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# Patterns in the Keller-Segel system with density cut-off

Benoît Perthame\*      Mingyue Zhang\*†

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## Abstract

The Patlak-Keller-Segel system with logistic sensitivity has been widely advocated as a model which avoids over-crowding and generates complex patterns. Here we also consider the general case of a nonlinear diffusion of porous medium type with exponent  $m$ . The complexity of the observed patterns makes it complicated to understand the processes at work. Here, we analyze the pattern formation ability of such a system, which depends highly on  $m$  and three different analyses are needed for  $m = 1$  (linear diffusion), for  $1 < m < 2$  and  $m \geq 2$ . Within these regimes the sensitivity also plays a crucial role as well as the conserved total mass. Typically small mass patterns exist for  $m > 2$  but not for  $m < 2$ . We focus specifically on the conditions for long term convergence to the constant solution, uniqueness of the steady state and on the contrary, existence of increasing steady solutions in dimension one. Our method is based on several tools, energy functional, reduction to a single equation, reduction to a first order equation. A major difficulty, in opposition to the case  $m = 1$ , is that solutions can vanish locally when  $m > 1$ .

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## Introduction

The Patlak-Keller-Segel system with logistic sensitivity and nonlinear diffusion has been advocated as a realistic and general model of cell movement. It is written

$$\begin{cases} \partial_t u - \Delta \frac{u^m}{m} + \operatorname{div}[\chi u(1-u)\nabla v] = 0, & x \in \Omega, \quad t > 0, \\ -D\Delta v + v = u, & x \in \Omega, \quad t > 0, \\ \nabla u \cdot \vec{n} = \nabla v \cdot \vec{n} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(0, x) = u^0(x), \quad 0 < u^0 < 1, & x \in \Omega. \end{cases} \quad (1)$$

Here,  $u(t, x)$  and  $v(t, x)$  represent the densities of cells and the chemical concentration, respectively. The constant parameter  $\chi > 0$  denotes the chemotactic sensitivity, and  $D > 0$  represents the chemical

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diffusion rate. The diffusivity exponent  $m \geq 1$  corresponds to slow diffusion. The domain  $\Omega \in \mathbb{R}^n$  is bounded with smooth boundary and  $\vec{n}$  is the unit normal vector. Integrating system (1) over  $\Omega$ , and using the Neumann boundary conditions, we obtain

$$M := \int_{\Omega} u(t, x) dx = \int_{\Omega} v(t, x) dx = \int_{\Omega} u^0(x) dx \leq |\Omega| = 1. \quad (2)$$

We only consider solutions satisfying

$$0 \leq u(t, x), v(t, x) \leq 1. \quad (3)$$

Thanks to the maximum principle, (3) can be ensured in  $[0, \infty) \times \Omega$  under the initial assumptions in (1).

We are more specifically interested in the pattern formation ability of such a system when  $m > 1$ . The case  $m = 1$  has been studied in [13, 21] and the complexity of possible patterns is remarkable; the solution evolves with different scales to stabilize to constant or oscillatory steady states. However, the parameter  $m$  plays a crucial role and three regimes occur. The case when  $m = 1$  (linear diffusion) appears to be very different and simpler than  $m > 1$ . For example steady states are positive for  $m = 1$ , but can be zero on a part of the domain for  $m > 1$ . But also the cases  $1 < m < 2$  and  $m = 2$  and  $m > 2$  turn out to be different. Typically small mass patterns exist for  $m \leq 2$  but not for  $m > 2$ . Within these regimes, the sensitivity  $\chi$  also plays a role as well as the conserved total mass. We study the conditions for long term convergence to the constant solution based on the energy of the system. Then we focus on conditions imposing that the constant  $M$  is the only steady state and, on the contrary, that increasing steady states exist in dimension one. Rather than bifurcation analysis as in the previous studies, our method is based on the reduction to a first order equation which solution generates an implicit.

Chemotaxis is the phenomenon where cells and organisms actively move along chemical gradients. This process has been modeled in a wide range of examples, including bacteria, slime molds, skin pigmentation patterns, and leukocytes, among others. One of the most well-known mathematical models for chemotaxis is the Patlak-Keller-Segel system, which was derived in [20, 15]. The simplicity of this model has made it widely applicable in various biological systems, see [17, 24, 12, 18]. It has also attracted the attention of mathematicians because, in dimension higher than two, solutions can exhibit global existence or blow-up, depending on the mass and many works are devoted to its analysis, see for instance [23, 4, 3].

Many modifications of the initial Keller-Segel system were introduced in order to improve the biophysical relevance of the model, including either a nonlinear diffusion coefficient or nonlinear chemotactic sensitivity. A large amount of literature assumes a nonlinear diffusion coefficient depending on  $u$ , often of porous medium type, for instance [16, 7, 6, 25] discuss the question of blow-up or global existence, uniqueness and the large time asymptotic behavior. Nonlinear sensitivities has also been studied, with the simplest nonlinear sensitivity function being the logistic case. Conditions for existence and blow-up are studied in [14, 10, 11], and [19] established the correlation between diffusion and sensitivity by using the volume filling mechanism. Numerical scheme preserving energy are proposed in [22, 8, 5]. It is numerically observed that complex patterns can be generated by such Keller-Segel systems and several theoretical studies confirm it [13, 21, 2, 9].

This paper is structured as follows. In Section 1, we present the energy and entropy identities, and the corresponding meaning for weak solutions of system (1). Based on energy, we prove for large enough mass, i.e.,  $M \approx 1$ , that the energy solutions converge to the constant solution  $M$  over time. Linear stability analysis confirms that the large mass condition is necessary. Section 2 is devoted to steady states and to reduce system (1) to a single equation with respect to  $v$  which requires some notations and in particular a function we call  $\Lambda$  which is instrumental for the sequel. These notations allow to establish the uniqueness of the steady state for large mass. Starting from Section 3, we study the steady state solutions in one dimension. We can now quantify, in terms of  $M$  and  $\chi$ , the uniqueness of the steady state. This is based on a further reduction to a first order equation on  $v$ . Then, in Section 4, we study the existence of the increasing steady state. The conditions depend heavily on the choice of parameters,  $m$ ,  $\chi$ ,  $D$  and  $M$ . We first establish a rough threshold for positive solutions for  $m \geq 1$ . Then, for  $m > 1$ , non-negative solutions, with  $u = 0$  within certain intervals, are built. Section 5 presents simulation results illustrating the patterns observed in the solutions.

## 1 Energy, time decay, stability

### 1.1 Energy, entropy

Classically, system (1) comes with an energy (see also [1]). We set

$$\Phi(u) = \int_0^u \varphi(w)dw, \quad \varphi(u) = \begin{cases} \int_0^u \frac{w^{m-2}}{1-w}dw, & m > 1, \\ \ln \frac{u}{1-u} = \int_{\frac{1}{2}}^u \frac{w^{-1}}{1-w}dw, & m = 1, \end{cases} \quad (4)$$

and notice that  $\varphi(u)$  is concave-convex for  $1 \leq m < 2$  and is convex for  $m \geq 2$  (see Fig. 1) since

$$\varphi''(u) = \frac{u^{m-3}}{(1-u)^2} (m-2 + (3-m)u). \quad (5)$$

One can check that smooth solutions satisfy the energy equality

$$\begin{cases} \mathcal{E}(u) = \int_{\Omega} [\Phi(u) - \chi uv + \frac{\chi D}{2} |\nabla v|^2 + \frac{\chi}{2} v^2] dx = \int_{\Omega} [\Phi(u) - \frac{\chi D}{2} |\nabla v|^2 - \frac{\chi}{2} v^2] dx, \\ \frac{d}{dt} \mathcal{E}(u(t)) = - \int_{\Omega} u(1-u) |\nabla(\varphi(u) - \chi v)|^2 dx = - \int_{\Omega} \frac{1}{u(1-u)} |\nabla \frac{u^m}{m} - \chi u(1-u) \nabla v|^2 dx. \end{cases} \quad (6)$$

An obvious observation is that, because  $u$ ,  $v$  are both less than 1, the energy is bounded from below

$$\mathcal{E}(u) \geq \mathcal{E}_{\min}(M, \chi).$$

From this, we conclude that

$$4 \int_0^\infty \int_{\Omega} |\nabla \frac{u^m}{m} - \chi u(1-u) \nabla v|^2 dx dt \leq \mathcal{E}(u^0) - \mathcal{E}_{\min}(M, \chi). \quad (7)$$

Furthermore, energy and energy dissipation determine the framework for weak solutions.

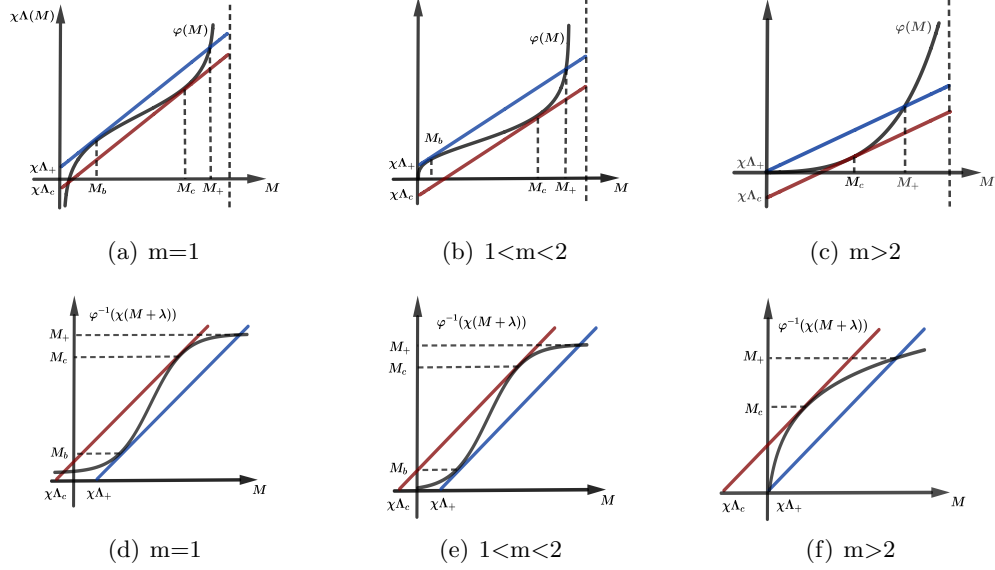


Figure 1: Top line, the function  $\varphi$  in (4) (black curve) and the construction of the mapping  $M \mapsto \varphi(M)$  in (21). Bottom line, the function  $\varphi^{-1}$ .

**Definition 1 (Weak solution)** *The couple  $(u, v)$  defined in  $[0, T) \times \Omega$  is said to be a weak solution of system (1) provided that for any  $T > 0$ ,*

$$0 \leq u, v \leq 1, \quad u^m \in L^2(0, T; H^1(\Omega)), \quad v \in L^\infty(0, T; H^1(\Omega)),$$

*and the equations hold in the sense of distributions; i.e., for any test functions  $\phi$  and  $\psi$  satisfying the Neumann boundary conditions and  $\phi(T, x) = 0$ , there exist*

$$\begin{cases} \int_0^T \int_\Omega (\nabla \frac{u^m}{m} \cdot \nabla \phi - \chi u(1-u) \nabla v \cdot \nabla \phi - u \phi_t) dx dt = \int_\Omega u^0(x) \phi(0, x) dx, & \forall \phi \in C^\infty([0, T) \times \Omega), \\ \int_\Omega (\nabla v \cdot \nabla \psi + v \psi - u \psi) dx = 0, & a.e. t \in [0, T), \quad \forall \psi \in H^1(\Omega). \end{cases}$$

*An energy solution is a weak solution that satisfies the energy inequality (7).*

We can also check that smooth solutions satisfy the entropy equality

$$\begin{cases} E(t) := \int_\Omega [u \ln u + (1-u) \ln(1-u)] dx, \\ \frac{dE}{dt} = \int_\Omega [-\varphi'(u) |\nabla u|^2 + \chi u(u-v)] dx. \end{cases} \quad (8)$$

Indeed, by (1)<sub>1</sub> we obtain

$$\begin{aligned} \frac{dE}{dt} &= \int_\Omega [\ln u - \ln(1-u)] u_t dx = - \int_\Omega \left( \frac{1}{u} + \frac{1}{1-u} \right) \nabla u \cdot \left[ \nabla \frac{u^m}{m} - \chi u(1-u) \nabla v \right] dx \\ &= - \int_\Omega [\varphi'(u) |\nabla u|^2 + \chi u \Delta v] dx. \end{aligned}$$

and we conclude using (1)<sub>2</sub>.

## 1.2 Long time behavior of solutions for large mass

For  $M$  large enough, we establish the long term convergence of the weak solution to  $M$ . We begin with introducing the  $L^p$  norm.

**Definition 2** For  $1 \leq p < +\infty$  and  $k \in \mathbb{N}^+$ , the  $L^p((k, k+1) \times \Omega)$  norm of a function  $f$  is

$$\|f\|_{L^p((k,k+1) \times \Omega)} = \left( \int_k^{k+1} \int_{\Omega} |f|^p dx dt \right)^{\frac{1}{p}}.$$

**Theorem 3 (Long time convergence for large mass)** Let  $(u, v)$  be an energy solution of system (1) for which we have, in the weak sense (see Def. 1). For  $M$  large enough, we have

$$\lim_{k \rightarrow \infty} \|(u - M)(t)\|_{L^p((k,k+1) \times \Omega)} = 0, \quad \lim_{t \rightarrow \infty} \|(v - M)(t)\|_{L^p(\Omega)} = 0, \quad p \in [1, \infty).$$

**Proof.** For an increasing sequence  $\{k\}$ ,  $k \in \mathbb{N}^+$ , we define the sequences  $\{u^k\}$  and  $\{v^k\}$  as

$$u^k(t, x) := u(k + t, x), \quad v^k(t, x) := v(k + t, x), \quad x \in \Omega, \quad 0 \leq t \leq T,$$

for which we have that  $(u^k, v^k)$  is the weak solution satisfying

$$\begin{cases} \partial_t u^k - \Delta \frac{(u^k)^m}{m} + \operatorname{div}[\chi u^k (1 - u^k) \nabla v^k] = 0, \\ -D\Delta v^k + v^k = u^k. \end{cases} \quad (9)$$

**Step 1 (Weak-\* convergence of  $(u^k, v^k)$ ).** Because of (3) and the energy dissipation (7), the solution  $(u^k, v^k)$  satisfies

$$0 \leq u^k, v^k \leq 1, \quad \|(u^k)^m\|_{L^2(0,T;H^1(\Omega))} \leq C, \quad \|v^k\|_{L^\infty(0,T;H^1(\Omega))} \leq C. \quad (10)$$

To obtain the last of these bounds, we multiply (9)<sub>2</sub> by  $v^k$  and integrate in  $\Omega$ , by the Young inequality, we get

$$D\|\nabla v^k\|_{L^2(\Omega)}^2 + \|v^k\|_{L^2(\Omega)}^2 = \|u^k v^k\|_{L^1(\Omega)} \leq \frac{1}{2}\|v^k\|_{L^2(\Omega)}^2 + \frac{1}{2}\|u^k\|_{L^2(\Omega)}^2,$$

which implies the announced bound.

Therefore, there are subsequences, still be denoted by  $\{u^k\}$  and  $\{v^k\}$ , and functions  $\bar{u} \in L^\infty((0, T) \times \Omega)$  and  $\bar{v} \in L^\infty(0, T; H^1(\Omega))$  such that

$$\begin{cases} u^k \rightharpoonup \bar{u} & \text{weakly star} & \text{in } L^\infty((0, T) \times \Omega), \\ v^k \rightharpoonup \bar{v} & \text{weakly star} & \text{in } L^\infty(0, T; H^1(\Omega)). \end{cases} \quad (11)$$

**Step 2 (Strong convergence of  $u^k$ ).** We verify that the solution  $(u^k, v^k)$  converges to  $(\bar{u}, \bar{v})$  strongly in  $L^2((0, T) \times \Omega)$  as  $k \rightarrow \infty$ . On the one hand, it follows from (7) that

$$\|\partial_t u^k\|_{L^2(0, T; H^{-1}(\Omega))} = \left\| \nabla \frac{(u^k)^m}{m} - \chi u^k (1 - u^k) \nabla v^k \right\|_{L^2((0, T) \times \Omega)} \leq C. \quad (12)$$

On the other hand, it follows from (7) and (10) that

$$\left\| \nabla \frac{(u^k)^m}{m} \right\|_{L^2((0, T) \times \Omega)} \leq \left\| \nabla \frac{(u^k)^m}{m} - \chi u^k (1 - u^k) \nabla v^k \right\|_{L^2((0, T) \times \Omega)} + C \leq C.$$

Therefore  $\{u^k\}$  is compact in space and we can apply the Aubin-Lions lemma to obtain the strong convergence, that is, up to a subsequence, we have for  $1 \leq p < \infty$ ,

$$u^k \rightarrow \bar{u} \quad \text{in } L^p((0, T) \times \Omega) \quad \text{as } k \rightarrow \infty. \quad (13)$$

Therefore, we conclude from (9) and (13) that  $(\bar{u}, \bar{v})$  is the weak solution satisfying

$$\begin{cases} \partial_t \bar{u} - \Delta \frac{\bar{u}^m}{m} + \operatorname{div}[\chi \bar{u} (1 - \bar{u}) \nabla \bar{v}] = 0, \\ -D \Delta \bar{v} + \bar{v} = \bar{u}. \end{cases} \quad (14)$$

**Step 3 (Strong convergence of  $v^k$ ).** We use (9) and (14), then

$$-D \Delta (v^k - \bar{v}) + (v^k - \bar{v}) = u^k - \bar{u}.$$

Multiplying by  $\nabla (v^k - \bar{v})$  and integrating in  $\Omega$  respect to  $x$ , by Young inequality, we have

$$\begin{aligned} D \|\nabla (v^k - \bar{v})\|_{L^2(\Omega)}^2 + \|v^k - \bar{v}\|_{L^2(\Omega)}^2 &= \|(u^k - \bar{u})(v^k - \bar{v})\|_{L^1(\Omega)} \\ &\leq \frac{1}{2} \|v^k - \bar{v}\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u^k - \bar{u}\|_{L^2(\Omega)}^2. \end{aligned}$$

Because of (11) and (13), we conclude that

$$v^k \rightarrow \bar{v} \quad \text{in } L^2(0, T; H^1(\Omega)) \quad \text{as } k \rightarrow \infty. \quad (15)$$

**Step 4 (Identifying  $(\bar{u}, \bar{v})$ ).** We are going to show that  $(\bar{u}, \bar{v}) = (M, M)$ . From the energy inequality (7), we conclude that

$$\sum_{k=1}^{\infty} \int_k^{k+1} \int_{\Omega} \left| \nabla \frac{u^m}{m} - \chi u (1 - u) \nabla v \right|^2 dx dt < \infty,$$

which implies

$$\int_0^T \int_{\Omega} \left| \nabla \frac{(u^k)^m}{m} - \chi u^k (1 - u^k) \nabla v^k \right|^2 dx dt \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

This, together with (10) and (13), by the Lebesgue dominated convergence theorem, for any test function  $\phi \in C_0^\infty((0, 1) \times \Omega)$ , yields that

$$\int_0^T \int_{\Omega} \bar{u} \partial_t \phi dx dt = \lim_{k \rightarrow \infty} \int_0^T \int_{\Omega} \left( \nabla \frac{(u^k)^m}{m} - \chi u^k (1 - u^k) \nabla v^k \right) \cdot \nabla \phi dx dt = 0.$$

Thus,  $\bar{u} = \bar{u}(x)$  is independent of  $t$ . From (14)<sub>2</sub>, it follows that  $\bar{v} = \bar{v}(x)$  is independent of  $t$ .

By the uniqueness of the steady state for large mass, which follows from Theorem 6, we obtain that  $\bar{u} = \bar{v} = M$ , hence the result for  $u$ .

**Step 5 (Pointwise convergence for  $v$ ).** Using (12), we infer that also  $\|\partial_t v^k\|_{L^2(0,T;H^{-1}(\Omega))}$  is bounded, and since  $\|v^k\|_{L^2(0,T;H^1(\Omega))} \leq C$ , we finally obtain that  $v^k$  converges in  $C(0,T;L^2(\Omega))$  and thus the pointwise limit

$$\lim_{t \rightarrow \infty} \|(v - M)(t)\|_{L^2(\Omega)} = 0.$$

By interpolation with  $L^1$  and  $L^\infty$ , this concludes the proof of Theorem 3.  $\square$

### 1.3 Linear stability results

The linearisation of system (1) around  $M$  is, with Neumann boundary conditions,

$$\begin{cases} u_t - M^{m-1}\Delta u + \chi M(1 - M)\Delta v = 0, & x \in \Omega, \quad t > 0, \\ -D\Delta v + v = u, & x \in \Omega, \quad t > 0. \end{cases} \quad (16)$$

As usual we consider an eigenvalue  $\nu_s$  of the Laplacian, with eigenfunction  $w(x)$ , i.e.,  $-\Delta w(x) = \nu_s w(x)$  with Neumann boundary condition. Using  $u = u_L e^{\nu_L t} w(x)$  and  $v = v_L e^{\nu_L t} w(x)$ , where  $\nu_L$  denotes the temporal growth rate. Then system (16) becomes

$$\begin{cases} \nu_L u_L + M^{m-1}\nu_s u_L - \chi M(1 - M)\nu_s v_L = 0, \\ D\nu_s v_L + v_L = u_L, \end{cases}$$

which gives

$$\nu_L = -M^{m-1}\nu_s + \frac{\chi M(1 - M)\nu_s}{D\nu_s + 1}.$$

The constant steady state  $(M, M)$  is linearly stable if and only if  $\nu_L$  is negative. Therefore, for  $\nu_s \neq 0$ , we arrive at

$$\varphi'(M) (1 + D\nu_s) = \frac{M^{m-2}(1 + D\nu_s)}{1 - M} > \chi \quad (\text{stability condition}) \quad (17)$$

and the best choice for  $\nu_s$  is the first non-zero eigenvalue.

This condition is compatible with the large mass condition in Theorem 3. It also corresponds to the uniqueness of the steady state for large mass proven in Theorem 6. In particular, for  $1 \leq m < 2$ , the pattern might not arise when  $M$  is too small.

When (17) is not fulfilled, the steady state is unstable, so we could expect pattern formation when  $M$  is small for  $m > 2$ . This is illustrated in Fig. 7.

## 2 Uniqueness of the steady state in $n$ dimensions and function $\Lambda(\cdot)$

Uniqueness of the (constant) steady state solutions is central in the proof of Theorem 3 and for pattern formation. We study them now.

### 2.1 Reduction to a single equation.

Thanks to energy dissipation, we can characterize positive steady state, with  $\varphi(u)$  defined by (4), we obtain that

$$\nabla \frac{u^m}{m} = \chi u(1 - u)\nabla v, \quad \frac{u^{m-2}}{1 - u}\nabla u = \nabla \varphi(u) = \chi \nabla v. \quad (18)$$



Therefore, there is a constant  $\lambda$  such that

$$\varphi(u) = \chi(v + \lambda). \quad (19)$$

Then, the positive steady states are solutions of the problem: find  $v$  and  $\lambda$  such that

$$\begin{cases} -D\Delta v + v = \varphi^{-1}(\chi(v + \lambda)), & x \in \Omega, \\ \nabla v \cdot \vec{n} = 0, & x \in \partial\Omega. \end{cases} \quad (20)$$

The constant solution  $M$  of (20) corresponds to the choice  $\lambda = \Lambda(M)$ , which is defined by

$$\varphi(M) = \chi(M + \Lambda(M)), \quad \Lambda(M) := \frac{\varphi(M)}{\chi} - M. \quad (21)$$

Notice that  $M \mapsto \Lambda(M)$  is not always one-to-one and the instability occurs in that range of  $M$  as we will show later.

## 2.2 Properties of $\Lambda(\cdot)$

Because  $\Lambda(\cdot)$  plays an important role later, we give some definitions and properties for future use. We first study the behavior of  $\Lambda'(\cdot)$ . We define, see Fig. 1, (4) and (5),

$$\begin{aligned} \chi_{\min} &:= \min\{\varphi'(u) \mid 0 \leq u \leq 1\}, \quad (22) \\ \chi_{\min} &= \begin{cases} \varphi'(\frac{m-2}{m-3}) = \frac{(3-m)^{3-m}}{(2-m)^{2-m}} & \text{for } 1 \leq m < 2, \\ \varphi'(0) = 1 & \text{for } m = 2, \\ \varphi'(0) = 0 & \text{for } m > 2. \end{cases} \quad (23) \end{aligned}$$

On the one hand, for  $\chi \leq \chi_{\min}$  the function  $\Lambda(\cdot)$  is monotonic non-decreasing since  $\Lambda'(M) = \frac{\varphi'(M)}{\chi} - 1$ . We also conclude that the constant solution  $M$  is stable according to (17) and there cannot be non-constant solutions of (20), as we see in Theorem 7.

On the other hand, for  $\chi > \chi_{\min}$ , the existence of non-constant solutions depends on roots  $M$  of the equation  $\Lambda'(M) = 0$ . We can define the one or two roots of  $\Lambda'(M) = 0$ , depending on  $m$  and  $\chi$ , as follows, (see Fig. 2)

**Definition 4 (Constants  $M_c, M_b, M_+, M_-, \Lambda_c, \Lambda_+$ )** For  $\chi > \chi_{\min}$ , the constants are defined by

1.  $M_c$  is the largest root of  $\Lambda'(M) = 0 \iff \varphi'(M) = \chi$ ,  $\Lambda_c := \Lambda(M_c)$ .
2.  $M_b < M_c$  is the smallest root of  $\varphi'(M) = \chi \iff \Lambda'(M) = 0$  and it exists if and only if  $1 \leq m < 2$ .
3.  $M_+ = \inf\{M \mid \Lambda(M) \text{ is one to one on } (M, 1]\}$ ,  $\Lambda_+ := \Lambda(M_+) = \Lambda(M_b) > 0$  for  $1 \leq m < 2$ ,  $\Lambda(M_+) = 0$  for  $m \geq 2$ .
4. For  $m < 2$ ,  $M_- = \sup\{M \mid \Lambda(M) \text{ is one to one on } [0, M]\}$ . Then,  $\Lambda_- := \Lambda(M_-) = \Lambda(M_c) > 0$ .

From Fig. 1, we observe that for  $\chi > \chi_{\min}$ , the constant  $M_+$  exists, where  $M_+ \geq M_c > 0$ , and it increases as  $\chi$  increases.

Next, we study the number of roots of  $\Lambda(M) = \lambda$ , for  $\chi > \chi_{\min}$ . We refer to Def. 4 and have

$$\text{for } 1 \leq m < 2 \quad \begin{cases} \Lambda'(M) > 0 \iff \varphi'(M) > \chi \iff M \in (0, M_b) \cup (M_c, 1), \\ \Lambda'(M) < 0 \iff \varphi'(M) < \chi \iff M \in (M_b, M_c). \end{cases} \quad (24)$$

Therefore, see Fig. 2, we conclude that, excluding the critical points and the remaining single root cases:

• **For**  $m = 1$  the equation  $\Lambda(M) = \lambda$  has three roots when  $\lambda \in (\Lambda_c, \Lambda_b)$ .

• **For**  $1 < m < 2$  the equation  $\Lambda(M) = \lambda$  has

$$\begin{cases} \text{three roots} & \text{when} & \lambda \in (\max\{0, \Lambda_c\}, \Lambda_b), \\ \text{two roots} & \text{when} & \Lambda_c < 0 \text{ and } \lambda \in (\Lambda_c, 0). \end{cases} \quad (25)$$

• **For**  $m \geq 2$  the equation  $\Lambda(M) = \lambda$  has two roots for  $\lambda \in (\Lambda_c, 0]$ .

Indeed, for  $m \geq 2$ ,  $\Lambda(0) = 0$ ,  $\Lambda'(0) < 0$ ,  $\Lambda''(M) = \varphi''(M) > 0$ , we have

$$\begin{cases} \Lambda'(M) < 0 \iff \varphi'(M) < \chi \iff M \in (0, M_c), \\ \Lambda'(M) > 0 \iff \varphi'(M) > \chi \iff M \in (M_c, 1). \end{cases} \quad (26)$$

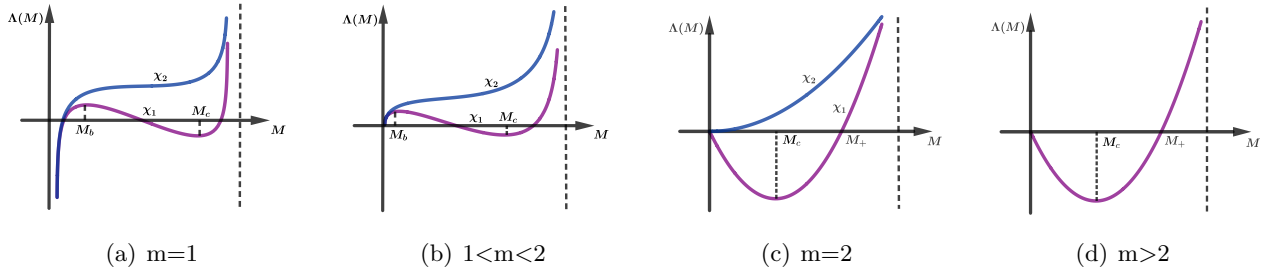


Figure 2: The mapping  $M \mapsto \Lambda(M)$  for two values of  $\chi$ , where  $\chi_1 > \chi_{\min} > \chi_2$  with  $\chi_{\min}$  defined by (22). When  $\chi_{\min} > \chi_2$ ,  $\Lambda$  is increasing. When  $m \geq 2$ , for all  $\chi > 0$ ,  $\Lambda(M)$  is convex and  $\Lambda'(0) \leq 0$ .

### 2.3 Uniqueness of steady state

Endowed with these notations, we can establish the uniqueness of the steady state for large mass by using the rule (19). We first verify the positivity of  $u(x)$  in the following proposition.

**Proposition 5 (Positivity of steady state for large mass)** *For  $M$  large enough, the steady state satisfies  $u \neq 0$  for any  $x \in \Omega$ .*

**Proof.** Because of (3), we conclude from (1)<sub>2</sub> that

$$\|\Delta v\|_{L^\infty(\Omega)} \leq \|v - u\|_{L^\infty(\Omega)} \leq 1.$$

By the elliptic regularity for the Neumann problem, we have  $v \in W^{2,p}(\Omega) \subset W^{1,\infty}(\Omega)$  for  $p$  large.

Using (18), we find that  $\|u^m\|_{W^{1,\infty}(\Omega)} \leq C$  because

$$\left\| \nabla \frac{u^m}{m} \right\|_{L^\infty(\Omega)} = \|\chi u(1-u)\nabla v\|_{L^\infty(\Omega)} \leq C.$$

Let  $a$  and  $b$  be positive and  $k \geq 1$ , there is a constant  $C_k > 0$  depending only on  $k$  such that

$$|b - a|^k \leq C_k |b^k - a^k|.$$

For  $m \geq 1$ , this leads to

$$\|u\|_{C^{0, \frac{1}{m}}} = \sup_{\substack{x_1, x_2 \in \Omega \\ x_1 \neq x_2}} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^{\frac{1}{m}}} \leq \sup_{\substack{x_1, x_2 \in \Omega \\ x_1 \neq x_2}} \sqrt[m]{\frac{|u(x_1)^m - u(x_2)^m|}{|x_1 - x_2|}} \leq C.$$

Assume  $u(x_0) = 0$ , for any  $\epsilon > 0$ , we get

$$\int_{B(x_0; \epsilon) \cap \Omega} u(x) dx = \int_{B(x_0; \epsilon) \cap \Omega} \frac{|u(x) - u(x_0)|}{|x - x_0|^{\frac{1}{m}}} |x - x_0|^{\frac{1}{m}} dx \lesssim \int_{B(x_0; \epsilon) \cap \Omega} |x - x_0|^{\frac{1}{m}} dx \lesssim \epsilon^{\frac{1}{m} + n}.$$

Given  $M > 1 - O(\epsilon^{n+1})$ , we may write

$$M = \int_{\Omega} u(x) dx = \int_{B(x_0; \epsilon) \cap \Omega} u(x) dx + \int_{\Omega \setminus B(x_0; \epsilon)} u(x) dx \lesssim 1 - O(\epsilon^n) + O(\epsilon^{\frac{1}{m} + n}) \lesssim 1 - O(\epsilon^n),$$

which is a contradiction.  $\square$

**Theorem 6 (Uniqueness of steady state)** *For  $M$  large enough, or for  $M$  small enough and  $m = 1$ , system (20) has a unique solution  $v = M$ . Consequently, system (1)-(2) has a unique steady state  $(u, v) = (M, M)$  with vanishing energy flux.*

**Proof.** We distinguish between the cases  $\lambda \geq \Lambda(M)$  and  $\lambda < \Lambda(M)$ .

**Step 1 (For  $\lambda \geq \Lambda(M)$ ).** From (20), knowing that  $\varphi$  is increasing, we can conclude that

$$-D\Delta v + v = \varphi^{-1}(\chi(v + \lambda)) \geq \varphi^{-1}(\chi(v + \Lambda(M))).$$

Set  $w := M - v$ , then we have

$$\begin{aligned} -D\Delta w + w &\leq M - \varphi^{-1}(\chi(v + \Lambda(M))) \\ &= \varphi^{-1}(\chi(M + \Lambda(M))) - \varphi^{-1}(\chi(v + \Lambda(M))). \end{aligned}$$

By multiplying both sides with  $w_+ := \max\{0, w\}$  and integrating, we obtain that

$$\begin{aligned} D \int_{\Omega} |\nabla w_+|^2 dx + \int_{\Omega} w_+^2 dx &= \int_{\Omega} [\varphi^{-1}(\chi(M + \Lambda(M))) - \varphi^{-1}(\chi(v + \Lambda(M)))] w_+ dx \\ &\leq \chi L_{\varphi^{-1}} \int_{\Omega} w_+^2 dx, \end{aligned}$$

where  $L_{\varphi^{-1}}$  is defined by

$$L_{\varphi^{-1}} := \sup_{\chi\Lambda(M) \leq z \leq \chi(1 + \Lambda(M))} (\varphi^{-1})'(z).$$

Using the Poincaré-Wirtinger inequality with  $\nu_s$  the second eigenvalue of  $-\Delta$ , we get

$$(D\nu_s + 1) \int_{\Omega} w_+^2 dx \leq \chi L_{\varphi^{-1}} \int_{\Omega} w_+^2 dx.$$

By using (21), we derive

$$\lim_{M \rightarrow 1} \Lambda(M) = +\infty, \quad m \geq 1, \quad \lim_{M \rightarrow 0} \Lambda(M) = -\infty, \quad m = 1. \quad (27)$$

Due to the assumption that  $u = \varphi^{-1}(z)$  is increasing with respect to  $z$  and bounded, we have

$$(\varphi^{-1})'(z) \rightarrow 0 \quad \text{as } z \rightarrow +\infty \quad m \geq 1, \quad (\varphi^{-1})'(z) \rightarrow 0 \quad \text{as } z \rightarrow -\infty \quad m = 1.$$

Consequently, for  $M$  large enough, or for  $M$  small and  $m = 1$ , we have  $L_{\varphi^{-1}} < \chi^{-1}$ , which implies

$$w_+ = 0 \iff w \leq 0,$$

and because of the mass conservation (2), we conclude  $v \equiv M$  in both cases.

**Step 2 (For  $\lambda < \Lambda(M)$ ).** The same method can be used via multiplying by  $w_- := \min\{0, w\}$ . Then we get  $w_- = 0$ , which implies  $v \equiv M$  in both cases.

**Step 3 (Solutions of system (1)-(2)).** For  $M$  large, we know from Proposition 5 that  $u$  is positive. Thus, system (1)-(2) is equivalent to system (20), and the unique steady state is  $(u, v) = (M, M)$ . When  $m = 1$ ,  $u$  and  $v$  are Lipschitz continuous. Then for  $x \in \Omega$  such that  $u(x) > 0$ , we get that  $\nabla \ln u(x) = \chi(1 - u(x))\nabla v(x)$  is bounded. Therefore,  $\ln u(x)$  is uniformly continuous, which implies  $u(x) > 0$  for any  $x \in \Omega$ .  $\square$

### 3 Uniqueness of the steady state in one dimension and function $G_\lambda$

In one dimension, the mass criteria for uniqueness can be estimated more quantitatively. We choose  $\Omega = (0, 1)$  and, based on (20), the positive solutions can be rewritten to find  $(v, \lambda)$  as follows

$$\begin{cases} -Dv'' + v = \varphi^{-1}(\chi(v + \lambda)), & 0 < x < 1, \\ v'(0) = v'(1) = 0. \end{cases} \quad (28)$$

We define, as given in (19),

$$u_\lambda := \varphi^{-1}(\chi(v + \lambda)) \in [0, 1), \quad \lambda_- := \max\{0, -\lambda\}. \quad (29)$$

Except for  $m = 1$ , the constant  $\lambda$  is constrained by the domain  $[0, +\infty)$  of  $\varphi^{-1}$ , which imposes

$$v + \lambda \geq 0, \quad \lambda_- \leq v(x) \leq 1 \quad \text{for } m > 1. \quad (30)$$

Multiplying (28) by  $v'$  and integrating, for a constant  $\mu$ , we find that the solutions satisfy, but are not equivalent to (28) when  $v'$  vanishes,

$$\begin{cases} D(v')^2 = G_\lambda(v) - \mu, & \lambda_- \leq v \leq 1, \\ G_\lambda(v) := v^2 - 2 \int_{\lambda_-}^v \varphi^{-1}(\chi(z + \lambda)) dz, \\ \mu = G_\lambda(v(0)) = G_\lambda(v(1)). \end{cases} \quad (31)$$

The shape of function  $G_\lambda(v)$  with specific parameters is depicted in Fig. 3.

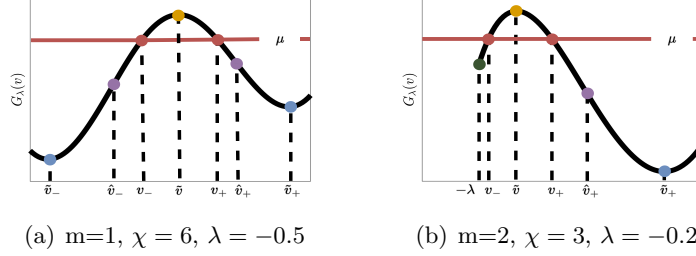


Figure 3: The profiles of  $G_\lambda(v)$  defined by (31) with  $\Lambda_c \leq \lambda \leq \Lambda_+$  and the notations defined in Notation 8.

### 3.1 Constant solution in one dimension

For the constant solution, from (21) and (31), we may define  $\bar{\mu}(M)$  with the relation

$$\begin{cases} M^2 - 2 \int_{\Lambda(M)_-}^M \varphi^{-1}(\chi(z + \Lambda(M))) dz = G_{\Lambda(M)}(M) =: \bar{\mu}(M), \\ 0 = M - \varphi^{-1}(\chi(M + \Lambda(M))) = \frac{1}{2} G'_{\Lambda(M)}(M). \end{cases} \quad (32)$$

We can now establish a result which improves Theorem 6.

**Theorem 7 (Uniqueness of steady state in one dimension)** *For  $n = 1$ , the only solution of (28) is the constant solution when one of the following conditions holds*

1.  $\chi < \chi_{\min}$  for  $1 \leq m \leq 2$ ,
2.  $M \geq M_+$  for  $m \geq 2$ .

**Proof.** We argue by contradiction.

**Step 1 (Generalities).** Assuming that  $v$  is not constant, from (28), we observe that  $G_\lambda(v) - \mu \geq 0$  is not always zero. Consequently,  $G_\lambda(v(x))$  has a maximum point  $v_1 = v(x_1)$ , where  $0 < x_1 < 1$  and

$$G_\lambda(v_1) - \mu > 0, \quad G'_\lambda(v_1)v'(x_1) = 0, \quad G''_\lambda(v_1)(v')^2(x_1) + G'_\lambda(v_1)v''(x_1) \leq 0.$$

It follows that  $v'(x_1) = \sqrt{G_\lambda(v_1) - \mu} \neq 0$  and thus

$$0 = G'_\lambda(v_1) = 2v_1 - 2\varphi^{-1}(\chi(v_1 + \lambda)) \iff \varphi(v_1) = \chi(v_1 + \lambda) \iff \lambda = \Lambda(v_1), \quad (33)$$

$$G''_\lambda(v_1) \leq 0. \quad (34)$$

**Step 2 (When  $m \geq 2$  and  $M \geq M_+$ ).** Let  $v$  be a non-constant solution. Integrating (28) over  $(0, 1)$  and using the concavity of  $\varphi^{-1}$  for  $m \geq 2$  and Jensen's inequality, we find

$$M = \varphi^{-1}(\chi(M + \Lambda(M))) = \int_0^1 \varphi^{-1}(\chi(v + \lambda)) dx < \varphi^{-1}(\chi(M + \lambda)), \quad (35)$$

which gives

$$\lambda > \Lambda(M) \geq 0. \quad (36)$$

By (33) and (36), we conclude that

$$\Lambda(v_1) > \Lambda(M) \geq \Lambda(M_+).$$

As a result, we find that  $v_1 > M_+ > M_c$ . By using (41), we can deduce that  $G''_\lambda(v_1) > 0$ . This leads to a contradiction with (34).

**Step 3 (When  $1 \leq m \leq 2$  and  $\chi < \chi_{\min}$ ).** From the definition of  $\chi_{\min}$  in (22), we have  $\chi < \varphi'(v_1)$ , which makes that (39) contradicts (34).  $\square$

However, it should be noted that the result 2 does not hold for  $M_c < M < M_+$ . Since for a given  $\lambda$ , it is possible that  $v_1$  be the smaller root of the equation  $\Lambda(v) = \lambda$  (refer to Fig. 1).

### 3.2 Profile of $G_\lambda(v)$

The existence of non-constant solutions of (28) imposes the existence of a value  $v(x_1) = v_1$  for which (33)–(34) hold, and thus it depends on the roots  $v$  of  $\Lambda(v) = \lambda$ , or equivalently, the roots of  $G'_\lambda(v) = 0$ . Moreover, the profile of  $G_\lambda(v)$  determines the type of non-constant solutions: positive solution or ‘non-negative’ solution (see Fig. 4). Before solving system (28), we study need some notations which describe  $G_\lambda(v)$ .

**Notation 8** We introduce the following notations when they exist (see also Fig. 3):

1.  $\tilde{v} = \tilde{v}(\lambda)$  and  $\tilde{v}_\pm = \tilde{v}_\pm(\lambda)$  denote the points satisfying  $\tilde{v}_- \leq \tilde{v} \leq \tilde{v}_+$ ,  $\Lambda(\tilde{v}) = \Lambda(\tilde{v}_\pm) = 0$  and  $G'_\lambda(\tilde{v}) = G'_\lambda(\tilde{v}_\pm) = 0$ ,  $G''_\lambda(\tilde{v}) < 0$  and  $G''_\lambda(\tilde{v}_\pm) \geq 0$ .
2.  $\hat{v}_\pm = \hat{v}_\pm(\lambda)$  denote the inflection points satisfying  $\hat{v}_- \leq \hat{v} \leq \hat{v}_+$  and  $G''_\lambda(\hat{v}_\pm) = 0$ .
3.  $\check{v}$  is the unique positive point such that  $G'''_\lambda(\check{v}) = 0 \iff \varphi''(\check{u}_\lambda) = 0$ ,  $\check{u}_\lambda = \frac{2-m}{3-m}$ .
4.  $\tilde{\mu} = G_\lambda(\tilde{v})$ .
5.  $\hat{\mu} = \max(G_\lambda(\hat{v}_+), G_\lambda(\hat{v}_-))$ .
6.  $\mu_c = \mu_c(\lambda)$  is defined by

$$\mu_c = \begin{cases} \max\{G_\lambda(\tilde{v}_-), G_\lambda(\tilde{v}_+)\} & \text{when } G'_\lambda(v) = 0 \text{ has three roots,} \\ \max\{G_\lambda(\lambda_-), G_\lambda(\tilde{v}_+)\} & \text{when } G'_\lambda(v) = 0 \text{ has two roots.} \end{cases}$$

7. For  $\mu \in (\tilde{\mu}, \mu_c)$ , let  $v_\pm = v_\pm(\lambda, \mu)$  denote the points satisfying  $G'_\lambda(v_-) > 0$  and  $G'_\lambda(v_+) < 0$ , and  $G_\lambda(v_\pm) = \mu$ .

Notice that  $\mu \in (\tilde{\mu}, \mu_c)$  is a necessary assumption for the existence of  $v_\pm$ . In particular, the points  $\tilde{v}_-$  and  $\hat{v}_-$  do not exist when  $m \geq 2$  or  $1 \leq m < 2$  with  $G'_\lambda(v) = 0$  has two roots.

**Properties of  $G'_\lambda(v)$  and  $G''_\lambda(v)$ .** We can calculate  $G'_\lambda$  and use the definition of  $u_\lambda$  in (29) to obtain

$$\begin{cases} G'_\lambda(v) = 2v - 2\varphi^{-1}(\chi(v + \lambda)), \\ G'_\lambda(v) = 0 \iff \varphi(v) = \chi(v + \lambda) \iff u_\lambda = v \iff \lambda = \Lambda(v). \end{cases} \quad (37)$$

In particular, we have

$$G'_\lambda(v) < 0 \iff \varphi(v) < \chi(v + \lambda) = \varphi(u_\lambda) \iff v < u_\lambda, \quad \tilde{v} < v < \tilde{v}_+. \quad (38)$$

Also, we calculate

$$G''_\lambda(v) = 2 - 2\chi(\varphi^{-1})'(\chi(v + \lambda)) = 2\left[1 - \frac{\chi}{\varphi'(u_\lambda)}\right]. \quad (39)$$

Then, we know from  $\varphi'(M_b) = \varphi'(M_c) = \chi$  that

$$G''_\lambda(v) = 0 \iff \varphi'(u_\lambda) = \chi = \varphi'(M_b) = \varphi'(M_c) \iff v = -\lambda + \frac{\varphi(M_c)}{\chi} \quad \text{or} \quad v = -\lambda + \frac{\varphi(M_b)}{\chi}, \quad (40)$$

and because of  $\varphi'(z) > 0$  for  $z \in (0, 1)$ , we have

$$G''_\lambda(v) > 0 \iff \varphi'(u_\lambda) > \chi \iff \varphi^{-1}(\chi(v + \lambda)) = u_\lambda > M_c \quad \text{or} \quad u_\lambda < M_b. \quad (41)$$

It is clear that when  $m \geq 2$  or  $1 < m < 2$  with  $G'_\lambda(v) = 0$  has two roots, since  $G''_\lambda(\widehat{v}_+) = 0$  and  $G'_\lambda(\widehat{v}_+) < 0$ , from (38) and (40), we obtain that

$$\widehat{v}_+ < u_\lambda = M_c. \quad (42)$$

The possible profiles  $G_\lambda(v)$  of are depicted in Fig. 4. In (a), we observe three extreme points corresponding to case 2 in Lemma 9 and case 2 in Lemma 10. In (b), we see two extreme points in accordance with case 3 in Lemma 10 and case 2 in Lemma 11.

**Lemma 9 (Profiles of  $G_\lambda(v)$  for  $m = 1$ )** For given  $\chi > \chi_{\min}$ , the equation  $G'_\lambda(v) = 0$  satisfies

1. for  $\lambda > \Lambda_+$  or  $\lambda < \Lambda_c$ , there is a unique root  $\tilde{v}$  and  $G''_\lambda(\tilde{v}) > 0$ ,  $G'_\lambda(v) < 0$  in  $(0, \tilde{v})$  and  $G'_\lambda(v) > 0$  in  $(\tilde{v}, 1)$ ;
2. for  $\Lambda_c \leq \lambda \leq \Lambda_+$ , there are three roots  $\tilde{v}_- < \tilde{v} < \tilde{v}_+$  such that  $G'_\lambda(v) < 0$  in  $(0, \tilde{v}_-) \cup (\tilde{v}, \tilde{v}_+)$  and  $G'_\lambda(v) > 0$  in  $(\tilde{v}_-, \tilde{v}) \cup (\tilde{v}_+, 1)$ . Only for  $\tilde{v}$  we have  $G''_\lambda(\tilde{v}) < 0$ .

**Lemma 10 (Profiles of  $G_\lambda(v)$  for  $1 < m < 2$ )** For given  $\chi > \chi_{\min}$ , the equation  $G'_\lambda(v) = 0$  satisfies

1. for  $\lambda > \Lambda_+$ , or  $\Lambda_c > 0$  and  $0 < \lambda < \Lambda_c$ , there is a unique root  $\tilde{v}$  such that  $G'_\lambda(v) < 0$  in  $(\lambda_-, \tilde{v})$  and  $G'_\lambda(v) > 0$  in  $(\tilde{v}, 1)$ ;
2. for  $\max\{0, \Lambda_c\} \leq \lambda \leq \Lambda_+$ , there are three roots  $\tilde{v}_- < \tilde{v} < \tilde{v}_+$  such that  $G'_\lambda(v) < 0$  in  $(0, \tilde{v}_-) \cup (\tilde{v}, \tilde{v}_+)$  and  $G'_\lambda(v) > 0$  in  $(\tilde{v}_-, \tilde{v}) \cup (\tilde{v}_+, 1)$ ;
3. for  $\Lambda_c < 0$  and  $\Lambda_c \leq \lambda < 0$ , there are two roots  $\tilde{v} < \tilde{v}_+$  such that  $G'_\lambda(v) < 0$  in  $(\tilde{v}, \tilde{v}_+)$  and  $G'_\lambda(v) > 0$  in  $(\lambda_-, \tilde{v}) \cup (\tilde{v}_+, 1)$ ;
4. for  $\Lambda_c \leq 0$  and  $\lambda < \Lambda_c$ , it holds  $G'_\lambda(v) \geq 0$  in  $(\lambda_-, 1)$ .

Notice that the case  $G'_\lambda(v) = 0$  with three roots only exists when  $\lambda \geq 0$ . A necessary condition of  $G'_\lambda(v) = 0$  with two roots is  $\lambda < 0$ .

**Lemma 11 (Profiles of  $G_\lambda(v)$  for  $m \geq 2$ )** For given  $\chi > \chi_{\min}$ , the equation  $G'_\lambda(v) = 0$  satisfies

1. for  $\lambda > \Lambda_+ = 0$ , there is a unique root  $\tilde{v}$  such that  $G'_\lambda(v) < 0$  in  $(\lambda_-, \tilde{v})$  and  $G'_\lambda(v) > 0$  in  $(\tilde{v}, 1)$ ;
2. for  $\Lambda_c \leq \lambda \leq \Lambda_+ = 0$ , there are two roots  $\tilde{v} < \tilde{v}_+$  such that  $G'_\lambda(v) < 0$  in  $(\tilde{v}, \tilde{v}_+)$  and  $G'_\lambda(v) > 0$  in  $(\lambda_-, \tilde{v}) \cup (\tilde{v}_+, 1)$ ;
3. for  $\lambda < \Lambda_c$ , it holds  $G'_\lambda(v) \geq 0$  in  $(\lambda_-, 1)$ .

## 4 Increasing steady states

So far, we have studied the uniqueness and the stability of the constant steady state. The next question is to understand pattern formation when the steady state is not unique, and the simplest

are the increasing solutions. With the same strategy, by symmetry and periodicity it is immediate to build other solutions.

First, we study the case  $m = 1$  where we have complete results regarding positive solutions. Second, we turn to the cases of positive solutions for  $m \geq 2$ . The conclusions for the case  $1 < m < 2$  are included within these two cases. Finally we study solutions that vanish over a certain interval for  $m > 1$ .

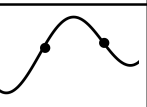



<div style="border: 1px solid black; border-radius: 10px; padding: 2px; display: inline-block;">           • denotes the points  <math>(\tilde{v}_{\pm}, G_{\lambda}(\tilde{v}_{\pm}))</math> </div>	$G_{\lambda}(v)$	$m$	$\chi$	positive solution	'non-negative' solution
	$m = 1$ $1 < m < 2$ with $\Lambda(v) = \Lambda(\tilde{v})$ has three roots	$\chi_1 \leq \chi$	Y	N	
	$m \geq 2$ $1 < m < 2$ with $\Lambda(v) = \Lambda(\tilde{v})$ has two roots	$\chi_1 < \chi \leq \chi_*$	Y	N	
	$m \geq 2$ $1 < m < 2$ with $\Lambda(v) = \Lambda(\tilde{v})$ has two roots	$\max\{\chi_1, \chi_*\} < \chi \leq \chi_{**}$			
	$m \geq 2$ $1 < m < 2$ with $\Lambda(v) = \Lambda(\tilde{v})$ has two roots	$\max\{\tilde{\chi}_1, \chi_{**}\} < \chi$	N	Y	

Figure 4: The table depicts the profiles of  $G_{\lambda}(v)$  with different values of  $m$ ,  $\tilde{v}$  and  $\chi$  (see also (58)), and the existence of positive increasing solutions and non-negative increasing solutions with  $u(x) = 0$  within certain interval. For given  $\tilde{v}$ , the value  $\chi_1 = \chi_1(\tilde{v})$  is given in (49) for positive solutions, and in (71) for 'non-negative' solutions. The constants  $\chi_* = \chi_*(\tilde{v})$  and  $\chi_{**} = \chi_{**}(\tilde{v})$  are such that  $G_{\lambda}(\lambda_-) = G_{\lambda}(\tilde{v}_+)$  and  $G_{\lambda}(\lambda_-) = G_{\lambda}(\tilde{v}_+)$ , respectively (see also Notation 15). We use 'Y' to denote that the system (31) or (66)-(67) has the corresponding solutions, and 'N' to indicate the non-existence of the corresponding solutions. The dots on  $G_{\lambda}(v)$  denote the inflection points  $(\tilde{v}_{\pm}, G_{\lambda}(\tilde{v}_{\pm}))$ , when there are two dots on  $G_{\lambda}(v)$ , and  $(\tilde{v}_+, G_{\lambda}(\tilde{v}_+))$  when there is only one dot.

#### 4.1 Strategy for positive steady state

The increasing steady state corresponds to  $v' > 0$  for  $x \in (0, 1)$ , and  $v'(0) = v'(1) = 0$ . To construct these solutions for a given  $(\lambda, \mu)$ , we reduce the problem to solving

$$\begin{cases} \sqrt{D} v'(x) = \sqrt{G_{\lambda}(v(x)) - \mu}, & v(0) = v_-, \quad v(1) = v_+, \\ G_{\lambda}(v_{\pm}) = \mu, \quad G_{\lambda}(v) > \mu, & v \in (v_-, v_+). \end{cases} \quad (43)$$

As in the proof of Theorem 7, Equations (33)-(34), there should be a value  $\tilde{v}$  such that

$$G_{\lambda}''(\tilde{v}) < 0, \quad G_{\lambda}'(\tilde{v}) = 0, \quad \lambda = \Lambda(\tilde{v}) \in (\Lambda(M_c), \Lambda_+), \quad \text{we set } \tilde{\mu} := G_{\lambda}(\tilde{v}). \quad (44)$$

Indeed, these conditions on  $G_{\lambda}$  impose that  $G_{\lambda}(\cdot)$  has two or three roots. Then we argue by decreasing  $\mu$  departing from  $\tilde{\mu}$ . This allows to define  $v_{\pm} = v_{\pm}(\lambda, \mu)$  by  $G_{\lambda}(v_{\pm}) = \mu$  as in Notation 8. Next, we



define  $X(\lambda, \mu)$  such that  $v(0) = v_-$  and  $v(X(\lambda, \mu)) = v_+$ . Solving (43), we get

$$\begin{aligned} X(\lambda, \mu) &= \int_0^{X(\lambda, \mu)} \frac{\sqrt{D}v'(x)dx}{\sqrt{G_\lambda(v) - \mu}} = \int_{v_-}^{v_+} \frac{\sqrt{D}dv}{\sqrt{G_\lambda(v) - \mu}} \\ &= \int_{v_-}^{\tilde{v}} \frac{\sqrt{D}dv}{\sqrt{G_\lambda(v) - \mu}} + \int_{\tilde{v}}^{v_+} \frac{\sqrt{D}dv}{\sqrt{G_\lambda(v) - \mu}} := X_I(\lambda, \mu) + X_{II}(\lambda, \mu). \end{aligned} \quad (45)$$

Notice that  $X(\lambda, \mu)$  is well defined for  $\mu \in (\mu_c, \tilde{\mu})$ , where  $\mu_c$  is defined in Def. 8 and  $\tilde{\mu}$  is defined in (44).

**Goals.** Our goals are as follows

1. To find  $(\lambda, \mu_0(\lambda))$  satisfying  $X(\lambda, \mu_0(\lambda)) = 1$ .
2. Evaluate the corresponding mass  $M(\lambda, \mu_0(\lambda))$  with

$$M(\lambda, \mu) := \int_0^{X(\lambda, \mu)} v(x)dx = \int_0^{X(\lambda, \mu)} u_\lambda(x)dx = \int_0^{X(\lambda, \mu)} \varphi^{-1}(\chi(v(x) + \lambda))dx. \quad (46)$$

Therefore, we now study the behaviour of  $X(\lambda, \mu)$  depending on the parameters  $\lambda$  and  $\mu$ .

**Proposition 12 (Properties of  $X(\lambda, \mu)$ )** *Being given  $(\lambda, \tilde{v}(\lambda))$  satisfying (44), it holds*

$$X(\lambda, \tilde{\mu}) := \lim_{\mu \rightarrow \tilde{\mu}} X(\lambda, \mu) = \frac{\sqrt{2D} \pi}{\sqrt{-G_\lambda''(\tilde{v})}}, \quad (47)$$

and with  $\hat{\mu}$  and  $\tilde{\mu}$  defined in Notation 8, we have

$$\frac{\partial X(\lambda, \mu)}{\partial \mu} > 0, \quad \mu \in (\hat{\mu}, \tilde{\mu}), \quad X(\lambda, \hat{\mu}) < X(\lambda, \tilde{\mu}). \quad (48)$$

As a consequence of (48),  $\hat{\mu}$  determines a minimal gap for the amplitude  $v_+ - v_-$  of patterns, because  $v_+ - v_- = \max\{v(x^1) - v(x^2) \mid x^1, x^2 \in \Omega\}$ .

**Proof. Step 1 (Proof of (47)).** Set

$$G_\lambda(v) - \mu = \mathcal{A}_\mu(v)(v_+ - v)(v - v_-), \quad \mathcal{A}_\mu(v) = \frac{G_\lambda(v) - \mu}{(v_+ - v)(v - v_-)}, \quad v \in (v_-, v_+).$$

We linearize  $\mathcal{A}_\mu(v)$  via the Taylor expansion of  $G_\lambda(v)$ , then we get

$$\lim_{\substack{\mu \rightarrow \tilde{\mu} \\ v \in (v_-(\mu), v_+(\mu))}} \mathcal{A}_\mu(v) = -\frac{1}{2}G_\lambda''(\tilde{v}).$$

Set  $v(z) = \frac{z}{2}(v_+(\mu) - v_-(\mu)) + \frac{1}{2}(v_+(\mu) + v_-(\mu))$ , we have

$$X(\lambda, \mu) = \sqrt{D} \int_{-1}^1 \frac{dz}{\sqrt{(1-z)(1+z)\mathcal{A}_\mu(v(z))}},$$

and then by the Lebesgue dominated convergence theorem, we have

$$\lim_{\mu \rightarrow \tilde{\mu}} X(\lambda, \mu) = \lim_{\mu \rightarrow \tilde{\mu}} \sqrt{D} \int_{-1}^1 \frac{\sqrt{2}dz}{\sqrt{-(1-z)(1+z)G_\lambda''(\tilde{v})}} = \frac{\sqrt{2D} \pi}{\sqrt{-G_\lambda''(\tilde{v})}}.$$

**Step 2 (Proof of (48)).** With  $v(z)$  defined as above, we have

$$\begin{aligned} \frac{\partial X(\lambda, \mu)}{\partial \mu} &= \int_{-1}^1 \frac{-1}{2\sqrt{(G_\lambda(v(z)) - \mu)^3}} \underbrace{\left[ G'_\lambda(v(z)) \left( \frac{z}{2} \left( \frac{\partial v_+}{\partial \mu} - \frac{\partial v_-}{\partial \mu} \right) + \frac{1}{2} \left( \frac{\partial v_+}{\partial \mu} + \frac{\partial v_-}{\partial \mu} \right) \right) - 1 \right]}_{:= \mathcal{B}_\mu(z)} dz \\ &= \int_{-1}^{\tilde{z}} + \int_{\tilde{z}}^1 := \frac{\partial X_I(\lambda, \mu)}{\partial \mu} + \frac{\partial X_{II}(\lambda, \mu)}{\partial \mu}, \end{aligned}$$

Because  $G_\lambda(v_\pm(\mu)) = \mu$  and thus  $G'_\lambda(v_\pm) \frac{\partial v_\pm}{\partial \mu} = 1$ , we know that  $\frac{\partial v_-}{\partial \mu} \geq 0$  and  $\frac{\partial v_+}{\partial \mu} \leq 0$ . Notice that  $\mathcal{B}_\mu(\pm 1) = 0$ , which implies  $X'_\lambda(\mu)$  is well defined.

For the part  $X_I$ , with  $G'_\lambda(v) > 0$ , we calculate

$$\mathcal{B}_\mu(z) \leq G'_\lambda(v(z)) \left( -\frac{1}{2} \left( \frac{\partial v_+}{\partial \mu} - \frac{\partial v_-}{\partial \mu} \right) + \frac{1}{2} \left( \frac{\partial v_+}{\partial \mu} + \frac{\partial v_-}{\partial \mu} \right) \right) - 1 = G'_\lambda(v(z)) \frac{\partial v_-}{\partial \mu} - 1, \quad z \in (-1, \tilde{z}),$$

and to ensure  $G'_\lambda(v(z)) \leq G'_\lambda(v_-)$ , we need that  $\mu \in (G_\lambda(\hat{v}_-), \tilde{\mu})$ . Then we get  $\mathcal{B}_\mu(z) \leq 0$ , which implies  $\frac{\partial X_I(\lambda, \mu)}{\partial \mu} \geq 0$ .

Similarly, for the part  $X_{II}$ , with  $G'_\lambda(v) < 0$ , we have

$$\mathcal{B}_\mu(z) \leq G'_\lambda(v(z)) \left( \frac{1}{2} \left( \frac{\partial v_+}{\partial \mu} - \frac{\partial v_-}{\partial \mu} \right) + \frac{1}{2} \left( \frac{\partial v_+}{\partial \mu} + \frac{\partial v_-}{\partial \mu} \right) \right) - 1 = G'_\lambda(v(z)) \frac{\partial v_+}{\partial \mu} - 1, \quad z \in (\tilde{z}, 1).$$

we set  $\mu \in (G_\lambda(\hat{v}_+), \tilde{\mu})$  to satisfy  $G'_\lambda(v_+) \leq G'_\lambda(v(z))$ , then we have  $\mathcal{B}_\mu(z) \leq 0$ , which implies  $\frac{\partial X_{II}(\lambda, \mu)}{\partial \mu} \geq 0$ .

Therefore, we conclude  $\frac{\partial X(\lambda, \mu)}{\partial \mu} \geq 0$  for  $\mu \in (\hat{\mu}, \tilde{\mu})$ .  $\square$

**Restriction on  $\tilde{v}$ .** To prove the existence of positive increasing solutions, our strategy requires an additional restriction on  $\tilde{v}$  beyond (44), namely (see (47))

$$X(\lambda, \tilde{\mu}) < 1 \iff G''_\lambda(\tilde{v}) < -2D\pi^2 \iff \chi_1(\tilde{v}) := \varphi'(\tilde{v})(D\pi^2 + 1) < \chi. \quad (49)$$

Since in one dimension,  $\nu_s = \pi^2$  is the principal eigenvalue for the Neumann problem, this condition excludes the stability condition (17) when  $\tilde{v} = M$ . For  $1 \leq m < 2$ , it excludes the choices of small  $\tilde{v}$ , and from this, also of small mass solutions. For  $m \geq 2$ , it allows to choose a small  $\tilde{v}$  and thus compatible with the pattern formation with small mass.

**Setting of  $\chi$ .** There is  $\tilde{v} \in (0, 1)$  satisfying (44), (49) if and only if

$$\chi > \chi_0 := \min_{\tilde{v} \text{ satisfying (44)}} \{ \chi_1(\tilde{v}) \mid \tilde{v} \in (0, 1) \} = \chi_{\min}(D\pi^2 + 1), \quad (50)$$

where  $\chi_{\min}$  is defined in (22). Notice that

$$\chi_0 = 4(D\pi^2 + 1) \quad \text{for } m = 1, \quad \chi_0 = D\pi^2 + 1 \quad \text{for } m = 2, \quad \chi_0 = 0 \quad \text{for } m > 2.$$

## 4.2 Positive increasing steady state for $m = 1$

We begin with the simplest case  $m = 1$  and prove two results.

**Theorem 13 (Existence of increasing solutions for  $m = 1$ )** *Let  $D > 0$  and  $\chi$  satisfy (50). For all  $\tilde{v}$  such that  $\chi_1(\tilde{v}) < \chi$ , there exist a solution  $(v, \lambda, \mu)$  of system (31) with  $\lambda = \Lambda(\tilde{v})$  satisfying  $v, v' > 0$  in  $(0, 1)$ . And thus  $(v, \lambda)$  solves system (28). Furthermore, we have  $\mu_c < \mu < \hat{\mu}$  and the mass  $M$  satisfies  $0 < \tilde{v}_- \leq M \leq \tilde{v}_+ < 1$ .*

**Theorem 14 (Constant solution for  $m = 1$ )** *For  $\chi > \chi_{\min}$ , and  $M < M_-$  or  $M > M_+$ , all solutions are constants.*

Before proving these theorems, we explain the existence of  $\tilde{v}_\pm$  and  $\hat{v}_\pm$  defined in Notation 8 for  $\tilde{v}$  satisfying (44) and (49) as follows.

**Existence of  $\tilde{v}_\pm$ .** Since  $G'_\lambda(\tilde{v}) = 0$  and  $G''_\lambda(\tilde{v}) < -2D\pi^2 < 0$ , there exists  $\delta > 0$  small enough such that  $G'_\lambda(\tilde{v} - \delta) > 0$  and  $G'_\lambda(\tilde{v} + \delta) < 0$ . Together with

$$G'_\lambda(0) = \frac{-2e^{\chi\lambda}}{1 + e^{\chi\lambda}} < 0 \quad \text{and} \quad G'_\lambda(1) = 2 - \frac{2e^{\chi(1+\lambda)}}{1 + e^{\chi(1+\lambda)}} > 0,$$

we can deduce from Lemma 9 that there are at most three roots of  $G'_\lambda(v) = 0$ . This implies the existence of  $0 < \tilde{v}_- < \tilde{v} < \tilde{v}_+ < 1$  satisfying  $G'_\lambda(\tilde{v}_\pm) = 0$  and

$$G'_\lambda(v) < 0, \quad v \in (0, \tilde{v}_-) \cup (\tilde{v}, \tilde{v}_+), \quad G'_\lambda(v) > 0, \quad v \in (\tilde{v}_-, \tilde{v}) \cup (\tilde{v}_+, 1).$$

**Existence of  $\hat{v}_\pm$ .** Calculate  $G''_\lambda(v)$  by

$$G''_\lambda(v) = 2\left(1 - \frac{\chi}{\varphi'(u)}\right) = 2\left(1 - \frac{\chi e^{\chi(v+\lambda)}}{(1 + e^{\chi(v+\lambda)})^2}\right).$$

This implies that when  $\chi > 4$ , there are two roots of  $G''_\lambda(v) = 0$  corresponding to condition (23), which are given by

$$\hat{v}_\pm = \frac{1}{\chi} \ln \frac{\chi - 2 \pm \sqrt{\chi^2 - 4\chi}}{2} - \lambda.$$

We obtain that  $G''_\lambda(v) < 0$  in  $(\hat{v}_-, \hat{v}_+)$  and  $G''_\lambda(v) > 0$  in  $(0, \hat{v}_-) \cup (\hat{v}_+, 1)$ .

**Proof of Theorem 13.** We prove the existence  $\mu_0(\lambda)$  such that  $X(\lambda, \mu_0(\lambda)) = 1$  by the middle value theorem. From (47), we have  $X(\lambda, \mu) < 1$  for  $\mu \approx \tilde{\mu}$ ,  $\mu < \tilde{\mu}$ . It remains to prove that there is  $\mu$  such that  $X(\lambda, \mu) > 1$ .

Without loss of generality, we assume  $G_\lambda(\tilde{v}_-) < G_\lambda(\tilde{v}_+)$ . Then we can infer from Def. 8 that when  $\mu = \mu_c$ , we have  $v_+ = \tilde{v}_+$  and thus  $G'_\lambda(v_+) = 0$  and  $G''_\lambda(v_+) > 0$ . Calculate

$$\begin{aligned} & \lim_{v \rightarrow v_+} (v_+ - v) \sqrt{\frac{1}{G_\lambda(v) - \mu_c}} \\ &= \lim_{v \rightarrow v_+} \frac{v_+ - v}{(G_\lambda(v_+) - \mu_c + G'_\lambda(v_+)(v_+ - v) + \frac{G''_\lambda(v_+)}{2}(v_+ - v)^2 + o((v_+ - v)^2))^{\frac{1}{2}}} \\ &= \sqrt{\frac{2}{G''_\lambda(v_+)}} > 0, \quad (\text{see } \hat{v}_\pm \text{ above}) \end{aligned}$$

which gives rise to  $X(\lambda, \mu_c) = +\infty$ . Therefore, by continuity, there exist  $\mu_0(\lambda) \in (\mu_c, \tilde{\mu})$  such that  $X(\lambda, \mu_0(\lambda)) = 1$  and Theorem 13 is proved.  $\square$

**Proof of Theorem 14.** For a non constant solution  $v$ , applying the argument in the proof of Theorem 7, there is  $\tilde{v} = v(x_1)$  for some  $0 < x_1 < 1$  such that  $G'_\lambda(\tilde{v}) = 0$  and  $G''_\lambda(\tilde{v}) \leq 0$ . Therefore, we are in the situation 2 of Lemma 9 and, from (37) and (39),

$$\Lambda_- \leq \lambda \leq \Lambda_+, \quad \lambda = \Lambda(\tilde{v}_\pm) = \Lambda(\tilde{v}). \quad (51)$$

From Fig. 2, we infer that  $\tilde{v}_- \in (M_-, M_b)$  and  $\tilde{v}_+ \in (M_c, M_+)$ .

Since  $G_\lambda(v(0)) = G_\lambda(v(1)) = \mu$  and  $G_\lambda(v(\cdot)) \geq \mu$ , we have (see Fig. 5)  $M_- \leq \tilde{v}_- < v(x) < \tilde{v}_+ \leq M_+$  for all  $x \in (0, 1)$ , and the result follows.  $\square$

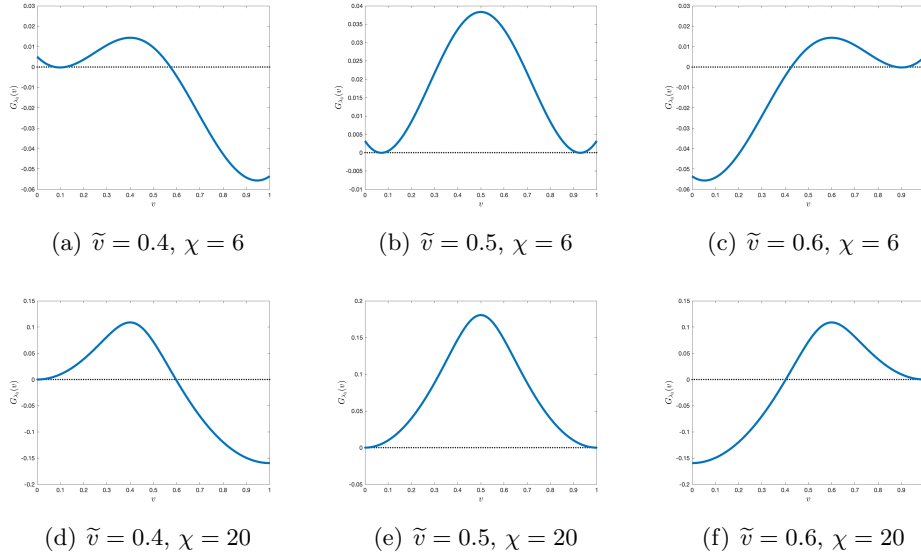


Figure 5: The profiles of  $G_\lambda(v)$  when  $m = 1$  with different value of  $\tilde{v}$  and  $\chi$ . The extreme points  $(\tilde{v}_\pm, G_\lambda(\tilde{v}_\pm))$  depends on both the density  $\tilde{v}$  and the parameter  $\chi$ . When  $\tilde{v} = 0.5$ , the figures are symmetric. For  $\tilde{v} = 0.4$ , we have  $G_\lambda(\tilde{v}_-) > G_\lambda(\tilde{v}_+)$ , whereas  $G_\lambda(\tilde{v}_-) < G_\lambda(\tilde{v}_+)$  when  $\tilde{v} = 0.6$ .

### 4.3 Increasing solutions for $1 < m < 2$

The argument in section 4.2 can be applied to the case  $1 < m < 2$  with  $\chi > \chi_0$ . The key is that  $\lambda = \Lambda(\tilde{v})$  should be chosen such that  $G'_\lambda(v) = 0$  has three roots. To this end, from Lemma 10, we know that  $\tilde{v}$  should be given as

$$\max\{0, \Lambda_c\} \leq \lambda = \Lambda(\tilde{v}) \leq \Lambda_+. \quad (52)$$

Then we conclude that there exist positive increasing solutions.

Furthermore, when  $\lambda = \Lambda(\tilde{v})$  is chosen such that  $G'_\lambda(v) = 0$  has two roots, that is

$$\Lambda_c \leq \lambda = \Lambda(\tilde{v}) < 0, \quad (53)$$

it falls under the same situation as the case  $m \geq 2$  that is discussed in the next section. Then, we get the existence of increasing solutions for a range of admissible values of  $\chi$ .

Once again, for the case with (52), the increasing steady state has a mass that is neither too large nor too small, while for two roots case with (53), it only exists when the mass is not too large.

#### 4.4 Positive increasing solutions for $m \geq 2$

We observe that one of the differences between the two cases  $m = 1$  and  $m \geq 2$  is the profile of  $G_\lambda(v)$ , which is expressed by the numbers of roots to  $G'_\lambda(v) = 0$ . For positive increasing solutions when  $m \geq 2$ , we again look for a  $\mu_0(\lambda)$  such that  $X(\lambda, \mu_0(\lambda)) = 1$  for  $m \geq 2$ . To do so, we introduce the following notations.

**Notation 15 (Constants  $\chi_*(\tilde{v})$  and  $\chi_{**}(\tilde{v})$ )** Choose  $\tilde{v}$  and set  $\lambda = \Lambda(\tilde{v})$ , let  $\chi_*(\tilde{v})$  and  $\chi_{**}(\tilde{v})$  be constants such that (see Fig. 4)

1. when  $G_\lambda(\lambda_-) = G_\lambda(\tilde{v}_+)$ , it holds  $\chi = \chi_*(\tilde{v})$ ;
2. when  $G_\lambda(\lambda_-) = G_\lambda(\hat{v}_+)$ , it holds  $\chi = \chi_{**}(\tilde{v}) > \chi_*(\tilde{v})$ . (See Lemma 17)

**Settings of  $\chi$ .** With  $\chi_0$  defined in (50), we also choose  $\chi > \chi_0$  to ensure the existence of  $\tilde{v}$  satisfying (49). It is clear that  $\chi_0 < \chi_1(\tilde{v})$ . However, due to technical limitations, we cannot explicitly compare  $\chi_1(\tilde{v})$  with  $\chi_*(\tilde{v})$  and  $\chi_{**}(\tilde{v})$ . Therefore, we can divide the problem into three different cases as follows (see also Fig. 4):

- (C1)  $\chi_1(\tilde{v}) < \chi < \chi_*(\tilde{v})$ ;
- (C2)  $\max\{\chi_1(\tilde{v}), \chi_*(\tilde{v})\} < \chi \leq \chi_{**}(\tilde{v})$ ;
- (C3)  $\max\{\chi_1(\tilde{v}), \chi_{**}(\tilde{v})\} < \chi$ .

Then, we study the existence of positive increasing solutions of system (31) in cases (C1) and (C3). The case (C2) is discussed later. Therefore, we can establish the following results.

**Theorem 16 (Existence of positive increasing solutions for  $m \geq 2$ )** Assume there is  $\tilde{v}$  such that  $\chi_1(\tilde{v}) < \chi_*(\tilde{v})$ , which is possible for  $D$  small. Then for  $\chi_1(\tilde{v}) < \chi < \chi_*(\tilde{v})$ , there exists a solution  $(v, \lambda, \mu)$  of system (31) with  $\lambda = \Lambda(\tilde{v})$  satisfying  $v, v' > 0$  on  $(0, 1)$ . Thus  $(v, \lambda)$  solves system (28). For  $\chi$  large enough and any  $\tilde{v}$  such that  $\max\{\chi_1(\tilde{v}), \chi_{**}(\tilde{v})\} < \chi$ , system (31) with  $\lambda = \Lambda(\tilde{v})$  has no solution  $(v, \lambda, \mu)$  satisfying  $v, v' > 0$  on  $(0, 1)$ .

In what follows, we begin by proving the uniqueness and the existence of  $\tilde{v}_+$  and  $\hat{v}_+$  with  $\tilde{v}$  satisfying (44).

**Existence of  $\tilde{v}_+$  and  $\hat{v}_+$ .** For  $\tilde{v}$  such that  $G'_\lambda(\tilde{v}) = 0$  and  $G''_\lambda(\tilde{v}) < 0$ , there is a small constant  $\delta > 0$  such that  $G'_\lambda(\tilde{v} + \delta) < 0$ .

From (37), and  $(\varphi^{-1})' > 0$ , we have

$$\varphi^{-1}(\chi(v + \lambda)) < \lim_{z \rightarrow +\infty} \varphi^{-1}(z) = 1, \quad \forall v \in (0, 1), \quad (54)$$

which implies  $G'_\lambda(1) > 0$ .

Therefore, there exist a value  $\tilde{v}_+ \in (\tilde{v} + \delta, 1)$  such that  $G'_\lambda(\tilde{v}_+) = 0$  and  $G''_\lambda(\tilde{v}_+) > 0$ . Combining with  $G''_\lambda(\tilde{v}) < 0$ , we obtain the existence of  $\hat{v}_+ \in (\tilde{v}, \tilde{v}_+)$ .

**Lemma 17** For given  $D$  and any  $\tilde{v} \in (0, M_+)$ , there exist unique  $\chi_*(\tilde{v})$  and  $\chi_{**}(\tilde{v})$  in  $(\frac{\tilde{v}^{m-2}}{1-\tilde{v}}, +\infty)$  and satisfy  $\chi_*(\tilde{v}) < \chi_{**}(\tilde{v})$ . Furthermore, for  $D$  fixed and  $\tilde{v}$  small enough or for  $\tilde{v}$  fixed and  $D$  small enough, we have  $\chi_1(\tilde{v}) < \chi_*(\tilde{v})$  and thus, (C1) and (C2) are well-defined.

**Proof.** We begin with considering the general case of  $\tilde{v} \in (0, 1)$ .

**Step 1 (Uniqueness of  $\chi_*(\tilde{v})$  and  $\chi_{**}(\tilde{v})$ ).** For given  $\tilde{v}$ , set

$$\mathcal{G}(\chi, v) := G_\lambda(v) - G_\lambda(-\lambda) = v^2 - \lambda^2 - 2 \int_{-\lambda}^v \varphi^{-1}(\chi(z + \lambda)) dz. \quad (55)$$

Differentiating (55) with respect to  $\chi$  gives

$$\frac{\partial \mathcal{G}(\chi, v)}{\partial \chi} = 2v \frac{\partial v}{\partial \chi} - 2 \int_{-\lambda}^v (\varphi^{-1})'(\chi(z + \lambda))(z + \lambda) dz - 2\varphi^{-1}(\chi(v + \lambda)) \frac{\partial v}{\partial \chi}.$$

From (30), we have

$$\frac{\partial v}{\partial \chi} = \frac{\partial v}{\partial \varphi^{-1}} \frac{\partial \varphi^{-1}}{\partial \chi} = \frac{v + \lambda}{\chi} \geq 0.$$

Thus, with  $v \leq \varphi^{-1}(\chi(v + \lambda))$  for  $v \in [\tilde{v}, \tilde{v}_+]$ , it holds

$$\frac{\partial \mathcal{G}(\chi, v)}{\partial \chi} \leq -2 \int_{-\lambda}^v (\varphi^{-1})'(\chi(z + \lambda))(z + \lambda) dz < 0. \quad (56)$$

As a result, we obtain that  $\mathcal{G}(\chi, \tilde{v}_+)$  and  $\mathcal{G}(\chi, \hat{v}_+)$  are decreasing with respect to  $\chi$ . Therefore, we directly get the uniqueness of  $\chi_*(\tilde{v})$  and  $\chi_{**}(\tilde{v})$ .

**Step 2 (Existence of  $\chi_*(\tilde{v})$  and  $\chi_{**}(\tilde{v})$ ).** On the one hand, being given  $\tilde{v}$  and  $\lambda = \Lambda(\tilde{v}) = \frac{\varphi(\tilde{v})}{\chi} - \tilde{v}$ , set  $\chi = \frac{\tilde{v}^{m-2}}{1-\tilde{v}}$ , we have

$$\varphi(v) \geq \chi(v + \lambda) \iff v \geq \varphi^{-1}(\chi(v + \lambda)), \quad \forall v \in (-\lambda, 1).$$

Calculate

$$\mathcal{G}(\chi, v) = v^2 - \lambda^2 - 2 \int_{-\lambda}^v \varphi^{-1}(\chi(z + \lambda)) dz \geq v^2 - \lambda^2 - 2 \int_{-\lambda}^v z dz = 0, \quad \forall v \in (-\lambda, 1),$$

which gives  $G_\lambda(-\lambda) \leq G_\lambda(v)$ .

On the other hand, because of (30) and (54), we have

$$\lim_{\chi \rightarrow +\infty} \mathcal{G}(\chi, v) = \lim_{\chi \rightarrow +\infty} [v^2 - \lambda^2 - 2 \int_{-\lambda}^v \varphi^{-1}(\chi(z + \lambda)) dz] = v^2 - \lambda^2 - 2(v + \lambda) < 0, \quad \forall v \in (-\lambda, 1),$$

which gives  $G_\lambda(-\lambda) \geq G_\lambda(v)$ .

Therefore, since  $\tilde{v}_+, \hat{v}_+ \in (-\lambda, 1)$ , there exist  $\chi_*(\tilde{v})$  and  $\chi_{**}(\tilde{v})$  satisfying  $G_\lambda(-\lambda) = G_\lambda(\tilde{v}_+)$  and  $G_\lambda(-\lambda) = G_\lambda(\hat{v}_+)$ , respectively.

**Step 3 (Compare  $\chi_*(\tilde{v})$  with  $\chi_{**}(\tilde{v})$ ).** For  $v \in (\tilde{v}, \tilde{v}_+)$ , we have

$$\frac{\partial \mathcal{G}(\chi, v)}{\partial v} = G'_\lambda(v) < 0. \quad (57)$$

Because  $\tilde{v} < \hat{v}_+ < \tilde{v}_+$ , we know that

$$G_\lambda(\hat{v}_+) > G_\lambda(\tilde{v}_+) \iff 0 = \mathcal{G}(\chi_{**}(\tilde{v}), \hat{v}_+) = \mathcal{G}(\chi_*(\tilde{v}), \tilde{v}_+) < \mathcal{G}(\chi_*(\tilde{v}), \hat{v}_+),$$

combining with (57), we obtain that  $\chi_*(\tilde{v}) < \chi_{**}(\tilde{v})$ .

Moreover, from (56), we can conclude, ignoring  $\chi_1(\tilde{v})$ , the possibilities

$$\begin{cases} G_\lambda(-\lambda) \leq G_\lambda(\tilde{v}_+), & \chi_*(\tilde{v}) \geq \chi, \\ G_\lambda(\tilde{v}_+) < G_\lambda(-\lambda) \leq G_\lambda(\hat{v}_+), & \chi_*(\tilde{v}) \leq \chi < \chi_{**}(\tilde{v}), \\ G_\lambda(-\lambda) > G_\lambda(\hat{v}_+), & \chi_{**}(\tilde{v}) < \chi, \end{cases} \quad (58)$$

which are depicted in Fig. 4.

**Step 4 (Proof of  $\chi_1(\tilde{v}) < \chi_*(\tilde{v})$ ).** When  $D$  is fixed and  $\tilde{v}$  is small enough, we know from (49) that  $\chi_1(\tilde{v})$  is small enough. Using (55), we compute

$$\lim_{\tilde{v} \rightarrow 0} \mathcal{G}(\chi_1(\tilde{v}), \tilde{v}_+) = \lim_{\tilde{v} \rightarrow 0} (\tilde{v}_+^2 - \lambda^2) = \tilde{v}_+^2 > 0.$$

In addition, we know

$$\mathcal{G}(\chi_*(\tilde{v}), \tilde{v}_+) = 0.$$

Therefore, since  $\mathcal{G}(\chi, v)$  is decreasing with respect  $\chi$  known in (56), we obtain that  $\chi_1(\tilde{v}) < \chi_*(\tilde{v})$ .

When  $\tilde{v}$  is fixed and  $D$  is small enough, due to  $\chi_*(\tilde{v}) \in (\frac{\tilde{v}^{m-2}}{1-\tilde{v}}, +\infty)$ , we conclude

$$\chi_1(\tilde{v}) = \frac{\tilde{v}^{m-2}(D\pi^2 + 1)}{1 - \tilde{v}} < \chi_*(\tilde{v}). \quad (59)$$

□

**Remark 18** When  $\tilde{v}$  is fixed and  $D$  is large, we cannot state analytical results since  $\chi_1(\tilde{v})$  and  $\chi_*(\tilde{v})$  can not be compared directly. The numerical results are shown in Fig. 8.

**Proof of Theorem 16. Step 1 (When  $\chi_1(\tilde{v}) < \chi < \chi_*(\tilde{v})$ ).** Since  $\chi_*(\tilde{v}) > \chi$ , according to (58), we have  $G_\lambda(-\lambda) < G_\lambda(\tilde{v}_+)$ . We use the same method as when  $m = 1$  in Theorem 13 of varying  $X(\lambda, \mu)$  when  $\mu$  decreases. Again, we have  $\lim_{\mu \rightarrow \tilde{\mu}} X(\lambda, \mu) < 1$  and  $X(\lambda, \mu_c) = \infty$ , therefore, we directly come to the conclusion.

**Step 2 (When  $\max\{\chi_1(\tilde{v}), \chi_{**}(\tilde{v})\} < \chi$ ).** Suppose there exists a solution  $(v, \lambda, \mu)$  with  $\lambda = \Lambda(\tilde{v})$  such that  $v, v' > 0$  and  $X(\lambda, \mu) = 1$ .

Under the assumption of  $\chi_1(\tilde{v}) < \chi$ , we know from Proposition 12 that  $\lim_{\mu \rightarrow \tilde{\mu}} X(\lambda, \mu) < 1$ . Consequently, there exists at least one value  $\mu$  such that

$$\frac{\partial X(\lambda, \mu)}{\partial \mu} < 0 \implies \mu_c < \mu < \hat{\mu}. \quad (60)$$

To guarantee the existence of  $v_\pm$  for  $\chi_{**}(\tilde{v}) < \chi$ , it is necessary to satisfy

$$G_\lambda(-\lambda) - \mu \leq 0 = G_\lambda(-\lambda) - \hat{\mu} < G_\lambda(-\lambda) - \tilde{\mu} \implies \hat{\mu} \leq \mu < \tilde{\mu},$$

which is a contradiction with (60). □

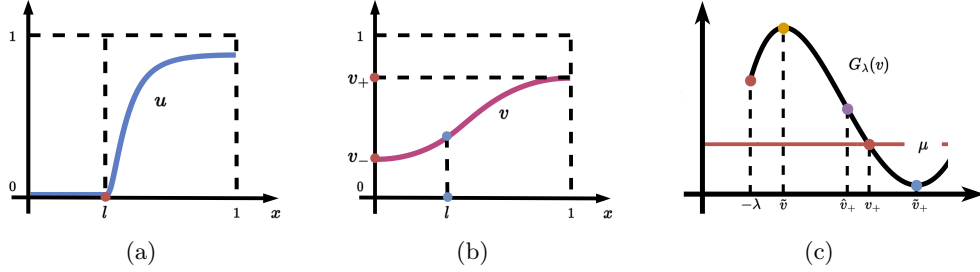


Figure 6: The profiles of the non-negative non-decreasing solutions  $(u(x), v(x))$  with  $u(x)$  vanishing in  $(0, l]$  (see (a) and (b)). The profile of  $G_\lambda(v)$  (see (c)) with  $m \geq 2$  or  $1 < m < 2$  and  $\chi$  and  $\tilde{v}$  such that  $\max\{\chi_1(\tilde{v}), \chi_{**}(\tilde{v})\} < \chi$  when  $G'_\lambda(v) = 0$  has two roots, where  $\lambda = \Lambda(\tilde{v})$ . The notations are defined in Notation 8 (see (b) and (c)).

#### 4.5 ‘Non-negative’ non-decreasing solutions

For  $m = 1$ , we have proved that the steady state cannot be zero in Theorem 6. However, it may be the case for  $m > 1$ , we want to know if the system of degenerate type has the non-negative solutions with intervals where  $u(x)$  vanishes (see section 5). To understand this, one immediate approach is to divide the interval  $(0, 1)$  in two or more sub-intervals and analyze each sub-interval separately and potentially utilize different techniques for each one. Assume there exists  $l \in (0, 1)$  such that we can divide the solutions as follows

$$u(x) \equiv 0, \quad x \in [0, l] \quad \text{and} \quad u(x) > 0, \quad x \in (l, 1]. \quad (61)$$

According to (19), there exists a constant  $\lambda$  to be determined such that

$$\begin{cases} u(x) = \varphi^{-1}(\chi(v + \lambda)), & x \in (l, 1], \\ u(l) = u'(1) = 0, \quad (u^m)'(l) = 0. \end{cases} \quad (62)$$

Based on our assumptions above, with  $u_\lambda(x)$  defined in (29), system (20) in one dimension becomes

$$\left\{ \begin{array}{ll} -Dv'' + v = 0, & x \in [0, l], \\ -Dv'' + v = \varphi^{-1}(\chi(v + \lambda)), & x \in (l, 1], \\ v'(0) = v'(1) = 0, \quad v(l^+) = v(l^-), \quad v'(l^+) = v'(l^-), \quad u_\lambda(l) = 0, \\ \int_0^1 v = M. \end{array} \right. \quad (63)$$

Solving the first equation of (63) with Neumann boundary conditions, we have

$$v(x) = v(0) \cosh \frac{x}{\sqrt{D}}, \quad x \in (0, l). \quad (64)$$

Notice that  $v(x) > v(0) \geq 0$  on  $(0, l)$  (see also Fig. 8). By continuity of  $u_\lambda(x)$ , we have

$$u_\lambda(l) = \varphi^{-1}(\chi(v(l) + \lambda)) = 0 \iff v(l) = -\lambda > 0, \quad (65)$$

which shows that  $\lambda_- = \max\{0, -\lambda\} = -\lambda$ . Notice that this is impossible when  $m = 1$ , because  $\varphi^{-1}(\chi(v(x) + \lambda)) > 0$  for any  $x \in (0, 1)$ . It is also not possible for  $1 < m < 2$  and  $\max\{0, \Lambda_c\} \leq \lambda \leq \Lambda_+$ ,



or equivalently when  $\tilde{v}$  is such that  $G'_\lambda(v) = 0$  has three roots, because in this case, we have  $\lambda \geq 0$  (see Fig. 2).

Multiplying (63)<sub>2</sub> by  $v'$  and integrating, the non-decreasing solutions satisfy

$$\begin{cases} \sqrt{D} v'(x) = \sqrt{G_\lambda(v) - \mu}, & x \in [l, 1], \\ v(l) = -\lambda > 0, \\ G_\lambda(v_+) - \mu = 0, \quad v_+ = v(1), \quad (\mathbf{i.e.} \ v'(1) = 0) \\ G_\lambda(-\lambda) = \lambda^2 > \mu. \end{cases} \quad (66)$$

and the matching conditions from (64), where we can eliminate  $v(0)$  as

$$v(l) = v(0) \cosh \frac{l}{\sqrt{D}}, \quad v'(l) = \frac{v(0)}{\sqrt{D}} \sinh \frac{l}{\sqrt{D}} \iff \frac{v(l)}{v'(l)} = \frac{1}{\sqrt{D}} \tanh \frac{l}{\sqrt{D}}. \quad (67)$$

We want to find  $l$  in (67) such that  $v(l) > 0$  and  $v'(l) > 0$  are the values at  $l$  of the solutions of (66). Our strategy is to solve (66) with  $\lambda$  and  $l$  as parameters by looking for  $\mu$  and then to match  $v(l)$  and  $v'(l)$  with (67) for a good choice of  $l$ .

**Solving (66) with  $\lambda$  and  $l$  given.** For a solution  $v(x)$  of (66), we know that  $v(l)$  satisfies

$$G_\lambda(v(l)) > \mu \quad \text{and} \quad \frac{1}{2} G'_\lambda(v(l)) = v(l) - \varphi^{-1}(\chi(v(l) + \lambda)) = v(l) > 0.$$

Since  $G_\lambda(v_+) = \mu$ , there should be  $\tilde{v} \in (v(l), v_+)$  satisfying

$$G'_\lambda(\tilde{v}) = 0 \iff \varphi(\tilde{v}) = \chi(\tilde{v} + \lambda) = \chi(\tilde{v} - v(l)) \quad \text{and} \quad G''_\lambda(\tilde{v}) < 0. \quad (68)$$

For given  $\chi$  and values of  $\lambda \in [\Lambda_c, \Lambda_+]$  (see Lemma 11), we can choose a  $\tilde{v}$  as in (68) and  $\lambda = \Lambda(\tilde{v})$ . Then as before, we reduce the problem to find  $\mu_0(\lambda)$  such that  $X(\lambda, \mu_0(\lambda)) = 1$ , where

$$X(\lambda, \mu) - l = \int_l^{X(\lambda, \mu)} \frac{\sqrt{D} v'(x) dx}{\sqrt{G_\lambda(v) - \mu}} = \int_{v(l)}^{v_+} \frac{\sqrt{D} dv}{\sqrt{G_\lambda(v) - \mu}},$$

and we define  $v_+ = v(X(\lambda, \mu))$ .

**Restriction on  $\tilde{v}$ .** Before solving (66) by the same technique as in Theorem 16, we point out that the restriction on  $\tilde{v}$  as in (49) is now written

$$1 > \lim_{\mu \rightarrow \tilde{\mu}} X(\lambda, \mu) = \frac{\sqrt{2D}\pi}{\sqrt{-G''_\lambda(\tilde{v})}} + l \iff G''_\lambda(\tilde{v}) < -2D \left( \frac{\pi}{1-l} \right)^2 \iff \frac{\tilde{v}^{m-2} (D(\frac{\pi}{1-l})^2 + 1)}{1-\tilde{v}} < \chi. \quad (69)$$

When  $l$  is chosen such that

$$l \leq \frac{1}{2}, \quad (70)$$

we obtain a sufficient condition for (69) to be independent of  $l$ , which can be written as

$$\tilde{\chi}_1(\tilde{v}) := \frac{\tilde{v}^{m-2} (D(2\pi)^2 + 1)}{1-\tilde{v}} < \chi. \quad (71)$$

Notice that for  $\tilde{v}$  such that  $\tilde{\chi}_1(\tilde{v}) < \chi$ , (70) gives a restriction on  $l$ .

With the settings above and the notations  $\chi_*(\tilde{v})$ ,  $\chi_{**}(\tilde{v})$  given in Notation 15, we establish the existence of solutions.

**Proposition 19 (Existence of solutions of system (66) for  $m \geq 2$ )** Given  $D$  and  $\chi$  large enough, we can choose  $\tilde{v}$  (or equivalently  $\lambda$ ) such that  $\max\{\tilde{\chi}_1(\tilde{v}), \chi_{**}(\tilde{v})\} < \chi$ . Then for  $l \leq \frac{1}{2}$ , there exists a solution  $(v, \lambda, \mu)$  of system (66) with  $\lambda = \Lambda(\tilde{v})$ .

**Remark 20 (Case  $1 < m < 2$ )** When  $\Lambda_c < 0$  and  $\Lambda_c \leq \lambda < 0$ , from Lemma 10, we know that  $G'_\lambda(v) = 0$  has two roots and (65) is true. Thus we can obtain the existence of solutions of system (66) with the same assumptions as in Proposition 19.

**Proof.** From Proposition 12, for each  $\mu \in [G_\lambda(-\lambda), \tilde{\mu}]$ , we have

$$X(\lambda, \mu) \leq \lim_{\mu \rightarrow \tilde{\mu}} X(\lambda, \mu) < 1. \quad (72)$$

Using the same technique as in Theorem 13, we conclude that  $X(\lambda, \mu_c) = +\infty$ . Therefore, there exists  $\mu_0(\lambda) \in (\mu_c, \tilde{\mu})$  such that  $X(\lambda, \mu_0(\lambda)) = 1$ .

We also have, thanks to the condition on  $\chi$ , that  $G_\lambda(\tilde{v}) < \lambda^2$  and thus  $G_\lambda(v) > \mu$  for  $v \in (-\lambda, v_+)$ .  $\square$

**Matching with condition (67).** With the same assumptions and parameters as in Proposition 19, there exists a solution  $v(x)$  of system (66). We now want to find  $l$  such that  $v(l)$  and  $v'(l)$  match with (67). The matching value  $v(l)$  satisfying (66) is determined by

$$\begin{cases} v(l) = -\lambda, \\ \sqrt{D} v'(l) = \sqrt{G_\lambda(v(l)) - \mu_0(\lambda)}. \end{cases} \quad (73)$$

Using (67), we get

$$\sqrt{1 - \frac{\mu_0(\lambda)}{\lambda^2}} = \frac{\sqrt{\lambda^2 - \mu_0(\lambda)}}{|\lambda|} = \tanh \frac{l}{\sqrt{D}} \iff l = \sqrt{D} \operatorname{arctanh} \sqrt{1 - \frac{\mu_0(\lambda)}{\lambda^2}}. \quad (74)$$

**Restriction on  $l$  in (70).** We are reduced to study the ranges of  $\tilde{v}$  such that  $l$  given in (74) satisfies (70), namely

$$l = \sqrt{D} \operatorname{arctanh} \sqrt{1 - \frac{\mu_0(\lambda)}{\lambda^2}} < \frac{1}{2}, \quad (75)$$

which can be rewritten and estimated as

$$\Lambda(\tilde{v}) = \lambda \geq -\sqrt{\frac{\mu_0(\lambda)}{1 - (\tanh \frac{1}{2\sqrt{D}})^2}} =: \Lambda_*. \quad (76)$$

Since  $M_c$  defines  $0 \geq \Lambda(\tilde{v}) \geq \Lambda(M_c)$  (see also in Fig. 2),

- when  $\Lambda(M_c) \geq \Lambda_*$  for any  $\tilde{v} \in (0, \frac{1}{2})$ , we have that (76) is true, and thus (70) is also true.
- when  $\Lambda(M_c) < \Lambda_*$  for some  $\tilde{v}$ , there exist  $\tilde{v}_a < \tilde{v}_b$  satisfying  $\Lambda(\tilde{v}_{a,b}) = \Lambda_*$ , and thus

$$\Lambda(\tilde{v}) \geq \Lambda_* \quad \text{for } \tilde{v} \in [0, \tilde{v}_a] \cup [\tilde{v}_b, M_+],$$

which implies (70) is true in  $[0, \tilde{v}_a] \cup [\tilde{v}_b, M_+]$ .

**Existence of  $\tilde{v}$ .** We want to know the existence of  $\tilde{v}$  satisfying

$$\begin{cases} \max\{\tilde{\chi}_1(\tilde{v}), \chi_{**}(\tilde{v})\} < \chi, \\ \tilde{v} \in [0, \tilde{v}_a] \cup [\tilde{v}_b, M_+]. \end{cases}$$

On the one hand, for  $\tilde{v}$  small enough, we know from (71) that  $\tilde{\chi}_1(\tilde{v}) < \chi$ .

On the other hand, because

$$\lambda = \Lambda(\tilde{v}) \rightarrow 0 \quad \text{as } \tilde{v} \rightarrow 0,$$

combining with (38), we have

$$\lim_{\tilde{v} \rightarrow 0} G_\lambda(-\lambda) - G_\lambda(\tilde{v}_+) = G_\lambda(\tilde{v} = 0) - \lim_{\lambda \rightarrow 0} G_\lambda(\tilde{v}_+) > 0,$$

which implies  $\chi_{**}(\tilde{v}) < \chi$ .

**Theorem 21 (Non-existence of solution of system (67)-(66) for  $m \geq 2$ )** *For given  $D$ , we can choose  $\tilde{v}$  small enough such that  $\tilde{\chi}_1(\tilde{v}) < \chi_*(\tilde{v})$ . Then for  $\tilde{\chi}_1(\tilde{v}) < \chi < \chi_*(\tilde{v})$  and for any  $l \in (0, 1)$ , system (67)-(66) with  $\lambda = \Lambda(\tilde{v})$  has no solution.*

**Proof.** For any  $l \in (0, 1)$ , if system (67)-(66) has a solution, then there is a solution  $(v, \lambda, \mu_0(\lambda))$  such that  $X(\lambda, \mu_0(\lambda)) = 1$ . Since  $\chi_*(\tilde{v}) > \chi$ , from (65), we have

$$G_\lambda(v(l)) = G_\lambda(-\lambda) < G_\lambda(\tilde{v}_+) < \mu_0(\lambda) \iff G_\lambda(v(l)) - \mu_0(\lambda) < 0,$$

which is a contradiction with  $v'(l) > 0$  for  $l > 0$  in (67).  $\square$

## 4.6 Transition between two types of solutions

For  $m \geq 2$ , we have proved the existence of positive increasing solution and of non-negative increasing solution vanishing in an interval. The next question is in which situation these two types solution would coincide when  $l \rightarrow 0$ .

**Proposition 22 (Transition between types of increasing solutions)** *For  $m \geq 2$ , assume there is  $\tilde{v}$  such that  $\tilde{\chi}_1(\tilde{v}) \leq \chi_*(\tilde{v})$ , which is possible for  $D$  small or  $\tilde{v}$  small. Then for  $\chi$  satisfying  $\chi_*(\tilde{v}) < \chi \leq \chi_{**}(\tilde{v})$  and  $l \leq \frac{1}{2}$  in system (66) with  $\lambda = \Lambda(\tilde{v})$ , we have*

1. when  $\chi_*(\tilde{v}) = \chi$ , there exist only positive increasing solutions;
2. when  $\chi_{**}(\tilde{v}) = \chi$ , there exist only the non-negative increasing solutions with  $u(x) = 0$ ,  $x \in [0, l]$ ;
3. there exists  $\chi(\tilde{v}) \in (\chi_*(\tilde{v}), \chi_{**}(\tilde{v}))$  such that  $l = 0$ .

**Proof. Step 1 (Case  $\chi_*(\tilde{v}) = \chi$ ).** We prove that (43) has a solution but (63) dose not. Indeed, using the same method with Theorem 13, we can obtain the existence of solutions of system (43).

If there is a solution of (66) with  $l \in (0, 1)$ , there should be  $\mu_0(\lambda) \in (G_\lambda(\tilde{v}_+), G_\lambda(-\lambda))$  such that  $X(\lambda, \mu_0(\lambda)) = 1$ . However, because

$$G_\lambda(\tilde{v}_+) = G_\lambda(-\lambda) \implies X(\lambda, \mu_0(\lambda)) = +\infty, \quad \chi_*(\tilde{v}) = \chi, \tag{77}$$

we get a contradiction. Thus, (63) has no increasing solution with  $l \in (0, 1)$ .

**Step 2 (Case  $\chi_{**}(\tilde{v}) = \chi$ ).** We prove that (63) has a solution but (43) does not. Using (48), we obtain that

$$G_\lambda(\hat{v}_+) = G_\lambda(-\lambda) \implies X(\lambda, \mu) < 1, \quad \chi_{**}(\tilde{v}) = \chi, \quad \forall \mu \in (G_\lambda(-\lambda), \tilde{\mu}], \quad (78)$$

which implies that (43) has no solution.

Combining (78) with  $X(\lambda, \mu_c) = +\infty$ , there exists  $\mu_0(\lambda) \in (\mu_c, G_\lambda(-\lambda))$  such that  $X(\lambda, \mu_0(\lambda)) = 1$ , which implies the existence of solutions of system (66). Apply the argument in section 4.5, the solutions satisfy the matching condition (67) for  $D$  small or  $\tilde{v}$  small. Therefore, we conclude the existence of solutions of system (63).

**Step 3 (Increasing solutions with  $l = 0$ ).** When  $l = 0$ , there should be  $\mu_0(\lambda) = G_\lambda(-\lambda)$ . Then we are reduced to prove the existence of  $\chi \in (\chi_*(\tilde{v}), \chi_{**}(\tilde{v}))$  such that  $X(\lambda, G_\lambda(-\lambda)) = 1$ . From (77) and (78), by continuity, we directly get the conclusion.  $\square$

## 5 Numerical simulations

We illustrate numerically the patterns of steady states  $(u(x), v(x))$  with different parameters and we compare the numerical results with the analysis presented earlier. More precisely, our analysis exhibits three types of steady state solutions: constant solutions, increasing positive solutions, and non-negative non-decreasing solutions with  $u(x) = 0$  in an interval. We show the occurrence of these types of steady state depending on the parameters in system (1). To this end, we solve (1) numerically using the implicit upwind scheme in [1].

**Mass  $M$ .** In Fig. 7, we depict the patterns with fixed  $\chi$  and  $D$ , varying  $m$  and  $M$ . We illustrate the conclusions of Theorem 7 regarding the uniqueness of a steady state of system (1) for large mass. As shown in (c), (f) and (i), there is only the constant solution  $(u, v) = (M, M)$  when  $M$  is large enough. For small mass, when  $m = 1$  and  $m = 1.5$ , we observe from (a) and (d) that only the constant solution exists. However, for  $m = 2$ , we observe patterns in (g) with small mass  $M = 0.1$ . These conclusions correspond to Theorem 6 and the linear stability analysis presented in Section 1.3.

**Diffusion coefficient  $m$ .** As we have shown, the parameter  $m$  has a decisive influence on the types of steady state.

- **Positive increasing solutions.** These are possible with specific parameters shown in Table 4. These conclusions are verified in Fig. 7 with  $m = 1$  and Fig. 8 with  $m > 1$ .
- **‘Non-negative’ solutions.** Non-negative solutions with  $u(x)$  vanishing in certain intervals only exist when  $m > 1$ . In Fig. 7, when  $m = 1.5$  and  $m = 2$ , we can observe such solutions. However, for  $m = 1$ , as we proved in Theorem 6, the steady state cannot be zero, indicating one can only observe positive steady states. This conclusion is also verified in our numerical results in Fig. 7.

**Parameters  $D$  and  $\chi$ .** Table 4 identifies the various types of steady states. For  $m > 1$ , we aim to determine the type of non-constant steady state for various values of  $D$  and  $\chi$  as depicted in Fig. 8. Notice that irrespective of the specific type of  $u(x)$ , the solution  $v(x)$  is positive on the interval  $(0, 1)$ .

- **Positive solutions.** As shown in Theorem 16, when  $D$  is small enough, it is possible to observe increasing positive solutions. This is also verified numerically in Fig. 8 (b), with a small value of  $D$  satisfying (59). For a large  $D$ , we do not have an explicit analytical result. However, it can be

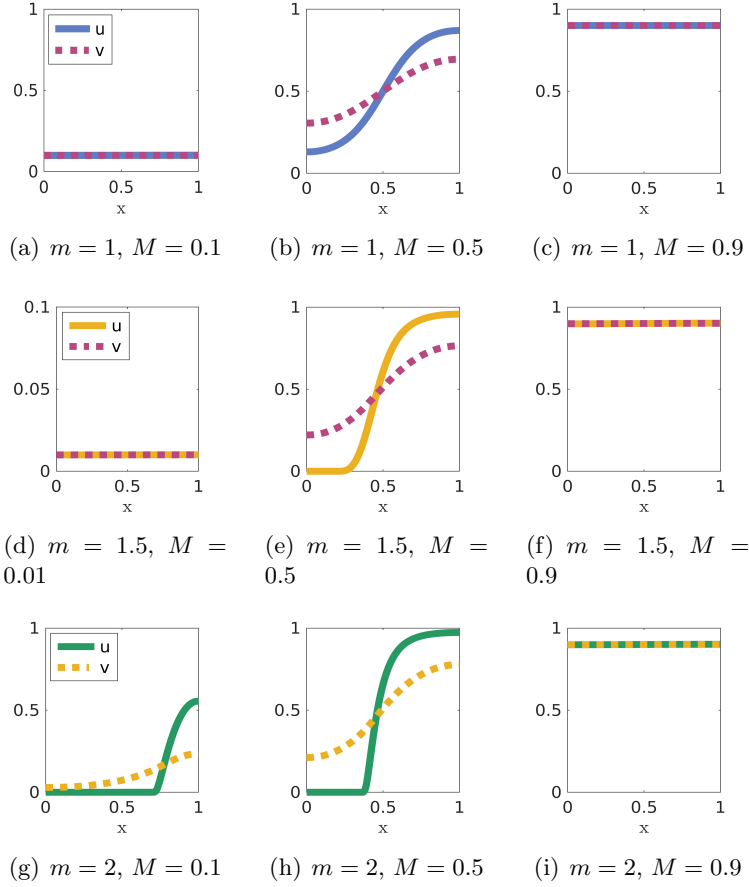


Figure 7: The steady state  $(u(x), v(x))$  of (1) in one dimension with  $\chi = 10$  and  $D = 0.1$ . The initial data is given by  $u^0 = M - 0.01 \cos(\pi x)$  and  $v^0 = M$ .

illustrated numerically. We simulate cases with  $D$  large. Then from (i) and (l) in Fig. 8, we observe that there also exist increasing positive solutions for some  $\chi$ .

• **‘Non-negative’ solutions.** In Fig. 8, fix  $\chi$  and  $D$  and choose  $\chi$  large enough, we can observe the ‘non-negative’ solutions, which correspond to the results in Section 4.5. Furthermore, Figures (g), (h), (k) and (l) in Fig. 8 show that the region with  $u(x)$  vanishing, denoted as  $[0, l]$  in (61), expands when a larger value of  $\chi$  is chosen.

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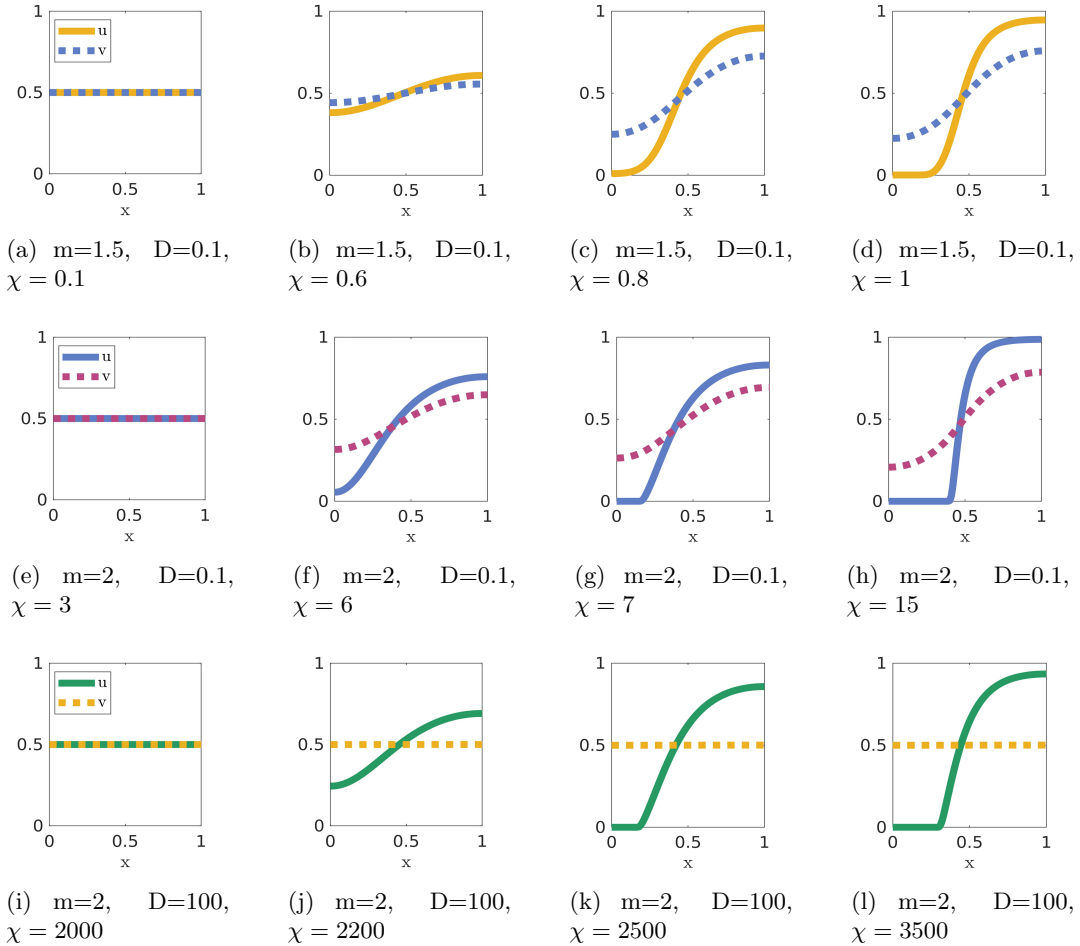


Figure 8: The steady state  $(u(x), v(x))$  of (1) in one dimension with  $m = 2$ . The mass  $M = 0.5$  and the initial data is given by  $u^0 = M - 0.1 \cos(\pi x)$  and  $v^0 = M$ .

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