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# The maximum number of homogeneous weights of linear codes over chain rings \*

Minjia Shi, TingTing Tong<sup>†</sup>, Thomas Honold<sup>‡</sup> & Patrick Solé<sup>§</sup>

## Abstract

The problem of determining the largest possible number of distinct Hamming weights in several classes of codes over finite fields was studied recently in several papers (Shi et al. 2019, Shi et al 2020, Chen et al. 2022). A further problem is to find the minimum length of codes meeting those bounds with equality.

These two questions are extended here to linear codes over chain rings for the homogeneous weight. An explicit upper bound is given for codes of given type and arbitrary length as a function of the residue field size. This bound is then shown to be tight by an argument based on Hjemslev geometries. The second question is studied for chain rings with residue field of order two.

**Keywords:** chain ring, homogeneous weight, Hjemslev geometry, maximum weight spectrum code

**MSC (2020):** 94 B27, 05 D99, 13 E99

## 1 Introduction

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From the viewpoint of combinatorial coding theory the following problem makes sense as an extremal parametric question:

How many distinct Hamming weights at most can a linear code of given length and dimension have?

One of the most important topics in coding theory is to characterize the weight distribution of a class of codes, but it is often difficult to describe this. In 1973, Delsarte [8] introduced four fundamental parameters of a linear code  $C$ : the minimum weight  $d$ , the weight spectrum cardinality  $s$ , the minimum weight  $d'$  of the dual code, and the weight spectrum cardinality  $s'$  of the dual code. And he also pointed out the combinatorial significance of these four fundamental parameters and showed that they have close connection with orthogonal arrays and combinatorial designs. In particular, he showed that  $s$  is intimately related to other parameters of the given  $C$ . Therefore, the weight spectrum cardinality of linear codes began to be extensively studied.

The minimum number of weights has been investigated for a long time; in particular one-weight codes have been characterized by Bonisoli [3] for finite fields and by Wood for  $\mathbb{Z}_N$  [25]. In contrast, linear codes of given dimension with the maximum number of weights have only started to be studied in [23]. Such codes are called MWS (Maximum Weight Spectrum) codes.

MWS codes over  $\mathbb{F}_q$  were characterized geometrically in [1], and it was shown that  $k$ -dimensional MWS codes have  $\frac{q^k-1}{q-1}$  distinct nonzero weights.<sup>1</sup>

The problem of determining the smallest length of an MWS code over  $\mathbb{F}_q$  of given dimension is still open in most cases. In [23, Theorem 1] it was shown that binary linear  $[2^k-1, k]$  with nonzero weights  $1, 2, \dots, 2^k-1$  exist, settling this problem for  $q = 2$ .<sup>2</sup> M. Kiermaier has determined computationally that the minimum lengths of  $[n, 3]$  MWS codes over  $\mathbb{F}_3$  and  $\mathbb{F}_4$  are 22 and 43, respectively.<sup>3</sup> Other studies on MWS codes are [2, 7].

The problem has been extended to codes with prescribed automorphisms: cyclic codes [21], quasicyclic codes [22].

In this paper we extend the problem in two directions: We replace the finite field by a finite (commutative) chain ring, and the Hamming weight by

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<sup>1</sup>Since nonzero multiples of a codeword have the same weight, this is clearly the maximum number possible.

<sup>2</sup>The codewords of weights  $1, 2, 2^2, \dots, 2^k-1$  in such a code must have disjoint supports, as is easily seen, proving also the uniqueness of these codes.

<sup>3</sup>M. Kiermaier, personal communication

the homogeneous weight.

Codes over *chain rings* have received a lot of attention in the last twenty years. Chain rings are special cases of quasi Frobenius rings, thus enjoying a MacWilliams formula [24]. Linear codes over these rings afford a canonical generator matrix as in the field case [19]. They form a bridge with Hjelmslev geometries, the chain ring analogue of Galois geometries over finite fields. See [12, 18, 19] for a general introduction and [16, 13] for special applications in the case of chain rings of order 4.

The *homogeneous weight* was introduced in [5] for the ring  $\mathbb{Z}_m$ , and for general finite rings in [9, 15]. It can be defined canonically, up to a multiplicative factor, from the lattice of ideals of the ring. It generalizes the Hamming metric over finite fields and the Lee metric over  $\mathbb{Z}_4$ , but differs from both in a general chain ring. It enjoys an organic connection with character sums defined from the additive group of the ring, which makes its manipulation easier than that of an arbitrary metric [19, Chap. 3 and 4]. A study of two-weight codes over chain rings is [20].

The material is arranged as follows. The next section collects basic definitions and notions needed for the other sections. Section 3 contains the main results of the paper. Section 4 discusses lower bounds for the lengths of MWS codes over chain rings. Section 5 concludes the article.

## 2 Preliminaries

### 2.1 Chain rings

Let  $R$  denote a (commutative<sup>4</sup>) chain ring with residue class field  $\mathbb{F}_q$ , length (nilpotency index)  $m$ , and maximal ideal  $M$  [19]. Thus  $R/M \cong \mathbb{F}_q$ , and  $M = (\theta)$  is principal ideal with generator  $\theta$ . The ideals of  $R$  then form a chain with respect to set inclusion:

$$\{0\} = M^m \subsetneq M^{m-1} \subsetneq \cdots \subsetneq M \subsetneq M^0 = R.$$

As a simple example, the reader can think of  $R = \mathbb{Z}_{p^m}$ ,  $q = \theta = p$ . Define the Teichmüller set  $T$  as a set of  $q$  elements of  $R$  such that  $T \equiv \mathbb{F}_q \pmod{M}$ .

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<sup>4</sup>Most of our results can be generalized to the non-commutative setting. However, in order to keep the presentation as [non-technical](#) as possible, we restrict ourselves to commutative chain rings.

Let  $T^* = T \setminus \{0\}$ . The  $\theta$ -adic expansion of an arbitrary element  $x \in R$  is  $x = a_1 + a_2\theta + \cdots + a_m\theta^{m-1}$ , with  $a_1, \dots, a_m$  unique elements in  $T$ .

A code  $C$  of length  $n$  is a nonempty subset of  $R^n$ . A linear code  $C$  of length  $n$  over  $R$  is an  $R$ -submodule of  $R^n$ . Any linear code  $C$  of length  $n$  over  $R$  is permutation-equivalent to a code with the following generator matrix :

$$\mathbf{G} = \begin{pmatrix} I_{k_1} & A_{11} & A_{12} & A_{13} & \cdots & A_{1,m-1} & A_{1,m} \\ 0 & \theta I_{k_2} & \theta A_{22} & \theta A_{23} & \cdots & \theta A_{2,m-1} & \theta A_{2,m} \\ 0 & 0 & \theta^2 I_{k_3} & \theta^2 A_{33} & \cdots & \theta^2 A_{3,m-1} & \theta^2 A_{3,m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \theta^{m-1} I_{k_m} & \theta^{m-1} A_{m,m} \end{pmatrix},$$

where  $I_{k_i}$  is the  $k_i \times k_i$  identity matrix and  $A_{i,j}$  are matrices over  $R$  for all  $i, j \in \{1, 2, \dots, m\}$ . A linear code  $C$  with a generator matrix in this form has  $q^K$  elements, where  $K = \log_q |C| = \sum_{i=0}^{m-1} (m-i)k_{i+1}$  is the composition length of the  $R$ -module  $C$ . We call the  $m$ -tuple  $\mathbf{k} = (k_1, k_2, \dots, k_m)$  of integers  $k_i \geq 0$  the *type* of  $C$ .

Alternatively, we can associate with  $C$  the integer partition

$$\lambda = \underbrace{(m, \dots, m)}_{k_1}, \underbrace{(m-1, \dots, m-1)}_{k_2}, \dots, \underbrace{(1, \dots, 1)}_{k_m}$$

of  $K$ , whose parts are the composition lengths of the cyclic  $R$ -modules appearing in a representation of  $C$  as a direct sum of cyclic  $R$  modules. This partition also determines  $C$  up to module isomorphism and is called the *shape* of  $C$ ; cf. [12]. The conjugate partition of  $\lambda$  (with reflected Ferrers diagram) is  $\mu = (k_1 + \cdots + k_m, k_1 + \cdots + k_{m-1}, \dots, k_1 + k_2, k_1)$ . It is called the *conjugate shape* of  $C$  in [12], and is sometimes easier to work with than type or shape.

These concepts apply to arbitrary finitely-generated modules over  $R$ , generalizing the well-known fact that isomorphism classes of finite abelian  $p$ -groups of exponent  $m$ , via direct-sum decomposition into cyclic groups, are parametrized by integer partitions with parts  $\leq m$ ; see, e.g., [17, [Chapter I-1.1](#)]. In fact, since every module over  $R$  can be embedded into a free module [12, [Theorem 2.6](#)], all isomorphism classes of finitely-generated modules over  $R$  are represented by linear codes.

For our purposes it is best to define the homogeneous weight on  $R$  as follows:

$$w_{\text{hom}}(x) = \begin{cases} q & \text{if } x \in (\theta^{m-1}) \setminus \{0\}, \\ q-1 & \text{if } x \in R \setminus (\theta^{m-1}), \\ 0 & \text{if } x = 0. \end{cases} \quad (1)$$

This definition uses the smallest normalization factor for which all weights are integers. For any vector  $x = (x_1, \dots, x_n) \in R^n$ , we define its homogeneous weight as  $w_{\text{hom}}(x) = \sum_{i=1}^n w_{\text{hom}}(x_i)$ . We denote by  $w_{\text{Ham}}(x)$  the Hamming weight of  $x$ , which also satisfies  $w_{\text{Ham}}(x) = \sum_{i=1}^n w_{\text{Ham}}(x_i)$ .

The *effective length*  $n(C)$  of a linear code is the number of positions where at least one codeword has a nonzero entry. It is well-known (see, e.g., [6, Lemma 3]) that the total homogeneous weight and the effective length of a linear code  $C$  over  $R$  are related by

$$\sum_{\mathbf{x} \in C} w_{\text{hom}}(\mathbf{x}) = \Gamma |C| n(C),$$

where  $\Gamma = |R|^{-1} \sum_{x \in R} w_{\text{hom}}(x)$ . For the chosen normalization  $\Gamma = q - 1$ .

For integers  $n \geq 1$  and  $k_1, k_2, \dots, k_m \geq 0$ , a linear code over  $R$  of type  $\mathbf{k} = (k_1, \dots, k_m)$  is denoted as “linear  $[n, \mathbf{k}]$  code” or, if the ring  $R$  matters, as “linear  $[n, \mathbf{k}]_R$  code”. Considering the ring  $R$  (and thus in particular  $q, m$ ) as fixed, we denote the maximum number of nonzero homogeneous weights a linear  $[n, \mathbf{k}]$  code over  $R$  can have by  $L(n, \mathbf{k})$ . Similarly,  $L(\mathbf{k})$  denotes the maximum number of nonzero homogeneous weights in a linear code over  $R$  of type  $\mathbf{k}$  but arbitrary length. Since  $\mathbf{k}$  determines  $|C|$ , the numbers  $L(\mathbf{k})$  are finite, and trivially we have  $L(n, \mathbf{k}) \leq L(\mathbf{k})$  for all  $n, \mathbf{k}$ .

**Proposition 1** *There is an integer  $N_0$  such that for all  $n \geq N_0$  we have*

$$L(n, \mathbf{k}) = L(\mathbf{k}),$$

*and the smallest such integer satisfies  $N_0 \geq \left\lceil \frac{L(\mathbf{k})}{q} \right\rceil$*

**Proof.** By definition of  $L(\mathbf{k})$ , there is a linear code  $C_0$  of some length  $N_0$  with  $L(\mathbf{k})$  nonzero homogeneous weights. If  $n > N_0$ , adding  $n - N_0$  zero columns to the generator matrix of  $C_0$  leads to

$$L(n, \mathbf{k}) \geq L(\mathbf{k}) = L(N_0, \mathbf{k}),$$

and from the trivial bound  $L(n, \mathbf{k}) \leq L(\mathbf{k})$  we then have  $L(n, \mathbf{k}) = L(\mathbf{k})$ . This proves the first assertion.

Since homogeneous weights in  $R$  are integers  $\leq q$ , we have

$$L(n, \mathbf{k}) \leq nq.$$

Letting  $N_0$  be such that  $L(N_0, \mathbf{k}) = L(\mathbf{k})$ , we have

$$L(\mathbf{k}) = L(N_0, \mathbf{k}) \leq N_0q.$$

The second assertion follows.  $\square$

The (coordinate) projective Hjelmslev geometry  $\text{PHG}(k-1, R)$  can be defined as the poset of all free submodules of a free  $R$ -module of rank  $k$ , e.g.,  $R^k$ . Free cyclic submodules of  $R^k$  are called *points* of  $\text{PHG}(k-1, R)$ , free submodules of  $R^k$  of rank 2 lines of  $\text{PHG}(k-1, R)$ , etc., and free submodules of  $R^k$  of rank  $k-1$  hyperplanes of  $\text{PHG}(k-1, R)$ . Hyperplanes are given by one linear equation  $a_1x_1 + \cdots + a_kx_k = 0$  where  $R(a_1, \dots, a_k)$  is a point of  $\text{PHG}(k-1, R)$ , and the so-defined correspondence between points and hyperplanes is 1-1. For more information on projective Hjelmslev geometry see, e.g., [14].

In [12] it is shown that there is a 1-1 correspondence between monomial equivalence classes of so-called fat<sup>5</sup> linear codes over a finite chain ring  $R$  and equivalence classes of multisets of points in the geometries  $\text{PHG}(k-1, R)$ .<sup>6</sup> A linear code  $C$  and its associated multiset  $\mathfrak{K}_C$  have the same *shape* in the sense that the  $R$ -module type of  $C$  and that of the span  $\langle \text{supp } \mathfrak{K}_C \rangle_R$ , which is also called *hull* of  $\mathfrak{K}_C$ , are parametrized by the same integer partition  $\lambda$ . The integer  $k$  can be chosen as the module rank of  $C$ . The homogeneous weight distribution of  $C$  can be obtained from geometric information about  $\mathfrak{K}_C$  (essentially the intersection patterns of  $\mathfrak{K}_C$  with the various hyperplanes of  $\text{PHG}(k-1, R)$ ).

The correspondence between fat linear codes and multisets of points can be extended to all linear codes, provided we include in  $\text{PHG}(k-1, R)$  also non-free cyclic submodules  $R(a_1, \dots, a_k) \neq 0$  of  $R^k$  (*degenerate points*), and the corresponding solution sets of  $a_1x_1 + \cdots + a_kx_k = 0$  (*degenerate hyperplanes*). We will denote  $\text{PHG}(k-1, R)$  extended in this way by  $\overline{\text{PHG}(k-1, R)}$ .

One of our main results depends on a combinatorial property of linear forms on  $\mathbb{Q}^n$ . Recall that linear map  $l: V \rightarrow K$  from a vector space to its underlying field of scalars (considered a vector space over itself) is called a linear form on  $V$ . Any linear form  $l$  on  $\mathbb{Q}^n$  has a unique representation  $l(x) = a_1x_1 + \cdots + a_nx_n$  with  $a_j \in \mathbb{Q}$ .

<sup>5</sup>A linear code is said to be fat (or regular) if for every coordinate position there exists a codeword with a '1' in that position.

<sup>6</sup>If  $R$  is non-commutative, one must associate left linear codes  $C \leq {}_R R^n$  with multisets of points in the "right" projective Hjelmslev geometry derived from  $R_R^k$ .

**Lemma 1** *Suppose  $l_1, \dots, l_N$  are finitely many nonzero linear forms on  $\mathbb{Q}^n$ . Then there exists  $x = (x_1, \dots, x_n) \in \mathbb{Q}^n$  with  $x_j > 0$  for  $1 \leq j \leq n$  and such that  $l_i(x) \neq 0$  for  $1 \leq i \leq N$ .*

In other words, finitely many nonzero linear forms on  $\mathbb{Q}^n$  have a simultaneous nonzero which is positive. We do not claim that this result is new—certainly it is well-known that finitely many nonzero linear forms on a vector space over an infinite field have a simultaneous nonzero, which amounts to the impossibility of covering the space by finitely many hyperplanes. The additional requirement “simultaneous positive nonzero” is an easy extension of this.

**Proof.** The proof is by induction on  $n$ , the case  $n = 1$  being trivial. Now suppose  $n \geq 2$ , and let  $l_i(x) = a_{i1}x_1 + \dots + a_{in}x_n$ . The linear forms  $l_i$  that do not involve  $x_n$  (i.e.,  $a_{in} = 0$ ) can be viewed as linear forms on  $\mathbb{Q}^{n-1}$  and, by induction, have a simultaneous positive nonzero  $(x_1, \dots, x_{n-1}) \in \mathbb{Q}^{n-1}$ . It remains to determine  $x_n \in \mathbb{Q}$ ,  $x_n > 0$ , in such a way that  $a_{in} \neq 0$  implies  $a_{i1}x_1 + \dots + a_{in}x_n \neq 0$ , i.e.,  $x_n \neq -\frac{1}{a_{in}}(a_{i1}x_1 + \dots + a_{i,n-1}x_{n-1})$ . Since this excludes only finitely many rational numbers and there is an infinite supply of positive rational numbers, this is always possible.  $\square$

Lemma 1 of course also implies the existence of a simultaneous positive integral nonzero.

### 3 Main results

If codewords  $x, y \in C$  generate the same cyclic submodule of  $C$ , we have  $y = ux$  for some unit  $u \in R^\times$  and hence  $w_{\text{hom}}(y) = w_{\text{hom}}(x)$ . Thus the number of distinct nonzero homogeneous weights of  $C$  cannot exceed the number of nonzero cyclic submodules of  $C$ . In the following theorem we make this bound precise by computing the number nonzero cyclic submodules of  $C$  as a function of  $q$  and the conjugate shape of  $C$ . The corresponding formula in terms of the type of  $C$  is more complicated to state and omitted. In the examples below it will be included.

**Theorem 1** *Let  $C$  be a linear code over  $R$  of type  $\mathbf{k}$  and conjugate shape  $\mu$ . The number of distinct nonzero homogeneous weights of  $C$ , and hence also the number  $L(\mathbf{k})$ , is bounded from above by*

$$\sum_{i=1}^m q^{\mu_1 + \dots + \mu_{i-1} - (i-1)} \cdot \frac{q^{\mu_i} - 1}{q - 1}. \quad (2)$$



**Proof.** Nonzero cyclic submodules of  $C$  have order  $q^i$ ,  $1 \leq i \leq m$ , and  $|Rx| = q^i$  if and only if  $x$  is of period  $\theta^i$  (cf. [12]), i.e.,  $\theta^i x = 0$ ,  $\theta^{i-1} x \neq 0$ . Hence, if we let

$$C[\theta^i] = \{x \in C; \theta^i x = 0\} \quad \text{for } 1 \leq i \leq m,$$

the set of codewords of exact period  $\theta^i$  is  $C[\theta^i] \setminus C[\theta^{i-1}]$ , and the number of cyclic submodules of  $C$  of order  $q^i$  is  $N_i := |C[\theta^i] \setminus C[\theta^{i-1}]|$  divided by the number  $q^i - q^{i-1}$  of generators of a fixed such module.

Since  $|C[\theta^i]| = q^{\mu_1 + \dots + \mu_i}$  (cf. [12]), we have

$$\begin{aligned} N_i &= q^{\mu_1 + \dots + \mu_i} - q^{\mu_1 + \dots + \mu_{i-1}} = q^{\mu_1 + \dots + \mu_{i-1}} (q^{\mu_i} - 1), \\ \frac{N_i}{q^i - q^{i-1}} &= \frac{N_i}{q^{i-1}(q-1)} = q^{\mu_1 + \dots + \mu_{i-1} - (i-1)} \cdot \frac{q^{\mu_i} - 1}{q-1}. \end{aligned}$$

Summing over  $i \in \{1, \dots, m\}$  we obtain that the number of nonzero cyclic submodules of  $C$  is given by (2). The theorem then follows from the remarks preceding it. □

**Remark 1** *The number of nonzero cyclic submodules of  $C$  can also be expressed as*

$$\begin{aligned} & \sum_{i=1}^m q^{\mu_1 + \dots + \mu_{i-1} - (i-1)} (1 + q + \dots + q^{\mu_i - 1}) \\ &= \sum_{i=1}^m \sum_{s=\mu_1 + \dots + \mu_{i-1} - (i-1)}^{\mu_1 + \dots + \mu_i - i} q^s \\ &= \left( \sum_{j=0}^{\mu_1 + \dots + \mu_m - m} q^j \right) + q^{\mu_1 - 1} + q^{\mu_1 + \mu_2 - 2} + \dots + q^{\mu_1 + \dots + \mu_{m-1} - (m-1)} \\ &= \frac{q^{\mu_1 + \dots + \mu_m - (m-1)} - 1}{q-1} + q^{\mu_1 - 1} + q^{\mu_1 + \mu_2 - 2} + \dots + q^{\mu_1 + \dots + \mu_{m-1} - (m-1)}. \end{aligned}$$

**Example 1** *The generator matrix of a linear code  $C$  over  $\mathbb{Z}_{p^2}$  (the case  $q = p$ ,  $m = 2$ ) has the following form:*

$$\mathbf{G} = \begin{pmatrix} I_{k_1} & A_{11} & A_{12} \\ 0 & pI_{k_2} & pA_{22} \end{pmatrix}.$$

Here  $(k_1, k_2)$  is the type of  $C$ , which is related to the conjugate shape  $(\mu_1, \mu_2)$  by  $\mu_1 = k_1 + k_2$ ,  $\mu_2 = k_1$ . [Theorem 1](#) and [Remark 1](#) give

$$\begin{aligned} L(k_1, k_2) &\leq \frac{p^{\mu_1 + \mu_2 - 1} - 1}{p - 1} + p^{\mu_1 - 1} \\ &= \frac{p^{2k_1 + k_2 - 1} - 1}{p - 1} + p^{k_1 + k_2 - 1}. \end{aligned}$$

**Example 2** The generator matrix of a linear code  $C$  over  $\mathbb{Z}_p^3$  ( $q = p$ ,  $m = 3$ ) has the following form:

$$\mathbf{G} = \begin{pmatrix} I_{k_1} & A_{11} & A_{12} & A_{13} \\ 0 & pI_{k_2} & pA_{22} & pA_{23} \\ 0 & 0 & p^2I_{k_3} & p^2A_{33} \end{pmatrix}.$$

Here the type  $(k_1, k_2, k_3)$  and the conjugate shape  $(\mu_1, \mu_2, \mu_3)$  are related by  $\mu_1 = k_1 + k_2 + k_3$ ,  $\mu_2 = k_1 + k_2$ ,  $\mu_3 = k_1$ , and so

$$\begin{aligned} L(k_1, k_2, k_3) &\leq \frac{p^{\mu_1 + \mu_2 + \mu_3 - 2} - 1}{p - 1} + p^{\mu_1 - 1} + p^{\mu_1 + \mu_2 - 2} \\ &= \frac{p^{3k_1 + 2k_2 + k_3 - 2} - 1}{p - 1} + p^{k_1 + k_2 + k_3 - 1} + p^{2k_1 + 2k_2 + k_3 - 2}. \end{aligned}$$

**Definition 1** A linear  $[n, \mathbf{k}]$  code  $C$  over  $R$  is called a Maximum Weight Spectrum (MWS) code, if  $C$  has the maximum number  $L(\mathbf{k})$  of (nonzero) homogeneous weights among all linear codes over  $R$  with the same module type  $\mathbf{k}$  as  $C$ .

**Remark 2** Let  $A(C) = (A_1, \dots, A_{nq^{m-1}})$  be the (nonzero) weight distribution of  $C$ , where  $A_i = |\{c \in C \mid w_h(c) = i\}|$ . Clearly we have that  $C$  is MWS if and only if the Hamming weight of  $A(C)$  is  $L(\mathbf{k})$ . If an  $[n, \mathbf{k}]$  MWS code exists then  $L(n, \mathbf{k}) = L(\mathbf{k})$ , of course.

Our next goal is to prove that the inequality in [Theorem 1](#) is tight, i.e., codewords of MWS codes have different homogeneous weight if they generate different cyclic submodules. First we provide some examples showing the existence of MWS codes over  $\mathbb{Z}_4$  meeting the bound in [Theorem 1](#).

**Example 3** We consider the 2-dimensional simplex code over  $\mathbb{Z}_4$  with the following generator matrix:

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 3 & 1 \end{pmatrix},$$

which is associated to the projective Hjelmslev line  $\text{PHG}(1, \mathbb{Z}_4)$ . The code is free of rank 2 and has 9 nonzero cyclic submodules.<sup>7</sup> In the following we list generators for these as rows of a matrix  $\mathbf{M} \in \mathbb{Z}_4^{9 \times 6}$  and the corresponding Lee weights (the homogeneous weight is equal to the Lee weight over  $\mathbb{Z}_4$ ) in another matrix  $\mathbf{W} \in \mathbb{Q}^{9 \times 6}$  “weight matrix”:

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 2 \\ 2 & 0 & 2 & 2 & 2 & 0 \\ 0 & 1 & 1 & 2 & 3 & 1 \\ 0 & 2 & 2 & 0 & 2 & 2 \\ 1 & 1 & 2 & 3 & 0 & 3 \\ 2 & 2 & 0 & 2 & 0 & 2 \\ 1 & 3 & 0 & 3 & 2 & 1 \\ 1 & 2 & 3 & 1 & 3 & 0 \\ 2 & 1 & 3 & 0 & 1 & 1 \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 2 \\ 2 & 0 & 2 & 2 & 2 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 2 & 2 & 0 & 2 & 2 \\ 1 & 1 & 2 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Now let us choose 6 positive multiplicities  $m_1, \dots, m_6$  for the 6 columns of  $\mathbf{G}$  such that the multiplicity vector  $\mathbf{m} = (m_1, \dots, m_6)$  satisfies the following:

The column vector  $\mathbf{W}\mathbf{m}^\top$  has distinct entries; equivalently, the linear forms on  $\mathbb{Q}^6$  determined by the rows of  $\mathbf{W}$  attain distinct values on  $\mathbf{m}$ . This is possible, since these 9 linear forms are distinct (i.e., the rows of  $\mathbf{W}$  are distinct), and hence their mutual differences are nonzero and must have a simultaneous positive integral nonzero by Lemma 1. Now let  $\overline{\mathbf{G}}$  be the matrix with Column  $j$  of  $\mathbf{G}$  repeated  $m_j$  times (in the same order). The matrix  $\overline{\mathbf{G}}$  generates a linear code  $C$  over  $\mathbb{Z}_4$ , whose corresponding matrix  $\overline{\mathbf{M}}$  arises from  $\mathbf{M}$  in the same way, and the column vector  $\mathbf{W}\mathbf{m}^\top$  contains the Lee weights of the rows of  $\overline{\mathbf{M}}$ , i.e., the nonzero Lee weights of  $C$ .

Thus we have a non-constructive proof (i.e., without specifying the multiplicities explicitly) that a  $\mathbb{Z}_4$ -linear code which is free of rank 2 and has the maximum number of 9 distinct Lee weights exists. The proof can easily be made constructive by finding a suitable multiplicity vector  $\mathbf{m}$ . For example, we can take  $\mathbf{m} = (1, 3, 3^2, 3^3, 3^4, 3^5)$ , because the rows of  $\mathbf{W}$  can be viewed as base-3 representations of certain integers. With this choice  $C$  has length 364 and 9 distinct Lee weights. However, there exists a free rank-2 code with 9 distinct weights and much smaller length, viz. 8. For this use only the first 3 columns of  $\mathbf{G}$ , which represent non-neighbor points in  $\text{PHG}(1, \mathbb{Z}_4)$ . The

<sup>7</sup>These correspond to the 6 points of  $\text{PHG}(1, \mathbb{Z}_4)$  and the 3 neighbor classes (“degenerate” points).

corresponding matrix pair is

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix},$$

and choosing  $\mathbf{m} = (1, 3, 4)$  produces a code with weights 5, 10, 7, 14, 12, 8, 4, 11, 9.

**Example 4** A good example for a non-free code over  $\mathbb{Z}_4$  is the code associated with a neighbor class of points in the projective Hjelmslev plane  $\text{PHG}(2, \mathbb{Z}_4)$ , which has generator matrix

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}.$$

This code has type  $(k_1, k_2) = (1, 2)$  and 11 nonzero cyclic submodules, of which 4 are free, with representatives

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 3 \\ 1 & 1 & 3 & 3 \\ 1 & 3 & 3 & 1 \end{pmatrix}.$$

Since all have 4 unit entries, they give rise to 4 identical rows in the corresponding weight matrix  $\mathbf{W}$ , and hence there is no way to select multiplicities to obtain a code of the desired form. The corresponding 4 rows will always have the same Lee weight. However, if we augment the generator matrix by columns corresponding to the 7 degenerate points, i.e.,

$$\left( \begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 0 & 2 \end{array} \right),$$

the problem goes away: The corresponding part of  $\mathbf{M}$  becomes

$$\left( \begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 2 & 2 & 2 \\ 1 & 3 & 1 & 3 & 2 & 2 & 0 & 2 & 2 & 0 & 0 \\ 1 & 1 & 3 & 3 & 2 & 0 & 2 & 2 & 0 & 2 & 0 \\ 1 & 3 & 3 & 1 & 2 & 2 & 2 & 0 & 0 & 0 & 2 \end{array} \right),$$

and gives rise to 4 distinct rows in  $\mathbf{W}$ . In fact the new weight matrix  $\mathbf{W}$  has 11 distinct rows, and the argument goes through.

Further, let us look at an example which is associated with the projective Hjelmslev plane  $\text{PHG}(2, \mathbb{Z}_4)$ . It can be shown that the 3-dimensional simplex code over  $\mathbb{Z}_4$ , which is free of rank 3, has length 28 and 35 nonzero cyclic submodules, gives rise to a weight matrix  $\mathbf{W} \in \mathbb{Q}^{35 \times 28}$ , whose rows are distinct and hence can be used to construct a free rank-3 code with 35 nonzero weights. The following example uses a “starter code” of much smaller length.

**Example 5** Consider the linear code over  $\mathbb{Z}_4$  with generator matrix

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The code has length 7, is free of rank 3, and hence has 35 nonzero cyclic submodules as well. The 7 columns of  $G$  represent a quadrangle in  $\text{PHG}(2, \mathbb{Z}_4)$  together with its 3 diagonal points. The rows of  $M$  and  $W$  (cf. the previous

examples) are

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 & 3 & 2 & 3 \\ 0 & 1 & 3 & 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 2 & 2 & 0 & 2 \\ 0 & 2 & 2 & 2 & 0 & 2 & 0 \\ 0 & 2 & 1 & 2 & 3 & 1 & 3 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 2 & 2 \\ 1 & 0 & 2 & 1 & 2 & 3 & 3 \\ 1 & 0 & 3 & 1 & 3 & 0 & 0 \\ 1 & 1 & 0 & 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 & 3 \\ 1 & 1 & 2 & 2 & 3 & 3 & 0 \\ 1 & 1 & 3 & 2 & 0 & 0 & 1 \\ 1 & 2 & 0 & 3 & 2 & 1 & 3 \\ 1 & 2 & 1 & 3 & 3 & 2 & 0 \\ 1 & 2 & 2 & 3 & 0 & 3 & 1 \\ 1 & 2 & 3 & 3 & 1 & 0 & 2 \\ 1 & 3 & 0 & 0 & 3 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 2 & 1 \\ 1 & 3 & 2 & 0 & 1 & 3 & 2 \\ 1 & 3 & 3 & 0 & 2 & 0 & 3 \\ 2 & 0 & 0 & 2 & 0 & 2 & 2 \\ 2 & 0 & 1 & 2 & 1 & 3 & 3 \\ 2 & 0 & 2 & 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 3 & 1 & 2 & 3 \\ 2 & 1 & 1 & 3 & 2 & 3 & 0 \\ 2 & 1 & 2 & 3 & 3 & 0 & 1 \\ 2 & 1 & 3 & 3 & 0 & 1 & 2 \\ 2 & 2 & 0 & 0 & 2 & 2 & 0 \\ 2 & 2 & 1 & 0 & 3 & 3 & 1 \\ 2 & 2 & 2 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 & 2 & 0 & 2 \\ 0 & 2 & 2 & 2 & 0 & 2 & 0 \\ 0 & 2 & 1 & 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 2 & 2 \\ 1 & 0 & 2 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 2 & 0 \\ 1 & 2 & 2 & 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 0 & 2 & 0 & 1 \\ 2 & 0 & 0 & 2 & 0 & 2 & 2 \\ 2 & 0 & 1 & 2 & 1 & 1 & 1 \\ 2 & 0 & 2 & 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 & 0 & 1 & 2 \\ 2 & 2 & 0 & 0 & 2 & 2 & 0 \\ 2 & 2 & 1 & 0 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Since the rows of  $\mathbf{W}$  are distinct vectors in  $\mathbb{Q}^7$ , it is possible to choose 7 multiplicities  $m_1, \dots, m_7 \geq 1$  for the 7 columns of  $\mathbf{G}$  such that the multiplicity vector  $\mathbf{m} = (m_1, \dots, m_7)$  gives rise to a column vector  $\mathbf{W}\mathbf{m}^\top$  with distinct entries. For example, we can take  $\mathbf{m} = (1, 2, 2^2, 2^3, 2^4, 2^5, 2^6)$ , as found by a computer search. The code of length 127 over  $\mathbb{Z}_4$  generated by the corresponding matrix  $\overline{\mathbf{G}}$  then has 35 distinct Lee weights.<sup>8</sup>

The geometric property of the point set  $\mathfrak{K}$  in  $\text{PHG}(2, \mathbb{Z}_4)$  represented by the columns of  $\mathbf{G}$  responsible for 35 distinct rows in  $\mathbf{W}$  is the following: The 7 points in  $\mathfrak{K}$  are pair-wise non-neighbors, and the 28 lines of  $\text{PHG}(2, \mathbb{Z}_4)$  have distinct  $\mathfrak{K}$ -types; see [12, Section 5].

The above example does not consider the length of MWS codes, the following example with type (2, 1) will discuss the shortest length problem. We will determine the appropriate multiplicity vector so that the length of MWS codes is minimal.

<sup>8</sup>It should be noted that the MWS property depends on the exact order in which the multiplicities are assigned to the columns of  $\mathbf{G}$ . For example, using  $\mathbf{m} = (1, 2^3, 2^5, 2, 2^2, 2^4, 2^6)$  instead produces a code with only 32 distinct Lee weights.

**Example 6** A good example with type  $(2,1)$  over  $\mathbb{Z}_4$  is the code associated with a neighbor class of points in the projective Hjelmslev plane  $\text{PHG}(2, \mathbb{Z}_4)$ , which has generator matrix

$$G = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

This code has 19 nonzero cyclic submodules. Through the program algorithm search, we take the multiplicity vector  $\mathbf{m} = (m_1, \dots, m_4) = (1, 3, 4, 8)$  (multiplicity vectors are not unique) to get a new generator matrix

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

The linear code generated by  $G_1$  has 19 different Lee weights: 4, 5, 7, 8, 9, 10, 11, 12, 14, 16, 20, 21, 23, 24, 25, 26, 27, 28, 30. In addition, the minimum value of  $m_1 + m_2 + m_3 + m_4$  searched by the program algorithm with  $G$  as the starting set is 16.

**Remark 3** The search algorithm for Example 6 is as follows: First, a matrix  $G = [v_1, \dots, v_n]$  is selected as a starting set, and the multiplicity of each column  $v_i$  is set to  $m_i$ ,  $1 \leq i \leq n$ . Second, we choose an appropriate positive integer  $N_0$ , where  $N_0$  must be large enough to ensure that the new matrix constructed is the generator matrix of MWS codes. Then we calculate the number of nonzero cyclic submodules of  $G$  as  $N$ . And  $S$  is denoted as the number of nonzero cyclic submodules of the linear code generated by new generator matrix. So we can formulate the following linear programming problem:

$$\begin{aligned} & \text{Minimize } m_1 + m_2 + \dots + m_n \\ & \text{subject to} \\ & S = N \\ & m_1 + m_2 + \dots + m_n \leq N_0 \\ & m_i \geq 1 \text{ for } 1 \leq i \leq n \end{aligned}$$

Let  $M$  be the optimal solution to the above linear programming problem. Then we get the shortest length of the corresponding MWS codes.

After working out the above examples the general picture became clear, and we were able to prove our main theorem:

**Theorem 2** *Let  $R$  be a chain ring with residue field  $\mathbb{F}_q$  and nilpotency index  $m$ . For all nonnegative integer  $m$ -tuples  $\mathbf{k} = (k_1, k_2, \dots, k_m)$  we have*

$$L(\mathbf{k}) = \sum_{i=1}^m q^{\mu_1 + \dots + \mu_{i-1} - (i-1)} \cdot \frac{q^{\mu_i} - 1}{q - 1},$$

where  $\mu = (\mu_1, \dots, \mu_m)$  is defined by  $\mu_i = k_1 + \dots + k_{m-i+1}$  for  $1 \leq i \leq m$ .

**Proof.** The rank of an  $R$ -module of type  $\mathbf{k}$  is  $k = k_1 + \dots + k_m = \mu_1$ . We choose a submodule  $U$  of  $R^k$  of type  $\mathbf{k}$  (hence conjugate shape  $\mu$ ), and include representatives of all nonzero cyclic submodules of  $U$  in the generator matrix  $\mathbf{G}$  as columns. Then the linear code  $C$  generated by  $\mathbf{G}$ , and likewise any other obtained from  $\mathbf{G}$  by repeating each column a positive number of times, will have type  $\mathbf{k}$  as well; see, e.g., [11, Corollary 1].<sup>9</sup> If the rows of the associated weight matrix  $\mathbf{W}$  (obtained in the same way as in our examples) are distinct, applying Lemma 1 to the mutual differences of the rational linear forms represented by the rows of  $\mathbf{W}$  shows the existence of a code of the given type with the indicated number of distinct nonzero weights.

Thus it remains to show that the rows of  $\mathbf{W}$  are distinct, and for this in turn it suffices to show that no two rows of  $\mathbf{W}$  have their zeros in the same positions or, equivalently, codewords of  $C$  with the same positions of zeros are unit multiples of each other.

Codewords of  $C$  are of the form  $\mathbf{xG}$ ,  $\mathbf{x} \in R^k$ , and a column  $\mathbf{u}$  of  $\mathbf{G}$  corresponds to a zero of  $\mathbf{xG}$  iff  $\mathbf{u}$  is in the kernel of the linear form  $R^k \rightarrow R$ ,  $\mathbf{y} \mapsto \mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \dots + x_k y_k$ . Hence we are reduced to showing that linear forms  $w_1, w_2$  on  $U$  with the same kernel  $K \neq U$  must be unit multiples of each other. But in this case  $w_1, w_2$  must have the same image, because by the Homomorphism Theorem the size of the image is the same and  $R$  has only 1 submodule of a given size. If the common image is  $(\theta^i)$ , there exists  $\mathbf{u}$  in  $U$  such that  $w_1(\mathbf{u}) = \theta^i$ , and then clearly  $U = K + R\mathbf{u}$ . On the other hand,  $w_2(\mathbf{u}) = r\theta^i$  for some unit  $r \in R^\times$ , and it follows easily that  $w_2 = r w_1$ .  $\square$

The proof of Theorem 2 is based on Lemma 1 and hence non-constructive, but it is easy to turn it into a construction using the interpretation of the rows of  $\mathbf{W}$  as base- $b$  representations of certain integers. To this end we can

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<sup>9</sup>In [11, Corollary 1] it is proved that the row and column module (“space”) of any matrix over  $R$  have the same type, provided  $R$  is commutative. In the non-commutative case this remains true for the pair “left row/right column module”.



set  $b = q + 1$ , which is larger than all entries of  $\mathbf{W}$ . Choosing the multiplicity vector  $\mathbf{m} = (1, b, b^2, \dots)$  will then result in a code meeting the bound of Theorem 2.

## 4 Minimum Length of MWS codes

From Section 3 we know the number of distinct weights of MWS codes of any given module type. In this section we will discuss the possible lengths of MWS codes in more detail.

Considering again  $R$  as fixed, let us denote the minimum length of a linear  $[n, \mathbf{k}]$  MWS code over  $R$  by  $n(\mathbf{k})$ . By Proposition 1, a trivial lower bound in terms of the residue field size  $q$  of  $R$  is given by

$$n(\mathbf{k}) \geq \left\lceil \frac{L(\mathbf{k})}{q} \right\rceil. \quad (3)$$

A corresponding upper bound may be inferred from the remark at the end of Section 3:

$$n(\mathbf{k}) \leq \sum_{i=0}^{L(\mathbf{k})-1} (q+1)^i = \frac{(q+1)^{L(\mathbf{k})} - 1}{q}, \quad (4)$$

since the number of columns of the weight matrix  $\mathbf{W}$  used in the proof of Theorem 2 is precisely  $L(\mathbf{k})$ .

In the following we will sharpen these bounds in the case  $q = m = 2$ . For simplicity we assume  $R = \mathbb{Z}_4$  in the discussion, although it mostly applies to the other chain ring of order 4, the ring  $\mathbb{F}_2[X]/(X^2)$  of dual numbers over  $\mathbb{F}_2$ , as well.

For  $q = m = 2$  we have  $L(\mathbf{k}) = 2^{\mu_1 + \mu_2 - 1} - 1 + 2^{\mu_1 - 1}$ , and the bound (3) takes the form  $n(\mathbf{k}) \geq 2^{\mu_1 + \mu_2 - 2} + 2^{\mu_1 - 2}$ .

**Theorem 3**  $n(\mathbf{k}) \geq 2^{\mu_1 + \mu_2 - 2} + 2^{\mu_1 - 1} - 1$ .

**Proof.** If  $C$  has conjugate shape  $\mu = (\mu_1, \mu_2)$ , the even-weight subcode  $C_0$  of  $C$  is either equal to  $C$  or, since it contains  $C[2]$ , of conjugate shape  $(\mu_1, \mu_2 - 1)$ . Thus the number of nonzero cyclic submodules of  $C_0$  is at least  $2^{\mu_1 + \mu_2 - 2} - 1 + 2^{\mu_1 - 1}$ . If  $C$  is MWS,  $C_0$  has at least  $2^{\mu_1 + \mu_2 - 2} - 1 + 2^{\mu_1 - 1}$  even weights, which is only possible if  $n \geq 2^{\mu_1 + \mu_2 - 2} - 1 + 2^{\mu_1 - 1}$ .  $\square$



with  $2a_1 = 2a_2$ ,  $2b_1 = 2b_2 = 2b_3 = 2b_4$ , etc.

Let  $\mathbf{c}_1, \dots, \mathbf{c}_k$  be the rows of  $\mathbf{G}$ , in order. Lemma 2 implies that all but the first entry of  $\mathbf{c}_1$  are in  $2\mathbb{Z}_4$ . Indeed, if  $a_1, a_2$  were units in  $\mathbb{Z}_4$ ,  $\text{supp}(2\mathbf{c}_1)$  would strictly contain the support of the 2nd row of  $\mathbf{G}$ , contradicting the lemma; similarly, if  $b_1, b_2, b_3, b_4$  were units,  $\text{supp}(2\mathbf{c}_1)$  would strictly contain the support of the 3rd row of  $\mathbf{G}$ , again contradicting the lemma; etc.

Then, since the subcode generated by  $\mathbf{c}_2, \dots, \mathbf{c}_k$  has all weights divisible by 4, the odd-weight subcode  $C_1$  of  $C$  must either have (i) all weights  $\equiv 1 \pmod{4}$  or (ii) all weights  $\equiv 3 \pmod{4}$ . Since  $C$  has exactly  $2^{k-1}$  odd weights and  $3 + (2^{k-1} - 1)4 = 2^{k+1} - 1$  exceeds twice the length of  $C$  (the maximum possible weight), Case (ii) cannot occur. This leaves odd weights  $1, 5, \dots, 2^{k+1} - 3$  as only possibility. Thus we may assume  $w_{\text{Lee}}(\mathbf{c}_1) = 1$ , i.e.,  $\mathbf{c}_1 = (1, 0, 0, \dots)$ , and the uniqueness proof is complete.  $\square$

Further we are able to settle the case  $k_1 = 2$ ,  $k_2 = 0$  corresponding to the projective Hjelmslev line  $\text{PHG}(1, \mathbb{Z}_4)$  completely:

**Theorem 5** *For  $R = \mathbb{Z}_4$  we have  $n(2, 0) = 8$ . Moreover, the  $[8, (2, 0)]$  MWS code is unique up to equivalence.*

**Proof.** We have  $L(2, 0) = 6 + 3 = 9$ , and the  $\mathbb{Z}_4$ -linear code generated by

$$\mathbf{G} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix},$$

whose nonzero codewords (up to scalar multiples) are listed in the 2nd column of the following table, is an  $[8, (2, 0)]$  MWS code:

$\mathbf{x}$	$\mathbf{xG}$	$w_{\text{Lee}}(\mathbf{xG})$
1 0	1 1 1 1 0 0 0 1	5
0 1	0 0 0 0 1 1 1 1	4
1 2	1 1 1 1 2 2 2 3	11
2 1	2 2 2 2 1 1 1 3	12
1 1	1 1 1 1 1 1 1 2	9
1 3	1 1 1 1 3 3 3 0	7
2 0	2 2 2 2 0 0 0 2	10
0 2	0 0 0 0 2 2 2 2	8
2 2	2 2 2 2 2 2 2 0	14

(5)

Now suppose  $C$  is an MWS code over  $\mathbb{Z}_4$  of type  $(2, 0)$ , i.e., free of rank 2, and minimum length  $n = n(2, 0)$ . Proving uniqueness of  $C$  amounts to showing

that the multiset  $\mathfrak{K}$  of points associated with  $C$  in the extended projective Hjelmslev line  $\overline{\text{PHG}}(1, \mathbb{Z}_4)$ , which contains the six points of  $\text{PHG}(1, \mathbb{Z}_4)$  and the three degenerate points  $\mathbb{Z}_4(2, 0)$ ,  $\mathbb{Z}_4(0, 2)$ ,  $\mathbb{Z}_4(2, 2)$ , must have the following form: There exist three mutually non-neighbor points  $P_1, P_2, P_3$  with multiplicities  $\mathfrak{K}(P_1) = 4$ ,  $\mathfrak{K}(P_2) = 3$ ,  $\mathfrak{K}(P_3) = 1$ , and  $\mathfrak{K}(P) = 0$  for the remaining points of  $\overline{\text{PHG}}(1, \mathbb{Z}_4)$ .

Since  $n \leq 8$ ,  $C$  must have at least one odd weight, implying that the even-weight subcode  $C_0$  of  $C$  has type  $(1, 1)$ . Thus  $C$  has 5 even weights, of which 3 are in  $C[2]$ , and 4 odd weights. Denoting the weights in  $C[2]$  by  $s_1 < s_2 < s_3$ , the remaining even weights by  $e_1 < e_2$ , and the odd weights by  $o_1 < o_2 < o_3 < o_4$ , we have

$$\begin{aligned} s_1 + s_2 + s_3 &= 4 \, \mathfrak{n}(C[2]), \\ s_1 + s_2 + s_3 + 2(e_1 + e_2) &= 8 \, \mathfrak{n}(C_0), \\ s_1 + s_2 + s_3 + 2(e_1 + e_2 + o_1 + o_2 + o_3 + o_4) &= 16 \, \mathfrak{n}(C); \end{aligned}$$

cf. [5]. Now we claim that

1. The multiplicities  $\mathfrak{K}([P_i])$  of the three point neighbor classes  $\mathfrak{K}([P_1])$ ,  $\mathfrak{K}([P_2])$ ,  $\mathfrak{K}([P_3])$  are distinct.
2. The multiplicities  $\mathfrak{K}(P_i), \mathfrak{K}(P'_i)$  of the two points  $P_i, P'_i$  in each neighbor class  $[P_i]$  are distinct.

The first assertion expresses the fact that the nonzero weights in  $C[2]$  are distinct; the second that the weights of  $\mathbf{xG}, \mathbf{x}'G$  for neighbor points  $\mathbb{Z}_4\mathbf{x}, \mathbb{Z}_4\mathbf{x}'$  of  $\text{PHG}(1, \mathbb{Z}_4)$  are distinct; cf. [12, Theorem 5.2].

This leaves the possibilities

$\mathfrak{n}(C[2])$	$\mathfrak{K}([P_1])$	$\mathfrak{K}([P_2])$	$\mathfrak{K}([P_3])$	$s_1$	$s_2$	$s_3$
6	3	2	1	6	8	10
7	4	2	1	6	10	12
8	5	2	1	6	12	14
8	4	3	1	8	10	14

(6)

For this note that, since  $2C = C[2]$ , the number of ordinary points of  $\mathfrak{K}$  is equal to  $\mathfrak{n}(C[2])$ .<sup>11</sup>

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<sup>11</sup>In general this number is equal to  $\mathfrak{n}(2C)$ .

Assuming, for the moment, that there are no degenerate points in  $\mathfrak{K}$ , the remaining even weights  $e_1, e_2$  are 4, 8 in the 1st case, 6, 8 in the 2nd case, and 6, 10 in the 3rd case. In the 4th case there are two possibilities, viz., 6, 10 for  $\{\mathfrak{K}(P_1), \mathfrak{K}(P'_1)\} = \{3, 1\}$ , and 4, 12 for  $\{\mathfrak{K}(P_1), \mathfrak{K}(P'_1)\} = \{4, 0\}$ . Only the last subcase produces a code with five distinct even weights.

Hence  $\{\mathfrak{K}(P_1), \mathfrak{K}(P'_1)\} = \{4, 0\}$  and, trivially,  $\{\mathfrak{K}(P_3), \mathfrak{K}(P'_3)\} = \{1, 0\}$ . In the case  $\{\mathfrak{K}(P_2), \mathfrak{K}(P'_2)\} = \{2, 1\}$  the odd weights are 7, 7, 9, 9, so that this case is also excluded. Thus necessarily  $\{\mathfrak{K}(P_2), \mathfrak{K}(P'_2)\} = \{3, 0\}$  and  $C$  is equivalent to the code in (5).

Finally it remains to exclude degenerate points in the cases with  $n(C[2]) < 8$ . Since the codewords in  $C_0 \setminus C[2]$  are neighbors, they have the same entries in the positions corresponding to degenerate points. Codewords in  $C[2]$  have zeros there. Thus by adding one or two degenerate points it is only possible to increase both  $e_1, e_2$  by 2 or 4, while  $s_1, s_2, s_3$  remain unchanged. But in Cases 1 and 2 above at least one of  $e_1 + 2, e_2 + 2$  and at least one of  $e_1 + 4$  and  $e_2 + 4$  are in  $\{s_1, s_2, s_3\}$ , contradiction.  $\square$

Determining the numbers  $n(2, 1)$  and  $n(3, 0)$  and classifying the corresponding minimum-length MWS codes seems to be considerably more difficult. A reasonable approach is to choose the support  $S$  of the associated multiset of points in  $\overline{\text{PHG}(2, \mathbb{Z}_4)}$  as small as possible, and try to determine corresponding multiplicities either by adhoc methods or computer search.

From Lemma 1 and the proof of [Theorem 2](#) that there exists a multiset  $\mathfrak{K}$  (of possibly large cardinality) that has  $\text{supp}(\mathfrak{K}) = S$  and is associated with an MWS code if and only if the rows of the weight matrix  $\mathbf{W}$  (cf. Section 3) corresponding to  $S$  are distinct. Such point sets will be called *starter sets* in the sequel. Since  $\mathfrak{K}$  and  $\text{supp} \mathfrak{K}$  have the same span, which in turn has the same module type  $\mathbf{k}$  as any associated linear code, we can speak of “starter sets of type  $\mathbf{k}$ ” (in place of the longer “starter sets for MWS codes of type  $\mathbf{k}$ ”).

**Example 7** *Inspecting the proof of [Theorem 4](#) we see that the points  $R(1, 0, 0, \dots)$ ,  $R(0, 2, 0, 0, \dots)$ ,  $\dots$ ,  $R(0, \dots, 0, 2)$  in  $\text{PHG}(k - 1, \mathbb{Z}_4)$  form a starter set  $S$  of type  $(1, k - 1)$ . It is clear that  $S$  is a minimal starter set for this type.*

**Example 8** *Minimal starter sets of type  $(2, 0)$  consist of three points on the projective Hjelmslev line  $\text{PHG}(1, \mathbb{Z}_4)$  that are mutually distant (i.e., non-neighbors). For example, we can take  $\mathbb{Z}_4(1, 0), \mathbb{Z}_4(0, 1), \mathbb{Z}_4(1, 1)$ .*

**Example 9** *The three points  $\mathbb{Z}_4(1, 0, 0)$ ,  $\mathbb{Z}_4(0, 1, 0)$ ,  $\mathbb{Z}_4(1, 1, 0)$  of the Hjelmslev plane  $\text{PHG}(2, \mathbb{Z}_4)$  together with the distant degenerate point  $\mathbb{Z}_4(0, 0, 2)$  of  $\overline{\text{PHG}(2, \mathbb{Z}_4)}$  form a minimal starter set of type  $(2, 1)$ .*

**Example 10** *Minimal starter sets of type  $(3, 0)$  contain six points in  $\text{PHG}(2, \mathbb{Z}_4)$  that are mutually non-neighbors.*

It turns out that in the case  $\mathbf{k} = (k, 0)$  starter sets are the same as so-called strong blocking sets, which are investigated by many authors these days because of their relation with minimal linear codes. We do not delve further into this subject, but just provide the definition and one recent reference, where interested readers can find more.

A set of points in a classical projective space  $\text{PG}(k - 1, \mathbb{F}_q)$  is said to be a *strong blocking set* (or *cutting blocking set*) if the intersection with any hyperplane is a spanning set of the hyperplane. This property is considerably stronger than that of an ordinary blocking set (with respect to hyperplanes), which only requires the intersection with any hyperplane to be nonempty. For a recent authoritative account of strong blocking sets we recommend [10].

In the plane case  $k = 3$  strong blocking sets are equivalent to 2-blocking sets (point sets meeting every line of  $\text{PG}(2, \mathbb{F}_q)$  in at least two points), and thus the unique minimal example in the Fano plane  $\text{PG}(2, \mathbb{F}_2)$  contains 6 points; compare with [Example 10](#) above. In  $\text{PG}(3, \mathbb{F}_2)$  minimal examples contain 9 points on three mutually disjoint lines.

**Theorem 6** *A set  $\mathfrak{K}$  of (ordinary) points in the projective Hjelmslev geometry  $\text{PHG}(k - 1, \mathbb{Z}_4)$  is a starter set of type  $(k, 0)$  if and only if the point neighbor classes  $[P]$  with  $\mathfrak{K}([P]) > 0$  form a strong blocking set in the quotient geometry  $\text{PG}(k - 1, \mathbb{F}_2)$ .*

Thus in the case  $R = \mathbb{Z}_4$  minimal starter sets  $S$  of type  $(k, 0)$  are obtained by selecting a set  $\overline{S}$  of point neighbor classes in the quotient geometry that forms a minimal strong blocking set and then arbitrarily one representative point from each class in  $\overline{S}$  to form the set  $S$ .

The rather technical proof of [Theorem 6](#) is omitted.

**Proposition 2** *When  $q = 2$ , let  $C$  be an  $[n, k_1, k_2]$  MWS code. Then an  $[2n + 1, k_1, k_2 + 1]$  MWS code exists.*

**Proof.** Let  $C$  be a linear code over a chain ring of length 2 with the following generator matrix

$$G_{k_1, k_2} = \begin{pmatrix} I_{k_1} & A_{11} & A_{12} \\ 0 & 2I_{k_2} & 2A_{22} \end{pmatrix},$$

where  $I_{k_i}$  is the  $k_i \times k_i$  identity matrix and  $A_{i,j}$  are matrices over a chain ring of length 2 for all  $i, j \in \{1, 2\}$ . Since  $C$  is an MWS code, we know that  $C$  has  $2^{2k_1+k_2-1} + 2^{k_1+k_2-1} - 1$  different weights by Theorem 2. Let us call it  $w_1 < w_2 < \dots < w_{2^{2k_1+k_2-1}+2^{k_1+k_2-1}-1}$ . The construction  $G_{k_1, k_2+1}$  by taking the following form

$$G_{k_1, k_2+1} = \left( \begin{array}{c|ccc} & 0 & \dots & 0 \\ & \vdots & & \vdots \\ G_{k_1, k_2} & & & \\ & 0 & \dots & 0 \\ \hline 0 \dots 0 & 2 & \dots & 2 \end{array} \right),$$

where the last row is made up of  $n$  0's and  $(n+1)$  2's. Thus the code spanned by the rows of  $G_{k_1, k_2+1}$  has new weights  $2(n+1) < 2(n+1) + w_1 < \dots < 2(n+1) + w_{2^{2k_1+k_2-1}+2^{k_1+k_2-1}-1}$ . The number of weights of the code generated by  $G_{k_1, k_2+1}$  is denoted as  $N_{k_1, k_2}$ . Since  $w_{2^{2k_1+k_2-1}+2^{k_1+k_2-1}-1} \leq 2n < 2(n+1)$ , then  $N_{k_1, k_2} \geq 2L_2(n, k_1, k_2) + 1$ . Hence, we have

$$N_{k_1, k_2} \geq 2^{2k_1+k_2} + 2^{k_1+k_2} - 1.$$

By Theorem 1, we have

$$\begin{aligned} N_{k_1, k_2} &\leq \frac{q^{k_1+k_2+1}(q^{k_1} - 1)}{q(q-1)} + \frac{q^{k_1+k_2+1} - 1}{q-1} \\ &= 2^{2k_1+k_2} + 2^{k_1+k_2} - 1. \end{aligned}$$

Thus,

$$N_{k_1, k_2} = 2^{2k_1+k_2} + 2^{k_1+k_2} - 1.$$

Hence, the code spanned by the rows of  $G_{k_1, k_2+1}$  is an  $[2n+1, k_1, k_2+1]$  MWS code.  $\square$

**Corollary 1** *When  $q = 2$ , let  $C$  be an  $[n, \mathbf{k}]$  MWS code. Then an  $[2n+1, \mathbf{k}+1]$  MWS code exists.*

**Proof.** Let  $C$  be a linear code over  $R$  with the following generator matrix

$$G_{k_1, k_2, \dots, k_m} = \begin{pmatrix} I_{k_1} & A_{11} & A_{12} & A_{13} & \dots & A_{1, m-1} & A_{1, m} \\ 0 & 2I_{k_2} & 2A_{22} & 2A_{23} & \dots & 2A_{2, m-1} & 2A_{2, m} \\ 0 & 0 & 2^2I_{k_3} & 2^2A_{33} & \dots & 2^2A_{3, m-1} & 2^2A_{3, m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2^{m-1}I_{k_m} & 2^{m-1}A_{m, m} \end{pmatrix},$$

where  $I_{k_i}$  is the  $k_i \times k_i$  identity matrix and  $A_{i, j}$  are matrices over  $R$  for all  $i, j \in \{1, 2, \dots, m\}$ . Since  $C$  is an MWS code, we know that  $C$  has  $L_2(k_1, k_2, \dots, k_m)$  different weights. Let us call it  $w_1 < w_2 < \dots < w_{L_2(k_1, k_2, \dots, k_m)}$ . The construction  $G_{k_1, k_2, \dots, k_{m+1}}$  by taking the following form

$$G_{k_1, k_2, \dots, k_{m+1}} = \left( \begin{array}{c|ccc} & 0 & \dots & 0 \\ G_{k_1, k_2, \dots, k_m} & \vdots & & \vdots \\ & 0 & \dots & 0 \\ \hline 0 \dots 0 & 2^{m-1} & \dots & 2^{m-1} \end{array} \right)$$

where the last row is made up of  $n$  0's and  $(n+1)$   $2^{m-1}$ 's. Thus the code spanned by the rows of  $G_{k_1, k_2, \dots, k_{m+1}}$  has new weights  $2^{m-1}(n+1) < 2^{m-1}(n+1) + w_1 < \dots < 2^{m-1}(n+1) + w_{L_2(k_1, k_2, \dots, k_m)}$ . The number of weights of the code generated by  $G_{k_1, k_2, \dots, k_{m+1}}$  is denoted as  $N_{k_1, k_2, \dots, k_m}$ . Since  $w_{L_2(k_1, k_2, \dots, k_m)} \leq 2^{m-1}n < 2^{m-1}(n+1)$ , then we have

$$\begin{aligned} N_{k_1, k_2, \dots, k_m} &\geq 2L_2(k_1, k_2, \dots, k_m) + 1 \\ &= L_2(k_1, k_2, \dots, k_m + 1). \end{aligned}$$

By Theorem 1, we have

$$N_{k_1, k_2, \dots, k_m} \leq L_2(k_1, k_2, \dots, k_m + 1).$$

Thus,

$$N_{k_1, k_2, \dots, k_m} = L_2(k_1, k_2, \dots, k_m + 1).$$

□

## 5 Conclusion and open questions

In this paper we have solved the problem of maximizing the number of distinct homogeneous weights that a linear code over a chain ring may have.



We determined in a non-constructive manner the number of distinct weights of an MWS code of any given module type over any finite chain ring. As we have remarked, it is not difficult to turn the proof into a code construction, but the resulting codes have extremely large lengths, and the determination of the minimum length of a  $[k_1, k_2, \dots, k_m]_q$  MWS code remains a challenging problem. We initiated work on this problem in Section 4, obtaining an upper bound on the length of MWS codes over chain rings with  $q = 2$ . This bound may still be rather weak, however. Moreover, some basic questions about the structure of MWS codes over chain rings remain unsolved. Is it possible to characterize these codes geometrically? Do there exist such codes with further desirable properties, such as regular and/or projective codes?

## 6 Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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