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Simulations of gamma distributions and multifactor gamma distributions

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Abstract

This article provides a general expression for infinitely divisible multivariate gamma distributions defined by their Laplace transforms, as well as the conditional Laplace transform of infinitely divisible multivariate gamma distributions.

We give algorithms for simulating infinitely divisible gamma distributions and infinitely divisible multifactor gamma distributions in dimension 2, 3, 4 and for all dimensions greater than 2 in the Markovian case. We give examples of simulations in dimension 2, 3, 4 and in dimension 5 in the Markovian case.

KEY WORDS: Conditional distribution, Lauricella function, Laplace transform, Markovian distribution

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1 Introduction

The aim of this paper is to extend simulations of bivariate gamma distributions, see [Bernardoff, 2023], to multivariate gamma distributions and multifactor gamma distributions defined by their Laplace transforms.

In this paper, we consider the following definitions given in [Bernardoff, 2018]. For more details see also [Bar-Lev et al., 1994], [Johnson et al., 2000] and [Bernardoff, 2006].

We use the extension of the classical univariate definition to \mathbb{R}^n obtained as follows: we consider an affine polynomial $P_n(\boldsymbol{\theta})$ in $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ where ‘affine’ means that, for $j = 1, \dots, n$, $\partial^2 P_n / \partial \theta_j^2 = 0$. We also assume that $P_n(\mathbf{0}) = 1$. For instance, for $n = 2$, we have $P_2(\theta_1, \theta_2) = 1 + p_{\{1\}}\theta_1 + p_{\{2\}}\theta_2 + p_{\{1,2\}}\theta_1\theta_2$.

We denote by $\mathfrak{P}_n = \mathfrak{P}([n])$ the family of all subsets of $[n] = \{1, \dots, n\}$ and \mathfrak{P}_n^* the family of non-empty subsets of $[n]$. For simplicity, if n is fixed and if there is no ambiguity, we denote these families by \mathfrak{P} and \mathfrak{P}^* , respectively. Similarly, we denote by $\mathfrak{P}_T = \mathfrak{P}(T)$ the family of all subsets of $T = \{1, \dots, n\}$ and \mathfrak{P}_T^* the family of non-empty subsets of T . Similarly, if there is no ambiguity, we denote P_n by P .

We denote by \mathbb{N} the set of non-negative integers. If $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{R}^n$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, then $\boldsymbol{\alpha}! = \alpha_1! \dots \alpha_n!$, $|\boldsymbol{\alpha}| = \alpha_1 + \dots + \alpha_n$, $a_{\boldsymbol{\alpha}} = a_{\alpha_1, \dots, \alpha_n}$ and

$$\mathbf{z}^{\boldsymbol{\alpha}} = \prod_{i=1}^n z_i^{\alpha_i} = z_1^{\alpha_1} \dots z_n^{\alpha_n}. \quad (1)$$

For T in \mathfrak{P}_n , we simplify the above notation by writing $\mathbf{z}^T = \prod_{t \in T} z_t$ instead of $\mathbf{z}^{\mathbf{1}^T}$ where

$$\mathbf{1}_T = (\alpha_1, \dots, \alpha_n) \text{ with } \alpha_i = 1 \text{ if } i \in T \text{ and } \alpha_i = 0 \text{ if } i \notin T. \quad (2)$$

We also write \mathbf{z}^{-T} for $\prod_{t \in T} 1/z_t$ if $z_t \neq 0$, $\forall t \in T$. For a mapping $a : \mathfrak{P} \rightarrow \mathbb{R}$, we shall use the notation $a : \mathfrak{P} \rightarrow \mathbb{R}$, $T \mapsto a_T$. In this notation, an affine polynomial with constant term equal to 1 is $P(\boldsymbol{\theta}) = \sum_{T \in \mathfrak{P}} p_T \boldsymbol{\theta}^T$, with $p_{\emptyset} = 1$. Now, if there is no ambiguity, for simplicity, we omit the braces and, if $T = \{t_1, \dots, t_k\}$, we denote $a_{\{t_1, \dots, t_k\}} = a_{t_1, \dots, t_k}$ and $a_{\emptyset} = a_0$.

The indicator function of a set S is denoted by $\mathbf{1}_S$, that is, $\mathbf{1}_S(x) = 1$ for $x \in S$ and 0 for $x \notin S$.

We fix $\lambda > 0$. If a random vector $\mathbf{X} = (X_1, \dots, X_n)$ on \mathbb{R}^n with probability distribution (pd) $\mu_{\mathbf{X}}$ is such that its Laplace transform (Lt) is

$$\mathbb{E} \{ \exp [- (\theta_1 X_1 + \dots + \theta_n X_n)] \} = [P(\boldsymbol{\theta})]^{-\lambda}, \quad (3)$$

where \mathbb{E} denotes the expectation, for a set of $\boldsymbol{\theta}$ with non-empty interior, then we denote $\mu_{\mathbf{X}} = \gamma_{(P, \lambda)}$, and $\gamma_{(P, \lambda)}$ will be called the *multivariate gamma distribution (mgd) associated with (P, λ)* . If \mathbf{X} has pd $\gamma_{(P, \lambda)}$, we denote it by $\mathbf{X} \sim \gamma_{(P, \lambda)}$, and P, λ is called respectively the scale parameter, the shape parameter. These mgds occur naturally in the classification of natural exponential families in \mathbb{R}^n [Bar-Lev et al., 1994].

The marginal distributions of the *mgd associated with (P, λ)* are univariate gamma distributions (ugd) of parameters (p_i, λ) for $i = 1, \dots, n$, with Lt

$$[P(0, \dots, 0, \theta_i, 0, \dots, 0)]^{-\lambda} = (1 + p_i \theta_i)^{-\lambda}, \quad (4)$$

and pd

$$\gamma_{(p_i, \lambda)}(dx) = x^{\lambda-1} p_i^{-\lambda} [\Gamma(\lambda)]^{-1} \exp(-x/p_i) \mathbf{1}_{(0, \infty)}(x) dx. \quad (5)$$

As in [Bernardoff, 2018], we extend the first definition to the *multifactor gamma distribution (mfgd) associated with $(P, \boldsymbol{\Lambda})$* where $\boldsymbol{\Lambda} = (\lambda, \lambda_1, \dots, \lambda_n)$ and $\lambda_i \geq \lambda > 0$ for all $i = 1, \dots, n$ by its Lt

$$\mathbb{E} \{ \exp [- (\theta_1 X_1 + \dots + \theta_n X_n)] \} = [P(\boldsymbol{\theta})]^{-\lambda} \prod_{i=1}^n (1 + p_i \theta_i)^{-(\lambda_i - \lambda)}. \quad (6)$$

By using (4), the marginal distributions of the *mgd associated with $(P, \boldsymbol{\Lambda})$* are ugd of parameters (p_i, λ_i) for $i = 1, \dots, n$, with Lt $(1 + p_i \theta_i)^{-\lambda_i}$, and pd

$\gamma_{(p_i, \lambda_i)}(dx) = x^{\lambda_i-1} p_i^{-\lambda_i} [\Gamma(\lambda_i)]^{-1} \exp(-x/p_i) \mathbf{1}_{(0, \infty)}(x) dx$. We can denote either $\gamma_{(p_i, \lambda_i)}$ or $\gamma_{(1+p_i \theta_i, \lambda_i)}$.

[Bernardoff, 2018] gives a proposition for building a random vector whose distribution is the mfgd associated with $(P, \boldsymbol{\Lambda})$.

Proposition 1 *A random vector \mathbf{X} with distribution $\gamma_{P,\Lambda}$ can be obtained in the following way:*

Let \mathbf{Y} be a random vector with distribution $\gamma_{(P,\lambda)}$. Let $\mathbf{Z} = (Z_1, \dots, Z_n)$ be a random vector of independent components for which its pdfs are $\gamma_{(p_i, \lambda_i - \lambda)}$, and such that \mathbf{Z} and \mathbf{Y} are independent random vectors. Then the random vector $\mathbf{X} = \mathbf{Y} + \mathbf{Z}$ has Lt (6), and consequently has the mfgd associated with (P, Λ) .

Remark 2 *This construction clearly allows us to simulate $\gamma_{(P,\Lambda)}$ by simulating $\mathbf{Z} \sim \gamma_{(P,\lambda)}$ and \mathbf{Y} .*

For the bivariate case, see also [Dussauchoy and Berland, 1972].

The Lt of mgd associated with (P, λ) and the Lt of mfgd associated with (P, Λ) are known by definition. But, its pdfs are unknown, except for the bivariate gamma distribution (bgd) associated with (P, λ) and the bivariate mfgd associated with $(P, (\lambda, \lambda_1, \lambda_2))$. Let us recall in the Proposition below, the known results.

Let F_m^p be the generalized hypergeometric function (see [Slater, 1966]) defined by

$$F_m^p(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_m; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_m)_k} \frac{z^k}{k!}, \quad (7)$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ for $a > 0$ and $k \in \mathbb{N}$ (or more generally by $\forall n \in \mathbb{N}, \forall a \in \mathbb{R}$, $(a)_0 = 1$, $(a)_{n+1} = (a+n)(a)_n$) is the Pochhammer's symbol. For simplification, we denote F_m^0 by F_m . [Bernardoff, 2006] gives the following proposition.

Proposition 3 *Let $P(\theta_1, \theta_2) = 1 + p_1\theta_1 + p_2\theta_2 + p_{12}\theta_1\theta_2$ be an affine polynomial where $p_1, p_2 > 0$ and $p_{1,2} > 0$. Let $\mu = \gamma_{(P,\lambda)}$ be the bgd associated with (P, λ) . The measure μ exists if and only if $c = (p_1p_2 - p_{1,2})/p_{1,2}^2 = p_1p_2/p_{1,2}^2\rho_{1,2} > 0$, where $\rho_{1,2} = 1 - p_{1,2}/(p_1p_2)$ is the correlation coefficient between margins. Then we have*

$$\gamma_{(P,\lambda)}(dx_1, dx_2) = \frac{p_{1,2}^{-\lambda}}{\Gamma(\lambda)^2} \mathbf{e}^{-\frac{p_2}{p_{12}}x_1 - \frac{p_1}{p_{12}}x_2} (x_1x_2)^{\lambda-1} F_1(\lambda; cx_1x_2) \mathbf{1}_{(0,\infty)^2}(\mathbf{x}) d\mathbf{x}. \quad (8)$$

with

$$F_1(\lambda; z) = \sum_{k=0}^{\infty} \frac{1}{(\lambda)_k} \frac{z^k}{k!} = \Gamma(\lambda) z^{-(\lambda-1)/2} I_{\lambda-1}(2\sqrt{z}),$$

where I_λ is the modified Bessel function of order λ .

For the case $\mathbf{\Lambda} = (\lambda, \lambda, \lambda_2)$, the *mfgd* associated with $(P, \mathbf{\Lambda})$ is named by [Chatelain et al., 2008] the *multisensor gamma distribution* associated with (P, λ, λ_2) and they have proved that its pd is given by the equality

$$\begin{aligned} \gamma_{(P, \mathbf{\Lambda})}(dx_1, dx_2) &= \frac{p_{12}^{-\lambda} p_2^{-(\lambda_2 - \lambda)} \mathbf{1}}{\Gamma(\lambda) \Gamma(\lambda_2)} x_1^{\lambda-1} x_2^{\lambda_2-1} \mathbf{e}^{-\frac{p_2}{p_{12}} x_1 - \frac{p_1}{p_{12}} x_2} \Phi_3(\lambda_2 - \lambda; \lambda_2; c \frac{p_{12}}{p_2} x_2; cx_1 x_2) \\ &\times \mathbf{1}_{(0, \infty)^2}(x_1, x_2) dx_1 dx_2, \end{aligned} \quad (9)$$

where

$$\Phi_3(a; b; x, y) = \sum_{m, n \geq 0} \frac{(a)_m}{(b)_{m+n}} \frac{x^m y^n}{m! n!} \quad (10)$$

is the Horn function.

For the bivariate general case, we have the following Theorem in [Bernardoff, 2018].

Let F_I be the function defined by

$$F_I(a, b, c, \mathbf{z}) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a)_{m_1} (b)_{m_2} (c)_{m_3}}{(a+c)_{m_1+m_3} (b+c)_{m_2+m_3}} \frac{z_1^{m_1} z_2^{m_2} z_3^{m_3}}{m_1! m_2! m_3!}, \quad (11)$$

it is a particular generalized Lauricella function defined, by example, in [Panda, 1973].

Theorem 4 *The pd of $\gamma_{(P, (\lambda, \lambda_1, \lambda_2))}$ is given by the equality*

$$\begin{aligned} \gamma_{(P, (\lambda, \lambda_1, \lambda_2))}(dx_1, dx_2) &= \frac{p_{12}^{-\lambda} p_1^{-(\lambda_1 - \lambda)} p_2^{-(\lambda_2 - \lambda)}}{\Gamma(\lambda_1) \Gamma(\lambda_2)} x_1^{\lambda_1-1} x_2^{\lambda_2-1} \mathbf{e}^{-\frac{p_2}{p_{12}} x_1 - \frac{p_1}{p_{12}} x_2} \\ &\times F_I\left(\lambda_1 - \lambda, \lambda_2 - \lambda, \lambda, \frac{p_{12}}{p_1} x_1, \frac{p_{12}}{p_2} x_2, cx_1 x_2\right) \mathbf{1}_{(0, \infty)^2}(x_1, x_2) dx_1 dx_2, \end{aligned} \quad (12)$$

If we get $\lambda_1 = \lambda$ in the equality (12), we obtain Chatelain and Tourneret's result (9)

because

$$\begin{aligned} F_I(0, \lambda_2 - \lambda, \lambda, z_1, z_2, z_3) &= \\ \sum_{m_2, m_3=0}^{\infty} \frac{(b)_{m_2}}{(b+c)_{m_2+m_3}} \frac{z_2^{m_2} z_3^{m_3}}{m_2! m_3!} &= \Phi_3(b; b+c; z_2, z_3). \end{aligned}$$

[Bernardoff, 2006] gives the following Proposition:

Proposition 5 Let μ be a mgd on \mathbb{R}^n associated with (P, λ) . Assume that μ is not concentrated on a linear subspace of \mathbb{R}^n of the form $\{\mathbf{x} \in \mathbb{R}^n; x_k = 0\}$ for some k in $[n] = \{1, \dots, n\}$. Then:

- (i) For all $i \in [n]$, $p_i \neq 0$.
- (ii) If $p_1, \dots, p_k < 0$ and $p_{k+1}, \dots, p_n > 0$, then $\text{Supp}(\mu) \subset (-\infty, 0]^k \times [0, \infty)^{n-k}$.
- (iii) If $p_1, \dots, p_n > 0$ then $p_{[n]} \geq 0$.

[Bernardoff, 2006] gives a necessary and sufficient condition for infinite divisibility of the mgd associated with (P, λ) , in the sense that the Lt of $\gamma_{(P, \lambda)}$ power t for all positive t is still the Lt of a positive measure, by the following theorem using the notation \tilde{b}_S from the notation b_S defined in [Bernardoff, 2003]:

Theorem 6 Let $\mu = \gamma_{P, \lambda}$ be a mgd associated with (P, λ) , where $\lambda > 0$ and $P(\boldsymbol{\theta}) = \sum_{T \in \mathfrak{P}_n} p_T \boldsymbol{\theta}^T$ is such that $p_i > 0$, for all $i \in [n]$, and $p_{[n]} > 0$. Let $\tilde{P}(\boldsymbol{\theta}) = \sum_{T \in \mathfrak{P}_n} \tilde{p}_T \boldsymbol{\theta}^T$ be the affine polynomial such that $\tilde{p}_T = -p_{\bar{T}}/p_{[n]}$ for all $T \in \mathfrak{P}_n$, where $\bar{T} = [n] \setminus T$. Let

$$\tilde{b}_S = b_S(\tilde{P}) = \sum_{k=1}^{|S|} (k-1)! \sum_{\mathcal{T} \in \Pi_S^k} \prod_{T \in \mathcal{T}} \tilde{p}_T, \quad (13)$$

with

$$b_S(P) = \sum_{k=1}^{|S|} (k-1)! \sum_{\mathcal{T} \in \Pi_S^k} \prod_{T \in \mathcal{T}} p_T, \quad (14)$$

where $|S|$ is the cardinality of the set S . Then the measure μ is infinitely divisible if and only if

$$\tilde{p}_i = \tilde{b}_{\{i\}} < 0 \text{ for all } i \in [n], \quad (15)$$

and

$$\tilde{b}_S \geq 0 \text{ for all } S \in \mathfrak{P}_n^* \text{ such that } |S| \geq 2. \quad (16)$$

Corollary 7 By the properties of infinite divisible distributions we conclude that the necessary and sufficient conditions for infinite divisibility of a mgd associated with (P, λ)

of theorem (6), are also necessary and sufficient conditions for infinite divisibility of mfgd associated with $(P, \mathbf{\Lambda})$.

To illustrate the difficulty to calculate the mfgd associated with (P, λ) we recall, for the trivariate gamma distribution (tgd), the following theorem given in [Bernardoff, 2018]. Let F_{II} be the function defined by

$$F_{II}(\lambda_1, \lambda_2, z_1, z_2, z_3, z_4) = \sum_{m_1, \dots, m_4=0}^{\infty} \frac{1}{(\lambda_1)_{m_1+m_2+m_3} (\lambda_2)_{2m_1+m_2+m_4}} \frac{z_1^{m_1} z_2^{m_2} z_3^{m_3} z_4^{m_4}}{m_1! m_2! m_3! m_4!}; \quad (17)$$

it is still a particular generalized Lauricella function.

We note that c in (8) is $\tilde{b}_{1,2}$.

Theorem 8 *In the case $n = 3$, $p_i > 0$ for $i \in [3]$, $p_{ij} > 0$ for $i, j \in [3]$, $\tilde{b}_{ij} = -\frac{b_k}{p_{123}} + \frac{p_{jk}p_{ik}}{p_{123}^2} \geq 0$ for $i \neq j$ and $\{i, j, k\} = [3]$, $p_{123} > 0$, and*

$$\tilde{b}_{123} = -\frac{1}{p_{123}} + \frac{p_{12}p_1}{p_{123}^2} + \frac{p_{13}p_2}{p_{123}^2} + \frac{p_{23}p_1}{p_{123}^2} + 2\frac{p_{12}p_{13}p_{23}}{p_{123}^3} \geq 0,$$

the infinitely divisible tgd $\gamma_{(P,\lambda)}$ associated with (P, λ) , is given by the formula

$$\begin{aligned} \gamma_{(P,\lambda)}(d\mathbf{x}) &= \frac{p_{123}^{-\lambda}}{[\Gamma(\lambda)]^3} \exp(\tilde{p}_1 x_1 + \tilde{p}_2 x_2 + \tilde{p}_3 x_3) (x_1 x_2 x_3)^{\lambda-1} \\ &\times F_{II}(\lambda, \lambda, \tilde{b}_{13} x_1 x_3, \tilde{b}_{23} x_2 x_3, \tilde{b}_{123} x_1 x_2 x_3, \tilde{b}_{12} x_1 x_2, \tilde{b}_{13} x_1 x_3 + \tilde{b}_{23} x_2 x_3) \mathbf{1}_{(0,\infty)^3}(\mathbf{x}) d\mathbf{x}. \end{aligned} \quad (18)$$

Remark 9 *The case $p_{123} = 0$ is solved by [Letac and Wesolowski, 2008].*

Remark 10 *If $\tilde{b}_{12} = \tilde{b}_{13} = \tilde{b}_{23} = 0$, Theorem 8 gives*

$$\gamma_{(P,\lambda)}(d\mathbf{x}) = \frac{p_{123}^{-\lambda}}{[\Gamma(\lambda)]^3} \exp(\tilde{p}_1 x_1 + \tilde{p}_2 x_2 + \tilde{p}_3 x_3) (x_1 x_2 x_3)^{\lambda-1} F_2\left(\lambda, \lambda; \tilde{b}_{123} x_1 x_2 x_3\right) \mathbf{1}_{(0,\infty)^3}(\mathbf{x}) d\mathbf{x}, \quad (19)$$

and if we put $\lambda = 1$ in this last equality, we obtain the Kibble and Moran distribution given in [Johnson et al., 2000].

After giving a general expression for the pd of infinitely divisible mgds, this paper provides their conditional Laplace transforms. These results allow us to extend the result of [Bernardoff, 2023] from the simulation of *bgds* to *mgds and mfgds*. These results allow us to achieve the aim of this paper for $n = 3, 4$ in the general case and, for two particular cases for any n .

This paper is organised as follows. Let $n \in \mathbb{N}^* \setminus \{1\}$ and let $\mathbf{X}_n = (X_1, \dots, X_n)$ be a real random vector of infinitely divisible mgd, Section 2 gives a general expression for the pd $\mu_{\mathbf{X}_n} = \gamma_{(P_n, \lambda)}$. Let $k \in \mathbb{N}^* \setminus \{1, n\}$ and $(x_1, \dots, x_k) \in \mathbb{R}^k$, Section 3 gives the conditional Lt of (X_k, \dots, X_n) given $(X_1, \dots, X_k) = (x_1, \dots, x_k)$ denoted by $L_{(X_k, \dots, X_n)}^{(X_1, \dots, X_k) = (x_1, \dots, x_k)}$. An important particular case is given. Section 4 gives a simpler expression of $L_{(X_2, \dots, X_n)}^{X_1 = x_1}$. Section 5 apply this last result for $n = 2, 3, 4$, and for the case $n = 2$, we obtain the result of [Bernardoff, 2023], see also [Walker, 2021]. For the case $n = 3, 4$, we obtain a new general result and algorithms for simulating tgds and quadrivariate gamma distributions. For $n = 5$, we could give an expression for $L_{(X_2, \dots, X_n)}^{X_1 = x_1}$ and apply the same method as for cases $n = 2, 3, 4$. Unfortunately, the computations seem long and arduous. Therefore, we study the simpler Markovian case in Section 6. Section 7 gives examples of simulations of mgds for $n = 2, 3, 4, 5$, and of mfgds for $n = 2, 3$. All simulations are performed using the R software, [R Core Team, 2023]. In order to facilitate the fluent presentation of the paper, proofs are collected in the appendix in order of appearance.

2 Probability distributions of multivariate gamma distributions

For a simple example of applying the main results, we will need the following example in dimension n .

Example 11 Let P_n be the affine polynomial defined in [Bernardoff, 2006] by

$$P_n(\boldsymbol{\theta}) = \frac{-q}{p} + \frac{1}{p} \prod_{i=1}^n (1 + p\theta_i) \quad (20)$$

where $0 < p = 1 - q < 1$. Let $\mu = \gamma_{P,1} = \varphi_{n,p,1}$ be the infinitely divisible mgd associated to $(P, 1)$. Let $\gamma_{P,\lambda} = \varphi_{n,p,\lambda}$ the mgd associated to (P, λ) . Then we have:

For $\mathbf{x} = (x_1, \dots, x_n)$ in \mathbb{R}^n ,

$$\gamma_{(P_n,\lambda)}(\mathbf{dx}) = \frac{p^{-(n-1)\lambda}}{(\Gamma(\lambda))^n} e^{-\frac{x_1+\dots+x_n}{p}} (\mathbf{x}^{[n]})^{\lambda-1} F_{n-1}(\lambda, \dots, \lambda; qp^{-n}\mathbf{x}^{[n]}) \mathbf{1}_{(0,\infty)^n}(\mathbf{x}) \mathbf{dx} \quad (21)$$

We will give an expression of the pd $\gamma_{(P_n,\lambda)}(\mathbf{dx})$ in the general case.

Let us denote $\boldsymbol{\theta}_{[n]} = (\theta_1, \dots, \theta_n)$, $P_n(\boldsymbol{\theta}_{[n]}) = \sum_{T \in \mathfrak{P}_n} p_T(P_n) \boldsymbol{\theta}_{[n]}^T$, for $1 \leq k < n$, $\boldsymbol{\theta}_{[k]} = (\theta_1, \dots, \theta_k)$, $P_k(\boldsymbol{\theta}_{[k]}) = \sum_{T \in \mathfrak{P}_k} p_T(P_k) \boldsymbol{\theta}_{[k]}^T = P_n((\theta_1, \dots, \theta_k, 0, \dots, 0))$, let us define for $T \in \mathfrak{P}_n$, $\tilde{p}_T(P_n) = -\frac{p_{[n] \setminus T}}{p_{\{n\}}}$, and for $T \in \mathfrak{P}_k$, $\tilde{p}_T(P_k) = -\frac{p_{[k] \setminus T}}{p_{\{k\}}}$, and $\boldsymbol{\theta}_{P_n} = (\tilde{p}_1(P_n), \dots, \tilde{p}_n(P_n))$, as well as $\boldsymbol{\theta}_{P_k} = (\tilde{p}_1(P_k), \dots, \tilde{p}_k(P_k))$. If there is no ambiguity, we denote $\boldsymbol{\theta}_{[n]} = \boldsymbol{\theta}$, $\boldsymbol{\theta}_{P_n} = \boldsymbol{\theta}_P$, $\tilde{p}_T(P_n) = \tilde{p}_T$, and $P_n(\boldsymbol{\theta}_n) = P(\boldsymbol{\theta})$. We also denote $\tilde{p}_i(P_n) = \tilde{p}_i(P) = \tilde{p}_i$, $i \in [n]$. We also denote by $\tilde{\mathbf{p}}$ the vector $\boldsymbol{\theta}_{P_n} = (\tilde{p}_1, \dots, \tilde{p}_n)$. If $n > 1$, let $\boldsymbol{\theta}_{P_n} = (\tilde{p}_1(P_n), \dots, \tilde{p}_n(P_n))$, so that $(\frac{\partial}{\partial \boldsymbol{\theta}})^{\overline{\{i\}}}(P_n)(\boldsymbol{\theta}_{P_n}) = 0, \forall i \in [n]$, since P_n is an affine polynomial, using the Taylor formula in $\boldsymbol{\theta}_{P_n}$, we get the following proposition which define the affine polynomial R_n .

Proposition 12 *With the above definition, we have*

$$P_n(\boldsymbol{\theta}_n) = p_{[n]}(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n]} \left\{ 1 - R_n \left[(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{-1} \right] \right\}. \quad (22)$$

with

$$R_n(\mathbf{z}_{[n]}) = R_n(z_1, \dots, z_n) = \sum_{T \in \mathfrak{P}_n, |T| \geq 2} r_T \mathbf{z}_{[n]}^T, \quad (23)$$

and

$$r_T = \frac{-1}{p_{[n]}} \left(\frac{\partial}{\partial \boldsymbol{\theta}} \right)^{\overline{T}} (P_n)(\boldsymbol{\theta}_{P_n}), T \in \mathfrak{P}_n, |T| \geq 2 \quad (24)$$

Since R_n depends of P_n , if necessary we will write $R_n(P_n)$. If there is no ambiguity, denoting $R_n(\mathbf{z}_{[n]})$ by $R(\mathbf{z})$, we have

$$P(\boldsymbol{\theta}) = p_{[n]}(\boldsymbol{\theta} - \boldsymbol{\theta}_P)^{[n]} \left\{ 1 - R \left[(\boldsymbol{\theta} - \boldsymbol{\theta}_P)^{-1} \right] \right\} \quad (25)$$

This last equality is still true for $n = 1$, with $R(z_1) = 0$

More specifically, let $T \in \mathfrak{P}_n$, $|T| \geq 2$, we have

$$r_T = \sum_{T' \in \mathfrak{P}_T} \tilde{p}_{T \setminus T'} \tilde{\mathbf{p}}^{T'}, \quad (26)$$

and for any $n \in \mathbb{N} \setminus \{0, 1\}$ if $|T| = 2 \leq n$,

$$r_T = \tilde{b}_T, \quad (27)$$

if $|T| = 3 \leq n$,

$$r_T = \tilde{b}_T, \quad (28)$$

if $|T| = 4 \leq n$,

$$r_T = \tilde{b}_T - \sum_{\{U, V\} \in \Pi_T^2, |U|=2, |V|=2} \tilde{b}_U \tilde{b}_V, \quad (29)$$

if $|T| = 5 \leq n$,

$$r_T = \tilde{b}_T - \sum_{\{U, V\} \in \Pi_T^2, |U|=3, |V|=2} \tilde{b}_U \tilde{b}_V, \quad (30)$$

Later, we will need the following definition and results. In the sequel, we suppose that $\forall T \in \mathfrak{P}_n$, $p_T \neq 0$.

Definition 13 Let $T \in \mathfrak{P}_n$, and $\bar{T} = [n] \setminus T$, and if there is no ambiguity, for simplicity we denote $S_T(\boldsymbol{\theta}_T)$ by S_T , the polynomial defined by: if $p_{\bar{T}} \neq 0$, then

$$S_T(\boldsymbol{\theta}_T) = \frac{1}{p_{\bar{T}}} \left(\frac{\partial}{\partial \boldsymbol{\theta}} \right)^{\bar{T}} [P_n[\boldsymbol{\theta}]]. \quad (31)$$

If q_U is the numbers such that $S_T(\boldsymbol{\theta}_T) = \sum_{U \in \mathfrak{P}_T} q_U \boldsymbol{\theta}^U$, we have

$$q_U = p_{\bar{T} \cup U} p_{\bar{T}}^{-1}. \quad (32)$$

Since S_T depends on P_n , if necessary we will write $S_T(P_n)$.

For example, if $n = 3$, we have $S_{2,3}(\theta_2, \theta_3) = 1 + \frac{p_{1,2}}{p_1} \theta_2 + \frac{p_{1,3}}{p_1} \theta_3 + \frac{p_{1,2,3}}{p_1} \theta_2 \theta_3$, $S_i(\theta_i) = 1 + \frac{p_{1,2,3}}{p_{[3] \setminus \{i\}}} \theta_i$, $i = 1, 2, 3$. We will need the following proposition.

Proposition 14 For $U \in \mathfrak{P}_T$, we have ,

$$S_U(S_T) = S_U(P_n). \quad (33)$$

We also have the following proposition.

Proposition 15 With the above definition, we have $\forall U \in \mathfrak{P}_T$

$$\tilde{q}_U = \tilde{p}_U, \quad (34)$$

therefore, we have

$$\tilde{b}_U(S_T) = \tilde{b}_U, \quad (35)$$

and

$$r_U(S_T) = r_U(P_n). \quad (36)$$

If $\gamma_{(P,\lambda)}$ is an infinitely divisible mgd, $\gamma_{(S_T,\lambda)}$ is also an infinitely divisible mgd from Theorem (6). We have the following equalities

$$S_T = (-\tilde{p}_T)^{-1} [(-\tilde{\mathbf{p}})^T \mathbf{S}^T - \sum_{T' \in \mathfrak{P}_T, |T'| > 1} r_{T'} (-\tilde{\mathbf{p}})^{T \setminus T'} \mathbf{S}^{T \setminus T'}] \quad (37)$$

or

$$(-\tilde{\mathbf{p}})^T \mathbf{S}^T = (-\tilde{p}_T) S_T + \sum_{T' \in \mathfrak{P}_T, |T'| > 1} r_{T'} (-\tilde{\mathbf{p}})^{T \setminus T'} \mathbf{S}^{T \setminus T'}. \quad (38)$$

Now, we can give the following expression of $\gamma_{P_n,\lambda}$ by the following theorem.

Theorem 16 Let $c_{\alpha,\lambda}(R)$ such that

$$[1 - R(\mathbf{z})]^{-\lambda} = \sum_{\alpha \in \mathbb{N}^n} c_{\alpha,\lambda}(R) \mathbf{z}^\alpha, \quad (39)$$

then

$$\gamma_{(P,\lambda)}(d\mathbf{x}) = \frac{p_{[n]}^{-\lambda}}{[\Gamma(\lambda)]^n} \exp(\boldsymbol{\theta}_P, \mathbf{x}) \mathbf{x}^{(\lambda-1)\mathbf{1}_n} \left[\sum_{\alpha \in \mathbb{N}^n} \frac{c_{\alpha,\lambda}(R)}{(\lambda)_\alpha} \mathbf{x}^\alpha \right] \mathbf{1}_{(0,\infty)^n}(\mathbf{x}) (d\mathbf{x}). \quad (40)$$

or more specifically

$$\gamma_{(P,\lambda)}(d\mathbf{x}) = \frac{p_{[n]}^{-\lambda}}{[\Gamma(\lambda)]^n} \exp(\boldsymbol{\theta}_P, \mathbf{x}) \mathbf{x}^{(\lambda-1)\mathbf{1}_n} \left[\sum_{\alpha \in \mathbb{N}^n, c_{\alpha,\lambda}(R) \neq 0} \frac{c_{\alpha,\lambda}(R)}{(\lambda)_\alpha} \mathbf{x}^\alpha \right] \mathbf{1}_{(0,\infty)^n}(\mathbf{x}) (d\mathbf{x}). \quad (41)$$

We deduce a result given in [Bernardoff, 2006] in the form of the following corollary.

Corollary 17 For $P_n(\boldsymbol{\theta}_{[n]}) = \frac{-q}{p} + \frac{1}{p} \prod_{i=1}^n (1 + p\theta_i)$, for $\mathbf{x} = (x_1, \dots, x_n)$ in \mathbb{R}^n , we have

$$\gamma_{(P_n, \lambda)}(\mathbf{dx}) = \frac{p^{-(n-1)\lambda}}{(\Gamma(\lambda))^n} e^{-\frac{x_1 + \dots + x_n}{p}} (\mathbf{x}^{[n]})^{\lambda-1} F_{n-1}(\lambda, \dots, \lambda; qp^{-n} \mathbf{x}^{[n]}) \mathbf{1}_{(0, \infty)^n}(\mathbf{x}) \mathbf{dx}. \quad (42)$$

This is the result (21).

For $n = 2$, we can give the following corollary.

Corollary 18 For $n = 2$, we have

$$R_2(\mathbf{z}) = \tilde{b}_{1,2} z_1 z_2, \quad (43)$$

and

$$c_{\alpha, \lambda}(R_2) = \frac{(\lambda)_l \tilde{b}_{1,2}^l}{l!}, \quad (44)$$

and $c_{\alpha, \lambda}(R_2) = 0$ otherwise. Hence, we have

$$\gamma_{(P_2, \lambda)}(\mathbf{dx}) = \frac{p_{1,2}^{-\lambda}}{[\Gamma(\lambda)]^2} \exp\left(-\frac{p_2}{p_{1,2}} x_1 - \frac{p_1}{p_{1,2}} x_2\right) (x_1 x_2)^{(\lambda-1)} F_1\left(\lambda, \tilde{b}_{1,2} x_1 x_2\right) \mathbf{1}_{(0, \infty)^2}(\mathbf{x}) (\mathbf{dx}). \quad (45)$$

We obtain the formula (8).

For $n = 3$, we can give the following corollary.

Corollary 19 For $n = 3$, we have

$$R_3(\mathbf{z}) = \tilde{b}_{1,2} z_1 z_2 + \tilde{b}_{1,3} z_1 z_3 + \tilde{b}_{2,3} z_2 z_3 + \tilde{b}_{1,2,3} z_1 z_2 z_3 \quad (46)$$

If $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, and $\max(\alpha_1, \alpha_2, \alpha_3) = \|\boldsymbol{\alpha}\|_\infty$, if $\|\boldsymbol{\alpha}\|_\infty \leq \frac{|\boldsymbol{\alpha}|}{2}$, we have

$$c_{\alpha, \lambda}(R_3) = \sum_{\|\boldsymbol{\alpha}\|_\infty \leq k \leq \frac{|\boldsymbol{\alpha}|}{2}, k \in \mathbb{N}} \frac{(\lambda)_k \tilde{b}_{1,2}^{k-\alpha_3} \tilde{b}_{1,3}^{k-\alpha_2} \tilde{b}_{2,3}^{k-\alpha_1} \tilde{b}_{1,2,3}^{\alpha_1 + \alpha_2 + \alpha_3 - 2k}}{(k-\alpha_3)! (k-\alpha_2)! (k-\alpha_1)! (\alpha_1 + \alpha_2 + \alpha_3 - 2k)!}. \quad (47)$$

and $c_{\alpha,\lambda}(R_3) = 0$ if $\|\alpha\|_\infty > \frac{|\alpha|}{2}$, in particular if $|\alpha| = 1$, $c_{\alpha,\lambda}(R_3) = 0$. Thus we have for $\mathbf{x} = (x_1, x_2, x_3)$

$$\begin{aligned} \gamma_{(P_3,\lambda)}(\mathbf{dx}) &= \frac{p_{[3]}^{-\lambda}}{[\Gamma(\lambda)]^3} \exp(\boldsymbol{\theta}_{P_3}, \mathbf{x}) \mathbf{x}^{(\lambda-1)\mathbf{1}_3} \\ &\times \left\{ \sum_{\alpha \in \mathbb{N}^3, \|\alpha\|_\infty \leq \frac{|\alpha|}{2}} \left[\sum_{\|\alpha\|_\infty \leq k \leq \frac{|\alpha|}{2}, k \in \mathbb{N}} \frac{(\lambda)_k \tilde{b}_{1,2}^{k-\alpha_3} \tilde{b}_{1,3}^{k-\alpha_2} \tilde{b}_{2,3}^{k-\alpha_1} \tilde{b}_{1,2,3}^{\alpha_1+\alpha_2+\alpha_3-2k}}{(k-\alpha_3)!(k-\alpha_2)!(k-\alpha_1)!(\alpha_1+\alpha_2+\alpha_3-2k)!} \frac{\mathbf{x}^\alpha}{(\lambda)_\alpha} \right] \mathbf{1}_{(0,\infty)^3}(\mathbf{x}) \right\} (\mathbf{dx}). \end{aligned} \quad (48)$$

Or, with for $\mathbf{z}_4 = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4$, and ${}_1\mathbf{F}_3$ is a generalized multivariate Lauricella function defined by

$${}_1\mathbf{F}_3(\lambda; \mathbf{z}_4) = \sum_{\mathbf{l} \in \mathbb{N}^4} \frac{(\lambda)_{l_1+l_2+l_3+l_4}}{(\lambda)_{l_2+l_3+l_4} (\lambda)_{l_1+l_3+l_4} (\lambda)_{l_1+l_2+l_4}} \frac{\mathbf{z}_4^{\mathbf{l}}}{\mathbf{l}!}. \quad (49)$$

$$\gamma_{(P_3,\lambda)}(\mathbf{dx}) = \frac{p_{[3]}^{-\lambda}}{[\Gamma(\lambda)]^3} \exp(\boldsymbol{\theta}_{P_3}, \mathbf{x}) \mathbf{x}^{(\lambda-1)\mathbf{1}_3} {}_1\mathbf{F}_3(\lambda; \tilde{b}_{1,2}x_1x_2, \tilde{b}_{1,3}x_1x_3, \tilde{b}_{2,3}x_2x_3, \tilde{b}_{1,2,3}x_1x_2x_3) \mathbf{1}_{(0,\infty)^3}(\mathbf{x}) (\mathbf{dx}). \quad (50)$$

We note that the latter result compared with (18) gives with $z_1 = \tilde{b}_{1,2}x_1x_2, z_2 = \tilde{b}_{1,3}x_1x_3, z_3 = \tilde{b}_{2,3}x_2x_3, z_4 = \tilde{b}_{1,2,3}x_1x_2x_3$:

$$\begin{aligned} {}_1\mathbf{F}_3(\lambda; z_1, z_2, z_3, z_4) &= \sum_{\mathbf{l} \in \mathbb{N}^4} \frac{(\lambda)_{l_1+l_2+l_3+l_4}}{(\lambda)_{l_2+l_3+l_4} (\lambda)_{l_1+l_3+l_4} (\lambda)_{l_1+l_2+l_4}} \frac{\mathbf{z}_4^{\mathbf{l}}}{\mathbf{l}!} \\ &= \sum_{\mathbf{l} \in \mathbb{N}^4} \frac{1}{(\lambda)_{l_1+l_3+l_4} (\lambda)_{l_2+2l_3+l_4}} \frac{(z_1, z_2 + z_3, z_2z_3, z_4)^{\mathbf{l}}}{\mathbf{l}!}. \end{aligned}$$

In the particular three cases $\tilde{b}_{1,2}, \tilde{b}_{1,3}, \tilde{b}_{2,3}, \tilde{b}_{1,2,3} > 0, \tilde{b}_{1,2}, \tilde{b}_{1,3}, \tilde{b}_{2,3} > 0, \tilde{b}_{1,2,3} = 0$ and $\tilde{b}_{1,2} = \tilde{b}_{1,3} = \tilde{b}_{2,3} = 0, \tilde{b}_{1,2,3} > 0$, we can give the following remarks.

Remark 20 If $\tilde{b}_{1,2}, \tilde{b}_{1,3}, \tilde{b}_{2,3}, \tilde{b}_{1,2,3} > 0$, if $\|\alpha\|_\infty \leq \frac{|\alpha|}{2}$,

$$c_{\alpha,\lambda}(R_3) = \sum_{\|\alpha\|_\infty \leq k \leq \frac{|\alpha|}{2}, k \in \mathbb{N}} \frac{(\lambda)_k \tilde{b}_{2,3}^{k-\alpha_1} \tilde{b}_{1,3}^{k-\alpha_2} \tilde{b}_{1,2}^{k-\alpha_3} \tilde{b}_{1,2,3}^{\alpha_1+\alpha_2+\alpha_3-2k}}{(k-\alpha_1)!(k-\alpha_2)!(k-\alpha_3)!(\alpha_1+\alpha_2+\alpha_3-2k)!} > 0 \quad (51)$$

and $c_{\alpha,\lambda}(R_3) = 0$ if $\|\alpha\|_\infty > \frac{|\alpha|}{2}$. Thus we have

$$\begin{aligned} \gamma_{(P_3,\lambda)}(\mathbf{dx}) &= \frac{p_{[3]}^{-\lambda}}{[\Gamma(\lambda)]^3} \exp(\boldsymbol{\theta}_P, \mathbf{x}) \mathbf{x}^{(\lambda-1)\mathbf{1}_3} \\ &\times \left\{ \sum_{\alpha \in \mathbb{N}^3, \|\alpha\|_\infty \leq \frac{|\alpha|}{2}} \left[\sum_{\|\alpha\|_\infty \leq k \leq \frac{|\alpha|}{2}, k \in \mathbb{N}} \frac{(\lambda)_k \tilde{b}_{2,3}^{k-\alpha_1} \tilde{b}_{1,3}^{k-\alpha_2} \tilde{b}_{1,2}^{k-\alpha_3} \tilde{b}_{1,2,3}^{\alpha_1+\alpha_2+\alpha_3-2k}}{(k-\alpha_1)!(k-\alpha_2)!(k-\alpha_3)!(\alpha_1+\alpha_2+\alpha_3-2k)!} \frac{\mathbf{x}^\alpha}{(\lambda)_\alpha} \right] \mathbf{1}_{(0,\infty)^3}(\mathbf{x}) \right\}(\mathbf{dx}). \end{aligned} \quad (52)$$

Remark 21 If $\tilde{b}_{1,2}, \tilde{b}_{1,3}, \tilde{b}_{2,3} > 0$ and $\tilde{b}_{1,2,3} = 0$, if $\|\alpha\|_\infty \leq \frac{|\alpha|}{2} = k \in \mathbb{N}$, then

$$c_{\alpha,\lambda}(R_3) = \frac{(\lambda)_k \tilde{b}_{2,3}^{k-\alpha_1} \tilde{b}_{1,3}^{k-\alpha_2} \tilde{b}_{1,2}^{k-\alpha_3}}{(k-\alpha_1)!(k-\alpha_2)!(k-\alpha_3)!} > 0, \quad (53)$$

and $c_{\alpha,\lambda}(R_3) = 0$ if $\|\alpha\|_\infty > \frac{|\alpha|}{2} \in \mathbb{N}$ or $\frac{|\alpha|}{2} \notin \mathbb{N}$. Thus we have

$$\gamma_{(P_3,\lambda)}(\mathbf{dx}) = \frac{p_{[3]}^{-\lambda}}{[\Gamma(\lambda)]^3} \exp(\boldsymbol{\theta}_P, \mathbf{x}) \mathbf{x}^{(\lambda-1)\mathbf{1}_3} \left[\sum_{\alpha \in \mathbb{N}^3, \|\alpha\|_\infty \leq \frac{|\alpha|}{2} = k \in \mathbb{N}} \frac{(\lambda)_k \tilde{b}_{2,3}^{k-\alpha_1} \tilde{b}_{1,3}^{k-\alpha_2} \tilde{b}_{1,2}^{k-\alpha_3}}{(k-\alpha_1)!(k-\alpha_2)!(k-\alpha_3)!} \frac{\mathbf{x}^\alpha}{(\lambda)_\alpha} \right] \mathbf{1}_{(0,\infty)^3}(\mathbf{x}) (\mathbf{dx}). \quad (54)$$

Remark 22 If $\tilde{b}_{1,2}, \tilde{b}_{1,3}, \tilde{b}_{2,3} = 0$, $\tilde{b}_{1,2,3} > 0$, then for $\alpha = k\mathbf{1}_3$, $k \in \mathbb{N}$,

$$c_{k\mathbf{1}_3,\lambda}(R_3) = \frac{(\lambda)_k \tilde{b}_{1,2,3}^k}{k!} \quad (55)$$

and $c_{\alpha,\lambda}(R_3) = 0$ otherwise. Thus we have

$$\gamma_{(P_3,\lambda)}(\mathbf{dx}) = \frac{p_{[3]}^{-\lambda}}{[\Gamma(\lambda)]^3} \exp(\boldsymbol{\theta}_P, \mathbf{x}) \mathbf{x}^{(\lambda-1)\mathbf{1}_3} F_2(\lambda, \lambda; \tilde{b}_{1,2,3} \mathbf{x}^{[3]}) \mathbf{1}_{(0,\infty)^3}(\mathbf{x}) (\mathbf{dx}) \quad (56)$$

3 Conditional Laplace transform

We now assume that P_n is an affine polynomial such that $P_n(\boldsymbol{\theta}) = \sum_{T \in \mathfrak{P}_n} p_T \boldsymbol{\theta}^T = 1 + \sum_{T \in \mathfrak{P}_n^*} p_T \boldsymbol{\theta}^T$, with $p_{[k]} > 0$ for $k = 1, \dots, n$, and is such that the mgd $\gamma_{(P_n,\lambda)}$ associated with (P_n, λ) is infinitely divisible. If $k \in [n]$, we denote by $\boldsymbol{\theta}_{[k]} = (\theta_1, \dots, \theta_k)$, $\boldsymbol{\theta}_{[n] \setminus [k]} = (\theta_{k+1}, \dots, \theta_n)$, and P_k the affine polynomial $P_k(\boldsymbol{\theta}_{[k]}) = \sum_{T \in \mathfrak{P}_k} p_T \boldsymbol{\theta}_{[k]}^T = P_n(\boldsymbol{\theta}_{[k]}, \mathbf{0}_{n-k})$. Similarly, if $T = \{t_1, \dots, t_k\} \subset [n]$, $t_1 < \dots < t_k$, we denote by $\boldsymbol{\theta}_T = (\theta_{t_1}, \dots, \theta_{t_k})$, and if $[n] \setminus T = \bar{T} = \{t_{k+1}, \dots, t_n\}$, $\boldsymbol{\theta}_{[n] \setminus T} = (\theta_{t_{k+1}}, \dots, \theta_{t_n})$, and P_T the affine polynomial

$P_T(\boldsymbol{\theta}_T) = \sum_{S \in \mathfrak{P}(T)} p_S \boldsymbol{\theta}_T^S = P_n(\boldsymbol{\theta}'_T)$, where $(\boldsymbol{\theta}'_T)_i = \theta_{t_i}$ if $t_i \in T$ and $(\boldsymbol{\theta}'_T)_i = 0$ if $t_i \notin T$.

For all $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we introduce the notation

$$\left(\frac{\partial}{\partial \boldsymbol{\theta}}\right)^\alpha = \frac{\partial^{|\alpha|}}{\partial \theta_1^{\alpha_1} \dots \partial \theta_n^{\alpha_n}}.$$

For all $T \in \mathfrak{P}_n$, we also define $(\partial/\partial \boldsymbol{\theta})^T = (\partial/\partial \boldsymbol{\theta})^{\mathbf{1}_T}$.

Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random real vector such that has distribution $\gamma_{(P_n, \lambda)}$, denoted by $\mathbf{X} \sim \gamma_{(P_n, \lambda)}$, we give a formula for the Lt of $X_{[n] \setminus [k]} = (X_{k+1}, \dots, X_n)$ given $\mathbf{X}_{[k]} = \mathbf{x}_{[k]}$, an important conditional distribution for the simulations of \mathbf{X} , in the following main theorem.

Theorem 23 *Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random real vector such that $\mathbf{X} \sim \gamma_{(P_n, \lambda)}$, with the notation of Theorem 16. Let $1 < n \in \mathbb{N}$, $1 \leq k < n$, and $Q_{[n] \setminus [k]}$ the affine polynomials with respect to the $n - k$ variables $\theta_{k+1}, \dots, \theta_n$ defined by*

$$Q_{[n] \setminus [k]}(\boldsymbol{\theta}_{[n] \setminus [k]}) = \prod_{i=k+1}^n [1 + \theta_i (-\boldsymbol{\theta}_{P_n})_i^{-1}] \quad (57)$$

If $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote $\mathbf{x}_{[k]} = (x_1, \dots, x_k) \in \mathbb{R}^k$. If $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$, let $\mathbf{y}_{[k]} \mathbf{x}_{[k]} = (y_1 x_1, \dots, y_k x_k)$, then the Lt of $\mathbf{X}_{[n] \setminus [k]}$ given $\mathbf{X}_{[k]} = \mathbf{x}_{[k]}$ is

$$L_{\mathbf{X}_{[n] \setminus [k]}}^{\mathbf{X}_{[k]} = \mathbf{x}_{[k]}}(\boldsymbol{\theta}_{[n] \setminus [k]}) = [Q_{[n] \setminus [k]}(\boldsymbol{\theta}_{[n] \setminus [k]})]^{-\lambda} \frac{\mathbf{F}_k(\lambda, R_n, \mathbf{x}_{[k]}, \boldsymbol{\theta}_{[n] \setminus [k]})}{\mathbf{F}_k(\lambda, R_n, \mathbf{x}_{[k]}, \mathbf{0}_{[n] \setminus [k]})} \quad (58)$$

with

$$\mathbf{F}_k(\lambda, R_n, \mathbf{x}_{[k]}, \boldsymbol{\theta}_{[n] \setminus [k]}) = \sum_{\boldsymbol{\alpha} = (\boldsymbol{\alpha}_{[k]}, \boldsymbol{\alpha}_{[n] \setminus [k]}) \in \mathbb{N}^n, c_{\boldsymbol{\alpha}, \lambda}(R_n) \neq 0} \frac{\mathbf{x}_{[k]}^{\boldsymbol{\alpha}_{[k]}}}{(\lambda)^{\boldsymbol{\alpha}_{[k]}}} c_{\boldsymbol{\alpha}, \lambda}(R_n) (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus [k]}^{-\boldsymbol{\alpha}_{[n] \setminus [k]}} \quad (59)$$

4 Conditional Laplace transform in the particular case $k = 1$

Another form of Theorem 23 can be given for $k = 1$ by the following theorem.

Theorem 24 Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random real vector such that $\mathbf{X} \sim \gamma_{(P_n, \lambda)}$, with the notation of Theorem 16. Let $1 < n \in \mathbb{N}$, $k = 1$, and S_{n-1} , B_{n-1} the affine polynomials with respect to the $n - 1$ variables $\theta_2, \dots, \theta_n$ defined by

$$S_{[n] \setminus [1]}(\boldsymbol{\theta}_{[n] \setminus [1]}) = \frac{p^{[n]}}{p_1} (\boldsymbol{\theta}_n - \boldsymbol{\theta}_{P_n})_{[n] \setminus [1]}^{\mathbf{1}_{[n] \setminus [1]}} \left(1 - R_n \left(0, (\boldsymbol{\theta}_n - \boldsymbol{\theta}_{P_n})_{[n] \setminus [1]}^{-1} \right) \right) \quad (60)$$

$$B_{n-1}(\boldsymbol{\theta}_{[n] \setminus [1]}) = \frac{p^{[n]}}{p_1} (\boldsymbol{\theta}_n - \boldsymbol{\theta}_{P_n})_{[n] \setminus [1]}^{\mathbf{1}_{[n] \setminus [1]}} \frac{\partial}{\partial z_1} R_n \left(0, (\boldsymbol{\theta}_n - \boldsymbol{\theta}_{P_n})_{[n] \setminus [1]}^{-1} \right) \quad (61)$$

we have

$$S_{[n] \setminus [1]}(\boldsymbol{\theta}_{[n] \setminus [1]}) = \frac{1}{p_1} \frac{\partial}{\partial \theta_1} P_n(\boldsymbol{\theta}_n), \quad (62)$$

$$B_{n-1}(\boldsymbol{\theta}_{[n] \setminus [1]}) = -\frac{1}{p_1} P_n(\tilde{p}_1, \boldsymbol{\theta}_{[n] \setminus [1]}) \quad (63)$$

and

$$S_{[n] \setminus [1]}(\boldsymbol{\theta}_{[n] \setminus [1]}) = \sum_{T \subset [n] \setminus [1]} \frac{p^{[1] \cup T}}{p^{[1]}} \boldsymbol{\theta}_{[n] \setminus [1]}^T. \quad (64)$$

Let \mathfrak{z}_{n-1} the function with respect to the $n - 1$ variables $\theta_2, \dots, \theta_n$ defined by

$$\begin{aligned} \mathfrak{z}_{n-1}(\boldsymbol{\theta}_{[n] \setminus [1]}) &= \frac{B_{n-1}(\boldsymbol{\theta}_{[n] \setminus [1]})}{S_{[n] \setminus [1]}(\boldsymbol{\theta}_{[n] \setminus [1]})} \\ &= \frac{\frac{\partial}{\partial z_1} R_n \left(0, (\boldsymbol{\theta}_n - \boldsymbol{\theta}_{P_n})_{[n] \setminus [1]}^{-1} \right)}{1 - R_n \left(0, (\boldsymbol{\theta}_n - \boldsymbol{\theta}_{P_n})_{[n] \setminus [1]}^{-1} \right)}. \end{aligned} \quad (65)$$

Let us define

$$I_1(R_n) = \{ \alpha_1 \in \mathbb{N}, \exists \boldsymbol{\alpha} = (\alpha_1, \boldsymbol{\alpha}_{[n] \setminus \{1\}}) \in \mathbb{N}^n, c_{\boldsymbol{\alpha}, \lambda}(R_n) \neq 0 \}, \quad (66)$$

then we define \mathbf{G} , a function with respect to the variable u_1 , by

$$\mathbf{G}(R_n, u_1) = \sum_{\alpha_1 \in I_1(R_n)} \frac{u_1^{\alpha_1}}{\alpha_1!} \quad (67)$$

If $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote $\mathbf{x}_{[k]} = (x_1, \dots, x_k) \in \mathbb{R}^k$. If $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$, let $\mathbf{y}_{[k]} \mathbf{x}_{[k]} = (y_1 x_1, \dots, y_k x_k)$, then the Lt of $\mathbf{X}_{[n] \setminus [1]} = (X_2, \dots, X_n)$ given $X_1 = x_1$ is

$$L_{\mathbf{X}_{[n] \setminus [1]}}^{X_1=x_1}(\boldsymbol{\theta}_{[n] \setminus [1]}) = [S_{[n] \setminus [1]}(\boldsymbol{\theta}_{[n] \setminus [1]})]^{-\lambda} \frac{\mathbf{G}(R_n, \mathfrak{z}_{n-1}(\boldsymbol{\theta}_{[n] \setminus [1]}) x_1)}{\mathbf{G}(R_n, \mathfrak{z}_{n-1}(\mathbf{0}_{[n] \setminus [1]}) x_1)} \quad (68)$$

Before prove Theorem 24, we give the following remark for the affine polynomial

$$S_{[n]\setminus[1]}(\boldsymbol{\theta}_{[n]\setminus[1]}) = \sum_{T \subset [n]\setminus[1]} \frac{p_{\{1\} \cup T}}{p_1} (\boldsymbol{\theta}_{[n]\setminus[1]})^T.$$

Remark 25 For $T \in \mathfrak{P}_{[n]\setminus[1]}$, we have $\tilde{p}_T(S_{[n]\setminus[1]}) = \tilde{p}_T(R_n)$, hence we have $\tilde{b}_T(S_{[n]\setminus[1]}) = \tilde{b}_T(P_n)$. Therefore, if $\gamma_{(P_n, \lambda)}$ is infinitely divisible $\gamma_{(S_{[n]\setminus[1]}, \lambda)}$ is also infinitely divisible by Theorem 6. We also have

$$r_T(S_{[n]\setminus[1]}) = r_T(P_n) = r_T \quad (69)$$

and therefore

$$S_{[n]\setminus[1]}(\boldsymbol{\theta}_{[n]\setminus[1]}) = \frac{p_{[n]}}{p_1} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n]\setminus\{1\}} \left\{ 1 - \sum_{T \subset [n]\setminus\{1\}} r_T (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{-T} \right\} \quad (70)$$

Now, we can prove Theorem 24. If there is no ambiguity, we denote $S_{[n]\setminus[1]}, B_{n-1}$ and \mathfrak{J}_{n-1} by S, B and \mathfrak{J} .

Before giving the main theorem, we prove by finite induction the following lemma.

Lemma 26 With the notations of Theorem (24), unless $R_n = 0$, we have,

$$I_1(R_n) = \{ \alpha_1 \in \mathbb{N}, \exists \boldsymbol{\alpha} = (\alpha_1, \boldsymbol{\alpha}_{[n]\setminus\{1\}}) \in \mathbb{N}^n, c_{\boldsymbol{\alpha}, \lambda}(R_n) \neq 0 \} = \mathbb{N},$$

therefore G defined in Theorem (24) by (67) is exp.

Now, we can give the following main Theorem.

Theorem 27 With the notations of Theorem (24), unless $R_n = 0$, we have

$$L_{\mathbf{X}_{[n]\setminus[1]}}^{X_1=x_1}(\boldsymbol{\theta}_{[n]\setminus[1]}) = S_{[n]\setminus[1]}^{-\lambda}(\boldsymbol{\theta}_{[n]\setminus[1]}) \exp\left\{-\left[\frac{P_n(0, \boldsymbol{\theta}_{[n]\setminus[1]})}{S_{[n]\setminus[1]}(\boldsymbol{\theta}_{[n]\setminus[1]})} - 1\right] \frac{x_1}{p_1}\right\} \quad (71)$$

$$= S_{[n]\setminus[1]}^{-\lambda}(\boldsymbol{\theta}_{[n]\setminus[1]}) \exp\left\{-\left[\frac{\sum_{T \in \mathfrak{P}([n]\setminus[1])} (p_T - \frac{p_{\{1\} \cup T}}{p_1}) \boldsymbol{\theta}_{[n]\setminus[1]}^T}{S_{[n]\setminus[1]}(\boldsymbol{\theta}_{[n]\setminus[1]})}\right] \frac{x_1}{p_1}\right\} \quad (72)$$

$$= S_{[n]\setminus[1]}^{-\lambda}(\boldsymbol{\theta}_{[n]\setminus[1]}) \quad (73)$$

$$\times \mathbf{e}^{\left\{\frac{p_{[n]}}{p_1} x_1 \sum_{T \subset [n]\setminus\{1\}, 0 < |T|} r_{\{1\} \cup T} (-\tilde{\mathbf{P}})^{\{[n]\setminus\{1\}\} \setminus T} [\mathbf{S}^{[n]\setminus\{1\} \setminus T}(\boldsymbol{\theta}_{[n]\setminus[1]}) S_{[n]\setminus[1]}^{-1}(\boldsymbol{\theta}_{[n]\setminus[1]}) - 1\right\}}$$

We denote by $\mathbf{S} = (S_1, \dots, S_n)$, with $S_i(\theta_i) = 1 + (-\tilde{p}_i)^{-1} \theta_i$, $i = 1, \dots, n$. For simplicity, we denote $L_{\mathbf{X}_{[n] \setminus [1]}^{X_1=x_1}}(\boldsymbol{\theta}_{[n] \setminus [1]})$ by $L_{\mathbf{X}_{[n] \setminus [1]}^{X_1=x_1}}$, $S_{[n] \setminus [1]}(\boldsymbol{\theta}_{[n] \setminus [1]})$ by $S_{[n] \setminus [1]}$, and $\mathbf{S}(\boldsymbol{\theta}_{[n] \setminus [1]})$ by \mathbf{S} and we can write

$$L_{\mathbf{X}_{[n] \setminus [1]}^{X_1=x_1}} = S_{[n] \setminus [1]}^{-\lambda} \mathbf{e}^{\{(-\tilde{p}_{[n] \setminus [1]})^{-1} x_1 \sum_{T \subset [n] \setminus \{1\}, 0 < |T|} r_{\{1\} \cup T}(-\tilde{\mathbf{p}})^{\{[n] \setminus \{1\}\} \setminus T} [\mathbf{S}_{[n] \setminus \{1\}}^{-1}]^{-1}\}}. \quad (74)$$

We can also write

$$L_{\mathbf{X}_{[n] \setminus [1]}^{X_1=x_1}} = S_{[n] \setminus [1]}^{-\lambda} \mathbf{e}^{\{(-\tilde{p}_{[n] \setminus [1]})^{-1} x_1 [\sum_{T \subset [n] \setminus \{1\}, 0 < |T|} r_{\{1\} \cup T}(-\tilde{\mathbf{p}})^{\{[n] \setminus \{1\}\} \setminus T} \mathbf{S}_{[n] \setminus \{1\}}^{-1}]^{-1} - C\}}, \quad (75)$$

where $C = \sum_{T \subset [n] \setminus \{1\}, 0 < |T|} r_{\{1\} \cup T}(-\tilde{\mathbf{p}})^{\{[n] \setminus \{1\}\} \setminus T}$ is such that $L_{\mathbf{X}_{[n] \setminus [1]}^{X_1=x_1}}(\mathbf{0}_{n-1}) = 1$.

5 Applications of the main results

5.1 A particular case in the case $k = 1$

Now let us apply the result of Corollary 17 to get the following Corollary.

Corollary 28 For $P_n(\boldsymbol{\theta}) = \frac{-q}{p} + \frac{1}{p} \prod_{i=1}^n (1 + p\theta_i)$, we have

$$L_{\mathbf{X}_{[n] \setminus [k]}^{\mathbf{X}^{[k]}=\mathbf{X}^{[k]}}}(\boldsymbol{\theta}_{[n] \setminus [k]}) = \left[\prod_{i=k+1}^n (1 + p\theta_i) \right]^{-\lambda} \frac{F_{k-1}(\lambda, \dots, \lambda, qp^{-k} \mathbf{X}_k^{[k]} \prod_{i=k+1}^n (1 + p\theta_i)^{-1})}{F_{k-1}(\lambda, \dots, \lambda, qp^{-k} \mathbf{X}_k^{[k]})} \quad (76)$$

The case $k = 1$ is simpler because $F_0 = \exp$, so we can give the following corollary.

Corollary 29 For $n > 1, k = 1$, we have

$$L_{\mathbf{X}_{[n] \setminus [1]}^{X_1=x_1}}(\boldsymbol{\theta}_{[n] \setminus [1]}) = \left[\prod_{i=2}^n (1 + p\theta_i) \right]^{-\lambda} \exp\{qp^{-1} x_1 [\prod_{i=2}^n (1 + p\theta_i)^{-1} - 1]\}, \quad (77)$$

or

$$L_{\mathbf{X}_{[n] \setminus [1]}^{X_1=x_1}}(\boldsymbol{\theta}_{[n] \setminus [1]}) = \sum_{k=0}^{\infty} \frac{(qp^{-1} x_1)^k}{k!} \exp(-qp^{-1} x_1) \left[\prod_{i=2}^n (1 + p\theta_i) \right]^{-(\lambda+k)}, \quad (78)$$

As a result, Formula (78) gives a simulation of $\mathbf{X}_{[n]}$. Let us denote by $\mathcal{P}(\mu)$ the Poisson distribution (Pd) of parameter μ , we derive the following theorem.

Theorem 30 Let $X_1 \sim \gamma_{(p,\lambda)}$, let $V_1 \sim \mathcal{P}(qp^{-1}X_1)$, let $\mathbf{X}_{[n]\setminus[1]} = (X_2, \dots, X_n)$, and $\mathbf{X}_{[n]\setminus[1]} | (X_1 = x_1) \sim \gamma_{(\prod_{i=2}^n (1+p\theta_i), \lambda+V_1)}$, then $\mathbf{X}_{[n]} = (X_1, X_2, \dots, X_n) \sim \gamma_{(P_n, \lambda)}$, with $P_n(\boldsymbol{\theta}_{[n]}) = \frac{-q}{p} + \frac{1}{p} \prod_{i=1}^n (1+p\theta_i)$.

We derive the following algorithm for simulating $\mathbf{X}_{[n]} \sim \gamma_{(P_n, \lambda)}$.

Algorithm 31 Simulation of an infinitely divisible mgd $\gamma_{(P_n, \lambda)}$, with $P_n(\boldsymbol{\theta}_{[n]}) = \frac{-q}{p} + \frac{1}{p} \prod_{i=1}^n (1+p\theta_i)$

1. Simulate $X_1 \sim \gamma_{(p, \lambda)}$
2. Simulate $V_1 \sim \mathcal{P}(qp^{-1}X_1)$
3. Simulate independently $X_i \sim \gamma_{(p, \lambda+V_1)}$
4. Then $\mathbf{X}_{[n]} = (X_1, X_2, \dots, X_n)$ simulate $\gamma_{(P_n, \lambda)}$.

5.2 The case $n = 2$ and $k = 1$

In this case, we give another proof of Theorem 14 in [Bernardoff, 2023]:

Theorem 32 Let $P_2(\theta_1, \theta_2) = 1 + p_1\theta_1 + p_2\theta_2 + p_{1,2}\theta_1\theta_2$, with $p_1 > 0$, $p_2 > 0$, $p_{1,2} > 0$, $\tilde{b}_{1,2} = \tilde{b}_{1,2}(P_2) = p_1p_2/p_{1,2}^2 - 1/p_{1,2} > 0$. Let $P_1(\theta_1) = P_2(\theta_1, 0)$, $X_1 \sim \gamma_{(P_1, \lambda)}$. Let $\alpha_1 = \frac{\tilde{b}_{1,2}}{(-p_2)}$, and $V_1 \sim \mathcal{P}(\alpha_1 X_1)$. We have $S_2(\theta_2) = 1 + \frac{p_{1,2}}{p_1}\theta_2$. Let $X_2 \sim \gamma_{(S_2, \lambda+V_1)}$, then $\mathbf{X}_{[2]} = (X_1, X_2) \sim \gamma_{(P_2, \lambda)}$.

We derive the following algorithm for simulating $\mathbf{X}_{[2]} = (X_1, X_2) \sim \gamma_{(P_2, \lambda)}$, see [Bernardoff, 2023]:

Algorithm 33 Simulation of an infinitely divisible bgd $\gamma_{(P_2, \lambda)}$

1. Simulate $X_1 \sim \gamma_{(p_1, \lambda)}$
2. Simulate $V_1 \sim \mathcal{P}(\frac{p_{1,2}}{p_1}\tilde{b}_{1,2}X_1)$
3. Simulate $X_2 \sim \gamma_{(\frac{p_{1,2}}{p_1}, \lambda+V_1)}$
4. Then $\mathbf{X}_{[2]} = (X_1, X_2)$ simulates $\gamma_{(P_2, \lambda)}$.

5.3 The case $n = 3$ and $k = 1$

In this case, we give the following theorem

Theorem 34 Let $P_3(\theta_1, \theta_2, \theta_3) = 1 + p_1\theta_1 + p_2\theta_2 + p_3\theta_3 + p_{1,2}\theta_1\theta_2 + p_{1,3}\theta_1\theta_3 + p_{2,3}\theta_2\theta_3 + p_{1,2,3}\theta_1\theta_2\theta_3$ with $p_i > 0, i = 1, 2, 3, p_{1,2} > 0, p_{2,3} > 0, p_{1,3} > 0, p_{1,2,3} > 0$, and $\tilde{b}_{1,2}(P_3) = p_{1,3}p_{2,3}/p_{1,2,3}^2 - p_3/p_{1,2,3} > 0, \tilde{b}_{1,3}(P_3) = p_{1,2}p_{2,3}/p_{1,2,3}^2 - p_2/p_{1,2,3} > 0, \tilde{b}_{2,3}(P_3) = p_{1,3}p_{1,2}/p_{1,2,3}^2 - p_1/p_{1,2,3} > 0$ ($\tilde{p}_1 = -p_{2,3}/p_{1,2,3} < 0, \tilde{p}_2 = -p_{1,3}/p_{1,2,3} < 0, \tilde{p}_3 = -p_{1,2}/p_{1,2,3} < 0$) and $\tilde{b}_{1,2,3}(P_3) = \tilde{p}_{1,2,3} + \tilde{p}_1\tilde{p}_{2,3} + \tilde{p}_2\tilde{p}_{1,3} + \tilde{p}_3\tilde{p}_{1,2} + 2\tilde{p}_1\tilde{p}_2\tilde{p}_3 = -1/p_{1,2,3} + p_{2,3}p_1/p_{1,2,3}^2 + p_{1,3}p_2/p_{1,2,3}^2 + p_{1,2}p_3/p_{1,2,3}^2 - 2p_{2,3}p_{1,3}p_{1,2}/p_{1,2,3}^3 > 0$. Let $P_1(\theta_1) = P_3(\theta_1, 0, 0) = 1 + p_1\theta_1$, let $X_1 \sim \gamma_{(P_1, \lambda)}$. Let $S_{2,3}(\theta_2, \theta_3) = 1 + \frac{p_{1,2}}{p_1}\theta_2 + \frac{p_{1,3}}{p_1}\theta_3 + \frac{p_{1,2,3}}{p_1}\theta_2\theta_3, S_2(\theta_2) = 1 + (-\tilde{p}_2)^{-1}\theta_2$ and $S_3(\theta_3) = 1 + (-\tilde{p}_3)^{-1}\theta_3$. Let $\alpha_1 = \frac{\tilde{b}_{1,2}}{(-\tilde{p}_2)}, \alpha_2 = \frac{\tilde{b}_{1,3}}{(-\tilde{p}_3)}, \alpha_3 = \frac{\tilde{b}_{1,2,3}}{(-\tilde{p}_{2,3})}, \alpha_4 = \frac{\tilde{b}_{1,2}\tilde{b}_{2,3}}{(-\tilde{p}_2)(-\tilde{p}_{2,3})}, \alpha_5 = \frac{\tilde{b}_{1,3}\tilde{b}_{2,3}}{(-\tilde{p}_3)(-\tilde{p}_{2,3})}$, and $V_i \sim \mathcal{P}(\alpha_i X_1), i \in [5]$ independent, with the notation $\mathbf{v} = (v_1, \dots, v_5) \in \mathbb{N}$, we have

$$L_{(X_2, X_3)}^{X_1=x_1} = \sum_{\mathbf{v} \in \mathbb{N}^5} \prod_{i=1}^5 \mathbf{P}(V_i = v_i) S_2^{-(v_1+v_4)} S_3^{-(v_2+v_5)} S_{2,3}^{-(\lambda+v_3+v_4+v_5)}. \quad (79)$$

Let $Y_2 \sim \gamma_{(S_2, V_1+V_4)}, Y_3 \sim \gamma_{(S_3, V_2+V_5)}, (Z_2, Z_3) \sim \gamma_{(S_{2,3}, \lambda+V_3+V_4+V_5)}$ independent, and let $(X_2, X_3) = (Y_2 + Z_2, Y_3 + Z_3)$, then $\mathbf{X}_{[3]} = (X_1, X_2, X_3) \sim \gamma_{(P_3, \lambda)}$.

From Theorem (34), we derive the following algorithm for simulating $\mathbf{X}_{[3]} = (X_1, X_2, X_3) \sim \gamma_{(P_3, \lambda)}$

Algorithm 35 Simulation of an infinitely divisible *tgd* $\gamma_{(P_3, \lambda)}$

1. Simulate $X_1 \sim \gamma_{(p_1, \lambda)}$.
2. Compute $\alpha_1 = \frac{\tilde{b}_{1,2}}{(-\tilde{p}_2)}, \alpha_2 = \frac{\tilde{b}_{1,3}}{(-\tilde{p}_3)}, \alpha_3 = \frac{\tilde{b}_{1,2,3}}{(-\tilde{p}_{2,3})}, \alpha_4 = \frac{\tilde{b}_{1,2}\tilde{b}_{2,3}}{(-\tilde{p}_2)(-\tilde{p}_{2,3})}, \alpha_5 = \frac{\tilde{b}_{1,3}\tilde{b}_{2,3}}{(-\tilde{p}_3)(-\tilde{p}_{2,3})}$.
Simulate independently $V_i \sim \mathcal{P}(\alpha_i X_1), i \in [5]$.
3. Simulate independently $Y_2 \sim \gamma_{((- \tilde{p}_2)^{-1}, V_1+V_4)}, Y_3 \sim \gamma_{((- \tilde{p}_3)^{-1}, V_2+V_5)}, (Z_2, Z_3) \sim \gamma_{(S_{2,3}, \lambda+V_3+V_4+V_5)}$.
4. Then $\mathbf{X}_{[3]} = (X_1, Y_2 + Z_2, Y_3 + Z_3)$ simulates $\gamma_{(P_3, \lambda)}$.

We can notice that $\mathbf{W} = (W_1, W_2, W_3) = (V_1 + V_4, V_2 + V_5, V_3 + V_4 + V_5)$ satisfied for $\mathbf{t} = (t_1, t_2, t_3) \in (0, \infty)^3$,

$$\mathbb{E}(\mathbf{t}^{\mathbf{W}} | X_1 = x_1) = \exp[(\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_3 + \alpha_4 t_1 t_3 + \alpha_5 t_2 t_3)x_1 - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)x_1],$$

and $\mathbf{W} | X_1 = x_1$ is a trivariate Poisson distribution, see [Dwass and Teicher, 1957].

From Theorem (32), we derive the following theorem and algorithm.

Theorem 36 *Let $X_1 \sim \gamma_{(p_1, \lambda)}$. Let $\alpha_1 = \frac{\tilde{b}_{1,2}}{(-\tilde{p}_2)}$, $\alpha_2 = \frac{\tilde{b}_{1,3}}{(-\tilde{p}_3)}$, $\alpha_3 = \frac{\tilde{b}_{1,2,3}}{(-\tilde{p}_{2,3})}$, $\alpha_4 = \frac{\tilde{b}_{1,2}\tilde{b}_{2,3}}{(-\tilde{p}_2)(-\tilde{p}_{2,3})}$, $\alpha_5 = \frac{\tilde{b}_{1,3}\tilde{b}_{2,3}}{(-\tilde{p}_3)(-\tilde{p}_{2,3})}$, $\alpha_6 = \frac{\tilde{b}_{2,3}}{(-\tilde{p}_3)}$, and $V_i \sim \mathcal{P}(\alpha_i X_1)$, $i \in [5]$. Let $Z'_2 \sim \gamma_{\left(\frac{p_{1,2}}{p_1}, \lambda + V_3 + V_4 + V_5\right)}$ and $V_6 \sim \mathcal{P}(\alpha_6 Z'_2)$, let $Y_2 \sim \gamma_{((-\tilde{p}_2)^{-1}, V_1 + V_4)}$, $Y_3 \sim \gamma_{((-\tilde{p}_3)^{-1}, V_2 + V_5)}$, $Z'_3 \sim \gamma_{((-\tilde{p}_3)^{-1}, \lambda + V_3 + V_4 + V_5 + V_6)}$ independent, we have $(Z'_2, Z'_3) \sim \gamma_{(S_{2,3}, \lambda + V_3 + V_4 + V_5)}$, then let $X_2, X_3 = (Y_2 + Z'_2, Y_3 + Z'_3)$ then $\mathbf{X}_{[3]} = (X_1, X_2, X_3) \sim \gamma_{(P_3, \lambda)}$.*

Algorithm 37 *Simulation of an infinitely divisible tgd*

1. Simulate $X_1 \sim \gamma_{(p_1, \lambda)}$.
2. Compute $\alpha_1 = \frac{\tilde{b}_{1,2}}{(-\tilde{p}_2)}$, $\alpha_2 = \frac{\tilde{b}_{1,3}}{(-\tilde{p}_3)}$, $\alpha_3 = \frac{\tilde{b}_{1,2,3}}{(-\tilde{p}_{2,3})}$, $\alpha_4 = \frac{\tilde{b}_{1,2}\tilde{b}_{2,3}}{(-\tilde{p}_2)(-\tilde{p}_{2,3})}$, $\alpha_5 = \frac{\tilde{b}_{1,3}\tilde{b}_{2,3}}{(-\tilde{p}_3)(-\tilde{p}_{2,3})}$, $\alpha_6 = \frac{\tilde{b}_{2,3}}{(-\tilde{p}_3)}$. Simulate independently $V_i \sim \mathcal{P}(\alpha_i X_1)$, $i \in [5]$.
3. Simulate $Z'_2 \sim \gamma_{\left(\frac{p_{1,2}}{p_1}, \lambda + V_3 + V_4 + V_5\right)}$.
4. Simulate $V_6 \sim \mathcal{P}(\alpha_6 Z'_2)$.
5. Simulate independently $Y_2 \sim \gamma_{((-\tilde{p}_2)^{-1}, V_4 + V_5)}$, $Y_3 \sim \gamma_{((-\tilde{p}_3)^{-1}, V_2 + V_5)}$, $Z'_3 \sim \gamma_{((-\tilde{p}_3)^{-1}, \lambda + V_3 + V_4 + V_5 + V_6)}$.
6. Then $\mathbf{X}_{[3]} = (X_1, X_2, X_3) = (X_1, Y_2 + Z'_2, Y_3 + Z'_3)$ simulates $\gamma_{(P_3, \lambda)}$.

We can give the following remark.

Remark 38 *In Theorem (34) or Algorithm (35), we use 3 univariate gamma distributions, 5 Poisson distributions and one bivariate distribution. In Theorem (36) or Algorithm (37), we use 5 univariate gamma distributions and 6 Poisson distributions. For the simulations themselves, we can use either method with the R software, [R Core Team, 2023].*

5.4 The case $n = 4$ and $k = 1$

In this case, we give the following theorem

Theorem 39 Let $P_4(\theta_4) = \sum_{T \in \mathfrak{P}_4} p_T \theta_4^T$ with $p_T > 0, T \in \mathfrak{P}_4^*$, and $\tilde{b}_T(P_4) > 0, T \in \mathfrak{P}_4^*, |T| > 1$, Let $P_1(\theta_1) = P_4(\theta_1, 0, 0, 0) = 1 + p_1 \theta_1$, let $X_1 \sim \gamma_{(P_1, \lambda)}$. Let $S_T, T \in \mathfrak{P}_4^*$ defined by (31), so that $S_2(\theta_2) = 1 + (-\tilde{p}_2)^{-1} \theta_2$ and $S_3(\theta_3) = 1 + (-\tilde{p}_3)^{-1} \theta_3$, $S_4(\theta_4) = 1 + (-\tilde{p}_4)^{-1} \theta_4$, $S_{2,3}(\theta_2, \theta_3) = 1 + \frac{p_{1,2,4}}{p_{1,4}} \theta_2 + \frac{p_{1,3,4}}{p_{1,4}} \theta_3 + \frac{p_{1,2,3,4}}{p_{1,4}} \theta_2 \theta_3$, $S_{2,4}(\theta_2, \theta_4) = 1 + \frac{p_{1,2,3}}{p_{1,3}} \theta_2 + \frac{p_{1,3,4}}{p_{1,3}} \theta_4 + \frac{p_{1,2,3,4}}{p_{1,3}} \theta_2 \theta_4$, $S_{3,4}(\theta_3, \theta_4) = 1 + \frac{p_{1,2,3}}{p_{1,2}} \theta_3 + \frac{p_{1,2,4}}{p_{1,2}} \theta_4 + \frac{p_{1,2,3,4}}{p_{1,2}} \theta_3 \theta_4$, and $S_{2,3,4}(\theta_2, \theta_3, \theta_4) = 1 + \frac{p_{1,2}}{p_1} \theta_2 + \frac{p_{1,3}}{p_1} \theta_3 + \frac{p_{1,4}}{p_1} \theta_4 + \frac{p_{1,2,3}}{p_1} \theta_2 \theta_3 + \frac{p_{1,2,4}}{p_1} \theta_2 \theta_4 + \frac{p_{1,3,4}}{p_1} \theta_3 \theta_4 + \frac{p_{1,2,3,4}}{p_1} \theta_2 \theta_3 \theta_4$. Let

$$\begin{aligned} \alpha_1 &= \frac{\tilde{b}_{1,2}}{(-\tilde{p}_2)}; \alpha_2 = \frac{\tilde{b}_{1,3}}{(-\tilde{p}_3)}; \alpha_3 = \frac{\tilde{b}_{1,4}}{(-\tilde{p}_4)}; \alpha_4 = \frac{\tilde{b}_{1,2,3}}{(-\tilde{p}_{2,3})}; \alpha_5 = \frac{\tilde{b}_{1,2,4}}{(-\tilde{p}_{2,4})}; \alpha_6 = \frac{\tilde{b}_{1,3,4}}{(-\tilde{p}_{3,4})}; \\ \alpha_7 &= \frac{\tilde{b}_{1,2} \tilde{b}_{2,3}}{(-\tilde{p}_2)(-\tilde{p}_{2,3})}; \alpha_8 = \frac{\tilde{b}_{1,2} \tilde{b}_{2,4}}{(-\tilde{p}_2)(-\tilde{p}_{2,4})}; \alpha_9 = \frac{\tilde{b}_{1,3} \tilde{b}_{2,3}}{(-\tilde{p}_3)(-\tilde{p}_{2,3})}; \alpha_{10} = \frac{\tilde{b}_{1,3} \tilde{b}_{3,4}}{(-\tilde{p}_3)(-\tilde{p}_{3,4})}; \\ \alpha_{11} &= \frac{\tilde{b}_{1,4} \tilde{b}_{2,4}}{(-\tilde{p}_4)(-\tilde{p}_{2,4})}; \alpha_{12} = \frac{\tilde{b}_{1,4} \tilde{b}_{3,4}}{(-\tilde{p}_4)(-\tilde{p}_{3,4})}; \alpha_{13} = \frac{\tilde{b}_{1,2,3,4}}{(-\tilde{p}_{2,3,4})}; \alpha_{14} = \frac{\tilde{b}_{1,2} \tilde{b}_{2,3,4} + \tilde{b}_{2,3} \tilde{b}_{1,2,4} + \tilde{b}_{2,4} \tilde{b}_{1,2,3}}{(-\tilde{p}_2)(-\tilde{p}_{2,3,4})}; \\ \alpha_{15} &= \frac{\tilde{b}_{1,3} \tilde{b}_{2,3,4} + \tilde{b}_{2,3} \tilde{b}_{1,3,4} + \tilde{b}_{3,4} \tilde{b}_{1,2,3}}{(-\tilde{p}_3)(-\tilde{p}_{2,3,4})}; \alpha_{16} = \frac{\tilde{b}_{1,4} \tilde{b}_{2,3,4} + \tilde{b}_{2,4} \tilde{b}_{1,3,4} + \tilde{b}_{3,4} \tilde{b}_{1,2,4}}{(-\tilde{p}_4)(-\tilde{p}_{2,3,4})}; \\ \alpha_{17} &= \frac{2\tilde{b}_{1,2} \tilde{b}_{2,3} \tilde{b}_{2,4}}{(-\tilde{p}_2)^2(-\tilde{p}_{2,3,4})}; \alpha_{18} = \frac{2\tilde{b}_{1,3} \tilde{b}_{2,3} \tilde{b}_{3,4}}{(-\tilde{p}_3)^2(-\tilde{p}_{2,3,4})}; \alpha_{19} = \frac{2\tilde{b}_{1,4} \tilde{b}_{2,4} \tilde{b}_{3,4}}{(-\tilde{p}_4)^2(-\tilde{p}_{2,3,4})}; \\ \alpha_{20} &= \frac{\tilde{b}_{2,3}(\tilde{b}_{1,2} \tilde{b}_{3,4} + \tilde{b}_{1,3} \tilde{b}_{2,4})}{(-\tilde{p}_2)(-\tilde{p}_3)(-\tilde{p}_{2,3,4})}; \alpha_{21} = \frac{\tilde{b}_{2,4}(\tilde{b}_{1,2} \tilde{b}_{3,4} + \tilde{b}_{1,4} \tilde{b}_{2,3})}{(-\tilde{p}_2)(-\tilde{p}_4)(-\tilde{p}_{2,3,4})}; \alpha_{22} = \frac{\tilde{b}_{3,4}(\tilde{b}_{1,3} \tilde{b}_{2,4} + \tilde{b}_{1,4} \tilde{b}_{2,3})}{(-\tilde{p}_3)(-\tilde{p}_4)(-\tilde{p}_{2,3,4})}; \\ \alpha_{23} &= \frac{\tilde{b}_{1,2,3} \tilde{b}_{2,3,4}}{(-\tilde{p}_{2,3})(-\tilde{p}_{2,3,4})}; \alpha_{24} = \frac{\tilde{b}_{1,2,4} \tilde{b}_{2,3,4}}{(-\tilde{p}_{2,4})(-\tilde{p}_{2,3,4})}; \alpha_{25} = \frac{\tilde{b}_{1,3,4} \tilde{b}_{2,3,4}}{(-\tilde{p}_{3,4})(-\tilde{p}_{2,3,4})}; \\ \alpha_{26} &= \frac{\tilde{b}_{2,3}(\tilde{b}_{1,2} \tilde{b}_{2,3,4} + \tilde{b}_{2,4} \tilde{b}_{1,2,3})}{(-\tilde{p}_2)(-\tilde{p}_{2,3})(-\tilde{p}_{2,3,4})}; \alpha_{27} = \frac{\tilde{b}_{2,4}(\tilde{b}_{1,2} \tilde{b}_{2,3,4} + \tilde{b}_{2,3} \tilde{b}_{1,2,4})}{(-\tilde{p}_2)(-\tilde{p}_{2,4})(-\tilde{p}_{2,3,4})}; \alpha_{28} = \frac{\tilde{b}_{2,3}(\tilde{b}_{1,3} \tilde{b}_{2,3,4} + \tilde{b}_{3,4} \tilde{b}_{1,2,3})}{(-\tilde{p}_3)(-\tilde{p}_{2,3})(-\tilde{p}_{2,3,4})}; \\ \alpha_{29} &= \frac{\tilde{b}_{3,4}(\tilde{b}_{1,3} \tilde{b}_{2,3,4} + \tilde{b}_{2,3} \tilde{b}_{1,3,4})}{(-\tilde{p}_3)(-\tilde{p}_4)(-\tilde{p}_{2,3,4})}; \alpha_{30} = \frac{\tilde{b}_{2,4}(\tilde{b}_{1,4} \tilde{b}_{2,3,4} + \tilde{b}_{3,4} \tilde{b}_{1,2,4})}{(-\tilde{p}_4)(-\tilde{p}_{2,4})(-\tilde{p}_{2,3,4})}; \alpha_{31} = \frac{\tilde{b}_{3,4}(\tilde{b}_{1,4} \tilde{b}_{2,3,4} + \tilde{b}_{2,4} \tilde{b}_{1,3,4})}{(-\tilde{p}_4)(-\tilde{p}_3)(-\tilde{p}_{2,3,4})}; \\ \alpha_{32} &= \frac{\tilde{b}_{1,2} \tilde{b}_{2,3}^2 \tilde{b}_{2,4}}{(-\tilde{p}_2)^2(-\tilde{p}_{2,3})(-\tilde{p}_{2,3,4})}; \alpha_{33} = \frac{\tilde{b}_{1,2} \tilde{b}_{2,3} \tilde{b}_{2,4}^2}{(-\tilde{p}_2)^2(-\tilde{p}_{2,4})(-\tilde{p}_{2,3,4})}; \alpha_{34} = \frac{\tilde{b}_{1,3} \tilde{b}_{2,3}^2 \tilde{b}_{3,4}}{(-\tilde{p}_3)^2(-\tilde{p}_{2,3})(-\tilde{p}_{2,3,4})}; \\ \alpha_{35} &= \frac{\tilde{b}_{1,3} \tilde{b}_{2,3} \tilde{b}_{3,4}^2}{(-\tilde{p}_3)^2(-\tilde{p}_{3,4})(-\tilde{p}_{2,3,4})}; \alpha_{36} = \frac{\tilde{b}_{1,4} \tilde{b}_{3,4} \tilde{b}_{2,4}^2}{(-\tilde{p}_4)^2(-\tilde{p}_{2,4})(-\tilde{p}_{2,3,4})}; \alpha_{37} = \frac{\tilde{b}_{1,4} \tilde{b}_{2,4} \tilde{b}_{3,4}^2}{(-\tilde{p}_4)^2(-\tilde{p}_{3,4})(-\tilde{p}_{2,3,4})}; \\ \alpha_{38} &= \frac{\tilde{b}_{2,3}^2(\tilde{b}_{1,2} \tilde{b}_{3,4} + \tilde{b}_{1,3} \tilde{b}_{2,4})}{(-\tilde{p}_2)(-\tilde{p}_3)(-\tilde{p}_{2,3})(-\tilde{p}_{2,3,4})}; \alpha_{39} = \frac{\tilde{b}_{2,4}^2(\tilde{b}_{1,2} \tilde{b}_{3,4} + \tilde{b}_{1,4} \tilde{b}_{2,3})}{(-\tilde{p}_2)(-\tilde{p}_4)(-\tilde{p}_{2,4})(-\tilde{p}_{2,3,4})}; \alpha_{40} = \frac{\tilde{b}_{3,4}^2(\tilde{b}_{1,3} \tilde{b}_{2,4} + \tilde{b}_{1,4} \tilde{b}_{2,3})}{(-\tilde{p}_3)(-\tilde{p}_4)(-\tilde{p}_{3,4})(-\tilde{p}_{2,3,4})}. \end{aligned}$$

Let $V_i \sim \mathcal{P}(\alpha_i X_1), i \in [40]$, independent Poisson distributions, we have with the defini-

tions given above and with $\mathbf{v} = (v_1, \dots, v_{40})$

$$\begin{aligned}
L_{(X_2, X_3, X_4)}^{X_1=x_1} &= \sum_{\mathbf{v} \in \mathbb{N}^{40}} \left[\prod_{i=1}^{40} \mathbf{P}(V_i = v_i) \right] \times \\
&S_2^{-(v_1+v_7+v_8+v_{14}+2v_{17}+v_{20}+v_{21}+v_{26}+v_{27}+2v_{32}+2v_{33}+v_{38}+v_{39})} \times \\
&S_3^{-(v_2+v_9+v_{10}+v_{15}+2v_{18}+v_{20}+v_{22}+v_{28}+v_{29}+2v_{34}+2v_{35}+v_{38}+v_{40})} \times \\
&S_4^{-(v_3+v_{11}+v_{12}+v_{16}+2v_{19}+v_{21}+v_{22}+v_{30}+v_{31}+2v_{36}+2v_{37}+v_{39}+v_{40})} \times \\
&S_{2,3}^{-(v_4+v_7+v_9+v_{23}+v_{26}+v_{28}+v_{32}+v_{34}+v_{38})} \times \\
&S_{2,4}^{-(v_5+v_8+v_{11}+v_{24}+v_{27}+v_{30}+v_{33}+v_{36}+v_{39})} \times \\
&S_{3,4}^{-(v_6+v_{10}+v_{12}+v_{25}+v_{29}+v_{31}+v_{35}+v_{37}+v_{40})} \times \\
&S_{2,3,4}^{-(\lambda+\sum_{i=13}^{40} v_i)}. \tag{80}
\end{aligned}$$

Let the following infinitely divisible independent random vectors defined by

$$X_1 \sim \gamma_{(P_1, \lambda)}, \tag{81}$$

$$Y_2 \sim \gamma_{(S_2, V_1+V_7+V_8+V_{14}+2V_{17}+V_{20}+V_{21}+V_{26}+V_{27}+2V_{32}+2V_{33}+V_{38}+V_{39})}, \tag{82}$$

$$Y_3 \sim \gamma_{(S_3, V_2+V_9+V_{10}+V_{15}+2V_{18}+V_{20}+V_{22}+V_{28}+V_{29}+2V_{34}+2V_{35}+V_{38}+V_{40})}, \tag{83}$$

$$Y_4 \sim \gamma_{(S_4, V_3+V_{11}+V_{12}+V_{16}+2V_{19}+V_{21}+V_{22}+V_{30}+V_{31}+2V_{36}+2V_{37}+V_{39}+V_{40})}, \tag{84}$$

$$(U_{1,2}, U_{1,3}) \sim \gamma_{(S_{2,3}, V_4+V_7+V_9+V_{23}+V_{26}+V_{28}+V_{32}+V_{34}+V_{38})}, \tag{85}$$

$$(U_{2,2}, U_{2,4}) \sim \gamma_{(S_{2,4}, V_5+V_8+V_{11}+V_{24}+V_{27}+V_{30}+V_{33}+V_{36}+V_{39})}, \tag{86}$$

$$(U_{3,3}, U_{3,4}) \sim \gamma_{(S_{3,4}, V_6+V_{10}+V_{12}+V_{25}+V_{29}+V_{31}+V_{35}+V_{37}+V_{40})}, \tag{87}$$

$$(W_2, W_3, W_4) \sim \gamma_{(S_{2,3,4}, \lambda+\sum_{i=13}^{40} V_i)}. \tag{88}$$

Let (X_2, X_3, X_4) defined by

$$X_2 = Y_2 + U_{1,2} + U_{2,2} + W_2, \tag{89}$$

$$X_3 = Y_3 + U_{1,3} + U_{3,3} + W_3, \tag{90}$$

$$X_4 = Y_4 + U_{2,4} + U_{3,4} + W_4, \tag{91}$$

then

$$\mathbf{X}_{[4]} = (X_1, X_2, X_3, X_4) \sim \gamma_{(P_{[4]}, \lambda)}. \tag{92}$$

From Theorem (39), we derive the following algorithm for simulating $\mathbf{X}_{[4]} = (X_1, X_2, X_3, X_4) \sim$

$\gamma_{(P_4, \lambda)}$

Algorithm 40 *Simulation of an infinitely divisible quadrivariate gamma distribution*

$\gamma_{(P_4, \lambda)}$

1. Simulate $X_1 \sim \gamma_{(P_1, \lambda)}$;
2. Compute $\alpha_i, i \in [40]$, with α_i defined in Theorem (39). Simulate independently $V_i \sim P(\alpha_i X_1), i \in [40], \alpha_i$ defined above;
3. Simulate independently

$$Y_2 \sim \gamma_{(S_2, V_1+V_7+V_8+V_{14}+2V_{17}+V_{20}+V_{21}+V_{26}+V_{27}+2V_{32}+2V_{33}+V_{38}+V_{39})},$$

$$Y_3 \sim \gamma_{(S_3, V_2+V_9+V_{10}+V_{15}+2V_{18}+V_{20}+V_{22}+V_{28}+V_{29}+2V_{34}+2V_{35}+V_{38}+V_{40})},$$

$$Y_4 \sim \gamma_{(S_4, V_3+V_{11}+V_{12}+V_{16}+2V_{19}+V_{21}+V_{22}+V_{30}+V_{31}+2V_{36}+2V_{37}+V_{39}+V_{40})},$$

$$(U_{1,2}, U_{1,3}) \sim \gamma_{(S_{2,3}, V_4+V_7+V_9+V_{23}+V_{26}+V_{28}+V_{32}+V_{34}+V_{38})},$$

$$(U_{2,2}, U_{2,4}) \sim \gamma_{(S_{2,4}, V_5+V_8+V_{11}+V_{24}+V_{27}+V_{30}+V_{33}+V_{36}+V_{39})},$$

$$(U_{3,3}, U_{3,4}) \sim \gamma_{(S_{3,4}, V_6+V_{10}+V_{12}+V_{25}+V_{29}+V_{31}+V_{35}+V_{37}+V_{40})},$$

$$(W_2, W_3, W_4) \sim \gamma_{(S_{2,3,4}, \lambda + \sum_{i=13}^{40} V_i)};$$

4. Compute $X_2 = Y_2 + U_{1,2} + U_{2,2} + W_2, X_3 = Y_3 + U_{1,3} + U_{3,3} + W_3$, and $X_4 = Y_4 + U_{2,4} + U_{3,4} + W_4$, then $\mathbf{X}_{[4]} = (X_1, X_2, X_3, X_4)$ simulates $\gamma_{(P_4, \lambda)}$.

Remark 41 *We need to simulate $\gamma_{(S_{2,3}, V_4+V_7+V_9+V_{23}+V_{26}+V_{28}+V_{32}+V_{34}+V_{38})}$,*

$\gamma_{(S_{2,4}, V_5+V_8+V_{11}+V_{24}+V_{27}+V_{30}+V_{33}+V_{36}+V_{39})}$, $\gamma_{(S_{3,4}, V_6+V_{10}+V_{12}+V_{25}+V_{29}+V_{31}+V_{35}+V_{37}+V_{40})}$, and $\gamma_{(S_{2,3,4}, \lambda + \sum_{i=13}^{40} V_i)}$.

To do this, we use 40 Pds and 4 ugds, 3 bgds and 1 tgd. In each time for bgd, we can use 1 Pd and 2 ugds, and for tgd we can use 6 Pds and 5 ugds. Finally, we can simulate $\gamma_{(P_{[4]}, \lambda)}$ with $40 + 3 + 6 = 49$ Pds and $4 + 6 + 5 = 15$ ugds.

We see that it is possible to simulate a mgd by induction. Unfortunately, the complexity of the computations seems enormous from $n = 5$ upwards. This is why we study the Markovian case, which is simpler.

6 Markovian multivariate gamma distributions

In this section, we use the results given in [Chatelain et al., 2009]. We suppose that $\mathbf{X}_{[n]} = (X_1, \dots, X_n) \sim \gamma_{(P_n, \lambda)}$, where P_n is an affine polynomial and $\lambda > 0$. We assume that $\gamma_{(P_n, \lambda)}$ is infinitely divisible and it satisfies the following first-order Markov property

$$\mathbf{P}(X_{i+1} \in B | X_i = x_i, \dots, X_1 = x_1) = \mathbf{P}(X_{i+1} \in B | X_i = x_i), \quad (93)$$

for any $1 \leq i \leq n-1$ and for any bounded set all $B \subset \mathbb{R}$. Such a distribution is called a Markovian mgd (Mmgd). Let $f_{\mathbf{X}_{[n]}}$ be the probability density function (pdf) of $\mathbf{X}_{[n]}$ on $(0, \infty)^n$, we have by Bayes' rule and Markov's property yields

$$\begin{aligned} f_{\mathbf{X}_{[n]}}(\mathbf{x}_{[n]}) &= f_{X_1}(x_1) \prod_{i=2}^{n-1} f_{X_i | X_{i-1}=x_{i-1}}(x_i) \\ &= f_{X_1}(x_1) \prod_{i=2}^{n-1} \frac{f_{(X_i, X_{i+1})}(x_i, x_{i+1})}{f_{X_i}(x_i)} \end{aligned}$$

where $f_{X_{i+1} | X_i=x_i}(x_{i+1})$ is the conditional pdf of X_{i+1} given $X_i = x_i$. The pdf of a Mmgd can then be derived from the pdfs of its univariate and bivariate marginal distributions.

We note that the pdf of $\mathbf{Y}_{[n]} = (p_1 X_1, \dots, p_n X_n)$ is

$$f_{\mathbf{Y}_{[n]}}(\mathbf{y}_{[n]}) = \frac{1}{\prod_{i=1}^n p_i} f_{\mathbf{X}_{[n]}}\left(\frac{y_1}{p_1}, \dots, \frac{y_n}{p_n}\right). \quad (94)$$

[Chatelain et al., 2009] use f_λ defined by

$$f_\lambda(z) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(\lambda + k)} = \frac{1}{\Gamma(\lambda)} F_1(\lambda, z)$$

For $p_i = 1, i \in [n]$, [Chatelain et al., 2009] give the result :

$$f_{\mathbf{X}_{[n]}}(\mathbf{x}_{[n]}) = \mathbf{e}^{\left(-\frac{x_1}{1-\rho_{1,2}} - \frac{x_n}{1-\rho_{n-1,n}} - \sum_{i=2}^{n-1} \frac{1-\rho_{i-1,i}\rho_{i,i+1}}{(1-\rho_{i-1,i})(1-\rho_{i,i+1})} x_i\right)} \frac{(x_1 \dots x_n)^n}{\Gamma(\lambda)} \prod_{i=2}^{n-1} \frac{f_\lambda\left(\frac{\rho_{i,i+1} x_i x_{i+1}}{(1-\rho_{i,i+1})^2}\right)}{(1-\rho_{i,i+1})^\lambda} \quad (95)$$

where $\rho_{i,i+1} = 1 - \frac{p_i p_{i+1}}{p_i p_{i+1}}$ is the correlation coefficient between X_i and X_{i+1} . From (5) and (8), we get $\tilde{b}_{i,i+1} = (p_i p_{i+1} - p_{i,i+1}) / p_{i,i+1}^2 = p_i p_{i+1} / p_{i,i+1}^2 \rho_{i,i+1} > 0$, where $\rho_{i,i+1} =$

$1 - \rho_{i,i+1}/(p_i p_{i+1})$ is the correlation coefficient between X_i and X_{i+1} .

$$\begin{aligned} f_{\mathbf{X}_{[n]}}(\mathbf{x}_{[n]}) &= \frac{\prod_{i=1}^{n-1} f_{(X_i, X_{i+1})}(x_i, x_{i+1})}{\prod_{i=2}^{n-1} f_{X_i}(x_i)} \\ &= \frac{(x_1 \cdots x_n)^{\lambda-1}}{\Gamma(\lambda)^n} \left(\prod_{i=1}^n p_i \right)^{-\lambda} \mathbf{e}^{\left(-\frac{1}{(1-\rho_{1,2})^{p_1}} x_1 - \frac{1}{(1-\rho_{n-1,n})^{p_n}} x_n - \sum_{i=2}^{n-1} \frac{1}{p_i} \frac{(1-\rho_{i-1,i} \rho_{i,i+1})}{(1-\rho_{i,i+1})(1-\rho_{i-1,i})} x_i \right)} \\ &\quad \prod_{i=1}^{n-1} \frac{F_1\left(\lambda; \frac{\rho_{i,i+1}}{p_i p_{i+1} (1-\rho_{i,i+1})^2} x_i x_{i+1}\right)}{(1-\rho_{i,i+1})^\lambda}. \end{aligned}$$

From (94) and (95), we have for $\mathbf{Y}_{[n]} = (p_1 X_1, \dots, p_n X_n)$

$$f_{\mathbf{Y}_{[n]}}(\mathbf{y}_{[n]}) = \frac{1}{\prod_{i=1}^n p_i} \frac{\left(\frac{y_1}{p_1} \cdots \frac{y_n}{p_n} \right)^{\lambda-1}}{\Gamma(\lambda)} \mathbf{e}^{\left(-\frac{1}{(1-\rho_{1,2})^{p_1}} \frac{y_1}{p_1} - \frac{1}{(1-\rho_{n-1,n})^{p_n}} \frac{y_n}{p_n} - \sum_{i=2}^{n-1} \frac{1}{p_i} \frac{(1-\rho_{i-1,i} \rho_{i,i+1})}{(1-\rho_{i,i+1})(1-\rho_{i-1,i})} \frac{y_i}{p_i} \right)}$$
(96)

$$\prod_{i=1}^{n-1} \frac{f_\lambda\left(\frac{\rho_{i,i+1}}{(1-\rho_{i,i+1})^2} \frac{y_i}{p_i} \frac{y_{i+1}}{p_{i+1}}\right)}{(1-\rho_{i,i+1})^\lambda}$$

or

$$f_{\mathbf{X}_{[n]}}(\mathbf{x}_{[n]}) = \left(\prod_{i=1}^n p_i \right)^{-\lambda} \frac{(y_1 \cdots y_n)^{\lambda-1}}{\Gamma(\lambda)^n} \mathbf{e}^{\left(-\frac{1}{(1-\rho_{1,2})^{p_1}} y_1 - \frac{1}{(1-\rho_{n-1,n})^{p_n}} y_n - \sum_{i=2}^{n-1} \frac{1}{p_i} \frac{(1-\rho_{i-1,i} \rho_{i,i+1})}{(1-\rho_{i,i+1})(1-\rho_{i-1,i})} y_i \right)}$$
(97)

$$\prod_{i=1}^{n-1} \frac{F_1\left(\lambda; \frac{\rho_{i,i+1}}{(1-\rho_{i,i+1})^2} \frac{y_i}{p_i} \frac{y_{i+1}}{p_{i+1}}\right)}{(1-\rho_{i,i+1})^\lambda}.$$

[Chatelain et al., 2009] also give the following Theorem in the case $p_i = 1, i \in [n]$.

Theorem 42 *Let $\mathbf{X}_{[n]}$ be a random vector distributed according to a Mmgd with shape parameter $\lambda > 0$. The Lt of $\mathbf{X}_{[n]}$ can be expressed for $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ as*

$$L_{\mathbf{X}_{[n]}}(\boldsymbol{\theta}) = \det(\mathbf{I}_n + \mathbf{D}_{\boldsymbol{\theta}} \mathbf{R}_{1/2})^{-\lambda} \quad (98)$$

where \mathbf{I}_n is the $n \times n$ identity matrix, $\mathbf{D}_{\boldsymbol{\theta}}$ is the diagonal matrix whose diagonal entries are the components of vector $\boldsymbol{\theta}$, and $\mathbf{R}_{1/2} = (a_{i,j})_{1 \leq i, j \leq n}$ is a correlation matrix such that

$$\begin{aligned} a_{i,i} &= 1, \forall 1 \leq i \leq n \\ a_{i,i+1} &= \sqrt{\rho_{i,i+1}}, \forall 1 \leq i \leq n-1 \\ a_{i,j} &= \sqrt{\rho_{i,j}} = \sqrt{\rho_{i,k}} \sqrt{\rho_{k,j}}, \forall 1 \leq i < k < j \leq n. \end{aligned} \quad (99)$$

We also have $(X_i, X_{i+l}) \sim \gamma_{(P_{[i,i+l]}, \lambda)}$ with $P_{[i,i+l]}(\theta_i, \theta_{i+1}) = 1 + \theta_i + \theta_{i+1} + (1 - a_{i,i+l}^2) \theta_i \theta_{i+1}$, and we have

$$\rho_{i,i+l} = \prod_{j=i}^{i+l-1} \rho_{j,j+1}, \text{ for all } 1 \leq i < i+l \leq n.$$

A straightforward consequence is that $(X_i, X_{i+1}) \sim \gamma_{(P_{[i,i+1]}, \lambda)}$ with $P_{[i,i+1]}(\theta_i, \theta_{i+1}) = 1 + \theta_i + \theta_{i+1} + (1 - \rho_{i,i+1}) \theta_i \theta_{i+1}$. We immediately derive the algorithm for the simulation of $\mathbf{X}_{[n]}$ from Algorithm 33 and the computations $\frac{\rho_{i,i+1}}{p_i} \tilde{b}_{i,i+1} = \frac{\rho_{i,i+1}}{1 - \rho_{i,i+1}}$ and $\frac{\rho_{i,i+1}}{p_i} = 1 - \rho_{i,i+1}$.

Algorithm 43 *Simulation of $\gamma_{(P_n, \lambda)}$, with $P_n = \det(\mathbf{I}_n + \mathbf{D}_\theta R_{1/2})$ under the conditions of Theorem 42*

1. Simulate $X_1 \sim \gamma_{(1, \lambda)}$;
2. For $i = 1$ to $n - 1$, do
 - simulate $V_i \sim \mathcal{P}\left(\frac{\rho_{i,i+1}}{1 - \rho_{i,i+1}} X_i\right)$,
 - simulate $X_{i+1} \sim \gamma_{(1 - \rho_{i,i+1}, \lambda + V_i)}$;
3. Then $\mathbf{X}_{[n]} = (X_1, \dots, X_n)$ simulate $\gamma_{(P_n, \lambda)}$.

We note that if we consider $\mathbf{Y}_{[n]} = (p_1 X_1, \dots, p_n X_n)$, then $\mathbf{Y}_{[n]} = (Y_1, \dots, Y_n) \sim \gamma_{(Q_n, \lambda)}$, where Q_n is an affine polynomial with

$$\begin{aligned} Q_n(\boldsymbol{\theta}_{[n]}) &= P_n(\mathbf{p}_{[n]} \boldsymbol{\theta}_{[n]}) \\ &= \det\left(\mathbf{I}_n + \mathbf{D}_{\mathbf{p}_{[n]} \boldsymbol{\theta}_{[n]}} R_{1/2}\right)^{-\lambda}. \end{aligned}$$

7 Simulations

7.1 Simulations in dimension 2

Let $P_2(\theta_1, \theta_2) = 1 + 3\theta_1 + 3\theta_2 + \theta_1\theta_2$ and $Q_2(\theta_1, \theta_2) = 1 + 15/13\theta_1 + 3/13\theta_2 + 1/13\theta_1\theta_2$, let $\mathbf{X}_{[2]} = (X_1, X_2) \sim \gamma_{(P_2, 2)}$ and $\mathbf{Y}_{[2]} = (Y_1, Y_2) \sim \gamma_{(Q_2, 2)}$ for which the correlation

coefficients ρ_{X_1, X_2} and ρ_{Y_1, Y_2} are respectively, $\rho_{X_1, X_2} = 1 - \frac{p_{1,2}}{p_1 p_2} = \frac{8}{9} = 0.889$ and $\rho_{Y_1, Y_2} = \frac{32}{45} = 0.711$.

Simulations for samples of size 1,000 of bgd $\gamma_{(P_2,2)}$ and $\gamma_{(Q_2,2)}$ are illustrated respectively by the graphical representations given in Figure 1.

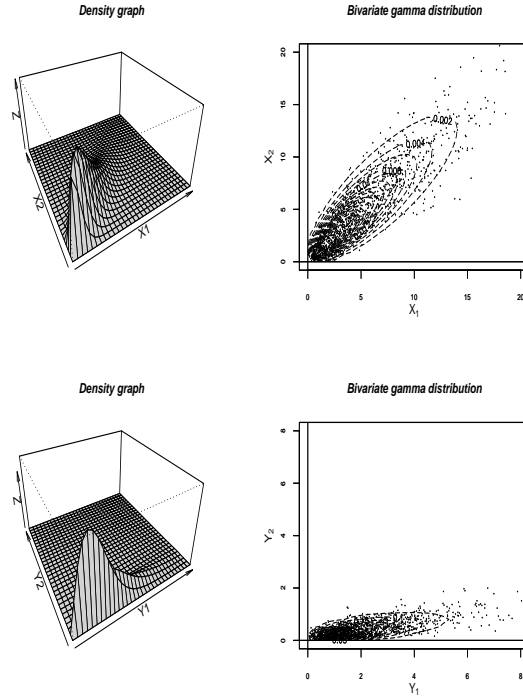


Figure 1: Distributions and simulations of $\mathbf{X}_{[2]}$ and $\mathbf{Y}_{[2]}$

Let $\mathbf{X}'_{[2]} = (X'_1, X'_2) \sim \gamma_{(P_2, (2,3,4))}$ and $\mathbf{Y}'_{[2]} = (Y'_1, Y'_2) \sim \gamma_{(Q_2, (2,3,4))}$ for which the correlation coefficients $\rho'_{X'_1, X'_2}$ and $\rho'_{Y'_1, Y'_2}$ are respectively (by the formula $\rho'_{X'_1, X'_2} = \frac{\lambda}{\sqrt{\lambda_1 \lambda_2}} \rho_{X_1, X_2}$) $\rho'_{X'_1, X'_2} = \frac{2}{\sqrt{12}} \frac{8}{9} = \frac{8}{27} \sqrt{3} \simeq 0.513$ and $\rho'_{Y'_1, Y'_2} = \frac{2}{\sqrt{12}} \frac{32}{45} = \frac{32}{135} \sqrt{3} \simeq 0.411$.

Simulations for samples of size 1,000 of mfgd $\gamma_{(P_2, (2,3,4))}$ and $\gamma_{(Q_2, (2,3,4))}$ are illustrated by the graphical representations given in Figure 2.

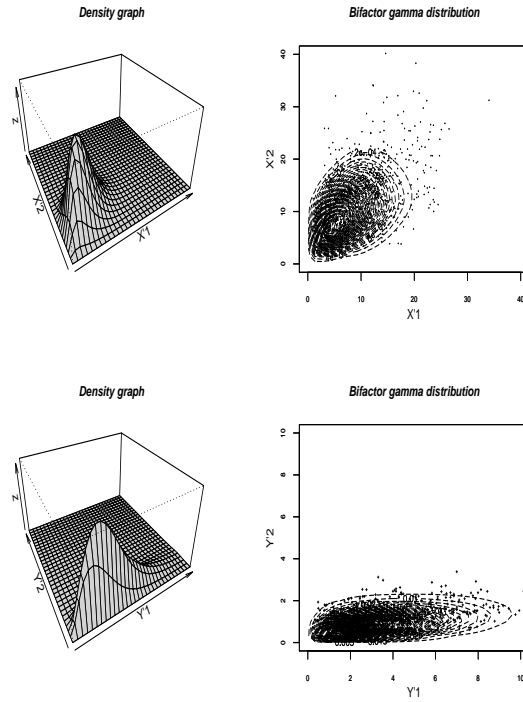


Figure 2: Distributions and simulations of $\mathbf{X}'_{[2]}$ and $\mathbf{Y}'_{[2]}$

7.2 Simulations in dimension 3

Let $P_3(\theta_1, \theta_2, \theta_3) = 1 + \theta_1 + \theta_2 + \theta_3 + 0.55\theta_1\theta_2 + 0.45\theta_1\theta_3 + 0.5\theta_2\theta_3 + 0.2\theta_1\theta_2\theta_3$ and $Q_3(\theta_1, \theta_2, \theta_3) = 1 + \theta_1 + 4\theta_2 + 5\theta_3 + 2.2\theta_1\theta_2 + 2.25\theta_1\theta_3 + 10\theta_2\theta_3 + 4\theta_1\theta_2\theta_3$, let $\mathbf{X}_{[3]} = (X_1, X_2, X_3) \sim \gamma_{(P_3,2)}$ and $\mathbf{Y}_{[3]} = (Y_1, Y_2, Y_3) \sim \gamma_{(Q_3,2)}$ with $(Y_1, Y_2, Y_3) = (X_1, 4X_2, 5X_3)$.

Simulations for samples of size 1,000 of mgd $\gamma_{(P_3,2)}$ and $\gamma_{(Q_3,2)}$ are illustrated by the graphical representations given in Figures 3,4 and Figures 5,6.

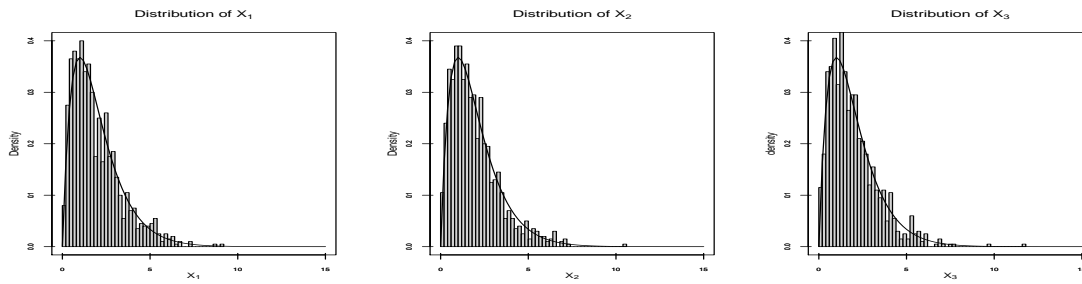


Figure 3: Distributions of X_1, X_2, X_3

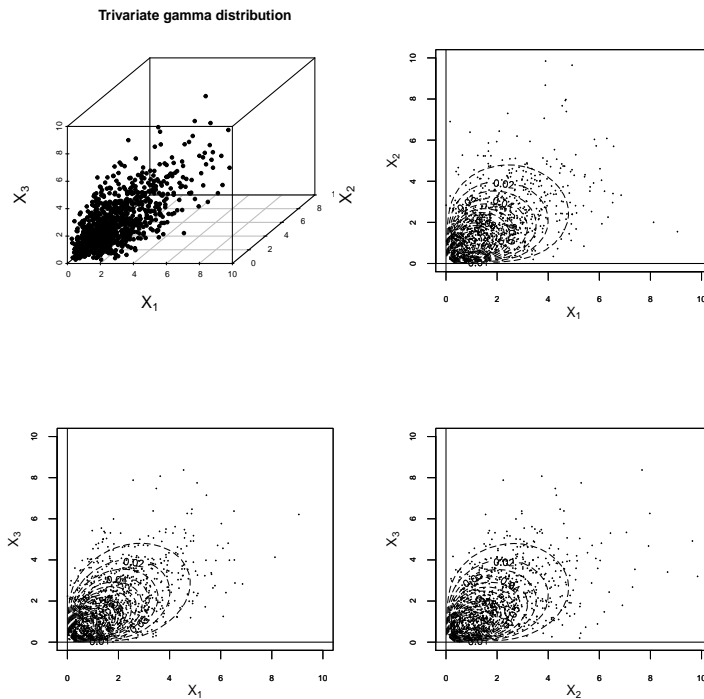


Figure 4: Distribution and simulation of $\mathbf{X}_{[3]}$

Let $\mathbf{X}'_{[3]} = (X'_1, X'_2, X'_3) \sim \gamma_{(P_{[3]}, (2,3,4,5))}$ and $\mathbf{Y}'_{[3]} = (Y'_1, Y'_2, Y'_3) \sim \gamma_{(Q_{[3]}, (2,3,4,5))}$.

Simulations for samples of size 1,000 of mfgd $\gamma_{(P_3, (2,3,4,5))}$ and $\gamma_{(Q_3, (2,3,4,5))}$ are illustrated by the graphical representations given in Figure 7 and Figure 8.

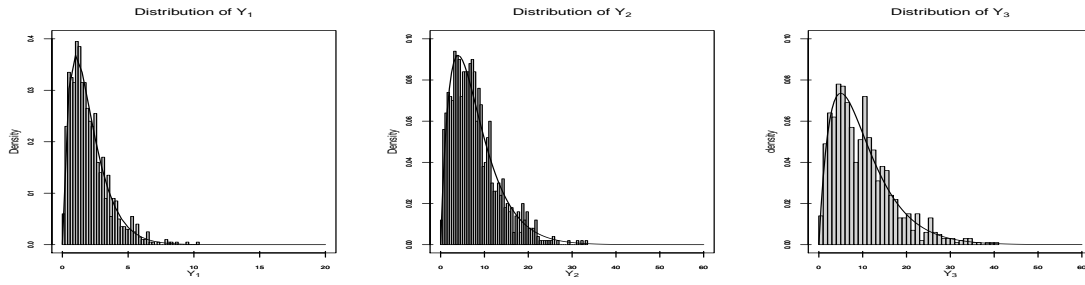


Figure 5: Distributions of Y_1, Y_2, Y_3

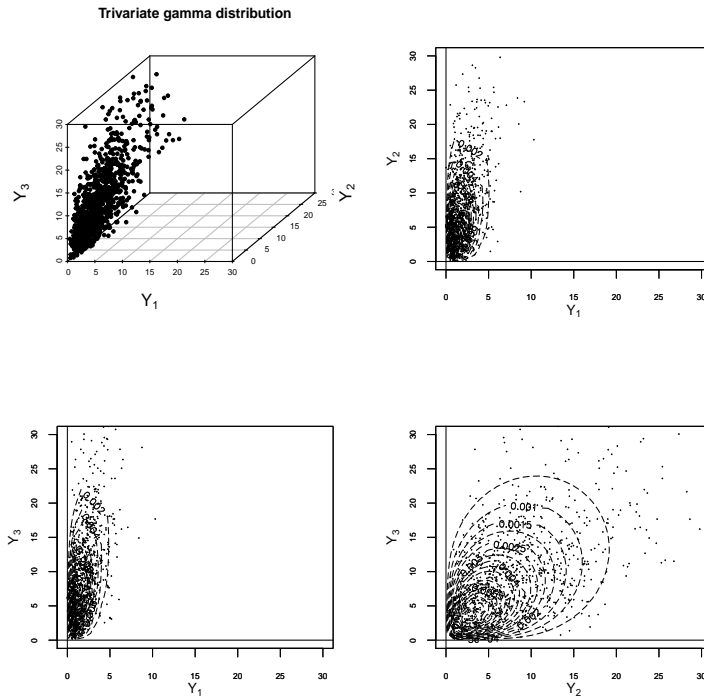


Figure 6: Distribution and simulation of $\mathbf{Y}_{[3]}$

7.3 Simulations in dimension 4

First, we look for the symmetric case where

$$P_4(\theta_1, \theta_2, \theta_3, \theta_4, \theta_4) = 1 + s_1(\theta_1 + \theta_2 + \theta_3 + \theta_4) + s_2(\theta_1\theta_2 + \theta_1\theta_3 + \theta_1\theta_4 + \theta_2\theta_3 + \theta_2\theta_4 + \theta_3\theta_4) \\ + s_3(\theta_1\theta_2\theta_3 + \theta_1\theta_2\theta_4 + \theta_1\theta_3\theta_4 + \theta_2\theta_3\theta_4) + s_4\theta_1\theta_2\theta_3\theta_4$$

According to formulas (3.13a) or (3.13b) in [Bernardoff, 2006] and [Comtet, 1974] pp.

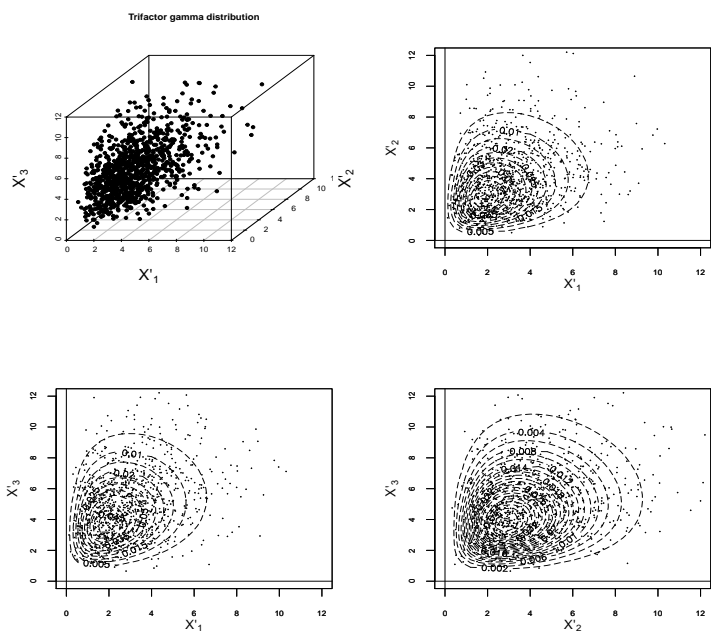


Figure 7: Distribution and simulation of $\mathbf{X}'_{[3]}$

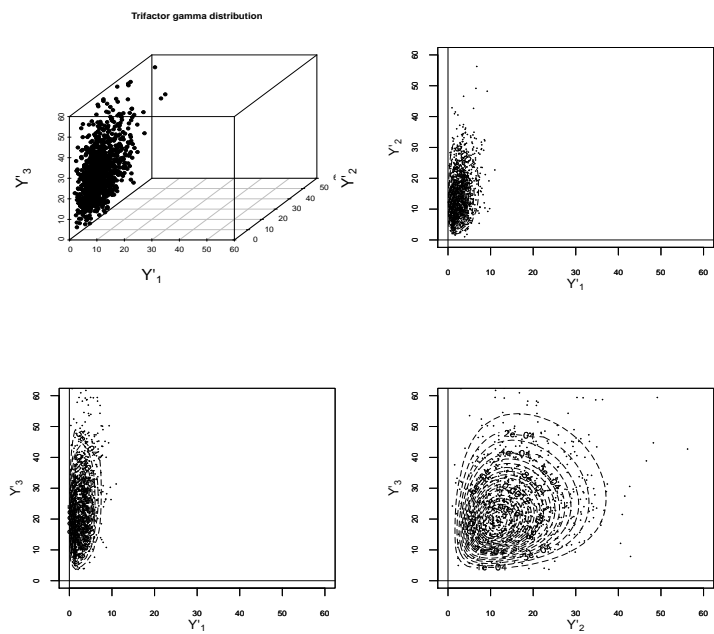


Figure 8: Distribution and simulation of $\mathbf{Y}'_{[3]}$

307-8, we respectively have for $n = 4$, and $|S| = 1, 2, 3, 4$, $\tilde{b}_S = -s_3/s_4, (-s_4s_2 + s_3^2)/s_4^2, -(s_1s_4^2 + 2s_3^3 - 3s_2s_3s_4)/s_4^3, (6s_3^4 - s_4^3 + 3s_2^2s_4^2 + 4s_1s_3s_4^2 - 12s_2s_3^2s_4)/s_4^4$. Without loss of generality, we can assume that $s_4 = 1$, so we respectively have for $n = 4$, and $|S| = 1, 2, 3, 4$, $\tilde{b}_S = -s_3, -s_2 + s_3^2, -s_1 + 3s_3s_2 - 2s_3^3, 6s_3^4 - 1 + 3s_2^2 + 4s_1s_3 - 12s_2s_3^2$. We must simultaneously check the conditions $-s_3 < 0, -s_2 + s_3^2 \geq 0, -s_1 + 3s_3s_2 - 2s_3^3 \geq 0$ and $6s_3^4 - 1 + 3s_2^2 + 4s_1s_3 - 12s_2s_3^2 \geq 0$ for the gamma distribution $\gamma_{(P_{[4]}, \lambda)}$ to be indefinitely divisible.

These conditions are equivalent to

$$s_3 > 0, s_2 \leq s_3^2, \frac{1 - 3s_2^2}{4s_3} - \frac{3}{2}s_3^3 + 3s_2s_3 \leq s_1 \leq 3s_3s_2 - 2s_3^3. \quad (100)$$

The last condition (100) is equivalent to $\frac{1-3s_2^2}{4s_3} + \frac{1}{2}s_3^3 + (3s_2s_3 - 2s_3^3) \leq s_1 \leq 3s_3s_2 - 2s_3^3$, which is only possible for $\frac{1-3s_2^2}{4s_3} + \frac{1}{2}s_3^3 \leq 0$. This gives us the following condition $\sqrt{\frac{1+2s_3^4}{3}} \leq s_2$ and (100) becomes

$$s_3 > 0, \sqrt{\frac{1 + 2s_3^4}{3}} \leq s_2 \leq s_3^2, \frac{1 - 3s_2^2}{4s_3} - \frac{3}{2}s_3^3 + 3s_2s_3 \leq s_1 \leq 3s_3s_2 - 2s_3^3. \quad (101)$$

For $s_3 = 2$, we get $3.3166248 \leq s_2 \leq 4$, and $s_2 = 3.5$ matches. For $s_3 = 2, s_2 = 3.5$, we get $4.53125 \leq s_1 \leq 5.0$ and $s_1 = 4.75$ is a possible value. We check that for $|S| = 1, 2, 3, 4$, we have respectively $\tilde{b}_S = -2, 0.5, 0.25, 1.75$. Let $\mathbf{X}_{[4]} = (X_1, X_2, X_3, X_4) \sim \gamma_{(P_{[4]}, 2)}$.

Simulations for samples of size 1,000 of mgd $\gamma_{(P_{[4]}, 2)}$ are illustrated by the graphical representations given in Figure 9 by four one-dimensional projections and Figure 10 by various three-dimensional projections.

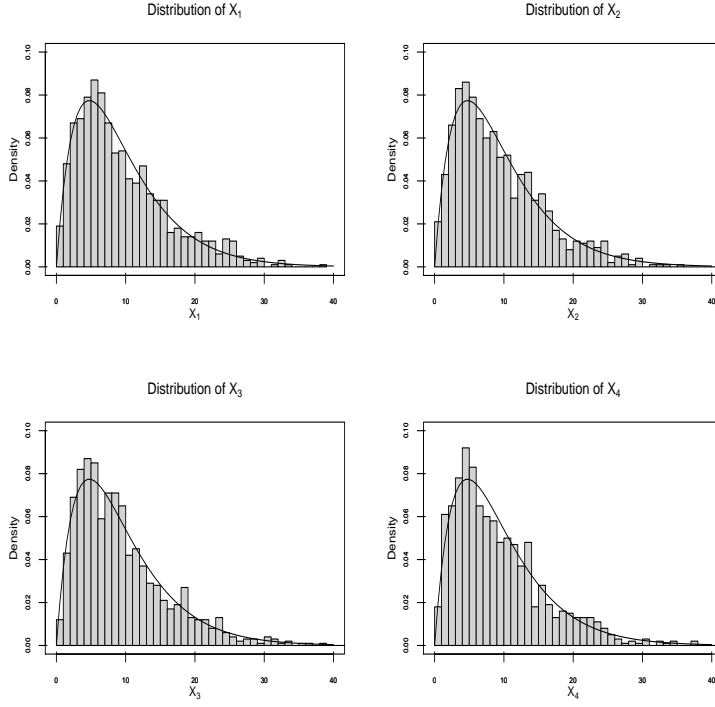



Figure 9: Distribution and simulation of X_1, X_2, X_3, X_4

Next, we search for the general case by slightly modifying the values of s_1, s_2, s_3, s_4 checking that the indefinite divisibility conditions of $\gamma_{(P_{[4]}, \lambda)}$ remain verified.

For example, we obtain $(p_1, p_2, p_3, p_4, p_{1,2}, p_{1,3}, p_{1,4}, p_{2,3}, p_{2,4}, p_{3,4}, p_{1,2,3}, p_{1,2,4}, p_{1,3,4}, p_{2,3,4}, p_{1,2,3,4}) =$
 $(4.75, 4.8, 4.85, 4.7, 3.5, 3.55, 3.6, 3.65, 3.45, 3.4, 2, 1.99, 2.02, 2.01, 1),$

with $(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4, \tilde{b}_{1,2}, \tilde{b}_{1,3}, \tilde{b}_{1,4}, \tilde{b}_{2,3}, \tilde{b}_{2,4}, \tilde{b}_{3,4}, \tilde{b}_{1,2,3}, \tilde{b}_{1,2,4}, \tilde{b}_{1,3,4}, \tilde{b}_{2,3,4}, \tilde{b}_{1,2,3,4}) =$
 $(-2.01, -2.02, -1.99, -2, 0.6602, 0.5499, 0.37, 0.4198, 0.49, 0.48, 0.111404, 0.2177, 0.3989, 0.5053, 1.590846)$

and $Q_4(\theta_1, \theta_2, \theta_3, \theta_4, \theta_4) = 1 + 4.75\theta_1 + 4.8\theta_2 + 4.85\theta_3 + 4.7\theta_4 + 3.5\theta_1\theta_2 + 3.55\theta_1\theta_3 +$
 $3.6\theta_1\theta_4 + 3.65\theta_2\theta_3 + 3.45\theta_2\theta_4 + 3.4\theta_3\theta_4 + 2\theta_1\theta_2\theta_3 + 1.99\theta_1\theta_2\theta_4 + 2.02\theta_1\theta_3\theta_4 + 2.01\theta_2\theta_3\theta_4 +$
 $\theta_1\theta_2\theta_3\theta_4.$ Let $\mathbf{Y}_{[4]} = (Y_1, Y_2, Y_3, Y_4) \sim \gamma_{(Q_4, 2)}$. Simulations for samples of size 1,000 of
mgd $\gamma_{(Q_4, 2)}$ are illustrated by the graphical representations given in Figure 11 by four
one-dimensional projections and Figure 12 by various three-dimensional projections.



Gdim3_gamma_E4b-eps-converted-to.pdf

Figure 10: Distribution and simulation of $\mathbf{X}_{[4]}$

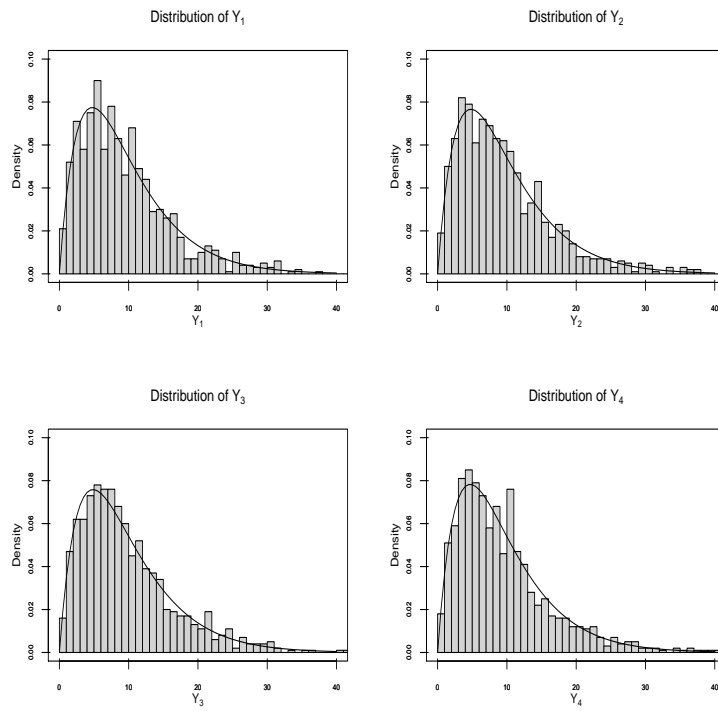


Figure 11: Distribution and simulation of Y_1, Y_2, Y_3, Y_4

7.4 Simulations in dimension 5

We apply the result of Theorem 42. For example, we obtain for $\rho_{1,2} = 0.9^2, \rho_{2,3} = 0.8^2, \rho_{3,4} = 0.7^2, \rho_{4,5} = 0.6^2,$



Figure 12: Distribution and simulation of $\mathbf{Y}_{[4]}$

$$\mathbf{R}_{1/2} = \begin{bmatrix} 1 & 0.9 & 0.72 & 0.504 & 0.3024 \\ 0.9 & 1 & 0.8 & 0.56 & 0.336 \\ 0.72 & 0.8 & 1 & 0.7 & 0.42 \\ 0.504 & 0.56 & 0.7 & 1 & 0.6 \\ 0.3024 & 0.336 & 0.42 & 0.6 & 1 \end{bmatrix} \text{ and}$$

$$\begin{aligned} P_5(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) &= \det(\mathbf{I}_5 + \mathbf{D}_{(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)} \mathbf{R}_{1/2}) \\ &= 1 + \theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 \\ &\quad + 0.19\theta_1\theta_2 + 0.4816\theta_1\theta_3 + 0.745984\theta_1\theta_4 + 0.90855424\theta_1\theta_5 + 0.36\theta_2\theta_3 \\ &\quad + 0.6864\theta_2\theta_4 + 0.887104\theta_2\theta_5 + 0.51\theta_3\theta_4 + 0.8236\theta_3\theta_5 + 0.64\theta_4\theta_5 \\ &\quad + 0.0684\theta_1\theta_2\theta_3 + 0.130416\theta_1\theta_2\theta_4 + 0.16854976\theta_1\theta_2\theta_5 + 0.245616\theta_1\theta_3\theta_4 \\ &\quad + 0.39664576\theta_1\theta_3\theta_5 + 0.47742976\theta_1\theta_4\theta_5 + 0.1836\theta_2\theta_3\theta_4 + 0.296496\theta_2\theta_3\theta_5 \\ &\quad + 0.439296\theta_2\theta_4\theta_5 + 0.3264\theta_3\theta_4\theta_5 + 0.034884\theta_1\theta_2\theta_3\theta_4 + 0.05633424\theta_1\theta_2\theta_3\theta_5 \\ &\quad + 0.08346624\theta_1\theta_2\theta_4\theta_5 + 0.15719424\theta_1\theta_3\theta_4\theta_5 + 0.117504\theta_2\theta_3\theta_4\theta_5 \\ &\quad + 0.02232576\theta_1\theta_2\theta_3\theta_4\theta_5. \end{aligned}$$

Algorithm 43 becomes

Algorithm 44 *Simulation of $\gamma_{(P_{[5]}, \lambda)}$, with $P_{[5]}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = \det(\mathbf{I}_5 + \mathbf{D}_{(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)} \mathbf{R}_{1/2})$*
 $\rho_{1,2} = 0.81, \rho_{2,3} = 0.64, \rho_{3,4} = 0.49, \rho_{4,5} = 0.36$

1. Simulate $X_1 \sim \gamma_{(1, \lambda)}$;
2. For $i = 1$ to 4, do
 - simulate $V_i \sim \mathcal{P}\left(\frac{\rho_{i, i+1}}{1 - \rho_{i, i+1}} X_i\right)$,
 - simulate $X_{i+1} \sim \gamma_{(1 - \rho_{i, i+1}, \lambda + V_i)}$;
3. Then $\mathbf{X}_{[5]} = (X_1, \dots, X_5)$ simulate $\gamma_{(P_{[5]}, \lambda)}$.

Let $\mathbf{X}_{[5]} = (X_1, X_2, X_3, X_4, X_5) \sim \gamma_{(P_5, 2)}$. Simulations for samples of size 1,000 of Mmgd $\gamma_{(P_5, 2)}$ are illustrated by the graphical representations given in Figure 13 by various one-dimensional projections and two-dimensional projections.

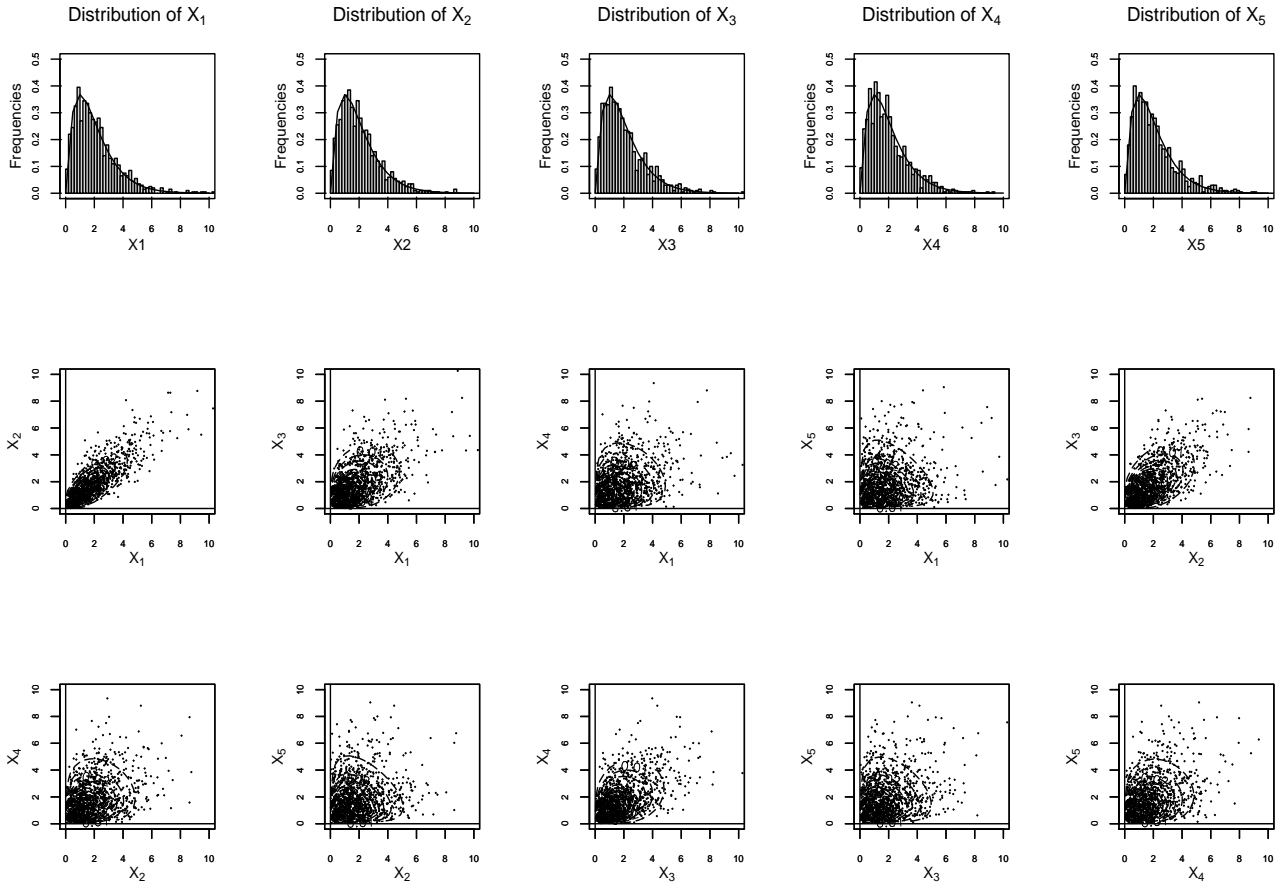


Figure 13: Distribution and simulation of $\mathbf{X}_{[5]}$

8 Appendix

Proofs of Corollaries, Lemmas, Propositions and Theorems

Proof of Proposition 12. Using the Taylor formula in $\boldsymbol{\theta}_{P_n}$, we get

$$\begin{aligned} P_n(\boldsymbol{\theta}_{[n]}) &= \sum_{T \in \mathfrak{P}_n} \left(\frac{\partial}{\partial \boldsymbol{\theta}} \right)^{\bar{T}} (P_n)(\boldsymbol{\theta}_{P_n}) (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^T \\ &= p_{[n]} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n]} \left(1 - \sum_{T \in \mathfrak{P}_n, |T| \geq 2} \frac{1}{p_{[n]}} \left(\frac{\partial}{\partial \boldsymbol{\theta}} \right)^{\bar{T}} (P_n)(\boldsymbol{\theta}_{P_n}) (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{-T} \right), \end{aligned}$$

Then by (24) and (23), we obtain (22).

Now we compute r_T for $T \in \mathfrak{P}_n, |T| \geq 2$, from (24), we have

$$\begin{aligned} r_T &= \frac{-1}{p_{[n]}} \left(\frac{\partial}{\partial \boldsymbol{\theta}} \right)^{\bar{T}} \left(\sum_{S \in \mathfrak{P}_n} p_S \boldsymbol{\theta}_n^S \right) (\boldsymbol{\theta}_{P_n}) \\ &= \sum_{S \in \mathfrak{P}_n} \tilde{p}_{\bar{S}} \boldsymbol{\theta}_n^{S \setminus \bar{T}} (\boldsymbol{\theta}_{P_n}) \\ &= \sum_{S \in \mathfrak{P}_n, S = \bar{T} \cup T', T' \subset [n] \setminus \bar{T}} \tilde{p}_{\bar{S}}^{S \setminus \bar{T}} \tilde{\mathbf{p}}^{T'} \\ &= \sum_{T' \in \mathfrak{P}_T} \tilde{p}_{T \setminus T'}^{T'} \tilde{\mathbf{p}}^{T'}. \end{aligned}$$

Hence if $|T| = 2$, Without loss of generality we compute r_T for $T = \{1, 2\}$

$$\begin{aligned} r_T &= \tilde{p}_{1,2} + \tilde{p}_1 \tilde{p}_2 + \tilde{p}_2 \tilde{p}_1 - \tilde{p}_1 \tilde{p}_2 \\ &= \tilde{p}_{1,2} + \tilde{p}_1 \tilde{p}_2 = \tilde{b}_{\{1,2\}} = \tilde{b}_T. \end{aligned}$$

If $|T| = 3$, Without loss of generality we compute r_T for $T = \{1, 2, 3\}$

$$\begin{aligned} r_T &= \tilde{p}_{1,2,3} + \tilde{p}_{1,2} \tilde{p}_3 + \tilde{p}_{1,3} \tilde{p}_2 + \tilde{p}_{2,3} \tilde{p}_1 + \tilde{p}_1 \tilde{p}_2 \tilde{p}_3 + \tilde{p}_2 \tilde{p}_1 \tilde{p}_3 + \tilde{p}_3 \tilde{p}_1 \tilde{p}_2 + \tilde{p}_1 \tilde{p}_2 \tilde{p}_3 - \tilde{p}_1 \tilde{p}_2 \tilde{p}_3 \\ &= \tilde{p}_{1,2,3} + \tilde{p}_{1,2} \tilde{p}_3 + \tilde{p}_{1,3} \tilde{p}_2 + \tilde{p}_{2,3} \tilde{p}_1 + 2 \tilde{p}_1 \tilde{p}_2 \tilde{p}_3 \\ &= \tilde{b}_{1,2,3} = \tilde{b}_T. \end{aligned}$$

If $|T| = 4$, Without loss of generality we compute r_T for $T = \{1, 2, 3, 4\}$, we can prove that

$$\begin{aligned} r_{\{1,2,3,4\}} &= \tilde{b}_{1,2,3,4} - \left(\tilde{b}_{1,2} \tilde{b}_{3,4} + \tilde{b}_{1,3} \tilde{b}_{2,4} + \tilde{b}_{1,4} \tilde{b}_{2,3} \right) \\ &= \tilde{b}_T - \sum_{\{U,V\} \in \Pi_T^2, |U|=2, |V|=2} \tilde{b}_U \tilde{b}_V. \end{aligned}$$

If $|T| = 5$, Without loss of generality we compute r_T for $T = \{1, 2, 3, 4, 5\}$, we can prove that

$$\begin{aligned}
r_{\{1,2,3,4,5\}} &= \tilde{b}_{1,2,3,4,5} - \tilde{b}_{1,2,3}\tilde{b}_{4,5} - \tilde{b}_{1,2,4}\tilde{b}_{3,5} - \tilde{b}_{1,2,5}\tilde{b}_{3,4} - \tilde{b}_{1,3,4}\tilde{b}_{2,4} - \tilde{b}_{1,3,5}\tilde{b}_{2,4} \\
&\quad - \tilde{b}_{1,4,5}\tilde{b}_{2,3} - \tilde{b}_{2,3,4}\tilde{b}_{1,5} - \tilde{b}_{2,3,5}\tilde{b}_{1,4} - \tilde{b}_{2,4,5}\tilde{b}_{1,3} - \tilde{b}_{3,4,5}\tilde{b}_{1,2} \\
&= \tilde{b}_T - \sum_{\{U,V\} \in \Pi_T^2, |U|=3, |V|=2} \tilde{b}_U \tilde{b}_V.
\end{aligned}$$

■

Proof of Proposition 14. Indeed, we have

$$\begin{aligned}
S_U(S_T) &= \frac{p_{\bar{T}}}{p_{\bar{T} \cup \bar{U}}} \frac{\partial}{\partial \boldsymbol{\theta}} \Big|_{\bar{U}} \left\{ \frac{1}{p_{\bar{T}}} \left(\frac{\partial}{\partial \boldsymbol{\theta}} \right)^{\bar{T}} [P_n(\boldsymbol{\theta})] \right\} \\
&= \frac{1}{p_{\bar{T} \cup \bar{U}}} \left(\frac{\partial}{\partial \boldsymbol{\theta}} \right)^{\bar{U} \cup \bar{T}} [P_n(\boldsymbol{\theta})] \\
&= \frac{1}{p_{\bar{U} \cap \bar{T}}} \left(\frac{\partial}{\partial \boldsymbol{\theta}} \right)^{\bar{U} \cap \bar{T}} [P_n(\boldsymbol{\theta})] \\
&= S_U(P_n).
\end{aligned}$$

■

Proof of Proposition 15. Equality (34) results from the following computation

$$\tilde{q}_U = \frac{p_{\bar{T} \cup T \setminus U}}{p_{\bar{T}}} / \frac{p_{\bar{T} \cup T}}{p_{\bar{T}}} = \frac{p_{\bar{U}}}{p_{[n]}}.$$

Equality (35) comes from (13) and (34). Equality (36) comes from (26) and (34). According to (22) for S_T and (36), we get

$$\begin{aligned}
S_T(\boldsymbol{\theta}_T) &= (-\tilde{p}_T)^{-1} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^T \left(1 - \sum_{T' \in \mathfrak{P}_T, |T'| > 1} r_{T'} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{-T'} \right) \\
&= (-\tilde{p}_T)^{-1} [(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^T - \sum_{T' \in \mathfrak{P}_T, |T'| > 1} r_{T'} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{T \setminus T'}]
\end{aligned}$$

and the equalities $\theta_i - \tilde{p}_i = (-\tilde{p}_i) S_i, i = 1, \dots, n$, gives (37). Equality (38) is a rewriting of (37). ■

Proof of Theorem 16. Let us remember some definitions introduced in [Bernardoff, 2006].

We construct certain measures on $[0, \infty)^n$ indexed by $I \in \mathfrak{P}_n^*$. For $i \in [n]$, define

$l_i(\mathbf{d}x_i) = \mathbf{1}_{(0,\infty)}(x_i) \mathbf{d}x_i$ if $i \in I$ and $l_i(\mathbf{d}x_i) = \delta_0(\mathbf{d}x_i)$ if $i \notin I$. We define the following measure on $[0, \infty)^n$:

$$h_I(\mathbf{d}\mathbf{x}) = \bigotimes_{i=1}^n l_i(\mathbf{d}x_i).$$

For instance, if $n = 3$ and $I = \{2, 3\}$, then

$$h_{\{2,3\}}(\mathbf{d}x_1, \mathbf{d}x_2, \mathbf{d}x_3) = \delta_0(\mathbf{d}x_1) \mathbf{1}_{\{0,\infty\}^2}(x_2, x_3) \mathbf{d}x_2 \mathbf{d}x_3$$

We denote by $\mathbf{1}_n$ the vector $(1, \dots, 1) \in \mathbb{R}^n$, and by $\mathbf{1}$ if there is no ambiguity, and by $\mathbf{0}_n$ the vector $(0, \dots, 0) \in \mathbb{R}^n$, and by $\mathbf{0}$ if there is no ambiguity. For $I \in \mathfrak{P}_n^*$, we write $\mathbb{N}_i^I = \mathbb{N}$ if $i \in I$, $\mathbb{N}_i^I = \{0\}$ if $i \notin I$, and $\mathbb{N}^I = \times_{i=1}^n \mathbb{N}_i^I$. For $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ with $\theta_i \neq 0$ for all $i \in [n]$, recall the notations $\boldsymbol{\theta}^{-1} = (\theta_1^{-1}, \dots, \theta_n^{-1})$ and for $\boldsymbol{\alpha} \in \mathbb{N}^n$, $\boldsymbol{\theta}^{-\boldsymbol{\alpha}} = (\boldsymbol{\theta}^{-1})^{\boldsymbol{\alpha}}$. For all $I \in \mathfrak{P}_n^*$, let

$$\mu_{\boldsymbol{\alpha}, I}(\mathbf{d}\mathbf{x}) = \frac{\mathbf{x}^{\boldsymbol{\alpha} - \mathbf{1}_I}}{(\boldsymbol{\alpha} - \mathbf{1}_I)!} h_I(\mathbf{d}\mathbf{x}) \quad (102)$$

Thus, for $\theta_1 > 0, \dots, \theta_n > 0$, the Lt of $\mu_{\boldsymbol{\alpha}, I}$ is $L_{\mu_{\boldsymbol{\alpha}, I}}(\boldsymbol{\theta}) = \boldsymbol{\theta}^{-\boldsymbol{\alpha}}$. More generally, for $-a_1 + \theta_1 > 0, \dots, -a_n + \theta_n > 0$, if $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$, then we have

$$L_{\exp(\mathbf{a}, \mathbf{x}) \mu_{\boldsymbol{\alpha}, I}}(\boldsymbol{\theta}) = (-\mathbf{a} + \boldsymbol{\theta})^{-\boldsymbol{\alpha}}. \quad (103)$$

The latter is still true if we replace $(\boldsymbol{\alpha} - \mathbf{1}_I)!$ in (102) by $\Gamma(\boldsymbol{\alpha}) = \prod_{i \in I} \Gamma(\alpha_i)$ if $\alpha_i > 0$, $i \in [n]$.

Using (25) and (39), we write

$$[P(\boldsymbol{\theta})]^{-\lambda} = p_{[n]}^{-\lambda} \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} c_{\boldsymbol{\alpha}, \lambda}(R) (-\boldsymbol{\theta}_P + \boldsymbol{\theta})^{-(\boldsymbol{\alpha} + \lambda \mathbf{1})}$$

Using (103) we get

$$[P(\boldsymbol{\theta})]^{-\lambda} = L_{p_{[n]}^{-\lambda} \exp(\boldsymbol{\theta}_P, \mathbf{x}) \mathbf{x}^{(\lambda-1)\mathbf{1}} (\sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} c_{\boldsymbol{\alpha}, \lambda}(R) \frac{\mathbf{x}^{\boldsymbol{\alpha}}}{\Gamma(\boldsymbol{\alpha} + \lambda \mathbf{1})}) h_{[n]}}(\boldsymbol{\theta})$$

and the equality $(\lambda)_{\boldsymbol{\alpha}} = \Gamma(\boldsymbol{\alpha} + \lambda \mathbf{1}) / [\Gamma(\lambda)]^n$ gives (40). ■

Proof of Corollary 17. We apply formula (41) of Theorem (16). We have $p_i = 1$, $i \in [n]$, $p_T = p^{|T|-1}$, $\tilde{p}_T = -p^{-|T|}$, for all $T \in \mathfrak{P}_n^*$. Then $\boldsymbol{\theta}_{P_n} = (-p^{-1}, \dots, -p^{-1}) =$

$-p^{-1}\mathbf{1}_n$. For $(\boldsymbol{\theta}_{P_n} + \boldsymbol{\theta}_{[n]})^{[n]} > qp^{-n}$ and $\theta_i > -p^{-1}$, $\forall i \in [n]$, we obtain

$$P_n(\boldsymbol{\theta}_{[n]}) = p^{n-1} \prod_{i=1}^n (p^{-1} + \theta_i) [1 - qp^{-n} \prod_{i=1}^n (p^{-1} + \theta_i)^{-1}],$$

and if $\mathbf{z} = (z_1, \dots, z_n)$, then

$$R_n(\mathbf{z}) = qp^{-n} \mathbf{z}^{[n]}.$$

Since

$$[1 - R_n(\mathbf{z})]^{-\lambda} = \sum_{l=0}^{\infty} \frac{(\lambda)_l}{l!} (qp^{-n})^l \mathbf{z}^{l\mathbf{1}_n},$$

we have $c_{l\mathbf{1}_n, \lambda}(R_n) = \frac{(\lambda)_l}{l!} (qp^{-n})^l$ if $l \in \mathbb{N}$, and $c_{\alpha, \lambda}(R_n) = 0$ if $\alpha \neq l\mathbf{1}_n$, $l \in \mathbb{N}$. Therefore formula (41) gives

$$\begin{aligned} \gamma_{(P, \lambda)}(\mathbf{dx}) &= \frac{(p^{n-1})^{-\lambda}}{[\Gamma(\lambda)]^n} \exp\left(-\frac{x_1 + \dots + x_n}{p}\right) (\mathbf{x}^{[n]})^{(\lambda-1)} \left[\sum_{l=0}^{\infty} \frac{1}{[(\lambda)_l]^{n-1} l!} (qp^{-n} \mathbf{x}^{[n]})^l \right] \mathbf{1}_{(0, \infty)^n}(\mathbf{x})(\mathbf{dx}), \\ &= \frac{(p^{n-1})^{-\lambda}}{[\Gamma(\lambda)]^n} \exp\left(-\frac{x_1 + \dots + x_n}{p}\right) (\mathbf{x}^{[n]})^{(\lambda-1)} F_{n-1}(\lambda, \dots, \lambda; qp^{-n} \mathbf{x}^{[n]}) \mathbf{1}_{(0, \infty)^n}(\mathbf{x})(\mathbf{dx}). \end{aligned}$$

and definition (7) of F_{n-1} gives (42). ■

Proof of Corollary 18. From (27), we have (43), and

$$[1 - R_2(\mathbf{z})]^{-\lambda} = \sum_{l=0}^{\infty} \frac{(\lambda)_l \tilde{b}_{1,2}^l}{l!} \mathbf{z}^{l\mathbf{1}_2}.$$

Hence, if $\boldsymbol{\alpha} = l\mathbf{1}_2 = (l, l)$, $l \in \mathbb{N}$,

$$c_{\boldsymbol{\alpha}, \lambda}(R_2) = \frac{(\lambda)_l \tilde{b}_{1,2}^l}{l!},$$

and $c_{\boldsymbol{\alpha}, \lambda}(R_2) = 0$ otherwise. Formula (41) therefore gives

$$\gamma_{(P_2, \lambda)}(\mathbf{dx}) = \frac{p_{1,2}^{-\lambda}}{[\Gamma(\lambda)]^2} \exp\left(-\frac{p_2}{p_{1,2}} x_1 - \frac{p_1}{p_{1,2}} x_2\right) (x_1 x_2)^{(\lambda-1)} \left(\sum_{l \in \mathbb{N}} \frac{1}{(\lambda)_l} \frac{(\tilde{b}_{1,2} x_1 x_2)^l}{l!} \right) \mathbf{1}_{(0, \infty)^2}(\mathbf{x})(\mathbf{dx}).$$

and definition (7) of F_1 gives (45). ■

Proof of Corollary 19. From (27) and (28), we have, $r_T = \tilde{b}_T$ for $T \in \mathfrak{P}_3$, $|T| = 2, 3$.

This proves the formula (46).

Now, for $\mathbf{z}_3 = (z_1, z_2, z_3) \in \mathbb{R}^3$, and $\lambda > 0$, we develop $[1 - R_3(\mathbf{z})]^{-\lambda}$ by

$$\begin{aligned}
[1 - R_3(\mathbf{z}_3)]^{-\lambda} &= [1 - (\tilde{b}_{1,2}z_1z_2 + \tilde{b}_{1,3}z_1z_3 + \tilde{b}_{2,3}z_2z_3 + \tilde{b}_{1,2,3}z_1z_2z_3)]^{-\lambda} \\
&= \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} (\tilde{b}_{1,2}z_1z_2 + \tilde{b}_{1,3}z_1z_3 + \tilde{b}_{2,3}z_2z_3 + \tilde{b}_{1,2,3}z_1z_2z_3)^k \\
&= \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} \sum_{l_1+l_2+l_3+l_4=k} \frac{k!}{l_1!l_2!l_3!l_4!} (\tilde{b}_{1,2}z_1z_2)^{l_1} (\tilde{b}_{1,3}z_1z_3)^{l_2} (\tilde{b}_{2,3}z_2z_3)^{l_3} (\tilde{b}_{1,2,3}z_1z_2z_3)^{l_4} \\
&= \sum_{k=0}^{\infty} \sum_{l_1+l_2+l_3+l_4=k} \frac{(\lambda)_k}{l_1!l_2!l_3!l_4!} \tilde{b}_{1,2}^{l_1} \tilde{b}_{1,3}^{l_2} \tilde{b}_{2,3}^{l_3} \tilde{b}_{1,2,3}^{l_4} z_1^{l_1+l_2+l_4} z_2^{l_1+l_3+l_4} z_3^{l_2+l_3+l_4} \\
&= \sum_{\mathbf{l}=(l_1,l_2,l_3,l_4) \in \mathbb{N}^4} \frac{(\lambda)_{l_1+l_2+l_3+l_4}}{l_1!l_2!l_3!l_4!} \tilde{b}_{1,2}^{l_1} \tilde{b}_{1,3}^{l_2} \tilde{b}_{2,3}^{l_3} \tilde{b}_{1,2,3}^{l_4} z_1^{l_1+l_2+l_4} z_2^{l_1+l_3+l_4} z_3^{l_2+l_3+l_4} \\
&= \sum_{\mathbf{l}=(l_1,l_2,l_3,l_4) \in \mathbb{N}^4} \frac{(\lambda)_{l_1+l_2+l_3+l_4}}{l_1!l_2!l_3!l_4!} (\tilde{b}_{1,2}z_1z_2)^{l_1} (\tilde{b}_{1,3}z_1z_3)^{l_2} (\tilde{b}_{2,3}z_2z_3)^{l_3} (\tilde{b}_{1,2,3}z_1z_2z_3)^{l_4}
\end{aligned}$$

Now, the conditions $k \in \mathbb{N}, l_1 + l_2 + l_3 + l_4 = k; \alpha_1 = l_1 + l_2 + l_4, \alpha_2 = l_1 + l_3 + l_4, \alpha_3 = l_2 + l_3 + l_4$ give $l_1 = k - \alpha_3 \geq 0, l_2 = k - \alpha_2 \geq 0, l_3 = k - \alpha_1 \geq 0, l_4 = \alpha_1 + \alpha_2 + \alpha_3 - 2k \geq 0$ and we get

$$\begin{aligned}
(1 - R_3(\mathbf{z}))^{-\lambda} &= \sum_{k \in \mathbb{N}} \sum_{\alpha=(\alpha_1,\alpha_2,\alpha_3) \in \mathbb{N}^3} \frac{(\lambda)_k \tilde{b}_{1,2}^{k-\alpha_3} \tilde{b}_{1,3}^{k-\alpha_2} \tilde{b}_{2,3}^{k-\alpha_1} \tilde{b}_{1,2,3}^{\alpha_1+\alpha_2+\alpha_3-2k}}{(k-\alpha_3)!(k-\alpha_2)!(k-\alpha_1)!(\alpha_1+\alpha_2+\alpha_3-2k)!} z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} \\
&= \sum_{\alpha \in \mathbb{N}^3} \left[\sum_{k \in \mathbb{N}, k \geq \alpha_1, k \geq \alpha_2, k \geq \alpha_3, k \leq \frac{\alpha_1+\alpha_2+\alpha_3}{2}} \frac{(\lambda)_k \tilde{b}_{1,2}^{k-\alpha_3} \tilde{b}_{1,3}^{k-\alpha_2} \tilde{b}_{2,3}^{k-\alpha_1} \tilde{b}_{1,2,3}^{\alpha_1+\alpha_2+\alpha_3-2k}}{(k-\alpha_3)!(k-\alpha_2)!(k-\alpha_1)!(\alpha_1+\alpha_2+\alpha_3-2k)!} \right] \mathbf{z}_3^\alpha \\
&= \sum_{\alpha \in \mathbb{N}^3, \|\alpha\|_\infty \leq \frac{|\alpha|}{2}} \left[\sum_{\|\alpha\|_\infty \leq k \leq \frac{|\alpha|}{2}, k \in \mathbb{N}} \frac{(\lambda)_k \tilde{b}_{1,2}^{k-\alpha_3} \tilde{b}_{1,3}^{k-\alpha_2} \tilde{b}_{2,3}^{k-\alpha_1} \tilde{b}_{1,2,3}^{\alpha_1+\alpha_2+\alpha_3-2k}}{(k-\alpha_3)!(k-\alpha_2)!(k-\alpha_1)!(\alpha_1+\alpha_2+\alpha_3-2k)!} \right] \mathbf{z}_3^\alpha
\end{aligned}$$

and we get the formula (47). The formula (40) gives (48).

Now, from (48), we have by $k \in \mathbb{N}, l_1 + l_2 + l_3 + l_4 = k; \alpha_1 = l_1 + l_2 + l_4, \alpha_2 =$

$$l_1 + l_3 + l_4, \alpha_3 = l_2 + l_3 + l_4$$

$$\begin{aligned}
\gamma_{(P_3, \lambda)}(\mathbf{dx}) &= \frac{p_{[3]}^{-\lambda}}{[\Gamma(\lambda)]^3} \exp(\boldsymbol{\theta}_P, \mathbf{x}) \mathbf{x}^{(\lambda-1)\mathbf{1}_3} \\
&\times \left\{ \sum_{\boldsymbol{\alpha} \in \mathbb{N}^3, \|\boldsymbol{\alpha}\|_\infty \leq \frac{|\boldsymbol{\alpha}|}{2}} \left[\sum_{\|\boldsymbol{\alpha}\|_\infty \leq k \leq \frac{|\boldsymbol{\alpha}|}{2}, k \in \mathbb{N}} \frac{(\lambda)_k \tilde{b}_{1,2}^{k-\alpha_3} \tilde{b}_{1,3}^{k-\alpha_2} \tilde{b}_{2,3}^{k-\alpha_1} \tilde{b}_{1,2,3}^{\alpha_1+\alpha_2+\alpha_3-2k}}{(k-\alpha_3)!(k-\alpha_2)!(k-\alpha_1)!(\alpha_1+\alpha_2+\alpha_3-2k)!} \frac{x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}}{(\lambda)_{\alpha_1} (\lambda)_{\alpha_2} (\lambda)_{\alpha_3}} \right] \right\} \\
&\times \mathbf{1}_{(0, \infty)^3}(\mathbf{x}) (\mathbf{dx}) \\
&= \frac{p_{[3]}^{-\lambda}}{[\Gamma(\lambda)]^3} \exp(\boldsymbol{\theta}_P, \mathbf{x}) \mathbf{x}^{(\lambda-1)\mathbf{1}_3} \\
&\times \left\{ \sum_{k=0}^{\infty} \sum_{l_1+l_2+l_3+l_4=k} \frac{(\lambda)_{l_1+l_2+l_3+l_4} \tilde{b}_{1,2}^{l_1} \tilde{b}_{1,3}^{l_2} \tilde{b}_{2,3}^{l_3} \tilde{b}_{1,2,3}^{l_4}}{(l_1)!(l_2)!(l_3)!(l_4)!} \frac{x_1^{l_1+l_2+l_4} x_2^{l_1+l_3+l_4} x_3^{l_2+l_3+l_4}}{(\lambda)_{l_1+l_2+l_4} (\lambda)_{l_1+l_3+l_4} (\lambda)_{l_2+l_3+l_4}} \right\} \mathbf{1}_{(0, \infty)^3}(\mathbf{x}) (\mathbf{dx}) \\
&= \frac{p_{[3]}^{-\lambda}}{[\Gamma(\lambda)]^3} \exp(\boldsymbol{\theta}_P, \mathbf{x}) \mathbf{x}^{(\lambda-1)\mathbf{1}_3} \\
&\times \left\{ \sum_{\mathbf{l}=(l_1, l_2, l_3, l_4) \in \mathbb{N}^4} \frac{(\lambda)_{l_1+l_2+l_3+l_4}}{(\lambda)_{l_1+l_2+l_4} (\lambda)_{l_1+l_3+l_4} (\lambda)_{l_2+l_3+l_4}} \frac{(\tilde{b}_{1,2} x_1 x_2)^{l_1} (\tilde{b}_{1,3} x_1 x_3)^{l_2} (\tilde{b}_{2,3} x_2 x_3)^{l_3} (\tilde{b}_{1,2,3} x_1 x_2 x_3)^{l_4}}{(l_1)!(l_2)!(l_3)!(l_4)!} \right\} \\
&\times \mathbf{1}_{(0, \infty)^3}(\mathbf{x}) (\mathbf{dx}) \\
&= \frac{p_{[3]}^{-\lambda}}{[\Gamma(\lambda)]^3} \exp(\boldsymbol{\theta}_P, \mathbf{x}) \mathbf{x}^{(\lambda-1)\mathbf{1}_3} \\
&\times {}_1\mathbf{F}_3(\lambda; \tilde{b}_{1,2} x_1 x_2, \tilde{b}_{1,3} x_1 x_3, \tilde{b}_{2,3} x_2 x_3, \tilde{b}_{1,2,3} x_1 x_2 x_3) \mathbf{1}_{(0, \infty)^3}(\mathbf{x}) (\mathbf{dx})
\end{aligned}$$

■

Proof of Remark 20. The only difference with (48) is that all $c_{\boldsymbol{\alpha}, \lambda}(R_3)$ defined by (51) for $\|\boldsymbol{\alpha}\|_\infty \leq \frac{|\boldsymbol{\alpha}|}{2}$ are positive and the formula (41) gives (52). ■

Proof of Remark 21. As $\tilde{b}_{1,2,3} = 0$, $\tilde{b}_{1,2,3}^{\alpha_1+\alpha_2+\alpha_3-2k} = 1$ only for $\frac{|\boldsymbol{\alpha}|}{2} = k \in \mathbb{N}$. In this case, if $\|\boldsymbol{\alpha}\|_\infty \leq \frac{|\boldsymbol{\alpha}|}{2} = k \in \mathbb{N}$,

$$c_{\boldsymbol{\alpha}, \lambda}(R_3) = \frac{(\lambda)_k \tilde{b}_{2,3}^{k-\alpha_1} \tilde{b}_{1,3}^{k-\alpha_2} \tilde{b}_{1,2}^{k-\alpha_3}}{(k-\alpha_1)!(k-\alpha_2)!(k-\alpha_3)!} > 0,$$

gives (53) and (41) gives (54). ■

Proof of Remark 22. If $\tilde{b}_{1,2}, \tilde{b}_{1,3}, \tilde{b}_{2,3} = 0$, $\tilde{b}_{1,2,3} > 0$,

$$c_{\boldsymbol{\alpha}, \lambda}(R_3) = \sum_{\|\boldsymbol{\alpha}\|_\infty \leq k \leq \frac{|\boldsymbol{\alpha}|}{2}, k \in \mathbb{N}} \frac{(\lambda)_k \tilde{b}_{2,3}^{k-\alpha_1} \tilde{b}_{1,3}^{k-\alpha_2} \tilde{b}_{1,2}^{k-\alpha_3} \tilde{b}_{1,2,3}^{\alpha_1+\alpha_2+\alpha_3-2k}}{(k-\alpha_1)!(k-\alpha_2)!(k-\alpha_3)!(\alpha_1+\alpha_2+\alpha_3-2k)!} \neq 0$$

only for $\boldsymbol{\alpha} = k\mathbf{1}_3$, $k \in \mathbb{N}$, and we obtain the formula (55), and $c_{\boldsymbol{\alpha},\lambda}(R_3) = 0$ otherwise.

Thus we have

$$\begin{aligned} \gamma_{(P_3,\lambda)}(\mathbf{d}\mathbf{x}) &= \frac{p_{[3]}^{-\lambda}}{[\Gamma(\lambda)]^3} \exp(\boldsymbol{\theta}_P, \mathbf{x}) \mathbf{x}^{(\lambda-1)\mathbf{1}_3} \left[\sum_{\boldsymbol{\alpha}=k\mathbf{1}_3, k \in \mathbb{N}} \frac{(\lambda)_k \tilde{b}_{1,2,3}^k}{k!} \frac{(\mathbf{x}^{[3]})^k}{[(\lambda)_{\mathbf{k}}]^3} \right] \mathbf{1}_{(0,\infty)^3}(\mathbf{x}) (\mathbf{d}\mathbf{x}) \\ &= \frac{p_{[3]}^{-\lambda}}{[\Gamma(\lambda)]^3} \exp(\boldsymbol{\theta}_P, \mathbf{x}) \mathbf{x}^{(\lambda-1)\mathbf{1}_3} \left[\sum_{\boldsymbol{\alpha}=k\mathbf{1}_3, k \in \mathbb{N}} \frac{1}{[(\lambda)_{\mathbf{k}}]^2} \frac{(\tilde{b}_{1,2,3} \mathbf{x}^{[3]})^k}{k!} \right] \mathbf{1}_{(0,\infty)^3}(\mathbf{x}) (\mathbf{d}\mathbf{x}) \\ &= \frac{p_{[3]}^{-\lambda}}{[\Gamma(\lambda)]^3} \exp(\boldsymbol{\theta}_P, \mathbf{x}) \mathbf{x}^{(\lambda-1)\mathbf{1}_3} F_2 \left(\lambda, \lambda, \tilde{b}_{1,2,3} \mathbf{x}^{[3]} \right) \mathbf{1}_{(0,\infty)^3}(\mathbf{x}) (\mathbf{d}\mathbf{x}) \end{aligned}$$

and (7) gives (56). ■

Proof of Theorem 23. If $\mathbf{1}_n = (1, \dots, 1) \in \mathbb{R}^n$, we denote $\mathbf{1}_k = (1, \dots, 1) \in \mathbb{R}^k$ and $\mathbf{1}_{n-k} = (1, \dots, 1) \in \mathbb{R}^{n-k}$. We note that $\mathbf{X}_{[k]}$ is a random real vector such that $\mathbf{X}_{[k]} \sim \gamma_{(P_k,\lambda)}$, with

$$P_k(\boldsymbol{\theta}_{[k]}) = \sum_{T \in \mathfrak{P}_k} p_T \boldsymbol{\theta}_{[k]}^T = \sum_{T \in \mathfrak{P}_k} p_T \boldsymbol{\theta}_{[k]}^T.$$

Using the equality (41), the Lt $L_{\mathbf{X}_{[n] \setminus [k]}^{\mathbf{X}_{[k]} = \mathbf{x}_{[k]}}$ is given by

$$\begin{aligned} &L_{\mathbf{X}_{[n] \setminus [k]}^{\mathbf{X}_{[k]} = \mathbf{x}_{[k]}}(\boldsymbol{\theta}_{[n] \setminus [k]}) \\ &= L \frac{\frac{p_{[n]}^{-\lambda}}{[\Gamma(\lambda)]^n} \exp(\boldsymbol{\theta}_{P_n}, \mathbf{x}) \mathbf{x}^{(\lambda-1)\mathbf{1}_n} (\sum_{\boldsymbol{\alpha} \in \mathbb{N}^n, c_{\boldsymbol{\alpha},\lambda}(R_n) \neq 0} \frac{c_{\boldsymbol{\alpha},\lambda}(R_n)}{(\lambda)_{\boldsymbol{\alpha}}} \mathbf{x}^{\boldsymbol{\alpha}})}{\frac{p_{[k]}^{-\lambda}}{[\Gamma(\lambda)]^k} \exp(\boldsymbol{\theta}_{P_k}, \mathbf{x}_{[k]}) \mathbf{x}_{[k]}^{(\lambda-1)\mathbf{1}_k} (\sum_{\boldsymbol{\alpha}_{[k]} \in \mathbb{N}^k, c_{\boldsymbol{\alpha}_{[k]},\lambda}(R_k) \neq 0} \frac{c_{\boldsymbol{\alpha}_{[k]},\lambda}(R_k)}{(\lambda)_{\boldsymbol{\alpha}_{[k]}}} \mathbf{x}_{[k]}^{\boldsymbol{\alpha}_{[k]}})} \mathbf{1}_{(0,\infty)^{n-k}}(\mathbf{x}_{[n] \setminus [k]}) (\mathbf{d}\mathbf{x}_{[n] \setminus [k]})} (\boldsymbol{\theta}_{[n] \setminus [k]}) \\ &= \frac{\left[\frac{p_{[n]}}{p_{[k]}} \right]^{-\lambda} \exp\left((\boldsymbol{\theta}_{P_n})_{[k]} - \boldsymbol{\theta}_{P_k}, \mathbf{x}_{[k]} \right)}{\sum_{\boldsymbol{\alpha}_{[k]} \in \mathbb{N}^k, c_{\boldsymbol{\alpha}_{[k]},\lambda}(R_k) \neq 0} \frac{c_{\boldsymbol{\alpha}_{[k]},\lambda}(R_k)}{(\lambda)_{\boldsymbol{\alpha}_{[k]}}} \mathbf{x}_{[k]}^{\boldsymbol{\alpha}_{[k]}}} \\ &\times \sum_{\boldsymbol{\alpha} = (\boldsymbol{\alpha}_{[k]}, \boldsymbol{\alpha}_{[n] \setminus [k]}) \in \mathbb{N}^n, c_{\boldsymbol{\alpha},\lambda}(R_n) \neq 0} \frac{c_{\boldsymbol{\alpha},\lambda}(R_n)}{(\lambda)_{\boldsymbol{\alpha}_{[k]}}} \mathbf{x}_{[k]}^{\boldsymbol{\alpha}_{[k]}} L_{\exp((\boldsymbol{\theta}_{P_n})_{[n] \setminus [k]}, \mathbf{x}_{[n] \setminus [k]}) \mu_{\boldsymbol{\alpha}_{[n] \setminus [k]} + \lambda \mathbf{1}_{n-k}, [n] \setminus [k]}}(\boldsymbol{\theta}_{[n] \setminus [k]}) \\ &= \frac{\left[\frac{p_{[n]}}{p_{[k]}} \right]^{-\lambda} \exp\left((\boldsymbol{\theta}_{P_n})_{[k]} - \boldsymbol{\theta}_{P_k}, \mathbf{x}_{[k]} \right)}{\sum_{\boldsymbol{\alpha}_{[k]} \in \mathbb{N}^k, c_{\boldsymbol{\alpha}_{[k]},\lambda}(R_k) \neq 0} \frac{c_{\boldsymbol{\alpha}_{[k]},\lambda}(R_k)}{(\lambda)_{\boldsymbol{\alpha}_{[k]}}} \mathbf{x}_{[k]}^{\boldsymbol{\alpha}_{[k]}}} \\ &\times \sum_{\boldsymbol{\alpha} = (\boldsymbol{\alpha}_{[k]}, \boldsymbol{\alpha}_{[n] \setminus [k]}) \in \mathbb{N}^n, c_{\boldsymbol{\alpha},\lambda}(R_n) \neq 0} \frac{\mathbf{x}_{[k]}^{\boldsymbol{\alpha}_{[k]}}}{(\lambda)_{\boldsymbol{\alpha}_{[k]}}} c_{\boldsymbol{\alpha},\lambda}(R_n) \left(\boldsymbol{\theta}_{[n] \setminus [k]} - (\boldsymbol{\theta}_{P_n})_{[n] \setminus [k]} \right)^{-\left(\boldsymbol{\alpha}_{[n] \setminus [k]} + \lambda \mathbf{1}_{n-k} \right)} \end{aligned}$$

We obtain

$$\begin{aligned}
L_{\mathbf{X}_{[n]\setminus[k]}^{\mathbf{X}_{[k]}=\mathbf{x}_{[k]}}}(\boldsymbol{\theta}_{[n]\setminus[k]}) &= \frac{\exp\left((\boldsymbol{\theta}_{P_n})_{[k]} - \boldsymbol{\theta}_{P_k}, \mathbf{x}_{[k]}\right)}{\sum_{\boldsymbol{\alpha}_{[k]} \in \mathbb{N}^k, c_{\boldsymbol{\alpha}_{[k]}, \lambda}(R_k) \neq 0} \frac{c_{\boldsymbol{\alpha}_{[k]}, \lambda}(R_k)}{(\lambda)_{\boldsymbol{\alpha}_{[k]}}} \mathbf{x}_{[k]}^{\boldsymbol{\alpha}_{[k]}}} \left(\frac{p_{[n]}}{p_{[k]}} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n]\setminus[k]}^{\mathbf{1}_{n-k}}\right)^{-\lambda} \\
&\times \sum_{\boldsymbol{\alpha}=(\boldsymbol{\alpha}_{[k]}, \boldsymbol{\alpha}_{[n]\setminus[k]}) \in \mathbb{N}^n, c_{\boldsymbol{\alpha}, \lambda}(R_n) \neq 0} \frac{\mathbf{x}_{[k]}^{\boldsymbol{\alpha}_{[k]}}}{(\lambda)_{\boldsymbol{\alpha}_{[k]}}} c_{\boldsymbol{\alpha}, \lambda}(R_n) (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n]\setminus[k]}^{-\boldsymbol{\alpha}_{[n]\setminus[k]}}. \tag{104}
\end{aligned}$$

Since $L_{\mathbf{X}_{[n]\setminus[k]}^{\mathbf{X}_{[k]}=\mathbf{x}_{[k]}}}$ is the Lt of a pd we have $L_{\mathbf{X}_{n-k}^{\mathbf{X}_k=\mathbf{x}_k}}(\mathbf{0}_{n-k}) = 1$ and we get

$$\begin{aligned}
&\frac{\exp\left((\boldsymbol{\theta}_{P_n})_{[k]} - \boldsymbol{\theta}_{P_k}, \mathbf{x}_{[k]}\right)}{\sum_{\boldsymbol{\alpha}_{[k]} \in \mathbb{N}^k, c_{\boldsymbol{\alpha}_{[k]}, \lambda}(R_k) \neq 0} \frac{c_{\boldsymbol{\alpha}_{[k]}, \lambda}(R_k)}{(\lambda)_{\boldsymbol{\alpha}_{[k]}}} \mathbf{x}_{[k]}^{\boldsymbol{\alpha}_{[k]}}} \tag{105} \\
&= \left(\frac{p_{[n]}}{p_{[k]}} (-\boldsymbol{\theta}_{P_n})_{[n]\setminus[k]}^{\mathbf{1}_{n-k}}\right)^\lambda \left[\sum_{\boldsymbol{\alpha}=(\boldsymbol{\alpha}_{[k]}, \boldsymbol{\alpha}_{[n]\setminus[k]}) \in \mathbb{N}^n, c_{\boldsymbol{\alpha}, \lambda}(R_n) \neq 0} \frac{\mathbf{x}_{[k]}^{\boldsymbol{\alpha}_{[k]}}}{(\lambda)_{\boldsymbol{\alpha}_{[k]}}} c_{\boldsymbol{\alpha}, \lambda}(R_n) (-\boldsymbol{\theta}_{P_n})_{[n]\setminus[k]}^{-\boldsymbol{\alpha}_{[n]\setminus[k]}} \right]^{-1}.
\end{aligned}$$

We carry (105) in (104) and we obtain

$$\begin{aligned}
&L_{\mathbf{X}_{[n]\setminus[k]}^{\mathbf{X}_{[k]}=\mathbf{x}_{[k]}}}(\boldsymbol{\theta}_{[n]\setminus[k]}) \\
&= \left[\frac{(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n]\setminus[k]}^{\mathbf{1}_{n-k}}}{(-\boldsymbol{\theta}_{P_n})_{[n]\setminus[k]}^{\mathbf{1}_{n-k}}}\right]^{-\lambda} \\
&\times \frac{\sum_{\boldsymbol{\alpha}=(\boldsymbol{\alpha}_{[k]}, \boldsymbol{\alpha}_{[n]\setminus[k]}) \in \mathbb{N}^n, c_{\boldsymbol{\alpha}, \lambda}(R_n) \neq 0} \frac{\mathbf{x}_{[k]}^{\boldsymbol{\alpha}_{[k]}}}{(\lambda)_{\boldsymbol{\alpha}_{[k]}}} c_{\boldsymbol{\alpha}, \lambda}(R_n) (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n]\setminus[k]}^{-\boldsymbol{\alpha}_{[n]\setminus[k]}}}{\sum_{\boldsymbol{\alpha}=(\boldsymbol{\alpha}_{[k]}, \boldsymbol{\alpha}_{[n]\setminus[k]}) \in \mathbb{N}^n, c_{\boldsymbol{\alpha}, \lambda}(R_n) \neq 0} \frac{\mathbf{x}_{[k]}^{\boldsymbol{\alpha}_{[k]}}}{(\lambda)_{\boldsymbol{\alpha}_{[k]}}} c_{\boldsymbol{\alpha}, \lambda}(R_n) (-\boldsymbol{\theta}_{P_n})_{[n]\setminus[k]}^{-\boldsymbol{\alpha}_{[n]\setminus[k]}}}. \tag{106}
\end{aligned}$$

As we have $(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n]\setminus[k]}^{\mathbf{1}_{n-k}} (-\boldsymbol{\theta}_{P_n})_{[n]\setminus[k]}^{-\mathbf{1}_{n-k}} = \prod_{i=k+1}^n [1 + \theta_i (-\boldsymbol{\theta}_{P_n})_i^{-1}]$, (106) gives (58). ■

Proof of Remark 25. We have $\tilde{p}_T(S_{[n]\setminus[1]}) = \frac{p_{\{1\} \cup T}}{p_1}$ and

$$\begin{aligned}
\tilde{p}_T(S_{[n]\setminus[1]}) &= -p_{([n]\setminus[1]) \setminus T}(S_{[n]\setminus[1]}) / \frac{p_{[n]}}{p_1} = -\frac{p_{\{1\} \cup ([n]\setminus[1]) \setminus T}}{p_1} / \frac{p_{[n]}}{p_1} \\
&= -\frac{p_{[n]\setminus T}}{p_{[n]}} = \tilde{p}_T(R_n).
\end{aligned}$$

We also have

$$\begin{aligned}
r_T(S_{[n]\setminus[1]}) &= -\frac{1}{\frac{p_{[n]}}{p_1}} \left(\frac{\partial}{\partial \boldsymbol{\theta}}\right)^{([n]\setminus\{1\}) \setminus T} \left(\frac{1}{p_1} \frac{\partial}{\partial \theta_1} P_n(\boldsymbol{\theta}_{P_n})\right) \\
&= -\frac{1}{p_{[n]}} \left(\frac{\partial}{\partial \boldsymbol{\theta}}\right)^{[n]\setminus T} P_n(\boldsymbol{\theta}_{P_n}) = r_T(P_n).
\end{aligned}$$

■

Proof of Theorem 24. We calculate, in five steps, another expression of (59) for $k = 1$ to obtain a new formula for $L_{\mathbf{X}_{[n] \setminus [1]}}^{X_1=x_1}(\boldsymbol{\theta}_{[n] \setminus [1]})$ from (58).

First step: For $\alpha_1 \in I_1(R_n)$, let $J_{[n] \setminus [1]}(\alpha_1) = \left\{ \boldsymbol{\alpha}_{[n] \setminus [1]} \in \mathbb{N}^{n-1}, c_{\alpha_1, \boldsymbol{\alpha}_{[n] \setminus [1]}, \lambda}(R_n) \neq 0 \right\}$, then we have by (59)

$$\mathbf{F}_1(\lambda, R_n, x_1, \boldsymbol{\theta}_{[n] \setminus [1]}) = \sum_{\alpha_1 \in I_1(R_n)} \frac{x_1^{\alpha_1}}{(\lambda)_{\alpha_1}} \sum_{\boldsymbol{\alpha}_{[n] \setminus [1]} \in J_{[n] \setminus [1]}(\alpha_1)} c_{\alpha_1, \boldsymbol{\alpha}_{[n] \setminus [1]}, \lambda}(R_n) (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus [1]}^{-\boldsymbol{\alpha}_{[n] \setminus [1]}}. \quad (107)$$

Second step: For $\alpha \in \mathbb{N}$ and $j \in \mathbb{N}$, we denote by $\langle \alpha \rangle_j$ the number $\alpha(\alpha-1)\dots(\alpha-j+1)$, in particular $\langle \alpha \rangle_\alpha = \alpha!$ and if $j > \alpha$, $\langle \alpha \rangle_j = 0$. For $\boldsymbol{\alpha}_{[k]} \in \mathbb{N}^{[k]}$, and $\alpha_i \geq \beta_i \in \mathbb{N}$, we denote by $\langle \boldsymbol{\alpha}_{[k]} \rangle_{\boldsymbol{\beta}_{[k]}}$ the number $\langle \alpha_1 \rangle_{\beta_1} \dots \langle \alpha_k \rangle_{\beta_k}$, in particular $\langle \boldsymbol{\alpha}_{[k]} \rangle_{\boldsymbol{\alpha}_{[k]}} = \boldsymbol{\alpha}_{[k]}!$ and if $\exists j \in [k], \beta_j > \alpha_j$, then $\langle \boldsymbol{\alpha}_{[k]} \rangle_{\boldsymbol{\beta}_{[k]}} = 0$. Let $\beta_1 \in \mathbb{N}$, we apply $\left(\frac{\partial}{\partial z_1}\right)^{\beta_1}$ to (39) with $k = 1$, because R_n is an affine polynomial, we obtain by [Comtet, 1974] p. 42

$$\begin{aligned} \left(\frac{\partial}{\partial z_1}\right)^{\beta_1} [1 - R(\mathbf{z})]^{-\lambda} &= \left(\frac{\partial}{\partial z_1}\right)^{\beta_1} \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} c_{\boldsymbol{\alpha}, \lambda}(R) \mathbf{z}^{\boldsymbol{\alpha}} \\ (\lambda)_{\beta_1} \left(\frac{\partial}{\partial z_1} R_n(\mathbf{z})\right)^{\beta_1} [1 - R_n(\mathbf{z})]^{-(\lambda + \beta_1)} \\ &= \sum_{\alpha_1 \in I_1(R_n)} \sum_{\boldsymbol{\alpha}_{[n] \setminus [1]} \in J_{[n] \setminus [1]}(\alpha_1), \alpha_1 \geq \beta_1} c_{\alpha_1, \boldsymbol{\alpha}_{[n] \setminus [1]}, \lambda}(R_n) \langle \alpha_1 \rangle_{\beta_1} z_1^{\alpha_1 - \beta_1} \mathbf{z}_{[n] \setminus [1]}^{\boldsymbol{\alpha}_{[n] \setminus [1]}}. \end{aligned} \quad (108)$$

Third step: Doing $z_1 = 0$ in Formula (108), we get

$$\begin{aligned} (\lambda)_{\beta_1} [1 - R_n(0, \mathbf{z}_{[n] \setminus [1]})]^{-(\lambda + \beta_1)} \left(\frac{\partial}{\partial z_1} R_n(0, \mathbf{z}_{[n] \setminus [1]})\right)^{\beta_1} \\ = \sum_{\boldsymbol{\alpha}_{[n] \setminus [1]} \in J_{[n] \setminus [1]}(\beta_1)} c_{\beta_1, \boldsymbol{\alpha}_{[n] \setminus [1]}, \lambda}(R_n) (\beta_1!) \mathbf{z}_{[n] \setminus [1]}^{\boldsymbol{\alpha}_{[n] \setminus [1]}}. \end{aligned} \quad (109)$$

Fourth step: Doing $\beta_1 = \alpha_1$ and $\mathbf{z}_{[n] \setminus [1]} = (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus [1]}^{-1}$ in Formula (109), we get

$$\begin{aligned} \sum_{\boldsymbol{\alpha}_{[n] \setminus [1]} \in J_{[n] \setminus [1]}(\alpha_1)} c_{\alpha_1, \boldsymbol{\alpha}_{[n] \setminus [1]}, \lambda}(R_n) (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus [1]}^{-\boldsymbol{\alpha}_{[n] \setminus [1]}} \\ = \frac{(\lambda)_{\alpha_1}}{\alpha_1!} [1 - R_n(0, \mathbf{z}_{[n] \setminus [1]})]^{-(\lambda + \alpha_1)} \left(\frac{\partial}{\partial z_1} R_n(0, (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus [1]}^{-1})\right)^{\alpha_1} \end{aligned} \quad (110)$$

Fifth step: Using \mathfrak{z} and S in Formula (110), we get

$$\begin{aligned} & \sum_{\alpha_{[n]\setminus[1]} \in J_{[n]\setminus[1]}(\alpha_1)} c_{\alpha_1, \alpha_{[n]\setminus[1]}, \lambda} (R_n) (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n]\setminus[1]}^{-\alpha_{[n]\setminus[1]}} \\ &= \left[\frac{(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n]\setminus[1]}^{-1_{n-1}}}{\frac{p_{[n]}}{p_1}} S(\boldsymbol{\theta}_{[n]\setminus[1]}) \right]^{-\lambda} \frac{(\lambda)_{\alpha_1}}{\alpha_1!} [\mathfrak{z}(\boldsymbol{\theta}_{[n]\setminus[1]})]^{\alpha_1}, \end{aligned} \quad (111)$$

hence (111) gives

$$\mathbf{F}_1(\lambda, R_n, x_1, \boldsymbol{\theta}_{[n]\setminus[1]}) = \left[\frac{(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n]\setminus[1]}^{-1_{n-1}}}{\frac{p_{[n]}}{p_1}} S(\boldsymbol{\theta}_{[n]\setminus[1]}) \right]^{-\lambda} \mathbf{G}(R_n, \mathfrak{z}(\boldsymbol{\theta}_{[n]\setminus[1]}), x_1) \quad (112)$$

Substituting (112) in Formula (58) for $k = 1$ and because $S(\mathbf{0}_{[n]\setminus[1]}) = 1$ from formula (62) and (57), we get

$$L_{\mathbf{X}_{[n]\setminus[1]}^{X_1=x_1}}(\boldsymbol{\theta}_{[n]\setminus[1]}) = [Q_{[n]\setminus[1]}(\boldsymbol{\theta}_{[n]\setminus[1]})]^{-\lambda} \frac{\left[\frac{(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n]\setminus[1]}^{-1_{n-1}}}{\frac{p_{[n]}}{p_1}} S(\boldsymbol{\theta}_{[n]\setminus[1]}) \right]^{-\lambda} \mathbf{G}(R_n, \mathfrak{z}(\boldsymbol{\theta}_{[n]\setminus[1]}), x_1)}{\left[\frac{(-\boldsymbol{\theta}_{P_n})_{[n]\setminus[1]}^{-1_{n-1}}}{\frac{p_{[n]}}{p_1}} S(\mathbf{0}_{[n]\setminus[1]}) \right]^{-\lambda} \mathbf{G}(R_n, \mathfrak{z}(\mathbf{0}_{[n]\setminus[1]}), x_1)} \quad (113)$$

Finally (113) and (57).gives (68). Formulas (62), (63) and (64) remain to be proven.

First, according to (25), we have

$$\frac{P_n(\boldsymbol{\theta}_{[n]})}{p_{[n]}(\theta_1 - \tilde{p}_1)} = \{1 - R_n [(\boldsymbol{\theta} - \boldsymbol{\theta}_P)^{-1}]\} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n]\setminus[1]}^{1_{n-1}}.$$

We make $\theta_1 \rightarrow \infty$ in the last equality and we get

$$\begin{aligned} & \lim_{\theta_1 \rightarrow \infty} \frac{P_n(\boldsymbol{\theta}_{[n]})}{p_{[n]}(\theta_1 - \tilde{p}_1)} \\ &= [1 - R_n(\mathbf{0}, (\boldsymbol{\theta} - \boldsymbol{\theta}_P)_{[n]\setminus[1]}^{-1})] (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n]\setminus[1]}^{1_{n-1}} \end{aligned} \quad (114)$$

and, because P is an affine polynomial, we have

$$\begin{aligned} & \lim_{\theta_1 \rightarrow \infty} \frac{P_n(\boldsymbol{\theta}_{[n]})}{p_{[n]}(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_1} \\ &= \lim_{\theta_1 \rightarrow \infty} \frac{P(0, \boldsymbol{\theta}_{[n]\setminus[1]}) + \theta_1 \frac{\partial}{\partial \theta_1} (0, \boldsymbol{\theta}_{[n]\setminus[1]})}{p_{[n]}(\theta_1 - \tilde{p}_1)} \\ &= \frac{\frac{\partial}{\partial \theta_1} P_n(\boldsymbol{\theta}_{[n]})}{p_{[n]}} \end{aligned} \quad (115)$$

Finally the equalities (114) and (115) prove the equality (62).

Secondly, according to (25), with $\mathbf{z} = (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{-1}$, that is $z_i = (\theta_i - \tilde{p}_i)^{-1}$ or $\theta_i = \tilde{p}_i + z_i^{-1}, i \in [n]$, we have

$$R_n(\mathbf{z}_{[n]}) = 1 - \frac{1}{p_{[n]}} \left(\prod_{j=2}^n z_j \right) z_1 P_n(\tilde{p}_1 + z_1^{-1}, \dots, \tilde{p}_n + z_n^{-1}).$$

By deriving with respect to the variable z_1 , we obtain

$$\begin{aligned} & \frac{\partial}{\partial z_1} R_n(z_1, z_2, \dots, z_n) \\ &= -\frac{1}{p_{[n]}} \left(\prod_{j=2}^n z_j \right) \left[P_n(\tilde{p}_1 + z_1^{-1}, \dots, \tilde{p}_n + z_n^{-1}) - z_1^{-1} \frac{\partial}{\partial \theta_1} [P_n(\tilde{p}_1 + z_1^{-1}, \dots, \tilde{p}_n + z_n^{-1})] \right]. \end{aligned} \quad (116)$$

Because R_n is an affine polynomial with respect to the n variables z_1, z_2, \dots, z_n , we know that $\frac{\partial}{\partial z_1} R_n(z_1, z_2, \dots, z_n)$ is an affine polynomial with respect to the $n - 1$ variables z_2, \dots, z_n . Putting $z_1 = 0$ in the left-hand side of (116) and making $z_1 \rightarrow \infty$ in the right-hand side of (116), we get

$$\begin{aligned} & \frac{\partial}{\partial z_1} R_n(0, z_2, \dots, z_n) \\ &= -\frac{1}{p_{[n]}} \left(\prod_{j=2}^n z_j \right) [P_n(\tilde{p}_1, \tilde{p}_2 + z_2^{-1}, \dots, \tilde{p}_n + z_n^{-1})]. \end{aligned}$$

and

$$\begin{aligned} & \left(\prod_{j=2}^n z_j^{-1} \right) \frac{\partial}{\partial z_1} R_n(0, z_2, \dots, z_n) \\ &= -\frac{1}{p_{[n]}} [P_n(\tilde{p}_1, \tilde{p}_2 + z_2^{-1}, \dots, \tilde{p}_n + z_n^{-1})]. \end{aligned}$$

Substituting $z_j = (\theta_j - \tilde{p}_j)^{-1}, j = 2, \dots, n$, we obtain

$$\begin{aligned} & \prod_{j=2}^n (\theta_j - \tilde{p}_j) \frac{\partial}{\partial z_i} R_n(0, (\theta_2 - \tilde{p}_2)^{-1}, \dots, (\theta_n - \tilde{p}_n)^{-1}) \\ &= -\frac{1}{p_{[n]}} [P_n(\tilde{p}_1, \theta_2, \dots, \theta_n)]. \end{aligned}$$

Multiplying by $\frac{p_{[n]}}{p_{[1]}}$ we obtain (63). Thirdly, applying $\frac{1}{p_{[1]}} \frac{\partial}{\partial \theta_1}$ to the equality

$$P_n(\boldsymbol{\theta}) = \sum_{T \subset \{2, \dots, n\}} p_{[1] \cup T} \boldsymbol{\theta}^{[1]} \boldsymbol{\theta}_{[n] \setminus [1]}^T + \sum_{T \in \mathfrak{P}_n, T \cap [1] \neq [1]} p_T \boldsymbol{\theta}^T$$

we obtain (64). ■

Proof of Lemma 26. First. If $\exists T \in \mathfrak{P}_n, |T| = 2; r_T \neq 0$, then without loss of generality, we can studies the case $T = [2]$, that is $r_{1,2} \neq 0$. For $\mathbf{z} = (z_1, z_2, \mathbf{0}_{n-2})$, we get

$$\begin{aligned} [1 - R_n(z_1, z_2, \mathbf{0}_{n-2})]^{-\lambda} &= \sum_{\alpha \in \mathbb{N}^2} c_{(\alpha_1, \alpha_2, \mathbf{0}_{n-2}), \lambda}(R_n) z_1^{\alpha_1} z_2^{\alpha_2} \\ &= (1 - r_{1,2} z_1 z_2)^{-\lambda} \\ &= \sum_{l=0}^{\infty} \frac{(\lambda)_l r_{1,2}^l}{l!} z_1^l z_2^l. \end{aligned}$$

Then we have $c_{(l, l, \mathbf{0}_{n-2}), \lambda} = \frac{(\lambda)_l r_{1,2}^l}{l!} \neq 0, l \in \mathbb{N}$. Finally, $\forall \alpha_1 \in \mathbb{N}, \exists \alpha_{[n] \setminus \{1\}} = (\alpha_1, \mathbf{0}_{n-2})$ such that $c_{(\alpha_1, \alpha_{[n] \setminus \{1\}}), \lambda} = \frac{(\lambda)_{\alpha_1} r_{1,2}^{\alpha_1}}{\alpha_1!} \neq 0$, and $I_1(R_n) = \mathbb{N}$.

Second. Let $k \in \mathbb{N}$, such that $\forall T \in \mathfrak{P}_n, |T| \leq k < n; r_T = 0$, if $\exists T \in \mathfrak{P}_n, |T| = k + 1; r_T \neq 0$, then without loss of generality, we can studies the case $T = [k + 1]$, that is $r_{1, \dots, k+1} \neq 0$. For $\mathbf{z} = (z_1, \dots, z_{k+1}, \mathbf{0}_{n-k-1})$, we get

$$\begin{aligned} [1 - R_n(z_1, \dots, z_{k+1}, \mathbf{0}_{n-k-1})]^{-\lambda} &= \sum_{\alpha \in \mathbb{N}^2} c_{(\alpha_1, \dots, \alpha_{k+1}, \mathbf{0}_{n-k-1}), \lambda}(R_n) z_1^{\alpha_1} \dots z_{k+1}^{\alpha_{k+1}} \\ &= (1 - r_{1, \dots, k+1} z_1 \dots z_{k+1})^{-\lambda} \\ &= \sum_{l=0}^{\infty} \frac{(\lambda)_l r_{1, \dots, k+1}^l}{l!} z_1^l \dots z_{k+1}^l. \end{aligned}$$

Then we have $c_{(l \mathbf{1}_{k+1}, \mathbf{0}_{n-k-1}), \lambda} = \frac{(\lambda)_l r_{1, \dots, k+1}^l}{l!} \neq 0, l \in \mathbb{N}$,

Finally, $\forall \alpha_1 \in \mathbb{N}, \exists \alpha_{[n] \setminus \{1\}} = (\alpha_1 \mathbf{1}_k, \mathbf{0}_{n-k-1})$ such that $c_{(\alpha_1, \alpha_{[n] \setminus \{1\}}), \lambda} = \frac{(\lambda)_l r_{1, \dots, k+1}^l}{l!} \neq 0$, and $I_1(R_n) = \mathbb{N}$.

Third. Now, if $\forall T \in \mathfrak{P}_n, |T| \leq n; r_T = 0$, then $R_n = 0$, and $P_n(\boldsymbol{\theta}_n) = p_{[n]} \prod_{i=1}^n (\theta_i - \tilde{p}_i) = \prod_{i=1}^n (1 + (-\tilde{p}_i)^{-1} \theta_i)$, because $P_n(\mathbf{0}_n) = 1$. In this case we have $\mathbf{X} = (X_1, \dots, X_n)$, with $X_i \sim \gamma_{(-\tilde{p}_i)^{-1}, \lambda}$, $X_i, i \in [n]$ being independent Therefore, $c_{(\alpha_1, \dots, \alpha_n), \lambda}(P_n) = 0$ unless $c_{(\mathbf{0}_n), \lambda}(P_n) = 1$, and $I_1(R_n) = \{0\}$

This completes the proof by finite induction. ■

Proof of Theorem 27. We have by (62), (63) and (65)

$$\begin{aligned} B(\boldsymbol{\theta}_{[n]\setminus[1]}) &= -\frac{1}{p_1} [P_n(0, \boldsymbol{\theta}_{[n]\setminus[1]}) + \tilde{p}_1 \frac{\partial P_n}{\partial \theta_1}(0, \boldsymbol{\theta}_{[n]\setminus[1]})] \\ &= -\tilde{p}_1 S_{[n]\setminus[1]}(\boldsymbol{\theta}_{[n]\setminus[1]}) - \frac{1}{p_1} P_n(0, \boldsymbol{\theta}_{[n]\setminus[1]}) \end{aligned}$$

We deduce

$$\mathfrak{z}_{n-1}(\boldsymbol{\theta}_{[n]\setminus[1]}) = -\tilde{p}_1 - \frac{1}{p_1} \frac{P_n(0, \boldsymbol{\theta}_{[n]\setminus[1]})}{S_{[n]\setminus[1]}(\boldsymbol{\theta}_{[n]\setminus[1]})}$$

We then obtain by (68),

$$L_{\mathbf{X}_{[n]\setminus[1]}}^{X_1=x_1}(\boldsymbol{\theta}_{[n]\setminus[1]}) = [S_{[n]\setminus[1]}(\boldsymbol{\theta}_{[n]\setminus[1]})]^{-\lambda} \frac{\exp\{-\tilde{p}_1 - \frac{1}{p_1} \frac{P_n(0, \boldsymbol{\theta}_{[n]\setminus[1]})}{S_{[n]\setminus[1]}(\boldsymbol{\theta}_{[n]\setminus[1]})} x_1\}}{\exp\{-\tilde{p}_1 - \frac{1}{p_1} x_1\}},$$

which gives (71). By (22) and (70), expanding $P_n(0, \boldsymbol{\theta}_{[n]\setminus[1]}) - S_{[n]\setminus[1]}(\boldsymbol{\theta}_{[n]\setminus[1]})$, we obtain (72).

By (22) and (70), we get

$$\begin{aligned} & - \left(\frac{P_n(0, \boldsymbol{\theta}_{[n]\setminus[1]})}{S_{[n]\setminus[1]}(\boldsymbol{\theta}_{[n]\setminus[1]})} - 1 \right) \frac{x_1}{p_1} \\ &= \left(1 - \frac{p_{[n]}(-\tilde{p}_1)(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n]\setminus\{1\}} \left\{ 1 - R_n \left[(-\tilde{p}_1), (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n]\setminus\{1\}} \right]^{-1} \right\}}{\frac{p_{[n]}}{p_1} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n]\setminus\{1\}} \left\{ 1 - R_{n-1}(S_{[n]\setminus[1]})(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n]\setminus\{1\}} \right\}^{-1}} \right) \frac{x_1}{p_1} \end{aligned}$$

By (22) and (70) we get

$$\begin{aligned}
& - \left(\frac{P_n(0, \boldsymbol{\theta}_{[n] \setminus \{1\}})}{S_{[n] \setminus \{1\}}(\boldsymbol{\theta}_{[n] \setminus \{1\}})} - 1 \right) \frac{x_1}{p_1} \\
& = \left(1 - \frac{p_{[n]}(-\tilde{p}_1)(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} \{1 - R_n[(-\tilde{p}_1), (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-1}\}}{p_1 (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} \{1 - R_{n-1}(S_{[n] \setminus \{1\}})[(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-1}\}} \right) \frac{x_1}{p_1} \\
& = (\tilde{p}_1 \{1 - R_n[(-\tilde{p}_1), (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-1}\}) \\
& + \frac{1}{p_1} \{1 - R_{n-1}(S_{[n] \setminus \{1\}})[(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-1}\} p_{[n]} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} \frac{x_1}{p_1} S_{[n] \setminus \{1\}}^{-1}(\boldsymbol{\theta}_{[n] \setminus \{1\}}) \\
& = (\tilde{p}_1 \{1 - \sum_{T \subset [n], |T| > 1} r_T[(-\tilde{p}_1), (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T}\}) \\
& + \frac{1}{p_1} \{1 - \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T[(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T}\} p_{[n]} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} \frac{x_1}{p_1} S_{[n] \setminus \{1\}}^{-1}(\boldsymbol{\theta}_{[n] \setminus \{1\}}) \\
& = (\tilde{p}_1 \{1 - \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}^{-T} - \sum_{T \subset [n] \setminus \{1\}, |T| > 0} r_{\{1\} \cup T} [(-\tilde{p}_1), (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-\{1\} \cup T}\}) \\
& + \frac{1}{p_1} \{1 - \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T[(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T}\} p_{[n]} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} \frac{x_1}{p_1} S_{[n] \setminus \{1\}}^{-1}(\boldsymbol{\theta}_{[n] \setminus \{1\}}) \\
& = (\tilde{p}_1 \{1 - \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T[(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T} - (-\tilde{p}_1)^{-1} \sum_{T \subset [n] \setminus \{1\}, |T| > 0} r_{\{1\} \cup T} [(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T}\}) \\
& + \frac{1}{p_1} \{1 - \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T[(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T}\} p_{[n]} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} \frac{x_1}{p_1} S_{[n] \setminus \{1\}}^{-1}(\boldsymbol{\theta}_{[n] \setminus \{1\}}) \\
& = (\{\tilde{p}_1 - \tilde{p}_1 \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T[(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T} + \sum_{T \subset [n] \setminus \{1\}, |T| > 0} r_{\{1\} \cup T} [(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T}\}) \\
& + \frac{1}{p_1} - \frac{1}{p_1} \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T[(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T} p_{[n]} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} \frac{x_1}{p_1} S_{[n] \setminus \{1\}}^{-1}(\theta_2, \dots, \theta_n)
\end{aligned}$$

then we have

$$\begin{aligned}
& - \left(\frac{P_n(0, \theta_2, \dots, \theta_n)}{S_{[n] \setminus \{1\}}(\theta_2, \dots, \theta_n)} - 1 \right) \frac{x_1}{p_1} \\
& = \left\{ \left(\tilde{p}_1 - \frac{1}{p_1} \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T [(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T} + \sum_{T \subset [n] \setminus \{1\}, |T| > 0} r_{\{1\} \cup T} [(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T} \right) \right. \\
& \quad \left. + \left(\frac{1}{p_1} - \frac{1}{p_1} \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T [(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T} \right) \right\} \\
& \times p_{[n]} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} \frac{x_1}{p_1} S_{[n] \setminus \{1\}}^{-1}(\theta_2, \dots, \theta_n) \\
& = \left[\left(\tilde{p}_1 + \frac{1}{p_1} \right) - \left(\tilde{p}_1 + \frac{1}{p_1} \right) \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T [(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T} \right. \\
& \quad \left. + \sum_{T \subset [n] \setminus \{1\}, |T| > 0} r_{\{1\} \cup T} [(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T} \right] \\
& \times p_{[n]} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} \frac{x_1}{p_1} S_{[n] \setminus \{1\}}^{-1}(\theta_2, \dots, \theta_n) \\
& \quad \left(\tilde{p}_1 + \frac{1}{p_1} \right) \left\{ 1 - \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T [(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T} \right\} \\
& = \left\{ \begin{aligned} & + \sum_{T \subset [n] \setminus \{1\}, |T|=1} r_{\{1\} \cup T} [(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T} \\ & + \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_{\{1\} \cup T} [(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T} \end{aligned} \right\} \\
& \times \frac{p_{[n]}}{p_1} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} x_1 S_{[n] \setminus \{1\}}^{-1}(\theta_2, \dots, \theta_n) \\
& \quad \left(\tilde{p}_1 + \frac{1}{p_1} \right) \left\{ 1 - \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T [(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T} \right\} \\
& = \left\{ \begin{aligned} & + \sum_{k=2}^n r_{\{1, k\}} (\theta_k - \tilde{p}_k)^{-1} \\ & + \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_{\{1\} \cup T} [(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T} \end{aligned} \right\} \\
& \times \frac{p_{[n]}}{p_1} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} x_1 S_{[n] \setminus \{1\}}^{-1}(\theta_2, \dots, \theta_n) \\
& = \left\{ \begin{aligned} & \left(\tilde{p}_1 + \frac{1}{p_1} \right) + \sum_{k=2}^n r_{\{1, k\}} (\theta_k - \tilde{p}_k)^{-1} \\ & + \sum_{T \subset [n] \setminus \{1\}, |T| > 1} \left[r_{\{1\} \cup T} - \left(\tilde{p}_1 + \frac{1}{p_1} \right) r_T \right] [(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-T} \end{aligned} \right\} \\
& \times \frac{p_{[n]}}{p_1} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} x_1 S_{[n] \setminus \{1\}}^{-1}(\theta_2, \dots, \theta_n)
\end{aligned}$$

and

$$\begin{aligned}
& - \left(\frac{P_n(0, \theta_2, \dots, \theta_n)}{S_{[n] \setminus [1]}(\theta_2, \dots, \theta_n)} - 1 \right) \frac{x_1}{p_1} \\
& = \left\{ \begin{aligned} & \left(\tilde{p}_1 + \frac{1}{p_1} \right) (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} + \sum_{k=2}^n r_{\{1, k\}} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1, k\}} \\ & + \sum_{T \subset [n] \setminus \{1\}, |T| > 1} \left[r_{\{1\} \cup T} - \left(\tilde{p}_1 + \frac{1}{p_1} \right) r_T \right] (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\} \setminus T} \end{aligned} \right\} \frac{p^{[n]} x_1 S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n)}{p_1}
\end{aligned}$$

and because

$$\begin{aligned}
& S_{[n] \setminus [1]}(\theta_2, \dots, \theta_n) \\
& = \frac{p^{[n]}}{p_1} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} \{1 - R_{n-1}(S_{[n] \setminus [1]}^{-1})[(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})_{[n] \setminus \{1\}}]^{-1}\} \\
& = \frac{p^{[n]}}{p_1} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} \left\{ 1 - \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T [(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})]^{-T} \right\} \\
& = \frac{p^{[n]}}{p_1} \left\{ (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} - \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} [(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})]^{-T} \right\} \\
& = \frac{p^{[n]}}{p_1} \left\{ (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} - \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\} \setminus T} \right\}
\end{aligned}$$

we get

$$\begin{aligned}
& (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} \\
&= \frac{p_1}{p^{[n]}} S_{[n] \setminus [1]}(\theta_2, \dots, \theta_n) + \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{\{[n] \setminus \{1\}\} \setminus T} \\
& (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n) \\
&= \frac{p_1}{p^{[n]}} + \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{\{[n] \setminus \{1\}\} \setminus T} S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n) \\
& \times (-\boldsymbol{\theta}_{P_n})^{-\{[n] \setminus \{1\}\}} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n) \\
&= \frac{p_1}{p^{[n]}} (-\boldsymbol{\theta}_{P_n})^{-\{[n] \setminus \{1\}\}} + \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T (-\boldsymbol{\theta}_{P_n})^{-\{[n] \setminus \{1\}\}} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{\{[n] \setminus \{1\}\} \setminus T} S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n) \\
& \times (-\boldsymbol{\theta}_{P_n})^{-\{[n] \setminus \{1\}\}} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\}} S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n) \\
&= \frac{p_1}{p^{[n]}} (-\boldsymbol{\theta}_{P_n})^{-\{[n] \setminus \{1\}\}} \\
& + \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T (-\boldsymbol{\theta}_{P_n})^{-T} (-\boldsymbol{\theta}_{P_n})^{-\{[n] \setminus \{1\}\} \setminus T} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{\{[n] \setminus \{1\}\} \setminus T} S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n) \\
& \times (1 + (-\tilde{p}_1)^{-1} \theta_1, \dots, 1 + (-\tilde{p}_n)^{-1} \theta_n)^{[n] \setminus \{1\}} S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n) \\
&= \frac{p_1}{p^{[n]}} (-\boldsymbol{\theta}_{P_n})^{-\{[n] \setminus \{1\}\}} \\
& + \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T (-\boldsymbol{\theta}_{P_n})^{-T} (1 + (-\tilde{p}_1)^{-1} \theta_1, \dots, 1 + (-\tilde{p}_n)^{-1} \theta_n)^{\{[n] \setminus \{1\}\} \setminus T} S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n) \\
& \times (1 + (-\tilde{p}_2)^{-1} \theta_2) \dots (1 + (-\tilde{p}_n)^{-1} \theta_n) S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n) \\
&= \frac{p_1}{p^{[n]}} (-\tilde{p}_2)^{-1} \dots (-\tilde{p}_n)^{-1} \\
& + \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T (-\boldsymbol{\theta}_{P_n})^{-T} (1 + (-\tilde{p}_1)^{-1} \theta_1, \dots, 1 + (-\tilde{p}_n)^{-1} \theta_n)^{\{[n] \setminus \{1\}\} \setminus T} S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n).
\end{aligned} \tag{117}$$

we have

$$\begin{aligned}
& - \left(\frac{P_n(0, \theta_2, \dots, \theta_n)}{S_{[n] \setminus [1]}(\theta_2, \dots, \theta_n)} - 1 \right) \frac{x_1}{p_1} \\
& = \left\{ \begin{aligned} & \left(\tilde{p}_1 + \frac{1}{p_1} \right) \left[\frac{p_1}{p_{[n]}} S_{[n] \setminus [1]}(\theta_2, \dots, \theta_n) + \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{\{[n] \setminus \{1\}\} \setminus T} \right] \\ & + \sum_{k=2}^n r_{\{1, k\}} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1, k\}} + \sum_{T \subset [n] \setminus \{1\}, |T| > 1} [r_{\{1\} \cup T} - \left(\tilde{p}_1 + \frac{1}{p_1} \right) r_T] (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\} \setminus T} \end{aligned} \right\} \\
& \times \frac{p_{[n]}}{p_1} x_1 S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n) \\
& = \left\{ \begin{aligned} & \left(\tilde{p}_1 + \frac{1}{p_1} \right) \frac{p_1}{p_{[n]}} S_{[n] \setminus [1]}(\theta_2, \dots, \theta_n) + \left(\tilde{p}_1 + \frac{1}{p_1} \right) \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_T (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{\{[n] \setminus \{1\}\} \setminus T} \\ & + \sum_{k=2}^n r_{\{1, k\}} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1, k\}} + \sum_{T \subset [n] \setminus \{1\}, |T| > 1} \left[r_{\{1\} \cup T} - \left(\tilde{p}_1 + \frac{1}{p_1} \right) r_T \right] (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\} \setminus T} \end{aligned} \right\} \\
& \times \frac{p_{[n]}}{p_1} x_1 S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n) \\
& = \left\{ \begin{aligned} & \left(\tilde{p}_1 + \frac{1}{p_1} \right) \frac{p_1}{p_{[n]}} S_{[n] \setminus [1]}(\theta_2, \dots, \theta_n) + \sum_{k=2}^n r_{\{1, k\}} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1, k\}} \\ & + \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_{\{1\} \cup T} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\} \setminus T} \end{aligned} \right\} \frac{p_{[n]}}{p_1} x_1 S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n) \\
& = x_1 \left(\tilde{p}_1 + \frac{1}{p_1} \right) + \sum_{k=2}^n r_{\{1, k\}} \frac{p_{[n]}}{p_1} x_1 (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1, k\}} S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n) \\
& + \sum_{T \subset [n] \setminus \{1\}, |T| > 1} r_{\{1\} \cup T} \frac{p_{[n]}}{p_1} x_1 (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\} \setminus T} S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n) \\
& = x_1 \left(\tilde{p}_1 + \frac{1}{p_1} \right) + \sum_{T \subset [n] \setminus \{1\}, |T| > 0} r_{\{1\} \cup T} \frac{p_{[n]}}{p_1} x_1 (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\} \setminus T} S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n)
\end{aligned}$$

and

$$\begin{aligned}
& L_{(X_2, \dots, X_n)}^{X_1 = x_1}(\theta_2, \dots, \theta_n) \\
& = [S_{[n] \setminus [1]}(\theta_2, \dots, \theta_n)]^{-\lambda} \\
& \exp \left\{ x_1 \left(\tilde{p}_1 + \frac{1}{p_1} \right) + \sum_{T \subset [n] \setminus \{1\}, |T| > 0} r_{\{1\} \cup T} \frac{p_{[n]}}{p_1} x_1 (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\} \setminus T} S_{[n] \setminus [1]}^{-1}(\theta_2, \dots, \theta_n) \right\}
\end{aligned}$$

Because

$$\begin{aligned}
1 &= L_{(X_2, \dots, X_n)}^{X_1=x_1}(0, \dots, 0) \\
&= \exp \left\{ x_1 \left(\tilde{p}_1 + \frac{1}{p_1} \right) + \sum_{T \subset [n] \setminus \{1\}, |T| > 0} r_{\{1\} \cup T} \frac{p_{[n]}}{p_1} x_1 (-\tilde{p}_2, \dots, -\tilde{p}_n)^{\{[n] \setminus \{1\}\} \setminus T} \right\} \\
&= \exp \left\{ \left(\tilde{p}_1 + \frac{1}{p_1} \right) x_1 \right\} \exp \left\{ \sum_{T \subset [n] \setminus \{1\}, |T| > 0} r_{\{1\} \cup T} \frac{p_{[n]}}{p_1} x_1 (-\tilde{p}_2, \dots, -\tilde{p}_n)^{\{[n] \setminus \{1\}\} \setminus T} \right\}
\end{aligned}$$

and

$$\begin{aligned}
&L_{(X_2, \dots, X_n)}^{X_1=x_1}(\boldsymbol{\theta}_{[n] \setminus [1]}) \\
&= S_{[n] \setminus [1]}^{-\lambda}(\boldsymbol{\theta}_{[n] \setminus [1]}) \\
&\times \mathbf{e}^{\{x_1(\tilde{p}_1 + \frac{1}{p_1}) + \sum_{T \subset [n] \setminus \{1\}, |T| > 0} r_{\{1\} \cup T} \frac{p_{[n]}}{p_1} x_1(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\} \setminus T} S_{[n] \setminus [1]}^{-1}(\boldsymbol{\theta}_2, \dots, \boldsymbol{\theta}_n)\}} \\
&= S_{[n] \setminus [1]}^{-\lambda}(\boldsymbol{\theta}_{[n] \setminus [1]}) \\
&\times \mathbf{e}^{\{\sum_{T \subset [n] \setminus \{1\}, |T| > 0} r_{\{1\} \cup T} \frac{p_{[n]}}{p_1} x_1(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\} \setminus T} S_{[n] \setminus [1]}^{-1}(\boldsymbol{\theta}_{[n] \setminus [1]}) - r_{\{1\} \cup T} \frac{p_{[n]}}{p_1} x_1(-\tilde{\mathbf{p}})^{\{[n] \setminus \{1\}\} \setminus T}\}} \\
&= S_{[n] \setminus [1]}^{-\lambda}(\boldsymbol{\theta}_{[n] \setminus [1]}) \\
&\times \mathbf{e}^{\{\sum_{T \subset [n] \setminus \{1\}, |T| > 0} r_{\{1\} \cup T} \frac{p_{[n]}}{p_1} x_1[(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\} \setminus T} S_{[n] \setminus [1]}^{-1}(\boldsymbol{\theta}_2, \dots, \boldsymbol{\theta}_n) - (-\tilde{\mathbf{p}})^{\{[n] \setminus \{1\}\} \setminus T}]\}} \\
&= S_{[n] \setminus [1]}^{-\lambda}(\boldsymbol{\theta}_{[n] \setminus [1]}) \\
&\times \mathbf{e}^{\{\sum_{T \subset [n] \setminus \{1\}, |T| > 0} r_{\{1\} \cup T} \frac{p_{[n]}}{p_1} x_1^{[n] \setminus \{1\} \setminus T} (-\tilde{\mathbf{p}})^{\{[n] \setminus \{1\}\} \setminus T} [(-\tilde{\mathbf{p}})^{-\{[n] \setminus \{1\} \setminus T\}} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{[n] \setminus \{1\} \setminus T} S_{[n] \setminus [1]}^{-1}(\boldsymbol{\theta}_2, \dots, \boldsymbol{\theta}_n) - 1]\}} \\
&= S_{[n] \setminus [1]}^{-\lambda}(\boldsymbol{\theta}_{[n] \setminus [1]}) \\
&\times \mathbf{e}^{\{\sum_{T \subset [n] \setminus \{1\}, |T| > 0} r_{\{1\} \cup T} \frac{p_{[n]}}{p_1} x_1(-\tilde{\mathbf{p}})^{\{[n] \setminus \{1\}\} \setminus T} [(1_n + (-\boldsymbol{\theta}_{P_n})^{-1} \boldsymbol{\theta}_{[n]})^{[n] \setminus \{1\} \setminus T} S_{[n] \setminus [1]}^{-1}(\boldsymbol{\theta}_{[n] \setminus [1]}) - 1]\}} \\
&= S_{[n] \setminus [1]}^{-\lambda}(\boldsymbol{\theta}_{[n] \setminus [1]}) \\
&\times \mathbf{e}^{\{\sum_{T \subset [n] \setminus \{1\}, 0 < |T|} r_{\{1\} \cup T} \frac{p_{[n]}}{p_1} x_1(-\boldsymbol{\theta}_{P_n})^{\{[n] \setminus \{1\}\} \setminus T} [(1_n + (-\boldsymbol{\theta}_{P_n})^{-1} \boldsymbol{\theta}_{[n]})^{[n] \setminus \{1\} \setminus T} S_{[n] \setminus [1]}^{-1}(\boldsymbol{\theta}_{[n] \setminus [1]}) - 1]\}}
\end{aligned}$$

Unless $Rn = 0$, for $T \in \mathfrak{P}([n] \setminus [1])$, $\bar{T} = [n] \setminus T$, and if there is no ambiguity, for simplicity we denote $S_T(\boldsymbol{\theta}_T)$ by S_T , the polynomial defined by

$$\begin{aligned}
S_T(\boldsymbol{\theta}_T) &= \frac{1}{p_{\bar{T}}} \left(\frac{\partial}{\partial \boldsymbol{\theta}} \right)^{\bar{T}} (P_n(\boldsymbol{\theta})) \\
&= \sum_{T' \in P(T)} \frac{p_{\bar{T} \cup T'}}{p_{\bar{T}}} \boldsymbol{\theta}^{T'}
\end{aligned}$$

$$\begin{aligned}
S_T(\boldsymbol{\theta}_T) &= \frac{1}{p_{\bar{T}}} \left(\frac{\partial}{\partial \boldsymbol{\theta}} \right)^{\bar{T}} (P_n(\boldsymbol{\theta})) \\
&= (-\tilde{p}_T)^{-1} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^T \left(1 - \sum_{T' \in \mathfrak{P}(T), |T'| > 1} r_{T'} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{-T'} \right) \\
&= (-\tilde{p}_T)^{-1} [(\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^T - \sum_{T' \in \mathfrak{P}(T), |T'| > 1} r_{T'} (\boldsymbol{\theta}_{[n]} - \boldsymbol{\theta}_{P_n})^{T \setminus T'}] \\
&= (-\tilde{p}_T)^{-1} [(-\boldsymbol{\theta}_{P_n})^T \mathbf{S}^T - \sum_{T' \in \mathfrak{P}(T), |T'| > 1} r_{T'} (-\boldsymbol{\theta}_{P_n})^{T \setminus T'} \mathbf{S}^{T \setminus T'}]
\end{aligned}$$

We have

$$S_T(\boldsymbol{\theta}_T) = \sum_{T' \in \mathfrak{P}(T)} \frac{p_{\bar{T} \cup T'}}{p_{\bar{T}}} \boldsymbol{\theta}^{T'} = \sum_{T' \in P(T)} q_{T'} \boldsymbol{\theta}^{T'},$$

with $q_{T'} = \frac{p_{\bar{T} \cup T'}}{p_{\bar{T}}}$, and for $T' \in \mathfrak{P}(T)$, we have

$$\tilde{q}_{T'} = -\frac{q_{T \setminus T'}}{q_T} = \frac{p_{\bar{T} \cup T \setminus T'}}{p_{\bar{T}}} = \frac{p_{\bar{T}'}}{p_{[n]}} = \tilde{p}_{T'}.$$

Therefore, we have

$$\tilde{b}_{T'}(S_T) = \tilde{b}_{T'},$$

and if $\gamma_{(P, \lambda)}$ is an infinitely divisible gamma distribution, $\gamma_{(S_T, \lambda)}$ is also an infinitely divisible gamma distribution.

$$S_T = (-\tilde{p}_T)^{-1} [(-\tilde{\mathbf{p}})^T \mathbf{S}^T - \sum_{T' \in \mathfrak{P}(T), |T'| > 1} r_{T'} (-\tilde{\mathbf{p}})^{T \setminus T'} \mathbf{S}^{T \setminus T'}]$$

or

$$(-\tilde{\mathbf{p}})^T \mathbf{S}^T = (-\tilde{p}_T) S_T + \sum_{T' \in \mathfrak{P}(T), |T'| > 1} r_{T'} (-\tilde{\mathbf{p}})^{T \setminus T'} \mathbf{S}^{T \setminus T'}$$

■

Proof of Corollary 28. From Proof of Corollary (17), we have $c_{\alpha, \lambda}(R_n) = 0$ unless $\alpha = k \mathbf{1}_n$, $k \in \mathbb{N}$ in which case $c_{k \mathbf{1}_n, \lambda}(R_n) = \frac{(\lambda)_k}{k!} (qp^{-n})^k$. We therefore have

$$\begin{aligned}
L_{\mathbf{X}_{[n] \setminus [k]}^{\mathbf{X}_{[k]} = \mathbf{x}_{[k]}}}(\boldsymbol{\theta}_{[n] \setminus [k]}) &= \left[\prod_{i=k+1}^n (1 + p\theta_i) \right]^{-\lambda} \frac{\sum_{k=0}^{\infty} \frac{(\mathbf{x}_k)^{k \mathbf{1}_k}}{\{(\lambda)_k\}^k} \frac{(\lambda)_k}{k!} (qp^{-n})^k \left(\boldsymbol{\theta}_{[n]} + \frac{1}{p} \mathbf{1}_n \right)_{n-k}^{-k \mathbf{1}_{n-k}}}{\sum_{k=0}^{\infty} \frac{(\mathbf{x}_k)^{k \mathbf{1}_k}}{\{(\lambda)_k\}^k} \frac{(\lambda)_k}{k!} (qp^{-n})^k \left(\frac{1}{p} \mathbf{1}_n \right)_{n-k}^{-k \mathbf{1}_{n-k}}} \\
&= \left[\prod_{i=k+1}^n (1 + p\theta_i) \right]^{-\lambda} \frac{\sum_{k=0}^{\infty} \frac{1}{\{(\lambda)_k\}^{k-1}} \frac{1}{k!} \left(qp^{-k} \mathbf{x}_k^{[k]} \prod_{i=k+1}^n (1 + p\theta_i)^{-1} \right)^k}{\sum_{k=0}^{\infty} \frac{1}{\{(\lambda)_k\}^k} \frac{(\lambda)_k}{k!} \left(qp^{-k} \mathbf{x}_k^{[k]} \right)^k}
\end{aligned}$$

and the definition of F_{k-1} gives formula (76). ■

Proof of Corollary 29. Doing $k = 1$ in Formula (76) we get (77). Formula (78) comes from the definition of \exp .

A second proof can be given by application of Theorem 24. We successively have $\mathbf{G}(R_n, u_1) = \sum_{\alpha_1 \in \mathbb{N}} \frac{u_1^{\alpha_1}}{\alpha_1!} = \exp(u_1)$, $S_{n-1}(\boldsymbol{\theta}_{[n] \setminus [1]}) = \prod_{i=2}^n (1 + p\theta_i)$, $B(\boldsymbol{\theta}_{[n] \setminus [1]}) = qp^{-1}$ and

$\mathfrak{z}(\boldsymbol{\theta}_{[n] \setminus [1]}) = qp^{-1}[\prod_{i=2}^n (1 + p\theta_i)]^{-1}$. As a result, by application of (68) we get (77).

Another proof of this result is given by (74) as follows. In this case, we have $P_n(\boldsymbol{\theta}) = \frac{-q}{p} + \frac{1}{p} \prod_{i=1}^n (1 + p\theta_i)$, $p_T = p^{|T|-1}$, $p_{[n]} = p^{n-1}$, $\tilde{p}_T = -p^{-|T|}$, $\tilde{p}_{\{i\}} = -p^{-1}$, $\boldsymbol{\theta}_{P_n} = \tilde{\mathbf{p}} = -p^{-1}\mathbf{1}_n$, $-\tilde{p}_{[n] \setminus [1]} = p^{-(n-1)}$, $S_{[n] \setminus \{1\}}(\boldsymbol{\theta}_{[n] \setminus [1]}) = \prod_{i=2}^n (1 + p\theta_i)$, and

$$P_n(\boldsymbol{\theta}) = p_{[n]} \prod_{i=1}^n (\theta_i + (-p^{-1})) [1 - qp^{-n} \prod_{i=1}^n (\theta_i + (-p^{-1}))^{-1}]$$

hence we have $r_T = 0$, if $1 < |T| < n$ and $r_{[n]} = qp^{-n}$. We deduce from (74)

$$\begin{aligned} L_{\mathbf{X}_{[n] \setminus [1]}}^{X_1=x_1} &= S_{[n] \setminus [1]}^{-\lambda} \mathbf{e}^{\{qp^{-1}x_1[S_{[n] \setminus [1]}^{-1}-1]\}} \\ &= \sum_{k=0}^{\infty} \frac{(qp^{-1}x_1)^k}{k!} \exp(-qp^{-1}x_1) S_{[n] \setminus [1]}^{-(\lambda+k)}. \end{aligned}$$

■

Proof of Theorem 30. Indeed, $X_1 \sim \gamma_{(p,\lambda)}$, let $V_1 \sim \mathcal{P}(qp^{-1}X_1)$, we have $P(V_1 = k | X_1 = x_1) = \frac{(qp^{-1}x_1)^k}{k!} \exp(-qp^{-1}x_1)$. Clearly the variable $X_i, i = 2 \dots, n$ are conditionally independent and $X_i | (X_1 = x_1) \sim \gamma_{(p,\lambda+V_1)}$. We have

$$L_{\mathbf{X}_{[n] \setminus [1]}}^{X_1=x_1}(\boldsymbol{\theta}_{[n] \setminus [1]}) = \sum_{v_1=0}^{\infty} P(V_1 = v_1 | X_1 = x_1) L_{\mathbf{X}_{[n] \setminus [1]}}^{X_1=x_1}(\boldsymbol{\theta}_{[n] \setminus [1]}),$$

and Formula (78) is verified. Finally we have $\mathbf{X}_{[n]} \sim \gamma_{(P_n,\lambda)}$, with $P_n(\boldsymbol{\theta}_{[n]}) = \frac{-q}{p} + \frac{1}{p} \prod_{i=1}^n (1 + p\theta_i)$. ■

Proof of Theorem 32. From (74), we deduce the Lt of the conditional distribution of $X_2 | X_1 = x_1$, for $(X_1, X_2) \sim \gamma_{(P_2,\lambda)}$, with $P_2(\theta_1, \theta_2) = 1 + p_1\theta_1 + p_2\theta_2 + p_{1,2}\theta_1\theta_2$ with $p_1 > 0$, $p_2 > 0$, $p_{1,2} > 0$, $\tilde{b}_{1,2} = \tilde{b}_{1,2}(P_2) = p_1p_2/p_{1,2}^2 - 1/p_{1,2} > 0$. We have $S_2(\theta_2) = 1 + \frac{p_{1,2}}{p_1}\theta_2$ and

$$L_{X_2}^{X_1=x_1} = S_2^{-\lambda} \exp\left\{\frac{\tilde{b}_{1,2}}{(-\tilde{p}_2)}x_1(S_2^{-1} - 1)\right\}. \quad (118)$$

By expanding the exponential function, we obtain the following expansion:

$$L_{X_2}^{X_1=x_1} = \sum_{v_1 \in \mathbb{N}} \mathbf{P}(V_1 = v_1) S_2^{-(\lambda+v_1)}. \quad (119)$$

Equality (119) proves that $X_2|X_1$ has distribution $\gamma_{(S_2, \lambda+V_1)}$, then $\mathbf{X}_{[2]} = (X_1, X_2) \sim \gamma_{(P_2, \lambda)}$. ■

Proof of Theorem 34. Hence for $n = 3$, we have from definition of $S_T, S_{2,3}(\theta_2, \theta_3) = 1 + \frac{p_{1,2}}{p_1}\theta_2 + \frac{p_{1,3}}{p_1}\theta_3 + \frac{p_{1,2,3}}{p_1}\theta_2\theta_3$, $S_2(\theta_2) = 1 + \frac{p_{1,2,3}}{p_{1,3}}\theta_2$, $S_3(\theta_3) = 1 + \frac{p_{1,2,3}}{p_{1,2}}\theta_3$, and from (75) and (28), we get

$$L_{(X_2, X_3)}^{X_1=x_1} = S_{2,3}^{-\lambda} \exp\{(-\tilde{p}_{2,3})^{-1} x_1 [\tilde{b}_{\{1,2\}}(-\tilde{p}_3) S_3 S_{2,3}^{-1} + \tilde{b}_{\{1,3\}}(-\tilde{p}_2) S_2 S_{2,3}^{-1} + \tilde{b}_{\{1,2,3\}} S_{2,3}^{-1} - C]\} \quad (120)$$

To compute another expression for $L_{(X_2, X_3)}^{X_1=x_1}$ we use (38), we then have

$$(-\tilde{p}_2)(-\tilde{p}_3) S_2 S_3 = (-\tilde{p}_{2,3}) S_{2,3} + \tilde{b}_{2,3}, \quad (121)$$

By respectively dividing (121) by $S_3 S_{2,3}$ and $S_2 S_{2,3}$, we successively obtain

$$(-\tilde{p}_3)(-\tilde{p}_2) S_2 S_{2,3}^{-1} = (-\tilde{p}_{2,3}) S_3^{-1} + \tilde{b}_{2,3} S_3^{-1} S_{2,3}^{-1}, \quad (122)$$

$$(-\tilde{p}_2)(-\tilde{p}_3) S_3 S_{2,3}^{-1} = (-\tilde{p}_{2,3}) S_2^{-1} + \tilde{b}_{2,3} S_2^{-1} S_{2,3}^{-1}, \quad (123)$$

Using (122) and (123) into (120), we get

$$L_{(X_2, X_3)}^{X_1=x_1} = S_{2,3}^{-\lambda} \exp\{x_1 [\alpha_1 S_2^{-1} + \alpha_2 S_3^{-1} + \alpha_3 S_{2,3}^{-1} + \alpha_4 S_2^{-1} S_{2,3}^{-1} + \alpha_5 S_3^{-1} S_{2,3}^{-1} - C]\},$$

and by the condition $L_{(X_2, X_3)}^{X_1=x_1}(0, 0) = 1$, we obtain

$$L_{(X_2, X_3)}^{X_1=x_1} = S_{2,3}^{-\lambda} \exp\{x_1 [\alpha_1 (S_2^{-1} - 1) + \alpha_2 (S_3^{-1} - 1) + \alpha_3 (S_{2,3}^{-1} - 1) + \alpha_4 (S_2^{-1} S_{2,3}^{-1} - 1) + \alpha_5 (S_3^{-1} S_{2,3}^{-1} - 1)]\}.$$

We expand exp, and we get (79). Equality (79) proves that $\mathbf{X}_{[3]} = (X_1, X_2, X_3) \sim \gamma_{(P_3, \lambda)}$.

■

Proof of Theorem 36. We use Theorem (32). Let $X'_2 \sim \gamma_{(\frac{p_{1,2}}{p_1}, \lambda+V_3+V_4+V_5)}$, let $\alpha_6 = \frac{\tilde{b}_{2,3}(S_{2,3})}{(-\tilde{p}_3(S_{2,3}))} = \frac{\tilde{b}_{2,3}}{(-\tilde{p}_3)}$, and $V_6 \sim \mathcal{P}(\alpha_6 X'_2)$, let $X'_3 \sim \gamma_{(S_3, \lambda+V_3+V_4+V_5+V_6)}$, then $(X'_2, X'_3) \sim \gamma_{(S_{2,3}, \lambda + V_3 + V_4 + V_5)}$. ■

Proof of Theorem 39. Hence for $n = 4$, from (75), (29) and (37), we get

$$(-\tilde{p}_{2,3}) S_{2,3} = [(-\tilde{p}_2) (-\tilde{p}_3) S_2 S_3 - \tilde{b}_{2,3}], \quad (124)$$

$$(-\tilde{p}_{2,4}) S_{2,4} = [(-\tilde{p}_2) (-\tilde{p}_4) S_2 S_4 - \tilde{b}_{2,4}], \quad (125)$$

$$(-\tilde{p}_{3,4}) S_{3,4} = [(-\tilde{p}_3) (-\tilde{p}_4) S_3 S_4 - \tilde{b}_{3,4}], \quad (126)$$

and

$$\begin{aligned} L_{(X_2, X_3, X_4)}^{X_1=x_1} = \\ S_{2,3,4}^{-\lambda} \exp\{(-\tilde{p}_{2,3,4})^{-1} x_1 [\tilde{b}_{1,2}(-\tilde{p}_{3,4}) S_{3,4} S_{2,3,4}^{-1} + \tilde{b}_{1,3}(-\tilde{p}_{2,4}) S_{2,4} S_{2,3,4}^{-1} + \tilde{b}_{1,4}(-\tilde{p}_{2,3}) S_{2,3} S_{2,3,4}^{-1} \\ + \tilde{b}_{1,2,3}(-\tilde{p}_4) S_4 S_{2,3,4}^{-1} + \tilde{b}_{1,2,4}(-\tilde{p}_3) S_3 S_{2,3,4}^{-1} + \tilde{b}_{1,3,4}(-\tilde{p}_2) S_2 S_{2,3,4}^{-1} + \tilde{b}_{1,2,3,4} S_{2,3,4}^{-1} - C]\} \end{aligned} \quad (127)$$

To compute $L_{(X_2, X_3, X_4)}^{X_1=x_1}$, we need to express the following expressions in terms of inverses of $S_T, T \in \mathfrak{P}(\{2, 3, 4\})$

$$S_{3,4} S_{2,3,4}^{-1}, S_{2,4} S_{2,3,4}^{-1}, S_{2,3} S_{2,3,4}^{-1}, S_4 S_{2,3,4}^{-1}, S_3 S_{2,3,4}^{-1}, S_2 S_{2,3,4}^{-1}.$$

From (38), we then successively have

$$(-\tilde{p}_2) (-\tilde{p}_3) S_2 S_3 = (-\tilde{p}_{2,3}) S_{2,3} + \tilde{b}_{2,3}, \quad (128)$$

$$(-\tilde{p}_2) (-\tilde{p}_4) S_2 S_4 = (-\tilde{p}_{2,4}) S_{2,4} + \tilde{b}_{2,4}, \quad (129)$$

$$(-\tilde{p}_3) (-\tilde{p}_4) S_3 S_4 = (-\tilde{p}_{3,4}) S_{3,4} + \tilde{b}_{3,4}, \quad (130)$$

From (37), we also successively have

$$\begin{aligned} S_{2,3,4} &= (-\tilde{p}_{2,3,4})^{-1} \left[\left((-\tilde{p}_2) (-\tilde{p}_3) S_2 S_3 - \tilde{b}_{2,3} \right) (-\tilde{p}_4) S_4 - \tilde{b}_{2,4} (-\tilde{p}_3) S_3 - \tilde{b}_{3,4} (-\tilde{p}_2) S_2 - \tilde{b}_{2,3,4} \right] \\ &= (-\tilde{p}_{2,3,4})^{-1} \left[(-\tilde{p}_{2,3}) S_{2,3} (-\tilde{p}_4) S_4 - \tilde{b}_{2,4} (-\tilde{p}_3) S_3 - \tilde{b}_{3,4} (-\tilde{p}_2) S_2 - \tilde{b}_{2,3,4} \right], \end{aligned} \quad (131)$$

$$\begin{aligned} S_{2,3,4} &= (-\tilde{p}_{2,3,4})^{-1} \left[\left((-\tilde{p}_2) (-\tilde{p}_4) S_2 S_4 - \tilde{b}_{2,4} \right) (-\tilde{p}_3) S_3 - \tilde{b}_{2,3} (-\tilde{p}_4) S_4 - \tilde{b}_{3,4} (-\tilde{p}_2) S_2 - \tilde{b}_{2,3,4} \right] \\ &= (-\tilde{p}_{2,3,4})^{-1} \left[(-\tilde{p}_{2,4}) S_{2,4} (-\tilde{p}_3) S_3 - \tilde{b}_{2,3} (-\tilde{p}_4) S_4 - \tilde{b}_{3,4} (-\tilde{p}_2) S_2 - \tilde{b}_{2,3,4} \right], \end{aligned} \quad (132)$$

$$\begin{aligned}
S_{2,3,4} &= (-\tilde{p}_{2,3,4})^{-1} \left[\left((-\tilde{p}_3) (-\tilde{p}_4) S_3 S_4 - \tilde{b}_{3,4} \right) (-\tilde{p}_2) S_2 - \tilde{b}_{2,3} (-\tilde{p}_4) S_4 - \tilde{b}_{2,4} (-\tilde{p}_3) S_3 - \tilde{b}_{2,3,4} \right] \\
&= (-\tilde{p}_{2,3,4})^{-1} \left[(-\tilde{p}_{3,4}) S_{3,4} (-\tilde{p}_2) S_2 - \tilde{b}_{2,3} (-\tilde{p}_4) S_4 - \tilde{b}_{2,4} (-\tilde{p}_3) S_3 - \tilde{b}_{2,3,4} \right], \quad (133)
\end{aligned}$$

and

$$(-\tilde{p}_4) (-\tilde{p}_{2,3}) S_4 S_{2,3} = (-\tilde{p}_{2,3,4}) S_{2,3,4} + \tilde{b}_{2,4} (-\tilde{p}_3) S_3 + \tilde{b}_{3,4} (-\tilde{p}_2) S_2 + \tilde{b}_{2,3,4}, \quad (134)$$

$$(-\tilde{p}_3) (-\tilde{p}_{2,4}) S_3 S_{2,4} = (-\tilde{p}_{2,3,4}) S_{2,3,4} + \tilde{b}_{2,3} (-\tilde{p}_4) S_4 + \tilde{b}_{3,4} (-\tilde{p}_2) S_2 + \tilde{b}_{2,3,4}, \quad (135)$$

$$(-\tilde{p}_2) (-\tilde{p}_{3,4}) S_2 S_{3,4} = (-\tilde{p}_{2,3,4}) S_{2,3,4} + \tilde{b}_{2,3} (-\tilde{p}_4) S_4 + \tilde{b}_{2,4} (-\tilde{p}_3) S_3 + \tilde{b}_{2,3,4}, \quad (136)$$

By respectively dividing (128) by $S_3 S_{2,3}$ and $S_2 S_{2,3}$, (129) by $S_4 S_{2,4}$ and $S_2 S_{2,4}$, and (130) by $S_4 S_{3,4}$ and $S_3 S_{3,4}$, we successively obtain

$$(-\tilde{p}_2) (-\tilde{p}_3) \frac{S_2}{S_{2,3}} = (-\tilde{p}_{2,3}) \frac{1}{S_3} + \tilde{b}_{2,3} \frac{1}{S_3 S_{2,3}}, \quad (137)$$

$$(-\tilde{p}_2) (-\tilde{p}_3) \frac{S_3}{S_{2,3}} = (-\tilde{p}_{2,3}) \frac{1}{S_2} + \tilde{b}_{2,3} \frac{1}{S_2 S_{2,3}}, \quad (138)$$

$$(-\tilde{p}_2) (-\tilde{p}_4) \frac{S_2}{S_{2,4}} = (-\tilde{p}_{2,4}) \frac{1}{S_4} + \tilde{b}_{2,4} \frac{1}{S_4 S_{2,4}}, \quad (139)$$

$$(-\tilde{p}_2) (-\tilde{p}_4) \frac{S_4}{S_{2,4}} = (-\tilde{p}_{2,4}) \frac{1}{S_2} + \tilde{b}_{2,4} \frac{1}{S_2 S_{2,4}}, \quad (140)$$

$$(-\tilde{p}_3) (-\tilde{p}_4) \frac{S_3}{S_{3,4}} = (-\tilde{p}_{3,4}) \frac{1}{S_4} + \tilde{b}_{3,4} \frac{1}{S_4 S_{3,4}}, \quad (141)$$

$$(-\tilde{p}_3) (-\tilde{p}_4) \frac{S_4}{S_{3,4}} = (-\tilde{p}_{3,4}) \frac{1}{S_3} + \tilde{b}_{3,4} \frac{1}{S_3 S_{3,4}}. \quad (142)$$

By respectively dividing (134) by $S_{2,3} S_{2,3,4}$ and $S_4 S_{2,3,4}$, (135) by $S_{2,4} S_{2,3,4}$ and $S_3 S_{2,3,4}$, and (136) by $S_{3,4} S_{2,3,4}$ and $S_2 S_{2,3,4}$, we successively obtain

$$(-\tilde{p}_4) (-\tilde{p}_{2,3}) \frac{S_4}{S_{2,3,4}} \quad (143)$$

$$= (-\tilde{p}_{2,3,4}) \frac{1}{S_{2,3}} + \tilde{b}_{2,4} (-\tilde{p}_3) \frac{1}{S_{2,3,4}} \frac{S_3}{S_{2,3}} + \tilde{b}_{3,4} (-\tilde{p}_2) \frac{1}{S_{2,3,4}} \frac{S_2}{S_{2,3}} + \tilde{b}_{2,3,4} \frac{1}{S_{2,3} S_{2,3,4}},$$

$$(-\tilde{p}_4) (-\tilde{p}_{2,3}) \frac{S_{2,3}}{S_{2,3,4}}$$

$$= (-\tilde{p}_{2,3,4}) \frac{1}{S_4} + \tilde{b}_{2,4} (-\tilde{p}_3) \frac{1}{S_4} \frac{S_3}{S_{2,3,4}} + \tilde{b}_{3,4} (-\tilde{p}_2) \frac{1}{S_4} \frac{S_2}{S_{2,3,4}} + \tilde{b}_{2,3,4} \frac{1}{S_4 S_{2,3,4}}, \quad (144)$$

$$(-\tilde{p}_3)(-\tilde{p}_{2,4}) \frac{S_3}{S_{2,3,4}} \quad (145)$$

$$= (-\tilde{p}_{2,3,4}) \frac{1}{S_{2,4}} + \tilde{b}_{2,3}(-\tilde{p}_4) \frac{1}{S_{2,3,4}} \frac{S_4}{S_{2,4}} + \tilde{b}_{3,4}(-\tilde{p}_2) \frac{1}{S_{2,3,4}} \frac{S_2}{S_{2,4}} + \tilde{b}_{2,3,4} \frac{1}{S_{2,4}S_{2,3,4}},$$

$$(-\tilde{p}_3)(-\tilde{p}_{2,4}) \frac{S_{2,4}}{S_{2,3,4}} \quad (146)$$

$$= (-\tilde{p}_{2,3,4}) \frac{1}{S_3} + \tilde{b}_{2,3}(-\tilde{p}_4) \frac{1}{S_3} \frac{S_4}{S_{2,3,4}} + \tilde{b}_{3,4}(-\tilde{p}_2) \frac{1}{S_3} \frac{S_2}{S_{2,3,4}} + \tilde{b}_{2,3,4} \frac{1}{S_3 S_{2,3,4}}$$

$$(-\tilde{p}_2)(-\tilde{p}_{3,4}) \frac{S_2}{S_{2,3,4}} \quad (147)$$

$$= (-\tilde{p}_{2,3,4}) \frac{1}{S_{3,4}} + \tilde{b}_{2,3}(-\tilde{p}_4) \frac{1}{S_{2,3,4}} \frac{S_4}{S_{3,4}} + \tilde{b}_{2,4}(-\tilde{p}_3) \frac{1}{S_{2,3,4}} \frac{S_3}{S_{3,4}} + \tilde{b}_{2,3,4} \frac{1}{S_{3,4}S_{2,3,4}},$$

$$(-\tilde{p}_2)(-\tilde{p}_{3,4}) \frac{S_{3,4}}{S_{2,3,4}} \quad (148)$$

$$= (-\tilde{p}_{2,3,4}) \frac{1}{S_2} + \tilde{b}_{2,3}(-\tilde{p}_4) \frac{1}{S_2} \frac{S_4}{S_{2,3,4}} + \tilde{b}_{2,4}(-\tilde{p}_3) \frac{1}{S_2} \frac{S_3}{S_{2,3,4}} + \tilde{b}_{2,3,4} \frac{1}{S_2 S_{2,3,4}},$$

Using (137) and (138) into (143), we get

$$(-\tilde{p}_4)(-\tilde{p}_{2,3}) \frac{S_4}{S_{2,3,4}} \quad (149)$$

$$= (-\tilde{p}_{2,3,4}) \frac{1}{S_{2,3}} + (-\tilde{p}_{2,3})(-\tilde{p}_2)^{-1} \tilde{b}_{2,4} \frac{1}{S_2 S_{2,3,4}} + (-\tilde{p}_{2,3})(-\tilde{p}_3)^{-1} \tilde{b}_{3,4} \frac{1}{S_3 S_{2,3,4}} \\ + \tilde{b}_{2,3,4} \frac{1}{S_{2,3} S_{2,3,4}} + (-\tilde{p}_2)^{-1} \tilde{b}_{2,3} \tilde{b}_{2,4} \frac{1}{S_2 S_{2,3} S_{2,3,4}} + (-\tilde{p}_3)^{-1} \tilde{b}_{2,3} \tilde{b}_{3,4} \frac{1}{S_3 S_{2,3} S_{2,3,4}}.$$

Using (139) and (140) into (145), we get

$$(-\tilde{p}_3)(-\tilde{p}_{2,4}) \frac{S_3}{S_{2,3,4}} \quad (150)$$

$$= (-\tilde{p}_{2,3,4}) \frac{1}{S_{2,4}} + (-\tilde{p}_{2,4})(-\tilde{p}_2)^{-1} \tilde{b}_{2,3} \frac{1}{S_2 S_{2,3,4}} + (-\tilde{p}_{2,4})(-\tilde{p}_4)^{-1} \tilde{b}_{3,4} \frac{1}{S_4 S_{2,3,4}} \\ + \tilde{b}_{2,3,4} \frac{1}{S_{2,4} S_{2,3,4}} + (-\tilde{p}_2)^{-1} \tilde{b}_{2,3} \tilde{b}_{2,4} \frac{1}{S_2 S_{2,4} S_{2,3,4}} + (-\tilde{p}_4)^{-1} \tilde{b}_{2,4} \tilde{b}_{3,4} \frac{1}{S_4 S_{2,4} S_{2,3,4}}$$

Using (141) and (142) into (147), we get

$$(-\tilde{p}_2)(-\tilde{p}_{3,4}) \frac{S_2}{S_{2,3,4}} \quad (151)$$

$$= (-\tilde{p}_{2,3,4}) \frac{1}{S_{3,4}} + (-\tilde{p}_{3,4})(-\tilde{p}_3)^{-1} \tilde{b}_{2,3} \frac{1}{S_3 S_{2,3,4}} + (-\tilde{p}_{3,4})(-\tilde{p}_4)^{-1} \tilde{b}_{2,4} \frac{1}{S_4 S_{2,3,4}} \\ + \tilde{b}_{2,3,4} \frac{1}{S_{3,4} S_{2,3,4}} + (-\tilde{p}_3)^{-1} \tilde{b}_{2,3} \tilde{b}_{3,4} \frac{1}{S_3 S_{3,4} S_{2,3,4}} + (-\tilde{p}_4)^{-1} \tilde{b}_{2,4} \tilde{b}_{3,4} \frac{1}{S_4 S_{3,4} S_{2,3,4}}$$

Using (151) and (150) into (144), we get

$$\begin{aligned}
& (-\tilde{p}_4)(-\tilde{p}_{2,3}) \frac{S_{2,3}}{S_{2,3,4}} \\
&= (-\tilde{p}_{2,3,4}) \frac{1}{S_4} + \tilde{b}_{2,3,4} \frac{1}{S_4 S_{2,3,4}} + (-\tilde{p}_{2,3,4})(-\tilde{p}_{2,4})^{-1} \tilde{b}_{2,4} \frac{1}{S_4 S_{2,4}} \\
&+ (-\tilde{p}_2)^{-1} \tilde{b}_{2,3} \tilde{b}_{2,4} \frac{1}{S_2 S_4 S_{2,3,4}} + (-\tilde{p}_4)^{-1} \tilde{b}_{3,4} \tilde{b}_{2,4} \frac{1}{S_4^2 S_{2,3,4}} \\
&+ (-\tilde{p}_{2,4})^{-1} \tilde{b}_{2,3,4} \tilde{b}_{2,4} \frac{1}{S_4 S_{2,4} S_{2,3,4}} + (-\tilde{p}_2)^{-1} (-\tilde{p}_{2,4})^{-1} \tilde{b}_{2,3} \tilde{b}_{2,4} \tilde{b}_{2,4} \frac{1}{S_2 S_4 S_{2,4} S_{2,3,4}} \\
&+ (-\tilde{p}_4)^{-1} (-\tilde{p}_{2,4})^{-1} \tilde{b}_{2,4} \tilde{b}_{3,4} \tilde{b}_{2,4} \frac{1}{S_4^2 S_{2,4} S_{2,3,4}} + (-\tilde{p}_{2,3,4})(-\tilde{p}_{3,4})^{-1} \tilde{b}_{3,4} \frac{1}{S_4 S_{3,4}} \\
&+ (-\tilde{p}_3)^{-1} \tilde{b}_{2,3} \tilde{b}_{3,4} \frac{1}{S_3 S_4 S_{2,3,4}} + (-\tilde{p}_4)^{-1} \tilde{b}_{2,4} \tilde{b}_{3,4} \frac{1}{S_4^2 S_{2,3,4}} \\
&+ (-\tilde{p}_{3,4})^{-1} \tilde{b}_{2,3,4} \tilde{b}_{3,4} \frac{1}{S_4 S_{3,4} S_{2,3,4}} + (-\tilde{p}_3)^{-1} (-\tilde{p}_{3,4})^{-1} \tilde{b}_{2,3} \tilde{b}_{3,4} \tilde{b}_{3,4} \frac{1}{S_3 S_4 S_{3,4} S_{2,3,4}} \\
&+ (-\tilde{p}_4)^{-1} (-\tilde{p}_{3,4})^{-1} \tilde{b}_{2,4} \tilde{b}_{3,4} \tilde{b}_{3,4} \frac{1}{S_4^2 S_{3,4} S_{2,3,4}} \tag{152}
\end{aligned}$$

Using (151) and (149) into (146), we get

$$\begin{aligned}
& (-\tilde{p}_3)(-\tilde{p}_{2,4}) \frac{S_{2,4}}{S_{2,3,4}} \\
&= (-\tilde{p}_{2,3,4}) \frac{1}{S_3} + \tilde{b}_{2,3,4} \frac{1}{S_3 S_{2,3,4}} \\
&+ (-\tilde{p}_{2,3,4})(-\tilde{p}_{2,3})^{-1} \tilde{b}_{2,3} \frac{1}{S_3 S_{2,3}} + (-\tilde{p}_2)^{-1} \tilde{b}_{2,4} \tilde{b}_{2,3} \frac{1}{S_2 S_3 S_{2,3,4}} + (-\tilde{p}_3)^{-1} \tilde{b}_{3,4} \tilde{b}_{2,3} \frac{1}{S_3^2 S_{2,3,4}} \\
&+ (-\tilde{p}_{2,3})^{-1} \tilde{b}_{2,3,4} \tilde{b}_{2,3} \frac{1}{S_3 S_{2,3} S_{2,3,4}} + (-\tilde{p}_2)^{-1} (-\tilde{p}_{2,3})^{-1} \tilde{b}_{2,3} \tilde{b}_{2,4} \tilde{b}_{2,3} \frac{1}{S_2 S_3 S_{2,3} S_{2,3,4}} \\
&+ (-\tilde{p}_3)^{-1} (-\tilde{p}_{2,3})^{-1} \tilde{b}_{2,3} \tilde{b}_{3,4} \tilde{b}_{2,3} \frac{1}{S_3^2 S_{2,3} S_{2,3,4}} \\
&+ (-\tilde{p}_{2,3,4})(-\tilde{p}_{3,4})^{-1} \tilde{b}_{3,4} \frac{1}{S_3 S_{3,4}} + (-\tilde{p}_3)^{-1} \tilde{b}_{2,3} \tilde{b}_{3,4} \frac{1}{S_3^2 S_{2,3,4}} + (-\tilde{p}_4)^{-1} \tilde{b}_{2,4} \tilde{b}_{3,4} \frac{1}{S_3 S_4 S_{2,3,4}} \\
&+ (-\tilde{p}_{3,4})^{-1} \tilde{b}_{2,3,4} \tilde{b}_{3,4} \frac{1}{S_3 S_{3,4} S_{2,3,4}} + (-\tilde{p}_3)^{-1} (-\tilde{p}_{3,4})^{-1} \tilde{b}_{2,3} \tilde{b}_{3,4} \tilde{b}_{3,4} \frac{1}{S_3^2 S_{3,4} S_{2,3,4}} \\
&+ (-\tilde{p}_4)^{-1} (-\tilde{p}_{3,4})^{-1} \tilde{b}_{2,4} \tilde{b}_{3,4} \tilde{b}_{3,4} \frac{1}{S_3 S_4 S_{3,4} S_{2,3,4}} \tag{153}
\end{aligned}$$

Using (150) and (149) into (148), we get

$$\begin{aligned}
& (-\tilde{p}_2)(-\tilde{p}_{3,4}) \frac{S_{3,4}}{S_{2,3,4}} \\
&= (-\tilde{p}_{2,3,4}) \frac{1}{S_2} + \tilde{b}_{2,3,4} \frac{1}{S_2 S_{2,3,4}} + \tilde{b}_{2,3} \frac{1}{S_2} \frac{(-\tilde{p}_4) S_4}{S_{2,3,4}} + \tilde{b}_{2,4} \frac{1}{S_2} \frac{(-\tilde{p}_3) S_3}{S_{2,3,4}} \\
&= (-\tilde{p}_{2,3,4}) \frac{1}{S_2} + \tilde{b}_{2,3,4} \frac{1}{S_2 S_{2,3,4}} \\
&+ [(-\tilde{p}_{2,3,4})(-\tilde{p}_{2,3})^{-1} \tilde{b}_{2,3} \frac{1}{S_2 S_{2,3}} + (-\tilde{p}_2)^{-1} \tilde{b}_{2,4} \tilde{b}_{2,3} \frac{1}{S_2^2 S_{2,3,4}} + (-\tilde{p}_3)^{-1} \tilde{b}_{3,4} \tilde{b}_{2,3} \frac{1}{S_2 S_3 S_{2,3,4}} \\
&+ (-\tilde{p}_{2,3})^{-1} \tilde{b}_{2,3,4} \tilde{b}_{2,3} \frac{1}{S_2 S_{2,3} S_{2,3,4}} + (-\tilde{p}_2)^{-1} (-\tilde{p}_{2,3})^{-1} \tilde{b}_{2,3} \tilde{b}_{2,4} \tilde{b}_{2,3} \frac{1}{S_2^2 S_{2,3} S_{2,3,4}} \\
&+ (-\tilde{p}_3)^{-1} (-\tilde{p}_{2,3})^{-1} \tilde{b}_{2,3} \tilde{b}_{3,4} \tilde{b}_{2,3} \frac{1}{S_2 S_3 S_{2,3} S_{2,3,4}}] + [(-\tilde{p}_{2,3,4})(-\tilde{p}_{2,4})^{-1} \tilde{b}_{2,4} \frac{1}{S_2 S_{2,4}} \\
&+ (-\tilde{p}_2)^{-1} \tilde{b}_{2,3} \tilde{b}_{2,4} \frac{1}{S_2^2 S_{2,3,4}} + (-\tilde{p}_4)^{-1} \tilde{b}_{2,4} \tilde{b}_{3,4} \frac{1}{S_2 S_4 S_{2,3,4}} + (-\tilde{p}_{2,4})^{-1} \tilde{b}_{2,3,4} \tilde{b}_{2,4} \frac{1}{S_2 S_{2,4} S_{2,3,4}} \\
&+ (-\tilde{p}_2)^{-1} (-\tilde{p}_{2,4})^{-1} \tilde{b}_{2,3} \tilde{b}_{2,4} \tilde{b}_{2,4} \frac{1}{S_2^2 S_{2,4} S_{2,3,4}} \\
&+ (-\tilde{p}_4)^{-1} (-\tilde{p}_{2,4})^{-1} \tilde{b}_{2,4} \tilde{b}_{3,4} \tilde{b}_{2,4} \frac{1}{S_2 S_4 S_{2,4} S_{2,3,4}}] \tag{154}
\end{aligned}$$

Using (151), (150), (149), (152),(153) and (154) into (127), by grouping terms of the same total degree and using the condition $L_{(X_2, X_3, X_4)}^{X_1=x_1}(0, 0, 0) = 1$, we obtain

$$\begin{aligned}
& L_{(X_2, X_3, X_4)}^{X_1=x_1}(\theta_2, \theta_3, \theta_4) \\
&= S_{2,3,4}^{-\lambda} \exp(x_1 \{ \alpha_1 [\frac{1}{S_2} - 1] + \alpha_2 [\frac{1}{S_3} - 1] + \alpha_3 [\frac{1}{S_4} - 1] + \alpha_4 [\frac{1}{S_{2,3}} - 1] + \alpha_5 [\frac{1}{S_{2,4}} - 1] + \alpha_6 [\frac{1}{S_{3,4}} - 1] \\
&\alpha_7 [\frac{1}{S_2 S_{2,3}} - 1] + \alpha_8 [\frac{1}{S_2 S_{2,4}} - 1] + \alpha_9 [\frac{1}{S_3 S_{2,3}} - 1] + \alpha_{10} [\frac{1}{S_3 S_{3,4}} - 1] + \alpha_{11} [\frac{1}{S_4 S_{2,4}} - 1] + \alpha_{12} [\frac{1}{S_4 S_{3,4}} - 1] \\
&+ \alpha_{13} [\frac{1}{S_{2,3,4}} - 1] + \alpha_{14} [\frac{1}{S_2 S_{2,3,4}} - 1] + \alpha_{15} [\frac{1}{S_3 S_{2,3,4}} - 1] + \alpha_{16} [\frac{1}{S_4 S_{2,3,4}} - 1] \\
&+ \alpha_{17} [\frac{1}{S_2^2 S_{2,3,4}} - 1] + \alpha_{18} [\frac{1}{S_3^2 S_{2,3,4}} - 1] + \alpha_{19} [\frac{1}{S_4^2 S_{2,3,4}} - 1] + \alpha_{20} [\frac{1}{S_2 S_3 S_{2,3,4}} - 1] + \alpha_{21} [\frac{1}{S_2 S_4 S_{2,3,4}} - 1] \\
&+ \alpha_{22} [\frac{1}{S_3 S_4 S_{2,3,4}} - 1] + \alpha_{23} [\frac{1}{S_{2,3} S_{2,3,4}} - 1] + \alpha_{24} [\frac{1}{S_{2,4} S_{2,3,4}} - 1] + \alpha_{25} [\frac{1}{S_{3,4} S_{2,3,4}} - 1] \\
&+ \alpha_{26} [\frac{1}{S_2 S_{2,3} S_{2,3,4}} - 1] + \alpha_{27} [\frac{1}{S_2 S_{2,4} S_{2,3,4}} - 1] + \alpha_{28} [\frac{1}{S_3 S_{2,3} S_{2,3,4}} - 1] + \alpha_{29} [\frac{1}{S_3 S_{3,4} S_{2,3,4}} - 1] \\
&+ \alpha_{30} [\frac{1}{S_4 S_{2,4} S_{2,3,4}} - 1] + \alpha_{31} [\frac{1}{S_4 S_{3,4} S_{2,3,4}} - 1] + \alpha_{32} [\frac{1}{S_2^2 S_{2,3} S_{2,3,4}} - 1] + \alpha_{33} [\frac{1}{S_2^2 S_{2,4} S_{2,3,4}} - 1] \\
&+ \alpha_{34} [\frac{1}{S_3^2 S_{2,3} S_{2,3,4}} - 1] + \alpha_{35} [\frac{1}{S_3^2 S_{3,4} S_{2,3,4}} - 1] + \alpha_{36} [\frac{1}{S_4^2 S_{2,4} S_{2,3,4}} - 1] + \alpha_{37} [\frac{1}{S_4^2 S_{3,4} S_{2,3,4}} - 1] \\
&+ \alpha_{38} [\frac{1}{S_2 S_3 S_{2,3} S_{2,3,4}} - 1] + \alpha_{39} [\frac{1}{S_2 S_4 S_{2,4} S_{2,3,4}} - 1] + \alpha_{40} [\frac{1}{S_3 S_4 S_{3,4} S_{2,3,4}} - 1] \})
\end{aligned}$$

Now we expand exp, and with $\mathbf{v} = (v_1, \dots, v_{40})$, we get

$$\begin{aligned}
L_{(X_2, X_3, X_4)}^{X_1=x_1} &= S_{2,3,4}^{-\lambda} \sum_{v_i \in \mathbb{N}^{40}} \left[\prod_{i=1}^{40} \mathbf{P}(V_i = v_i) \right] \\
&\times \left(\frac{1}{S_2^{v_1}} \frac{1}{S_3^{v_2}} \frac{1}{S_4^{v_3}} \frac{1}{S_{2,3}^{v_4}} \frac{1}{S_{2,4}^{v_5}} \frac{1}{S_{3,4}^{v_6}} \frac{1}{(S_2 S_{2,3})^{v_7}} \frac{1}{(S_2 S_{2,4})^{v_8}} \frac{1}{(S_3 S_{2,3})^{v_9}} \right. \\
&\times \frac{1}{(S_3 S_{3,4})^{v_{10}}} \frac{1}{(S_4 S_{2,4})^{v_{11}}} \frac{1}{(S_4 S_{3,4})^{v_{12}}} \frac{1}{S_{2,3,4}^{v_{13}}} \frac{1}{(S_2 S_{2,3,4})^{v_{14}}} \frac{1}{(S_3 S_{2,3,4})^{v_{15}}} \frac{1}{(S_4 S_{2,3,4})^{v_{16}}} \\
&\times \frac{1}{(S_2^2 S_{2,3,4})^{v_{17}}} \frac{1}{(S_3^2 S_{2,3,4})^{v_{18}}} \frac{1}{(S_4^2 S_{2,3,4})^{v_{19}}} \frac{1}{(S_2 S_3 S_{2,3,4})^{v_{20}}} \frac{1}{(S_2 S_4 S_{2,3,4})^{v_{21}}} \\
&\times \frac{1}{(S_3 S_4 S_{2,3,4})^{v_{22}}} \frac{1}{(S_{2,3} S_{2,3,4})^{v_{23}}} \frac{1}{(S_{2,4} S_{2,3,4})^{v_{24}}} \frac{1}{(S_{3,4} S_{2,3,4})^{v_{25}}} \frac{1}{(S_2 S_{2,3} S_{2,3,4})^{v_{26}}} \\
&\times \frac{1}{(S_2 S_{2,4} S_{2,3,4})^{v_{27}}} \frac{1}{(S_3 S_{2,3} S_{2,3,4})^{v_{28}}} \frac{1}{(S_3 S_{3,4} S_{2,3,4})^{v_{29}}} \frac{1}{(S_4 S_{2,4} S_{2,3,4})^{v_{30}}} \\
&\times \frac{1}{(S_4 S_{3,4} S_{2,3,4})^{v_{31}}} \frac{1}{(S_2^2 S_{2,3} S_{2,3,4})^{v_{32}}} \frac{1}{(S_2^2 S_{2,4} S_{2,3,4})^{v_{33}}} \frac{1}{(S_3^2 S_{2,3} S_{2,3,4})^{v_{34}}} \\
&\times \frac{1}{(S_3^2 S_{3,4} S_{2,3,4})^{v_{35}}} \frac{1}{(S_4^2 S_{2,4} S_{2,3,4})^{v_{36}}} \frac{1}{(S_4^2 S_{3,4} S_{2,3,4})^{v_{37}}} \frac{1}{(S_2 S_3 S_{2,3} S_{2,3,4})^{v_{38}}} \\
&\times \left. \frac{1}{(S_2 S_4 S_{2,4} S_{2,3,4})^{v_{39}}} \frac{1}{(S_3 S_4 S_{3,4} S_{2,3,4})^{v_{40}}} \right)
\end{aligned}$$

Combining, we obtain (80). Definitions (81), (82), (83), (84), (85), (86), (87), (88), and (89), (90), (91) with Equality (80) give (92). This completes the proof of Theorem (39).

■

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