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## K3 SURFACES AND WEYL GROUP OF TYPE $E_6$

by

Cédric Bonnafé

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**Abstract.** — Adapting methods of previous papers by A. Sarti and the author, we construct K3 surfaces from invariants of the Weyl group of type  $E_6$ . We study in details one of these surfaces, which turns out to have Picard number 20: for this example, we describe an elliptic fibration (and its singular fibers) and the Picard lattice.

This paper can be seen as a continuation of previous works by A. Sarti and the author on the construction of *singular* K3 surfaces by using invariants of finite reflection groups [BoSa1, BoSa3]. We consider here quotients of complete intersections defined by fundamental invariants of a group of rank 6, namely the Weyl group of type  $E_6$ .

Let  $W$  be the Weyl group of type  $E_6$  acting on  $V = \mathbb{C}^6$  and let  $W'$  denote its derived subgroup (which has index 2 in  $W$  and is equal to  $W \cap \mathbf{SL}_{\mathbb{C}}(V)$ ). Then the algebra  $\mathbb{C}[V]^{W'}$  of polynomial functions on  $V$  invariant under the action of  $W'$  is generated by six homogeneous and algebraically independent polynomials  $f_2, f_5, f_6, f_8, f_9$  and  $f_{12}$  (of respective degrees 2, 5, 6, 8, 9 and 12).

We denote by  $\mathcal{X} \subset \mathbb{P}(V)$  the surface defined by  $f_2 = f_6 = f_8 = 0$  and, for  $\lambda, \mu \in \mathbb{C}^2$ , we denote by  $\mathcal{Y}_{\lambda, \mu}$  the surface defined by  $f_5 = f_6 + \lambda f_2^3 = f_8 + \mu f_2^4 = 0$ . It turns out that  $\mathcal{X}$  is smooth and that  $\mathcal{Y}_{\lambda, \mu}$  is smooth for generic values of  $(\lambda, \mu)$ . Our first main result in this paper is the following (see Theorem 3.2):

### Theorem A.

- (a) *The minimal resolution of the singular surface  $\mathcal{X}/W'$  is a smooth K3 surface.*
- (b) *If  $\lambda, \mu$  be such that  $\mathcal{Y}_{\lambda, \mu}$  has at most ADE singularities<sup>(\*)</sup>, then the minimal resolution of  $\mathcal{Y}_{\lambda, \mu}/W'$  is a smooth K3 surface.*

In the rest of the paper, we investigate further properties of  $\mathcal{X}/W'$  and of its minimal resolution  $\tilde{\mathcal{X}}$ . The second main result of the paper is the following (see Theorem 5.1):

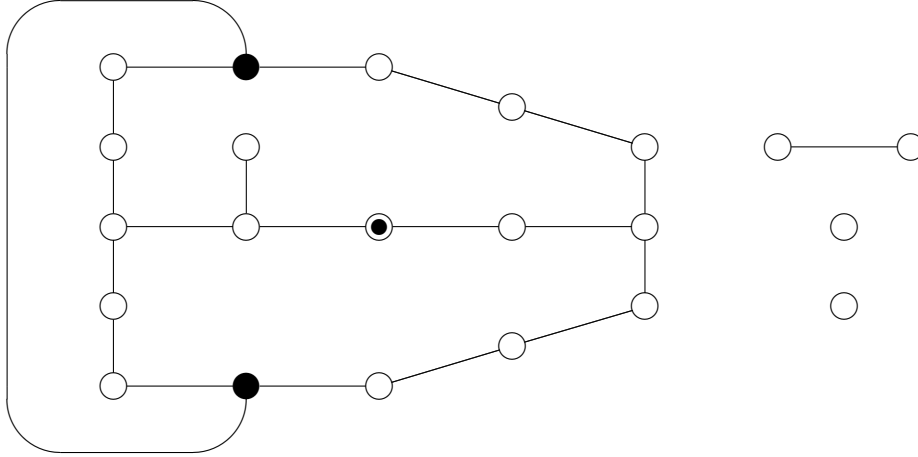
### Theorem B.

- (a) *The K3 surface  $\tilde{\mathcal{X}}$  admits an elliptic fibration  $\tilde{\varphi} : \tilde{\mathcal{X}} \rightarrow \mathbb{P}^1(\mathbb{C})$  whose singular fibers are of type  $E_7 + E_6 + A_2 + 2A_1$ .*
- (b) *The Picard lattice of  $\tilde{\mathcal{X}}$  has rank 20 and discriminant  $-228 = -2^2 \cdot 3 \cdot 19$ .*
- (c) *The transcendental lattice of  $\tilde{\mathcal{X}}$  is given by the matrix  $\begin{pmatrix} 2 & 0 \\ 0 & 114 \end{pmatrix}$ .*

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<sup>(\*)</sup>We do not know if there are values of  $(\lambda, \mu)$  such that  $\mathcal{Y}_{\lambda, \mu}$  has more complicated singularities.

In the course of the proof of Theorem B, we obtain a complete description of the Picard lattice: it is generated by 22 smooth rational curves whose intersection graph is given by<sup>(†)</sup>



In this graph, the two black disks correspond to sections of the elliptic fibration  $\tilde{\varphi}$  mentioned in Theorem B(a). The disk with a smaller black disk inside corresponds to a double section. The singular fibers of  $\tilde{\varphi}$  of type  $E_7$  and  $E_6$  are given by the white disks of the big connected subgraph.

The paper is organized as follows. In Section 1, we fix general notation and prove some preliminary results about group actions on tangent spaces. In Section 2, we fix the context, recall properties of the Weyl group of type  $E_6$  and recall results from Springer theory [Spr]. Section 3 is mainly devoted to the proof of Theorem A. Section 4 gathers many geometric properties of the quotient variety  $\mathcal{X}/W'$  (singularities, smooth rational curves, explicit equations in a weighted projective space): in a first reading of this paper, we believe the reader might skip the details of this very computational section and remember only the concrete results. Theorem B is proved in Section 5. As a complement to all these data, the last section 6 contains the decomposition of the character of the representation  $H^2(\mathcal{X}, \mathbb{C})$  of  $W$  into a sum of irreducible characters (note that  $\dim H^2(\mathcal{X}) = 9\,502$ ).

Many of the results of this paper (namely, the ones from Section 4 to the end) rely on computer calculations done with the software MAGMA [BCP97]. To simplify the exposition, we have removed the codes of these computations: they are available on the arXiv version of this paper [Arxiv], and precise references to this arXiv version will be given at each stage such a computational check is needed.

## 1. General notation, preliminaries

All vector spaces, all algebras, all algebraic varieties will be defined over  $\mathbb{C}$ . Algebraic varieties will always be reduced and quasi-projective, but not necessarily irreducible. If  $\mathcal{X}$  is an algebraic variety and if  $x \in \mathcal{X}$ , we denote by  $T_x(\mathcal{X})$  the tangent space of  $\mathcal{X}$  at  $x$ . If  $\mathcal{X}$  is moreover affine, we denote by  $\mathbb{C}[\mathcal{X}]$  its algebra of regular functions.

<sup>(†)</sup>This means that two smooth rational curves  $C$  and  $C'$  in this list intersect if and only if they are joined by an edge in the above graph, and that, if so, then  $C \cdot C' = 1$ .

We fix a square root  $i$  of  $-1$  in  $\mathbb{C}$ . If  $V$  is a vector space,  $g$  is an element of  $\text{End}_{\mathbb{C}}(V)$  and  $\zeta \in \mathbb{C}$ , we denote by  $V(g, \zeta)$  the  $\zeta$ -eigenspace of  $g$ . The list of eigenvalues of an element of  $\text{End}_{\mathbb{C}}(V)$  will always be given with multiplicities (and will be seen as a multiset: a multiset will be always written with double brackets, e.g.  $\{\{a, b, \dots\}\}$ ). If  $d \in \mathbb{N}^*$ , we denote by  $\mu_d$  the group of  $d$ -th roots of unity in  $\mathbb{C}^\times$  and we set  $\zeta_d = \exp(2i\pi/d)$ .

If  $V$  is a finite dimensional vector space and if  $v \in V \setminus \{0\}$ , we denote by  $[v]$  its image in the projective space  $\mathbb{P}(V)$ . If  $p \in \mathbb{P}(V)$ , we denote by  $G_p$  its stabilizer in  $G \subset \text{GL}_{\mathbb{C}}(V)$ . In other words,  $G_{[v]}$  is the set of elements of  $G$  admitting  $v$  as an eigenvector.

If  $X$  is a subset of  $V$  and if  $G$  is a subgroup of  $\text{GL}_{\mathbb{C}}(V)$ , we denote by  $G_X^{\text{set}}$  (resp.  $G_X^{\text{pt}}$ ) the setwise (resp. pointwise) stabilizer of  $X$  and we set  $G[X] = G_X^{\text{set}}/G_X^{\text{pt}}$  (so that  $G[X]$  acts faithfully on the set  $X$  and on the vector space generated by  $X$ ).

If  $d_1, \dots, d_n$  are positive integers, we denote by  $\mathbb{P}(d_1, \dots, d_n)$  the associated weighted projective space. If  $f_1, \dots, f_r \in \mathbb{C}[X_1, \dots, X_n]$  are homogeneous (with  $X_i$  endowed with the degree  $d_i$ ), we denote by  $\mathcal{X}(f_1, \dots, f_r)$  the (possibly non-reduced) closed subscheme defined by  $f_1 = \dots = f_r = 0$ .

The next three lemmas will be used throughout this paper. The first one is trivial, but very useful [BoSa1, Lem. 2.2]:

**Lemma 1.1.** — *Let  $V$  be a finite dimensional vector space, let  $f \in \mathbb{C}[V]$ , let  $g \in \text{GL}_{\mathbb{C}}(V)$  and let  $v \in V \setminus \{0\}$  be such that:*

- (1)  $g(v) = \xi v$ , with  $\xi \in \mathbb{C}^\times$ .
- (2)  $f$  is homogeneous of degree  $d$  with  $\xi^d \neq 1$ .
- (3)  $f$  is  $g$ -invariant.

Then  $f(v) = 0$ .

The next one is certainly well-known and might have its own interest:

**Lemma 1.2.** — *Let  $V$  be a finite dimensional vector space, let  $n = \dim_{\mathbb{C}} V$ , let  $G$  be a subgroup of  $\text{GL}_{\mathbb{C}}(V)$ , let  $v \in V \setminus \{0\}$  and let  $f_1, \dots, f_r \in \mathbb{C}[V]$  be  $G$ -invariant and homogeneous of respective degrees  $d_1, \dots, d_r$ . We assume that:*

- (1)  $\mathcal{X}(f_1, \dots, f_r)$  is a global complete intersection in  $\mathbb{P}(V)$ .
- (2)  $G$  stabilizes the line  $[v]$  (let  $\theta_v : G \rightarrow \mathbb{C}^\times$  denote the linear character defined by  $g(v) = \theta_v(g)v$  for all  $g \in G$ ).
- (3)  $[v]$  is a smooth point of  $\mathcal{X}(f_1, \dots, f_r)$ .

Then the  $r$ -dimensional semisimple representation  $\theta_v^{d_1-1} \oplus \dots \oplus \theta_v^{d_r-1}$  is isomorphic to a subrepresentation  $E$  of the  $(n-1)$ -dimensional representation  $(V/[v])^*$  of  $G$  and, as representations of  $G$ , we have an isomorphism

$$\mathbb{T}_{[v]}(\mathcal{X}(f_1, \dots, f_r)) \simeq E^\perp \otimes \theta_v^{-1},$$

where  $E^\perp = \{x \in V/[v] \mid \forall \varphi \in E, \varphi(x) = 0\}$ .

**Remark 1.3.** — Keep the notation of the proposition. Recall that we have a natural identification  $\mathbb{T}_{[v]}(\mathbb{P}(V)) \simeq V/[v]$  but that this identification is not  $G$ -equivariant. We will recall the construction of this isomorphism in the proof of the proposition, so that we can follow the action of  $G$ : as we will see, we have a natural isomorphism of  $G$ -representations  $\mathbb{T}_{[v]}(\mathbb{P}(V)) \simeq$

$(V/[v]) \otimes \theta_v^{-1}$ . Therefore,  $\Gamma_{[v]}(\mathcal{X})$  is naturally a subrepresentation of  $(V/[v]) \otimes \theta_v^{-1}$ , and this is the aim of this proposition to identify this subrepresentation. ■

*Proof.* — Write  $\mathcal{X} = \mathcal{X}(f_1, \dots, f_r)$  and  $p = [v]$ . The point  $p$  of  $\mathbb{P}(V)$  corresponds to the homogeneous ideal  $\mathfrak{p}$  of  $\mathbb{C}[V]$  generated by  $v^\perp \subset V^*$ . The local ring  $\mathcal{O}$  of  $\mathbb{P}(V)$  at  $p$  is

$$\mathbb{C}[V]_{(\mathfrak{p})} = \{a/b \mid a, b \in \mathbb{C}[V], \text{ homogeneous and } b \notin \mathfrak{p}\}.$$

We denote by  $\mathfrak{m}$  the unique maximal ideal of  $\mathcal{O}$ . Then

$$\mathfrak{m} = \{a/b \mid a, b \in \mathbb{C}[V], \text{ homogeneous, } a \in \mathfrak{p} \text{ and } b \notin \mathfrak{p}\}.$$

The map

$$\begin{aligned} d_v : \mathfrak{m} &\longrightarrow V^* \\ f &\longmapsto d_v f \end{aligned}$$

induces an isomorphism

$$\delta : T_p(\mathbb{P}(V))^* = \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} v^\perp = (V/[v])^*.$$

(Recall that, if  $a \in \mathfrak{p}$  is homogeneous of degree  $m$  and  $b \notin \mathfrak{p}$  is homogeneous, then  $d_v(a/b)(v) = (d_v a)(v)/b(v) = 0$  because  $(d_v a)(v) = ma(v) = 0$  by Euler's identity). Note, however, that  $\delta$  is not  $G$ -equivariant for the classical action: indeed, as  $g(v) = \theta_v(g)v$  for all  $g \in G$ ,  $\delta$  induces an isomorphism of  $G$ -modules

$$\delta : T_p(\mathbb{P}(V))^* \xrightarrow{\sim} (V/[v])^* \otimes \theta_v.$$

Through this isomorphism,

$$T_p(\mathcal{X})^* \simeq ((V/[v])^*/E) \otimes \theta_v,$$

where

$$E = \sum_{k=1}^r \mathbb{C}\delta(f_k).$$

Now, since  $\mathcal{X}$  is smooth at  $p$  and  $\mathcal{X}$  is a global complete intersection, this means that the elements  $\delta(f_1), \dots, \delta(f_r)$  are linearly independent. To conclude the proof of the proposition, it remains to notice that  $g(\delta(f_k)) = \theta_v(g)^{d_k-1} \delta(f_k)$  for all  $g \in G$ , which follows immediately from the  $G$ -invariance of  $f_k$ . □

The next corollary is an immediate consequence of Lemma 1.2 in the case where  $G$  is a cyclic group.

**Corollary 1.4.** — *Let  $V$  be a finite dimensional vector space, let  $n = \dim_{\mathbb{C}} V$ , let  $g \in \mathrm{GL}_{\mathbb{C}}(V)$ , let  $v \in V \setminus \{0\}$  and let  $f_1, \dots, f_r \in \mathbb{C}[V]$  be  $g$ -invariant and homogeneous of respective degrees  $d_1, \dots, d_r$ . Let  $\{\{\xi_1, \dots, \xi_n\}\}$  be the multiset of eigenvalues of  $g$ . We assume that:*

- (1)  $\mathcal{X}(f_1, \dots, f_r)$  is a global complete intersection in  $\mathbb{P}(V)$ .
- (2)  $g(v) = \xi_n v$ .
- (3)  $[v]$  is a smooth point of  $\mathcal{X}(f_1, \dots, f_r)$ .

*Then  $\{\{\xi_n^{-d_1}, \dots, \xi_n^{-d_r}\}\}$  is contained in the multiset  $\{\{\xi_n^{-1}\xi_1, \dots, \xi_n^{-1}\xi_{n-1}\}\}$  and the list of eigenvalues of  $g$  for its action on the tangent space  $\Gamma_{[v]}(\mathcal{X}(f_1, \dots, f_r))$  is the multiset*

$$\{\{\xi_n^{-1}\xi_1, \dots, \xi_n^{-1}\xi_{n-1}\}\} \setminus \{\{\xi_n^{-d_1}, \dots, \xi_n^{-d_r}\}\}.$$

**Remark 1.5.** — Keep the notation of Lemma 1.2 and assume moreover that  $G$  is a closed reductive subgroup of  $\mathrm{GL}_{\mathbb{C}}(V)$  (for instance,  $G$  might be finite), that  $v \in V^G$  (so that  $\theta_v = 1$ ) and that  $\mathbb{P}(V^G) \subset \mathcal{X}(f_1, \dots, f_r)$ . We want to show that

(#)  $[v]$  is a singular point of  $\mathcal{X}(f_1, \dots, f_r)$ .

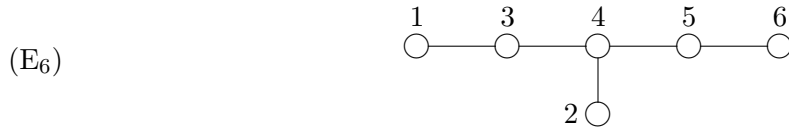
For this, assume that  $[v]$  is smooth and let  $E$  denote the  $G$ -stable subspace of  $(V/[v])^*$  of dimension  $r$  defined in the proof of Lemma 1.2 and such that the  $G$ -module  $T_{[v]}(\mathcal{X}(f_1, \dots, f_r))$  is identified with  $E^{\perp}$ . By hypothesis,  $V^G/[v] \subset T_{[v]}(\mathcal{X}(f_1, \dots, f_r))$  so, since  $G$  acts semisimply on  $V$ , its orthogonal  $F$  in  $(V/[v])^*$  satisfies  $F^G = 0$ . This contradicts the fact that  $E \subset F$ . This shows (#).

If  $G$  is finite,  $\dim_{\mathbb{C}} V^G = 1$  (resp.  $\dim_{\mathbb{C}} V^G = 2$ ) and  $r = 1$ , we retrieve [BoSa1, Coro. 2.4] (resp. [BoSa1, Coro. 2.9]). ■

## 2. Set-up

For the classical theory of Coxeter groups, reflection groups, Dynkin diagrams,... we mainly refer to Bourbaki's book [Bou] or Broué's book [Bro]. For Springer theory (and its enhancement by Lehrer-Springer), we refer to [Spr], [LeSp1], [LeSp2] and [LeMi].

Let  $V = \mathbb{C}^6$  be endowed with its canonical basis  $(e_1, e_2, e_3, e_4, e_5, e_6)$ . Through this basis, we identify  $\mathrm{GL}_{\mathbb{C}}(V)$  and  $\mathrm{GL}_6(\mathbb{C})$  and we let it act on  $V$  on the left. We denote by  $(X_1, X_2, X_3, X_4, X_5, X_6)$  the dual basis of  $(e_1, e_2, e_3, e_4, e_5, e_6)$ , so that the algebra  $\mathbb{C}[V]$  will be identified with the polynomial algebra  $\mathbb{C}[X_1, X_2, X_3, X_4, X_5, X_6]$ . We endow it with the symmetric bilinear form  $\langle \cdot, \cdot \rangle$  attached to the Dynkin diagram of type  $E_6$  (we follow the strange numbering of nodes given by Bourbaki [Bou, Chap. 6, Planche V]):



Recall that this means that

$$\begin{cases} \langle e_k, e_k \rangle = 1 & \text{if } 1 \leq k \leq 6, \\ \langle e_k, e_l \rangle = -1/2 & \text{if } 1 \leq k \neq l \leq 6 \text{ and } \{k, l\} \text{ is an edge of the graph (E}_6\text{)}, \\ \langle e_k, e_l \rangle = 0 & \text{if } 1 \leq k \neq l \leq 6 \text{ and } \{k, l\} \text{ is not an edge of the graph (E}_6\text{)}. \end{cases}$$

Recall that  $\langle \cdot, \cdot \rangle$  is non-degenerate (in fact, when restricted to  $\mathbb{R}^6$ , it is positive definite). For  $1 \leq k \leq 6$ , we denote by  $s_k$  the orthogonal reflection such that  $s_k(e_k) = -e_k$ . We set

$$W = \langle s_1, s_2, s_3, s_4, s_5, s_6 \rangle.$$

We denote by  $O(V)$  the orthogonal group of  $V$ , with respect to the bilinear form  $\langle \cdot, \cdot \rangle$ . By construction,  $W$  is a subgroup of  $O(V)$  and is called a *Weyl group of type  $E_6$* .

**2.A. First properties of  $W$ .** — The group  $W$  is finite and acts irreducibly on  $V$ . Moreover,

$$(2.1) \quad |W| = 51\,840.$$

Also, the center of  $W$  is trivial, so  $W$  acts faithfully on  $\mathbb{P}(V)$ . Let  $\varepsilon : W \rightarrow \mu_2 = \{-1, 1\}$ ,  $w \mapsto \det(w)$ . Recall that  $\text{Ker } \varepsilon$  is the derived (i.e. commutator) subgroup of  $W$ , which will be denoted by  $W'$ . In particular,

$$(2.2) \quad |W'| = 25\,920.$$

We denote by  $\text{Deg}(W)$  (resp.  $\text{Codeg}(W)$ ) the *degrees* (resp. the *codegrees*) of  $W$ , as defined in [Bro, Chap. 4]. It turns out that

$$(2.3) \quad \text{Deg}(W) = \{2, 5, 6, 8, 9, 12\} \quad \text{and} \quad \text{Codeg}(W) = \{0, 3, 4, 6, 7, 10\}.$$

In particular, this means that there exist 6 homogeneous  $W$ -invariant polynomials  $f_2, f_5, f_6, f_8, f_9$  and  $f_{12}$ , of respective degrees 2, 5, 6, 8, 9 and 12, such that

$$(2.4) \quad \mathbb{C}[V]^W = \mathbb{C}[f_2, f_5, f_6, f_8, f_9, f_{12}].$$

We set  $\mathbf{f} = (f_2, f_5, f_6, f_8, f_9, f_{12})$  and we recall that  $\mathbf{f}$  is not uniquely determined. Thanks to (2.4), we get that

$$\begin{aligned} \pi_{\mathbf{f}} : \mathbb{P}(V) &\longrightarrow \mathbb{P}(2, 5, 6, 8, 9, 12) \\ [v] &\longmapsto [f_2(v), f_5(v), f_6(v), f_8(v), f_9(v), f_{12}(v)] \end{aligned}$$

is well-defined and induces an isomorphism of varieties

$$(2.5) \quad \mathbb{P}(V)/W \xrightarrow{\sim} \mathbb{P}(2, 5, 6, 8, 9, 12).$$

The graded algebra associated with the weighted projective space  $\mathbb{P}(2, 5, 6, 8, 9, 12)$  will be denoted by  $\mathbb{C}[Z_2, Z_5, Z_6, Z_8, Z_9, Z_{12}]$ , with  $Z_d$  being given the degree  $d$  (for all  $d \in \text{Deg}(W)$ ).

**Remark 2.6.** — The quadratic form  $Q : V \rightarrow \mathbb{C}$ ,  $v \mapsto \langle v, v \rangle$  is  $W$ -invariant. Hence  $f_2$  is a scalar multiple of  $Q$ . Since  $Q$  is positive definite when restricted to  $\mathbb{R}^6$ , we have in particular that  $f_2(e_1) \neq 0$ . ■

Let  $\text{Ref}(W)$  be the set of reflections of  $W$  and let  $\mathcal{A}$  be the hyperplane arrangement of  $W$  (i.e.  $\mathcal{A} = \{V^s \mid s \in \text{Ref}(W)\}$ ). Then

$$(2.7) \quad |\text{Ref}(W)| = |\mathcal{A}| = \sum_{d \in \text{Deg}(W)} (d - 1) = 36.$$

If  $H \in \mathcal{A}$ , we denote by  $\alpha_H$  an element of  $V^*$  such that  $H = \text{Ker } \alpha_H$ . We set

$$\text{Jac} = \prod_{H \in \mathcal{A}} \alpha_H \in \mathbb{C}[V].$$

Since all the  $\alpha_H$  are well-defined up to a non-zero scalar,  $\text{Jac}$  is also well-defined up to a non-zero scalar. Then

$$(2.8) \quad {}^w \text{Jac} = \varepsilon(w) \text{Jac}$$

for all  $w \in W$  and

$$(2.9) \quad \mathbb{C}[V]^{W'} = \mathbb{C}[f_2, f_5, f_6, f_8, f_9, f_{12}, \text{Jac}].$$

Moreover,  $\text{Jac}$  is homogeneous of degree 36 by (2.7). Also, since  $\text{Jac}^2 \in \mathbb{C}[V]^W$  by (2.8), there exists a unique homogeneous polynomial  $P_{\mathbf{f}} \in \mathbb{C}[Z_2, Z_5, Z_6, Z_8, Z_9, Z_{12}]$  such that

$$\text{Jac}^2 = P_{\mathbf{f}}(f_2, f_5, f_6, f_8, f_9, f_{12}).$$

Of course,  $P_{\mathbf{f}}$  depends heavily on the choice of the family  $\mathbf{f}$  and, up to a non-zero scalar, on the choice of the  $\alpha_H$ 's. It turns out that this relation generates the ideal of relations between the functions  $f_2, f_5, f_6, f_8, f_9, f_{12}$  and  $\text{Jac}$ . In particular, the map

$$\begin{aligned} \pi'_{\mathbf{f}} : \mathbb{P}(V) &\longrightarrow \mathbb{P}(2, 5, 6, 8, 9, 12, 36) \\ [v] &\longmapsto [f_2(v) : f_5(v) : f_6(v) : f_8(v) : f_9(v) : f_{12}(v) : \text{Jac}(v)] \end{aligned}$$

is well-defined and induces an isomorphism of varieties

$$(2.10) \quad \mathbb{P}(V)/W' \xrightarrow{\sim} \{[z_2 : z_5 : z_6 : z_8 : z_9 : z_{12} : j] \in \mathbb{P}(2, 5, 6, 8, 9, 12, 36) \mid j^2 = P_{\mathbf{f}}(z_2, z_5, z_6, z_8, z_9, z_{12})\}.$$

Finally, we denote by  $\omega : \mathbb{P}(V)/W' \rightarrow \mathbb{P}(V)/W$  the natural morphism, which is just the quotient map by  $\mu_2 \simeq W/W'$ . Through the isomorphisms (2.5) and (2.10), the action of the non-trivial element of  $\mu_2$  is given by the involutive automorphism  $\sigma$  given by

$$\sigma[z_2 : z_5 : z_6 : z_8 : z_9 : z_{12} : j] = [z_2 : z_5 : z_6 : z_8 : z_9 : z_{12} : -j]$$

and  $\omega$  is given by

$$\omega[z_2 : z_5 : z_6 : z_8 : z_9 : z_{12} : j] = [z_2 : z_5 : z_6 : z_8 : z_9 : z_{12}].$$

**2.B. Eigenspaces, Springer theory.** — As in [BoSa1], an important role is played by Springer and Lehrer-Springer theory. We recall briefly the results we will need (this subsection is a simplified version of [BoSa1, §3.3], adapted to the particular case of our Weyl group  $W$ ). All the results stated here can be found in [Spr], [LeSp1], [LeSp2]. Note that some of the proofs have been simplified in [LeMi]. Let us fix now a natural number  $e$ . We set

$$\begin{aligned} \Delta(e) &= \{d \in \text{Deg}(W) \mid e \text{ divides } d\}, \\ \Delta^*(e) &= \{d^* \in \text{Codeg}(W) \mid e \text{ divides } d^*\}, \\ \delta(e) &= |\Delta(e)| \quad \text{and} \quad \delta^*(e) = |\Delta^*(e)|. \end{aligned}$$

With this notation, we have

$$(2.11) \quad \delta(e) = \max_{w \in W} (\dim V(w, \zeta_e)).$$

In particular,  $\zeta_e$  is an eigenvalue of some element of  $W$  if and only if  $\delta(e) \neq 0$  that is, if and only if  $e \in \{1, 2, 3, 4, 5, 6, 8, 9, 12\}$ . In this case, we fix an element  $w_e$  of  $W$  of minimal order such that

$$\dim V(w_e, \zeta_e) = \delta(e).$$

We set for simplification  $V(e) = V(w_e, \zeta_e)$  and  $W(e) = W[V(e)] = W_{V(e)}^{\text{set}}/W_{V(e)}^{\text{pt}}$ .

If  $f \in \mathbb{C}[V]$ , we denote by  $f^{[e]}$  its restriction to  $V(e)$ . Note that if  $d \notin \Delta(e)$ , then  $f_d^{[e]} = 0$  by Lemma 1.1.

**Theorem 2.12 (Springer, Lehrer-Springer).** — *Assume that  $\delta(e) \neq 0$ . Then:*

- (a) *If  $w \in W$ , then there exists  $x \in W$  such that  $x(V(w, \zeta_e)) \subset V(e)$ .*
- (b)  *$W(e)$  acts (faithfully) on  $V(e)$  as a group generated by reflections.*
- (c) *The family  $(f_d^{[e]})_{d \in \Delta(e)}$  is a family of fundamental invariants of  $W(e)$ . In particular, the list of degrees of  $W(e)$  consists of the degrees of  $W$  which are divisible by  $e$ .*
- (d) *We have*

$$\bigcup_{w \in W} V(w, \zeta_e) = \bigcup_{x \in W} x(V(e)) = \{v \in V \mid \forall d \in \text{Deg}(W) \setminus \Delta(e), f_d(v) = 0\}.$$



- (e)  $\delta^*(e) \geq \delta(e)$  with equality if and only if  $W_{V(e)}^{\text{pt}} = 1$ .  
(f) If  $\delta^*(e) = \delta(e)$ , then  $w_e$  has order  $e$ ,  $W(e) = W_{V(e)}^{\text{set}} = C_W(w_e)$  and the multiset of eigenvalues (with multiplicity) of  $w_e$  is equal to  $\{\{\zeta_e^{1-d}\}\}_{d \in \text{Deg}(W)}$ . Moreover, if  $w$  is such that  $\dim V(w, \zeta_e) = \delta(e)$ , then  $w$  is conjugate to  $w_e$ .

**Example 2.13.** — Let  $e \in \{8, 9, 12\}$ . Then  $\delta(e) = \delta^*(e) = 1$ . So  $V(e)$  is a line in  $V$ , and can be viewed as an element of  $\mathbb{P}(V)$ . It follows from the above results that

$$W_{V(e)} = \langle w_e \rangle$$

(see for instance [BoSa1, Rem. 3.14] for a proof).

Moreover, by Theorem 2.12(f), we have  $\det(w_e) = \zeta_e^{-36}$ , so

$$w_9, w_{12} \in W' \quad \text{and} \quad w_8 \notin W'.$$

In particular, if we denote by  $p_e$  (resp.  $q_e$ ) the image of  $V(e)$  in  $\mathbb{P}(V)/W'$  (resp.  $\mathbb{P}(V)/W$ ), then the morphism  $\omega$  is unramified (resp. ramified) over  $q_9$  and  $q_{12}$  (resp.  $p_8$ ). This implies also that  $\sigma(p_9) \neq p_9$  and  $\sigma(p_{12}) \neq p_{12}$ . ■

**Example 2.14.** — Note that  $\delta^*(5) = 2 > \delta(5) = 1$  while, if  $e \in \{1, 2, 3, 4, 6, 8, 9, 12\}$ , then  $\delta^*(e) = \delta(e)$ . If  $x$  is an element of order 5, then  $x$  admits a primitive fifth root of unity as eigenvalue, so admits  $\zeta_5$  as an eigenvalue because  $x$  is represented by a matrix with rational coefficients. So we can take  $w_5 = x$ , and so  $w_5$  has order 5.

The above argument shows that the multiset of eigenvalues of  $w_5$  is  $\{\{1, 1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4\}\}$ . Now, if  $w$  is an element of  $W$  such that  $w(V(5)) = V(5)$ , then, since  $w$  is defined over  $\mathbb{Q}$ , it stabilizes the  $\zeta_5^k$ -eigenspace of  $w$  for  $k \in \{1, 2, 3, 4\}$ . Therefore, it stabilizes the sum  $E$  of these 4 eigenspaces. But  $E = \text{Ker}(\text{Id}_V + w_5 + w_5^2 + w_5^3 + w_5^4)$  is defined over  $\mathbb{Q}$  (and so over  $\mathbb{R}$ ) so its orthogonal  $E^\perp$  in  $V$  with respect to the bilinear form  $\langle \cdot, \cdot \rangle$  satisfies  $E \oplus E^\perp = V$  (because its restriction to  $\mathbb{R}^6$  is positive definite). This shows that  $w$  and  $w_5$  stabilize also  $E^\perp$ , which is necessarily equal to  $\text{Ker}(w_5 - \text{Id}_V)$ . In particular  $w$  centralizes  $w_5$ . But, by the fourth line of [Arxiv, Comput. I.12], we have  $C_{W'}(w_5) = \langle w_5 \rangle$ . So we have proved that

$$W'_{V(5)} = \langle w_5 \rangle.$$

This fact will be used in the proof of Theorem 3.2. ■

### 3. K3 surfaces

Let

$$\mathcal{X} = \mathcal{X}(f_2, f_6, f_8)$$

and, for  $\lambda, \mu \in \mathbb{C}$ , set

$$\mathcal{Y}_{\lambda, \mu} = \mathcal{X}(f_5, f_6 + \lambda f_2^3, f_8 + \mu f_2^4).$$

Note that the invariants  $f_2$  and  $f_5$  are uniquely defined up to a scalar and that, up to a scalar, every fundamental invariant of degree 6 (resp. 8) is of the form  $f_6 + \lambda f_2^3$  (resp.  $f_8 + \mu f_2^4 + \nu f_6 f_2$ ) for some  $\lambda \in \mathbb{C}$  (resp.  $\mu, \nu \in \mathbb{C}$ ). But  $\mathcal{X}(f_2, f_6 + \lambda f_2^3, f_8 + \mu f_2^4 + \nu f_6 f_2) = \mathcal{X}$  and  $\mathcal{X}(f_5, f_6 + \lambda f_2^3, f_8 + \mu f_2^4 + \nu f_6 f_2) = \mathcal{Y}_{\lambda, \mu - \lambda \nu}$ . So this shows in particular that  $\mathcal{X}$  does not depend on the choices of the fundamental invariants of  $W$ . By construction,  $\mathcal{X}$  and  $\mathcal{Y}_{\lambda, \mu}$  are  $W$ -stable.

**Proposition 3.1.** — *With the above notation, we have:*

- (a)  $\mathcal{X}$  is a smooth irreducible surface, which is a complete intersection in  $\mathbb{P}(V)$ .
- (b) If  $\lambda, \mu \in \mathbb{C}$ , then  $\mathcal{Y}_{\lambda,\mu}$  has pure dimension 2, and so is a complete intersection in  $\mathbb{P}(V)$ . If it is smooth or has only ADE singularities, then it is irreducible.
- (c) The set of  $(\lambda, \mu) \in \mathbb{C}^2$  such that  $\mathcal{Y}_{\lambda,\mu}$  is smooth is a non-empty open subset of  $\mathbb{C}^2$ .

*Proof.* — Through the isomorphism (2.5), we have

$$\mathcal{X}/W \simeq \mathbb{P}(5, 9, 12) \quad \text{and} \quad \mathcal{Y}_{\lambda,\mu}/W \simeq \mathbb{P}(2, 9, 12),$$

so  $\mathcal{X}/W$  and  $\mathcal{Y}_{\lambda,\mu}/W$  are irreducible of dimension 2. Since  $W$  is finite, this implies that  $\mathcal{X}$  and  $\mathcal{Y}_{\lambda,\mu}$  are of pure dimension 2 and so are complete intersections. In particular, they are connected [Har, Chap. II, Exer. 8.4(c)].

(a) By [Arxiv, Comput. I.1], the open affine chart of  $\mathcal{X}$  defined by  $x_6 \neq 0$  is smooth. This shows that the singular locus  $\mathcal{S}$  of  $\mathcal{X}$  is contained in the projective hyperplane  $\mathbb{P}(H)$ , where  $H = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{C}^6 \mid x_6 = 0\}$ . Since  $W$  acts on  $\mathcal{X}$ , we have that  $\mathcal{S}$  is contained in  $\mathbb{P}(\cap_{w \in W} w(H))$ . But  $\cap_{w \in W} w(H)$  is a  $W$ -stable proper subspace of  $V$ , so it is equal to  $\{0\}$  since  $W$  acts irreducibly on  $V$ . Hence  $\mathcal{S} = \emptyset$ . This shows that  $\mathcal{X}$  is smooth and, in particular, irreducible (because it is connected).

(b) We already know that  $\mathcal{Y}_{\lambda,\mu}$  is connected. If moreover it admits only ADE singularities, then it is necessarily irreducible.

(c) Let  $U$  be the set of  $(\lambda, \mu) \in \mathbb{C}^2$  such that  $\mathcal{Y}_{\lambda,\mu}$  is smooth. It is clear that  $U$  is open, so we only need to prove that it is non-empty. With the particular choice of fundamental invariants given in [Arxiv, App. I], we have that the open affine chart of  $\mathcal{Y}_{0,0}$  defined by  $x_6 \neq 0$  is smooth by [Arxiv, Comput. I.1]. The same argument as in (a) allows to conclude that  $\mathcal{Y}_{0,0}$  is smooth and irreducible.  $\square$

We are now ready to state the first main result of this paper:

**Theorem 3.2.** — *Let  $\lambda, \mu \in \mathbb{C}$  be such that  $\mathcal{Y}_{\lambda,\mu}$  admits only ADE singularities and let  $\mathcal{V}$  be one of the varieties  $\mathcal{X}$  or  $\mathcal{Y}_{\lambda,\mu}$ . Then  $\mathcal{V}/W'$  is a K3 surface with only ADE singularities and its minimal smooth resolution  $\tilde{\mathcal{V}}$  is a smooth K3 surface.*

*Proof.* — The proof will be given in the next subsections, following the same lines as [BoSa1, Theo. 5.4]. More precisely, if  $\mathcal{V}$  denotes one of the above varieties  $\mathcal{X}$  or  $\mathcal{Y}_{\lambda,\mu}$ , we will prove in the next subsections the following three facts:

- (A) The smooth locus of  $\mathcal{V}/W'$  admits a symplectic form (see Lemma 3.5);
- (B) The variety  $\mathcal{V}/W'$  has only ADE singularities (see Lemma 3.7);
- (C) The Euler characteristic of  $\mathcal{V}/W'$  is positive (see Lemma 3.10).

Therefore, by (A) and (B), the variety  $\mathcal{V}/W'$  is a symplectic singularity and its minimal smooth resolution  $\tilde{\mathcal{V}}$  is a crepant resolution, i.e. a symplectic resolution. So  $\tilde{\mathcal{V}}$  admits a symplectic form. By the classification of surfaces, this forces  $\tilde{\mathcal{V}}$  to be a smooth K3 surface or an abelian variety. But, by (B), the Euler characteristic of  $\tilde{\mathcal{V}}$  is greater than or equal to the one of  $\mathcal{V}/W'$ , so the Euler characteristic of  $\tilde{\mathcal{V}}$  is positive by (C). Since an abelian variety has Euler characteristic 0, this shows that  $\tilde{\mathcal{V}}$  is a smooth K3 surface.  $\square$

So it remains to prove the three facts (A), (B) and (C) used in the above proof.

**Notation.** For the rest of this section, we fix  $\lambda, \mu$  in  $\mathbb{C}$  such that  $\mathcal{Y}_{\lambda, \mu}$  admits only ADE singularities and we denote by  $\mathcal{V}$  a variety which can be  $\mathcal{X}$  or  $\mathcal{Y}_{\lambda, \mu}$ .

Recall that this implies that  $\mathcal{V}$  is irreducible and normal and, in particular, that  $\mathcal{V}/W'$  is irreducible and normal too.

**3.A. Symplectic form.** — Through the isomorphism (2.10), we have

$$\mathcal{X}/W' \simeq \{[z_5 : z_9 : z_{12} : j] \in \mathbb{P}(5, 9, 12, 36) \mid j^2 = P_{\mathbf{f}}(0, z_5, 0, 0, z_9, z_{12})\}$$

and  $\mathcal{Y}_{\lambda, \mu}/W' \simeq \{[z_2 : z_9 : z_{12} : j] \in \mathbb{P}(2, 9, 12, 36) \mid j^2 = P_{\mathbf{f}}(z_2, 0, -\lambda z_2^3, -\mu z_2^4, z_9, z_{12})\}$ .

But note that  $\mathbb{P}(5, 9, 12, 36) = \mathbb{P}(5, 3, 4, 12)$ , so there exists a unique homogeneous polynomial  $Q_{\mathbf{f}} \in \mathbb{C}[Y_5, Y_3, Y_4]$  of degree 24 (where  $Y_i$  is endowed with the degree  $i$ ) such that  $P_{\mathbf{f}}(0, z_5, 0, 0, z_9, z_{12}) = Q_{\mathbf{f}}(z_5^3, z_9, z_{12})$ . Hence,

$$(3.3) \quad \mathcal{X}/W' \simeq \{[y_5 : y_3 : y_4 : j] \in \mathbb{P}(5, 3, 4, 12) \mid j^2 = Q_{\mathbf{f}}(y_5, y_3, y_4)\}.$$

So the degree of the equation defining  $\mathcal{X}/W'$  (namely, 24) is equal to the sum of the weights of the projective space (namely,  $5 + 3 + 4 + 12$ ). By [BoSa1, Lem. A.1], this implies that the smooth locus of  $\mathcal{X}/W'$  is endowed with a symplectic form.

On the other hand,  $\mathbb{P}(2, 9, 12, 36) = \mathbb{P}(1, 9, 6, 18) = \mathbb{P}(1, 3, 2, 6)$  so there exists a unique homogeneous polynomial  $Q_{\mathbf{f}}^{\lambda, \mu} \in \mathbb{C}[Y_1, Y_3, Y_2]$  of degree 12 (where  $Y_i$  is endowed with the degree  $i$ ) such that  $P_{\mathbf{f}}(z_2, 0, -\lambda z_2^3, -\mu z_2^4, z_9, z_{12}) = Q_{\mathbf{f}}^{\lambda, \mu}(z_2^3, z_9, z_{12})$ . Hence,

$$(3.4) \quad \mathcal{Y}_{\lambda, \mu}/W' \simeq \{[y_1 : y_3 : y_2 : j] \in \mathbb{P}(1, 3, 2, 6) \mid j^2 = Q_{\mathbf{f}}^{\lambda, \mu}(y_1, y_3, y_2)\}.$$

So the degree of the equation defining  $\mathcal{Y}_{\lambda, \mu}/W'$  (namely, 12) is equal to the sum of the weights of the projective space (namely,  $1 + 3 + 2 + 6$ ). By [BoSa1, Lem. 5.4], this implies that the smooth locus of  $\mathcal{X}/W'$  is endowed with a symplectic form. Therefore, we have proved the following lemma, which corresponds to the Fact (A) stated in the proof of Theorem 3.2:

**Lemma 3.5.** — *The smooth locus of  $\mathcal{V}/W'$  admits a symplectic form.*

**Remark 3.6.** — In both cases, the ramification locus of the morphism  $\omega : \mathcal{V}/W' \rightarrow \mathcal{V}/W$  is given by the equation  $j = 0$  (in the models given by equations (3.3) or (3.4)). ■

**3.B. Singularities.** — We aim to prove here the following lemma, which corresponds to the Fact (B) stated in the proof of Theorem 3.2:

**Lemma 3.7.** — *The surface  $\mathcal{V}/W'$  has only ADE singularities.*

*Proof.* — First, as  $\mathcal{V}$  has only ADE singularities and  $W'$  has index 2 in  $W$ , every point of  $\mathcal{V}/W'$  lying above a smooth point of  $\mathcal{V}/W$  is smooth or is an ADE singularity by [BoSa1, Coro. B.7]. So it remains only to study the points lying above the singular points of  $\mathcal{V}/W$ .

The singular points of  $\mathcal{X}/W \simeq \mathbb{P}(5, 3, 4)$  are  $q_5 = [1 : 0 : 0]$ ,  $q_9 = [0 : 1 : 0]$  and  $q_{12} = [0 : 0 : 1]$ . The singular points of  $\mathcal{Y}_{\lambda, \mu}/W \simeq \mathbb{P}(1, 3, 2)$  are  $q_9 = [0 : 1 : 0]$  and  $q_{12} = [0 : 0 : 1]$ . Note that the notation  $q_9$  and  $q_{12}$  is consistent with Example 2.13, as they correspond to the points  $q_9$  and  $q_{12}$  defined in this example through the embeddings  $\mathcal{V}/W \hookrightarrow \mathbb{P}(V)/W \simeq \mathbb{P}(2, 5, 6, 8, 9, 12)$ . Still by Example 2.13, the morphism  $\mathcal{V}/W' \rightarrow \mathcal{V}/W$  is unramified above  $q_9$  and  $q_{12}$ . Therefore, the points  $p_9$  and  $p'_9 = \sigma(p_9)$  (resp.  $p_{12}$  and  $p'_{12} = \sigma(p_{12})$ ) of  $\mathcal{V}/W'$  are distinct and have the same type of singularities than the point  $q_9$  (resp.  $q_{12}$ ) of  $\mathcal{V}/W$ . But  $q_9$  is a singular point of type  $A_2$  of  $\mathbb{P}(5, 3, 4)$  or  $\mathbb{P}(1, 3, 2)$  and  $q_{12}$  is a singular point of type  $A_3$  (resp.  $A_1$ ) of  $\mathbb{P}(5, 3, 4)$  (resp.  $\mathbb{P}(1, 3, 2)$ ). This shows that the following results holds:

**Lemma 3.8.** — *We have:*

- (a) *The points  $p_9$  and  $p'_9$  are  $A_2$  singularities of  $\mathcal{X}/W'$  and the points  $p_{12}$  and  $p'_{12}$  are  $A_3$  singularities of  $\mathcal{X}/W'$ .*
- (b) *The points  $p_9$  and  $p'_9$  are  $A_2$  singularities of  $\mathcal{Y}_{\lambda, \mu}/W'$  and the points  $p_{12}$  and  $p'_{12}$  are  $A_1$  singularities of  $\mathcal{Y}_{\lambda, \mu}/W'$ .*

Therefore, it remains to prove that the points of  $\mathcal{X}/W'$  lying above  $q_5$  are ADE singularities. For this, note that

$$\pi^{-1}(q_5) = \{x \in \mathbb{P}(V) \mid f_2(x) = f_6(x) = f_8(x) = f_9(x) = f_{12}(x) = 0\}.$$

But, by (2.11) and Theorem 2.12(d) applied to the case where  $e = 5$ , we get that  $V(5)$  is a line in  $V$ , so may be viewed as a point of  $\mathbb{P}(V)$  and  $\pi_f^{-1}(q_5)$  is the  $W$ -orbit of  $V(5)$ . Now,  $\delta^*(5) = 2 > \delta(5) = 1$ , so it follows from Theorem 2.12(f) that  $W_{V(5)}^{\text{pt}} \neq 1$ . By Steinberg's Theorem (see for instance [Bro, Theo. 4.7]), this shows that  $W_{V(5)}^{\text{pt}}$  contains a reflection, and so the stabilizer  $G$  of  $V(5)$  in  $W$  contains a reflection. In particular,  $G$  is not contained in  $W'$ . This proves that the morphism  $\mathcal{X}/W' \rightarrow \mathcal{X}/W$  is ramified above  $q_5$ : we denote by  $p_5$  the unique point of  $\mathcal{X}/W'$  lying above  $q_5$ .

Now, Example 2.14 shows that the stabilizer of  $V(5)$  in  $W'$  is  $\langle w_5 \rangle$ . So, in order to determine the type of singularity of  $\mathcal{X}/W'$  at  $p_5$ , we only need to determine the two eigenvalues of  $w_5$  for its action on  $T_{V(5)}(\mathcal{X})$ . This is easily done thanks to Corollary 1.4: the two eigenvalues are  $\zeta_5$  and  $\zeta_5^{-1}$ . We have thus proved the following result:

**Lemma 3.9.** — *The point  $p_5$  is an  $A_4$  singularity of  $\mathcal{X}/W'$ .*

This completes the proof of Lemma 3.7. □

**3.C. Euler characteristic.** — Since  $\mathcal{V}$  is a complete intersection which is smooth or has only ADE singularities, its cohomology (with coefficients in  $\mathbb{C}$ ) is concentrated in even degree [Dim, Theo. 2.1, Lem. 3.2 and Example 3.3]. Now,  $H^k(\mathcal{V}/W', \mathbb{C}) \simeq H^k(\mathcal{V}, \mathbb{C})^{W'}$ , so the cohomology of  $\mathcal{V}/W'$  is concentrated in even degree. This implies the next lemma, which corresponds to the Fact (C) stated in the proof of Theorem 3.2, and completes the proof of Theorem 3.2:

**Lemma 3.10.** — *The Euler characteristic of  $\mathcal{V}/W'$  is positive.*

**Notation.** *The minimal smooth resolution of  $\mathcal{X}/W'$  will be denoted by  $\rho : \tilde{\mathcal{X}} \rightarrow \mathcal{X}/W'$ . Theorem 3.2 says  $\tilde{\mathcal{X}}$  is a smooth projective K3 surface.*

#### 4. Some numerical data for the surface $\mathcal{X}/W'$

We complete here the qualitative result given by Theorem 3.2 with some concrete results concerning the surface  $\mathcal{X}/W'$  (type of singularities, equation, coordinates of singular points, cohomology,...). These informations will be used in the next section to obtain further properties of the K3 surface  $\tilde{\mathcal{X}}$  (Picard lattice, elliptic fibration,...).

**4.A. Singularities.** — In the course of the proof of Theorem 3.2, we have obtained some quantitative results (see Lemmas 3.8 and 3.9). We complete them by determining all the singularities of  $\mathcal{X}/W'$ :

**Proposition 4.1.** — *The surface  $\mathcal{X}/W'$  admits  $A_4 + 2 A_3 + 3 A_2 + 2 A_1$  singularities.*

*Proof.* — For proving this proposition, we must investigate fixed points under various elements of  $W'$ , up to conjugacy. Note also that we have already found  $A_4 + 2 A_3 + 2 A_2$  singularities in  $\mathcal{X}/W'$  (see Lemma 3.8 and 3.9), given by the points  $p_5, p_9, p'_9, p_{12}$  and  $p'_{12}$  lying respectively above the points  $q_5 = [1 : 0 : 0]$ ,  $q_9 = [0 : 1 : 0]$  and  $q_{12} = [0 : 0 : 1]$  of  $\mathcal{X}/W \simeq \mathbb{P}(5, 3, 4)$ .

Note also that, for any  $w \in W$ , the fixed point scheme  $\mathcal{X}^w$  is also smooth and in particular is reduced (so, in the MAGMA computations necessary in the proof of the proposition, we do not need to compute its reduced subscheme for determining the exact number of points). So let us now investigate the fixed points subscheme, by letting the order of  $w$  increasing.

**Lemma 4.2 (Elements of order 2).** — *There are two conjugacy classes of elements of order 2: the one of the element  $w_2$  defined after Theorem 2.12 with eigenvalues  $\{1, 1, -1, -1, -1, -1\}$  and the one of an element  $v_2$  admitting  $\{1, 1, 1, 1, -1, -1\}$  as a list of eigenvalues. Moreover:*

- (a) *The image of  $\mathcal{X}^{w_2}$  in  $\mathcal{X}/W'$  is contained in  $\{p_{12}, p'_{12}\}$ .*
- (b) *The image of  $\mathcal{X}^{v_2}$  in  $\mathcal{X}/W'$  consists in two elements  $p_1$  and  $p'_1$  which are both  $A_1$  singularities.*

*Proof of Lemma 4.2.* — The first statement is proved in [Arxiv, Comput. I.2].

(a) By [Arxiv, Comput. I.3],  $\mathcal{X}^{w_2}$  has dimension 0 and is contained in  $\mathcal{X}(f_5, f_9)$ , so its image in  $\mathcal{X}/W$  is  $q_{12}$ . This proves (a).

(b) By the first four commands in [Arxiv, Comput. I.4], we know that  $\mathcal{X}^{v_2}$  has dimension 0 and consists of 96 points. The fifth command shows that, if  $g \in W' \setminus \{1, v_2\}$ , then

(♣) 
$$\mathcal{X}^g \cap \mathcal{X}^{v_2} = \emptyset.$$

If the reader wants to check this computation, he or she must be aware that it takes about 10 minutes on a standard computer. We now need to understand how many  $W'$ -orbits meet  $\mathcal{X}^{v_2}$ . For this, let  $w \in W'$  and  $x \in \mathcal{X}^{v_2}$  be such that

$w(x) \in \mathcal{X}^{v_2}$ . Then  $w^{-1}v_2w(x) = x$  and so it follows from  $(\clubsuit)$  that  $w^{-1}v_2w = v_2$  or, in other words, that  $w \in C_{W'}(v_2)$ .

But  $|C_{W'}(v_2)| = 96$  thanks to the last two commands of [Arxiv, Comput. I.4]. Since  $C_{W'}(v_2)/\langle v_2 \rangle$  acts freely on  $\mathcal{X}^{v_2}$  by  $(\clubsuit)$ , there are two  $C_{W'}(v_2)$ -orbits in  $\mathcal{X}^{v_2}$ , and we denote by  $p_1$  and  $p'_1$  their image in  $\mathcal{X}/W'$ . Still by  $(\clubsuit)$ , they are both singularities of type  $A_1$  (indeed, if  $x \in \mathcal{X}^{v_2}$ , the action of  $v_2$  on  $T_x(\mathcal{X})$  has no eigenvalue equal to 1, because  $x$  is isolated; thus,  $v_2$  acts as  $-\text{Id}_{T_x(\mathcal{X})}$ ).  $\square$

**Lemma 4.3 (Elements of order 3).** — *There are four conjugacy classes of elements of order 3: the one of the element  $w_3$  defined after Theorem 2.12 with eigenvalues  $\{\{\zeta_3, \zeta_3, \zeta_3, \zeta_3^2, \zeta_3^2, \zeta_3^2\}\}$ , the one of its inverse  $w_3^{-1}$ , the one of an element  $v_3$  with eigenvalues  $\{\{1, 1, \zeta_3, \zeta_3, \zeta_3^2, \zeta_3^2\}\}$  and the one of an element  $u_3$  with eigenvalues  $\{\{1, 1, 1, 1, \zeta_3, \zeta_3^2\}\}$ . Moreover:*

- (a)  $\mathcal{X}^{w_3} = \mathcal{X}^{w_3^{-1}}$  has pure dimension 1.
- (b)  $\mathcal{X}^{v_3}$  contains 12 points. Moreover, if  $x \in \mathcal{X}^{v_3}$ , then  $W'_x$  has order 9 and acts as a reflection group on  $T_x(\mathcal{X})$ . Hence, the image of  $x$  in  $\mathcal{X}/W'$  is smooth.
- (c)  $\mathcal{X}^{u_3}$  contains 96 points. Moreover, if  $x \in \mathcal{X}^{u_3}$ , then one of the following holds:
  - (c1)  $W'_x = \langle u_3 \rangle$  and  $u_3$  acts on  $T_x(\mathcal{X})$  with eigenvalues  $\zeta_3$  and  $\zeta_3^2$ . Hence the image of  $x$  in  $\mathcal{X}/W'$  is an  $A_2$  singularity.
  - (c2)  $W'_x$  has order 9 and acts as a reflection group on  $T_x(\mathcal{X})$ . Hence, the image of  $x$  in  $\mathcal{X}/W'$  is smooth.

Also, all the points  $x$  satisfying (c1) belong to the same  $W'$ -orbit, and their image  $p_2$  in  $\mathcal{X}/W'$  is not equal to  $p_9$  or  $p'_9$ .

*Proof of Lemma 4.3.* — The first statement and (a) are both proved in [Arxiv, Comput. I.5].

(b) The first three commands of [Arxiv, Comput. I.6] show that  $\mathcal{X}^{v_3}$  has dimension 0. The next two commands of [Arxiv, Comput. I.6] show that the set  $G(v_3)$  of elements  $w \in W'$  such that  $\mathcal{X}^{v_3} \cap \mathcal{X}^w \neq \emptyset$  has cardinality 9. The next three commands of [Arxiv, Comput. I.6] show that  $G(v_3)$  is a group isomorphic to  $\mu_3 \times \mu_3$ . The last command of [Arxiv, Comput. I.6] show that  $\mathcal{X}^{v_3} \subset \mathcal{X}^w$  for all  $w \in G(v_3)$ .

This implies that, if  $x \in \mathcal{X}^{v_3}$ , then its stabilizer in  $W'$  is isomorphic to  $\mu_3 \times \mu_3$ . But, up to conjugacy, the only subgroup of  $\text{GL}_{\mathbb{C}}(T_x(\mathcal{X})) \simeq \text{GL}_2(\mathbb{C})$  isomorphic to  $\mu_3 \times \mu_3$  is  $\langle \text{diag}(\zeta_3, 1), \text{diag}(1, \zeta_3) \rangle$ , which is a reflection group.

(c) The first four commands of [Arxiv, Comput. I.7] show that  $\mathcal{X}^{u_3}$  has dimension 0 and contains 96 points. The next two commands of [Arxiv, Comput. I.7] show that the normalizer  $N(u_3) = N_{W'}(\langle u_3 \rangle)$  has order 216. Note that  $N(u_3)$  acts on  $\mathcal{X}^{u_3}$ . Also, the scheme  $\mathcal{X}^{u_3}$  is naturally defined over  $\mathbb{Q}$  (we denote by  $\mathcal{X}_{\mathbb{Q}}^{u_3}$  its  $\mathbb{Q}$ -form) and computations in MAGMA are performed over  $\mathbb{Q}$ . Therefore, the last three commands of [Arxiv, Comput. I.7] show that the scheme  $\mathcal{X}_{\mathbb{Q}}^{u_3}$  has three irreducible components  $\mathcal{F}_1$ ,  $\mathcal{F}_2$  and  $\mathcal{F}_3$ , admitting respectively 72, 12 and 12 points over  $\mathbb{C}$ : we set  $A = \mathcal{F}_1(\mathbb{C})$  and  $B = \mathcal{F}_2(\mathbb{C}) \cup \mathcal{F}_3(\mathbb{C})$ . In particular,  $|A| = 72$  and  $|B| = 24$ . Note that the group  $N(u_3)/\langle u_3 \rangle$  acts on  $A$  and  $B$ .

(c1) The first three lines of [Arxiv, Comput. I.8] show that, if  $w \in W' \setminus \langle u_3 \rangle$ , then  $A \cap \mathcal{X}^w = \emptyset$ . This shows that  $N(u_3)/\langle u_3 \rangle$  acts freely on  $A$ . Since  $|A| = 72 = |N(u_3)/\langle u_3 \rangle|$ , this also shows that  $N(u_3)$  acts transitively: we denote by  $p'_3$  the unique point in the image of  $A$  in  $\mathcal{X}/W'$ . Note that  $p_2 \notin \{p_9, p'_9\}$  by the last line of [Arxiv, Comput. I.8].

Since the stabilizer of any point  $x$  in  $A$  is equal to  $\langle u_3 \rangle$ , we need to determine the eigenvalues of  $u_3$  for its action on  $T_x(\mathcal{X})$ . Now, let  $v \in V \setminus \{0\}$  be such that  $[v] \in A$ . Then  $u_3(v) = \xi v$  for some  $\xi \in \mu_3$ . Since  $x$  is a smooth point of  $\mathcal{X}$ , it follows from Corollary 1.4 that the multiset  $\{\{\xi^{-2}, \xi^{-6}, \xi^{-8}\}\}$  is contained in the multiset  $\{\{\xi^{-1}, \xi^{-1}, \xi^{-1}, \xi^{-1}, \xi^{-1}\zeta_3, \xi^{-1}\zeta_3^2\}\}$ . This forces  $\xi = 1$  and, still by Corollary 1.4, this implies that the eigenvalues of  $u_3$  for its action on  $T_x(\mathcal{X})$  are  $\zeta_3$  and  $\zeta_3^2$ . This shows that  $p_2$  is an  $A_2$  singularity.

(c2) The first three lines of [Arxiv, Comput. I.9] show that the set  $T_B$  of elements  $w$  of  $W'$  such that  $B \cap \mathcal{X}^w \neq \emptyset$  contains 15 elements. By the next three lines of [Arxiv, Comput. I.9], the group  $G_B$  generated by  $T_B$  has order 27 and is contained in  $N(u_3)$ . Now, let  $x \in B$ . Then its stabilizer  $W'_x$  is contained in  $G_B$ . Since  $|N(u_3)| = 216$  and  $|B| = 24$ , this forces  $|W'_x| \geq 9$ . But  $W'_x \neq G_B$  because there are elements  $w \in G_B$  such that  $B \cap \mathcal{X}^w = \emptyset$  (see the seventh line of [Arxiv, Comput. I.9] we have that  $G_B \simeq \mu_3 \times \mu_3 \times \mu_3$ , so  $W'_x \simeq \mu_3 \times \mu_3$ . Now, as in (b), we conclude that the image of  $x$  in  $\mathcal{X}/W'$  is smooth because the unique subgroup of  $\text{GL}_2(\mathbb{C})$  isomorphic to  $\mu_3 \times \mu_3$  is generated by reflections.  $\square$

**Lemma 4.4 (Elements of order 4).** — *There are two conjugacy classes of elements of order 4: the one of the element  $w_4$  defined after Theorem 2.12 with eigenvalues  $\{\{1, 1, i, i, -i, -i\}\}$  and the one of an element  $v_4$  admitting  $\{\{1, 1, -1, -1, i, -i\}\}$  as a list of eigenvalues. Moreover:*

- (a) *The image of  $\mathcal{X}^{w_4}$  in  $\mathcal{X}/W'$  is  $\{p_{12}, p'_{12}\}$ .*
- (b)  *$\mathcal{X}^{v_4} = \emptyset$ .*

*Proof of Lemma 4.4.* — This follows immediately from [Arxiv, Comput. I.11].  $\square$

**Lemma 4.5 (Elements of order 5).** — *There is a unique conjugacy class of elements of order 5, namely the one of the element  $w_5$  defined after Theorem 2.12, which has  $\{\{1, 1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4\}\}$  as list of eigenvalues. Moreover,  $\mathcal{X}^{w_5}$  has dimension 0 and its image in  $\mathcal{X}/W'$  is the point  $p_5$  defined in the proof of Lemma 3.7 (and recall that it is an  $A_4$  singularity by Lemma 3.9).*

*Proof of Lemma 4.5.* — The first statement is proved in the first two lines of [Arxiv, Comput. I.12]. The last three lines of [Arxiv, Comput. I.12] show that  $\mathcal{X}^{w_5}$  has dimension 0 and contains 4 points.

Now, for  $1 \leq k \leq 4$ , let  $v_k \in V \setminus \{0\}$  be such that  $w_5(v_k) = \zeta_5^k v_k$ . Then  $f_2(v_k) = f_6(v_k) = f_8(v_k) = f_9(v_k) = f_{12}(v_k) = 0$  by Lemma 1.1, so  $[v_k] \in \mathcal{X}^{w_5}$ . Therefore,

$$\mathcal{X}^{w_5} = \{[v_1], [v_2], [v_3], [v_4]\}$$

and the last statement now follows from the proof of Lemma 3.7.  $\square$

**Lemma 4.6 (Elements of order 6).** — *There are six conjugacy classes of elements of order 6: the one of the element  $w_6$  defined after Theorem 2.12 with eigenvalues  $\{\{\zeta_6, \zeta_6, \zeta_6^{-1}, \zeta_6^{-1}, \zeta_3, \zeta_3^{-1}\}\}$ , the one of  $w_6^{-1}$ , and four others whose elements  $w$  satisfies  $\dim \text{Ker}(w - \zeta_6 \text{Id}_V) \leq 1$ . Moreover:*

- (a) *The image of  $\mathfrak{X}^{w_6} = \mathfrak{X}^{w_6^{-1}}$  in  $\mathfrak{X}/W'$  is  $\{p_{12}, p'_{12}\}$ .*
- (b) *If  $w$  is an element of order 6 such that  $\dim \text{Ker}(w - \zeta_6 \text{Id}_V) \leq 1$ , then  $\mathfrak{X}^w = \emptyset$ .*

*Proof of Lemma 4.6.* — The first statement follows from the first 8 commands of [Arxiv, Comput. I.13]. (a) follows from the next two commands of [Arxiv, Comput. I.13]. (b) follows from the last command of [Arxiv, Comput. I.13].  $\square$

**Lemma 4.7 (Elements of order 9 or 12).** — *There are two conjugacy classes of elements of order 9 (resp. 12): the one of the element  $w_9$  (resp.  $w_{12}$ ) defined after Theorem 2.12 and the one of  $w_9^{-1}$  (resp.  $w_{12}^{-1}$ ).*

- (a) *The image of  $\mathfrak{X}^{w_9} = \mathfrak{X}^{w_9^{-1}}$  is equal to  $\{p_9, p'_9\}$ .*
- (b) *The image of  $\mathfrak{X}^{w_{12}} = \mathfrak{X}^{w_{12}^{-1}}$  is equal to  $\{p_{12}, p'_{12}\}$ .*

*Proof of Lemma 4.7.* — This is proved in [Arxiv, Comput. I.14], following the same arguments as in the proof of Lemma 4.5.  $\square$

Since the order of an element of  $W'$  belongs to  $\{1, 2, 3, 4, 5, 6, 9, 12\}$  by [Arxiv, Comput. I.15], we have investigated all the possibilities. So the proposition follows from the series of Lemmas 4.2-4.7.  $\square$

**Remark 4.8.** — The fact that  $w_3$  and  $w_3^{-1}$  are not conjugate in  $W'$  can also be checked without computer: in fact, they are not conjugate in  $\text{SO}(V)$  by [Bon2, Lemma 1.7].  $\blacksquare$

For the singular points of  $\mathfrak{X}/W'$ , we keep the notation of the proof of Proposition 4.1:

- The two singular points of type  $A_1$  are denoted by  $p_1$  and  $p'_1$ . Note that they are both  $\sigma$ -fixed because they lie above smooth points of  $\mathfrak{X}/W$  (so they must lie on the ramification locus of  $\omega : \mathfrak{X}/W' \rightarrow \mathfrak{X}/W$ ).
- The singular point of type  $A_2$  coming from the fixed point locus  $\mathfrak{X}^{u_3}$ , where  $u_3$  has order 3 and satisfies  $\dim V^{u_3} = 4$ , is denoted by  $p_2$ . Again,  $\sigma(p_2) = p_2$ .
- The singular points  $p_5, p_9, p'_9, p_{12}$  and  $p'_{12}$  involved in Lemma 3.7 are of type  $A_4, A_2, A_2, A_3$  and  $A_3$  respectively.



**4.B. Two smooth rational curves.** — Recall that  $V(3)$  is the eigenspace of  $w_3$  for the eigenvalue  $\zeta_3$  and denote by  $V^-(3)$  the eigenspace of  $w_3$  for the eigenvalue  $\zeta_3^{-1}$ . By (2.11), they both have dimension 3 and so

$$(4.9) \quad V = V(3) \oplus V^-(3).$$

Let  $\mathcal{C}^+ = \mathbb{P}(V(3)) \cap \mathcal{X}$  and  $\mathcal{C}^- = \mathbb{P}(V^-(3)) \cap \mathcal{X}$ . By Theorem 2.12(f),  $w_3$  and  $w_3^{-1}$  are conjugate in  $W$ , but it follows from Lemma 4.3 that they are not conjugate in  $W'$ . Fix  $g \in W$  be such that  $w_3^{-1} = gw_3g^{-1}$ . Then  $g \notin W'$  and  $g(V(3)) = V^-(3)$ . This shows that

$$(4.10) \quad \mathcal{C}^- = g(\mathcal{C}^+).$$

**Lemma 4.11.** — *The schemes  $\mathcal{C}^+$  and  $\mathcal{C}^-$  are smooth irreducible curves of genus 10.*

*Proof.* — By (4.9),  $\mathbb{P}(V)^{w_3} = \mathbb{P}(V(3)) \dot{\cup} \mathbb{P}(V^-(3))$ , where  $\dot{\cup}$  means a disjoint union. So

$$(4.12) \quad \mathcal{X}^{w_3} = \mathcal{C}^+ \dot{\cup} \mathcal{C}^-.$$

Since  $\mathcal{X}^{w_3}$  is smooth, this implies that  $\mathcal{C}^+$  and  $\mathcal{C}^-$  are smooth. Now, by Lemma 1.1, the restriction of  $f_2, f_5$  and  $f_8$  to  $V(3)$  are equal to 0. Therefore,

$$\mathcal{C}^+ = \{x \in \mathbb{P}(V(3)) \mid f_6(v) = 0\}$$

This shows that  $\mathcal{C}^+$  is a connected curve of  $\mathbb{P}(V(3))$ . Since it is smooth, it must be irreducible. Moreover, it is of degree 6, so its genus is equal to 10.  $\square$

Let  $\mathcal{C}_5^+$  (resp.  $\mathcal{C}_5^-$ ) denote the image of  $\mathcal{C}^+$  (resp.  $\mathcal{C}^-$ ) in  $\mathcal{X}/W'$  and let  $\mathcal{C}_5$  denote the image of  $\mathcal{C}^+$  in  $\mathcal{X}/W \simeq \mathbb{P}(5, 3, 4)$ . Note that  $\mathcal{C}_5$  is also the image of  $\mathcal{C}^-$  by (4.10). Moreover,

$$(4.13) \quad \mathcal{C}_5 = \{[y_5, y_3, y_4] \in \mathbb{P}(5, 3, 4) \mid y_5 = 0\} = \mathbb{P}(3, 4) = \mathbb{P}^1(\mathbb{C}).$$

Indeed,  $\mathcal{C}_5$  is irreducible, of dimension 1 and contained in  $\{[y_5 : y_3 : y_4] \in \mathbb{P}(5, 3, 4) \mid y_5 = 0\}$  by Lemma 1.1 (see also Theorem 2.12(d)).

**Proposition 4.14.** — *We have  $\mathcal{C}_5^- = \sigma(\mathcal{C}_5^+) \neq \mathcal{C}_5^+$ . Moreover,  $\mathcal{C}_5^+$  and  $\mathcal{C}_5^-$  are both isomorphic to  $\mathbb{P}^1(\mathbb{C})$ , intersect transversely along only one point and satisfy*

$$\mathcal{C}_5^+ \cup \mathcal{C}_5^- = \{[y_5 : y_3 : y_4 : j] \in \mathcal{X}/W' \mid y_5 = 0\}.$$

*Proof.* — The fact that  $\mathcal{C}_5^- = \sigma(\mathcal{C}_5^+)$  follows from (4.10). Now, the irreducibility of  $\mathcal{C}^+$  and  $\mathcal{C}^-$  implies the irreducibility of  $\mathcal{C}_5^+$  and  $\mathcal{C}_5^-$ . Also, by (4.13),

$$\mathcal{C}_5^+ \cup \mathcal{C}_5^- \simeq \{[y_3 : y_4 : j] \in \mathbb{P}(3, 4, 12) \mid j^2 = Q_f(0, y_3, y_4)\}.$$

But  $\mathbb{P}(3, 4, 12) = \mathbb{P}(3, 1, 3) = \mathbb{P}(1, 1, 1) = \mathbb{P}^2(\mathbb{C})$ , so there exists a polynomial  $Q_3 \in \mathbb{C}[T_3, T_4]$ , which is homogenous of degree 2 (with  $T_k$  of degree 1) and such that  $Q_f(0, y_3, y_4) = Q_3(y_3^4, y_4^3)$ . Therefore,

$$\mathcal{C}_5^+ \cup \mathcal{C}_5^- \simeq \{[t_3 : t_4 : j] \in \mathbb{P}^2(\mathbb{C}) \mid j^2 = Q_3(t_3, t_4)\}.$$

It just remains to prove that the polynomial  $Q_3$  is the square of a linear form: in other words, we only need to prove that  $\mathcal{C}_5^+ \neq \mathcal{C}_5^-$ .

Since  $\mathcal{C}^+$  is irreducible of degree 6, it cannot be contained in a union of projective lines of  $\mathbb{P}(V(3))$ . Since moreover  $\delta^*(3) = \delta(3) = 3$ , Theorem 2.12(f) thus implies that there exists  $v \in V(3) \setminus \{0\}$  such that  $[v] \in \mathcal{C}^+$  and  $W_v = 1$ . We now only need to prove that  $\pi_f'([v]) \notin \mathcal{C}^-$ .

So assume that  $\pi'_f([v]) \in \mathcal{C}^-$ . This would imply that there exists  $w \in W'$  such that  $w(v) \in V^-(3)$ . Consequently,

$$w^{-1}w_3ww_3(v) = v,$$

and so  $w^{-1}w_3ww_3 = 1$  since  $W_v = 1$ . Hence,  $w_3$  and  $w_3^{-1}$  are conjugate in  $W'$ , which is impossible by Lemma 4.3.  $\square$

By exchanging  $p_9$  and  $p'_9$  (resp.  $p_{12}$  and  $p'_{12}$ ) if necessary, we may assume that

$$p_9, p_{12} \in \mathcal{C}_5^+ \quad \text{and} \quad p'_9, p'_{12} \in \mathcal{C}_5^-.$$

**4.C. Singular points, equation.** — Of course, the equation giving  $\mathcal{X}/W'$  and the coordinates of the singular points depend on a model for  $W$  and on a choice of fundamental invariants. With the choices made in [Arxiv, App. I], we get:

**Lemma 4.15.** — *The coordinates of the singular points in  $\mathbb{P}(5, 3, 4, 12)$  are given by*

$$\begin{aligned} p_5 &= [1 : 0 : 0 : 0], & p_9, p'_9 &= [0 : 1 : 0 : \pm \frac{27}{64}\sqrt{-3}], & p_{12}, p'_{12} &= [0 : 0 : 1 : \pm \frac{16}{243}\sqrt{-3}] \\ p_1, p'_1 &= [\frac{1}{9}(\eta 536\sqrt{19} + 2336) : \frac{1}{9}(-\eta 60\sqrt{19} - 260) : \frac{1}{3}(\eta 130\sqrt{19} + 565) : 0], & \text{with } \eta &= \pm 1, \\ p_2 &= [4/9 : 10/9 : -5/12 : 0]. \end{aligned}$$

*Proof.* — The coordinates of  $p_5$  are given by Lemma 1.1. Let  $v_9$  (resp.  $v_{12}$ ) be a generator of the line  $\text{Ker}(w_9 - \zeta_9 \text{Id}_V)$  (resp.  $\text{Ker}(w_{12} - \zeta_{12} \text{Id}_V)$ ). The computation of the images  $p_9$  and  $p_{12}$  of  $[v_9]$  and  $[v_{12}]$  are done in [Arxiv, Comput. I.16] (note that we already know that  $f_5(v_9) = f_5(v_{12}) = f_9(v_{12}) = f_{12}(v_9) = 0$ ). To renormalize  $p_9$  (resp.  $p_{12}$ ), we just need to compute the corresponding value of  $j^2/y_3^8$  (resp.  $j^2/y_4^6$ ) and we get the result.

For the coordinates of  $p'_1$  and  $p''_1$ , we compute the image of  $\mathcal{X}^{v_2}$  and renormalize the solutions (see [Arxiv, Comput. I.17]). We do a similar work for  $p_2$ , by using the set  $A$  involved in the proof of Lemma 4.3 (see [Arxiv, Comput. I.18]).  $\square$

Let us complete this result by computing the coordinates of the intersection point  $p$  of  $\mathcal{C}_5^+$  and  $\mathcal{C}_5^-$ . For this, note that  $f_5(x) = \text{Jac}(x) = 0$  for all  $x \in \mathcal{X}^{v_3}$  (see the first two lines of [Arxiv, Comput. I.19]). Therefore,  $p$  is the image of  $\mathcal{X}^{v_3}$  under the map  $\pi'_f$ . This image is computed in [Arxiv, Comput. I.19] and we get

$$(4.16) \quad p := [0 : 2/9 : 1/4 : 0].$$

We are now ready to compute the equation of  $\mathcal{X}/W'$ . A direct way would be to compute the polynomial  $P_f$  involved in (2.10). However, this is a too long computation and our computer collapsed... So we need another approach.

First, it follows from Lemma 4.14 and its proof that there exist  $\alpha$  and  $\beta$  in  $\mathbb{C}$  such that

$$Q_f(y_5, y_3, y_4) \equiv (\alpha y_3^4 + \beta y_4^3)^2 \pmod{y_5}.$$

To obtain the coefficients  $\alpha$  and  $\beta$ , we just apply this equation to the coordinates of  $p_9$ ,  $p'_9$ ,  $p_{12}$ ,  $p'_{12}$  and  $p$  computed in Lemma 4.15 and in (4.16). We then get

$$Q_f(y_5, y_3, y_4) \equiv -3 \left( \frac{27}{64} y_3^4 - \frac{16}{243} y_4^3 \right)^2 \pmod{y_5}.$$

By investigating all possible monomials of degree 19 in  $y_5, y_3, y_4$ , we get that there exist complex numbers  $a, b, c, d, e$  such that

$$Q_f(y_5, y_3, y_4) \equiv -3\left(\frac{27}{64}y_3^4 - \frac{16}{243}y_4^3\right)^2 - y_5R$$

with

$$R = y_3y_4(ay_3^4 + by_4^3) + cy_5y_3^2y_4^2 + dy_5^2y_3^3 + ey_5^4y_4.$$

Using the fact that  $p_1, p'_1, p_2$  belong to  $\mathcal{X}/W'$ , this gives three equations for  $a, b, c, d, e$ . Computing the image  $\mathcal{X}^{s_2} \cap \mathbb{P}(\text{Ker}(x_1))$  in  $\mathcal{X}/W'$  gives two Galois conjugate points  $r$  and  $r'$  with coordinates in  $\mathbb{Q}(\sqrt{6})$  (see [Arxiv, Comput. I.20]). This leads to two other equations for  $a, b, c, d, e$ . Solving this system with five equations and five unknowns (see [Arxiv, Comput. I.21]) yields a unique solution, which is given by  $(a, b, c, d, e) = (\frac{207}{32}, \frac{800}{729}, \frac{1375}{81}, -\frac{3125}{864}, -\frac{3125}{108})$ . In other words:

$$(4.17) \quad \mathcal{X}/W' = \{[y_5 : y_3 : y_4 : j] \in \mathbb{P}(5, 3, 4, 12) \mid j^2 = -3\left(\frac{27}{64}y_3^4 - \frac{16}{243}y_4^3\right)^2 - y_5R(y_5, y_3, y_4)\}$$

with

$$(4.18) \quad R = y_3y_4\left(\frac{207}{32}y_3^4 + \frac{800}{729}y_4^3\right) + \frac{1375}{81}y_5y_3^2y_4^2 - \frac{3125}{864}y_5^2y_3^3 - \frac{3125}{108}y_5^3y_4.$$

**4.D. Affine charts.** — In the sequel, we will need to compute in the standard open affine charts of  $\mathbb{P}(5, 3, 4, 12)$ . If  $k \in \{3, 4, 5\}$ , we denote by  $\mathcal{U}_k$  the open subset defined by

$$\mathcal{U}_k = \{[y_5 : y_3 : y_4 : j] \in \mathbb{P}(5, 3, 4, 12) \mid y_k \neq 0\}.$$

Since  $\mathcal{X}/W'$  is contained in  $\mathcal{U}_5 \cup \mathcal{U}_3 \cup \mathcal{U}_4$ , we do not need to work in the affine chart defined by  $j \neq 0$ .

*4.D.1. The affine chart  $\mathcal{U}_3$ .* — Let us first start with  $\mathcal{U}_3$ . Setting  $y_3 = 1$ , the new variables are  $a_3 = y_5^3, b_3 = y_4^3, c_3 = y_5y_4$  and  $j$ . So

$$\mathcal{U}_3 = \{(a_3, b_3, c_3, j) \in \mathbb{A}^4(\mathbb{C}) \mid c_3^3 = a_3b_3\}.$$

Then it follows from (4.17) that

$$(4.19) \quad (\mathcal{X}/W') \cap \mathcal{U}_3 = \{(a_3, b_3, c_3, j) \in \mathbb{A}^4(\mathbb{C}) \mid c_3^3 = a_3b_3 \text{ and } j^2 = -3\left(\frac{27}{64} - \frac{16}{243}b_3\right)^2 - \frac{207}{32}c_3 - \frac{800}{729}b_3c_3 - \frac{1375}{81}c_3^2 + \frac{3125}{864}a_3 + \frac{3125}{108}a_3c_3\}.$$

*4.D.2. The affine chart  $\mathcal{U}_4$ .* — Let us now turn to  $\mathcal{U}_4$ . Setting  $y_4 = 1$ , the new variables are  $a_4 = y_5^4, b_4 = y_3^4, c_4 = y_5y_3$  and  $j$ . So

$$\mathcal{U}_4 = \{(a_4, b_4, c_4, j) \in \mathbb{A}^4(\mathbb{C}) \mid c_4^4 = a_4b_4\}.$$

Then it follows from (4.17) that

$$(4.20) \quad (\mathcal{X}/W') \cap \mathcal{U}_4 = \{(a_4, b_4, c_4, j) \in \mathbb{A}^4(\mathbb{C}) \mid c_4^4 = a_4b_4 \text{ and } j^2 = -3\left(\frac{27}{64}b_4 - \frac{16}{243}\right)^2 - \frac{207}{32}b_4c_4 - \frac{800}{729}c_4 - \frac{1375}{81}c_4^2 + \frac{3125}{864}c_4^3 + \frac{3125}{108}a_4\}.$$

4.D.3. *The affine chart  $\mathcal{U}_5$ .* — Obtaining the equation of  $(\mathcal{X}/W') \cap \mathcal{U}_5$  is much more complicated. This is done in [Arxiv, §I.E]). The result is given by [Arxiv, Comput. I.26]: setting  $y_5 = 1$  and

$$Y_3 = y_3^5, \quad Y_4 = y_4^5, \quad h_1 = y_3 j, \quad h_2 = y_3^2 y_4, \quad h_3 = y_4^2 j \quad \text{and} \quad h_4 = y_3 y_4^3,$$

we have that  $(\mathcal{X}/W') \cap \mathcal{U}_5$  may be identified with the closed subvariety of  $\mathbb{A}^6(\mathbb{C})$  consisting of points  $(Y_3, Y_4, h_1, h_2, h_3, h_4)$  such that

$$(4.21) \quad \begin{cases} h_1 h_4 = h_2 h_3, \\ Y_4 h_2 = h_4^2, \\ Y_4 h_1 = h_3 h_4, \\ Y_3 h_4 = h_2^3, \\ Y_3 h_3 = h_1 h_2^2, \\ Y_3 Y_4 = h_2^2 h_4, \\ Y_3^2 + \frac{2944}{243} Y_3 h_2 - \frac{2048}{6561} Y_3 h_4 - \frac{400000}{59049} Y_3 + \frac{4096}{2187} h_1^2 \\ \quad + \frac{5632000}{177147} h_2^2 + \frac{3276800}{1594323} h_2 h_4 - \frac{3200000}{59049} h_2 + \frac{1048576}{43046721} h_4^2 = 0, \\ Y_3 h_2 h_4 + \frac{1048576}{43046721} Y_4^2 + \frac{3276800}{1594323} Y_4 h_4 - \frac{3200000}{59049} Y_4 \\ \quad + \frac{2944}{243} h_2^2 h_4 - \frac{2048}{6561} h_2 h_4^2 - \frac{400000}{59049} h_2 h_4 + \frac{4096}{2187} h_3^2 + \frac{5632000}{177147} h_4^2 = 0, \\ Y_3 h_2^2 + \frac{2944}{243} Y_3 h_4 + \frac{1048576}{43046721} Y_4 h_4 + \frac{4096}{2187} h_1 h_3 \\ \quad - \frac{2048}{6561} h_2^2 h_4 - \frac{400000}{59049} h_2^2 + \frac{5632000}{177147} h_2 h_4 + \frac{3276800}{1594323} h_4^2 - \frac{3200000}{59049} h_4 = 0. \end{cases}$$

4.E. *Two other smooth rational curves in  $\mathcal{X}/W'$ .* — For  $k \in \{3, 4\}$ , we set

$$\mathcal{C}_k = \{[y_5 : y_3 : y_4 : j] \in \mathcal{X}/W' \mid y_k = 0\}.$$

**Proposition 4.22.** — *The schemes  $\mathcal{C}_3$  and  $\mathcal{C}_4$  are reduced, irreducible and are smooth rational curves. They intersect transversely at  $p_5 = [1 : 0 : 0 : 0]$ .*

*Proof.* — From the equation (4.17), we get that

$$\mathcal{C}_3 \simeq \{[y_5 : y_4 : j] \in \mathbb{P}(5, 4, 12) \mid j^2 = -\frac{2^8}{3^9} y_4^6 + \frac{3125}{108} y_5^4 y_4\}.$$

But  $\mathbb{P}(5, 4, 12) = \mathbb{P}(5, 1, 3)$ , so

$$\mathcal{C}_3 \simeq \{[y_5 : y_4 : j] \in \mathbb{P}(5, 1, 3) \mid j^2 = -\frac{2^8}{3^9} y_4^6 + \frac{3125}{108} y_5 y_4\}.$$

The open subset  $\mathcal{C}_3 \cap \mathcal{U}_4$  is clearly isomorphic to  $\mathbb{A}^1(\mathbb{C})$ .

Similarly, from the equation (4.17), we get that

$$\mathcal{C}_4 \simeq \{[y_5 : y_3 : j] \in \mathbb{P}(5, 3, 12) \mid j^2 = -\frac{3^7}{2^{12}} y_3^6 + \frac{3125}{864} y_5^3 y_3^3\}.$$

But  $\mathbb{P}(5, 3, 12) = \mathbb{P}(5, 1, 4)$ , so

$$\mathcal{C}_4 \simeq \{[y_5 : y_3 : j] \in \mathbb{P}(5, 1, 4) \mid j^2 = -\frac{3^7}{2^{12}} y_3^6 + \frac{3125}{864} y_5 y_3^3\}.$$

The open subset  $\mathcal{C}_4 \cap \mathcal{U}_3$  is clearly isomorphic to  $\mathbb{A}^1(\mathbb{C})$ .

So it remains to prove that  $\mathcal{C}_3$  and  $\mathcal{C}_4$  are smooth and intersect transversely. Both questions are local around the point  $p_5 = [1 : 0 : 0 : 0]$ , so we must work in the affine chart  $\mathcal{U}_5$ . Keep the

notation from (4.21). Then  $\mathcal{C}_3 \cap \mathcal{U}_5$  (resp.  $\mathcal{C}_4 \cap \mathcal{U}_5$ ) is the subvariety of  $(\mathcal{X}/W') \cap \mathcal{U}_5$  defined by  $Y_3 = h_1 = h_2 = h_4 = 0$  (resp.  $Y_4 = h_2 = h_3 = h_4 = 0$ ). Using Equations (4.21), we get

$$\mathcal{C}_3 \cap \mathcal{U}_5 = \{(Y_3, Y_4, h_1, h_2, h_3, h_4) \in \mathbb{A}^6(\mathbb{C}) \mid Y_3 = h_1 = h_2 = h_4 = \lambda_3 Y_4^2 + \mu_3 Y_4 + \nu_3 h_3^2 = 0\}$$

and

$$\mathcal{C}_4 \cap \mathcal{U}_5 = \{(Y_3, Y_4, h_1, h_2, h_3, h_4) \in \mathbb{A}^6(\mathbb{C}) \mid Y_4 = h_2 = h_3 = h_4 = \lambda_4 Y_3^2 + \mu_4 Y_3 + \nu_4 h_1^2 = 0\}$$

for some non-zero rational numbers  $\lambda_3, \mu_3, \nu_3, \lambda_4, \mu_4$  and  $\nu_4$ . The smoothness of  $\mathcal{C}_3 \cap \mathcal{U}_5$  and  $\mathcal{C}_4 \cap \mathcal{U}_5$  follows immediately, as well as the fact they intersect transversely at  $p_5$  (which corresponds to the point  $(0, 0, 0, 0, 0, 0) \in \mathbb{A}^6(\mathbb{C})$ ).  $\square$

## 5. The K3 surface $\tilde{\mathcal{X}}$

Recall that  $\rho : \tilde{\mathcal{X}} \rightarrow \mathcal{X}/W'$  denotes the minimal smooth resolution. We will deduce several properties of  $\tilde{\mathcal{X}}$  (Picard lattice, elliptic fibration,...) from the list of properties of  $\mathcal{X}/W'$  given in the previous section. Note that since  $\tilde{\mathcal{X}}$  is obtained from  $\mathcal{X}/W'$  by successively blowing-up the singular locus, the automorphism  $\sigma$  of  $\mathcal{X}/W'$  lift to an automorphism of  $\tilde{\mathcal{X}}$  (which will still be denoted by  $\sigma$ ).

We denote by  $\Delta_1$  and  $\Delta'_1$  the two smooth rational curves of  $\tilde{\mathcal{X}}$  lying above  $p_1$  and  $p'_1$  respectively. For  $e \in \{2, 5, 9, 12\}$ , we denote by  $\Delta_e^1, \dots, \Delta_e^{r_e}$  the smooth rational curves of  $\tilde{\mathcal{X}}$  lying above  $p_e$  (here,  $r_e$  is the Milnor number of the singularity  $p_e$ <sup>(†)</sup>), and we assume that they are numbered in such a way that  $\Delta_e^k \cap \Delta_e^{k+1} \neq \emptyset$ . For  $e \in \{9, 12\}$ , the smooth rational curves of  $\tilde{\mathcal{X}}$  lying above  $p'_e$  are then given by  $\sigma(\Delta_e^1), \dots, \sigma(\Delta_e^{r_e})$ .

Finally, we denote by  $\tilde{\mathcal{C}}_5^\pm$  the strict transform of  $\mathcal{C}_5^\pm$  in  $\tilde{\mathcal{X}}$ . Of course,  $\tilde{\mathcal{C}}_5^- = \sigma(\tilde{\mathcal{C}}_5^+)$ . As  $\tilde{\mathcal{X}}$  is obtained from  $\mathcal{X}/W'$  by successive blow-ups of points,  $\tilde{\mathcal{C}}_5^+$  and  $\tilde{\mathcal{C}}_5^-$  are smooth rational curves. Also, we denote by  $\tilde{\mathcal{C}}_3$  and  $\tilde{\mathcal{C}}_4$  the strict transforms of  $\mathcal{C}_3$  and  $\mathcal{C}_4$ : for the same reason, they are also smooth rational curves.

One of the aims of this section is to determine the intersection graph of the 22 smooth rational curves  $\tilde{\mathcal{C}}_3, \tilde{\mathcal{C}}_4, \tilde{\mathcal{C}}_5^\pm, \Delta_1, \Delta'_1, (\Delta_e^k)_{e \in \{2, 5, 9, 12\}, 1 \leq k \leq r_e}$  and  $(\sigma \Delta_e^k)_{e \in \{9, 12\}, 1 \leq k \leq r_e}$ . For this, we will use the construction of an elliptic fibration on  $\tilde{\mathcal{X}}$ .

Recall that, for a K3 surface, an *elliptic fibration* is just a morphism to  $\mathbb{P}^1(\mathbb{C})$  such that at least one fiber is a smooth elliptic curve. Since  $\mathcal{X}/W'$  has  $A_4 + 2A_3 + 3A_2 + 2A_1$  singularities, its Picard number is greater than or equal to  $1 + (4 + 2 \cdot 3 + 3 \cdot 2 + 1) = 19$  (in fact, we will see later that it has Picard number 20). Therefore, it admits an elliptic fibration (because every K3 surface with Picard number  $\geq 5$  admits an elliptic fibration [Huy, Chap. 11, Prop. 1.3(ii)]). Another aim of this section is to construct at least one such fibration. For this, let

$$\begin{aligned} \varphi : (\mathcal{X}/W') \setminus \{p_5\} &\longrightarrow \mathbb{P}^1(\mathbb{C}) \\ [y_5 : y_3 : y_4 : j] &\longmapsto [y_3^4 : y_4^3]. \end{aligned}$$

Then  $\varphi$  is a well-defined morphism of varieties, so the map  $\varphi \circ \rho : \tilde{\mathcal{X}} \setminus \rho^{-1}(p_5) \rightarrow \mathbb{P}^1(\mathbb{C})$  is also a well-defined morphism of varieties. We are now ready to prove the second main result of our paper:

<sup>(†)</sup>For the definition of the Milnor number of an isolated hypersurface singularity, see [Mil, §7]: recall that the Milnor number of a singularity of type  $A_k, D_k$  or  $E_k$  is equal to  $k$ .

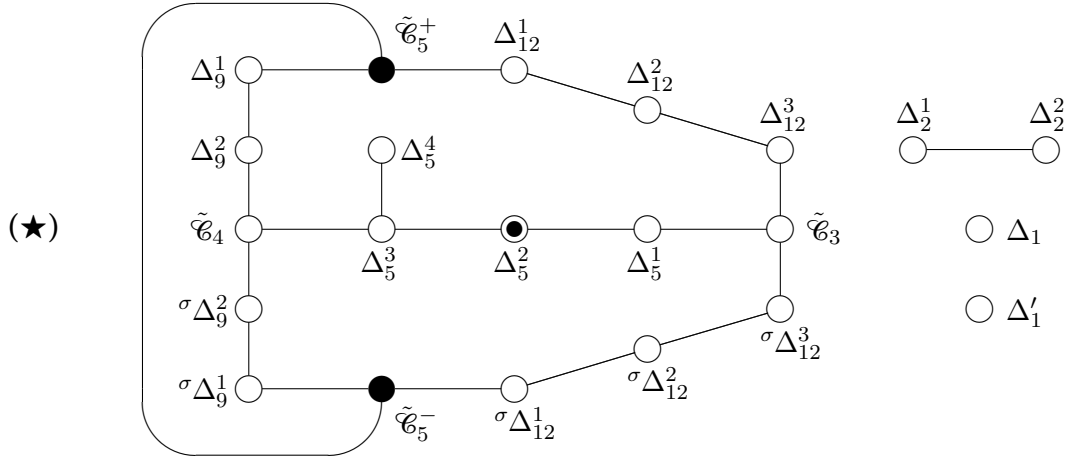
**Theorem 5.1.** — *With the above notation, we have:*

- (a) *The morphism  $\varphi \circ \rho : \tilde{\mathcal{X}} \setminus \rho^{-1}(p_5) \rightarrow \mathbb{P}^1(\mathbb{C})$  extends to a unique morphism*

$$\tilde{\varphi} : \tilde{\mathcal{X}} \rightarrow \mathbb{P}^1(\mathbb{C}),$$

*which is an elliptic fibration whose singular fibers are of type  $E_7 + E_6 + A_2 + 2A_1$ .*

- (b) *There exists a way of numbering the smooth rational curves lying above singular points of  $\mathcal{X}/W'$  such that the intersection graph of the 22 smooth rational curves  $\tilde{\mathcal{C}}_3, \tilde{\mathcal{C}}_4, \tilde{\mathcal{C}}_5^\pm, \Delta_1, \Delta'_1, (\Delta_e^k)_{e \in \{2,5,9,12\}, 1 \leq k \leq r_e}$  and  $(\sigma \Delta_e^k)_{e \in \{9,12\}, 1 \leq k \leq r_e}$  is given by*



*In this graph:*

- (b1) *The union of the singular fibers of  $\tilde{\varphi}$  of type  $E_7$  and  $E_6$  is given by the white disks in the big connected subgraph of (★).*
- (b2) *The singular fibers of  $\tilde{\varphi}$  of type  $A_1$  are  $\tilde{\mathcal{C}}_1 \cup \Delta_1$  and  $\tilde{\mathcal{C}}'_1 \cup \Delta'_1$  for some smooth rational curves  $\tilde{\mathcal{C}}_1$  and  $\tilde{\mathcal{C}}'_1$ .*
- (b3) *The singular fiber of  $\tilde{\varphi}$  of type  $A_2$  is  $\tilde{\mathcal{C}}_2 \cup \Delta_2^1 \cup \Delta_2^2$  for some smooth rational curve  $\tilde{\mathcal{C}}_2$ .*
- (b4) *The curves marked with full black disks in (★) are sections of  $\tilde{\varphi}$ .*
- (b5) *The curve  $\Delta_5^2$  is a double section of  $\tilde{\varphi}$ .*
- (c) *The 22 smooth rational curves in this intersection graph generate the Picard lattice  $\text{Pic}(\tilde{\mathcal{X}})$ . More precisely,  $\text{Pic}(\tilde{\mathcal{X}})$  is generated by the list obtained from these 22 smooth rational curves by removing  $\Delta_5^2$  and  $\Delta_5^4$ . Its discriminant is  $-228 = -2^2 \cdot 3 \cdot 19$ .*
- (d) *The Mordell-Weil group of  $\tilde{\varphi}$  is isomorphic to  $\mathbb{Z}^2$ .*
- (e) *The transcendental lattice is given by the matrix  $\begin{pmatrix} 2 & 0 \\ 0 & 114 \end{pmatrix}$ .*

**Remark 5.2.** — *There are several possible types of singular fibers of type  $A_1$  (resp.  $A_2$ ) in elliptic fibrations. In the above Theorem 5.1, singular fibers of type  $A_1$  (resp.  $A_2$ ) are of type  $I_2$  (resp.  $I_3$ ) in Kodaira's classification. ■*

*Proof.* — *The (very computational) proof of the statements (a) and (b) is given in Appendix I.*

For (c), let  $M$  denote the incidence matrix of the 18 smooth rational curves belonging to the big connected subgraph of (★). Then  $M$  has rank 16 and the greatest common divisor of the diagonal minors of  $M$  is equal to 19 by [Arxiv, Comput. IV.1]. Moreover, the diagonal minor corresponding to the curves  $\tilde{\mathcal{C}}_3, \tilde{\mathcal{C}}_4, \tilde{\mathcal{C}}_5^\pm, \Delta_5^1, \Delta_5^3, (\Delta_e^k)_{e \in \{9,12\}, 1 \leq k \leq r_e}$  and

$(\sigma \Delta_e^k)_{e \in \{9,12\}, 1 \leq k \leq r_e}$  is equal to  $-19$  (see the last command of [Arxiv, Comput. IV.1]). So, if we denote by  $\Lambda$  the lattice generated by these 16 curves together with  $\Delta_1, \Delta'_1, \Delta_2^1$  and  $\Delta_2^2$ , then  $\Lambda$  has rank 20 and discriminant  $-228 = -2^2 \cdot 3 \cdot 19$ . This shows that  $\text{Pic}(\tilde{\mathcal{X}})$  has rank  $\geq 20$ , and so has rank 20 as a K3 surface has always Picard number  $\leq 20$ . If we denote by  $n$  the index of  $\Lambda$  in  $\text{Pic}(\tilde{\mathcal{X}})$ , then  $n^2$  divides 228, which shows that  $n \in \{1, 2\}$ .

But if we denote by  $T(\tilde{\mathcal{X}})$  the transcendental lattice of  $\tilde{\mathcal{X}}$ , then  $T(\tilde{\mathcal{X}})$  has rank  $22 - 20 = 2$ , is even and definite positive, with discriminant  $228/n^2$ . Hence it can be represented by a matrix of the form

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

with  $a, c > 0$  and  $4ac - b^2 = 228/n^2$ . But  $4ac - b^2 \equiv 0$  or  $3 \pmod{4}$ , so  $4ac - b^2 \neq 57$ . This shows that  $n = 1$  and that  $\text{Pic}(\tilde{\mathcal{X}})$  is generated by  $\tilde{\mathcal{C}}_3, \tilde{\mathcal{C}}_4, \tilde{\mathcal{C}}_5^\pm, \Delta_5^1, \Delta_5^3, (\Delta_e^k)_{e \in \{9,12\}, 1 \leq k \leq r_e}$  and  $(\sigma \Delta_e^k)_{e \in \{9,12\}, 1 \leq k \leq r_e}$ , as expected. This concludes the proof of (c).

(d) Since we have determined the Picard lattice of  $\tilde{\mathcal{X}}$  in (c), the structure of the Mordell-Weil group follows (note that it has no torsion, which is compatible with [Shi, Table 1, entry 2420]).

(e) The transcendental lattice of  $\tilde{\mathcal{X}}$  is given by a matrix of the form  $\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$  whose underlying quadratic form is definite positive and has discriminant 228 by (c). The classification of even integral binary quadratic forms [Bue, Theo. 2.3], shows that there are only four such matrices, up to equivalence, namely:

$$M_1 = \begin{pmatrix} 2 & 0 \\ 0 & 114 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 6 & 0 \\ 0 & 38 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 4 & 2 \\ 2 & 58 \end{pmatrix} \quad \text{and} \quad M_4 = \begin{pmatrix} 12 & 6 \\ 6 & 22 \end{pmatrix}.$$

Let  $P = \text{Pic}(\tilde{\mathcal{X}})$  and  $T = T(\tilde{\mathcal{X}})$ . Let

$$P^\perp = \{v \in \mathbb{Q} \otimes_{\mathbb{Z}} P \mid \forall v' \in P, \langle v, v' \rangle \in \mathbb{Z}\}$$

and let us define  $T^\perp$  similarly. Then the quadratic forms on  $\mathbb{Q} \otimes_{\mathbb{Z}} P$  and  $\mathbb{Q} \otimes_{\mathbb{Z}} T$  induce well-defined maps

$$q_P : P^\perp/P \longrightarrow \mathbb{Q}/2\mathbb{Z} \quad \text{and} \quad q_T : T^\perp/T \longrightarrow \mathbb{Q}/2\mathbb{Z}.$$

Since  $H^2(\tilde{\mathcal{X}}, \mathbb{Z})$  is unimodular of signature  $(3, 19)$ , it turns out that there is an isomorphism  $\iota : T^\perp/T \xrightarrow{\sim} P^\perp/P$  such that  $q_T = -q_P \circ \iota$  (see [Nik, Prop. 1.6.1]). In particular, the set of values of  $q_T$  and  $-q_P$  coincide: the subset  $-q_P(P^\perp/P)$  of  $\mathbb{Q}/2\mathbb{Z}$  can easily be computed thanks to (b) and (c) using MAGMA, and we only need to compare the corresponding sets for the four rank 2 lattices determined by  $M_1, M_2, M_3$  and  $M_4$ . This comparison gives the result (see [Arxiv, §IV.B] for detailed computations).  $\square$

**Remark 5.3.** — Theorem 5.1(e) show that the K3 surface  $\tilde{\mathcal{X}}$  is not a Kummer surface [Huy, Chap. 14, Cor. 3.20]. It also shows that  $\tilde{\mathcal{X}}$  is not isomorphic to any of the singular K3 surfaces constructed by Barth-Sarti in [BaSa] or to any of the singular K3 surfaces constructed by Brandhorst-Hashimoto in [BrHa] (see also [BoSa2] for a description of some of these).  $\blacksquare$

## 6. Complement: action of $W$ on the cohomology of $\mathfrak{X}$

The group  $W$  acts on  $\mathfrak{X}$  so it acts on the cohomology groups  $H^k(\mathfrak{X}, \mathbb{C})$ . Since  $\mathfrak{X}$  is a complete intersection in  $\mathbb{P}^5(\mathbb{C})$ , with defining equations of degree 2, 6, and 8, we have:

$$(6.1) \quad \dim_{\mathbb{C}} H^k(\mathfrak{X}, \mathbb{C}) = \begin{cases} 1 & \text{if } k \in \{0, 4\}, \\ 9502 & \text{if } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The action of  $W$  on  $H^0(\mathfrak{X}, \mathbb{C})$  and  $H^4(\mathfrak{X}, \mathbb{C})$  is trivial. The aim of this subsection is to determine the character of the representation of  $W$  afforded by  $H^2(\mathfrak{X}, \mathbb{C})$ .

For this, we first need to parametrize the irreducible characters of  $W$ . If  $\chi \in \text{Irr}(W)$ , we denote by  $b_\chi$  the minimal number  $k$  such that  $\chi$  occurs in the character of the symmetric power  $\text{Sym}^k(V)$  of the natural representation  $V$  of  $W$ . For instance, if we denote by  $\mathbf{1}_W$  the trivial character of  $W$  and by  $\chi_V$  the character afforded by the natural representation  $V$ , then

$$(6.2) \quad b_{\mathbf{1}_W} = 0, \quad b_{\chi_V} = 1 \quad \text{and} \quad b_\varepsilon = |\mathcal{A}| = 36$$

(recall that  $\varepsilon$  denotes the restriction of the determinant to  $W$ ). Indeed, the first two equalities are immediate from the definition and the last one follows from [Bou, Chap. V, §5, Prop. 5] and (2.7). Recall from Molien's formula that the number  $b_\chi$  can be computed as follows: let  $t$  be an indeterminate and let

$$F_\chi(t) = \frac{\prod_{d \in \text{Deg}(W)} (1 - t^d)}{|W|} \sum_{w \in W} \frac{\chi(w^{-1})}{\det(1 - tw)} \in \mathbb{C}(t).$$

It is a classical fact [Bro, §4.5.2] that  $F_\chi(t) \in \mathbb{N}[t]$ , that  $F_\chi(1) = \chi(1)$  and

$$(6.3) \quad b_\chi = \text{val } F_\chi(t).$$

The polynomial  $F_\chi(t)$  is called the *fake degree* of  $\chi$ .

A particular feature of the Weyl group of type E<sub>6</sub> is that the map

$$(6.4) \quad \begin{array}{ccc} \text{DB} : \text{Irr}(W) & \longrightarrow & \mathbb{N} \times \mathbb{N} \\ \chi & \longmapsto & (\chi(1), b_\chi) \end{array}$$

is injective (see for instance the first command of [Arxiv, Comput. II.2]). We denote by  $\mathcal{DB}(W)$  the image of DB and, if  $(d, b) \in \mathcal{DB}(W)$ , we denote by  $\phi_{d,b}$  its inverse image in  $\text{Irr}(W)$ . Note that  $\phi_{d,b}$  is the character afforded by an irreducible representation of dimension  $d$ . For instance, by (6.2), we get

$$(6.5) \quad \phi_{1,0} = \mathbf{1}_W, \quad \phi_{6,1} = \chi_V \quad \text{and} \quad \phi_{1,36} = \varepsilon.$$

By the last line of [Arxiv, Comput. II.2], we have that  $|\text{Irr}(W)| = 25$  and that

$$(6.6) \quad \begin{aligned} \mathcal{DB}(W) = \{ & (1, 0); (1, 36); (6, 1); (6, 25); (10, 9); (15, 17); (15, 4); (15, 16); (15, 5); \\ & (20, 20); (20, 10); (20, 2); (24, 6); (24, 12); (30, 3); (30, 15); (60, 11); \\ & (60, 8); (60, 5); (64, 13); (64, 4); (80, 7); (81, 6); (81, 10); (90, 8) \}. \end{aligned}$$

For  $i \geq 0$ , let  $\chi_{\mathfrak{X}}^{(i)}$  denote the character afforded by the  $W$ -module  $H^i(\mathfrak{X}, \mathbb{C})$ . We set

$$\chi_{\mathfrak{X}} = \sum_{i \geq 0} (-1)^i \chi_{\mathfrak{X}}^{(i)}.$$

By (6.1), we have

$$(6.7) \quad \chi_{\mathfrak{X}} = \chi_{\mathfrak{X}}^{(0)} + \chi_{\mathfrak{X}}^{(2)} + \chi_{\mathfrak{X}}^{(4)} = 2 \cdot \mathbf{1}_W + \chi_{\mathfrak{X}}^{(2)}.$$



We determine  $\chi_{\mathcal{X}}^{(2)}$  in the next proposition:

**Proposition 6.8.** — *With the above notation, we have*

$$\begin{aligned} \chi_{\mathcal{X}}^{(2)} = & \mathbf{1}_W + 3\varepsilon + 8\phi_{6,25} + 2\phi_{10,9} + 7\phi_{15,17} + \phi_{15,4} + 9\phi_{15,16} + \phi_{15,5} \\ & + 14\phi_{20,20} + 4\phi_{20,10} + 2\phi_{24,6} + 8\phi_{24,12} + 14\phi_{30,15} + 18\phi_{60,11} + 12\phi_{60,8} \\ & + 4\phi_{60,5} + 26\phi_{64,13} + 2\phi_{64,4} + 12\phi_{80,7} + 7\phi_{81,6} + 21\phi_{81,10} + 12\phi_{90,8}. \end{aligned}$$

*Proof.* — Since  $\mathcal{X}$  is smooth and  $W$  is finite, it follows from Lefschetz fixed point formula that  $\chi_{\mathcal{X}}(w)$  is equal to the Euler characteristic of the fixed point subvariety  $\mathcal{X}^w$ . If  $\dim(\mathcal{X}^w) \geq 1$ , then  $w$  is conjugate to 1,  $s_1$  or  $w_3$  (see [Arxiv, Comput. II.3]). But:

- $\chi_{\mathcal{X}}(1) = 9\,504$  by (6.1).
- Note that  $\mathbb{P}(V)^{s_1} = [e_1] \cup \mathbb{P}(V^{s_1})$ . Since  $[e_1] \notin \mathcal{X}$  by Remark 2.6, we have that

$$\mathcal{X}^{s_1} = \mathcal{X} \cap \mathbb{P}(V^{s_1}).$$

So  $\mathcal{X}^{s_1}$  is a smooth complete intersection in  $\mathbb{P}(V^{s_1}) \simeq \mathbb{P}^4(\mathbb{C})$  defined by equations of degree 2, 6 and 8 (the restrictions of  $f_2$ ,  $f_6$  and  $f_8$  to  $V^{s_1}$ ), so it has Euler characteristic  $-2 \cdot 6 \cdot 8 \cdot (2 + 6 + 8 - 4 - 1) = -1\,056$ . Hence,  $\chi_{\mathcal{X}}(s_1) = -1\,056$ .

- Recall that  $\mathcal{X}^{w_3}$  is the disjoint union of two smooth curves of genus 10 by Lemma 4.11. So  $\chi_{\mathcal{X}}(w_3) = -36$ .

If  $\dim(\mathcal{X}^w) \leq 0$ , then  $\chi_{\mathcal{X}}(w)$  is just the cardinality of  $\mathcal{X}^w$  (it might be equal to 0). These last values of  $\chi_{\mathcal{X}}$  as well as the decomposition of  $\chi_{\mathcal{X}}$  as a sum of irreducible characters are computed in [Arxiv, Comput. II.4]. The result then follows from (6.7).  $\square$

To be fair, knowing the exact character is not that interesting, but at least we will use it for making a sanity check for Proposition 4.1. Indeed,  $H^k(\mathcal{X}, \mathbb{C})^{W'}$  is the direct sum of  $H^k(\mathcal{X}, \mathbb{C})^W$  and the  $\varepsilon$ -isotypic component of  $H^k(\mathcal{X}, \mathbb{C})$ . Then Proposition 6.8 together with (6.1) shows that

$$\sum_{k \in \mathbb{Z}} (-1)^k \dim_{\mathbb{C}} H^k(\mathcal{X}, \mathbb{C})^{W'} = 6.$$

In other words, the Euler characteristic of  $\mathcal{X}/W'$  is equal to 6. But the fiber of the map  $\tilde{\mathcal{X}} \rightarrow \mathcal{X}/W'$  above an  $A_k$  singularity is the union of  $k$  smooth rational curves in  $A_k$ -configuration, and this union has Euler characteristic  $k + 1$ . So the Euler characteristic of  $\tilde{\mathcal{X}}$  is the Euler characteristic of  $\mathcal{X}/W'$  plus the sum of all the Milnor numbers of singularities of  $\mathcal{X}/W'$ . So, by Proposition 4.1, the Euler characteristic of  $\tilde{\mathcal{X}}$  is

$$6 + 2 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 + 4 = 24,$$

as expected for a K3 surface.

**Remark 6.9.** — Since  $\mathcal{X}$  is a smooth complete intersection, its Hodge numbers can be computed from the degrees of the equations and we get that

$$h^{2,0}(\mathcal{X}) = h^{0,2}(\mathcal{X}) = 1\,591 \quad \text{and} \quad h^{1,1}(\mathcal{X}) = 6\,320.$$

However, we do not know how to compute the character of the representations  $H^{p,q}(\mathcal{X}, \mathbb{C})$  of  $W$ , for  $p + q = 2$ .  $\blacksquare$

## Appendix I

### Proof of Theorem 5.1(a) and (b)

**I.A. Extension of  $\varphi \circ \rho$ .** — Let  $\hat{\rho} : \hat{\mathcal{X}} \rightarrow \mathcal{X}/W'$  denote the blow-up of  $\mathcal{X}/W'$  at  $p_5$ . Since  $p_5$  is an  $A_4$ -singularity,  $\hat{\mathcal{X}}$  contains a unique singular point above  $p_5$ , which will be denoted by  $\hat{p}_5$ . It is an  $A_2$ -singularity of  $\hat{\mathcal{X}}$  and we denote by  $\hat{\pi} : \hat{\mathcal{X}} \rightarrow \hat{\mathcal{X}}$  the blow-up of  $\hat{\mathcal{X}}$  at  $\hat{p}_5$  and we set  $\hat{\hat{\rho}} = \hat{\rho} \circ \hat{\pi}$ . Since  $\tilde{\mathcal{X}}$  is obtained from  $\mathcal{X}/W'$  by successive blow-ups of the singular points, there is a morphism  $\tilde{\mathcal{X}} \rightarrow \hat{\mathcal{X}}$ . Therefore, in order to prove that  $\varphi \circ \rho$  extends to  $\tilde{\mathcal{X}}$ , we only need to prove that the map

$$\varphi \circ \hat{\rho} : \hat{\mathcal{X}} \setminus \hat{\rho}^{-1}(p_5) \rightarrow \mathbb{P}^1(\mathbb{C})$$

extends to a morphism  $\hat{\mathcal{X}} \rightarrow \mathbb{P}^1(\mathbb{C})$ .

For this, it is sufficient to work in the affine chart  $\mathcal{U}_5$ . Let  $\varphi_5$  denote the restriction of  $\varphi$  to  $((\mathcal{X}/W') \cap \mathcal{U}_5) \setminus \{p_5\}$ . Keeping the notation of §4.D,  $\varphi_5$  is defined by

$$\varphi_5(Y_3, Y_4, h_1, h_2, h_3, h_4) = \begin{cases} [Y_3 : h_4] & \text{if } (Y_3, h_4) \neq (0, 0), \\ [h_2^2 : Y_4] & \text{if } (h_2, Y_4) \neq (0, 0). \end{cases}$$

Note that if  $(Y_3, Y_4, h_1, h_2, h_3, h_4) \in (\mathcal{X}/W') \cap \mathcal{U}_5$  is such that  $Y_3 = Y_4 = h_2 = h_4 = 0$ , then  $h_1 = h_3 = 0$ . So  $\varphi_5$  is indeed well-defined by the above formula on  $((\mathcal{X}/W') \cap \mathcal{U}_5) \setminus \{p_5\}$ .

We denote by  $\hat{\mathcal{X}}_5$  the inverse image of  $(\mathcal{X}/W') \cap \mathcal{U}_5$  in  $\hat{\mathcal{X}}$  (so that  $\hat{\mathcal{X}}_5$  is the blow-up of  $(\mathcal{X}/W') \cap \mathcal{U}_5$  at  $p_5$ ). So  $\hat{\mathcal{X}}_5$  is embedded in  $\mathbb{A}^6(\mathbb{C}) \times \mathbb{P}^5(\mathbb{C})$ . If  $t \in \{Y_3, Y_4, h_1, h_2, h_3, h_4\}$ , we denote by  $t'$  the corresponding variable in  $\mathbb{P}^5(\mathbb{C})$  and by  $\hat{\mathcal{X}}_5^{(t)}$  the open affine chart of  $\hat{\mathcal{X}}_5$  defined by  $t' \neq 0$ . Then:

**Lemma I.1.** — *The point  $\hat{p}_5$  belongs to the affine chart  $\hat{\mathcal{X}}_5^{(h_1)}$  and does not belong to the others.*

*Proof.* — This follows from [Arxiv, Comput. III.2] □

To prove that  $\varphi \circ \hat{\rho}$  extends to  $\hat{\mathcal{X}}$ , we first prove the following lemma:

**Lemma I.2.** — *The morphism  $\varphi \circ \hat{\rho} : \hat{\mathcal{X}} \setminus \hat{\rho}^{-1}(p_5) \rightarrow \mathbb{P}^1(\mathbb{C})$  extends to a morphism  $\hat{\mathcal{X}} \setminus \{\hat{p}_5\} \rightarrow \mathbb{P}^1(\mathbb{C})$ .*

*Proof.* — If  $(Y_3, h_4) \neq (0, 0)$ , then  $[Y_3 : h_4] = [Y'_3 : h'_4]$  and so the map  $\varphi_5 \circ \hat{\rho}$  extends to the affine charts  $\hat{\mathcal{X}}_5^{(Y_3)}$  and  $\hat{\mathcal{X}}_5^{(h_4)}$ . Similarly, if  $Y_4 \neq 0$ , then  $[h_2^2 : Y_4] = [h_2 h'_2 : Y'_4]$  and so  $\varphi_5 \circ \hat{\rho}$  extends to the affine chart  $\hat{\mathcal{X}}_5^{(Y_4)}$ .

Now, it follows from [Arxiv, Comput. III.4] that the two affine charts  $\hat{\mathcal{X}}_5^{(h_2)}$  and  $\hat{\mathcal{X}}_5^{(h_3)}$  can be embedded in  $\mathbb{A}^4(\mathbb{C})$ , and that maps  $\hat{\mathcal{X}}_5^{(h_2)} \rightarrow (\mathcal{X}/W') \cap \mathcal{U}_5$  and  $\hat{\mathcal{X}}_5^{(h_3)} \rightarrow (\mathcal{X}/W') \cap \mathcal{U}_5$  are given by

$$\begin{aligned} \hat{\mathcal{X}}_5^{(h_2)} &\longrightarrow (\mathcal{X}/W') \cap \mathcal{U}_5 \\ (a, b, c, d) &\longmapsto (a^2 d, a d^3, a b d, a d, a c d, a d^2) \end{aligned}$$

and

$$\begin{aligned} \hat{\mathcal{X}}_5^{(h_3)} &\longrightarrow (\mathcal{X}/W') \cap \mathcal{U}_5 \\ (a, b, c, d) &\longmapsto (a b, a c, a d, a c d^2, a, a c d). \end{aligned}$$

So the restriction of  $\varphi_5 \circ \hat{\rho}$  to  $\hat{\mathcal{X}}_5^{(h_2)} \setminus \hat{\rho}^{-1}(p_5)$  (resp.  $\hat{\mathcal{X}}_5^{(h_3)} \setminus \hat{\rho}^{-1}(p_5)$ ) is given by  $(a, b, c, d) \mapsto [a^2d : ad^2]$  (resp.  $(a, b, c, d) \mapsto [a^2c^2d^4 : ac]$ ). Now,  $(a, b, c, d) \mapsto [acd^4 : 1]$  is well-defined on  $\hat{\mathcal{X}}_5^{(h_2)}$  and extends  $\varphi_5 \circ \hat{\rho}$ . Moreover,  $(a, b, c, d) \mapsto [a : d]$  is well-defined on  $\mathcal{B}$  (thanks to the command `Dimension (Scheme (Xhat5 [4] [1], [a [1], a [4]]))`); in [Arxiv, Comput. III.4], which shows that the subvariety of  $\hat{\mathcal{X}}_5^{(h_3)}$  defined by  $a = d = 0$  is empty) and extends  $\varphi_5 \circ \hat{\rho}$ .

Therefore, it remains to understand what happens on the open affine subset of  $\hat{\mathcal{X}}_5^{(h_1)}$ . Then it follows from [Arxiv, Comput. III.6] that

$$(I.3) \quad \hat{\mathcal{X}}_5^{(h_1)} = \{(a, b, c, d) \in \mathbb{A}^4(\mathbb{C}) \mid ac^2 = bd \quad \text{and} \\ abcd - \frac{512}{6561}bd^3 - \frac{6561}{2048}ab^2 - \frac{621}{16}abc - \frac{1600}{243}bd^2 - \frac{2750}{27}bd - 6a + \frac{3125}{144}b + \frac{3125}{18}c = 0\}$$

and that  $\hat{\rho}(a, b, c, d) = (ab, acd^2, a, ac, ad, acd)$ .

Note also that the last command of [Arxiv, Comput. III.4] shows that  $\hat{p}_5$  corresponds to the points  $(0, 0, 0, 0)$  in  $\hat{\mathcal{X}}_5^{(h_1)} \subset \mathbb{A}^4(\mathbb{C})$ .

In particular, the restriction of  $\varphi \circ \hat{\rho}$  to  $\hat{\mathcal{X}}_5^{(h_1)} \setminus \hat{\rho}^{-1}(p_5)$  is given by

$$(a, b, c, d) \mapsto \begin{cases} [ab : acd] & \text{if } (ab, acd) \neq (0, 0), \\ [a^2c^2 : acd^2] & \text{if } (a^2c^2, acd^2) \neq (0, 0). \end{cases}$$

But this can be extended to a morphism  $\hat{\varphi} : \hat{\mathcal{X}}_5^{(h_1)} \setminus \{(0, 0, 0, 0)\} \rightarrow \mathbb{P}^1(\mathbb{C})$  by the formula

$$(a, b, c, d) \mapsto \begin{cases} [b : cd] & \text{if } (b, cd) \neq (0, 0), \\ [ac : d^2] & \text{if } (ac, d^2) \neq (0, 0). \end{cases}$$

Indeed, if  $(a, b, c, d) \in \hat{\mathcal{X}}_5^{(h_1)}$  satisfies  $b = d = ac = 0$ , then  $a = b = c = d = 0$  (see the equations of  $\hat{\mathcal{X}}_5^{(h_1)}$ ).  $\square$

Therefore, it remains to prove that  $\hat{\varphi} \circ \hat{\pi} : \hat{\mathcal{D}} \setminus \hat{\pi}^{-1}(\hat{p}_5) \rightarrow \mathbb{P}^1(\mathbb{C})$  can be extended to  $\hat{\mathcal{D}}$ , where  $\hat{\mathcal{D}} = \hat{\pi}^{-1}(\hat{\mathcal{X}}_5^{(h_1)})$  is an open subset of  $\hat{\mathcal{X}}$ . Note that  $\hat{\mathcal{D}}$  is also the blow-up of  $\hat{\mathcal{X}}_5^{(h_1)}$  at  $\hat{p}$ . But [Arxiv, Comput. III.8] shows that  $\hat{\mathcal{D}}$  is embedded in  $\mathbb{A}^4(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$  and is the variety of points  $((a, b, c, d), [b' : c' : d']) \in \mathbb{A}^4(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$  such that<sup>(§)</sup>

$$(I.4) \quad \begin{cases} bc' = cb', \\ cd' = dc', \\ bd' = db', \\ ac'^2 = d'b', \\ acc' = db', \\ ac^2 = bd, \\ abcd - \frac{512}{6561}bd^3 - \frac{6561}{2048}ab^2 - \frac{621}{16}abc - \frac{1600}{243}bd^2 - \frac{2750}{27}bd - 6a + \frac{3125}{144}b + \frac{3125}{18}c = 0. \end{cases}$$

<sup>(§)</sup>A priori,  $\hat{\mathcal{D}}$  should be embedded in  $\mathbb{A}^4(\mathbb{C}) \times \mathbb{P}^3(\mathbb{C})$ , but the second equation giving  $\hat{\mathcal{X}}_5^{(h_1)}$  in (I.3) has a non-trivial homogeneous component of degree 1, which allows to eliminate one variable.

If  $(a, b, c, d, b', c', d') \in \mathbb{C}^7$ , we set

$$\begin{aligned} \alpha'(a, b, c, d, b', c', d') = & \left( abcd' - \frac{512}{6561}bd^2d' - \frac{6561}{2048}abb' - \frac{621}{16}abc' \right. \\ & \left. - \frac{1600}{243}bdd' - \frac{2750}{27}bd' + \frac{3125}{144}b' + \frac{3125}{18}c' \right) / 6. \end{aligned}$$

Then, if  $((a, b, c, d), [b' : c' : d']) \in \hat{\mathcal{D}}$ , it is immediately checked from the equations (I.4) that

$$(I.5) \quad \begin{cases} \alpha'(a, b, c, d, b', c', d')b = ab', \\ \alpha'(a, b, c, d, b', c', d')c = ac', \\ \alpha'(a, b, c, d, b', c', d')d = ad', \end{cases}$$

Therefore, if  $(b, cd) \neq (0, 0)$ , then  $[b : cd] = [b' : cd']$  and, if  $(ac, d^2) \neq (0, 0)$ , then  $[ac : d^2] = [\alpha'(a, b, c, d, b', c', d')c' : d'^2]$ . But if  $b' = d' = \alpha'(a, b, c, d, b', c', d')c' = 0$ , then  $b' = c' = d' = 0$ , which is impossible. Therefore, the formula

$$\hat{\varphi}((a, b, c, d), [b', c', d']) = \begin{cases} [b' : cd'] & \text{if } (b', cd') \neq (0, 0), \\ [\alpha'(a, b, c, d, b', c', d')c' : d'^2] & \text{if } (\alpha'(a, b, c, d, b', c', d')c', d'^2) \neq (0, 0), \end{cases}$$

gives a well-defined morphism  $\hat{\mathcal{D}} \rightarrow \mathbb{P}^1(\mathbb{C})$  which extends  $\hat{\varphi} \circ \hat{\pi}$ .

This shows that the morphism  $\tilde{\varphi} : \tilde{\mathcal{X}} \rightarrow \mathbb{P}^1(\mathbb{C})$  expected by Theorem 5.1(a) is well-defined.

**I.B. Elliptic fibration.** — By the work done in the previous subsection, it remains to show that at least one fiber of  $\tilde{\varphi}$  is a smooth elliptic curve. For this, let  $\mathcal{S}$  be the set of singular points of  $\mathcal{X}/W'$  different from  $p_5$  and let  $\mathcal{S}' = \varphi(\mathcal{S}) \subset \mathbb{P}^1(\mathbb{C})$ . Note that  $\mathcal{S}$  and  $\mathcal{S}'$  are finite.

Among the four rational curves  $\Delta_5^k$ ,  $1 \leq k \leq 4$ , some of them are mapped to a point under  $\tilde{\varphi}$  and some of them are mapped to the whole  $\mathbb{P}^1(\mathbb{C})$ . We denote by  $\mathcal{S}''$  the set of points which are image of the rational curves contracted by  $\tilde{\varphi}$ . We set  $U = \mathbb{P}^1(\mathbb{C}) \setminus (\mathcal{S}' \cup \mathcal{S}'')$ . We denote by  $U'$  the open subset of  $\mathbb{P}^1(\mathbb{C})$  such that the restriction of  $\tilde{\varphi}$  to  $\tilde{\varphi}^{-1}(U') \rightarrow U'$  is smooth. Finally, we set  $U^+ = U \cap U'$ .

Let  $x \in U^+$ . Since  $[0 : 1]$  and  $[1 : 0]$  belong to  $\mathcal{S}'$ , we can write  $x = [1 : b]$  for some non-zero complex number  $b$ . Since  $[1 : b] \notin \mathcal{S}''$ ,  $\varphi^{-1}([1 : b])$  meets  $\mathcal{U}_3$  and  $\varphi^{-1}([1 : b]) \cap \mathcal{U}_3$  is a dense open subset of  $\tilde{\varphi}^{-1}([1 : b])$  meets  $\mathcal{U}_3$ . But it follows from (4.19) that

$$\begin{aligned} \varphi^{-1}([1 : b]) \cap \mathcal{U}_3 & \simeq \{(c_3, j) \in \mathbb{A}^3(\mathbb{C}) \mid \\ j^2 & = -3\left(\frac{27}{64} - \frac{16}{243}b\right)^2 - \frac{207}{32}c_3 - \frac{800}{729}bc_3 - \frac{1375}{81}c_3^2 + \frac{3125}{864b}c_3^3 + \frac{3125}{108b}c_3^4\}. \end{aligned}$$

This equation shows that  $\varphi^{-1}([1 : b]) \cap \mathcal{U}_3$  has genus 1, and so  $\varphi^{-1}([1 : b])$  is a smooth elliptic curve.

This shows that the morphism  $\tilde{\varphi} : \tilde{\mathcal{X}} \rightarrow \mathbb{P}^1(\mathbb{C})$  is an elliptic fibration.

**I.C. Local study at  $p_9$  and  $p_{12}$ .** — Let us first study the situation at  $p_9$ . For this, we work in the affine chart  $\mathcal{U}_3$  and use the equations (4.19). In this chart, which is a closed subvariety of  $\mathbb{A}^4(\mathbb{C})$ , the point  $p_9$  has coordinates  $(0, 0, 0, 27\sqrt{-3}/64)$ . Moving this point to the origin by working with the coordinate  $j_3 = j - 27\sqrt{-3}/64$ , we get from (4.19) that

$$\begin{aligned} (\mathcal{X}/W') \cap \mathcal{U}_3 & = \{(a_3, b_3, c_3, j_3) \in \mathbb{A}^4(\mathbb{C}) \mid c_3^3 = a_3b_3 \text{ and} \\ (j_3 + \sqrt{-3}\frac{27}{64})^2 & = -3\left(\frac{27}{64} - \frac{16}{243}b_3\right)^2 - \frac{207}{32}c_3 - \frac{800}{729}b_3c_3 - \frac{1375}{81}c_3^2 + \frac{3125}{864}a_3 + \frac{3125}{108}a_3c_3\}, \end{aligned}$$

$$\mathcal{C}_4 = \{(a_3, b_3, c_3, j_3) \in \mathbb{A}^4(\mathbb{C}) \mid b_3 = c_3 = 0 \text{ and } j_3^2 + \sqrt{-3} \frac{27}{32} j_3 = \frac{3125}{864} a_3\},$$

$$\mathcal{C}_5^+ = \{(a_3, b_3, c_3, j_3) \in \mathbb{A}^4(\mathbb{C}) \mid a_3 = c_3 = 0 \text{ and } j_3 = -\sqrt{-3} \frac{16}{243} b_3\}$$

and

$$p_9 = (0, 0, 0, 0) \in \mathcal{C}_4 \cap \mathcal{C}_5^+.$$

Working in the completed local ring of  $\mathcal{U}_3$  at  $p_9$ , we can replace the variable  $j_3$  by  $j'_3 = j_3^2 + \sqrt{-3} \frac{27}{32} j_3$ , so that, in  $\hat{\mathcal{O}}_{\mathcal{X}/W', p_9}$ ,  $j'_3$  can be expressed in terms of the variables  $a_3, b_3, c_3$ . In other words,

$$\hat{\mathcal{O}}_{\mathcal{X}/W', p_9} \simeq \mathbb{C}[[a_3, b_3, c_3]] / \langle c_3^3 - a_3 b_3 \rangle$$

and

$$\hat{\mathcal{O}}_{\mathcal{C}_4, p_9} \simeq \hat{\mathcal{O}}_{\mathcal{X}/W', p_9} / \langle b_3, c_3 \rangle \quad \text{and} \quad \hat{\mathcal{O}}_{\mathcal{C}_5^+, p_9} \simeq \hat{\mathcal{O}}_{\mathcal{X}/W', p_9} / \langle a_3, c_3 \rangle.$$

The computation of the intersections of  $\tilde{\mathcal{C}}_5^+, \tilde{\mathcal{C}}_4, \Delta_9^1$  and  $\Delta_9^2$  can be done with these completed local rings, and we get that, after exchanging  $\Delta_9^1$  and  $\Delta_9^2$  if necessary, the following result holds:

**Lemma I.6.** — *The curve  $\tilde{\mathcal{C}}_5^+$  (resp.  $\tilde{\mathcal{C}}_4$ ) intersects  $\Delta_9^1$  (resp.  $\Delta_9^2$ ) transversely at one point and does not intersect  $\Delta_9^2$  (resp.  $\Delta_9^1$ ).*

The situation at  $p_{12}$  is very similar to the situation at  $p_9$ . Indeed, we work in the affine chart  $\mathcal{U}_4$  and use the equations (4.20) and we obtain that

$$\hat{\mathcal{O}}_{\mathcal{X}/W', p_{12}} \simeq \mathbb{C}[[a_4, b_4, c_4]] / \langle c_4^4 - a_4 b_4 \rangle$$

and

$$\hat{\mathcal{O}}_{\mathcal{C}_3, p_{12}} \simeq \hat{\mathcal{O}}_{\mathcal{X}/W', p_{12}} / \langle b_4, c_4 \rangle \quad \text{and} \quad \hat{\mathcal{O}}_{\mathcal{C}_5^+, p_{12}} \simeq \hat{\mathcal{O}}_{\mathcal{X}/W', p_{12}} / \langle a_4, c_4 \rangle.$$

Therefore, after exchanging  $\Delta_{12}^1$  and  $\Delta_{12}^3$  if necessary, the following result holds:

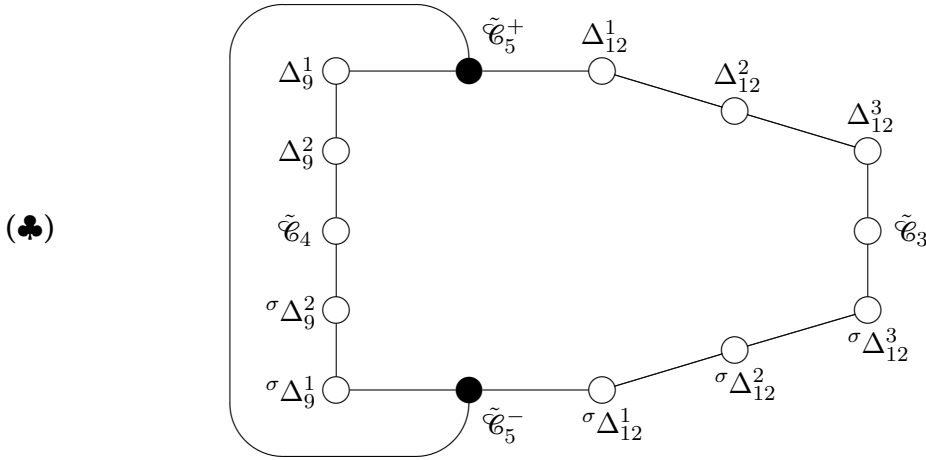
**Lemma I.7.** — *The curve  $\tilde{\mathcal{C}}_5^+$  (resp.  $\tilde{\mathcal{C}}_3$ ) intersect  $\Delta_{12}^1$  (resp.  $\Delta_{12}^3$ ) transversely at one point and does not intersect  $\Delta_{12}^2 \cup \Delta_{12}^3$  (resp.  $\Delta_{12}^1 \cup \Delta_{12}^2$ ).*

Finally, since  $\mathcal{C}_5^+$  and  $\mathcal{C}_5^-$  intersect transversely at a smooth point of  $\mathcal{X}/W$ , we get:

**Lemma I.8.** — *The curves  $\tilde{\mathcal{C}}_5^+$  and  $\tilde{\mathcal{C}}_5^-$  intersect transversely at only one point.*

Using the automorphism  $\sigma$ , Lemmas I.6, I.7 and I.8 show that the intersection graph of the 14 smooth rational curves  $\tilde{\mathcal{C}}_3, \tilde{\mathcal{C}}_4, \tilde{\mathcal{C}}_5^+, \tilde{\mathcal{C}}_5^-, (\Delta_9^k)_{1 \leq k \leq 2}, {}^\alpha \Delta_9^k)_{1 \leq k \leq 2}, (\Delta_{12}^k)_{1 \leq k \leq 3}$  and

$(\sigma \Delta_{12}^k)_{1 \leq k \leq 3}$  is given by:



This shows part of Theorem 5.1(b).

**I.D. Singular fibers.** — The study of singular fibers will be done in several steps:

*I.D.1. Singular fiber above  $[0 : 1]$ .* — The fiber  $\tilde{\varphi}^{-1}([0 : 1])$  contains  $\Delta_{12}^1, \Delta_{12}^2, \Delta_{12}^3, \sigma \Delta_{12}^1, \sigma \Delta_{12}^2, \sigma \Delta_{12}^3, \tilde{\mathcal{C}}_3$  and maybe some of the curves  $\Delta_5^k$ . Since the curves  $\Delta_5^k$  do not intersect the curves  $\Delta_{12}^l$  and  $\sigma \Delta_{12}^l$ , the Kodaira classification of singular fibers (together with the graph (♣)) shows that  $\tilde{\varphi}^{-1}([0 : 1])$  contains exactly one of the curves  $\Delta_5^k$  and is a singular fiber of type  $E_7$ .

*I.D.2. Singular fiber above  $[1 : 0]$ .* — The fiber  $\tilde{\varphi}^{-1}([1 : 0])$  contains  $\Delta_9^1, \Delta_9^2, \sigma \Delta_9^1, \sigma \Delta_9^2, \tilde{\mathcal{C}}_4$  and maybe some of the curves  $\Delta_5^k$ . Since the curves  $\Delta_5^k$  do not intersect the curves  $\Delta_9^l$  and  $\sigma \Delta_9^l$ , the Kodaira classification of singular fibers (together with the graph (♣)) shows that  $\tilde{\varphi}^{-1}([1 : 0])$  contains exactly two of the curves  $\Delta_5^k$  and is a singular fiber of type  $E_6$ .

*I.D.3. Singular fibers above  $\varphi(p_1), \varphi(p'_1)$  and  $\varphi(p'_2)$ .* — Let  $\mathcal{E}_1 = \tilde{\varphi}^{-1}(\varphi(p_1)), \mathcal{E}'_1 = \tilde{\varphi}^{-1}(\varphi(p'_1))$  and  $\mathcal{E}_2 = \tilde{\varphi}^{-1}(\varphi(p_2))$ . Since  $\mathcal{E}_1$  contains  $\Delta_1$  and at least one other irreducible curve, the Kodaira classification of singular fibers shows that the Euler characteristic  $\chi(\mathcal{E}_1)$  of  $\mathcal{E}_1$  is  $\geq 2$ , and is equal to 2 if and only if it is of type  $I_2$  in Kodaira's notation. The same hold for  $\mathcal{E}'_1$ . Also,  $\mathcal{E}_2$  contains  $\Delta_2^1 \cup \Delta_2^2$  and at least one other irreducible curve. So  $\chi(\mathcal{E}_2) \geq 3$ , with equality if and only if  $\mathcal{E}_2$  is of type  $I_2$  in Kodaira's notation.

But, by the two previous paragraphs, we have

$$\chi(\tilde{\varphi}^{-1}([0 : 1])) = 9 \quad \text{and} \quad \chi(\tilde{\varphi}^{-1}([1 : 0])) = 8.$$

Since the sum of the Euler characteristics of the singular fibers is equal to 24, this forces  $\chi(\mathcal{E}_1) = \chi(\mathcal{E}'_1) = 2$  and  $\chi(\mathcal{E}_2) = 3$  and that there is no other singular fiber for  $\tilde{\varphi}$ . This completes the proof of Theorem 5.1(a).

**I.E. Local study at  $p_5$ .** — The fiber  $\hat{\rho}^{-1}(p_5)$  is the union of two smooth rational curves. By (I.3), its intersection with  $\hat{\mathcal{X}}_5^{(h_1)}$  is given by the following equations:

$$\hat{\rho}^{-1}(\hat{p}_5) \cap \hat{\mathcal{X}}_5^{(h_1)} = \{(a, b, c, d) \in \mathbb{A}^4(\mathbb{C}) \mid a = bd = b + 8c = 0\}.$$

It contains two irreducible components  $\hat{\Delta}_5^b$  and  $\hat{\Delta}_5^d$  which are respectively given by the equations  $a = b = c = 0$  and  $a = d = b + 8c = 0$ . The restriction of  $\hat{\varphi}$  to  $\hat{\Delta}_5^b \setminus \{\hat{p}_5\}$  is constant and equal to  $[0 : 1]$ , while its restriction to  $\hat{\Delta}_5^d$  is constant and equal to  $[1 : 0]$ . We denote by  $\hat{\Delta}_5^1$  (resp.  $\hat{\Delta}_5^2$ ) the closure of  $\hat{\Delta}_5^a$  (resp.  $\hat{\Delta}_5^b$ ) in  $\hat{\mathcal{X}}$ .

We denote by  $\hat{\mathcal{C}}_3$  and  $\hat{\mathcal{C}}_4$  the respective strict transforms of  $\mathcal{C}_3$  and  $\mathcal{C}_4$  in the blow-up  $\hat{\mathcal{X}}$ .

**Lemma I.9.** — *With the above notation, the following holds:*

- (a) *The curve  $\hat{\mathcal{C}}_3$  meets  $\hat{\Delta}_5^1$  and does not meet  $\hat{\Delta}_5^2$ .*
- (b) *The curve  $\hat{\mathcal{C}}_4$  meets  $\hat{\Delta}_5^1 \cup \hat{\Delta}_5^2$  only at  $\hat{p}_5$ .*

*Proof.* — Let  $k \in \{3, 4\}$ . Since  $\mathcal{C}_k$  is a smooth rational curve, its strict transform meets  $\hat{\rho}^{-1}(p_5)$  (which is identified with the tangent cone of  $\mathcal{X}/W'$  at  $p_5$ ) at only one point (the line given by the tangent space of  $\mathcal{C}_k$  at  $p_5$ ).

As  $\mathcal{C}_3 \cap \mathcal{U}_5$  is the subscheme of  $(\mathcal{X}/W') \cap \mathcal{U}_5$  defined by the equations  $Y_3 = h_1 = h_2 = h_4 = 0$ , its strict transform does not meet the open chart  $\hat{\mathcal{X}}_5^{(h_1)}$ . So  $\hat{p}_5 \notin \hat{\mathcal{C}}_3$ . So  $\hat{\mathcal{C}}_3$  meets  $\hat{\Delta}_5^1$  or  $\hat{\Delta}_5^2$  but does not meet both. Since  $\hat{\varphi}(\hat{\mathcal{C}}_3 \setminus \{\hat{p}_5\}) = \hat{\varphi}(\hat{\Delta}_5^1 \setminus \{\hat{p}_5\}) = [0 : 1] \neq [1 : 0] = \hat{\varphi}(\hat{\Delta}_5^2 \setminus \{\hat{p}_5\})$ , we get (a).

As  $\mathcal{C}_4 \cap \mathcal{U}_5$  is the subscheme of  $(\mathcal{X}/W') \cap \mathcal{U}_5$  defined by the equations  $Y_4 = h_2 = h_3 = h_4 = 0$  and since  $\hat{\rho}(a, b, c, d) = (ab, acd^2, a, ac, ad, acd)$ ,  $\hat{\mathcal{C}}_4 \cap \hat{\mathcal{X}}_5^{(h_1)}$  is the subscheme of  $\hat{\mathcal{X}}_5^{(h_1)}$  defined by the equations  $c = d = 0$ . So it contains  $\hat{p}_5$ .  $\square$

We denote by  $\hat{\Delta}_5^1$  and  $\hat{\Delta}_5^4$  the respective strict transforms of  $\hat{\Delta}_5^1$  and  $\hat{\Delta}_5^2$  in  $\hat{\mathcal{X}}$ . Also, we denote by  $\hat{\Delta}_5^2$  and  $\hat{\Delta}_5^3$  the two smooth rational curves contained in  $\hat{\pi}^{-1}(\hat{p}_5)$ , numbered in such a way that  $\hat{\Delta}_5^1 \cap \hat{\Delta}_5^2 \neq \emptyset$ . Finally, we denote by  $\hat{\mathcal{C}}_4$  the strict transform of  $\hat{\mathcal{C}}_4$  in  $\hat{\mathcal{X}}$ . These are all smooth rational curves. Using the equations of  $\hat{\mathcal{D}}$  given in (I.4), we get that:

- $\hat{\Delta}_5^1 \cap \hat{\mathcal{D}} = \{((0, 0, 0, d), [0 : 0 : 1]) \mid d \in \mathbb{C}\}$  and  
 $\tilde{\varphi}((0, 0, 0, d), [0 : 0 : 1]) = [0 : 1]$ .
- $\hat{\Delta}_5^2 \cap \hat{\mathcal{D}} = \hat{\Delta}_5^2 = \{((0, 0, 0, 0), [0 : x : y]) \mid [x : y] \in \mathbb{P}^1(\mathbb{C})\}$  and  
 $\tilde{\varphi}((0, 0, 0, 0), [0 : x : y]) = [(3125/108)x^2 : y^2]$ .
- $\hat{\Delta}_5^3 \cap \hat{\mathcal{D}} = \hat{\Delta}_5^3 = \{((0, 0, 0, 0), [x : y : 0]) \mid [x : y] \in \mathbb{P}^1(\mathbb{C})\}$  and  
 $\tilde{\varphi}((0, 0, 0, 0), [x : y : 0]) = [1 : 0]$ .
- $\hat{\Delta}_5^4 \cap \hat{\mathcal{D}} = \{((0, -8c, c, 0), [-8 : 1 : 0]) \mid c \in \mathbb{C}\}$  and  
 $\tilde{\varphi}((0, -8c, c, 0), [-8 : 1 : 0]) = [1 : 0]$ .
- $\hat{\mathcal{C}}_4 = \{((a, b, 0, 0), [1 : 0 : 0]) \mid \frac{6561}{2048}ab^2 + 6a - \frac{3125}{144}b = 0\}$  and  
 $\tilde{\varphi}((a, b, 0, 0), [1 : 0 : 0]) = [1 : 0]$ .

This allows to check easily that the intersection graph given in Theorem 5.1 is correct.

**I.F. Sections.** — The fact that  $\tilde{\mathcal{C}}_5^+$  and  $\tilde{\mathcal{C}}_5^-$  are sections of  $\tilde{\varphi}$  follows from their explicit description given in the proof of Proposition 4.14. The fact that  $\Delta_5^2$  is a double section follows from the computations done at the end of the previous subsection.

The proof of Theorem 5.1(b) is now complete.

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