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Gourab Mukhopadhyay

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# SUMS OF CUBES OVER ODD-INDEX $q$ -FIBONACCI NUMBERS

GOURAB MUKHOPADHYAY

ABSTRACT. In this paper we study sums of the form  $\sum_{k=1}^n F_{mk}^3$  and  $\sum_{k=1}^n (-1)^k F_{mk}^3$  where  $m$  is an odd integer for the generalised Fibonacci numbers defined as  $qF_{n-1} + F_{n-2}$ .

## 1. INTRODUCTION

Fibonacci numbers are given by the recursion equation

$$F_n = \begin{cases} 0 & \text{when } n = 0 \\ 1 & \text{when } n = 1 \\ F_{n-1} + F_{n-2} & \text{when } n \geq 2 \end{cases}$$

and has played a significant role in mathematics. It relates to numerous things in mathematics, for example golden ratio, Binet's formula, Lucas Numbers etc. It also finds applications of algorithms of computers for instance Fibonacci Cubes, Fibonacci search techniques, selection etc. have usages of Fibonacci Numbers. In [PFM<sup>+</sup>24] a Fibonacci sequence approach has been used for the algorithm. Fibonacci numbers help in understanding various things like loops, recursion loops and other programming skills that are basic. In [IMS97], certain characterization of Fibonacci has been explored.

It has far reaching applications in various fields. For example, S. Sinha in his research [Sin17] tried to display the various applications of Fibonacci numbers.

From a purely mathematical perspective, one can study a slightly modified recursion relation; we call it the generalized Fibonacci numbers (or  $q$ -Fibonacci numbers). Fix a non-zero integer  $q$  and define

$$F_n = F_n^{(q)} = \begin{cases} 0 & \text{when } n = 0 \\ 1 & \text{when } n = 1 \\ qF_{n-1} + F_{n-2} & \text{when } n \geq 2. \end{cases}$$

Note that, since this entire article involves working with  $q$ -Fibonacci numbers (and  $q$  remains fixed throughout) we drop it from the notation for ease.

The article [Fro18] by R. Frontczak gave the summation formula for the form  $\sum_{k=1}^n F_{mk}^3$  when  $m$  is an odd integer for the regular Fibonacci series. In particular they showed that

$$\begin{aligned} \sum_{k=1}^n F_{mk}^3 &= \frac{F_m^2}{(L_{3m})} (F_{3mn+3m} + F_{3mn} - F_{3m}) \\ &\quad - \frac{1}{(L_m)} (F_{mn+n}^3 + F_{mn}^3 - F_m^3) \\ \text{and } \sum_{k=1}^n (-1)^k F_{mk}^3 &= \frac{F_m^2}{(L_{3m})} ((-1)^n F_{3mn+3m} + (-1)^{n+1} F_{3mn} - F_{3m}) \\ &\quad - \frac{1}{(L_m)} \left( (-1)^n F_{m(n+1)}^3 + (-1)^{n+1} F_{mn}^3 - F_m^3 \right). \end{aligned}$$

In this paper, we study a generalization of the question to  $q$ -Fibonacci numbers for the form  $\sum_{k=1}^n F_{mk}^3$ . It is also possible to study the generalization of this question to  $(q, p)$ -Fibonacci numbers and  $(q$  and  $p$  remain fixed though-out and they are non-zero integers. So, in that case  $F_n = qF_{n-1} + pF_{n-2}$ ) which is now in progress.

The expression we obtain involves another recursion series, namely the  $q$ -Lucas numbers which are defined as follows

$$L_n = L_n^{(q)} = \begin{cases} 2 & \text{when } n = 0 \\ 1 & \text{when } n = 1 \\ qL_{n-1} + L_{n-2} & \text{when } n \geq 2. \end{cases}$$

**Previous Work:** There have been numerous studies on the sum formulas of various forms for generalized Fibonacci, Lucas, Oresme etc numbers. An article by Y.Soykan showed sum formulas  $\sum_{k=0}^n kx^k W_k^3$  and  $\sum_{k=1}^n kx^k W_{-k}^3$  for the cubes of the square of the terms has been given terms. Soykan presented sum formulas for the squares of generalized Fibonacci numbers too in [S<sup>+</sup>20]. In order to find closed forms of the sum formulas  $\sum_{k=0}^n x^k W_{mk+j}^2$  for generalized Fibonacci, there has been research too. Some formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers as special cases are there too. Apart from that, for generalized Oresme numbers, the sum formulas  $\sum_{k=0}^n x^k W_{mk+j}^3$  is presented in a research [Soy21a] by Soykan. In another [Soy21b] study close form formulas  $\sum_{k=0}^n kx^k W_k^2$ ,  $\sum_{k=0}^n kx^k W_{k+2} W_k$  and  $\sum_{k=0}^n kx^k W_{k+1} W_k$  for the squares of generalized Tribonacci numbers are shown. In [How99], F. T. Howard has some generalizations of  $F_{m+n} = L_m F_n + (-1)^{m-1} F_{n-m}$  involving the Fibonacci and the Lucas numbers. In a research [Yay11] O.Yayenie showed new identities of generalized Fibonacci numbers and some good properties. The research [KPF23] by M.Kumari, K.Prasad and R.Frontczak gave the summation formulas and the relation of k-Fibonacci and k-Lucas spinors. There are also some other papers also that deal with similar kind of things. But there have not been any studies that involve generalization of the question to q-Fibonacci numbers for the form  $\sum_{k=1}^n F_{mk}^3$ . In this paper, we study a generalization of the question to q-Fibonacci numbers for the form  $\sum_{k=1}^n F_{mk}^3$  which is different from other previous studies.

The main result of this paper is a generalization of the result of Frontczak.

**Theorem.** Fix a positive integer  $q$  and an odd (positive) integer  $m$ . Write  $F_t$  to denote the  $t$ -th term of  $q$ -Fibonacci number. Then

$$\begin{aligned} \sum_{k=1}^n F_{mk}^3 &= \frac{F_m^2}{(L_{3m} + F_{3m}(q-1))} (F_{3mn+3m} + F_{3mn} - F_{3m}) \\ &\quad - \frac{1}{(L_m + F_m(q-1))} (F^3 mn + n + F^3 mn - F^3 m) \\ \text{and } \sum_{k=1}^n (-1)^k F_{mk}^3 &= \frac{F_m^2}{(L_{3m} + F_{3m}(q-1))} ((-1)^n F_{3mn+3m} + (-1)^{n+1} F_{3mn} - F_{3m}) \\ &\quad - \frac{1}{(L_m + F_m(q-1))} ((-1)^n F_{m(n+1)}^3 + (-1)^{n+1} F_{mn}^3 - F_m^3). \end{aligned}$$

*Organization:* Including this Introduction the article has *four* sections. Section 2 is preliminary in nature. In this section, we express the  $q$ -Fibonacci and  $q$ -Lucas numbers in terms of solutions of their characteristic series. In the Section 3, we perform technical calculations concerning  $q$ -Fibonacci numbers and  $q$ -Lucas numbers that are required for proving the main result(s) in Section 4.

## 2. BACKGROUND AND SETUP

**Definition 2.1.** The characteristic equation of a recurrence relation is an equation that a number  $\lambda$  should satisfy in order for the geometric series  $(\lambda^n)_{n \in \mathbb{N}}$  to be a solution of the recurrence relation.

For example, the characteristic equation of the (usual) Fibonacci numbers, i.e. when  $q = 1$ , is  $x^2 - x - 1 = 0$ . This means that if  $\lambda$  is a solution of the characteristic equation, then  $\lambda^n$  is a solution of the recurrence relation.

Similarly, the characteristic equation of the 1-Lucas numbers is  $y^2 - y - 1 = 0$ . In this case as well, if  $\lambda$  is a solution of the characteristic equation, then  $\lambda^n$  is a solution of the recurrence relation.

### 2.1. Preliminary calculations with generalized Fibonacci numbers.

**Lemma 2.2.** For  $F_n = qF_{n-1} + F_{n-2}$ , the characteristic equation is  $x^2 - qx - 1 = 0$ . In particular, the solutions of the characteristic equation are  $x = \frac{q \pm \sqrt{q^2 + 4}}{2}$

It is easy to obtain

$$(1) \quad F_n = \frac{\left(\frac{q+\sqrt{q^2+4}}{2}\right)^n - \left(\frac{q-\sqrt{q^2+4}}{2}\right)^n}{\sqrt{q^2+4}} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where we have defined

$$\alpha = \left(\frac{q + \sqrt{q^2 + 4}}{2}\right) \text{ and } \beta = \left(\frac{q - \sqrt{q^2 + 4}}{2}\right).$$

**2.2. Preliminary calculations with generalized Lucas numbers.** For the generalized  $q$ -Lucas number  $L_n = qL_{n-1} + L_{n-2}$ , one can derive that the characteristic equation is  $x^2 - qx - 1 = 0$  in the similar way as in Lemma 2.2. The initial conditions for this recurrence relation are  $L_0 = 2$  and  $L_1 = 1$ . Once again, the solutions of this equation are  $y = \frac{q \pm \sqrt{q^2 + 4}}{2}$ . We set

$$\alpha = \left(\frac{q + \sqrt{q^2 + 4}}{2}\right) \text{ and } \beta = \left(\frac{q - \sqrt{q^2 + 4}}{2}\right).$$

We will now prove an analogue of (1) for these generalized Lucas numbers. Since  $\alpha \neq \beta$ , we know from the principle of superposition that there exist real numbers  $A$  and  $B$  such that for all  $n \geq 2$ , the general form of the recurrence relation is given by

$$(2) \quad L_n = A\alpha^n + B\beta^n.$$

**Lemma 2.3.** *With notation as introduced above*

$$L_n = \frac{(\alpha^n + \beta^n)(\alpha - \beta) - (\alpha^n - \beta^n)(q - 1)}{\alpha - \beta}.$$

It is easy to obtain

$$B = \frac{q + \sqrt{q^2 + 4} - 1}{\sqrt{q^2 + 4}} \text{ and } A = \frac{\sqrt{q^2 + 4} - q + 1}{\sqrt{q^2 + 4}}.$$

Using the recurrence relation  $L_n = A\alpha^n + B\beta^n$ , we have

$$(3) \quad L_n = \frac{(\sqrt{q^2 + 4} - q + 1)\alpha^n}{\sqrt{q^2 + 4}} + \frac{(q + \sqrt{q^2 + 4} - 1)\beta^n}{\sqrt{q^2 + 4}} = \frac{(\alpha^n + \beta^n)(\alpha - \beta) - (\alpha^n - \beta^n)(q - 1)}{\alpha - \beta}.$$

### 3. PRELIMINARY LEMMAS

In this section, we record technical manipulations involving Fibonacci numbers and Lucas numbers. Even though they might appear ad-hoc at the moment, they will be crucially used in proving our main result(s).

**Lemma 3.1.** *With notation as above,*

$$(4) \quad F_{n+1} + F_{n-1} = \alpha^n + \beta^n.$$

*Proof.* Using the definition of the generalized Fibonacci numbers

$$\begin{aligned} F_{n+1} + F_{n-1} &= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + \frac{\alpha^n \beta - \beta^n \alpha}{(\alpha - \beta)\alpha\beta} \\ &= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} - \frac{\alpha^n \beta - \beta^n \alpha}{(\alpha - \beta)} \quad \text{where we use that } \alpha\beta = -1 \\ &= \frac{\alpha^{n+1} - \beta^{n+1} - \alpha^n \beta + \beta^n \alpha}{(\alpha - \beta)} \\ &= \frac{\alpha^n(\alpha - \beta) + \beta^n(\alpha - \beta)}{\alpha - \beta} \\ &= \frac{(\alpha - \beta)(\alpha^n + \beta^n)}{\alpha - \beta} \\ &= \alpha^n + \beta^n. \end{aligned}$$

This completes the proof of the lemma. □

Substituting (4) and (1) in Lemma 2.3, we obtain an expression for the  $n$ -th term of the  $q$ -Lucas sequences in terms of three consecutive terms of the  $q$ -Fibonacci sequence. More precisely,

$$(5) \quad L_n = F_{n+1} - F_n(q-1) + F_{n-1}$$

Next we record a lemma pertaining to products of two (generalized) Fibonacci numbers

**Lemma 3.2.** *Keep the notation introduced above. For all  $n \geq 2$*

$$(6) \quad F_{n-1}F_{n+1} = F_n^2 + (-1)^n.$$

*Proof.* Recall we have shown in (1) that  $F_n$  can be expressed in terms of  $\alpha, \beta$ . We use this expression for the following calculations.

$$\begin{aligned} F_{n-1}F_{n+1} &= \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} \times \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \\ &= \frac{1}{(\alpha - \beta)^2} \left( \alpha^{2n} + \beta^{2n} - \alpha^n \beta^n \left( \frac{\alpha}{\beta} + \frac{\beta}{\alpha} \right) \right) \\ &= \frac{1}{(\alpha - \beta)^2} \left( \alpha^{2n} + \beta^{2n} - \alpha^n \beta^n \left( \frac{\alpha^2 + \beta^2}{\alpha\beta} \right) \right) \\ &= \frac{1}{(\alpha - \beta)^2} \left( \alpha^{2n} + \beta^{2n} - \alpha^n \beta^n \left( \frac{q^2 - 2\alpha\beta}{\alpha\beta} \right) \right) \quad \text{using } \alpha + \beta = q \\ &= \frac{1}{(\alpha - \beta)^2} \left( \alpha^{2n} + \beta^{2n} + \alpha^n \beta^n (q^2 + 2) \right) \quad \text{using } \alpha\beta = -1 \\ &= \frac{1}{(\alpha - \beta)^2} \left( \alpha^{2n} + \beta^{2n} - 2\alpha^n \beta^n + (q^2 + 4) \alpha^n \beta^n \right) \\ &= \frac{\alpha^{2n} + \beta^{2n} - 2\alpha^n \beta^n}{(\alpha - \beta)^2} + \frac{(q^2 + 4)\alpha^n \beta^n}{(\alpha - \beta)^2} \\ &= F_n^2 + \alpha^n \beta^n \quad \text{using } \alpha - \beta = \sqrt{q^2 + 4} \\ &= F_n^2 + (-1)^n \quad \text{using } \alpha\beta = -1. \end{aligned} \quad \square$$

**Lemma 3.3.** *With notation as above,*

$$(q^2 + 1)F_n^3 + 3F_nF_{n+1}F_{n-1} = F_{3n}.$$

*Proof.* To prove the above equation, we first simplify each term on the left hand side in terms of  $\alpha$  and  $\beta$ . Simplifying  $F_n^3$  we get

$$\begin{aligned} F_n^3 &= F_n^2 \times F_n \\ &= \frac{\alpha^{2n} + \beta^{2n} - 2\alpha^n \beta^n}{(\alpha - \beta)^2} \times \frac{\alpha^n - \beta^n}{\alpha - \beta} \\ &= \frac{\alpha^{3n} - \beta^{3n} - 3\alpha^{2n} \beta^n + 3\alpha^n \beta^{2n}}{(\alpha - \beta)^3}. \end{aligned}$$

Next, we simplify the product of three consecutive Fibonacci numbers.

$$\begin{aligned} F_nF_{n+1}F_{n-1} &= \frac{\alpha^n - \beta^n}{\alpha - \beta} \times \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \times \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} \\ &= \frac{\alpha^n - \beta^n}{\alpha - \beta} \times \frac{1}{(\alpha - \beta)^2} \left( \alpha^{2n} + \beta^{2n} + \alpha^n \beta^n (q^2 + 2) \right) \quad \text{using } \alpha\beta = -1 \end{aligned}$$

as obtained in proof of Lemma 3.2

$$= \frac{\alpha^{3n} - \beta^{3n} + (q^2 + 1)\alpha^{2n} \beta^n - (q^2 + 1)\alpha^n \beta^{2n}}{(\alpha - \beta)^3}.$$

Putting this together, we obtain

$$\begin{aligned}
& (q^2 + 1)F_n^3 + 3F_nF_{n+1}F_{n-1} \\
&= \frac{(q^2 + 1)\alpha^{3n} - (q^2 + 1)\beta^{3n} - 3(q^2 + 1)\alpha^{2n}\beta^n + 3(q^2 + 1)\alpha^n\beta^{2n}}{(\alpha - \beta)^3} \\
&+ \frac{3\alpha^{3n} - 3\beta^{3n} + 3(q^2 + 1)\alpha^{2n}\beta^n - 3(q^2 + 1)\alpha^n\beta^{2n}}{(\alpha - \beta)^3} \\
&= \frac{(q^2 + 4)\alpha^{3n} - (q^2 + 4)\beta^{3n}}{(\alpha - \beta)^3} \\
&= (q^2 + 4) \times \frac{\alpha^{3n} - \beta^{3n}}{(\alpha - \beta)^3} \\
&= (q^2 + 4) \times \frac{\alpha^{3n} - \beta^{3n}}{(q^2 + 4)(\alpha - \beta)} \text{ using } \alpha - \beta = \sqrt{q^2 + 4} \\
&= \frac{\alpha^3n - \beta^3n}{\alpha - \beta} \\
&= F_{3n}.
\end{aligned}$$

□

We record a result that will be useful for us

**Theorem 3.4.** *With notation as introduced previously,*

$$F_n = F_n^{(q)} = \begin{cases} qF_{n-1} + F_{n-2} & \text{when } n \geq 2. \end{cases}$$

We obtain,

$$\begin{aligned}
F_n &= qF_{n-1} + F_{n-2} \\
F_n - F_{n-2} &= qF_{n-1} \\
F_{n+1} - F_{n-2+1} &= qF_n
\end{aligned}$$

$$(7) \quad F_{n+1} - F_{n-1} = qF_n$$

#### 4. PROOF OF THE MAIN RESULT

In this section, we will prove the theorem mentioned in the introduction.

**Theorem 4.1.** *Fix an integer  $q$  and an odd integer  $m$ . Write  $F_t$  to denote the  $t$ -th term of  $q$ -Fibonacci number. Then*

$$\begin{aligned}
\sum_{k=1}^n F_{mk}^3 &= \frac{F_m^2}{(L_{3m} + F_{3m}(q-1))} (F_{3mn+3m} + F_{3mn} - F_{3m}) \\
&\quad - \frac{1}{(L_m + F_m(q-1))} (F_{mn+n}^3 + F_{mn}^3 - F_m^3) \\
\sum_{k=1}^n (-1)^k F_{mk}^3 &= \frac{F_m^2}{(L_{3m} + F_{3m}(q-1))} ((-1)^n F_{3mn+3m} + (-1)^{n+1} F_{3mn} - F_{3m}) \\
&\quad - \frac{1}{(L_m + F_m(q-1))} \left( (-1)^n F_{m(n+1)}^3 + (-1)^{n+1} F_{mn}^3 - F_m^3 \right).
\end{aligned}$$

We now begin proving this theorem via a series of steps.

**Lemma 4.2.** *Fix  $q, m$  as in the statement of Theorem 4.1. Let  $k$  be any positive integer. Then*

$$F_{k+m}^3 + (-1)^{3m} F_{k-m}^3 = (L_m + F_m(q-1))(F_m^2 F_{3k} + (-1)^m F_k^3).$$

*Proof.* With notation as above

$$F_{k+m}^3 + (-1)^{3m} F_{k-m}^3 = (F_k F_{m-1} + F_m F_{k+1})^3 + (F_k F_{m+1} - F_m F_{k+1})^3.$$

For ease of notation, set

$$E = (F_k F_{m-1} + F_m F_{k+1})^3 + (F_k F_{m+1} - F_m F_{k+1})^3.$$

We begin by opening the brackets.

$$\begin{aligned} E &= (F_k F_{m-1})^3 + (F_m F_{k+1})^3 + 3(F_k F_{k+1} F_{m-1} F_m)(F_k F_{m-1} + F_m F_{k+1}) \\ &\quad + (F_k F_{m+1})^3 - (F_m F_{k+1})^3 + 3(F_k F_{k+1} F_m F_{m+1})(-F_k F_{m+1} + F_m F_{k+1}) \\ &= F_k^3 (F_{m-1}^3 + F_{m+1}^3) + 3F_k^2 F_m F_{k+1} (F_{m-1}^2 - F_{m+1}^2) + 3F_k F_m^2 F_{k+1}^2 (F_{m-1} + F_{m+1}) \\ &= F_k^3 (F_{m+1} + F_{m-1})(F_{m+1}^2 + F_{m-1}^2 - F_{m-1} F_{m+1}) + 3F_k^2 F_m F_{k+1} (F_{m-1} - F_{m+1})(F_{m-1} + F_{m+1}) \\ &\quad + 3F_k F_m^2 F_{k+1}^2 (F_{m-1} + F_{m+1}) \\ &= F_k^3 (L_m + F_m(q-1))((F_{m+1} - F_{m-1})^2 + F_{m-1} F_{m+1}) + 3F_k^2 F_m F_{k+1} (F_{m-1} - F_{m+1})(L_m + F_m(q-1)) \\ &\quad + 3F_k F_m^2 F_{k+1}^2 (L_m + F_m(q-1)). \end{aligned}$$

Here, we have used (5) and completed squares to obtain the last equation. Continuing with the manipulations, and using (7), we obtain that

$$\begin{aligned} E &= F_k^3 (L_m + F_m(q-1))((qF_m)^2 + F_{m-1} F_{m+1}) + 3F_k^2 F_m F_{k+1} (-qF_m)(L_m + F_m(q-1)) \\ &\quad + 3F_k F_m^2 F_{k+1}^2 (L_m + F_m(q-1)). \end{aligned}$$

Using the expression in (6) we can write

$$\begin{aligned} E &= F_k^3 (L_m + F_m(q-1))(q^2 F_m^2 + F_m^2 + (-1)^m) + 3F_k^2 F_m F_{k+1} (-qF_m)(L_m + F_m(q-1)) \\ &\quad + 3F_k F_m^2 F_{k+1}^2 (L_m + F_m(q-1)) \\ &= F_k^3 (L_m + F_m(q-1))((q^2 + 1)F_m^2 + (-1)^m) + 3F_k^2 F_m F_{k+1} (L_m + F_m(q-1))(-qF_m) \\ &\quad + 3F_k F_m^2 F_{k+1} (L_m + F_m(q-1))(F_{k+1}) \\ &= (q^2 + 1)F_k^3 (L_m + F_m(q-1))F_m^2 + (-1)^m F_k^3 (L_m + F_m(q-1)) + 3F_k F_{k+1} F_m^2 (L_m + F_m(q-1))(-qF_k + F_{k+1}) \\ &= (q^2 + 1)F_k^3 (L_m + F_m(q-1))F_m^2 + (-1)^m F_k^3 (L_m + F_m(q-1)) + 3F_k F_{k+1} F_m^2 (L_m + F_m(q-1))(F_{k-1}) \text{ by (7)} \\ &= (L_m + F_m(q-1))F_m^2 ((q^2 + 1)F_k^3 + 3F_k F_{k+1} F_{k-1}) + (-1)^m F_k^3 (L_m + F_m(q-1)) \\ &= (L_m + F_m(q-1))F_m^2 F_{3k} + (-1)^m F_k^3 (L_m + F_m(q-1)) \quad \text{using Lemma 3.3.} \end{aligned}$$

Equivalently,

$$F_{k+m}^3 + (-1)^{3m} F_{k-m}^3 = (L_m + F_m(q-1))(F_m^2 F_{3k} + (-1)^m F_k^3).$$

This completes the proof of the lemma. □

Replacing  $k$  by  $mk$  in Lemma 4.2 we get,

$$(8) \quad F_{mk+m}^3 - F_{mk-m}^3 = (L_m + F_m(q-1))[F_m^2 F_{3mk} - F_{mk}^3].$$

**Lemma 4.3.** *Keeping the notation as before,*

$$\sum_{k=1}^n F_{mk} = \frac{1}{(L_m + F_m(q-1))} (F_{mn+m} + F_{mn} - F_m).$$

*Proof.*

$$\begin{aligned}
\sum_{k=1}^n F_{mk} &= \sum_{k=1}^n \frac{\alpha^{mk} - \beta^{mk}}{\alpha - \beta} = \frac{(\alpha^m + \alpha^{2m} + \alpha^{3m} + \dots + \alpha^{nm}) - (\beta^m + \beta^{2m} + \beta^{3m} + \dots + \beta^{nm})}{\alpha - \beta} \\
&= \frac{1}{(\alpha - \beta)} \left( \alpha^m \frac{(1 - \alpha^{mn})}{1 - \alpha^m} - \beta^m \frac{(1 - \beta^{mn})}{1 - \beta^m} \right) \text{ by geometric progression} \\
&= \frac{1}{(\alpha - \beta)} \left( \frac{\alpha^m - \beta^m - \alpha^{mn+m} + \beta^{mn+m} - \alpha^{mn} + \beta^{mn}}{-\alpha^m - \beta^m} \right) \text{ using } \alpha\beta = -1; \text{ and } m \text{ is odd.} \\
&= \frac{1}{(\alpha - \beta)} \left( \frac{\alpha^m - \beta^m - \alpha^{mn+m} + \beta^{mn+m} - \alpha^{mn} + \beta^{mn}}{-(\alpha^m + \beta^m)} \right) \\
&= \frac{1}{(\alpha - \beta)} \left( \frac{\alpha^m - \beta^m - \alpha^{mn+m} + \beta^{mn+m} - \alpha^{mn} + \beta^{mn}}{-(L_m + F_m(q-1))} \right) \\
&= \frac{1}{-(L_m + F_m(q-1))} \left( \frac{\alpha^m - \beta^m - \alpha^{mn+m} + \beta^{mn+m} - \alpha^{mn} + \beta^{mn}}{(\alpha - \beta)} \right) \\
&= \frac{1}{-(L_m + F_m(q-1))} \left( \frac{(\alpha^m - \beta^m)}{(\alpha - \beta)} - \frac{(\alpha^{mn+m} - \beta^{mn+m})}{(\alpha - \beta)} - \frac{(\alpha^{mn} - \beta^{mn})}{(\alpha - \beta)} \right) \\
&= \frac{1}{-(L_m + F_m(q-1))} (F_m - F_{mn+m} - F_{mn}) \\
&= \frac{1}{(L_m + F_m(q-1))} (F_{mn+m} + F_{mn} - F_m). \quad \square
\end{aligned}$$

**Lemma 4.4.** *With notation as before,*

$$\sum_{k=1}^n (-1)^k F_{mk} = \frac{1}{(L_m + F_m(q-1))} ((-1)^n F_{mn+m} + (-1)^{n+1} F_{mn} - F_m).$$

*Proof.* We begin by rewriting  $q$ -Fibonacci number in terms of  $\alpha, \beta$  as in (1) and simplifying the expression

$$\begin{aligned}
\sum_{k=1}^n (-1)^k F_{mk} &= \sum_{k=1}^n (-1)^k \frac{\alpha^{mk} - \beta^{mk}}{\alpha - \beta} \\
&= \frac{(-\alpha^m + \alpha^{2m} - \alpha^{3m} + \dots + (-1)^n \alpha^{nm}) - (-\beta^m - \beta^{2m} - \beta^{3m} + \dots + (-1)^n \beta^{nm})}{\alpha - \beta} \\
&= \frac{1}{(\alpha - \beta)} \left( (-\alpha^m) \frac{(1 - (-1)^n \alpha^{mn})}{1 + \alpha^m} - (-\beta^m) \frac{(1 - (-1)^n \beta^{mn})}{1 + \beta^m} \right) \text{ by geometric progression.}
\end{aligned}$$

By cross multiplying the terms within the bracket and using that  $\alpha\beta = -1$  and that  $m$  is odd we obtain

$$\begin{aligned}
\sum_{k=1}^n (-1)^k F_{mk} &= \frac{1}{(\alpha - \beta)} \left( \frac{-\alpha^m + \beta^m + (-1)^n \alpha^{mn+m} - (-1)^n \beta^{mn+m} + (-1)^n \alpha^{mn} \alpha^m \beta^m - (-1)^n \beta^{mn} \alpha^m \beta^m}{\alpha^m + \beta^m} \right) \\
&= \frac{1}{(\alpha - \beta)} \left( \frac{-\alpha^m + \beta^m + (-1)^n \alpha^{mn+m} - (-1)^n \beta^{mn+m} + (-1)^n \alpha^{mn} \alpha^m \beta^m - (-1)^n \beta^{mn} \alpha^m \beta^m}{(L_m + F_m(q-1))} \right) \\
&= \frac{1}{(L_m + F_m(q-1))} \left( \frac{-\alpha^m + \beta^m + (-1)^n \alpha^{mn+m} - (-1)^n \beta^{mn+m} + (-1)^n \alpha^{mn} \alpha^m \beta^m - (-1)^n \beta^{mn} \alpha^m \beta^m}{(\alpha - \beta)} \right) \\
&= \frac{1}{(L_m + F_m(q-1))} \left( -\frac{(\alpha^m - \beta^m)}{(\alpha - \beta)} + (-1)^n \frac{(\alpha^{mn+m} - \beta^{mn+m})}{(\alpha - \beta)} + (-1)^{n+1} \frac{(\alpha^{mn} - \beta^{mn})}{(\alpha - \beta)} \right) \\
&= \frac{1}{(L_m + F_m(q-1))} (-F_m + (-1)^n F_{mn+m} + (-1)^{n+1} F_{mn}) \\
&= \frac{1}{(L_m + F_m(q-1))} ((-1)^n F_{mn+m} + (-1)^{n+1} F_{mn} - F_m). \quad \square
\end{aligned}$$

**Lemma 4.5.** *With notation as before,*

$$(9) \quad \sum_{k=1}^n F_{mk}^3 = F_m^2 \sum_{k=1}^n F_{3mk} - \frac{F_{m(n+1)}^3 + F_{mn}^3 - F_m^3}{L_m + F_m(q-1)}$$

$$(10) \quad \sum_{k=1}^n (-1)^k F_{mk}^3 = F_m^2 \sum_{k=1}^n (-1)^k F_{3mk} - \frac{(-1)^n F_{m(n+1)}^3 + (-1)^{n+1} F_{mn}^3 - F_m^3}{L_m + F_m(q-1)}.$$

*Proof.* We prove the two expressions separately.

(a) Let  $f$  be any function. For fixed integers  $m, j$  we know from [Fro18, Theorem 1, p. 5] that

$$\sum_{k=1}^n [f(m(k+j)) - f(m(k-j))] = \sum_{k=n+1-j}^{n+j} f(mk) - \sum_{k=1-j}^j f(mk).$$

Note that the above expression holds for all positive integers  $m$  and does not require the hypothesis that  $m$  is odd.

Set  $j = 1$  and  $f(t) = F_t^3$  in the above equation. This gives

$$\begin{aligned} \sum_{k=1}^n [F_{m(k+1)}^3 - F_{m(k-1)}^3] &= \sum_{k=n}^{n+1} F_{mk}^3 - \sum_{k=0}^1 F_{mk}^3 \\ &= F_{mn}^3 + F_{m(n+1)}^3 - F_0^3 - F_m^3 \\ &= F_{mn}^3 + F_{m(n+1)}^3 - F_m^3. \end{aligned}$$

Replacing the expression on the left hand side of the equality with (8) we obtain

$$\frac{F_{m(n+1)}^3 + F_{mn}^3 - F_m^3}{(L_m + F_m(q-1))} = \sum_{k=1}^n F_m^2 F_{3mk} - \sum_{k=1}^n F_{mk}^3.$$

Rearranging the terms we get the desired expression

$$\sum_{k=1}^n F_{mk}^3 = F_m^2 \sum_{k=1}^n F_{3mk} - \frac{F_{m(n+1)}^3 + F_{mn}^3 - F_m^3}{(L_m + F_m(q-1))}.$$

(b) Again, let  $f$  be any function. For fixed integers  $m, j$ , we know from [Fro18, Theorem 1, p. 5] that

$$\sum_{k=1}^n (-1)^{k-1} [f(m(k+j)) - f(m(k-j))] = \sum_{k=n+1-j}^{n+j} (-1)^{k+j-1} f(mk) - \sum_{k=1-j}^j (-1)^{k+j-1} f(mk),$$

Once again, set  $f(t) = F_t^3$  and  $j = 1$  in the above equation to obtain,

$$\frac{(-1)^n F_{m(n+1)}^3 + (-1)^{n+1} F_{mn}^3 - F_m^3}{(L_m + F_m(q-1))} = \sum_{k=1}^n (-1)^k F_m^2 F_{3mk} - \sum_{k=1}^n (-1)^k F_{mk}^3.$$

Rearranging the terms,

$$\sum_{k=1}^n (-1)^k F_{mk}^3 = F_m^2 \sum_{k=1}^n (-1)^k F_{3mk} - \frac{(-1)^n F_{m(n+1)}^3 + (-1)^{n+1} F_{mn}^3 - F_m^3}{(L_m + F_m(q-1))}. \quad \square$$

By replacing  $m$  with  $3m$  in Lemma 4.3 and using (9) we obtain

$$\sum_{k=1}^n F_{mk}^3 = \frac{F_m^2}{(L_{3m} + F_{3m}(q-1))} (F_{3mn+3m} + F_{3mn} - F_{3m}) - \frac{F_{mn+n}^3 + F_{mn}^3 - F_m^3}{(L_m + F_m(q-1))}.$$

Similarly, using (10) we obtain

$$\begin{aligned} \sum_{k=1}^n (-1)^k F_{mk}^3 &= \frac{F_m^2}{(L_{3m} + F_{3m}(q-1))} ((-1)^n F_{3mn+3m} + (-1)^{n+1} F_{3mn} - F_{3m}) \\ &\quad - \frac{(-1)^n F_{m(n+1)}^3 + (-1)^{n+1} F_{mn}^3 - F_m^3}{(L_m + F_m(q-1))}. \end{aligned}$$

Then by rearranging we obtain,

$$\sum_{k=1}^n F_{mk}^3 = \frac{F_m^2}{(L_{3m} + F_{3m}(q-1))} (F_{3mn+3m} + F_{3mn} - F_{3m})$$

$$- \frac{1}{(L_m + F_m(q-1))} (F_{mn+n}^3 + F_{mn}^3 - F_m^3)$$

and

$$\sum_{k=1}^n (-1)^k F_{mk}^3 = \frac{F_m^2}{(L_{3m} + F_{3m}(q-1))} ((-1)^n F_{3mn+3m} + (-1)^{n+1} F_{3mn} - F_{3m})$$

$$- \frac{1}{(L_m + F_m(q-1))} ((-1)^n F_{m(n+1)}^3 + (-1)^{n+1} F_{mn}^3 - F_m^3).$$

We have now completed the proof of the main theorem.

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#### REFERENCES

- [Fro18] Robert Frontczak. Sums of cubes over odd-index Fibonacci numbers. *Integers*, 18:9 pages, 04 2018.
- [How99] FT Howard. Generalizations of a Fibonacci identity. In *Applications of Fibonacci Numbers: Volume 8*, pages 201–211. Springer, 1999.
- [IMS97] Costas S Iliopoulos, Dennis Moore, and William F Smyth. A characterization of the squares in a Fibonacci string. *Theoretical Computer Science*, 172(1-2):281–291, 1997.
- [KPF23] Munesh Kumari, Kalika Prasad, and Robert Frontczak. On the  $k$ -Fibonacci and  $k$ -Lucas spinors. *Notes on Number Theory and Discrete Mathematics*, 29(2):322–335, 2023.
- [PFM+24] João Luiz Junho Pereira, Matheus Brendon Francisco, Benedict Jun Ma, Guilherme Ferreira Gomes, and Ana Carolina Lorena. Golden lichtenberg algorithm: a Fibonacci sequence approach applied to feature selection. *Neural Computing and Applications*, pages 1–19, 2024.
- [S+20] Yüksel Soykan et al. Closed formulas for the sums of squares of generalized Fibonacci numbers. *Asian Journal of Advanced Research and Reports*, 9(1):23–39, 2020.
- [Sin17] Sudipta Sinha. The Fibonacci numbers and its amazing applications. *International Journal of Engineering Science Invention*, 6(9):7–14, 2017.
- [Soy21a] Yüksel Soykan. On the sum of the cubes of generalized oresme numbers: The sum formula  $\sum_{k=0}^n x^k w_{m k+j}^3$ . *Asian Research Journal of Current Science*, pages 295–308, 2021.
- [Soy21b] Yüksel Soykan. A study on the sums of squares of generalized tribonacci numbers: Closed form formulas of  $\sum_{k=0}^n k^2$ . *Journal of Scientific Perspectives*, 5(1):1–23, 2021.
- [Yay11] Omer Yayenie. A note on generalized fibonacci sequences. *Applied Mathematics and computation*, 217(12):5603–5611, 2011.

(GM) UNIVERSITY OF TEXAS - RIO GRANDE VALLEY, USA  
*Email address:* gourab.mukhopadhyay01@utrgv.edu