) + ra^{n-1}(0,\cdot), & \text{in } \Omega, \\ r\partial_t\phi^n(1,\cdot) = f_1 - f^n(1,\cdot) + ra^{n-1}(1,\cdot), & \text{in } \Omega, \\ \partial_n\phi^n = 0, & \text{on } [0,1]\times\partial\Omega, \end{cases} \tag{17}$$

  with $\psi^n := (f^n, (f^n\mathbf{v})^n)$ and $q^{n-1} := (a^{n-1}, b^{n-1})$.
- Find $q^n := (a^n, b^n)$ such that

$$\inf_{(a^n,b^n)\in K} \big(a(t,\mathbf{x}) - \alpha^n(t,\mathbf{x})\big)^2 + \big|b(t,\mathbf{x}) - \beta^n(t,\mathbf{x})\big|^2,$$

  with $p^n(t,\mathbf{x}) := \big(\alpha^n(t,\mathbf{x}), \beta^n(t,\mathbf{x})\big) = \nabla_{t,\mathbf{x}}\phi^n(t,\mathbf{x}) + \frac{\phi^n(t,\mathbf{x})}{r}$ and

$$K := \Big\{(a,b) \;\Big|\; a(t,\mathbf{x}) + \frac{|b(t,\mathbf{x})|^2}{2} \leq 0,\ \forall t,\mathbf{x}\Big\}.$$

  This turns out to be a simple projection step and it can be computed analytically by using Cardano's formula.
- Perform the update

$$\psi^{n+1} = \psi^n + r(\nabla_{t,\mathbf{x}}\phi^n - q^n).$$

The stopping criterion is:

$$\sqrt{\frac{\int_0^1\int_\Omega f^n|\text{res}^n|\, d\mathbf{x}\, dt}{\int_0^1\int_\Omega f^n|\nabla_{\mathbf{x}}\phi^n|^2\, d\mathbf{x}\, dt}} < \varepsilon,$$

for $\varepsilon > 0$, where

$$\text{res}^n := \partial_t\phi^n + \frac{|\nabla_{\mathbf{x}}\phi^n|^2}{2}.$$

5 Implementation Details and Numerical Results

The algorithm is in space-time variables, so we discretize the time interval $[0,1]$ using a uniform grid $t_1 = 0 < t_2 < \ldots < t_N = 1$ for time and a finite difference regular mesh for the space variable. We solve the equation on ϕ^n by using a simple first order finite differences and we use the same grid both for the solver and the projection steps (to update ψ and compute q). We compute f which represents an "interpolated" intensity map between f_0 and f_1 and the optical flow $\mathbf{u}^n$. It is possible to extract $\mathbf{u}$ at convergence as the desired optical flow, but we choose to approximate, hopefully better, the solution from the transport map by using

$$\mathbf{u}(\mathbf{x}) := \chi(1,\mathbf{x}) - \chi(0,\mathbf{x}) = \int_0^1 \dot\chi(s,\mathbf{x})\, ds = \int_0^1 \mathbf{u}\big(s,\chi(s,\mathbf{x})\big)\, ds,$$

and an accurate numerical integration formulae. Next, we compute

$$m_t(\mathbf{x}) := -\text{div}\,\mathbf{u}(\mathbf{x}).$$

Remark 5.1. We do not use an advanced and accurate method to avoid computing the divergence in this article as the numerical results are satisfactory

The implementation of the algorithms is done in the programming language Python and can be found at [19].

In Figure 1 and Figure 2 we give the initial and final images f_0 and f_1 and the associated *ground-truth* optical flow without luminosity variations for several examples from [2]. We also present the computed vectors fields for the classical method of Horn-Schunck and optimal transport optical flows with the parameters, respectively, $\alpha = 0.1$ and $\lambda = 0.2$ [15] and $N = 8$ points for the time's discretization, $r = 1$ in (17) and $\varepsilon = 0.1$. We observe that the results given by the optimal transport for the vector fields are sparse and more accurate and even if the mean square L^2 error is small for both models, the location of the motion is captured better with the optimal transport computations. A zoom on the results on m_t, almost zero in this case of no varying illumination, shows a "homogenization" of m_t in the classical method whereas in the optimal transport model m_t as a divergence of $\mathbf{u}$ reflects the (small) variations (jumps) of $\mathbf{u}$ in locations where the motion takes place.

In the case with no varying illumination we may notice that the computed flow with optimal transport is accurate and sparse (no regularizing effect). The variation of the illumination is almost zero for both models, however, it may be noted that with optimal transport small scales effects (details near the edges) appear in m_t which is normal as the optical flow is not smoothed.

In Figure 3 and Figure 4 we present the same simulations results when the varying illumination is non zero. The results in this case confirm what observe when no variations of luminosity occur. The sparsity and the accuracy seems to outperform those of the classical approach. Both the locations and the magnitude of the optical flow and m_t are captured with high accuracy. The mean square L^2 errors shows a homogenization effect with the classical method (as expected from the regularization) while the error is concentrated on the tight locations of the motion for the optimal transport model. In the case of varying illumination we still have very accurate results compared to the classical approach.

At the contrary, the time of computations as expected is higher for the optimal transport approach mainly because of the space-time Laplacian computations. This gap may be reduced by using other existing numerical algorithms of optimal mass transportation.

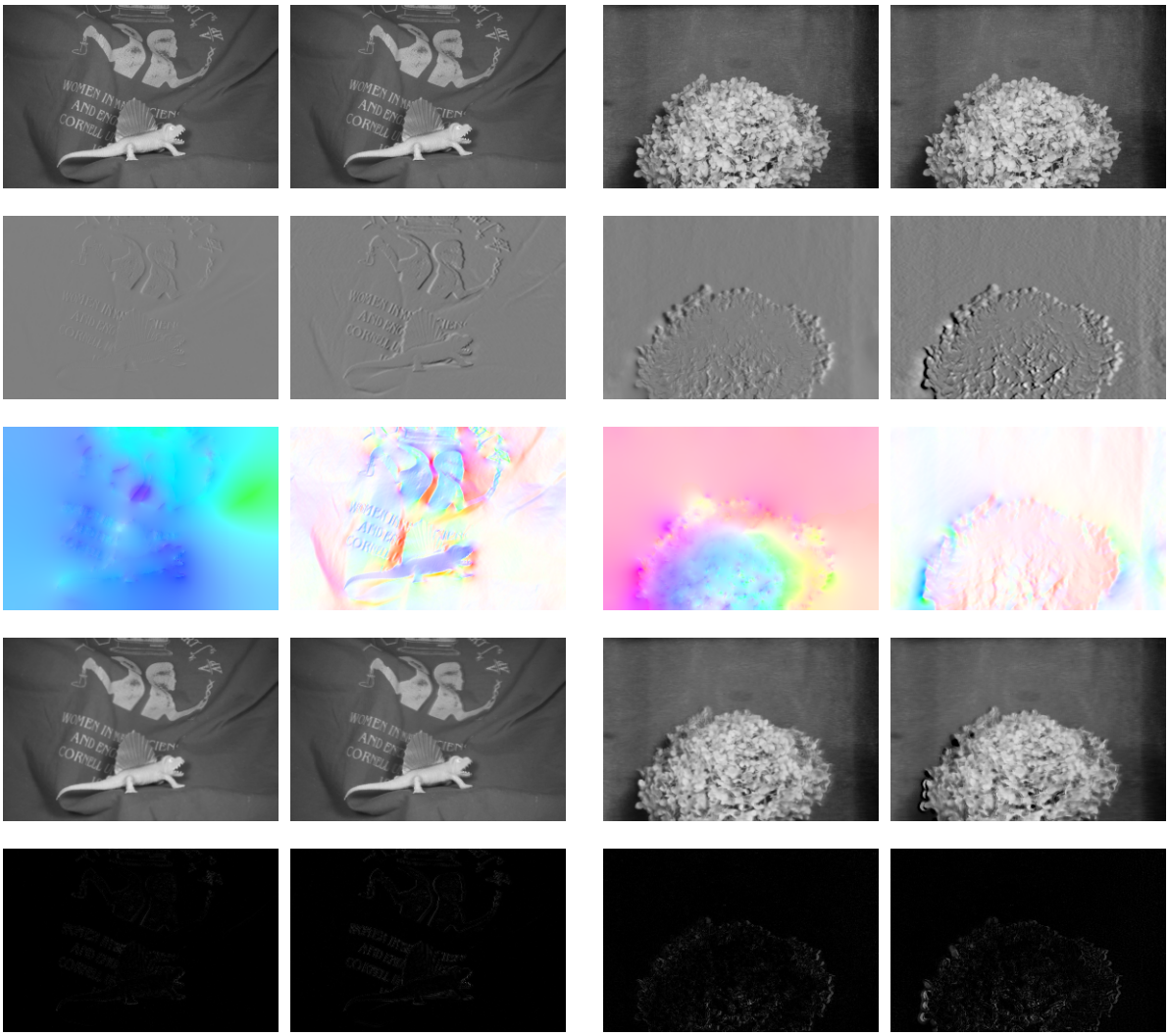

Figure 1: Optical Flow Benchmark for *Dimetrodon*, *Hydrangea*; First row: f_0 (1^{st} and 3^{rd} column) and f_1 (2^{nd} and 4^{th} column). Second row: m with classical method (1^{st} and 3^{rd} column) and m with the optimal transport (2^{nd} and 4^{th} column). Third row: u with classical method (1^{st} and 3^{rd} column) and u with the optimal transport (2^{nd} and 4^{th} column). Fourth row: reconstruction x with classical method (1^{st} and 3^{rd} column) and x with the optimal transport (2^{nd} and 4^{th} column). Fifth row: $(f_1 - x)^2$ with classical method (1^{st} and 3^{rd} column) and $(f_1 - x)^2$ with the optimal transport (2^{nd} and 4^{th} column).

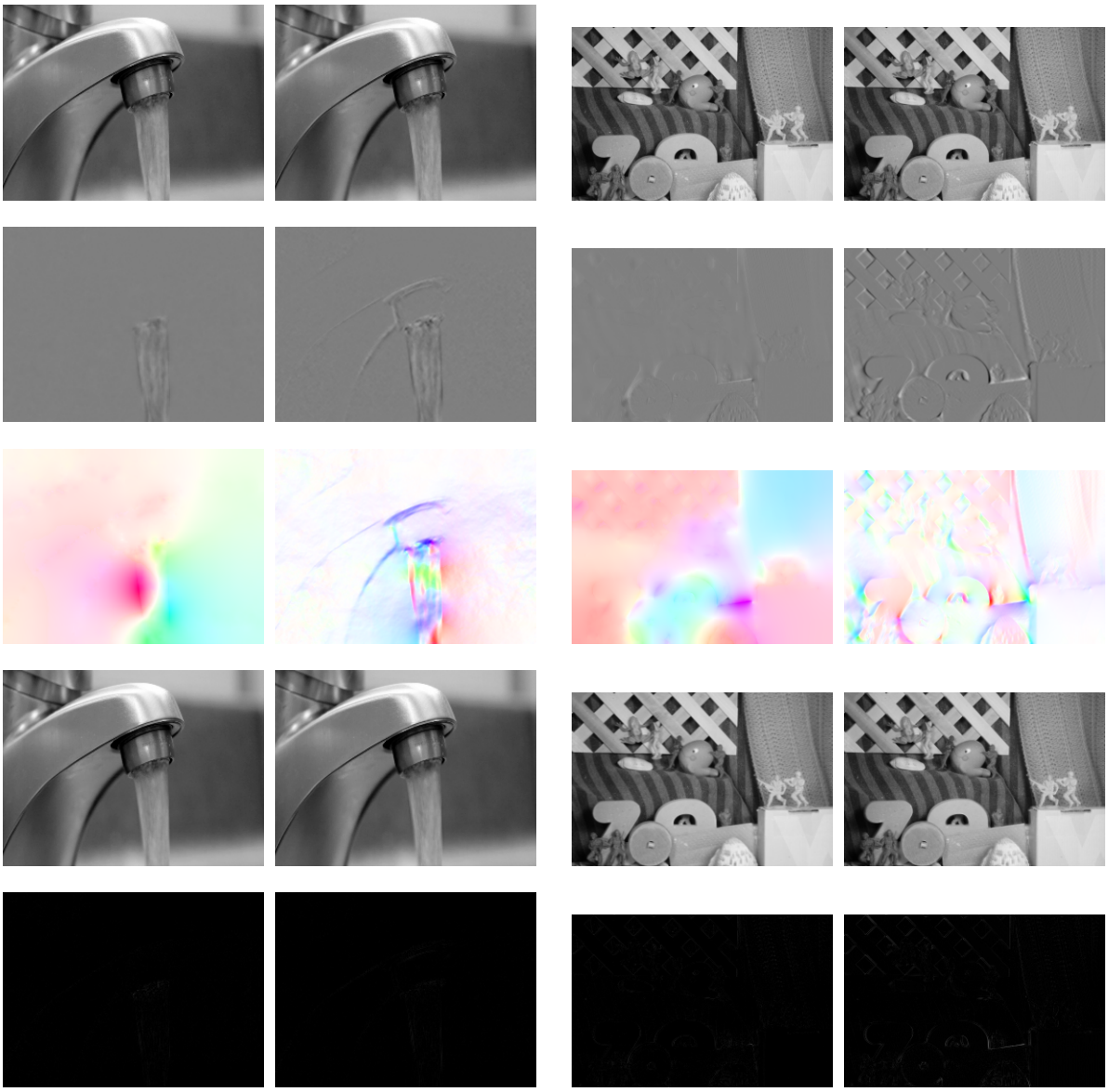

Figure 2: Optical Flow Benchmark for *Sink*, *RubberWhale*; First row: f_0 (1^{st} and 3^{rd} column) and f_1 (2^{nd} and 4^{th} column). Second row: m with classical method (1^{st} and 3^{rd} column) and m with the optimal transport (2^{nd} and 4^{th} column). Third row: u with classical method (1^{st} and 3^{rd} column) and u with the optimal transport (2^{nd} and 4^{th} column). Fourth row: reconstruction x with classical method (1^{st} and 3^{rd} column) and x with the optimal transport (2^{nd} and 4^{th} column). Fifth row: $(f_1 - x)^2$ with classical method (1^{st} and 3^{rd} column) and $(f_1 - x)^2$ with the optimal transport (2^{nd} and 4^{th} column).

Conclusion

In this article, we considered the optical flow estimation from the point of view of optimal mass transportation which leads to new models in this field for both varying and no varying illumination cases. The models obtained do not require any regularization and a priori assumptions on the geometry of the optical flow and give in turn accurate results. Theoretically, regarding the optical flow determination as determining constant speed geodesics in $(\mathcal{P}_2, W_2)$ comes with several consequences such as optical flow estimation between manifolds, denoising with generalized incompressible flows and the last but not the least solutions with complex and rich geometric properties among which the sparsity,

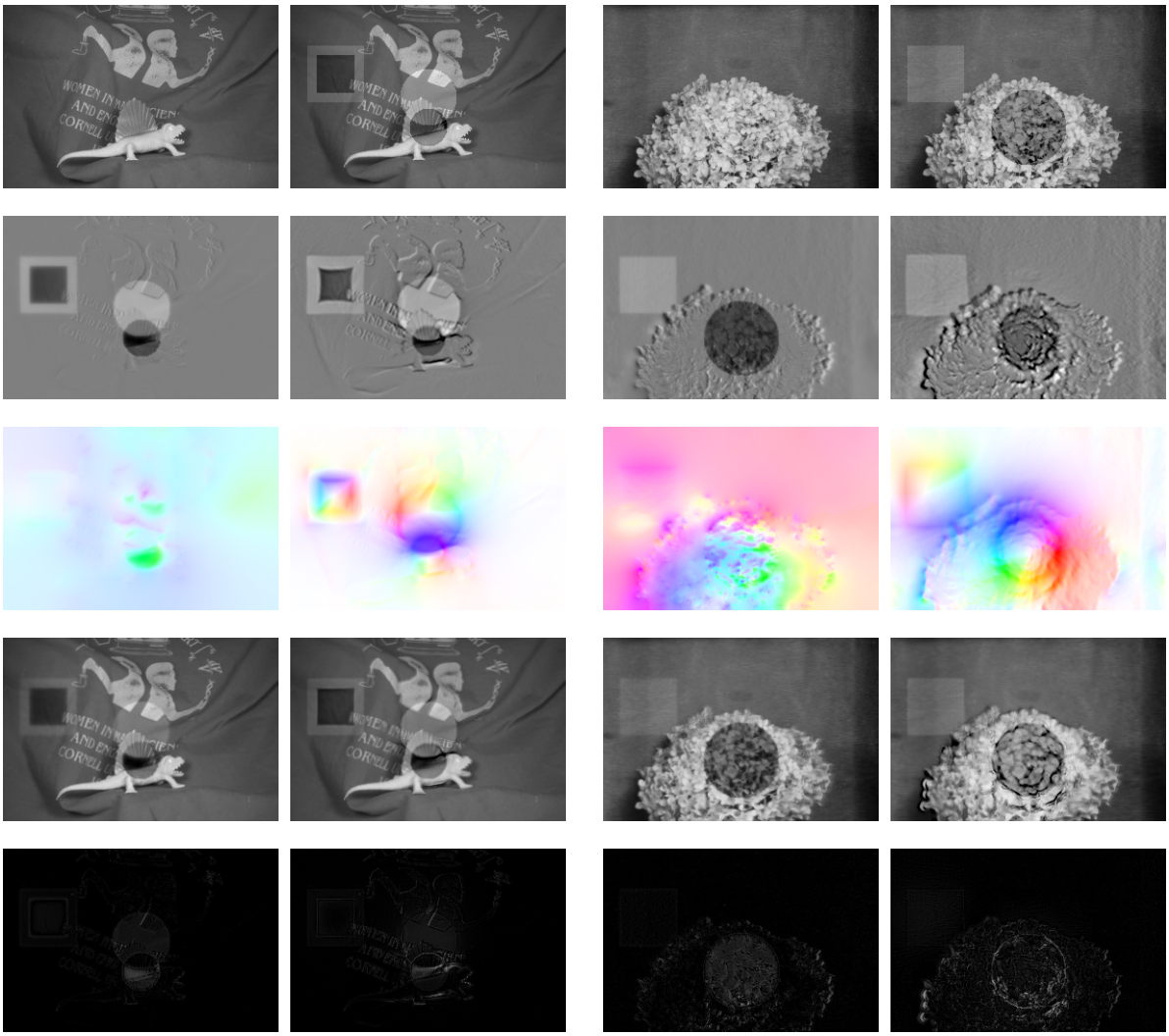

Figure 3: Optical Flow Benchmark for *Dimetrodon*, *Hydrangea* with illumination; First row: f_0 (1^{st} and 3^{rd} column) and f_1 (2^{nd} and 4^{th} column). Second row: m with classical method (1^{st} and 3^{rd} column) and m with the optimal transport (2^{nd} and 4^{th} column). Third row: u with classical method (1^{st} and 3^{rd} column) and u with the optimal transport (2^{nd} and 4^{th} column). Fourth row: reconstruction x with classical method (1^{st} and 3^{rd} column) and x with the optimal transport (2^{nd} and 4^{th} column). Fifth row: $(f_1 - x)^2$ with classical method (1^{st} and 3^{rd} column) and $(f_1 - x)^2$ with the optimal transport (2^{nd} and 4^{th} column).

the presence of singularities and textures (oscillations). Moreover, the approach is very general and restricted to the transport of discrete measures include and complete the classical methods of pixel-wise "transport".

This optimal transport approach seems to us the right way of modeling the motion in a scene, particularly when varying illumination occurs and gives a "physical" meaning to such variations. Actually, the price to pay using this approach is the numerical side where the cost of computations is expensive as we solve a space-time problems and future works should improve this aspect for some practical applications more focused in speed of computations than motion analysis.

We emphasize that the optimal transport point of view based on weak transport and unbalanced transport open promising research perspectives for complex motion analysis problems (e.g. occlusion/disocclusion -object appearing/disappearing- in a scene).

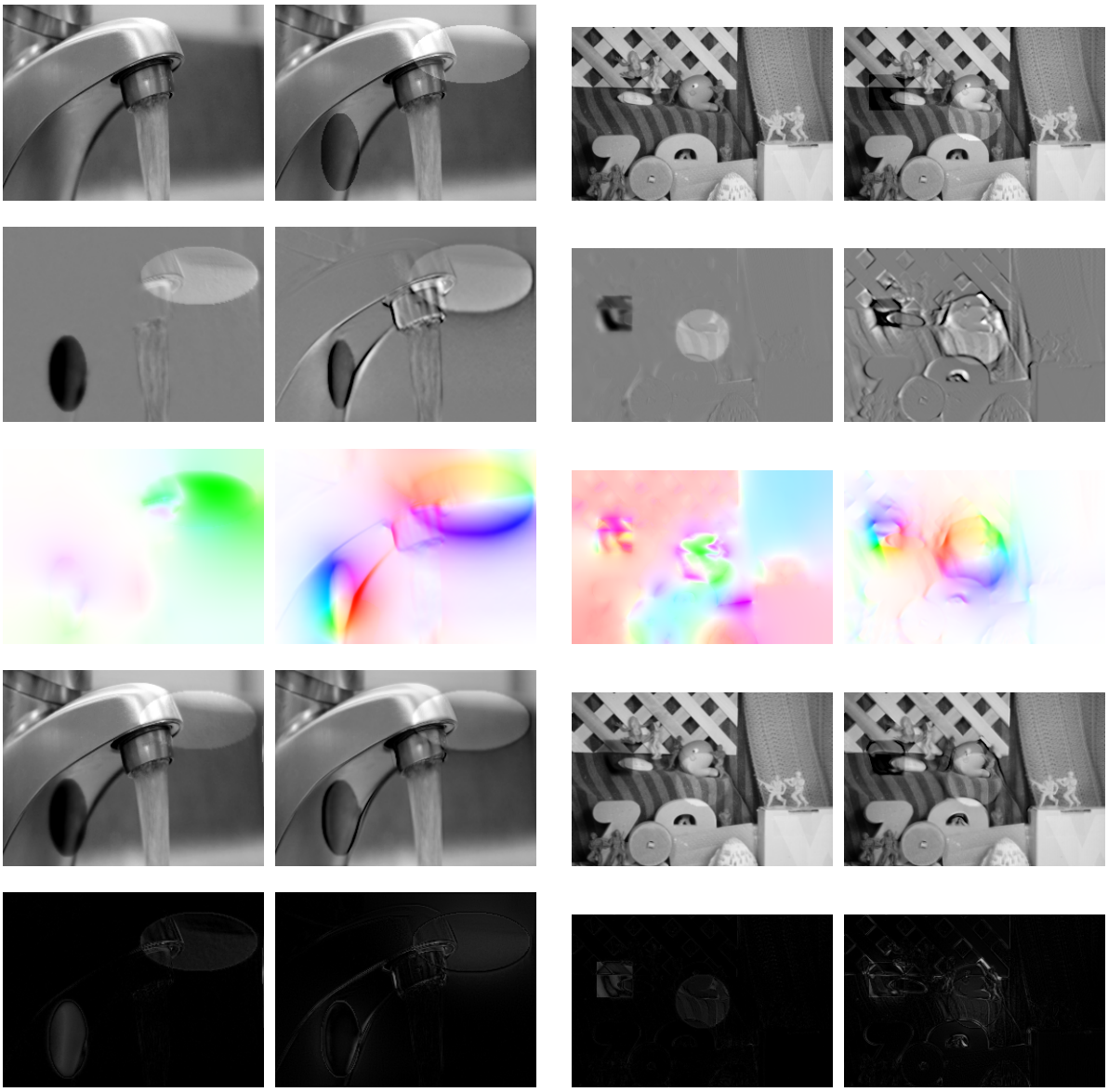

Figure 4: Optical Flow Benchmark for *Sink*, *RubberWhale* with illumination; First row: f_0 (1^{st} and 3^{rd} column) and f_1 (2^{nd} and 4^{th} column). Second row: m with classical method (1^{st} and 3^{rd} column) and m with the optimal transport (2^{nd} and 4^{th} column). Third row: u with classical method (1^{st} and 3^{rd} column) and u with the optimal transport (2^{nd} and 4^{th} column). Fourth row: reconstruction x with classical method (1^{st} and 3^{rd} column) and x with the optimal transport (2^{nd} and 4^{th} column). Fifth row: $(f_1 - x)^2$ with classical method (1^{st} and 3^{rd} column) and $(f_1 - x)^2$ with the optimal transport (2^{nd} and 4^{th} column).

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