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Cutoff for the mixing time of the Facilitated Exclusion Process

Brune Massoulié

Abstract

We compute the mixing time of the Facilitated Exclusion Process (FEP) and obtain cutoff and pre-cutoff in different regimes. The main tool to obtain this result is a new bijective, deterministic mapping between the joint law of an ergodic FEP and its current through the origin, and the joint law of a SSEP and its current through the origin. This mapping is interesting in itself, as it remains valid in the non-ergodic regime where it gives a coupling between the position of a tagged particle in the FEP and the current through the origin in a SSEP with traps.

1 Introduction

The Facilitated Exclusion Process (FEP) is an interacting particle system on a lattice that was introduced in the physics literature by [1, 2] as a model with an active-absorbing-state phase transition. It is defined as follows: there is at most one particle per lattice site and each particle tries to jump to a neighbouring site at rate 1. The jump is not allowed if the target site is occupied (exclusion constraint), or if the particle is isolated, i.e. it has no nearest neighbour occupied site (kinetic constraint). The kinetic constraint makes the FEP's behaviour very different from that of the Simple Symmetric Exclusion Process (SSEP), which is defined similarly but without the kinetic constraint. Indeed, unlike the SSEP, the FEP is non-reversible, non-attractive and has transient configurations and absorbing states (when no particles have neighbours, the system freezes).

The FEP has mostly been studied on one-dimensional lattices, such as \mathbb{Z} [3, 4, 5, 6], a closed segment [7], a segment connected to reservoirs [8] and the discrete circle $\mathbb{T}_N := \mathbb{Z}/N\mathbb{Z}$ [9, 10, 11, 7, 12]. In this paper we will study the mixing time for the FEP on a discrete circle. A first important observation is that, depending on the number K of particles, the FEP on \mathbb{T}_N has very different long-time behaviours :

- if $K \leq \frac{N}{2}$ we say the FEP is *subcritical*. In this case all particles end up becoming isolated and therefore the system ultimately becomes frozen, i.e. no jumps are possible anymore;
- if $K > \frac{N}{2}$ we say the FEP is *supercritical*. In this case the system never reaches a frozen configuration, but at some point each empty site becomes surrounded by particles and this property is preserved forever: we then say the system has reached its ergodic component. This is due to the fact that the dynamics can separate 2 neighbouring empty sites but not make them join each other (this would require a jump of an isolated particle).

Our main result is to show cutoff for the mixing time of a supercritical FEP on \mathbb{T}_N . More precisely, taking N to infinity the ε -mixing time (namely the time needed for a Markov chain to be ε -close to its invariant measure) does not depend at first order on ε (see Theorem 2.1). This means that the worst total variation distance as a function of time goes very abruptly from almost 1 to almost 0 around the mixing time. This type of phenomena was first found by [13, 14] in the context of card shuffling, and was since shown in many Markov chains (see for an introduction [15, Chapter 18]). We mention in particular that cutoff for the SSEP on the circle was shown in [16], as we will use some of its tools in this paper. The mixing time of FEP was

previously studied in [7] and [12]. In [7] pre-cutoff for the mixing time, and cutoff if restricted to the ergodic component were proved for FEP on the segment. For the FEP on the circle, [7] showed the mixing time is of order $N^2 \log N$ with some conditions on the initial configuration. This result was extended to any initial configuration in [12]. Moreover, [12] proves cutoff for the transience time and computes the transience time as a function of the number of particles, where the *transience time* is the time needed to reach either a frozen configuration (in the subcritical case) or the ergodic component (in the supercritical case).

An account of our main results and ideas follows. First we prove that the worst mixing time over all initial supercritical configurations exhibits cutoff, provided the number of particles K is not too close to the degenerate cases $K = N/2$ or $K = N$ (see Theorem 2.1). We also study the mixing time as a function of the number of particles K , and obtain bounds that imply cutoff in certain regimes, such as the close to critical regime (e.g. $K = \frac{N}{2} + \log N$) and the macroscopically supercritical regime (e.g. $K = \rho N$, $\frac{1}{2} < \rho < 1$), and that imply pre-cutoff in the intermediary case (e.g. $K = \frac{N}{2} + N^\alpha$, $0 < \alpha < 1$) (see Theorem 2.2). Our main tool is a new bijective mapping from the pair of an ergodic FEP and its current through the origin, to the pair of a SSEP and its current through the origin. This is a natural extension of the mapping introduced in [12]. By using this mapping together with a height function representation of the SSEP and its current, obtained adapting the work of [16], we obtain sharp results on the mixing time of FEP started in the ergodic component. Then we combine this result with the sharp results on the transience time we obtained in [12] to get our Theorems 2.1 and 2.2.

The use of mappings to study the FEP is not novel. We recall the mapping to a zero-range process [2, 10], the interpretation of the ergodic FEP as an exclusion process with objects of size 2 [9, 7], more recently a lattice path representation [7], and a mapping to a SSEP with traps [12] which is the particle system version of the latter. These processes are more convenient to study than the FEP, indeed the zero-range process and the SSEP with traps are attractive. The novelty of our mapping is that it is bijective. Let us emphasize that our mapping, besides being a key tool for the study of the mixing time, has other interesting consequences. For instance, the position of a tagged particle in the ergodic FEP can be exactly coupled to the current through the origin in an associated SSEP. More generally, the position of a tagged particle in the FEP can be deterministically mapped to the current through the origin of a SSEP with traps. This mapping would still be valid by changing jump rates, and could be extended to the case of the full integer line and perhaps the closed segment.

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Notation and conventions

In this article, we will work with three different processes, that can be mapped one to another: the Facilitated Exclusion Process (FEP), the Simple Symmetric Exclusion Process (SSEP) and the Corner-Flip Dynamics (CFD). Although, thanks to the mappings we will introduce, these three processes can be defined on the same probability space, we will use different notations for their distributions, to highlight which process we are working on. We will also choose different typical names for the configurations and the variables according to which process we are looking at. These conventions are summarised below:

Process	FEP	SSEP	CFD
Total number of sites	N	K	K
Position of a site	x	k	k
Number of particles	K	P	P uphill slopes
Typical configuration name	$\eta = (\eta_x)_{x \in \mathbb{T}_N}$	$\sigma = (\sigma_k)_{k \in \mathbb{T}_K}$	$\zeta = (\zeta_k)_{k \in \mathbb{T}_K}$
Distribution of the process	\mathbb{P}_η	\mathbf{P}_σ	\mathbb{Q}_ζ

Here are some other conventions we will use:

- We will write the elements of $\mathbb{T}_N := \mathbb{Z}/N\mathbb{Z}$ as $\{1, \dots, N\}$ so we start at 1 and finish at N .
- We will often consider intervals of sites of the periodic lattice. Throughout the paper, these intervals are considered clockwise, more precisely, for $x, y \in \mathbb{T}_N$, if $1 \leq y < x \leq N$, then $[x, y] = \{x, x+1, \dots, N\} \cup \{1, \dots, y\}$. It should be clear from context whether we are working modulo N or K , so we won't explicit this size parameter in the clockwise intervals.
- We will write $f(N) = \mathcal{O}_\varepsilon(g(N))$ to indicate that there exists a constant $C_\varepsilon > 0$, depending on ε , such that for all N , $f(N) \leq C_\varepsilon g(N)$.

2 Model and results

We study the Facilitated Exclusion Process (FEP) on the discrete circle $\mathbb{T}_N = \{1, \dots, N\}$. It is a particle system with an exclusion constraint, so on each site there is at most 1 particle. Therefore, its configurations belong to $\Gamma_N := \{0, 1\}^{\mathbb{T}_N}$, and we denote a configuration by $\eta = (\eta_x)_{x \in \mathbb{T}_N}$, where $\eta_x = 1$ if there is a particle at site x and $\eta_x = 0$ if site x is empty.

The FEP has the following dynamics: each particle tries jumping at rate 2 to the left or to the right with probability $\frac{1}{2}$ (equivalently, each jump direction is attempted at rate 1). However, the jump is cancelled if one of the two following constraints is not satisfied:

- The exclusion constraint: if the target site is already occupied, the jump is forbidden.
- The kinetic constraint: if, before the jump, the particle is isolated (meaning it has no neighbour), the jump is forbidden.

In other words, the FEP on \mathbb{T}_N is the continuous time Markov process on Γ_N with generator given by

$$\mathcal{L}_N^{\text{FEP}} f(\eta) = \sum_{x \in \mathbb{T}_N} \sum_{z = \pm 1} \eta_x \eta_{x-z} (1 - \eta_{x+z}) (f(\eta^{x, x+z}) - f(\eta)), \quad (1)$$

where

$$\eta_y^{x, x+z} = \begin{cases} \eta_y & \text{if } y \notin \{x, x+z\} \\ \eta_{x+z} & \text{if } y = x \\ \eta_x & \text{if } y = x+z, \end{cases} \quad (2)$$

which corresponds to a jump from site x to $x+z$ if $\eta_{x-z} \eta_x (1 - \eta_{x+z}) = 1$.

Furthermore, the system is conservative (the number of particles is preserved by the dynamics), so it will be convenient to study the FEP for a given number of particles K . We thus define the set of exclusion configurations on \mathbb{T}_N with K particles:

$$\Gamma_{N, K} = \{\eta \in \{0, 1\}^{\mathbb{T}_N} : |\eta| = K\}. \quad (3)$$

Depending on the number of particles K , the FEP has different long-time behaviours. If $K \leq \frac{N}{2}$ (the subcritical case), the system ends up becoming frozen: all particles are isolated and

no jumps are possible. On the other hand, if $K > \frac{N}{2}$ (the supercritical case), the system never freezes, but eventually reaches an absorbing set of configurations, the ergodic component (the set of configurations with every empty site surrounded by particles).

Therefore, the FEP is nonreversible and has transient states: in the subcritical case, they are the configurations that aren't frozen, and in the supercritical case, they are the configurations that aren't ergodic (with at least one pair of neighbouring empty sites).

In this article, we study the FEP's mixing time, and therefore focus on the regime $K \geq \frac{N}{2}$, which is the only one where a non frozen stationary state is reached. We therefore define, for $K \geq \frac{N}{2}$,

$$\mathcal{E}_{N,K} = \{\eta \in \Gamma_{N,K} : \forall x \in \mathbb{T}_N, \eta_x + \eta_{x+1} \geq 1\} \quad (4)$$

the set of ergodic configurations of the FEP. Set also

$$\mathcal{T}_{N,K} = \Gamma_{N,K} \setminus \mathcal{E}_{N,K} \quad (5)$$

the set of transient configurations of the FEP on \mathbb{T}_N with K particles. Then, the invariant law of the FEP on \mathbb{T}_N with K particles is given by

$$\pi_{N,K}^{\text{FEP}} = \mathcal{U}(\mathcal{E}_{N,K}). \quad (6)$$

For $\eta \in \Gamma_{N,K}$, we denote by $\mathbb{P}_\eta(\eta(t) \in \cdot)$ the distribution of the FEP started from η at time t . Then, for all $\varepsilon \in (0, 1)$, define the ε -mixing time

$$\tau_{N,K}^{\text{FEP}}(\varepsilon) = \inf\{t \geq 0 : \forall \eta \in \Gamma_{N,K}, d_{\text{TV}}(\mathbb{P}_\eta(\eta(t) \in \cdot), \pi_{N,K}^{\text{FEP}}) \leq \varepsilon\}. \quad (7)$$

Our main results concern the behaviour of $\tau_{N,K}^{\text{FEP}}(\varepsilon)$ as N and K go to infinity. We define here the cutoff and pre-cutoff phenomena:

Definition 1 (Cutoff and pre-cutoff). *A sequence of mixing times $(\tau_N)_{N \geq 0}$ exhibits cutoff if*

$$\forall \varepsilon \in (0, 1), \lim_{N \rightarrow \infty} \frac{\tau_N(\varepsilon)}{\tau_N(\frac{1}{4})} = 1. \quad (8)$$

A sequence of mixing times $(\tau_N)_{N \geq 0}$ exhibits pre-cutoff if there exist $C, C' > 0$ such that

$$\forall \varepsilon \in (0, 1), C \leq \liminf_{N \rightarrow \infty} \frac{\tau_N(\varepsilon)}{\tau_N(\frac{1}{4})} \leq \limsup_{N \rightarrow \infty} \frac{\tau_N(\varepsilon)}{\tau_N(\frac{1}{4})} \leq C'. \quad (9)$$

Theorem 2.1 (K -uniform cutoff for the mixing time). *Let $(a_N)_{N \geq 1}$ be a sequence such that $1 \ll a_N \ll \log \log N$. Then, for all $0 < \varepsilon < 1$, there exists C_ε such that for all N ,*

$$\left| \max_{\frac{N}{2} + a_N \leq K \leq N - a_N} \tau_{K,N}^{\text{FEP}}(\varepsilon) - \frac{1}{4\pi^2} N^2 \log(N) \right| \leq C_\varepsilon N^2 \log \log N. \quad (10)$$

This is an estimate on the mixing time starting from the worst configuration, over all possible values of K . To be precise, not all values of K are included, because for technical reasons we require $K - N/2$ and $N - K$ to go to infinity. However we still cover a wide range of configurations, as we can take (a_N) going "slowly" to infinity, for example $\forall N, a_N = \sqrt{\log \log N}$.

Interestingly, the dominant term of this worst mixing time is exactly the critical FEP's transience time from [12]. In fact, the worst mixing time is achieved in the close to critical regime, when $K - N/2$ is small, and the transience time is very long and dominates. This study of the mixing time as a function of K is the object of the following Theorem.

Theorem 2.2 (Cutoff and pre-cutoff as a function of K). For all $0 < \varepsilon < 1$, for all sequence $K = K(N)$ such that $\forall N, N/2 < K < N$, and $(2K - N) \wedge (N - K) \rightarrow +\infty$,

- If $\frac{\log(2K-N)}{\log K} \rightarrow 0$ (e.g. $K = N/2 + \log N$), then there exists C_ε such that

$$\left| \tau_{N,K}^{\text{FEP}}(\varepsilon) - \frac{1}{\pi^2} K^2 \log K \right| \leq C_\varepsilon N^2 (\log \log N) \vee (\log(2K - N)). \quad (11)$$

- If $\frac{\log(2K-N)}{\log K} \rightarrow \alpha \in (0, 1)$ (e.g. $K = N/2 + N^\alpha$), then

$$\frac{1-\alpha}{\pi^2} \vee \frac{1-\alpha/2}{8\pi^2} \leq \liminf_{N \rightarrow \infty} \frac{\tau_{N,K}^{\text{FEP}}(\varepsilon)}{K^2 \log K} \leq \limsup_{N \rightarrow \infty} \frac{\tau_{N,K}^{\text{FEP}}(\varepsilon)}{K^2 \log K} \leq \frac{1-\alpha}{\pi^2} + \frac{\alpha}{8\pi^2}. \quad (12)$$

- If $\frac{\log((2K-N) \wedge (N-K))}{\log K} \rightarrow 1$ (e.g. $K/N \rightarrow \rho \in (\frac{1}{2}, 1)$), then there exists C_ε such that

$$\left| \tau_{N,K}^{\text{FEP}}(\varepsilon) - \frac{1}{8\pi^2} K^2 \log K \right| \leq C_\varepsilon K^2. \quad (13)$$

Remark 1. In the case where $\frac{\log(N-K)}{\log K} \rightarrow 0$, for example if $K = N - \log N$, the mixing time is of order N^2 .

The FEP dynamics can be decomposed in 2 phases: first the system needs to reach the ergodic component, this is the *transience time*, then once ergodic the FEP continues evolving to become mixed, we call this the *ergodic mixing time*. The three regimes above can then be understood in the following way. When K is close to $N/2$, the transience time dominates the ergodic mixing time, and the cutoff follows from the transience time cutoff from [12]. When $K - N/2$ is for example a power of N , both times have the same order so we only obtain pre-cutoff. Last, when $K - N/2$ is for example a positive fraction of N , the ergodic mixing time dominates the transience time, and corresponds to a related SSEP's mixing time. Then the cutoff follows from adapting the proof of [16] for the cutoff of the SSEP.

The results in Theorem 2.2 are formulated in an asymptotic way, but we also have non-asymptotic estimates which are given in Section 3.

3 Structure of the proof

For $N \geq 2$ and $\frac{N}{2} < K < N$, we set for all $\varepsilon \in (0, 1)$,

$$\theta_{N,K}(\varepsilon) = \inf \{ t \geq 0 : \forall \eta \in \Gamma_{N,K}, \mathbb{P}_\eta(\eta(t) \in \mathcal{T}_{N,K}) \leq \varepsilon \}, \quad (14)$$

the transience time of the FEP on \mathbb{T}_N with K particles. We also define

$$\tau_{\mathcal{E}_{N,K}}^{\text{FEP}}(\varepsilon) = \inf \{ t \geq 0 : \forall \eta \in \mathcal{E}_{N,K}, d_{\text{TV}}(\mathbb{P}_\eta(\eta(t) \in \cdot), \pi_{N,K}^{\text{FEP}}) \leq \varepsilon \} \quad (15)$$

the mixing time of a FEP started from $\mathcal{E}_{N,K}$: it is already ergodic, whereas $\tau_{N,K}^{\text{FEP}}(\varepsilon)$ is the worst mixing time over all initial states, including transient ones.

It is not hard to see that

Proposition 3.1. For all ε, N and $N/2 < K < N$,

$$\theta_{N,K}(\varepsilon) \vee \tau_{\mathcal{E}_{N,K}}^{\text{FEP}}(\varepsilon) \leq \tau_{N,K}^{\text{FEP}}(\varepsilon) \leq \theta_{N,K} \left(\frac{\varepsilon}{2} \right) + \tau_{\mathcal{E}_{N,K}}^{\text{FEP}} \left(\frac{\varepsilon}{2} \right). \quad (16)$$

We then just need to combine estimates on the transience time from [12] and study $\tau_{\mathcal{E}_{N,K}}^{\text{FEP}}(\varepsilon)$. We already know well $\theta_{N,K}(\varepsilon)$:

Proposition 3.2 (Transience time estimate from [12, Lemma 4.5, Remark 9]). *For all ε , there exists $C_\varepsilon > 0$ such that for all N , for all $\frac{N}{2} < K < N$,*

$$-C_\varepsilon K^2 \leq \theta_{N,K}(\varepsilon) - \frac{K^2}{\pi^2} \log \frac{K}{2K-N} \leq C_\varepsilon K^2 \log \log K. \quad (17)$$

If $K = K(N)$ is such that $\frac{\log(2K-N)}{\log K} \rightarrow 1$, there exists C'_ε such that

$$\theta_{N,K}(\varepsilon) \leq C'_\varepsilon K^2 \quad (18)$$

So it remains to study $\tau_{\mathcal{E}_{N,K}}^{\text{FEP}}(\varepsilon)$.

Section 5 is devoted to proving the following result:

Proposition 3.3 (Upper bound). *For all $0 < \varepsilon < 1$, for all sequence $K = K(N)$ such that $\forall N, N/2 < K < N$, and $\min(2K - N, N - K) \rightarrow +\infty$, there exists $C_\varepsilon > 0$ such that*

$$\tau_{\mathcal{E}_{N,K}}^{\text{FEP}}(\varepsilon) \leq \frac{K^2}{8\pi^2} \log(2K - N) \wedge (N - K) + C_\varepsilon K^2 \quad (19)$$

Remark 2. *In the conditions of Proposition 3.3, by [16], the right hand side of (19) is equivalent to the ε -mixing time of a SSEP on \mathbb{T}_K with $2K - N$ particles.*

Section 6 is devoted to proving the following result:

Proposition 3.4 (Lower bound). *For all $0 < \varepsilon < 1$, $N \geq 2$ and $N/2 < K < N$,*

$$\frac{K^2}{4\pi^2} \log \frac{K}{\sqrt{(2K - N) \wedge (N - K)}} + \mathcal{O}_\varepsilon(K^2) \leq \tau_{\mathcal{E}_{N,K}}^{\text{FEP}}(\varepsilon). \quad (20)$$

Remark 3. *In the case where $K \sim \rho N$ and $\rho > \frac{1}{2}$, the left hand side of (20) corresponds to the mixing time of a SSEP on \mathbb{T}_K with $2K - N$ particles. Depending on the behaviour of K , this isn't always true.*

Proof of Theorem 2.1 and 2.2. Combining (16), (17) with Propositions 3.1,3.2,3.3 and 3.4 gives the following bounds:

$$\frac{K^2}{\pi^2} \left(\log \frac{K}{2K - N} + \frac{1}{4} \log \frac{K}{\sqrt{(2K - N) \wedge (N - K)}} \right) + \mathcal{O}_\varepsilon(K^2) \leq \tau_{N,K}^{\text{FEP}}(\varepsilon) \quad (21)$$

$$\tau_{N,K}^{\text{FEP}}(\varepsilon) \leq \frac{K^2}{\pi^2} \left(\log \frac{K}{2K - N} + \frac{1}{8} \log(2K - N) \wedge (N - K) \right) + \mathcal{O}_\varepsilon(K^2 \log \log K) \quad (22)$$

and we can remove the $\log \log K$ in (22) if $\frac{\log(2K-N)}{\log K} \rightarrow 1$. Then, Theorem 2.2 follows by distinguishing on the behaviour of $\frac{\log(2K-N)}{\log K}$, and we obtain Theorem 2.1 by noticing that the maximum of the upper bound in (22) is achieved for the smallest values of K . \square

The core of our paper will be proving Propositions 3.3 and 3.4.

4 A crucial mapping

We introduce here a mapping between the ergodic FEP and the SSEP which is a key ingredient for the proof of both Propositions 3.3 and 3.4. We define it for the ergodic FEP but it can directly be extended to all FEP configurations, that are then mapped to appropriate SSEP with traps configurations, as exploited in [12].

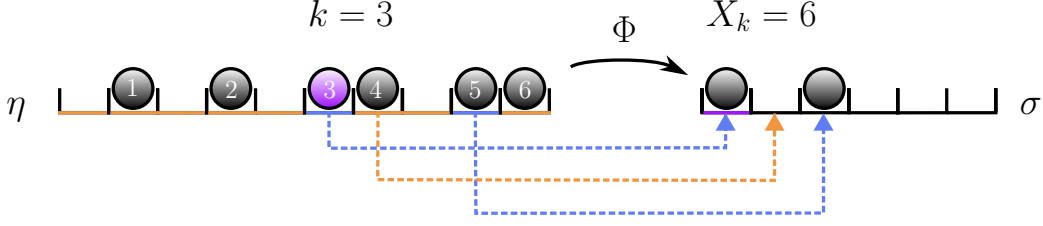


Figure 1: Illustration of the static mapping. The rank of the purple particle in η is k and its position is X_k . Each site of σ is in correspondence with a particle of η : the first site of σ is related to the k^{th} particle of η , the second site of σ to the $(k+1)^{\text{th}}$ particle of η , etc. If a particle of η is followed by another particle, it is underlined in blue in η , and there is a particle on the corresponding site of σ . If a particle of η is followed by an empty site, it is underlined in orange in η , and the corresponding site of σ is empty.

Static mapping. For the mapping to be bijective, we do not directly associate a SSEP configuration to a FEP configuration, but a couple made of the position of a tagged particle in the FEP and a SSEP configuration, to the couple made of the rank of this tagged particle in the FEP and a FEP configuration. More precisely, for all FEP configuration $\eta \in \Gamma_{N,K}$, for all $k \in \mathbb{T}_K$, set

$$x_k(\eta) = \inf\{x \in \mathbb{T}_N : \sum_{y=1}^x \eta_y = k\}, \quad (23)$$

the position of the k^{th} particle in η . Then,

Proposition 4.1 (Static mapping).

$$\Phi : \begin{cases} \mathbb{T}_K \times \mathcal{E}_{N,K} \rightarrow \mathbb{T}_N \times \Gamma_{K,2K-N} \\ (k, \eta) \mapsto (x_k(\eta), \sigma^{(k,\eta)}), \end{cases} \quad (24)$$

where $\sigma^{(k,\eta)} \in \Gamma_{K,2K-N}$ is defined by, for all $l \in \mathbb{T}_K$,

$$\sigma_l^{(k,\eta)} = 2 - x_{k+l}(\eta) + x_{k+l-1}(\eta), \quad (25)$$

is bijective.

Proof. Definition. It is easy to check that $\sigma^{(k,\eta)}$, as defined by (25), is an exclusion configuration: by ergodicity of η , $1 \leq x_j(\eta) - x_{j-1}(\eta) \leq 2$ for all $j \in \mathbb{T}_K$. To see that it has $2K - N$ particles, we can sum (25) for $1 \leq l \leq K$.

Surjectivity. Now we show that, given a couple $(x, \sigma) \in \mathbb{T}_N \times \Gamma_{K,2K-N}$, we can find a couple $(k, \eta) \in \mathbb{T}_K \times \mathcal{E}_{N,K}$ such that $\Phi(k, \eta) = (x, \sigma)$. We construct a set of sites in \mathbb{T}_N

$$I = \left\{ x + \sum_{j=1}^l (2 - \sigma_j) \pmod N, 0 \leq l \leq K - 1 \right\}, \quad (26)$$

where by convention the empty sum is zero. We set η the configuration on \mathbb{T}_N such that its occupied sites are the sites of I , ie $\eta_y = \mathbb{1}_{\{y \in I\}}$ for $y \in \mathbb{T}_N$, and finally set

$$k = \sum_{y=1}^x \eta_y. \quad (27)$$

Then, we show $\eta \in \mathcal{E}_{N,K}$. It has K particles, indeed the elements of I are distinct: for all $0 \leq j < l \leq K$,

$$x + \sum_{i=1}^l (2 - \sigma_i) - \left(x + \sum_{i=1}^j (2 - \sigma_i) \right) = \sum_{i=j+1}^l (2 - \sigma_i). \quad (28)$$

The r.h.s. of (28) is clearly positive, we now show that it is strictly less than N :

- If $l - j \leq N - K$, (28) is upper-bounded by $2(l - j) \leq 2(N - K) < N$.
- If $l - j > N - K$, notice that σ has $N - K$ empty sites, so σ has at least $l - j - (N - K)$ particles in the segment $[j+1, l]$. Hence (28) is upper-bounded by $2(l - j) - (l - j - (N - K)) = l - j + N - K < N$ because $l - j < K$.

In all cases, the difference between two positions indexed by different j and l is strictly between 0 and N , so taking modulo N the positions are still distinct, so I has cardinality K . Finally, η is ergodic because by the definition of I , the maximum distance between two consecutive particles is 2.

So $(k, \eta) \in \mathbb{T}_K \times \mathcal{E}_{N,K}$, and we just need to show that $\Phi(k, \eta) = (x, \sigma)$. First, $\sum_{y=1}^x \eta_y = k$ so $x_k(\eta) \leq x$. Then, $\eta_x = 1$ because $x \in I$, so $\forall 1 \leq y \leq x - 1$, $\sum_{z=1}^y \eta_z < k$. So $x_k(\eta) = x$. Similarly, for all $l \in \mathbb{T}_K$,

$$x_{k+l}(\eta) = x + \sum_{j=1}^l (2 - \sigma_j). \quad (29)$$

So, for all $l \in \mathbb{T}_K$,

$$\sigma_l^{(k,\eta)} = 2 - x_{k+l}(\eta) + x_{k+l-1}(\eta) = 2 - (2 - \sigma_l) = \sigma_l. \quad (30)$$

This concludes the proof that $\Phi(k, \eta) = (x, \sigma)$.

Injectivity. Let $(k, \eta), (k', \eta')$ such that $\Phi(k, \eta) = \Phi(k', \eta')$. Then,

$$x_k(\eta) = x_{k'}(\eta') \quad (31)$$

$$\forall l \in \mathbb{T}_K, \quad x_{k+l}(\eta) - x_{k+l-1}(\eta) = x_{k'+l}(\eta') - x_{k'+l-1}(\eta'). \quad (32)$$

Therefore, for all $l \in \mathbb{T}_K$, $x_{k+l}(\eta) = x_{k'+l}(\eta')$ so η and η' have the same particle positions, hence $\eta = \eta'$. Then, since $x_k(\eta) = x_{k'}(\eta')$, $k = k'$. \square

Notice that this static mapping allows to easily compute the cardinality of $\mathcal{E}_{N,K}$, which was already known by [9]:

$$|\mathcal{E}_{N,K}| = \frac{N}{K} |\Gamma_{K,2K-N}| = \frac{N}{K} \binom{K}{2K-N}. \quad (33)$$

We could thus have shown only injectivity or surjectivity and used the cardinality of $\mathcal{E}_{N,K}$ to obtain the bijectivity of Φ , but we chose this approach as it yields a nice proof of the cardinality of $\mathcal{E}_{N,K}$.

Dynamic mapping. We now consider the effect of the FEP dynamics on the mapping.

Let $(k, \eta) \in \mathbb{T}_K \times \mathcal{E}_{N,K}$, we set $k(0) = k$ and $\eta(0) = \eta$. Let $x = x_k(\eta)$, we set $X(0) = x$ and consider the joint dynamics of the FEP and a tagged particle, started from (x, η) , given by the following generator:

$$\begin{aligned} \mathcal{L}^{\text{FEP}, \text{tag}} f(x, \eta) = & \sum_{\substack{1 \leq y \leq N \\ y \neq x}} \eta_y \sum_{z \in \{-1, 1\}} \eta_{y-z} (1 - \eta_{y+z}) (f(x, \eta^{y, y+z}) - f(x, \eta)) \\ & + \eta_x \sum_{z \in \{-1, 1\}} \eta_{x-z} (1 - \eta_{x+z}) (f(x+z, \eta^{x, x+z}) - f(x, \eta)). \end{aligned} \quad (34)$$

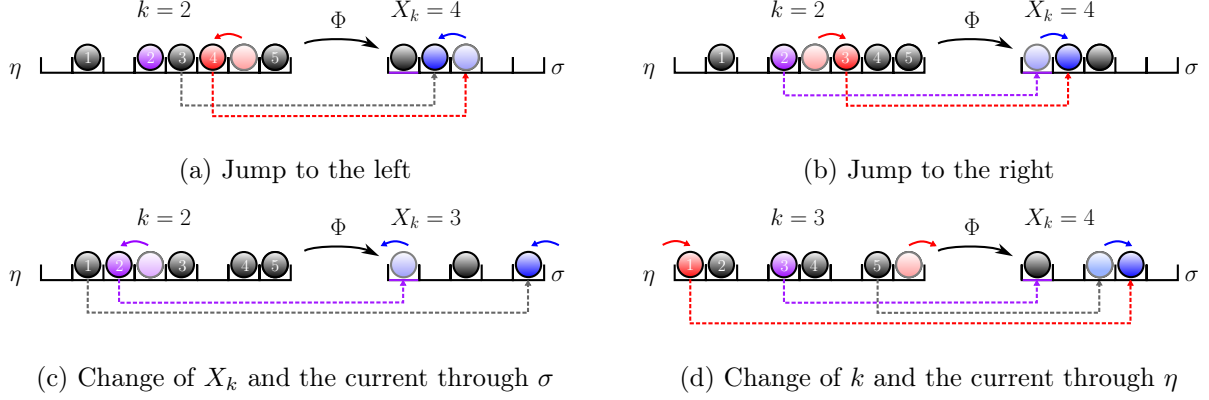


Figure 2: Different kinds of jumps. The tagged particle is coloured in purple.

We denote by $\eta(t)$ the FEP at time t and $X(t)$ the position of the tagged particle in $\eta(t)$ at time t . We also set

$$k(t) = \sum_{y=1}^{X(t)} \eta_y(t) \quad (35)$$

the rank of the particle at site $X(t)$ in $\eta(t)$, it is a deterministic function of $\eta(t)$ and $X(t)$. Defining the current through an edge as the total number of particles having crossed it from left to right minus the total number of particles having crossed it from right to left, $k(t) - k(0)$ is equal to the current through edge $(N, 1)$ from times 0 to t in $(\eta(s))_{0 \leq s \leq t}$, modulo K . Last, we set for all $t \geq 0$

$$(Y(t), \sigma(t)) = \Phi(k(t), \eta(t)). \quad (36)$$

A first observation is that for all $t \geq 0$,

$$Y(t) = X(t). \quad (37)$$

Indeed, $k(t)$ is defined as the rank of the tagged particle, whose position is $X(t)$, so it is clear that $Y(t) = x_{k(t)}(\eta(t)) = X(t)$ at all times $t \geq 0$.

Then we have the following:

Proposition 4.2 (Properties of the dynamic mapping).

1. The process $(\sigma(t))_{t \geq 0}$ is a SSEP started from $\sigma(0)$.
2. For all $t \geq 0$, $X(t) - X(0)$ is equal to the total current through edge $(K, 1)$ from times 0 to t in $(\sigma(s))_{0 \leq s \leq t}$, modulo N .

Proof. We list every possible transition of (X, η) and their effect on $(X, \sigma) = \Phi(k, \eta)$. Notice that, aside from the evolution of X , this is the same proof as the dynamic mapping in [12], but we focus here on the ergodic component. The transitions are summarised in Figure 2.

Jumps that don't affect X . We consider jumps of particles other than the k^{th} particle.

Consider a particle in η of rank $k+l \pmod K$ with $l \neq 0$, and set $y = x_{k+l}(\eta)$ its position. If it has a right neighbour and an empty site to its left, as in Figure 2a, then $\sigma_l = 1$ and $\sigma_{l-1} = 0$. Indeed, $x_{k+l+1}(\eta) = x_{k+l}(\eta) + 1$, and since η is ergodic, $x_{k+l-1}(\eta) = x_{k+l}(\eta) - 2$.

Let $\eta^{y,y-1}$ the configuration after the particle jumps to the left: then, $x_{k+l}(\eta^{y,y-1}) = x_{k+l}(\eta) - 1$, $x_{k+l+1}(\eta^{y,y-1}) = x_{k+l}(\eta^{y,y-1}) + 2$ and $x_{k+l-1}(\eta^{y,y-1}) = x_{k+l}(\eta^{y,y-1}) - 1$.

Therefore, $\Phi(k, \eta^{y,y-1}) = (Y, \sigma^{l,l-1})$ where $\sigma_{l-1}^{l,l-1} = \sigma_l = 1$ and $\sigma_l^{l,l-1} = \sigma_{l-1} = 0$.

Similarly, consider a particle in η of rank $k+l \pmod K$ with $l \neq 0$ and position y . If it has a left neighbour and an empty site to its right, as in Figure 2b, then $\sigma_{l-1} = 1$ and $\sigma_l = 0$.

Let $\eta^{y,y+1}$ the configuration after the particle jumps to the right: then, $x_{k+l}(\eta^{y,y+1}) = x_{k+l}(\eta) + 1$, $x_{k+l+1}(\eta^{y,y+1}) = x_{k+l}(\eta^{y,y+1}) + 1$ and $x_{k+l-1}(\eta^{y,y+1}) = x_{k+l}(\eta^{y,y+1}) - 2$.

Therefore, $\Phi(k, \eta^{y,y+1}) = (X, \sigma^{l-1,l})$ where $\sigma_{l-1}^{l-1,l} = \sigma_l = 0$ and $\sigma_l^{l-1,l} = \sigma_{l-1} = 1$.

Jumps that change X. If $\eta_{X+1} = 1$ and $\eta_{X-1} = 0$, like in Figure 2c, then $\sigma_1 = 1$, $\sigma_K = 0$. As before, the jump $\eta \rightarrow \eta^{X,X-1}$ induces a jump $\sigma \rightarrow \sigma^{1,K}$. Such a jump reduces the current through $(K, 1)$ in the SSEP by 1.

If $\eta_{X+1} = 0$ and $\eta_{X-1} = 1$, then $\sigma_1 = 0$, $\sigma_K = 1$. Then, the jump $\eta \rightarrow \eta^{X,X+1}$ induces a jump $\sigma \rightarrow \sigma^{K,1}$, which increases the current through $(K, 1)$ in the SSEP by 1.

In both cases, the current through $(K, 1)$ in σ has the same evolution as X .

To conclude, all of these transitions occur at rate 1, so $(\sigma(t))$ has the law of a SSEP started from $\sigma(0)$, and $(X(t))$ evolves like the current through $(K, 1)$ in this SSEP. \square

Corollary 4.3. *The trajectory of a tagged particle in an ergodic FEP can be coupled in a deterministic way to the total current through the origin in the corresponding SSEP, taken modulo N .*

Remark 4. *This can be generalised to any starting configuration for the FEP: the trajectory of a tagged particle in the FEP can be coupled in a deterministic way to the total current in an associated SSEP with traps, introduced in [12].*

For $\sigma \in \Gamma_{K,2K-N}$, we denote by $\mathbf{P}_\sigma(\sigma(t) \in \cdot)$ the distribution of the SSEP started from σ at time t . Set

$$\pi_{K,2K-N}^{\text{SSEP}} = \mathcal{U}(\Gamma_{K,2K-N}) \quad (38)$$

its invariant law. Then,

Proposition 4.4 (Stationary measures and effect of the mapping). *The process $(X(t), \sigma(t))$ is a Markov chain, with stationary law $\nu_{N,K}^{c,\text{SSEP}} := \mathcal{U}(\mathbb{T}_N) \otimes \pi_{K,2K-N}^{\text{SSEP}}$.*

Furthermore, if $(X, \sigma) \sim \mathcal{U}(\mathbb{T}_N) \otimes \pi_{K,2K-N}^{\text{SSEP}}$, then $\Phi^{-1}(X, \sigma) \sim \mathcal{U}(\mathbb{T}_K) \otimes \pi_{N,K}^{\text{FEP}} :=: \mu_{N,K}^{r,\text{FEP}}$.

Proof. It is clear that $(X(t), \sigma(t))$ is a Markov chain, of generator:

$$\mathcal{L}^{c,\text{SSEP}} f(x, \sigma) = \sum_{k=1}^{K-1} \left(f(x, \sigma^{k,k+1}) - f(x, \sigma) \right) + \left(f(x + \sigma_K - \sigma_1, \sigma^{K,1}) - f(x, \sigma) \right), \quad (39)$$

and for all (x, σ) , $\mathcal{L}^{c,\text{SSEP}} \nu_{N,K}^{c,\text{SSEP}}(x, \sigma) = 0$. Last, for any (k, η) in $\mathbb{T}_K \times \mathcal{E}_{N,K}$,

$$\nu_{N,K}^{c,\text{SSEP}}(\Phi(k, \eta)) = \frac{1}{N} \frac{1}{\binom{K}{2K-N}} = \frac{1}{K} \frac{1}{|\mathcal{E}_{N,K}|} = \mu_{N,K}^{r,\text{FEP}}(k, \eta) \quad (40)$$

\square

In the following proofs, we focus on the case where $2K - N \leq K/2$, so when we study the SSEP it has less particles than empty sites, but the exact same proofs could be done for $2K - N > K/2$ by replacing all of the $2K - N$ by $N - K$ (this would correspond to viewing the particles in the SSEP as empty sites and vice versa).

5 Proof of the upper bound

We give ourselves an initial ergodic configuration $\eta \in \mathcal{E}_{N,K}$, and an initial rank

$$k(0) \sim \mathcal{U}(\mathbb{T}_K), \quad (41)$$

independent from the rest. For all $t \geq 0$, set

$$(X(t), \sigma(t)) = \Phi(k(t), \eta(t)). \quad (42)$$

Our strategy will be to find a time such that $(X(t), \sigma(t))$ has probably been coupled with a (X', σ') whose law is close enough to $\mathcal{U}(\mathbb{T}_K) \otimes \pi_{K, 2K-N}^{\text{SSEP}}$. Then, by Proposition 4.4, $d_{\text{TV}}(\mathbb{P}_\eta(\eta(t) \in \cdot), \pi_{N,K}^{\text{FEP}})$ will be small. To lighten notation, we give a name to the number of particles $2K - N$ in σ and set

$$P = 2K - N. \quad (43)$$

5.1 The height function representation

We now introduce the representation of the couple $(X(t), \sigma(t))$ as a height function. This is a very convenient way to keep track of the joint law of the two coordinates. We inspire ourselves from [16] where this is used as a tool to develop a coupling between the SSEP and its equilibrium measure. In the case of [16], only the SSEP part is looked at, but here we will use to our advantage the fact that when height functions couple, not only the SSEP parts but also the first coordinates have coupled. We now give definitions and some useful properties from [16].

Definition 2 (Height function associated with a couple). *Let $(Y, \sigma) \in \mathbb{Z} \times \Gamma_{K,P}$. The map to a height function is defined as*

$$\Psi : \begin{cases} \mathbb{Z} \times \Gamma_{K,P} \rightarrow \mathbb{Z}^{\mathbb{T}_K} \\ (Y, \sigma) \mapsto \zeta, \end{cases} \quad (44)$$

where

$$\begin{cases} \zeta_0 = \zeta_K = -Y \\ \zeta_k = \zeta_{k-1} + \sigma_k - \frac{P}{K} \quad \forall 1 \leq k < K. \end{cases} \quad (45)$$

Notice that we allow the first coordinate to be any integer and don't take it modulo N yet, this will be useful for us to put height functions one above another. Notice also that when σ has a particle at site k , ζ goes up by $1 - \frac{P}{N}$, otherwise it goes down by $\frac{P}{N}$. Set

$$\Omega_{K,P} = \Psi(\mathbb{Z} \times \Gamma_{K,P}) \quad (46)$$

the set of possible height functions, then Ψ is bijective from $\mathbb{Z} \times \Gamma_{K,P}$ to $\Omega_{K,P}$.

We define a Markov chain on $\Omega_{K,P}$ called the corner-flip dynamics, following the definition from [16, Section 5]:

Definition 3 (Corner-flip dynamics). *For $\zeta \in \Omega_{K,P}$ and $k \in \mathbb{T}_K$, define ζ^k the configuration with a flip at k such that*

$$\begin{cases} \zeta_l^k = \zeta_l \quad \forall l \neq k \\ \zeta_k^k = \zeta_{k+1} + \zeta_{k-1} - \zeta_k. \end{cases} \quad (47)$$

Then, the corner-flip dynamics on $\Omega_{K,P}$ is the Markov chain such that for all k , ζ goes to ζ^k at rate 1 and other transitions aren't possible.

Notice that this dynamics corresponds to turning a local maximum into a local minimum and vice-versa, hence the name. Then we have the following property:

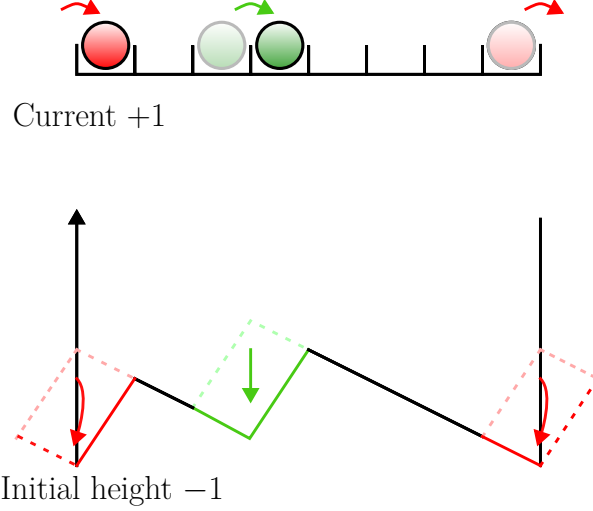


Figure 3: Illustration of the dynamic mapping and the link between current and initial height.

Proposition 5.1 (Dynamic mapping with the height function). *If $\sigma(t)$ is a SSEP started from σ , and $Y(t) - Y(0) \in \mathbb{Z}$ is the total current that went through edge $(K, 1)$ in $(\sigma(s))_{s \leq t}$, then setting $\zeta(t) = \Psi(Y(t), \sigma(t))$, $(\zeta(t))_{t \geq 0}$ follows corner-flip dynamics as defined above.*

Proof. We just need to analyse all possible transitions. It is classical that a particle jump induces a corner flip. In the particular case of a jump across the origin, if this jump increases the current, we go from a local maximum to a local minimum, so ζ_0 decreases by 1: it therefore has the same evolution as $-Y(t)$. Similarly, if a jump decreases the current, it means we go from a local minimum to a local maximum at 0, so that ζ_0 increases by 1, and again ζ_0 has the same evolution as $-Y(t)$. This is illustrated in Figure 3. \square

Proposition 5.2 (Fluctuations of the density in the SSEP). [16, Proposition 3.2] *There exists $c > 0$ such that for N big enough, for all $\sigma \in \Gamma_{K,P}$ and $s \geq 16$, if $t \geq \frac{1}{8\pi^2} K^2 \log P$,*

$$\mathbf{P}_\sigma \left(\exists k, l \in \mathbb{T}_K, \left| \sum_{j=k+1}^l \left(\sigma_j(t) - \frac{P}{K} \right) \right| \geq s\sqrt{P} \right) \leq 2 \exp(-cs^2). \quad (48)$$

Proposition 5.3 (Fluctuations of the density in the stationary SSEP). [16, Remark 4.3] *There exists $c > 0$ such that for N big enough, for all $s \geq 0$, if $\sigma \sim \pi_{K,P}^{\text{SSEP}}$,*

$$\pi_{K,P}^{\text{SSEP}} \left(\exists k, l \in \mathbb{T}_K, \left| \sum_{j=k+1}^l \left(\sigma_j - \frac{P}{K} \right) \right| \geq s\sqrt{P} \right) \leq 2 \exp(-cs^2). \quad (49)$$

Last, we will use the coupling from [16, Section 5.3] and the time needed to couple from [16, Proposition 5.3].

Proposition 5.4 (Monotonous coupling). *Let $\zeta, \zeta', \zeta'' \in \Omega_{K,P}$ such that $\zeta \leq \zeta' \leq \zeta''$. There exists a coupling \mathbb{Q} of $(\zeta(t), \zeta'(t), \zeta''(t))$ such that $\zeta(t)$ (resp. $\zeta'(t), \zeta''(t)$) has the law of corner-flip dynamics started from ζ (resp. ζ', ζ'') at time t , and $\forall t \geq 0$, $\zeta(t) \leq \zeta'(t) \leq \zeta''(t)$ \mathbb{Q} -a.s.*

Proposition 5.5 (Coupling of two height functions). *Let $\sigma \sim \pi_{K,P}^{\text{SSEP}}$, let $x \in \mathbb{Z}$ and $H \geq 0$. Let $\zeta^{(1)} = \zeta^{(1),H} = \Psi(x + H, \sigma)$ and $\zeta^{(2)} = \zeta^{(2),H} = \Psi(x - H, \sigma)$. Notice that $\zeta^{(1)} \geq \zeta^{(2)}$. For all $s > 0$ and $0 < \varepsilon < 1$, there exists $C(s, \varepsilon) > 0$ such that if $H \leq s\sqrt{P}$, for all $t \geq C(s, \varepsilon)K^2$,*

$$\mathbb{Q}_{(\zeta^{(1)}, \zeta^{(2)})}(\zeta^{(1)}(t) \neq \zeta^{(2)}(t)) \leq \varepsilon, \quad (50)$$

using the same coupling as before.

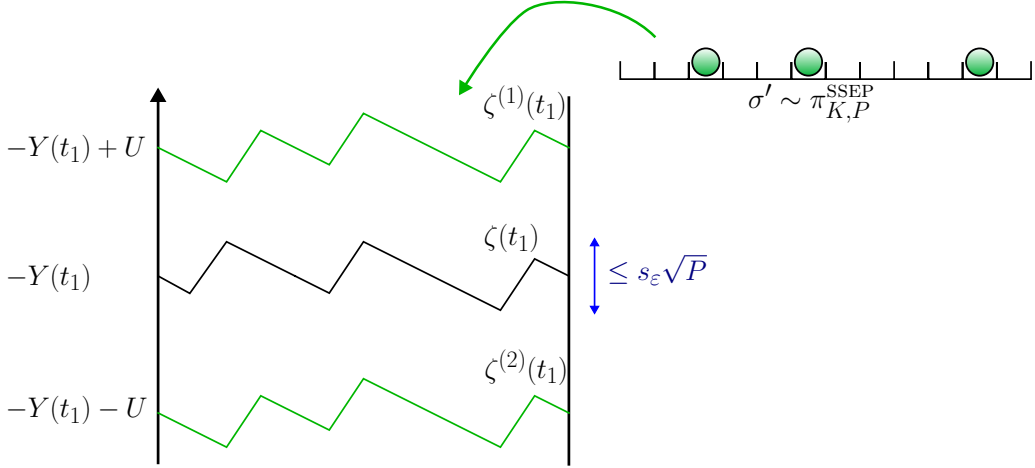


Figure 4: Summary of the coupling strategy. The maximum height difference of ζ is controlled by Proposition 5.2. This gives us the inequality $\zeta^{(2)} \leq \zeta \leq \zeta^{(1)}$. Lemma 5.8 tells us the initial heights of $\zeta^{(1)}$ and $\zeta^{(2)}$ modulo N are close to being uniform. So when all height functions are coupled, $(X(t), \sigma(t))$ is close to $\mathcal{U}(\mathbb{T}_N) \otimes \pi_{K,P}^{\text{SSEP}}$.

5.2 Coupling with near-equilibrium

The idea of the coupling is summarised in Figure 4. Recall we consider, for all t , $(X(t), \sigma(t))$ where $X(t) \in \mathbb{T}_N$ and $X(t) - X(0)$ gives the total current (modulo N) that went through edge $(K, 1)$ in $(\sigma(s))_{s \leq t}$. It will actually be more convenient for our purpose to work with $(Y(t), \sigma(t))$, where $Y(0) = X(0)$ and for all t , $Y(t) - Y(0) \in \mathbb{Z}$ is the total current through the origin in $(\sigma(s))_{s \leq t}$, not modulo N . It is then easy to go back to $X(t)$ by taking $Y(t)$ modulo N .

Now, for all t , we set

$$\zeta(t) = \Psi(Y(t), \sigma(t)) \quad (51)$$

the associated height function. Since we considered the current in \mathbb{Z} and not modulo N , we can use the dynamic mapping with the height function from Proposition 5.1.

We set $t_1 = \frac{1}{8\pi^2} K^2 \log P$. By Proposition 5.2 we can give ourselves $s = s_\varepsilon$ such that

$$\mathbf{P}_\sigma \left(\exists k, l, \left| \sum_{j=k}^l \left(\sigma_j(t_1) - \frac{P}{K} \right) \right| \geq s\sqrt{P} \right) \leq \varepsilon/4. \quad (52)$$

Notice that this is equivalent to

$$\mathbb{Q} \left(\max_{k, l \in \mathbb{T}_K} \zeta_k(t_1) - \zeta_l(t_1) > s\sqrt{P} \right) \leq \varepsilon/4. \quad (53)$$

Definition 4. Take $\sigma' \sim \pi_{K,P}^{\text{SSEP}}$ independent of the rest, $s' = \frac{2s}{\varepsilon}$ and $U \sim \mathcal{U} \left(\left[\lceil s'\sqrt{P} \rceil, 2\lceil s'\sqrt{P} \rceil \right] \right)$ independent from the rest. Then we set

$$\zeta^{(1)}(t_1) = \Psi(Y(t_1) - U, \sigma') \quad \text{and} \quad \zeta^{(2)}(t_1) = \Psi(Y(t_1) + U, \sigma'). \quad (54)$$

Lemma 5.6 (Inequality between height functions). *With probability greater than $1 - \varepsilon$,*

$$\zeta^{(2)}(t_1) \leq \zeta(t_1) \leq \zeta^{(1)}(t_1). \quad (55)$$

Proof. For any height function ξ , we set $\Delta(\xi) = \max_{k, l \in \mathbb{T}_K} \xi_k(t) - \xi_l(t)$. Since $s' \geq 2s_\varepsilon$,

$$\mathbb{Q} \left(\zeta(t_1) \leq \zeta^{(1)}(t_1) \right) \geq \mathbb{Q} \left(\Delta(\zeta(t_1)) < s\sqrt{P}, \Delta(\zeta^{(1)}(t_1)) < s\sqrt{P} \right). \quad (56)$$

So, $\mathbb{Q}(\zeta(t_1) \not\leq \zeta^{(1)}(t_1)) \leq \mathbb{Q}(\Delta(\zeta(t_1)) \geq s\sqrt{P}) + \mathbb{Q}(\Delta(\zeta^{(1)}(t_1)) \geq s\sqrt{P}) \leq \frac{\varepsilon}{2}$ by (53), and we conclude similarly for $\zeta^{(2)}(t_1)$. \square

Now, for $t \geq t_1$ we make the height functions evolve simultaneously under the corner-flip dynamics with the coupling \mathbb{Q} .

Lemma 5.7 (Time to merge). *There exists $C(s', \varepsilon)$ such that, setting $t_2 = C(s', \varepsilon)K^2$,*

$$\mathbb{Q}\left(\zeta^{(1)}(t_1 + t_2) = \zeta^{(2)}(t_1 + t_2) = \zeta(t_1 + t_2)\right) \geq 1 - 2\varepsilon \quad (57)$$

Proof. This follows from applying Markov property at time t_1 and combining Lemma 5.6 and Propositions 5.4 and 5.5. \square

Lemma 5.8 (Distribution of the initial height). *For $i \in \{1, 2\}$, set $X^{(i)}(t_1) = -\zeta_0^{(i)}(t_1) \pmod N$. Then,*

$$d_{\text{TV}}\left(\mathbb{Q}(X^{(i)}(t_1) \in \cdot), \mathcal{U}(\mathbb{T}_N)\right) \leq \varepsilon. \quad (58)$$

This result is more difficult and the proof is postponed to the end of the Section. In the meantime, we will explain how the combination of these results leads to an upper-bound on the FEP mixing time.

Lemma 5.9. *For all $t \geq t_1$, set $(Y^{(i)}(t), \sigma^{(i)}(t)) = \Psi^{-1}(\zeta^{(i)}(t))$, and recall that $X^{(i)}(t) = Y^{(i)}(t) \pmod N$. Then, $\forall t \geq t_1$,*

$$d_{\text{TV}}\left(\mathbb{Q}\left((X^{(i)}, \sigma^{(i)})(t) \in \cdot\right), \mathcal{U}(\mathbb{T}_N) \otimes \pi_{K,P}^{\text{SSEP}}\right) \leq \varepsilon. \quad (59)$$

Proof. By Lemma 5.8 and the independence of $X^{(i)}(t_1)$ and $\sigma^{(i)}(t_1)$ (see Definition 4), we have $d_{\text{TV}}\left(\mathbb{Q}\left((X^{(i)}, \sigma^{(i)})(t_1) \in \cdot\right), \mathcal{U}(\mathbb{T}_N) \otimes \pi_{K,P}^{\text{SSEP}}\right) \leq \varepsilon$. Then, under \mathbb{Q} , $(X^{(i)}(t), \sigma^{(i)}(t))_{t \geq t_1}$ is distributed as the joint law of a SSEP and its current started from $(X^{(i)}(t_1), \sigma^{(i)}(t_1))$. Since this is a Markov chain, the total variation distance to its invariant measure decreases. \square

Combining all the previous results, we obtain

Proposition 5.10 (Distance of $(X(t), \sigma(t))$ to equilibrium). *For all $t \geq t_1 + t_2$,*

$$d_{\text{TV}}\left(\mathbf{P}_{(X(0), \sigma)}\left((X(t), \sigma(t)) \in \cdot\right), \mathcal{U}(\mathbb{T}_N) \otimes \pi_{K,P}^{\text{SSEP}}\right) \leq 3\varepsilon. \quad (60)$$

Proof. Let $A \subset \mathbb{T}_N \times \Gamma_{K,P}$ and recall $\nu_{N,K}^{c, \text{SSEP}} = \mathcal{U}(\mathbb{T}_N) \otimes \pi_{K,P}^{\text{SSEP}}$.

$$\begin{aligned} \mathbf{P}_{(X, \sigma)}\left((X(t), \sigma(t)) \in A\right) &\leq \mathbf{P}_{(X, \sigma)}\left((X^{(1)}(t), \sigma^{(1)}(t)) \in A\right) + \mathbb{Q}\left(\zeta^{(1)}(t) \neq \zeta(t)\right) \\ &\leq \nu_{N,K}^{c, \text{SSEP}}(A) + \varepsilon + 2\varepsilon \end{aligned} \quad (61)$$

\square

We conclude the proof of Proposition 3.3, by showing that

$$\forall t \geq \frac{K^2}{8\pi^2} \log P + C_\varepsilon K^2, \quad d_{\text{TV}}\left(\mathbb{P}_\eta(\eta(t) \in \cdot), \pi_{N,K}^{\text{FEP}}\right) \leq \varepsilon. \quad (62)$$

Let $A \subset \mathcal{E}_{N,K}$ and $t \geq t_1 + t_2(\varepsilon/3) = \frac{K^2}{8\pi^2} \log P + C(s'_{\varepsilon/3}, \varepsilon/3)K^2$, with $C(s, \varepsilon)$ from Proposition 5.5. Let $(X^U, \sigma^U) \sim \mathcal{U}(\mathbb{T}_N) \otimes \pi_{K,P}^{\text{SSEP}}$ and $(k^U, \eta^U) \sim \mathcal{U}(\mathbb{T}_K) \otimes \pi_{N,K}^{\text{FEP}}$, recall by Proposition 4.4

that $\Phi(k^U, \eta^U)$ has the same law as (X^U, σ^U) .

$$\begin{aligned}
\mathbb{P}_\eta(\eta(t) \in A) &= \mathbb{P}_{(k(0), \eta)}((k(t), \eta(t)) \in \mathbb{T}_K \times A) \\
&= \mathbf{P}_{(X(0), \sigma)}((X(t), \sigma(t)) \in \Phi(\mathbb{T}_K \times A)) \\
&\leq \mathbf{P}((X^U, \sigma^U) \in \Phi(\mathbb{T}_K \times A)) + \varepsilon \\
&= \mathbb{P}((k^U, \eta^U) \in \mathbb{T}_K \times A) + \varepsilon \\
&= \pi_{N, K}^{\text{FEP}}(A) + \varepsilon,
\end{aligned} \tag{63}$$

which concludes.

5.3 Distribution of the initial height

We now prove Lemma 5.8. We will only show (58) for $X^{(2)}(t_1) = Y(t_1) + U \pmod N$, since the proof is the same for $Y(t_1) - U$. We will need this useful property:

Proposition 5.11 (Distribution of the first coordinates). *If $k(0) \sim \mathcal{U}(\mathbb{T}_K)$ and is independent from $(\eta(t))_{t \geq 0}$, then $\forall t \geq 0$,*

- $k(t) \sim \mathcal{U}(\mathbb{T}_K)$ and is independent from $\eta(t)$.
- Conditionally on $\eta(t)$, $X(t)$ is uniformly distributed over the occupied sites of $\eta(t)$.

Let $x \in \mathbb{T}_N$. Let $U \sim \mathcal{U}(\llbracket \lceil s'\sqrt{P} \rceil, 2\lceil s'\sqrt{P} \rceil \rrbracket)$ independent of $(k(t_1), \eta(t_1))$. Recall that we consider segments as clockwise modulo N intervals (or modulo K in the context of \mathbb{T}_K).

$$\begin{aligned}
\mathbb{P}(Y(t_1) + U = x \pmod N) &= \mathbb{P}(X(t_1) + U = x \pmod N) \\
&= \sum_{u=\lceil s'\sqrt{P} \rceil}^{2\lceil s'\sqrt{P} \rceil} \sum_{k=1}^K \mathbb{P}(k(t) = k, U = u, x_k(\eta(t_1)) = x - u \pmod N)
\end{aligned}$$

then since $U \perp\!\!\!\perp k(t_1)$ and $(U, k(t_1)) \perp\!\!\!\perp \eta(t_1)$,

$$\begin{aligned}
&= \frac{1}{K} \frac{1}{\lceil s'\sqrt{P} \rceil} \sum_{u=\lceil s'\sqrt{P} \rceil}^{2\lceil s'\sqrt{P} \rceil} \sum_{k=1}^K \mathbb{P}(x_k(\eta(t_1)) = x - u \pmod N) \\
&= \frac{1}{K} \frac{1}{\lceil s'\sqrt{P} \rceil} \sum_{u=\lceil s'\sqrt{P} \rceil}^{2\lceil s'\sqrt{P} \rceil} \mathbb{P}(\eta_{x-u}(t_1) = 1) \\
&= \frac{1}{K} \frac{1}{\lceil s'\sqrt{P} \rceil} \mathbb{E}[|\eta_{\llbracket x-2\lceil s'\sqrt{P} \rceil, x-\lceil s'\sqrt{P} \rceil \rrbracket}(t_1)|] \\
&= \frac{1}{K} \frac{1}{|I|} \mathbb{E}[|\eta_I(t_1)|],
\end{aligned} \tag{64}$$

where $I = \llbracket x - 2\lceil s'\sqrt{P} \rceil, x - \lceil s'\sqrt{P} \rceil \rrbracket$.

Proposition 5.12 (Link between particle density in the SSEP and the FEP). *For all $s, P \geq 1$,*

$$\begin{aligned}
\mathbb{P}_\eta \left(\exists \text{ a segment } I \subset \mathbb{T}_N, \left| |\eta_I(t)| - \frac{K}{N}|I| \right| \geq s \frac{K}{N} \sqrt{P} \right) \\
\leq \mathbf{P}_\sigma \left(\exists k, l, \left| \sum_{j=k}^{l+1} \left(\sigma_j(t_1) - \frac{P}{K} \right) \right| \geq s\sqrt{P} - 1 \right)
\end{aligned} \tag{65}$$

Proof. Notice that a segment J of size j in $\sigma(t_1)$ corresponds to j consecutive particles in $\eta(t_1)$, which are contained in a segment of size $\sum_{k \in J} 2 - \sigma_k(t_1)$. Indeed, if there is a particle at one site in $\sigma(t_1)$, it means there is no space between the corresponding particle in $\eta(t_1)$ and its right neighbour, so this corresponding particle occupies a space of 1 in $\eta(t_1)$. If there is no particle at a site of $\sigma(t_1)$, it means the corresponding particle in $\eta(t_1)$ is followed by an empty site in $\eta(t_1)$, so it occupies 2 spaces in $\eta(t_1)$.

So given a segment J in the SSEP, there is a corresponding segment in the FEP, starting with a particle, of length $\sum_{k \in J} (2 - \sigma_j(t_1))$. Now, replacing P by $2K - N$, notice that:

$$\begin{aligned} \mathbf{P}_\sigma \left(\exists k, l : \left| \sum_{j=k}^{l+1} \left(\sigma_j(t_1) - \frac{2K - N}{K} \right) \right| \geq s\sqrt{P} \right) \\ = \mathbf{P}_\sigma \left(\exists k, l, \left| \frac{N}{K}(l - k + 2) - \sum_{j=k}^{l+1} (2 - \sigma_j(t_1)) \right| \geq s\sqrt{P} \right), \end{aligned} \quad (66)$$

which is therefore the probability that in the FEP, there exists a segment I starting with a particle such that $|\eta_I(t_1)| - \frac{K}{N}|I| \geq s\frac{K}{N}\sqrt{P}$.

If there exists a segment $I = [x, y]$ such that $\eta_x = 0$ and $|\eta_I(t_1)| - \frac{K}{N}|I| \geq s\frac{K}{N}\sqrt{P}$ holds, then for $I' = [x + 1, y]$, we have $|\eta_{I'}(t_1)| - \frac{K}{N}|I'| \geq \frac{K}{N}(s\sqrt{P} - 1)$ and $\eta_{x+1} = 1$. So,

$$\begin{aligned} & \mathbb{P}_\eta \left(\exists \text{ a segment } I, \left| |\eta_I(t_1)| - \frac{K}{N}|I| \right| \geq s\frac{K}{N}\sqrt{P} \right) \\ & \leq \mathbb{P}_\eta \left(\exists \text{ a segment } I \text{ starting with a particle, } \left| |\eta_I(t_1)| - \frac{K}{N}|I| \right| \geq \frac{K}{N}(s\sqrt{P} - 1) \right) \\ & = \mathbf{P}_\sigma \left(\exists \text{ a segment } J \subset \mathbb{T}_K, \left| \sum_{k \in J} \left(\sigma_k(t_1) - \frac{P}{K} \right) \right| \geq s\sqrt{P} - 1 \right). \end{aligned} \quad (67)$$

□

Remark 5. The same result can be shown when considering $\sigma' = 1 - \sigma$ and $P' = N - K$, to prove the upper bound when $N - K \leq 2K - N$.

Let s such that $\mathbf{P}_\sigma \left(\exists k, l : \left| \sum_{j=k}^{l+1} \left(\sigma_j(t_1) - \frac{P}{K} \right) \right| \geq s\sqrt{P} - 1 \right) \leq \frac{\varepsilon}{4}$. Then, the probability that $|\eta_I(t_1)| \leq |I|\frac{K}{N} + s\frac{K}{N}\sqrt{P}$ is greater than $1 - \frac{\varepsilon}{4}$.

So, setting $I = [x - 2\lceil s'\sqrt{P} \rceil, x - \lceil s'\sqrt{P} \rceil]$, since $|\eta_I(t_1)| \leq |I|$ we have

$$\begin{aligned} \mathbb{P}(X(t_1) + U = x) &= \frac{1}{K} \frac{1}{|I|} \mathbb{E} [|\eta_I(t_1)|] \\ &\leq \frac{1}{K} \frac{1}{|I|} \left(|I|\frac{K}{N} + s\frac{K}{N}\sqrt{P} + \frac{\varepsilon}{4}|I| \right) \\ &= \frac{1}{N} + \frac{1}{N} \frac{s\sqrt{P}}{|I|} + \frac{\varepsilon}{4K} \\ &\leq \frac{1}{N} + \frac{1}{N} \frac{s}{s'} + \frac{\varepsilon}{2N} \end{aligned} \quad (68)$$

Setting $s' = \frac{2}{\varepsilon}s$, we have $\mathbb{P}(X(t_1) + U = x) \leq \frac{1}{N} + \frac{\varepsilon}{N}$, so $d_{\text{TV}}(\mathbb{P}(X(t_1) + U \in \cdot), \mathcal{U}(\mathbb{T}_N)) \leq \varepsilon$.

6 Proof of the lower bound

In this Section, we will also assume that $2K - N \leq N - K$ and use the shorthand $P = 2K - N$. To show the lower bound for $2K - N > N - K$, simply replace σ by $\sigma' = 1 - \sigma$ and P by $P' = N - K$.

We choose a specific initial configuration

$$\eta = \underbrace{\bullet \bullet \dots \bullet \bullet \bullet \bullet \bullet \bullet}_{2K-N} \underbrace{\circ \circ \dots \circ \circ \circ \circ \circ \circ}_{2(N-K)}. \quad (69)$$

We wish to show that for $t < \frac{1}{4\pi^2} K^2 \log \frac{K}{\sqrt{P}} - C_\varepsilon K^2$, there is an event A that has small probability under the invariant measure but there is a big probability that $\eta(t) \in A$. More precisely, for $s > 0$, set

$$A_s = \left\{ \eta' \in \mathcal{E}_{N,K} : \exists x, y, \left| |\eta|_{[x,y-1]} - \frac{K}{N}(y-x) \right| \geq s\sqrt{P} \right\}. \quad (70)$$

Then, we will show that

Lemma 6.1. *For all $0 < \varepsilon < 1$, there exist $C_\varepsilon > 0$ and $s = s_\varepsilon > 0$ such that for all $t \leq \frac{K^2}{4\pi^2} \log \frac{K}{\sqrt{P}} - C_\varepsilon K^2$,*

$$\pi_{N,K}^{\text{FEP}}(A_s) < \varepsilon' \quad (71)$$

$$\mathbb{P}_\eta(\eta(t) \in A_s) > \varepsilon + \varepsilon', \quad (72)$$

where $\varepsilon' = \min(\varepsilon, \frac{1-\varepsilon}{2})$.

To show (71), we can use Proposition 5.12, taking t to infinity, to relate $\pi_{N,K}^{\text{FEP}}(A_s)$ to the probability a similar event for the SSEP under its invariant measure. Then, using Proposition 5.2, we can find a $s = s_\varepsilon$, such that $\pi_{N,K}^{\text{FEP}}(A_s) \leq \varepsilon'$.

Now, to show (72), we will use this result, in the spirit of Proposition 5.12, but giving a lower bound:

Proposition 6.2 (Relating fluctuations in the FEP to fluctuations in the SSEP, lower bound).

$$\begin{aligned} \mathbb{P}_\eta \left(\exists \text{ a segment } I \subset \mathbb{T}_N, \left| |\eta|_I(t) - \frac{K}{N}|I| \right| \geq s \frac{K}{N} \sqrt{P} \right) \\ \geq \mathbf{P}_\sigma \left(\exists k, l, \left| \sum_{j=k}^{l+1} \left(\sigma_j(t) - \frac{P}{K} \right) \right| \geq s\sqrt{P} \right). \end{aligned} \quad (73)$$

Proof. We start from the same observation as in the proof of Proposition 5.12: in the FEP, there exists a segment I starting with a particle that contains m particles iff in the corresponding SSEP, there exists a segment J of size m such that $\sum_{k \in J} (2 - \sigma_k) = |I|$. So,

$$\begin{aligned} \left| |\eta|_I(t) - \frac{K}{N}|I| \right| &= \left| |J| - \frac{K}{N} \sum_{k \in J} (2 - \sigma_k(t)) \right| \\ &= \frac{K}{N} \left| \sum_{k \in J} \left(\frac{N}{K} - 2 + \sigma_k(t) \right) \right| \\ &= \frac{K}{N} \left| \sum_{k \in J} \left(\sigma_k(t) - \frac{2K - N}{K} \right) \right|. \end{aligned} \quad (74)$$

So,

$$\begin{aligned}
& \mathbb{P}_\eta \left(\exists \text{ a segment } I, \left| |\eta_I(t)| - \frac{K}{N}|I| \right| \geq s \frac{K}{N} \sqrt{P} \right) \\
& \geq \mathbb{P}_\eta \left(\exists \text{ a segment } I, \eta_{\inf I}(t) = 1, \left| |\eta_I(t)| - \frac{K}{N}|I| \right| \geq s \frac{K}{N} \sqrt{P} \right) \\
& = \mathbf{P}_\sigma \left(\exists \text{ a segment } J, \left| \sum_{k \in J} \sigma_k(t) - \frac{P}{K} \right| \geq s \sqrt{P} \right), \tag{75}
\end{aligned}$$

which concludes. \square

Remark 6. As mentioned in Remark 5, the same result can be shown for $\sigma' = 1 - \sigma$ and $P' = N - K$, which is the setting when $N - K \geq 2K - N$.

We now show (72). Recall that η is defined in (69), we set $k(0) = 1$ and for all $t \geq 0$,

$$(X(t), \sigma(t)) = \Phi(k(t), \eta(t)), \tag{76}$$

where Φ is the mapping defined in (24). In particular, we have $\sigma(0) = \mathbb{1}_{[1, P]}$. We set $J = [1, P]$. Then, we will study $\mathbb{E}_\sigma[|\sigma_J(t)|] - \frac{P}{K}|J|$.

For all $k \in \mathbb{T}_K$, set $u_k(t) = \mathbb{E}_\sigma[\sigma_k(t)]$. By Kolmogorov equation, we have for all k and t

$$\frac{du_k}{dt}(t) = \Delta u_k(t), \tag{77}$$

with Δ the discrete Laplacian, such that $\Delta u_k(t) = u_{k-1}(t) - 2u_k(t) + u_{k+1}(t)$ and $u(0) = \mathbb{1}_J$.

Set $\phi_0 : k \mapsto 1$ and for all $1 \leq l \leq \lfloor K/2 \rfloor$ and $0 \leq k \leq K - 1$

$$\phi_l(k) = \sqrt{2} \cos \frac{2\pi l k}{K} \tag{78}$$

$$\psi_l(k) = \sqrt{2} \sin \frac{2\pi l k}{K}. \tag{79}$$

Then

$$\Delta \phi_0 = 0 \tag{80}$$

and for all $1 \leq l \leq \lfloor K/2 \rfloor$,

$$\Delta \phi_l = -\mu_l \phi_l \tag{81}$$

$$\Delta \psi_l = -\mu_l \psi_l, \tag{82}$$

with

$$\mu_l = 2 \left(1 - \cos \frac{2\pi l}{K} \right) \tag{83}$$

The $(\phi_l)_{0 \leq l \leq \lfloor K/2 \rfloor}$ and $(\psi_l)_{1 \leq l \leq \lfloor \frac{K-1}{2} \rfloor}$ form an orthonormal basis for the following scalar product:

$$\langle f, g \rangle = \frac{1}{K} \sum_{k=0}^{K-1} f(k)g(k). \tag{84}$$

We set $\chi_0 = \phi_0$, $\lambda_0 = 0$ and for $0 \leq k \leq \lfloor K/2 \rfloor - 1$, $\chi_{2k+1} = \phi_k$, $\chi_{2k+2} = \psi_k$, $\lambda_{2k+1} = \lambda_{2k+2} = \mu_k$. Note that $\psi_{K/2} = \chi_K = 0$ if K is even, so our orthonormal basis is given by the $(\chi_k)_{0 \leq k \leq K-1}$. Then, denoting by $c_l = \langle \chi_l, u(0) \rangle$, we have for all t

$$u(t) = \sum_{l=0}^{K-1} c_l e^{-\lambda_l t} \chi_l \tag{85}$$

$$= \frac{P}{K} \mathbb{1}_{\mathbb{T}_K} + \sum_{l=1}^{K-1} c_l e^{-\lambda_l t} \chi_l. \tag{86}$$

Then, for all t ,

$$\begin{aligned}
\mathbb{E}_\sigma[|\sigma_{|J}(t)|] &= K \langle u(t), u(0) \rangle \\
&= K \sum_{l=0}^{K-1} c_l^2 e^{-\lambda_l t} \\
&= \frac{P^2}{K} + K \sum_{l=1}^{K-1} c_l^2 e^{-\lambda_l t}.
\end{aligned} \tag{87}$$

Therefore,

$$K c_1^2 e^{-\lambda_1 t} \leq \mathbb{E}_\sigma[|\sigma_{|J}(t)|] - \frac{P^2}{K} \leq e^{-\lambda_1 t} K \sum_{l=1}^{K-1} c_l^2, \tag{88}$$

and since $\sum_{l=1}^{K-1} c_l^2 \leq \langle u(0), u(0) \rangle = \frac{P}{K}$, for all t ,

$$\frac{1}{\lambda_1} \log \frac{K c_1^2}{\mathbb{E}_\sigma[|\sigma_{|J}(t)|] - \frac{P^2}{K}} \leq t \leq \frac{1}{\lambda_1} \log \frac{P}{\mathbb{E}_\sigma[|\sigma_{|J}(t)|] - \frac{P^2}{K}}. \tag{89}$$

By Cantelli inequality, for $\lambda < 0$,

$$\mathbf{P}_\sigma(|\sigma_{|J}(t)| - \mathbb{E}_\sigma[|\sigma_{|J}(t)|] \geq \lambda) \geq \frac{\lambda^2}{\lambda^2 + \text{Var}(|\sigma_{|J}(t)|)}. \tag{90}$$

Using negative dependence (see for example [17, Lemma 4]), this becomes

$$\mathbf{P}_\sigma(|\sigma_{|J}(t)| - \mathbb{E}_\sigma[|\sigma_{|J}(t)|] \geq \lambda) \geq \frac{\lambda^2}{\lambda^2 + \mathbb{E}_\sigma[|\sigma_{|J}(t)|]}. \tag{91}$$

Let $s' \geq s_\varepsilon$ such that $(s' - s_\varepsilon)^2 > \frac{\varepsilon + \varepsilon'}{1 - \varepsilon - \varepsilon'} (1 + s')$. Then, since $\frac{P}{K} \leq 1$ and $\frac{1}{\sqrt{P}} \leq 1$,

$$\frac{P}{K} + \frac{s'}{\sqrt{P}} < \left(\frac{1}{\varepsilon + \varepsilon'} - 1 \right) (s' - s_\varepsilon)^2, \tag{92}$$

so that

$$\frac{1}{1 + \frac{1}{(s' - s_\varepsilon)^2} \left(\frac{P}{K} + \frac{s'}{\sqrt{P}} \right)} > \varepsilon + \varepsilon'. \tag{93}$$

So, if we choose t^* such that $\mathbb{E}_\sigma[|\sigma_{|J}(t^*)|] = \frac{P^2}{K} + s' \sqrt{P}$, and set $\lambda = -(s' - s_\varepsilon) \sqrt{P}$,

$$\begin{aligned}
\mathbf{P}_\sigma(|\sigma_{|J}(t^*)| - \mathbb{E}_\sigma[|\sigma_{|J}(t^*)|] \geq \lambda) &= \mathbf{P}_\sigma \left(|\sigma_{|J}(t^*)| \geq \frac{P^2}{K} + s' \sqrt{P} - (s' - s_\varepsilon) \sqrt{P} \right) \\
&= \mathbf{P}_\sigma \left(|\sigma_{|J}(t^*)| \geq \frac{P^2}{K} + s_\varepsilon \sqrt{P} \right) \\
&\geq \frac{\lambda^2}{\lambda^2 + \frac{P^2}{K} + s' \sqrt{P}} \text{ by (91)} \\
&= \frac{1}{1 + \frac{1}{(s' - s_\varepsilon)^2} \left(\frac{P}{K} + \frac{s'}{\sqrt{P}} \right)} \\
&> \varepsilon + \varepsilon'.
\end{aligned} \tag{94}$$

We now estimate the corresponding time t^* :

$$\frac{1}{\lambda_1} \log \frac{K c_1^2}{s' \sqrt{P}} \leq t^* \leq \frac{1}{\lambda_1} \log \frac{P}{s' \sqrt{P}} \tag{95}$$

Since $\frac{1}{\lambda_1} = \frac{K^2}{4\pi^2} + \mathcal{O}(1)$ and c_1 doesn't go to zero, for all $t \leq \frac{K^2}{4\pi^2} \log \frac{K}{\sqrt{P}} + \mathcal{O}_\varepsilon(K^2)$, $t \leq \tau_{\mathcal{E}_{N,K}}^{\text{FEP}}(\varepsilon)$.

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