# Γ-convergence and stochastic homogenization of integral functionals defined on measures

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# Γ-CONVERGENCE AND STOCHASTIC HOMOGENIZATION OF INTEGRAL FUNCTIONALS DEFINED ON MEASURES

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Abstract. We study the Γ-convergence of nonconvex integral functionals on vector measures, investigating both Γ-convergence and stochastic homogenization. By setting abstract conditions on the behavior of adapted minimization problems associated with these functionals, we establish an integral representation of the Γ-limit. This representation is then used to prove stochastic homogenization theorems, resulting in new homogenization formulas.

## 1. Introduction

Let $N, l \in \mathbb{N}^*$ be two positive integers. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let $\mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ be the space of $\mathbb{R}^l$-valued Radon measures and $\mathcal{B}(\Omega)$ the set of all Borel subsets of $\Omega$. In this paper, we study the Γ-convergence with respect to the weak$^*$ convergence in $\mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ of integrals $\{I_\varepsilon\}_{\varepsilon>0}$ with $I_\varepsilon : \mathcal{M}\left(\Omega;\mathbb{R}^l\right) \times \mathcal{B}(\Omega) \to [0,\infty]$ defined by

$$I_\varepsilon(\nu, A) := \begin{cases} \displaystyle\int_A f_\varepsilon\left(x, \frac{d\nu}{d\mu_\varepsilon}(x)\right) d\mu_\varepsilon(x) & \text{if } \nu \ll \mu_\varepsilon \\ \infty & \text{otherwise,} \end{cases}$$

where $\{\mu_\varepsilon\}_{\varepsilon>0}$ is a family of positive Radon measures on $\Omega$ satisfying $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu_0$ in $\mathcal{M}_+(\Omega)$ for some positive Radon measure $\mu_0$, and the integrands $f_\varepsilon : \Omega \times \mathbb{R}^l \to [0,\infty]$ are Borel measurable and not necessarily convex.

The Γ-convergence of the family $\{I_\varepsilon\}_{\varepsilon>0}$ has been previously studied using convex duality methods in [Bou87, AB98, BF91], for relaxation and integral representation problems see [BV88, But89, DGAB87, BB93, BB90, BB92, AB88]. While these convexity duality techniques are powerful, providing very general integral representation theorems for abstract functionals defined on measures over a locally compact and separable metric space $\Omega$ (see [BB92]), they do not readily yield tractable formulas for the limit integrands.

In this work, with the stochastic homogenization of such functionals in mind, the Γ-convergence analysis is conducted using the behavior of the family of local minimization problems $\{m_\varepsilon\}_{\varepsilon>0}$, where each $m_\varepsilon : \mathcal{M}\left(\Omega;\mathbb{R}^l\right) \times \mathcal{B}(\Omega) \to [0,\infty]$ is defined by

$$m_\varepsilon(\nu, B) := \inf\left\{ I_\varepsilon(\lambda, B) : \mathcal{M}\left(\Omega;\mathbb{R}^l\right) \ni \lambda \ll \mu_\varepsilon, \text{ and } \lambda(B) = \nu(B) \right\}. \tag{1}$$

We first present the following Γ-convergence result by assuming linear growth on the integrands; we also state and prove a version of this result without linear growth (see Theorem 2.1).

**Theorem 1.1.** *Assume that*

($C_1$) *there exist* $c, C > 0$ *such that for every* $\varepsilon > 0$ *and every* $(x, v) \in \Omega \times \mathbb{R}^l$

$$c|v| \le f_\varepsilon(x, v) \le C(1 + |v|);$$

($C_2$) $f_\varepsilon(x, 0) = 0$ *for all* $\varepsilon > 0$ *and all* $x \in \Omega$;

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($\mathrm{H_{reg}}$) *for every* $(u,O)\in L^1_{\mu_0}\left(\Omega;\mathbb{R}^l\right)\times\mathcal{O}(\Omega)$, *it holds*

$$\overline{\lim_{\rho\to 0}}\,\underline{\lim_{\varepsilon\to 0}}\,\frac{m_\varepsilon\left(u(x)\,\mu_0,B_\rho(x)\right)}{\mu_0\left(B_\rho(x)\right)}\geq\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\frac{m_\varepsilon\left(u(x)\,\mu_0,B_\rho(x)\right)}{\mu_0\left(B_\rho(x)\right)}\quad\mu_0\text{-a.e. in } O;$$

($\mathrm{H_{sing}}$) *for every* $(\nu,O)\in\mathcal{M}\left(\Omega;\mathbb{R}^l\right)\times\mathcal{O}(\Omega)$, *and every* $v\in L^1_{|\nu_0^s|}\left(\Omega;\mathbb{S}^{l-1}\right)$, *it holds*

$$\underline{\lim_{\rho\to 0}}\,\underline{\lim_{\varepsilon\to 0}}\,\frac{m_\varepsilon\left(v(x)\,|\nu_0^s|,B_\rho(x)\right)}{|\nu_0^s|\left(B_\rho(x)\right)}\geq\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\frac{m_\varepsilon\left(v(x)\,|\nu_0^s|,B_\rho(x)\right)}{|\nu_0^s|\left(B_\rho(x)\right)}\quad|\nu_0^s|\text{-a.e. in } O,$$

*where* $\nu_0^s=\nu-\frac{d\nu}{d\mu_0}\mu_0$ *is the singular part in the Lebesgue decomposition of* $\nu$ *with respect to the measure* $\mu_0$.

($\mathrm{M_0}$) *the limit measure* $\mu_0\in\mathcal{M}_+(\Omega)$ *satisfies*

$$\lim_{t\to 1^-}\,\underline{\lim_{r\to 0}}\,\frac{\mu_0\left(B_{tr}(x)\right)}{\mu_0\left(B_r(x)\right)}=1\quad\text{for all } x\in\Omega.\tag{2}$$

*Then for every* $O\in\mathcal{O}(\Omega)$ *with* $\mu_0(\partial O)=0$, *the family of integral functionals* $\{I_\varepsilon(\cdot,O)\}_\varepsilon$ $\Gamma(w^*)$*-converges to* $I_0(\cdot,O)$ *with*

$$I_0(\nu,O)=\int_O f_0\left(x,\frac{d\nu}{d\mu_0}(x)\right)d\mu_0(x)+\int_O f_s\left(x,\frac{d\nu}{d|\nu_0^s|}(x)\right)d|\nu_0^s|(x)$$

*for all* $\nu\in\mathcal{M}\left(\Omega;\mathbb{R}^l\right)$, *where* $f_0:\Omega\times\mathbb{R}^l\to[0,\infty]$ *and* $f_s:\Omega\times\mathbb{S}^{l-1}\to[0,\infty]$ *are respectively defined by :*

$$f_0(x,v):=\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\frac{m_\varepsilon\left(v\mu_0,B_\rho(x)\right)}{\mu_0\left(B_\rho(x)\right)},$$

*and*

$$f_s(x,w):=\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\frac{m_\varepsilon\left(w|\nu_0^s|,B_\rho(x)\right)}{|\nu_0^s|\left(B_\rho(x)\right)}.$$

To derive homogenization theorems, it suffices to verify ($\mathrm{H_{reg}}$) and ($\mathrm{H_{sing}}$) along with additional assumptions on the integrands. Thus, we are able to study the stochastic homogenization of integrals $\{I_\varepsilon\}_{\varepsilon>0}$ where each $I_\varepsilon:\mathcal{M}\left(\Omega;\mathbb{R}^l\right)\times\mathcal{B}(\Omega)\times\Sigma\to[0,\infty]$ is defined by

$$I_\varepsilon(\nu,A,\omega):=\begin{cases}\displaystyle\int_A f\left(\frac{x}{\varepsilon},\frac{d\nu}{d\mu_\varepsilon}(x),\omega\right)d\mu_\varepsilon(x) & \text{if } \nu\ll\mu_\varepsilon\\ \infty & \text{otherwise,}\end{cases}$$

where $\left(\Sigma,\mathcal{T},\mathbb{P},(\tau_z)_{z\in\mathbb{Z}^N}\right)$ is a measurable dynamical system with $\{\tau_z:\Sigma\to\Sigma\}_{z\in\mathbb{Z}^N}$ a group of $\mathbb{P}$-preserving transformations on the measurable space $(\Sigma,\mathcal{T})$. The family of measures $\{\mu_\varepsilon\}_{\varepsilon>0}$ is defined by

$$\mu_\varepsilon(\cdot):=\varepsilon^N\mu\left(\frac{1}{\varepsilon}\cdot\right)$$

with $\mu\in\mathcal{M}_+\left(\mathbb{R}^N\right)$ a 1-periodic measure, i.e. satisfying $\mu(e_i+A)=\mu(A)$ for all $i\in\{1,\dots,N\}$ and all $A\in\mathcal{B}\left(\mathbb{R}^N\right)$, where $\{e_i\}_{i=1}^N$ is the canonical basis of $\mathbb{R}^N$. Observe that $\mu_\varepsilon\overset{*}{\rightharpoonup}\mu_0=p_0\mathcal{L}_N$ in $\mathcal{M}_+\left(\mathbb{R}^N\right)$ as $\varepsilon\to 0$, where $p_0=\mu(\mathbb{Y})$ and $\mathcal{L}_N$ denotes the Lebesgue measure on $\mathbb{R}^N$. By using a subadditive theorem (see Section 4) we obtain explicit formulas for the homogenized integrands. A version of this result without assuming linear growth on the integrand is also provided, as stated in Theorem 4.2.

**Theorem 1.2.** *Let* $f:\mathbb{R}^N\times\mathbb{R}^l\times\Sigma\to[0,\infty]$ *be a* $\left(\mathcal{B}\left(\mathbb{R}^N\right)\otimes\mathcal{B}\left(\mathbb{R}^l\right)\otimes\mathcal{T},\,\mathcal{B}\left(\overline{\mathbb{R}}_+\right)\right)$*-measurable integrand satisfying the following four assumptions*

($\mathrm{C_1^\omega}$) *there exist* $c,C>0$ *such that for every* $\omega\in\Sigma$ *and for every* $(x,v)\in\mathbb{R}^N\times\mathbb{R}^l$

$$c|v|\leq f(x,v,\omega)\leq C\left(1+|v|\right);$$

($\mathrm{C_2^\omega}$) $f(x,0,\omega)=0$ *for all* $x\in\mathbb{R}^N$ *and all* $\omega\in\Sigma$;

($\mathrm{C}_3^\omega$) *for all* $\omega \in \Sigma$ *and for every* $(z,x,v) \in \mathbb{Z}^N \times \mathbb{R}^N \times \mathbb{R}^l$

$$f(x+z,v,\omega) = f(x,v,\tau_z\omega);$$

($\mathrm{C}_4^\omega$) *there exist* $T>0$, $\beta>0$ *and* $r \in ]0,1[$ *such that for every* $\omega \in \Sigma$, *for every* $t>T$, *every* $v \in \mathbb{S}^{l-1}$ *and every* $x \in \mathbb{R}^N$ *we have*

$$\left|\frac{f(x,tv,\omega)}{t} - f^\infty(x,v,\omega)\right| \leq \frac{\beta}{t^r} \quad \text{where} \quad f^\infty(x,v,\omega) := \overline{\lim_{t\to\infty}} \frac{f(x,tv,\omega)}{t}.$$

*Then for every* $O \in \mathcal{O}(\Omega)$ *with* $\mu_0(\partial O) = 0$, *and for* $\mathbb{P}$*-a.e.* $\omega \in \Sigma$ *the family* $\{I_\varepsilon(\cdot,O,\omega)\}_\varepsilon$ $\Gamma(w^*)$*-converges to* $I_0(\cdot,O,\omega)$ *with*

$$I_0(\nu,O,\omega) = \int_O f^{\mathrm{hom}}\left(\frac{d\nu}{d\mu_0}(x),\omega\right) d\mu_0(x) + \int_O f_s^{\mathrm{hom}}\left(\frac{d\nu}{d|\nu_0^s|}(x),\omega\right) d|\nu_0^s|(x)$$

*for all* $\nu \in \mathcal{M}(\Omega;\mathbb{R}^l)$, *where* $f^{\mathrm{hom}} : \mathbb{R}^l \times \Sigma \to [0,\infty]$ *and* $f_s^{\mathrm{hom}} : \mathbb{S}^{l-1} \times \Sigma \to [0,\infty]$ *are respectively given by:*

$$f^{\mathrm{hom}}(v,\omega) := \inf_{k\in\mathbb{N}^*} \mathbb{E}^{\mathcal{I}} \inf\left\{\frac{1}{p_0 k^N}\int_{k\mathbb{Y}} f(z,v+\psi(z),\cdot)\,d\mu(z) : \psi \in \mathcal{A}_0(k\mathbb{Y})\right\}(\omega),$$

*and*

$$f_s^{\mathrm{hom}}(v,\omega) := \inf_{k\in\mathbb{N}^*} \mathbb{E}^{\mathcal{I}} \inf\left\{\frac{1}{p_0 k^N}\int_{k\mathbb{Y}} f^\infty(z,v+\psi(z),\cdot)\,d\mu(z) : \psi \in \mathcal{A}_0(k\mathbb{Y})\right\}(\omega),$$

*where* $\mathcal{A}_0(k\mathbb{Y}) := \left\{\psi \in L^1(k\mathbb{Y};\mathbb{R}^l) : \fint_{k\mathbb{Y}} \psi d\mathcal{L}_N = 0\right\}$.

*When* $(\Sigma,\mathcal{T},\mathbb{P},\{\tau_z\}_{z\in\mathbb{Z}^N})$ *is ergodic then the homogenized integrands are deterministic, i.e.*

$$f^{\mathrm{hom}}(v,\omega) = f^{\mathrm{hom}}(v) := \inf_{k\in\mathbb{N}^*} \int_\Sigma \inf\left\{\frac{1}{p_0 k^N}\int_{k\mathbb{Y}} f(z,v+\psi(z),\omega)\,d\mu(z) : \psi \in \mathcal{A}_0(k\mathbb{Y})\right\} d\mathbb{P}(\omega),$$

$$f_s^{\mathrm{hom}}(v,\omega) = f_s^{\mathrm{hom}}(v) := \inf_{k\in\mathbb{N}^*} \int_\Sigma \inf\left\{\frac{1}{p_0 k^N}\int_{k\mathbb{Y}} f^\infty(z,v+\psi(z),\omega)\,d\mu(z) : \psi \in \mathcal{A}_0(k\mathbb{Y})\right\} d\mathbb{P}(\omega).$$

The paper is structured as follows. In Section 2, we provide the proof of the abstract $\Gamma$-convergence result, Theorem 1.1. We begin by proving the result in the case where the initial integrands have linear growth. Subsequently, through a standard approximation procedure, we establish and prove an abstract $\Gamma$-convergence theorem without assuming growth on the integrands. Section 3 focuses on some examples. Subsection 3.1 highlights an example where $f_\varepsilon$ is convex and independent of both $x$ and $\varepsilon$. In Subsection 3.2, we apply our result to the relaxation problem for $x$-dependent integrands, complementing the general result of [DGAB87, Theorem 2.4.] by obtaining additional insights on the limiting integrands. Section 4 deals with the proof of the stochastic homogenization result, Theorem 1.2. Section 5 is dedicated to proving Propositions 2.1 and 2.2. These propositions provide bounds under integral form for the $\Gamma$-$\underline{\lim}_{\varepsilon\to0} I_\varepsilon(\cdot,O)$, $\Gamma$-$\overline{\lim}_{\varepsilon\to0} I_\varepsilon(\cdot,O)$, and local bounds of the average of $\overline{\lim}_{\varepsilon\to0} m_\varepsilon$ over small balls. The proof of Theorem 1.1 is then an easy and direct application of these previous propositions. Finally, Section 6 concludes with the proofs of some auxiliary results that are necessary for the overall arguments.

**Notation.** We denote by $\mathcal{M}_+(\Omega)$ the set of all positive Radon measures on $\Omega$ and by $\mathcal{M}(\Omega;\mathbb{R}^l)$ the space of all $\mathbb{R}^l$-valued Radon measures on $\Omega$ with $l \in \mathbb{N}^*$.

For $\mu \in \mathcal{M}_+(\Omega)$, if $w \in L^1_\mu(\Omega;\mathbb{R}^l)$ then $w\mu \in \mathcal{M}(\Omega;\mathbb{R}^l)$ where

$$w\mu(B) := \int_B w d\mu \quad \text{for all Borel set } B \subset \Omega.$$

In particular, for a Borel set $A \subset \Omega$, $\mathbb{1}_A \mu$ is the measure defined by

$$\mathbb{1}_A \mu(B) = \int_B \mathbb{1}_A \, d\mu = \mu(A \cap B) \qquad \text{for all Borel set } B \subset \Omega.$$

When we write $\nu \ll \mu$ for $\mu \in \mathcal{M}_+(\Omega)$ and $\nu \in \mathcal{M}\left(\Omega; \mathbb{R}^l\right)$, we mean that the total variation $|\nu|$ of $\nu$ is absolutely continuous with respect to $\mu$, i.e. $|\nu| \ll \mu$.

The Lebesgue decomposition of $\lambda \in \mathcal{M}\left(\Omega; \mathbb{R}^l\right)$ with respect to $\mu$ is given by

$$\lambda = \frac{d\lambda}{d\mu}(\cdot)\, \mu + \lambda^s$$

where $\frac{d\lambda}{d\mu}(\cdot) \in L^1_\mu\left(\Omega; \mathbb{R}^l\right)$ is given by $\frac{d\lambda}{d\mu}(x) = \lim_{\rho \to 0} \frac{\lambda(B_\rho(x))}{\mu(B_\rho(x))}$ $\mu$-a.e. in $\Omega$, and where $\lambda^s$ and $\mu$ are mutually singular, denoted by $\lambda^s \perp \mu$, this means that there exists a Borel set $X \subset \Omega$ such that $\mu(\Omega \setminus X) = 0$ and $|\lambda^s|(X) = 0$.

We also introduce the following notation:

- $\mathcal{O}_0(\Omega) := \{O \in \mathcal{O}(\Omega) : \mu_0(\partial O) = 0\}$.
- $R^x_\mu := \left\{\rho \in ]0, \infty[ : \mu(\partial B_\rho(x)) > 0\right\}$.
- $\mathbb{Y} := ]0,1[^N$ and $Y := \left]-\frac{1}{2}, \frac{1}{2}\right[^N$.
- $\mathbb{S}^{l-1}$ denotes the unit sphere of $\mathbb{R}^l$ centered at 0, i.e. $\mathbb{S}^{l-1} := \{v \in \mathbb{R}^l : |v| = 1\}$.

## 2. Proof of Γ-convergence theorems 1.1 and 2.1

For every $(\nu, O) \in \mathcal{M}\left(\Omega; \mathbb{R}^l\right) \times \mathcal{O}(\Omega)$ we set:

$$\left(\Gamma(w^*)\text{-}\overline{\lim_{\varepsilon \to 0}} I_\varepsilon\right)(\nu, O) := \inf\left\{\overline{\lim_{\varepsilon \to 0}} I_\varepsilon(\nu_\varepsilon, O) : \mathcal{M}\left(\Omega; \mathbb{R}^l\right) \ni \nu_\varepsilon \overset{*}{\rightharpoonup} \nu \text{ in } \mathcal{M}\left(O; \mathbb{R}^l\right)\right\};$$

$$\left(\Gamma(w^*)\text{-}\underline{\lim_{\varepsilon \to 0}} I_\varepsilon\right)(\nu, O) := \inf\left\{\underline{\lim_{\varepsilon \to 0}} I_\varepsilon(\nu_\varepsilon, O) : \mathcal{M}\left(\Omega; \mathbb{R}^l\right) \ni \nu_\varepsilon \overset{*}{\rightharpoonup} \nu \text{ in } \mathcal{M}\left(O; \mathbb{R}^l\right)\right\}.$$

For each $O \in \mathcal{O}(\Omega)$, if

$$\left(\Gamma(w^*)\text{-}\underline{\lim_{\varepsilon \to 0}} I_\varepsilon\right)(\cdot, O) = \left(\Gamma(w^*)\text{-}\overline{\lim_{\varepsilon \to 0}} I_\varepsilon\right)(\cdot, O)$$

then we say that $\{I_\varepsilon(\cdot, O)\}_{\varepsilon > 0}$ Γ-*converges to*

$$I_0(\cdot, O) := \left(\Gamma(w^*)\text{-}\underline{\lim_{\varepsilon \to 0}} I_\varepsilon\right)(\cdot, O) = \left(\Gamma(w^*)\text{-}\overline{\lim_{\varepsilon \to 0}} I_\varepsilon\right)(\cdot, O).$$

For more details on the theory of Γ-convergence we refer to [DM93] (see also [BD98]).

### 2.1. Bounds for $\Gamma(w^*)\text{-}\underline{\lim}_{\varepsilon \to 0} I_\varepsilon(\cdot, O)$, $\Gamma(w^*)\text{-}\overline{\lim}_{\varepsilon \to 0} I_\varepsilon(\cdot, O)$ and $m_\varepsilon$.

The proof of Theorem 1.1 is based on the following two propositions below. Proposition 2.1 provides lower (resp. upper) bound for the Γ-liminf (resp. Γ-limsup) under integral form.

**Proposition 2.1.** *Let $O \in \mathcal{O}(\Omega)$.*

*(i) For every $\nu \in \mathcal{M}\left(\Omega; \mathbb{R}^l\right)$, there exists a sequence $\{\nu_\varepsilon\}_{\varepsilon > 0} \subset \mathcal{M}\left(\Omega; \mathbb{R}^l\right)$ such that*

$$\sup_{\varepsilon > 0} I_\varepsilon(\nu_\varepsilon, O) < \infty, \quad \nu_\varepsilon \overset{*}{\rightharpoonup} \nu \text{ in } \mathcal{M}\left(O; \mathbb{R}^l\right)$$

*and*

$$\Gamma(w^*)\text{-}\underline{\lim_{\varepsilon \to 0}} I_\varepsilon(\nu, O) \geq \sum_{\sigma \in \{\mu_0, |\nu_0^s|\}} \int_O \overline{\lim_{t \to 1^-}}\, \overline{\lim_{\rho \to 0}}\, \underline{\lim_{\varepsilon \to 0}} \frac{I_\varepsilon(\nu_\varepsilon, B_{t\rho}(x))}{\sigma(B_\rho(x))} d\sigma(x). \tag{3}$$

*(ii) Assume that* $(C_1)$ *holds. If $\mu_0(\partial O) = 0$ then for every $\nu \in \mathcal{M}\left(\Omega; \mathbb{R}^l\right)$*

$$\Gamma(w^*)\text{-}\overline{\lim_{\varepsilon \to 0}} I_\varepsilon(\nu, O) \leq \sum_{\sigma \in \{\mu_0, |\nu_0^s|\}} \int_O \lim_{\rho \to 0}\, \overline{\lim_{\varepsilon \to 0}} \frac{m_\varepsilon(\nu, B_\rho(x))}{\sigma(B_\rho(x))} d\sigma(x). \tag{4}$$

The proof of Proposition 2.1 $(i)$ is standard and involves weak convergence and Lebesgue decomposition of measures, using the blow-up method outlined in [FM92, FM93]. The main challenge lies in the proof of Proposition 2.1 $(ii)$, which is divided into two steps. Firstly, we show that the $\Gamma$-limsup is lower than the lower Vitali envelope of the set function $\overline{\lim}_{\varepsilon\to 0} m_\varepsilon(\cdot, O)$. Then, we establish (see Lemma 6.3) that the lower Vitali envelope admits an integral representation with density the derivative of the set function $\overline{\lim}_{\varepsilon\to 0} m_\varepsilon(\cdot, O)$.

**Proposition 2.2.** *Let $O \in \mathcal{O}(\Omega)$. Assume that $(\mathrm{C}_1)$ holds. Then:*

*$(i)$ for every $\nu \in \mathcal{M}\left(\Omega;\mathbb{R}^l\right)$, every sequence $\{\nu_\varepsilon\}_{\varepsilon>0} \subset \mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ and every $\theta \in \mathcal{M}_+(O)$ satisfying*

$$\nu_\varepsilon \overset{*}{\rightharpoonup} \nu \ \text{ in } \mathcal{M}\left(O;\mathbb{R}^l\right) \quad \text{and} \quad |\nu_\varepsilon| \overset{*}{\rightharpoonup} \theta \ \text{ in } \mathcal{M}_+(O)$$

*we have:*

$$\overline{\lim_{\rho\to 0}}\,\underline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon\left(\frac{d\nu}{d\mu_0}(x)\,\mu_0, B_\rho(x)\right)}{\mu_0(B_\rho(x))} \le \overline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\underline{\lim_{\varepsilon\to 0}} \frac{I_\varepsilon(\nu_\varepsilon, B_{\tau\rho}(x))}{\mu_0(B_\rho(x))} \quad \mu_0\text{-a.e. in } O; \tag{5}$$

$$\underline{\lim_{\rho\to 0}}\,\underline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon\left(\frac{d\nu}{d|\nu_0^s|}(x)\,|\nu_0^s|, B_\rho(x)\right)}{|\nu_0^s|(B_\rho(x))} \le \overline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\underline{\lim_{\varepsilon\to 0}} \frac{I_\varepsilon(\nu_\varepsilon, B_{\tau\rho}(x))}{|\nu_0^s|(B_\rho(x))} \quad |\nu_0^s|\text{-a.e. in } O; \tag{6}$$

*$(ii)$ for every $\nu \in \mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ and every $\sigma \in \{\mu_0, |\nu_0^s|\}$,*

$$\underline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon(\nu, B_\rho(x))}{\sigma(B_\rho(x))} \le \overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon\left(\frac{d\nu}{d\sigma}(x)\,\sigma, B_\rho(x)\right)}{\sigma(B_\rho(x))} \quad \sigma\text{-a.e. in } O. \tag{7}$$

The proof of Theorem 1.1 consists in using the inequalities of Propositions 2.1, 2.2 and the conditions $(\mathrm{H}_{\mathrm{reg}})$ and $(\mathrm{H}_{\mathrm{sing}})$ to bridge the gap between the $\Gamma$-liminf and $\Gamma$-limsup.

2.2. **Proof of Theorem 1.1.** Let $\nu \in \mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ and $O \in \mathcal{O}_0(\Omega)$. Combining Proposition 2.1, Proposition 2.2 and condition $(\mathrm{H}_{\mathrm{reg}})$, we can write, for $\mu_0$-a.e. $x \in O$, that

$$\begin{aligned}
\underline{\lim_{r\to 0}} \frac{\left(\Gamma(w^*)\text{-}\underline{\lim_{\varepsilon\to 0}} I_\varepsilon\right)(\nu, B_r(x))}{\mu_0(B_r(x))} &\overset{(3)}{\ge} \overline{\lim_{t\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\underline{\lim_{\varepsilon\to 0}} \frac{I_\varepsilon(\nu_\varepsilon, B_{t\rho}(x))}{\mu_0(B_\rho(x))} \\
&\overset{(5)}{\ge} \overline{\lim_{\rho\to 0}}\,\underline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon\left(\frac{d\nu}{d\mu_0}(x)\,\mu_0, B_\rho(x)\right)}{\mu_0(B_\rho(x))} \\
&\overset{(\mathrm{H}_{\mathrm{reg}})}{\ge} \overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon\left(\frac{d\nu}{d\mu_0}(x)\,\mu_0, B_\rho(x)\right)}{\mu_0(B_\rho(x))} \\
&\overset{(7)}{\ge} \underline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon(\nu, B_\rho(x))}{\mu_0(B_\rho(x))} \overset{(4)}{\ge} \overline{\lim_{r\to 0}} \frac{\left(\Gamma(w^*)\text{-}\overline{\lim_{\varepsilon\to 0}} I_\varepsilon\right)(\nu, B_r(x))}{\mu_0(B_r(x))}.
\end{aligned}$$

It follows, since $\left(\Gamma\left(w^*\right)\text{-}\underline{\lim}_{\varepsilon\to 0} I_\varepsilon\right)(\nu,\cdot) \le \left(\Gamma\left(w^*\right)\text{-}\overline{\lim}_{\varepsilon\to 0} I_\varepsilon\right)(\nu,\cdot)$, that for $\mu_0$-a.e. $x\in O$ we have

$$\begin{aligned}
\lim_{r\to 0}\frac{\left(\Gamma\left(w^*\right)\text{-}\underline{\lim}_{\varepsilon\to 0} I_\varepsilon\right)(\nu,B_r(x))}{\mu_0(B_r(x))} &= \lim_{r\to 0}\frac{\left(\Gamma\left(w^*\right)\text{-}\overline{\lim}_{\varepsilon\to 0} I_\varepsilon\right)(\nu,B_r(x))}{\mu_0(B_r(x))}\\
&= \overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon\left(\frac{d\nu}{d\mu_0}(x)\,\mu_0,B_\rho(x)\right)}{\mu_0(B_\rho(x))}\\
&= \lim_{\rho\to 0}\,\overline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon(\nu,B_\rho(x))}{\mu_0(B_\rho(x))}\\
&= \overline{\lim_{\rho\to 0}}\,\underline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon\left(\frac{d\nu}{d\mu_0}(x)\,\mu_0,B_\rho(x)\right)}{\mu_0(B_\rho(x))},
\end{aligned}$$

which shows that all these functions are Borel measurable, as ensured by Lemma 6.3, which guarantees the measurability of the following function:

$$\Omega\ni x\longmapsto \lim_{\rho\to 0}\,\overline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon(\nu,B_\rho(x))}{\mu_0(B_\rho(x))}.$$

Similarly by using ($\mathrm{H_{sing}}$), we have for $|\nu_0^s|$-a.e. $x\in O$

$$\begin{aligned}
\lim_{r\to 0}\frac{\left(\Gamma\left(w^*\right)\text{-}\underline{\lim}_{\varepsilon\to 0} I_\varepsilon\right)(\nu,B_r(x))}{|\nu_0^s|(B_r(x))} &\overset{(3)}{\ge} \overline{\lim_{t\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\underline{\lim_{\varepsilon\to 0}}\frac{I_\varepsilon(\nu_\varepsilon,B_{t\rho}(x))}{|\nu_0^s|(B_\rho(x))}\\
&\overset{(6)}{\ge} \lim_{\rho\to 0}\,\underline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon\left(\frac{d\nu}{d|\nu_0^s|}(x)\,|\nu_0^s|,B_\rho(x)\right)}{|\nu_0^s|(B_\rho(x))}\\
&\overset{(\mathrm{H_{sing}})}{\ge} \overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon\left(\frac{d\nu}{d|\nu_0^s|}(x)\,|\nu_0^s|,B_\rho(x)\right)}{|\nu_0^s|(B_\rho(x))}\\
&\overset{(7)}{\ge} \lim_{\rho\to 0}\,\overline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon(\nu,B_\rho(x))}{|\nu_0^s|(B_\rho(x))} \overset{(4)}{\ge} \overline{\lim_{r\to 0}}\frac{\left(\Gamma\left(w^*\right)\text{-}\overline{\lim}_{\varepsilon\to 0} I_\varepsilon\right)(\nu,B_r(x))}{|\nu_0^s|(B_r(x))}.
\end{aligned}$$

It follows that for $|\nu_0^s|$-a.e. $x\in O$

$$\begin{aligned}
\lim_{r\to 0}\frac{\left(\Gamma\left(w^*\right)\text{-}\underline{\lim}_{\varepsilon\to 0} I_\varepsilon\right)(\nu,B_r(x))}{|\nu_0^s|(B_r(x))} &= \lim_{r\to 0}\frac{\left(\Gamma\left(w^*\right)\text{-}\overline{\lim}_{\varepsilon\to 0} I_\varepsilon\right)(\nu,B_r(x))}{|\nu_0^s|(B_r(x))}\\
&= \overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon\left(\frac{d\nu}{d|\nu_0^s|}(x)\,|\nu_0^s|,B_\rho(x)\right)}{|\nu_0^s|(B_\rho(x))}\\
&= \lim_{\rho\to 0}\,\overline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon(\nu,B_\rho(x))}{|\nu_0^s|(B_\rho(x))} = \overline{\lim_{\rho\to 0}}\,\underline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon\left(\frac{d\nu}{d|\nu_0^s|}(x)\,|\nu_0^s|,B_\rho(x)\right)}{|\nu_0^s|(B_\rho(x))}.
\end{aligned}$$

From the above, we see that for all $\sigma\in\{\mu_0,|\nu_0^s|\}$ and $\sigma$-a.e. $x\in O$,

$$\overline{\lim_{t\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\underline{\lim_{\varepsilon\to 0}}\frac{I_\varepsilon(\nu_\varepsilon,B_{t\rho}(x))}{\sigma(B_\rho(x))} = \lim_{\rho\to 0}\,\overline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon(\nu,B_\rho(x))}{\sigma(B_\rho(x))} = \overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon\left(\frac{d\nu}{d\sigma}(x)\,\sigma,B_\rho(x)\right)}{\sigma(B_\rho(x))}.$$

Using Proposition 2.1 we conclude that

$$\left(\Gamma(w^*)\text{-}\underline{\lim}_{\varepsilon\to 0} I_\varepsilon\right)(\nu,O) \geq \sum_{\sigma\in\{\mu_0,|\nu_0^s|\}} \int_O \overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon\left(\frac{d\nu}{d\sigma}(x)\,\sigma, B_\rho(x)\right)}{\sigma(B_\rho(x))} d\sigma(x)$$
$$\geq \left(\Gamma(w^*)\text{-}\overline{\lim_{\varepsilon\to 0}} I_\varepsilon\right)(\nu,O),$$

which completes the proof. ■

2.3. **Γ-convergence without linear growth.** In the case where the integrands $f_\varepsilon$ do not have linear growth, set $f_\varepsilon^j$ the Moreau-Yosida envelope of $f_\varepsilon$ of index 1 defined by

$$\Omega\times\mathbb{R}^l \ni (x,v) \longmapsto f_\varepsilon^j(x,v) := \inf\left\{f_\varepsilon(x,w) + j|v-w| : w\in\mathbb{R}^l\right\},$$

and for every $\nu\in\mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ and every $B\in\mathcal{B}(\Omega)$, the associated integrals

$$I_\varepsilon^j(\nu,B) := \begin{cases} \displaystyle\int_B f_\varepsilon^j\left(x,\frac{d\lambda}{d\mu_\varepsilon}(x)\right)d\mu_\varepsilon(x) & \text{if } \lambda\ll\mu_\varepsilon \\ \infty & \text{otherwise,}\end{cases}$$

with the corresponding local minimisation problems

$$m_\varepsilon^j(\nu,B) := \inf\left\{I_\varepsilon^j(\lambda,B) : \mathcal{M}\left(\Omega;\mathbb{R}^l\right)\ni\lambda\ll\mu_\varepsilon, \text{ and } \lambda(B)=\nu(B)\right\}$$

for all $(j,\varepsilon)\in\mathbb{N}\times\mathbb{R}_+^*$.

The conditions $(\mathrm{H}_{\mathrm{reg}})$ and $(\mathrm{H}_{\mathrm{sing}})$ become, for each $j\in\mathbb{N}$:

$(\mathrm{H}_{\mathrm{reg}}^j)$ for every $(u,O)\in L^1_{\mu_0}\left(\Omega;\mathbb{R}^l\right)\times\mathcal{O}(\Omega)$, it holds

$$\overline{\lim_{\rho\to 0}}\,\underline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon^j(u(x)\,\mu_0, B_\rho(x))}{\mu_0(B_\rho(x))} \geq \overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon^j(u(x)\,\mu_0, B_\rho(x))}{\mu_0(B_\rho(x))} \quad \mu_0\text{-a.e. in } O;$$

$(\mathrm{H}_{\mathrm{sing}}^j)$ for every $(\nu,O)\in\mathcal{M}\left(\Omega;\mathbb{R}^l\right)\times\mathcal{O}(\Omega)$, and every $v\in L^1_{|\nu_0^s|}\left(\Omega;\mathbb{S}^{l-1}\right)$, it holds

$$\underline{\lim_{\rho\to 0}}\,\underline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon^j(v(x)\,|\nu_0^s|, B_\rho(x))}{|\nu_0^s|(B_\rho(x))} \geq \overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon^j(v(x)\,|\nu_0^s|, B_\rho(x))}{|\nu_0^s|(B_\rho(x))} \quad |\nu_0^s|\text{-a.e. in } O.$$

**Theorem 2.1.** *Suppose* $(\mathrm{M}_0)$ *and*

$(\mathrm{C}_1')$ *there exists* $c>0$ *such that for every* $\varepsilon>0$ *and every* $(x,v)\in\Omega\times\mathbb{R}^l$

$$c|v| \leq f_\varepsilon(x,v).$$

*Assume that* $(\mathrm{H}_{\mathrm{reg}}^j)$ *and* $(\mathrm{H}_{\mathrm{sing}}^j)$ *hold for all* $j\in\mathbb{N}$. *Then for every* $O\in\mathcal{O}_0(\Omega)$, *the family of integral functionals* $\{I_\varepsilon(\cdot,O)\}_\varepsilon$ $\Gamma(w^*)$*-converges to* $I_0(\cdot,O)$ *with*

$$I_0(\nu,O) = \int_O f_0\left(x,\frac{d\nu}{d\mu_0}(x)\right)d\mu_0(x) + \int_O f_s\left(x,\frac{d\nu}{d|\nu_0^s|}(x)\right)d|\nu_0^s|(x)$$

*for all* $\nu\in\mathcal{M}\left(\Omega;\mathbb{R}^l\right)$, *where* $f_0:\Omega\times\mathbb{R}^l\to[0,\infty]$ *and* $f_s:\Omega\times\mathbb{S}^{l-1}\to[0,\infty]$ *are respectively defined by:*

$$f_0(x,v) := \sup_{j\in\mathbb{N}}\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon^j(v\mu_0, B_\rho(x))}{\mu_0(B_\rho(x))},$$

*and*

$$f_s(x,w) := \sup_{j\in\mathbb{N}}\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon^j(w|\nu_0^s|, B_\rho(x))}{|\nu_0^s|(B_\rho(x))}.$$

**Proof of Theorem 2.1.** Let $\varepsilon > 0$. First, observe, by using $(C_2)$, that for every $x \in \Omega$ and every $j \in \mathbb{N}$

$$f_\varepsilon^j(x,v) \leq j(1+|v|),$$

moreover, we also see that $f_\varepsilon^j(x,v) \geq \min(c,j)|v|$ for all $j \in \mathbb{N}^*$, where $c$ is the constant appearing in $(C_1)$. Thus $(C_1)$ holds for each of $\{f_\varepsilon^j\}_{\varepsilon>0}$. Fix $O \in \mathcal{O}_0(\Omega)$. Applying Theorem 1.1 to each of $\{f_\varepsilon^j\}_{\varepsilon>0}$, the family of integral functionals $\{I_\varepsilon^j(\cdot,O)\}_\varepsilon$ $\Gamma(w^*)$-converges to $I_0^j(\cdot,O)$ with

$$I_0^j(\nu,O) = \int_O f_0^j\left(x, \frac{d\nu}{d\mu_0}(x)\right) d\mu_0(x) + \int_O f_s^j\left(x, \frac{d\nu}{d|\nu_0^s|}(x)\right) d|\nu_0^s|(x)$$

for all $\nu \in \mathcal{M}(\Omega;\mathbb{R}^l)$, where $f_0^j : \Omega \times \mathbb{R}^l \to [0,\infty]$ and $f_s^j : \Omega \times \mathbb{S}^{l-1} \to [0,\infty]$ are respectively given by:

$$f_0^j(x,v) := \overline{\lim_{\rho\to 0}}\, \overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon^j(v\mu_0, B_\rho(x))}{\mu_0(B_\rho(x))},$$

and

$$f_s^j(x,w) := \overline{\lim_{\rho\to 0}}\, \overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon^j(w|\nu_0^s|, B_\rho(x))}{|\nu_0^s|(B_\rho(x))}.$$

Let $(\nu,O) \in \mathcal{M}(\Omega;\mathbb{R}^l) \times \mathcal{O}(\Omega)$. On one hand, since $f_\varepsilon \geq f_\varepsilon^j$, we have

$$\left(\Gamma(w^*)\text{-}\underline{\lim_{\varepsilon\to 0}} I_\varepsilon\right)(\nu,O) \geq \Gamma(w^*)\text{-}\underline{\lim_{\varepsilon\to 0}} I_\varepsilon^j(\nu,O) = I_0^j(\nu,O) \qquad \text{for all } j \in \mathbb{N}.$$

On the other hand, applying Lemma 6.1

$$\sup_{j\in\mathbb{N}} I_0^j(\nu,O) = \left(\Gamma(w^*)\text{-}\overline{\lim_{\varepsilon\to 0}} I_\varepsilon\right)(\nu,O).$$

It follows that $\left(\Gamma(w^*)\text{-}\underline{\lim_{\varepsilon\to 0}} I_\varepsilon\right)(\nu,O) \geq \sup_{j\in\mathbb{N}} I_0^j(\nu,O) = \left(\Gamma(w^*)\text{-}\overline{\lim_{\varepsilon\to 0}} I_\varepsilon\right)(\nu,O)$, which means that $\{I_\varepsilon(\cdot,O)\}_\varepsilon$ $\Gamma(w^*)$-converges to $I_0(\cdot,O) = \sup_{j\in\mathbb{N}} I_0^j(\cdot,O)$. We conclude the proof by applying the monotone convergence theorem to derive the formulas for the limit integrands. ■

## 3. Examples

3.1. **Γ-convergence with convex integrands not depending on $(x,\varepsilon)$.** Let $\Phi : \mathbb{R}^l \to \overline{\mathbb{R}}$ be a convex and lower semicontinuous function satisfying for some $c > 0$

$$c|v| \leq \Phi(v) \quad \text{for all } v \in \mathbb{R}^l \quad \text{and } \Phi(0) = 0. \tag{8}$$

For each $\varepsilon > 0$ define $I_\varepsilon : \mathcal{M}(\Omega;\mathbb{R}^l) \times \mathcal{B}(\Omega) \to [0,\infty]$ by

$$I_\varepsilon(\nu,A) := \begin{cases} \displaystyle\int_A \Phi\left(\frac{d\nu}{d\mu_\varepsilon}\right) d\mu_\varepsilon & \text{if } \nu \ll \mu_\varepsilon \\ \infty & \text{otherwise,} \end{cases}$$

where $\{\mu_\varepsilon\}_{\varepsilon>0} \subset \mathcal{M}_+(\Omega)$ is a family of positive Radon measures. ‘

As a consequence of Theorem 2.1, we retrieve the following result (see [Bou87, AB98]). It is worth noting that we make the additional assumption $(M_0)$, which can be explained by the “nonconvex” approach used to prove Theorem 2.1 (see Subsection 6.4 for the proof).

**Theorem 3.1.** *Let $\mu_0 \in \mathcal{M}_+(\Omega)$ satisfy $(M_0)$. Let $O \in \mathcal{O}_0(\Omega)$. If $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu_0$ in $\mathcal{M}_+(\Omega)$ then $\{I_\varepsilon(\cdot,O)\}_\varepsilon$ $\Gamma(w^*)$-converges to $I_0(\cdot,O)$ with*

$$I_0(\nu,O) = \int_O \Phi\left(\frac{d\nu}{d\mu_0}\right) d\mu_0 + \int_O \Phi^\infty\left(\frac{d\nu}{d|\nu_0^s|}\right) d|\nu_0^s|$$

*for all $\nu \in \mathcal{M}(\Omega;\mathbb{R}^l)$, where $\Phi^\infty(\cdot) := \sup_{t>0} \frac{1}{t}\Phi(t\cdot)$ is the recession function of $\Phi$, and $\nu_0^s = \nu - \frac{d\nu}{d\mu_0}\mu_0$.*

As an illustration of Theorem 3.1 we consider the example of [Bou87, Exemple 4.1, pp. 138]. Let $\mu \in \mathcal{M}_+(\Omega)$ be a positive Radon measure. Let $\{a_\varepsilon\}_{\varepsilon>0} \subset L^1_\mu(\Omega)$ be such that for some $c, C > 0$ we have for every $\varepsilon > 0$,

$$c \leq a_\varepsilon \leq C.$$

Let $p > 1$ and $q = \left(1 - \frac{1}{p}\right)^{-1}$. Define the integrals $J_\varepsilon : \mathcal{M}\left(\Omega; \mathbb{R}^l\right) \times \mathcal{B}(\Omega) \to [0, \infty]$ by

$$J_\varepsilon(\nu, A) := \begin{cases} \displaystyle\int_A a_\varepsilon \left|\frac{d\nu}{d\mu}\right|^p d\mu & \text{if } \nu \ll \mu \\ \infty & \text{otherwise.} \end{cases}$$

Setting $\mu_\varepsilon := \frac{1}{a_\varepsilon^{q-1}}\mu$, we can rewrite $J_\varepsilon$ as

$$J_\varepsilon(\nu, A) := \begin{cases} \displaystyle\int_A \left|\frac{d\nu}{d\mu_\varepsilon}\right|^p d\mu_\varepsilon & \text{if } \nu \ll \mu_\varepsilon \\ \infty & \text{otherwise.} \end{cases}$$

Now, under the assumptions of Theorem 3.1 the family $\{J_\varepsilon(\cdot, O)\}_\varepsilon$ $\Gamma$-converges at each $\nu \in \mathcal{M}\left(\Omega; \mathbb{R}^l\right)$ to $J_0(\cdot, O)$ given by

$$J_0(\nu, O) = \int_O \left|\frac{d\nu}{d\mu_0}\right|^p d\mu_0 + \int_O \chi_{\{0\}}\left(\frac{d\nu}{d|\nu_0^s|}\right) d|\nu_0^s|,$$

where $\chi_{\{0\}}$ is the indicator function of $\{0\}$, i.e.

$$\chi_{\{0\}}(v) := \begin{cases} 0 & \text{if } v = 0 \\ \infty & \text{otherwise. } \blacksquare \end{cases}$$

3.2. **Relaxation with integrand dependent on $x$.** Let $\Phi : \Omega \times \mathbb{R}^l \to [0, \infty]$ be a Borel function satisfying for some $c > 0$

$$\Phi(x, 0) = 0 \quad \text{and} \quad c|v| \leq \Phi(x, v) \quad \text{for all } (x, v) \in \Omega \times \mathbb{R}^l. \tag{9}$$

Define $I : \mathcal{M}\left(\Omega; \mathbb{R}^l\right) \times \mathcal{B}(\Omega) \to [0, \infty]$ by

$$I(\nu, A) := \begin{cases} \displaystyle\int_A \Phi\left(x, \frac{d\nu}{d\mu_0}\right) d\mu_0 & \text{if } \nu \ll \mu_0 \\ \infty & \text{otherwise,} \end{cases}$$

where $\mu_0 \in \mathcal{M}_+(\Omega)$ is a positive Radon measure. The relaxed functional with respect to the weak star convergence of measures is given by

$$\overline{I}(\nu, A) = \inf\left\{\underline{\lim}_{\varepsilon\to 0} I(\nu_\varepsilon, A) : \nu_\varepsilon \overset{*}{\rightharpoonup} \nu \text{ in } \mathcal{M}\left(\Omega; \mathbb{R}^l\right)\right\}$$

for all $\nu \in \mathcal{M}\left(\Omega; \mathbb{R}^l\right)$. The following result was proved in [DGAB87, Theorem 2.4] (see also [AB88, Theorem 2.4] and [But89, Theorem 3.3.1, pp. 99])

**Theorem 3.2.** *There exists a Borel function $\varphi : \Omega \times \mathbb{R}^l \to [0, \infty]$ such that for $\mu_0$-a.e. in $x \in \Omega$ the function $\varphi(x, \cdot)$ is convex and lower semicontinuous and*

$$\overline{I}(\nu, A) = \int_A \varphi\left(x, \frac{d\nu}{d\mu_0}\right) d\mu_0 + \int_A \varphi^\infty\left(x, \frac{d\nu}{d|\nu_0^s|}\right) d|\nu_0^s|.$$

*for all $(\nu, A) \in \mathcal{M}\left(\Omega; \mathbb{R}^l\right) \times \mathcal{B}(\Omega)$. Moreover, the recession function $\varphi^\infty$ defined by*

$$\varphi^\infty(x, v) := \overline{\lim_{t\to\infty}} \frac{\varphi(x, tv)}{t}$$

*is lower semicontinuous in $(x, v)$.*

The following result provides, in the case where the measure $\mu_0$ satisfies $(\mathrm{M}_0)$, a representation for the relaxed integrands. This complements the result of [DGAB87, Theorem 2.4]. By combining Theorem 3.2 and Theorem 2.1 we have:

**Theorem 3.3.** *Assume that $\mu_0$ satisfies $(\mathrm{M}_0)$. Let $O \in \mathcal{O}_0(\Omega)$. Then for every $\nu \in \mathcal{M}\left(O;\mathbb{R}^l\right)$*

$$\overline{I}(\nu,O)=\int_O \Phi_0\left(x,\frac{d\nu}{d\mu_0}\right)d\mu_0+\int_O \Phi_s\left(x,\frac{d\nu}{d|\nu_0^s|}\right)d|\nu_0^s|$$

*where $\Phi_0:\Omega\times\mathbb{R}^l\to[0,\infty]$ and $\Phi_s:\Omega\times\mathbb{S}^{l-1}\to[0,\infty]$ are respectively given by:*

$$\Phi_0(x,u):=\sup_{j\in\mathbb{N}}\overline{\lim_{\rho\to 0}}\inf_{\substack{w\in L^1_{\mu_0}\left(B_\rho(x);\mathbb{R}^l\right)\\ \int_{B_\rho(x)} w d\mu_0=0}}\fint_{B_\rho(x)}\Phi^j\left(z,u+w(z)\right)d\mu_0(z);$$

*and*

$$\Phi_s(x,v):=\sup_{j\in\mathbb{N}}\overline{\lim_{\rho\to 0}}\inf_{\substack{w\in L^1_{\mu_0}\left(B_\rho(x);\mathbb{R}^l\right)\\ \int_{B_\rho(x)} w d\mu_0=0}}\frac{1}{|\nu_0^s|\left(B_\rho(x)\right)}\int_{B_\rho(x)}\Phi^j\left(z,\frac{|\nu_0^s|\left(B_\rho(x)\right)}{\mu_0\left(B_\rho(x)\right)}v+w(z)\right)d\mu_0(z).$$

*Moreover, for $\mu_0$-a.e. in $x\in\Omega$ the function $\Phi_0(x,\cdot)$ is convex and lower semicontinuous, and $\Phi_s$ is lower semicontinuous in $(x,v)$ and $\Phi_s(x,\cdot)=\Phi_0^\infty(x,\cdot)$ $|\nu_0^s|$-a.e. $x\in\Omega$.*

**Proof of Theorem 3.3.** We see, by setting $f_\varepsilon(x,v):=\Phi(x,v)$ for all $(x,v)\in\Omega\times\mathbb{R}^l$, that $(\mathrm{C}_1)$ and $(\mathrm{C}_2)$ are fulfilled since (9).

Let $x\in\Omega$. Let $\nu\in\mathcal{M}\left(\Omega;\mathbb{R}^l\right)$. For each $j\in\mathbb{N}$ we denote by $\Phi^j$ the Moreau-Yosida envelope of $\Phi$ of index 1 defined by

$$\Phi^j(x,w):=\inf_{v\in\mathbb{R}^l}\Phi(x,w+v)+j|v|.$$

Combining Lemma 6.3 and Corollary 6.1, we see that for every $j\in\mathbb{N}$, every $(\nu,O)\in\mathcal{M}\left(\Omega;\mathbb{R}^l\right)\times\mathcal{O}(\Omega)$, and every $v\in L^1_{|\nu_0^s|}\left(\Omega;\mathbb{S}^{l-1}\right)$, the following limit exists

$$\lim_{\rho\to 0}\frac{m^j\left(v(x)|\nu_0^s|,B_\rho(x)\right)}{|\nu_0^s|\left(B_\rho(x)\right)}\quad |\nu_0^s|\text{-a.e. in } O,$$

where $\nu_0^s=\nu-\frac{d\nu}{d\mu_0}\mu_0$ is the singular part in the Lebesgue decomposition of $\nu$ with respect to the measure $\mu_0$, which means that $(\mathrm{H}^j_{\mathrm{sing}})$ hold. We apply Theorem 2.1 and the properties of convexity and lower semicontinuity of the relaxed formulas are given by Theorem 3.2. ■

## 4. Stochastic homogenization of integrals defined on measures

4.1. **Subadditive theorem.** Let $(\Sigma,\mathcal{T},\mathbb{P})$ be a probability space. Let $\{\tau_z:\Sigma\to\Sigma\}_{z\in\mathbb{Z}^N}$ be a group of $\mathbb{P}$-preserving transformations on $(\Sigma,\mathcal{T})$, i.e.

- $(\mathscr{T}_1)$ $\tau_z$ is $\mathcal{T}$-measurable for all $z\in\mathbb{Z}^N$ (measurability);
- $(\mathscr{T}_2)$ $\tau_z\circ\tau_{z'}=\tau_{z+z'}$ and $\tau_{-z}=\tau_z^{-1}$ for all $z,z'\in\mathbb{Z}^N$ (group property);
- $(\mathscr{T}_3)$ $\mathbb{P}(\tau_z(E))=\mathbb{P}(E)$ for all $E\in\mathcal{T}$ and all $z\in\mathbb{Z}^N$ (mass invariance).

The quadruplet $(\Sigma,\mathcal{T},\mathbb{P},\{\tau_z\}_{z\in\mathbb{Z}^N})$ is called a *measurable dynamical system*. We denote by $\mathcal{I}:=\left\{E\in\mathcal{T}:\forall z\in\mathbb{Z}^N\quad \mathbb{P}(\tau_z(E)\,\Delta E)=0\right\}$ the $\sigma$-algebra of invariant sets with respect to $(\Sigma,\mathcal{T},\mathbb{P},\{\tau_z\}_{z\in\mathbb{Z}^N})$. When $\mathbb{P}(E)\in\{0,1\}$ for all $E\in\mathcal{I}$, the measurable dynamical system $(\Sigma,\mathcal{T},\mathbb{P},\{\tau_z\}_{z\in\mathbb{Z}^N})$ is said to be *ergodic*. Denote by $\mathcal{B}_b\left(\mathbb{R}^N\right)$ the set of all bounded Borel sets of $\mathbb{R}^N$.

**Definition 4.1.** *We say that $\mathcal{S}:\mathcal{B}_b\left(\mathbb{R}^N\right)\to L^1(\Sigma,\mathcal{T},\mathbb{P})$ is a subadditive process if*

- $(\mathscr{S}_1)$ *$\mathcal{S}(A\cup B)\le\mathcal{S}(A)+\mathcal{S}(B)$ for all $A,B\in\mathcal{B}_b\left(\mathbb{R}^N\right)$ such that $A\cap B=\emptyset$* (subadditivity)*;*
- $(\mathscr{S}_2)$ *$\mathcal{S}(A+z)=\mathcal{S}(A)\circ\tau_z$ for all $A\in\mathcal{B}_b\left(\mathbb{R}^N\right)$ and all $z\in\mathbb{Z}^N$* (stationarity)*.*

*If in addition the measurable dynamical system $(\Sigma,\mathcal{T},\mathbb{P},\{\tau_z\}_{z\in\mathbb{Z}^N})$ is ergodic, we say that $\mathcal{S}$ is an ergodic subadditive process.*

For a bounded Borel set $A \in \mathcal{B}_b\left(\mathbb{R}^N\right)$ we set

$$R(A) := \sup\left\{\rho \in \mathbb{R}_+ : \exists x \in A \quad \overline{B}_\rho(x) \subset A\right\}.$$

A sequence of bounded Borel sets $\{A_n\}_{n\in\mathbb{N}}$ is called regular if there exist an increasing sequence of intervals $\{J_n\}_{n\in\mathbb{N}}$ whose the bounds belongs to $\mathbb{Z}^N$ and $C > 0$ such that $\mathcal{L}_N(J_n) \leq C\mathcal{L}_N(A_n)$ for all $n \in \mathbb{N}$. We denote by $\mathbb{Y}$ the unit cell $[0,1[^N$.

**Theorem 4.1** ([LM02, Theorem 4.1, p. 33])**.** *Let $\mathcal{S} : \mathcal{B}_b\left(\mathbb{R}^N\right) \to L^1(\Sigma, \mathcal{T}, \mathbb{P})$ be a subadditive process such that*

- *($\mathcal{S}_1$) $\mathcal{S}$ is a nonnegative subadditive process, i.e. $\mathcal{S}(A)(\omega) \geq 0$ for all $A \in \mathcal{B}_b\left(\mathbb{R}^N\right)$ and all $\omega \in \Sigma$;*
- *($\mathcal{S}_2$) there exists $h \in L^1(\Sigma, \mathcal{T}, \mathbb{P})$ such that $\mathcal{S}(A) \leq h$ for all Borel set $A \subset \mathbb{Y}$.*

*Then there exists $\Sigma' \in \mathcal{T}$ with $\mathbb{P}(\Sigma') = 1$ such that for every $\omega \in \Sigma'$ and every regular sequence of Borel convex sets $\{C_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_b\left(\mathbb{R}^N\right)$ satisfying $\lim_{n\to\infty} R(C_n) = \infty$ we have*

$$\lim_{n\to\infty} \frac{\mathcal{S}(C_n)(\omega)}{\mathcal{L}_N(C_n)} = \inf_{k\in\mathbb{N}^*} \mathbb{E}^{\mathcal{I}}\left[\frac{\mathcal{S}(k\mathbb{Y})}{k^N}\right](\omega).$$

*If $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_z\}_{z\in\mathbb{Z}^N})$ is ergodic then for $\mathbb{P}$-a.e. $\omega \in \Sigma$*

$$\lim_{n\to\infty} \frac{\mathcal{S}(C_n)(\omega)}{\mathcal{L}_N(C_n)} = \inf_{k\in\mathbb{N}^*} \int_\Sigma \frac{\mathcal{S}(k\mathbb{Y})(\omega)}{k^N} d\mathbb{P}(\omega).$$

4.2. **Properties of subadditive processes associated with $f$ and $f^\infty$.** For the proof of Theorem 1.2 we will need the following lemma which is proved in Subsections 6.5 and 6.6.

**Lemma 4.1.** *For each $u \in \mathbb{R}^l$, each $A \in \mathcal{B}_b\left(\mathbb{R}^N\right)$ and each $\omega \in \Sigma$ we set*

$$\mathcal{S}_u(A)(\omega) := \inf_{\substack{\mathcal{M}\left(\Omega;\mathbb{R}^l\right)\ni\lambda\ll\mu \\ \lambda(A)=u\mu_0(A)}} \int_A f\left(x, \frac{d\lambda}{d\mu}(x), \omega\right) d\mu(x).$$

- *($i$) For each $u \in \mathbb{R}^l$ the map $\mathcal{S}_u : \mathcal{B}_b\left(\mathbb{R}^N\right) \to L^1(\Sigma, \mathcal{T}, \mathbb{P})$ is a nonnegative subadditive process satisfying for some $C > 0$, for $\mathbb{P}$-a.e. $\omega \in \Sigma$ and for every Borel set $A \subset [0,1[^N$*

$$\mathcal{S}_u(A)(\omega) \leq C(1 + |u|).$$

- *($ii$) There exists $\Sigma' \in \mathcal{T}$ with $\mathbb{P}(\Sigma') = 1$ such that for every $\omega \in \Sigma'$, every $u \in \mathbb{R}^l$ and every cube $B \subset \mathbb{R}^N$*

$$\lim_{\varepsilon\to 0} \frac{\mathcal{S}_u\left(\frac{1}{\varepsilon}B\right)(\omega)}{\mathcal{L}_N\left(\frac{1}{\varepsilon}B\right)} = \inf_{k\in\mathbb{N}^*} \mathbb{E}^{\mathcal{I}}\left[\frac{\mathcal{S}_u\left([0,k[^N\right)}{k^N}\right](\omega).$$

We will also need the following lemma, the proof of which is similar to that of Lemma 4.1, which is stated just above.

**Lemma 4.2.** *For each $w \in \mathbb{S}^{l-1}$, each $A \in \mathcal{B}_b\left(\mathbb{R}^N\right)$ and each $\omega \in \Sigma$ we set*

$$\mathcal{S}^\infty_w(A)(\omega) := \inf_{\substack{\mathcal{M}\left(\Omega;\mathbb{R}^l\right)\ni\lambda\ll\mu \\ \lambda(A)=w\mu_0(A)}} \int_A f^\infty\left(x, \frac{d\lambda}{d\mu}(x), \omega\right) d\mu.$$

- *($i$) For each $w \in \mathbb{S}^{l-1}$ the map $\mathcal{S}^\infty_w : \mathcal{B}_b\left(\mathbb{R}^N\right) \to L^1(\Sigma, \mathcal{T}, \mathbb{P})$ is a nonnegative subadditive process satisfying for some $C > 0$, for $\mathbb{P}$-a.e. $\omega \in \Sigma$ and for every Borel set $A \subset [0,1[^N$*

$$\mathcal{S}^\infty_w(A)(\omega) \leq C|w|.$$

- *($ii$) There exists $\Sigma' \in \mathcal{T}$ with $\mathbb{P}(\Sigma') = 1$ such that for every $\omega \in \Sigma'$, every $w \in \mathbb{S}^{l-1}$ and every cube $B \subset \mathbb{R}^N$*

$$\lim_{\varepsilon\to 0} \frac{\mathcal{S}^\infty_w\left(\frac{1}{\varepsilon}B\right)(\omega)}{\mathcal{L}_N\left(\frac{1}{\varepsilon}B\right)} = \inf_{k\in\mathbb{N}^*} \mathbb{E}^{\mathcal{I}}\left[\frac{\mathcal{S}^\infty_w\left([0,k[^N\right)}{k^N}\right](\omega).$$

4.3. **Proof of Theorem 1.2.** We recall that the measure $\mu \in \mathcal{M}_+\left(\mathbb{R}^N\right)$ is a 1-periodic measure which satisfies

$$\mu_\varepsilon \overset{*}{\rightharpoonup} \mu_0 = p_0 \mathcal{L}_N \text{ in } \mathcal{M}_+\left(\mathbb{R}^N\right) \text{ as } \varepsilon \to 0,$$

where $p_0 = \mu\left(\mathbb{Y}\right)$ with $\mathbb{Y} := ]0,1[^N$. The proof consists in verifying the assumptions of Theorem 1.1. Since $(\mathrm{C}_1^\omega)$, we note that for $\mathbb{P}$-a.e. $\omega \in \Sigma$ the family of integrands

$$\left\{ f_\varepsilon\left(\cdot,\cdot\right) := f\left(\frac{\cdot}{\varepsilon}, \cdot, \omega\right) \lfloor_\Omega \right\}_\varepsilon$$

satisfies $(\mathrm{C}_1)$. It remains to show that $(\mathrm{H}_{\mathrm{reg}})$ and $(\mathrm{H}_{\mathrm{sing}})$ hold, i.e. for every $O \in \mathcal{O}\left(\Omega\right)$, for $\mathbb{P}$-a.e. $\omega \in \Sigma$, for every $u \in L^1_{\mu_0}\left(\Omega; \mathbb{R}^l\right)$ and every $v \in L^1_{|\nu_0^s|}\left(\Omega; \mathbb{S}^{l-1}\right)$, it holds

$$\overline{\lim_{\rho\to 0}} \underline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon\left(u\left(x\right)\mu_0, B_\rho\left(x\right), \omega\right)}{\mu_0\left(B_\rho\left(x\right)\right)} \geq \overline{\lim_{\rho\to 0}} \overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon\left(u\left(x\right)\mu_0, B_\rho\left(x\right), \omega\right)}{\mu_0\left(B_\rho\left(x\right)\right)} \quad \mu_0\text{-a.e. in } O; \tag{10}$$

$$\underline{\lim_{\rho\to 0}} \underline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon\left(u\left(x\right)|\nu_0^s|, B_\rho\left(x\right), \omega\right)}{|\nu_0^s|\left(B_\rho\left(x\right)\right)} \geq \overline{\lim_{\rho\to 0}} \overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon\left(u\left(x\right)|\nu_0^s|, B_\rho\left(x\right), \omega\right)}{|\nu_0^s|\left(B_\rho\left(x\right)\right)} \quad |\nu_0^s|\text{-a.e. in } O, \tag{11}$$

where $\nu_0^s = \nu - \frac{d\nu}{d\mu_0}\mu_0$, and where for every $r > 0$, every $\varepsilon > 0$, every $x \in \Omega$ and every $\omega \in \Sigma$

$$m_\varepsilon\left(u\left(x\right)\mu_0, B_r\left(x\right), \omega\right) := \inf_{\substack{\mathcal{M}\left(\Omega;\mathbb{R}^l\right)\ni\lambda\ll\mu_\varepsilon \\ \lambda(B_r(x))=u(x)\mu_0(B_r(x))}} \int_{B_r(x)} f\left(\frac{y}{\varepsilon}, \frac{d\lambda}{d\mu_\varepsilon}\left(y\right), \omega\right) d\mu_\varepsilon\left(y\right);$$

$$m_\varepsilon\left(v\left(x\right)|\nu_0^s|, B_r\left(x\right), \omega\right) := \inf_{\substack{\mathcal{M}\left(\Omega;\mathbb{R}^l\right)\ni\lambda\ll\mu_\varepsilon \\ \lambda(B_r(x))=v(x)|\nu_0^s|(B_r(x))}} \int_{B_r(x)} f\left(\frac{y}{\varepsilon}, \frac{d\lambda}{d\mu_\varepsilon}\left(y\right), \omega\right) d\mu_\varepsilon\left(y\right).$$

*Proof of* (10). Note that for a measure $\lambda \in \mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ with $\lambda \ll \mu$

$$\frac{d\lambda}{d\mu_\varepsilon}\left(\varepsilon x\right) = \frac{1}{\varepsilon^N} \lim_{\rho\to 0} \frac{\lambda\left(\varepsilon B_{\rho/\varepsilon}\left(x\right)\right)}{\mu\left(B_{\rho/\varepsilon}\left(x\right)\right)} = \frac{d\lambda_{1/\varepsilon}}{d\mu}\left(x\right) \quad \mu\text{-a.e. in } \Omega,$$

where $\lambda_{1/\varepsilon}\left(\cdot\right) := \varepsilon^{-N}\lambda\left(\varepsilon\cdot\right)$ for all $\varepsilon > 0$. Fix $\omega \in \Sigma$. For every $\rho > 0$ and every $u \in L^1\left(\Omega;\mathbb{R}^l\right)$ we can write for $\mu_0$-a.e. $x \in \Omega$

$$\begin{aligned}
\frac{m_\varepsilon\left(u\left(x\right)\mu_0, B_\rho\left(x\right), \omega\right)}{\mu_0\left(B_\rho\left(x\right)\right)} &= \inf_{\substack{\mathcal{M}\left(\Omega;\mathbb{R}^l\right)\ni\lambda\ll\mu_\varepsilon \\ \lambda(B_\rho(x))=u(x)\mu_0(B_\rho(x))}} \frac{1}{\mu_0\left(B_\rho\left(x\right)\right)} \int_{B_\rho(x)} f\left(\frac{y}{\varepsilon}, \frac{d\lambda}{d\mu_\varepsilon}\left(y\right), \omega\right) d\mu_\varepsilon\left(y\right) \\
&= \inf_{\substack{\mathcal{M}\left(\Omega;\mathbb{R}^l\right)\ni\lambda\ll\mu_\varepsilon \\ \lambda(B_\rho(x))=u(x)\mu_0(B_\rho(x))}} \frac{1}{\mu_0\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)} \int_{\frac{1}{\varepsilon}B_\rho(x)} f\left(s, \frac{d\lambda}{d\mu_\varepsilon}\left(\varepsilon s\right), \omega\right) d\mu\left(s\right) \\
&= \inf_{\substack{\mathcal{M}\left(\Omega;\mathbb{R}^l\right)\ni\lambda\ll\mu_\varepsilon \\ \lambda(B_\rho(x))=u(x)\mu_0(B_\rho(x))}} \frac{1}{\mu_0\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)} \int_{\frac{1}{\varepsilon}B_\rho(x)} f\left(s, \frac{d\lambda_{1/\varepsilon}}{d\mu}\left(s\right), \omega\right) d\mu\left(s\right) \\
&= \inf_{\substack{\mathcal{M}\left(\Omega;\mathbb{R}^l\right)\ni\lambda\ll\mu \\ \lambda\left(\frac{1}{\varepsilon}B_\rho(x)\right)=u(x)\mu_0\left(\frac{1}{\varepsilon}B_\rho(x)\right)}} \frac{1}{\mu_0\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)} \int_{\frac{1}{\varepsilon}B_\rho(x)} f\left(s, \frac{d\lambda}{d\mu}\left(s\right), \omega\right) d\mu\left(s\right) \\
&= \frac{\mathcal{S}_{u(x)}\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)\left(\omega\right)}{\mu_0\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)} = \frac{1}{p_0} \frac{\mathcal{S}_{u(x)}\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)\left(\omega\right)}{\mathcal{L}_N\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)}.
\end{aligned}$$

Applying Lemma 4.1 $(ii)$, we obtain for $\mathbb{P}$-a.e. $\omega \in \Sigma$

$$\lim_{\varepsilon\to 0} \frac{m_\varepsilon\left(u\left(x\right)\mu_0, B_\rho\left(x\right), \omega\right)}{\mu_0\left(B_\rho\left(x\right)\right)} = \lim_{\varepsilon\to 0} \frac{\mathcal{S}_{u(x)}\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)\left(\omega\right)}{\mu_0\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)} = \frac{1}{p_0} \inf_{k\in\mathbb{N}^*} \mathbb{E}^{\mathcal{I}}\left[\frac{\mathcal{S}_{u(x)}\left([0,k[^N\right)}{k^N}\right]\left(\omega\right),$$

and so by letting $\rho \to 0$

$$\lim_{\rho\to 0}\lim_{\varepsilon\to 0}\frac{m_\varepsilon\left(u\left(x\right)\mu_0, B_\rho\left(x\right),\omega\right)}{\mu_0\left(B_\rho\left(x\right)\right)} = \inf_{k\in\mathbb{N}^*}\mathbb{E}^{\mathcal{I}}\left[\frac{\mathcal{S}_{u(x)}\left([0,k[^N\right)}{p_0k^N}\right](\omega),$$

which means that (10) holds.

*Proof of* (11). Fix $\omega\in\Sigma$. Let $x\in O$ be such that

$$\lim_{\rho\to 0} t_\rho^x = \infty \text{ where } t_\rho^x := \frac{|\nu_0^s|\left(B_\rho\left(x\right)\right)}{\mu_0\left(B_\rho\left(x\right)\right)}.$$

For every $\rho>0$, every $\varepsilon>0$ and every $v\in L^1_{|\nu_0^s|}\left(\Omega;\mathbb{S}^{l-1}\right)$ we can write

$$\begin{aligned}
&\frac{m_\varepsilon\left(v\left(x\right)|\nu_0^s|, B_\rho\left(x\right),\omega\right)}{|\nu_0^s|\left(B_\rho\left(x\right)\right)}\\
&= \inf_{\substack{\mathcal{M}\left(\Omega;\mathbb{R}^l\right)\ni\lambda\ll\mu_\varepsilon\\ \lambda(B_\rho(x))=v(x)|\nu_0^s|(B_\rho(x))}} \frac{1}{|\nu_0^s|\left(B_\rho\left(x\right)\right)}\int_{B_\rho(x)} f\left(\frac{y}{\varepsilon},\frac{d\lambda}{d\mu_\varepsilon}\left(y\right),\omega\right)d\mu_\varepsilon\left(y\right)\\
&= \inf_{\substack{\mathcal{M}\left(\Omega;\mathbb{R}^l\right)\ni\lambda\ll\mu_\varepsilon\\ \frac{1}{t_\rho^x}\lambda(B_\rho(x))=v(x)|\nu_0^s|(B_\rho(x))}} \frac{\varepsilon^N}{|\nu_0^s|\left(B_\rho\left(x\right)\right)}\int_{\frac{1}{\varepsilon}B_\rho(x)} f\left(s,\frac{d\lambda}{d\mu_\varepsilon}\left(\varepsilon s\right),\omega\right)d\mu\left(s\right)\\
&= \inf_{\substack{\mathcal{M}\left(\Omega;\mathbb{R}^l\right)\ni\lambda\ll\mu_\varepsilon\\ \frac{1}{t_\rho^x}\lambda(B_\rho(x))=v(x)\mu_0(B_\rho(x))}} \frac{1}{\mu_0\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)}\int_{\frac{1}{\varepsilon}B_\rho(x)} \frac{1}{t_\rho^x} f\left(s,\frac{d\lambda_{1/\varepsilon}}{d\mu}\left(s\right),\omega\right)d\mu\left(s\right)\\
&= \inf_{\substack{\mathcal{M}\left(\Omega;\mathbb{R}^l\right)\ni\lambda\ll\mu\\ \frac{1}{t_\rho^x}\lambda\left(\frac{1}{\varepsilon}B_\rho(x)\right)=v(x)\mu_0\left(\frac{1}{\varepsilon}B_\rho(x)\right)}} \frac{1}{\mu_0\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)}\int_{\frac{1}{\varepsilon}B_\rho(x)} \frac{1}{t_\rho^x} f\left(s,\frac{d\lambda}{d\mu}\left(s\right),\omega\right)d\mu\left(s\right)\\
&= \inf_{\substack{\mathcal{M}\left(\Omega;\mathbb{R}^l\right)\ni\lambda\ll\mu\\ \lambda\left(\frac{1}{\varepsilon}B_\rho(x)\right)=v(x)\mu_0\left(\frac{1}{\varepsilon}B_\rho(x)\right)}} \frac{1}{\mu_0\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)}\int_{\frac{1}{\varepsilon}B_\rho(x)} \frac{1}{t_\rho^x} f\left(s,t_\rho^x\frac{d\lambda}{d\mu}\left(s\right),\omega\right)d\mu\left(s\right).
\end{aligned}$$

Fix $\omega\in\Sigma'$ with $\Sigma'\in\mathcal{T}$ given by Lemma 4.2 $(ii)$. Using $(\mathrm{C}_4^\omega)$ and taking $t_\rho^x := \frac{|\nu_0^s|(B_\rho(x))}{\mu_0(B_\rho(x))} > T$, we have

$$\frac{m_\varepsilon\left(v\left(x\right)|\nu_0^s|, B_\rho\left(x\right),\omega\right)}{|\nu_0^s|\left(B_\rho\left(x\right)\right)} \le \frac{\beta}{\left(t_\rho^x\right)^r}\frac{\mu\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)}{\mu_0\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)} + \frac{\mathcal{S}^\infty_{v(x)}\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)(\omega)}{\mu_0\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)},$$

letting $\varepsilon\to 0$, we obtain

$$\overline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon\left(v\left(x\right)|\nu_0^s|, B_\rho\left(x\right),\omega\right)}{|\nu_0^s|\left(B_\rho\left(x\right)\right)} \le \frac{\beta p_0}{\left(t_\rho^x\right)^r} + \overline{\lim_{\varepsilon\to 0}}\frac{\mathcal{S}^\infty_{v(x)}\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)(\omega)}{\mu_0\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)}.$$

Applying Lemma 4.2 $(ii)$ and passing to the limit $\rho\to 0$, we have

$$\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon\left(v\left(x\right)|\nu_0^s|, B_\rho\left(x\right),\omega\right)}{|\nu_0^s|\left(B_\rho\left(x\right)\right)} \le \frac{1}{p_0}\inf_{k\in\mathbb{N}^*}\mathbb{E}^{\mathcal{I}}\left[\frac{\mathcal{S}^\infty_{v(x)}\left([0,k[^N\right)}{k^N}\right](\omega).$$

On the other hand

$$\frac{m_\varepsilon\left(v\left(x\right)|\nu_0^s|, B_\rho\left(x\right),\omega\right)}{|\nu_0^s|\left(B_\rho\left(x\right)\right)} \ge -\frac{\beta}{\left(t_\rho^x\right)^r}\frac{\mu\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)}{\mu_0\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)} + \frac{\mathcal{S}^\infty_{v(x)}\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)(\omega)}{\mu_0\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)},$$

letting $\varepsilon\to 0$, we obtain

$$\underline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon\left(v\left(x\right)|\nu_0^s|, B_\rho\left(x\right),\omega\right)}{|\nu_0^s|\left(B_\rho\left(x\right)\right)} \ge \underline{\lim_{\varepsilon\to 0}}\frac{\mathcal{S}^\infty_{v(x)}\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)(\omega)}{\mu_0\left(\frac{1}{\varepsilon}B_\rho\left(x\right)\right)} - \frac{\beta p_0}{\left(t_\rho^x\right)^r}$$

which gives by letting $\rho \to 0$

$$\underline{\lim}_{\rho\to 0}\underline{\lim}_{\varepsilon\to 0}\frac{m_\varepsilon\left(v\left(x\right)|\nu_0^s|,B_\rho\left(x\right),\omega\right)}{|\nu_0^s|\left(B_\rho\left(x\right)\right)}\geq\frac{1}{p_0}\inf_{k\in\mathbb{N}^*}\mathbb{E}^{\mathcal{I}}\left[\frac{\mathcal{S}^\infty_{v(x)}\left([0,k[^N\right)}{k^N}\right](\omega).$$

We finally obtain

$$\begin{aligned}\underline{\lim}_{\rho\to 0}\underline{\lim}_{\varepsilon\to 0}\frac{m_\varepsilon\left(v\left(x\right)|\nu_0^s|,B_\rho\left(x\right),\omega\right)}{|\nu_0^s|\left(B_\rho\left(x\right)\right)}&=\overline{\lim}_{\rho\to 0}\overline{\lim}_{\varepsilon\to 0}\frac{m_\varepsilon\left(v\left(x\right)|\nu_0^s|,B_\rho\left(x\right),\omega\right)}{|\nu_0^s|\left(B_\rho\left(x\right)\right)}\\&=\inf_{k\in\mathbb{N}^*}\mathbb{E}^{\mathcal{I}}\left[\frac{\mathcal{S}^\infty_{v(x)}\left([0,k[^N\right)}{p_0k^N}\right](\omega).\ \blacksquare\end{aligned}$$

The following result is a stochastic homogenization theorem without assuming linear growth on $f$. The proof follows the same strategy as that in Theorem 2.1.

**Theorem 4.2** (**Stochastic homogenization without linear growth**). *Let $f:\mathbb{R}^N\times\mathbb{R}^l\times\Sigma\to[0,\infty]$ a $\left(\mathcal{B}\left(\mathbb{R}^N\right)\otimes\mathcal{B}\left(\mathbb{R}^l\right)\otimes\mathcal{T},\mathcal{B}\left(\overline{\mathbb{R}}_+\right)\right)$-measurable integrand satisfy. Assume that $(\mathrm{C}_2^\omega)$, $(\mathrm{C}_3^\omega)$ and $(\mathrm{C}_4^\omega)$ hold. Assume that*

$(\mathrm{C}_1^\omega)'$ *there exists $c>0$ such that for every $\omega\in\Sigma$ and every $(x,v)\in\mathbb{R}^N\times\mathbb{R}^l$*

$$c|v|\leq f\left(x,v,\omega\right).$$

*For every $O\in\mathcal{O}_0\left(\Omega\right)$ and for $\mathbb{P}$-a.e. $\omega\in\Sigma$ the family of integral functionals $\{I_\varepsilon\left(\cdot,O,\omega\right)\}_\varepsilon$ $\Gamma\left(w^*\right)$-converges to $I_0\left(\cdot,O,\omega\right)$ with*

$$I_0\left(\nu,O,\omega\right)=\int_O f^{\mathrm{hom}}\left(\frac{d\nu}{d\mu_0}\left(x\right),\omega\right)d\mu_0\left(x\right)+\int_O f_s^{\mathrm{hom}}\left(\frac{d\nu}{d|\nu_0^s|}\left(x\right),\omega\right)d|\nu_0^s|\left(x\right)$$

*for all $\nu\in\mathcal{M}\left(\Omega;\mathbb{R}^l\right)$, where $f^{\mathrm{hom}}:\mathbb{R}^l\times\Sigma\to[0,\infty]$ and $f_s^{\mathrm{hom}}:\mathbb{S}^{l-1}\times\Sigma\to[0,\infty]$ are respectively given by:*

$$f^{\mathrm{hom}}\left(v,\omega\right):=\sup_{j\in\mathbb{N}}\inf_{k\in\mathbb{N}^*}\mathbb{E}^{\mathcal{I}}\left[\inf\left\{\frac{1}{p_0k^N}\int_{k\mathbb{Y}}f^j\left(z,v+\psi\left(z\right),\cdot\right)d\mu\left(z\right):\psi\in\mathcal{A}_0\left(k\mathbb{Y}\right)\right\}\right](\omega),$$

*and*

$$f_s^{\mathrm{hom}}\left(v,\omega\right):=\sup_{j\in\mathbb{N}}\inf_{k\in\mathbb{N}^*}\mathbb{E}^{\mathcal{I}}\left[\inf\left\{\frac{1}{p_0k^N}\int_{k\mathbb{Y}}\left(f^j\right)^\infty\left(z,v+\psi\left(z\right),\cdot\right)d\mu\left(z\right):\psi\in\mathcal{A}_0\left(k\mathbb{Y}\right)\right\}\right](\omega),$$

*where $\mathcal{A}_0\left(k\mathbb{Y}\right):=\left\{\psi\in L^1\left(k\mathbb{Y};\mathbb{R}^l\right):\fint_{k\mathbb{Y}}\psi d\mathcal{L}_N=0\right\}$.*

*When $\left(\Sigma,\mathcal{T},\mathbb{P},\{\tau_z\}_{z\in\mathbb{Z}^N}\right)$ is ergodic then:*

$$\begin{aligned}f^{\mathrm{hom}}\left(v,\omega\right)&=f^{\mathrm{hom}}\left(v\right):=\\&\sup_{j\in\mathbb{N}}\inf_{k\in\mathbb{N}^*}\int_\Sigma\inf\left\{\frac{1}{p_0k^N}\int_{k\mathbb{Y}}f^j\left(z,v+\psi\left(z\right),\omega\right)d\mu\left(z\right):\psi\in\mathcal{A}_0\left(k\mathbb{Y}\right)\right\}d\mathbb{P}\left(\omega\right),\\f_s^{\mathrm{hom}}\left(v,\omega\right)&=f_s^{\mathrm{hom}}\left(v\right):=\\&\sup_{j\in\mathbb{N}}\inf_{k\in\mathbb{N}^*}\int_\Sigma\inf\left\{\frac{1}{p_0k^N}\int_{k\mathbb{Y}}\left(f^j\right)^\infty\left(z,v+\psi\left(z\right),\omega\right)d\mu\left(z\right):\psi\in\mathcal{A}_0\left(k\mathbb{Y}\right)\right\}d\mathbb{P}\left(\omega\right).\end{aligned}$$

## 5. Proof of Propositions 2.2 and 2.1

For a measure $\lambda\in\mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ or $\lambda\in\mathcal{M}_+\left(\Omega\right)$ we set for every $x\in\Omega$, every $\rho>0$ and every $\sigma\in\mathcal{M}_+\left(\Omega\right)$

$$\left[\frac{d\lambda}{d\sigma}(x)\right]_\rho:=\frac{\lambda\left(B_\rho\left(x\right)\right)}{\sigma\left(B_\rho\left(x\right)\right)}.$$

The following lemma is used in the proof of Proposition 2.2 $(i)$.

**Lemma 5.1.** *Let $\sigma \in \mathcal{M}_+(\Omega)$, $\nu \in \mathcal{M}(\Omega;\mathbb{R}^l)$, $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $O \in \mathcal{O}(\Omega)$. Let $\{\nu_\varepsilon\}_\varepsilon \subset \mathcal{M}(\Omega;\mathbb{R}^l)$ be such that $\nu_\varepsilon \overset{*}{\rightharpoonup} \nu$ in $\mathcal{M}(O;\mathbb{R}^l)$ and $|\nu_\varepsilon| \overset{*}{\rightharpoonup} \theta$ in $\mathcal{M}_+(O)$ as $\varepsilon \to 0$. Let $x \in O$ and $\rho > 0$ with $B_\rho(x) \subset O$. Let $t \in ]0,1[$ and $r \in [t^2\rho, t\rho[$. Let $\varphi_r \in \mathcal{C}_c(B_\rho(x);[0,1])$ be such that $\varphi_r \equiv 1$ in $\overline{B_r(x)}$ and $\varphi_r \equiv 0$ in $\overline{B_\rho(x)} \setminus B_{t\rho}(x)$. For each $\varepsilon > 0$ we set $\nu_\varepsilon^r := (\varphi_r \nu_{\varepsilon,1}, \cdots, \varphi_r \nu_{\varepsilon,l}) \in \mathcal{M}(O;\mathbb{R}^l)$ where $\nu_\varepsilon = (\nu_{\varepsilon,1}, \ldots, \nu_{\varepsilon,i}, \ldots, \nu_{\varepsilon,l})$. Then*

$$\overline{\lim_{\varepsilon\to 0}} \left| \frac{m_\varepsilon\left(\frac{d\nu}{d\sigma}(x)\,\sigma; B_\rho(x)\right)}{\sigma(B_\rho(x))} - \frac{m_\varepsilon(\nu_\varepsilon^r; B_\rho(x))}{\sigma(B_\rho(x))} \right| \leq 2C \left( \left[\frac{d|\nu|}{d\sigma}(x)\right]_\rho - \frac{\sigma\left(B_{t^2\rho}(x)\right)}{\sigma(B_\rho(x))} \left[\frac{d|\nu|}{d\sigma}(x)\right]_{t^2\rho} \right) + C \left| \frac{d\nu}{d\sigma}(x) - \left[\frac{d\nu}{d\sigma}(x)\right]_\rho \right|,$$

*where $C > 0$ is the constant growth appearing in $(\mathrm{C}_1)$ and*

$$\left[\frac{d|\nu|}{d\sigma}(x)\right]_{s\rho} := \frac{|\nu|(B_{s\rho}(x))}{\sigma(B_{s\rho}(x))} \quad \text{for all } s \in ]0,1].$$

*Proof.* We see that $\nu_\varepsilon^r \overset{*}{\rightharpoonup} \nu^r$ in $\mathcal{M}(O;\mathbb{R}^l)$ and $|\nu_\varepsilon^r| \overset{*}{\rightharpoonup} \theta^r$ in $\mathcal{M}_+(O)$ as $\varepsilon \to 0$, where $\theta^r := \varphi_r \mu$ verifies $\theta^r(\partial B_\rho(x)) = 0$ since the support of $\varphi_r$ is include in $\overline{B}_{t\rho}(x)$. So, by [FL07, Corollary 1.204, pp. 131] we have $\lim_{\varepsilon\to 0} \nu_\varepsilon^r(B_\rho(x)) = \nu^r(B_\rho(x))$. Using Lemma 6.2, we have for every $\varepsilon > 0$

$$\begin{aligned} \left| \frac{m_\varepsilon(\nu, B_\rho(x))}{\sigma(B_\rho(x))} - \frac{m_\varepsilon(\nu_\varepsilon^r; B_\rho(x))}{\sigma(B_\rho(x))} \right| &\leq C \left| \frac{\nu(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{\nu_\varepsilon^r(B_\rho(x))}{\sigma(B_\rho(x))} \right| \\ &\leq C \left| \frac{\nu(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{\nu^r(B_\rho(x))}{\sigma(B_\rho(x))} \right| \\ &\quad + C \left| \frac{\nu^r(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{\nu_\varepsilon^r(B_\rho(x))}{\sigma(B_\rho(x))} \right|. \end{aligned}$$

Passing to the limit $\varepsilon \to 0$ we obtain

$$\overline{\lim_{\varepsilon\to 0}} \left| \frac{m_\varepsilon(\nu, B_\rho(x))}{\sigma(B_\rho(x))} - \frac{m_\varepsilon(\nu_\varepsilon^r; B_\rho(x))}{\sigma(B_\rho(x))} \right| \leq C \left| \frac{\nu(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{\nu^r(B_\rho(x))}{\sigma(B_\rho(x))} \right|. \tag{12}$$

Since $\nu(B_r) = \nu^r(B_r)$, $|\nu^r| \leq |\nu|$ and $B_r(x) \supset B_{t^2\rho}(x)$ it follows that

$$\begin{aligned} \left| \frac{\nu(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{\nu^r(B_\rho(x))}{\sigma(B_\rho(x))} \right| = \left| \frac{\nu(B_\rho(x) \setminus B_r)}{\sigma(B_\rho(x))} - \frac{\nu^r(B_\rho(x) \setminus B_r)}{\sigma(B_\rho(x))} \right| &\leq 2 \frac{|\nu|(B_\rho(x) \setminus B_r)}{\sigma(B_\rho(x))} \\ &\leq 2 \frac{|\nu|\left(B_\rho(x) \setminus B_{t^2\rho}(x)\right)}{\sigma(B_\rho(x))}. \end{aligned} \tag{13}$$

Moreover, we can write

$$\frac{|\nu|\left(B_\rho(x) \setminus B_{t^2\rho}(x)\right)}{\sigma(B_\rho(x))} = \frac{|\nu|(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{\sigma\left(B_{t^2\rho}(x)\right)}{\sigma(B_\rho(x))} \frac{|\nu|\left(B_{t^2\rho}(x)\right)}{\sigma\left(B_{t^2\rho}(x)\right)}.$$

The assertion of the lemma follows by combining (12) and (13). ■

5.1. **Proof of Proposition 2.2** $(i)$**.** Let $O \in \mathcal{O}(\Omega)$. Let $\nu \in \mathcal{M}(\Omega;\mathbb{R}^l)$, and let $\{\nu_\varepsilon\}_{\varepsilon>0} \subset \mathcal{M}(\Omega;\mathbb{R}^l)$ be a sequence satisfying

$$\nu_\varepsilon \overset{*}{\rightharpoonup} \nu \text{ in } \mathcal{M}\left(O;\mathbb{R}^l\right) \quad \text{and} \quad |\nu_\varepsilon| \overset{*}{\rightharpoonup} \theta \text{ in } \mathcal{M}_+(O).$$

Fix $\sigma \in \{\mu_0, |\nu_0^s|\}$. Fix $x \in O$ satisfying

$$0 = \lim_{r\to 0} \left| \frac{d\nu}{d\sigma}(x) - \frac{\nu(B_r(x))}{\sigma(B_r(x))} \right| \tag{14}$$

$$0 = \lim_{r\to 0} \left| \frac{d(\mu_0 + \theta + |\nu|)}{d\sigma}(x) - \frac{(\mu_0 + \theta + |\nu|)(B_r(x))}{\sigma(B_r(x))} \right|. \tag{15}$$

Let $t \in ]0,1[$ and $\rho \in ]0,1[$ with $B_\rho(x) \subset O$. Let $K \in \mathbb{N}^*$. For each $i \in \{0,1,\dots,K\}$, we set

$$B_i := B_{t^2\rho + (1-t)t\rho\frac{i}{K}}(x) \quad \left(\text{note that } B_0 = B_{t^2\rho}(x) \subset \cdots \subset B_i \subset \cdots \subset B_K = B_{t\rho}(x)\right).$$

For every $i \in \{1,\dots,K\}$, let $\varphi_i \in \mathcal{C}_c(B_i;[0,1])$ be a cut-off function between the following two closed sets $\overline{B_{i-1}}$ and $O \setminus B_i$, which means that $\varphi_i \equiv 1$ on $\overline{B_{i-1}}$ and $\varphi_i \equiv 0$ on $\overline{O} \setminus B_i$. Set $\nu_\varepsilon^i := \varphi_i \nu_\varepsilon \in \mathcal{M}(\Omega;\mathbb{R}^l)$. We have

$$\begin{aligned}
\frac{m_\varepsilon(\nu_\varepsilon^i, B_\rho(x))}{\sigma(B_\rho(x))} &\le \frac{I_\varepsilon(\nu_\varepsilon^i, B_\rho(x))}{\sigma(B_\rho(x))} \\
&= \frac{1}{\sigma(B_\rho(x))}\left(I_\varepsilon(\nu_\varepsilon, B_{i-1}) + I_\varepsilon(\nu_\varepsilon^i, B_i \setminus B_{i-1}) + I_\varepsilon(0, B_\rho(x) \setminus B_i)\right) \\
&\le \frac{I_\varepsilon(\nu_\varepsilon, B_{t\rho}(x))}{\sigma(B_\rho(x))} + \frac{I_\varepsilon(\nu_\varepsilon^i, B_i \setminus B_{i-1})}{\sigma(B_\rho(x))} \\
&\le \frac{I_\varepsilon(\nu_\varepsilon, B_{t\rho}(x))}{\sigma(B_\rho(x))} + C\frac{\mu_\varepsilon(B_i \setminus B_{i-1}) + |\nu_\varepsilon|(B_i \setminus B_{i-1})}{\sigma(B_\rho(x))}.
\end{aligned}$$

By averaging over every layer, i.e. summing over all $i \in \{1,\dots,K\}$ and dividing by $K$, there exists $i_K \in \{1,\dots,K\}$ such that

$$\begin{aligned}
\frac{m_\varepsilon(\nu_\varepsilon^{i_K}, B_\rho(x))}{\sigma(B_\rho(x))} &\le \frac{1}{K}\sum_{i=1}^K \frac{m_\varepsilon(\nu_\varepsilon^i, B_\rho(x))}{\sigma(B_\rho(x))} \\
&\le \frac{I_\varepsilon(\nu_\varepsilon, B_{t\rho}(x))}{\sigma(B_\rho(x))} + \frac{C}{K}\frac{\mu_\varepsilon(B_{t\rho}(x) \setminus B_{t^2\rho}(x))}{\sigma(B_\rho(x))} + \frac{C}{K}\frac{|\nu_\varepsilon|(B_{t\rho}(x) \setminus B_{t^2\rho}(x))}{\sigma(B_\rho(x))}.
\end{aligned}$$

Passing to the limit $\varepsilon \to 0$

$$\begin{aligned}
\underline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon(\nu_\varepsilon^{i_K}, B_\rho(x))}{\sigma(B_\rho(x))} &\le \underline{\lim_{\varepsilon\to 0}} \frac{I_\varepsilon(\nu_\varepsilon, B_{t\rho}(x))}{\sigma(B_\rho(x))} \\
&\quad + \frac{C}{K}\overline{\lim_{\varepsilon\to 0}} \frac{\mu_\varepsilon(B_{t\rho}(x) \setminus B_{t^2\rho}(x))}{\sigma(B_\rho(x))} + \frac{C}{K}\overline{\lim_{\varepsilon\to 0}} \frac{|\nu_\varepsilon|(B_{t\rho}(x) \setminus B_{t^2\rho}(x))}{\sigma(B_\rho(x))} \\
&\le \underline{\lim_{\varepsilon\to 0}} \frac{I_\varepsilon(\nu_\varepsilon, B_{t\rho}(x))}{\sigma(B_\rho(x))} + \frac{C}{K}\left(\left[\frac{d\mu_0}{d\sigma}(x)\right]_\rho + \left[\frac{d\theta}{d\sigma}(x)\right]_\rho\right).
\end{aligned} \tag{16}$$

By using Lemma 5.1 with $r = t^2\rho + t(1-t)\rho^{\frac{i_K}{K}}$, we have

$$\begin{aligned}
\underline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon\left(\frac{d\nu}{d\sigma}(x)\,\sigma, B_\rho(x)\right)}{\sigma(B_\rho(x))} &\le \underline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon(\nu_\varepsilon^{i_K}, B_\rho(x))}{\sigma(B_\rho(x))} \\
&\quad + 2C\left(-\frac{\sigma(B_{t^2\rho}(x))}{\sigma(B_\rho(x))}\left[\frac{d|\nu|}{d\sigma}(x)\right]_{t^2\rho} + \left[\frac{d|\nu|}{d\sigma}(x)\right]_\rho\right) \\
&\quad + C\left|\frac{d\nu}{d\sigma}(x) - \left[\frac{d\nu}{d\sigma}(x)\right]_\rho\right|
\end{aligned} \tag{17}$$

where $C>0$ is given by $(\mathrm{C}_1)$. Therefore combining (16) and (17)

$$\begin{aligned}
&\varliminf_{\varepsilon\to 0}\frac{m_\varepsilon\left(\frac{d\nu}{d\sigma}(x)\,\sigma, B_\rho(x)\right)}{\sigma(B_\rho(x))}\\
\leq&\varliminf_{\varepsilon\to 0}\frac{I_\varepsilon(\nu_\varepsilon, B_{t\rho}(x))}{\sigma(B_\rho(x))}+\frac{C}{K}\left(\left[\frac{d\mu_0}{d\sigma}(x)\right]_\rho+\left[\frac{d\theta}{d\sigma}(x)\right]_\rho\right)+C\left|\frac{d\nu}{d\sigma}(x)-\left[\frac{d\nu}{d\sigma}(x)\right]_\rho\right|\\
&+2C\left(-\frac{\sigma\left(B_{t^2\rho}(x)\right)}{\sigma(B_\rho(x))}\left[\frac{d|\nu|}{d\sigma}(x)\right]_{t^2\rho}+\left[\frac{d|\nu|}{d\sigma}(x)\right]_\rho\right)\\
\leq&-2C\frac{\sigma\left(B_{t^2\rho}(x)\right)}{\sigma(B_\rho(x))}\frac{d|\nu|}{d\sigma}(x)+\varliminf_{\varepsilon\to 0}\frac{I_\varepsilon(\nu_\varepsilon, B_{t\rho}(x))}{\sigma(B_\rho(x))}+\frac{C}{K}\left(\left[\frac{d\mu_0}{d\sigma}(x)\right]_\rho+\left[\frac{d\theta}{d\sigma}(x)\right]_\rho\right)\\
&+2C\left[\frac{d|\nu|}{d\sigma}(x)\right]_\rho+C\left|\frac{d\nu}{d\sigma}(x)-\left[\frac{d\nu}{d\sigma}(x)\right]_\rho\right|+2C\left|\frac{d|\nu|}{d\sigma}(x)-\left[\frac{d|\nu|}{d\sigma}(x)\right]_{t^2\rho}\right|.
\end{aligned}$$

We show only the case $\sigma=|\nu_0^s|$; the case with $\sigma=\mu_0$ is similar and follows the same lines. So, in case $\sigma=|\nu_0^s|$, letting $\rho\to 0$ and using Lemma 5.2

$$\begin{aligned}
&\varliminf_{\rho\to 0}\varliminf_{\varepsilon\to 0}\frac{m_\varepsilon\left(\frac{d\nu}{d|\nu_0^s|}(x)\,|\nu_0^s|, B_\rho(x)\right)}{|\nu_0^s|(B_\rho(x))}\\
\leq&\varliminf_{\rho\to 0}-2C\frac{\sigma\left(B_{t^2\rho}(x)\right)}{\sigma(B_\rho(x))}\frac{d|\nu|}{d\sigma}(x)+\overline{\lim_{\rho\to 0}}\varliminf_{\varepsilon\to 0}\frac{I_\varepsilon(\nu_\varepsilon, B_{t\rho}(x))}{\sigma(B_\rho(x))}+\frac{C}{K}\left(\frac{d\mu_0}{d\sigma}(x)+\frac{d\theta}{d\sigma}(x)\right)+2C\frac{d|\nu|}{d\sigma}(x)\\
\leq&2C\left(1-t^{2N}\right)\frac{d|\nu|}{d\sigma}(x)+\frac{C}{K}\left(\frac{d\mu_0}{d\sigma}(x)+\frac{d\theta}{d\sigma}(x)\right)+\overline{\lim_{\rho\to 0}}\varliminf_{\varepsilon\to 0}\frac{I_\varepsilon(\nu_\varepsilon, B_{t\rho}(x))}{\sigma(B_\rho(x))},
\end{aligned}$$

and letting $t\to 1$ and $K\to\infty$ one obtains (6). ■

5.2. **Proof of Proposition 2.2** $(ii)$**.** Fix $\nu\in\mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ and $\sigma\in\{\mu_0,|\nu_0^s|\}$. Let $x\in O$ be such that

$$\lim_{r\to 0}\frac{\nu(B_r(x))}{\sigma(B_r(x))}=\frac{d\nu}{d\sigma}(x)<\infty.$$

Let $\rho>0$. It is easy to deduce (7) from the following inequality

$$\sup_{\varepsilon>0}\left|\frac{m_\varepsilon\left(\frac{d\nu}{d\sigma}(x)\,\sigma, B_\rho(x)\right)}{\sigma(B_\rho(x))}-\frac{m_\varepsilon(\nu, B_\rho(x))}{\sigma(B_\rho(x))}\right|\leq C\left|\frac{d\nu}{d\sigma}(x)-\frac{\nu(B_\rho(x))}{\sigma(B_\rho(x))}\right|,$$

which is a consequence of Lemma 6.2. ■

5.3. **Proof of Proposition 2.1** $(i)$**.** Let $O\in\mathcal{O}(\Omega)$. Let $\nu\in\mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ be such that $I_-(\nu,O)<\infty$. There exists a sequence $\{\nu_\varepsilon\}_{\varepsilon>0}\subset\mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ such that $\nu_\varepsilon\overset{*}{\rightharpoonup}\nu$ in $\mathcal{M}\left(O;\mathbb{R}^l\right)$ as $\varepsilon\to 0$, and

$$I_-(\nu,O)=\varliminf_{\varepsilon\to 0}I_\varepsilon(\nu_\varepsilon,O)<\infty\qquad\text{and}\qquad\sup_{\varepsilon>0}I_\varepsilon(\nu_\varepsilon,O)<\infty.\tag{18}$$

For each $\varepsilon>0$ we set $\Theta_\varepsilon:=f_\varepsilon\left(\cdot,\frac{d\nu_\varepsilon}{d\mu_\varepsilon}(\cdot)\right)\mu_\varepsilon\lfloor_O\in\mathcal{M}_+(O)$. Using (18) there exists a subsequence $\{\Theta_{\varepsilon_n}\}_{n\in\mathbb{N}}\subset\mathcal{M}_+(O)$ and $\Theta\in\mathcal{M}_+(O)$ such that $\Theta_{\varepsilon_n}\overset{*}{\rightharpoonup}\Theta$ in $\mathcal{M}_+(O)$ as $n\to\infty$. The Lebesgue decomposition theorem gives that

$$\Theta=\frac{d\Theta}{d\mu_0}\mu_0+\frac{d\Theta}{d|\nu_0^s|}|\nu_0^s|+\Theta^s$$

where $\nu_0^s$ is the singular part of the Lebesgue decomposition of $\nu$ with respect to $\mu_0$, i.e. $\nu = \frac{d\nu}{d\mu_0}(\cdot)\,\mu_0 + \nu_0^s$, and $\Theta^s$ the singular part of the Lebesgue decomposition of $\Theta$ with respect to $\mu_0 + |\nu_0^s|$. Therefore we have

$$I_-(\nu, O) = \lim_{\varepsilon\to 0} I_\varepsilon(\nu_\varepsilon, O) = \lim_{n\to\infty} I_{\varepsilon_n}(\nu_{\varepsilon_n}, O)$$
$$= \lim_{n\to\infty} \Theta_{\varepsilon_n}(O) \geq \Theta(O) \geq \sum_{\sigma\in\{\mu_0,|\nu_0^s|\}} \int_O \frac{d\Theta}{d\sigma}(x)\, d\sigma(x). \tag{19}$$

Moreover, for every $t \in ]0,1[$,

$$\frac{d\Theta}{d\mu_0}(x) = \lim_{\rho\to 0} \frac{\Theta(B_\rho(x))}{\mu_0(B_\rho(x))} \geq \overline{\lim_{\rho\to 0}} \frac{\Theta\left(\overline{B_{t\rho}(x)}\right)}{\mu_0(B_\rho(x))} \geq \overline{\lim_{\rho\to 0}}\, \overline{\lim_{n\to\infty}} \frac{\Theta_{\varepsilon_n}(B_{t\rho}(x))}{\mu_0(B_\rho(x))} \quad \mu_0\text{-a.e. in } O,$$

hence for $\mu_0$-a.e. $x \in O$

$$\frac{d\Theta}{d\mu_0}(x) \geq \overline{\lim_{t\to 1^-}}\, \overline{\lim_{\rho\to 0}}\, \underline{\lim_{\varepsilon\to 0}} \frac{I_\varepsilon(\nu_\varepsilon, B_{t\rho}(x))}{\mu_0(B_\rho(x))}.$$

Similarly, we have

$$\frac{d\Theta}{d|\nu_0^s|}(x) = \lim_{\rho\to 0} \frac{\Theta(B_\rho(x))}{|\nu_0^s|(B_\rho(x))} \geq \overline{\lim_{\rho\to 0}} \frac{\Theta\left(\overline{B_{t\rho}(x)}\right)}{|\nu_0^s|(B_\rho(x))} \geq \overline{\lim_{\rho\to 0}}\, \overline{\lim_{n\to\infty}} \frac{\Theta_{\varepsilon_n}(B_{t\rho}(x))}{|\nu_0^s|(B_\rho(x))} \quad |\nu_0^s|\text{-a.e. in } O,$$

and then

$$\frac{d\Theta}{d|\nu^s|}(x) \geq \overline{\lim_{t\to 1^-}}\, \overline{\lim_{\rho\to 0}}\, \underline{\lim_{\varepsilon\to 0}} \frac{I_\varepsilon(\nu_\varepsilon, B_{t\rho}(x))}{|\nu_0^s|(B_\rho(x))} \quad |\nu_0^s|\text{-a.e. in } O.$$

To finish the proof, it suffices to verify that for $\sigma = \mu_0$ or $\sigma = |\nu_0^s|$ the function

$$O \ni x \longmapsto h_\sigma(x) := \overline{\lim_{t\to 1^-}}\, \overline{\lim_{\rho\to 0}}\, \underline{\lim_{\varepsilon\to 0}} \frac{I_\varepsilon(\nu_\varepsilon, B_{t\rho}(x))}{\sigma(B_\rho(x))}$$

is Borel measurable. For this, we need the following result (for a proof, see [FM93, Lemma 2.13, pp. 46] and [Alb93, Theorem 5.8, pp. 33], see also [ADM92]).

**Lemma 5.2.** *Let $\sigma$ be a positive locally finite measure on $\Omega$. For $\sigma$-a.e. $x \in \Omega$ and for every $t \in ]0,1[$, we have*

$$\overline{\lim_{\rho\to 0}} \frac{\sigma(B_{t\rho}(x))}{\sigma(B_\rho(x))} \geq t^N.$$

We have by using Lemma 5.2 for $\sigma$-a.e. $x \in O$ and every $t \in ]0,1[$

$$\frac{d\Theta}{d\sigma}(x) \geq \overline{\lim_{\rho\to 0}}\, \underline{\lim_{\varepsilon\to 0}} \frac{I_\varepsilon(\nu_\varepsilon, B_{t\rho}(x))}{\sigma(B_\rho(x))} \geq \overline{\lim_{\rho\to 0}} \frac{\Theta(B_{t\rho}(x))}{\sigma(B_\rho(x))} = \overline{\lim_{\rho\to 0}} \frac{\Theta(B_{t\rho}(x))}{\sigma(B_{t\rho}(x))} \frac{\sigma(B_{t\rho}(x))}{\sigma(B_\rho(x))} \geq t^N \frac{d\Theta}{d\sigma}(x)$$

letting $t \to 1$ we obtain $h_\sigma(x) = \frac{d\Theta}{d\sigma}(x)$ $\sigma$-a.e. in $O$. Thus $h_\sigma$ is Borel measurable. ■

*Remark* 5.1. Since (19) we can write

$$I_-(\nu, O) \geq \sum_{\sigma\in\{\mu_0,|\nu_0^s|\}} \int_O \frac{d\Theta}{d\sigma}(x)\, d\sigma(x) = \sum_{\sigma\in\{\mu_0,|\nu_0^s|\}} \int_O \overline{\lim_{t\to 1^-}}\, \overline{\lim_{\rho\to 0}}\, \underline{\lim_{\varepsilon\to 0}} \frac{I_\varepsilon(\nu_\varepsilon, B_{t\rho}(x))}{\sigma(B_\rho(x))} d\sigma(x).$$

5.4. **Proof of Proposition 2.1 (*ii*).** Taking Lemma 6.3 into account, which gives an integral representation of the lower Vitali envelope with density the derivative of the set function

$$\overline{m}(\nu, \cdot) := \overline{\lim_{\varepsilon\to 0}}\, m_\varepsilon(\nu, \cdot),$$

we see that it is sufficient to show that for every $(\nu, O) \in \mathcal{M}\left(\Omega; \mathbb{R}^l\right) \times \mathcal{O}_0(\Omega)$

$$\left(\Gamma(w^*) \text{-} \overline{\lim_{\varepsilon\to 0}} I_\varepsilon\right)(\nu, O) \leq \overline{m}_-^*(\nu, O).$$

Let $(\nu, O) \in \mathcal{M}\left(\Omega;\mathbb{R}^l\right) \times \mathcal{O}_0(\Omega)$ be such that $\overline{m}^*_-(\nu, O) < \infty$ with

$$\overline{m}^*_-(\nu, O) = \liminf_{\delta\to 0} \left\{ \sum_{i\in I} \overline{m}(\nu, B_i) : \{B_i\}_{i\in I} \in \mathcal{V}^\delta(O) \right\},$$

where, for every $\delta > 0$,

$$\mathcal{V}^\delta(O) := \Bigg\{ \{B_i\}_{i\in I} : B_i \text{ is an open ball, } \pi_0(\partial B_i) = 0, \overline{B}_i \subset O, \operatorname{diam}(B_i) \in ]0,\delta[$$
$$I \text{ is countable, } \pi_0\left(O \setminus \bigcup_{i\in I} B_i\right) = 0, \text{ and } \overline{B}_i \cap \overline{B}_j = \emptyset \text{ for all } i \neq j \Bigg\}$$

with $\pi_0 := \mu_0 + |\nu_0^s|$.

Fix $\delta \in ]0,1[$. There exists $\left\{B_i^\delta\right\}_{i\in I_\delta} \in \mathcal{V}^\delta(O)$ such that

$$\overline{m}^*_-(\nu, O) + \delta \geq \sum_{i\in I_\delta} \overline{m}\left(\nu, B_i^\delta\right). \tag{20}$$

Using $(C_1)$, we have for every $\varepsilon > 0$, every $\nu \in \mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ and every $A \in \mathcal{B}(\Omega)$

$$0 \leq m_\varepsilon(\nu, A) \leq \widehat{\mu}_\varepsilon(A) + C|\nu|(A),$$

where $\widehat{\mu}_\varepsilon := C\mu_\varepsilon$ with $C > 0$ given by $(C_1)$.

**Step 1: we prove that $\overline{\lim}_{\varepsilon\to 0} \sum_{i\in I_\delta} m_\varepsilon\left(\nu, B_i^\delta\right) \leq \sum_{i\in I_\delta} \overline{m}\left(\nu, B_i^\delta\right)$.** Set $\widehat{\mu_0} := C\mu_0$. We apply the Fatou lemma to the nonnegative function

$$I_\delta \ni i \longmapsto \left(\widehat{\mu}_\varepsilon + C|\nu|\right)\left(B_i^\delta\right) - m_\varepsilon\left(\nu, B_i^\delta\right),$$

so, we have

$$\begin{aligned}
\underline{\lim}_{\varepsilon\to 0} \sum_{i\in I_\delta} \left(\widehat{\mu_\varepsilon} + C|\nu|\right)\left(B_i^\delta\right) - m_\varepsilon\left(\nu, B_i^\delta\right) &\geq \sum_{i\in I_\delta} \underline{\lim}_{\varepsilon\to 0} \left(\left(\widehat{\mu}_\varepsilon + C|\nu|\right)\left(B_i^\delta\right) - m_\varepsilon\left(\nu, B_i^\delta\right)\right) \\
&\geq \sum_{i\in I_\delta} \left(\widehat{\mu_0} + C|\nu|\right)\left(B_i^\delta\right) - \overline{\lim}_{\varepsilon\to 0} m_\varepsilon\left(\nu, B_i^\delta\right) \\
&\geq \left(\widehat{\mu_0} + C|\nu|\right)\left(O^\delta\right) - \sum_{i\in I_\delta} \overline{\lim}_{\varepsilon\to 0} m_\varepsilon\left(\nu, B_i^\delta\right)
\end{aligned}$$

where $O^\delta := \bigcup_{i\in I_\delta} B_i^\delta$. On the other hand, we have

$$\underline{\lim}_{\varepsilon\to 0} \sum_{i\in I_\delta} \left(\widehat{\mu}_\varepsilon + C|\nu|\right)\left(B_i^\delta\right) - m_\varepsilon\left(\nu, B_i^\delta\right) \leq \widehat{\mu_0}\left(\overline{O^\delta}\right) + C|\nu|\left(O^\delta\right) - \overline{\lim}_{\varepsilon\to 0} \sum_{i\in I_\delta} m_\varepsilon\left(\nu, B_i^\delta\right).$$

Hence

$$\overline{\lim}_{\varepsilon\to 0} \sum_{i\in I_\delta} m_\varepsilon\left(\nu, B_i^\delta\right) \leq \sum_{i\in I_\delta} \overline{m}\left(\nu, B_i^\delta\right) + C\mu_0(\partial O) = \sum_{i\in I_\delta} \overline{m}\left(\nu, B_i^\delta\right)$$

since $\mu_0(\partial O) = 0$.

**Step 2: we prove that there exists $\left\{\nu_\varepsilon^\delta\right\}_{\varepsilon>0,\delta>0} \subset \mathcal{M}\left(O;\mathbb{R}^l\right)$ such that $\overline{\lim}_{\delta\to 0} \overline{\lim}_{\varepsilon\to 0} I_\varepsilon\left(\nu_\varepsilon^\delta, O\right) \leq \sum_{i\in I_\delta} \overline{m}\left(\nu, B_i^\delta\right)$.** Fix $\varepsilon \in ]0,1[$. For each $i \in I_\delta$ there exists $\nu_{i,\varepsilon}^\delta \in \mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ such that $\nu_{i,\varepsilon}^\delta\left(B_i^\delta\right) = \nu\left(B_i^\delta\right)$, $\nu_{i,\varepsilon}^\delta \ll \mu_\varepsilon$ and

$$m_\varepsilon\left(\nu, B_i^\delta\right) + \varepsilon \frac{\pi_0\left(B_i^\delta\right)}{\pi_0(O)} \geq I_\varepsilon\left(\nu_{i,\varepsilon}^\delta, B_i^\delta\right). \tag{21}$$

Set $\nu_\varepsilon^\delta := \sum_{i\in I_\delta} \nu_{i,\varepsilon}^\delta \mathbb{1}_{B_i^\delta}$. Using $(C_1)$ we have $\nu_\varepsilon^\delta \in \mathcal{M}\left(O;\mathbb{R}^l\right)$, indeed

$$\left|\nu_\varepsilon^\delta\right|(O) \le \sum_{i\in I_\delta}\left(\left|\nu_{i,\varepsilon}^\delta \mathbb{1}_{B_i^\delta}\right|(O)\right) = \sum_{i\in I_\delta}\int_{B_i^\delta}\left|\frac{d\nu_{i,\varepsilon}^\delta}{d\mu_\varepsilon}\right| d\mu_\varepsilon \le \frac{1}{c}\sum_{i\in I_\delta} m_\varepsilon\left(\nu, B_i^\delta\right) + \varepsilon\frac{\pi_0\left(B_i^\delta\right)}{\pi_0(O)}$$
$$\le \frac{1}{c}\left(\sup_{\varepsilon>0}\widehat{\mu}_\varepsilon(O) + C|\nu|(O) + 1\right). \quad (22)$$

By using *Step 1* and (20) we have

$$\overline{\lim_{\delta\to 0}}\,\overline{\lim_{\varepsilon\to 0}} I_\varepsilon\left(\nu_\varepsilon^\delta, O\right) = \overline{\lim_{\delta\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\sum_{i\in I_\delta} I_\varepsilon\left(\nu_{i,\varepsilon}^\delta, B_i^\delta\right) \le \overline{\lim_{\delta\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\sum_{i\in I_\delta} m_\varepsilon\left(\nu, B_i^\delta\right) \le \overline{\lim_{\delta\to 0}}\sum_{i\in I_\delta}\overline{m}\left(\nu, B_i^\delta\right)$$
$$\le \overline{m}_-^*(\nu,\cdot)(O),$$

moreover

$$\sup_{\delta>0}\overline{\lim_{\varepsilon\to 0}}\left(\widehat{\mu}_\varepsilon + |\nu| + \pi_0\right)\left(O\setminus\bigcup_{i\in I_\delta} B_i^\delta\right) = 0.$$

Using a diagonalization argument, there exists $\{\delta(\varepsilon)\}_{\varepsilon>0}$ increasing with $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$ such that

$$\overline{\lim_{\varepsilon\to 0}} I_\varepsilon(\nu_\varepsilon, O) \le \overline{m}_-^*(\nu,\cdot)(O) \quad \text{and} \quad \lim_{\varepsilon\to 0}\left(\widehat{\mu}_\varepsilon + |\nu| + \pi_0\right)\left(O\setminus\bigcup_{i\in I_{\delta(\varepsilon)}} B_i^{\delta(\varepsilon)}\right) = 0. \quad (23)$$

where $\nu_\varepsilon := \nu_\varepsilon^{\delta(\varepsilon)}$. To finish the proof we need to show that $\nu_\varepsilon \overset{*}{\rightharpoonup} \nu$ in $\mathcal{M}\left(O;\mathbb{R}^l\right)$.

**Step 3: we prove that $\boldsymbol{\nu_\varepsilon^{\delta(\varepsilon)} \overset{*}{\rightharpoonup} \nu}$ in $\boldsymbol{\mathcal{M}\left(\Omega;\mathbb{R}^l\right)}$.** Let $U\subset O$ be an open set. We consider the sets

$$U^\delta := \bigcup_{i\in\left\{i\in I_\delta : B_i^\delta\cap U\neq\emptyset\right\}} B_i^\delta \quad \text{and} \quad U_\delta := \bigcup_{i\in\left\{i\in I_\delta : B_i^\delta\subset U\right\}} B_i^\delta. \quad (24)$$

Setting $K_\delta := \left\{i\in I_\delta : B_i^\delta\cap U\neq\emptyset \text{ and } B_i^\delta\cap O\setminus U\neq\emptyset\right\}$, we have

$$\left|\nu_\varepsilon^\delta\right|\left(U^\delta\setminus U_\delta\right) \le \sum_{i\in K_\delta}\left|\nu_{i,\varepsilon}^\delta\right|\left(B_i^\delta\right) \le \frac{1}{c}\sum_{i\in K_\delta} I_\varepsilon\left(\nu_{i,\varepsilon}^\delta, B_i^\delta\right) \le \frac{1}{c}\left(\sum_{i\in K_\delta} m_\varepsilon\left(\nu, B_i^\delta\right) + \varepsilon\frac{\pi_0\left(B_i^\delta\right)}{\pi_0(O)}\right)$$
$$\le C'\left(\widehat{\mu}_\varepsilon + |\nu| + \pi_0\right)\left(U^\delta\setminus U_\delta\right) \quad (25)$$

where $C' := \max\left(\frac{1}{c}, \frac{1}{c\pi_0(O)}, C\right)$.

Let $\{\varepsilon_n\}_{n\in\mathbb{N}}\subset ]0,1[$ be a sequence satisfying $\lim_{n\to\infty}\varepsilon_n = 0$. We have to prove that $\nu^n \overset{*}{\rightharpoonup} \nu$ in $\mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ where $\nu^n := \nu_{\varepsilon_n}^{\delta(\varepsilon_n)}$. Since (22), consider $f\in\mathcal{C}_c\left(O;\mathbb{R}^l\right)$, we need to show that

$$\lim_{n\to\infty}\int_O f d\nu^n = \int_O f d\nu \quad \text{where} \quad \int_O f d\nu^n = \sum_{s=1}^l\int_O f_s d\nu_s^n \ \text{ for all } n\in\mathbb{N},$$

where $f = (f_1,\dots,f_s,\dots,f_l)$, $\nu^n = (\nu_1^n,\dots,\nu_s^n,\dots,\nu_l^n)$, $f_s\in\mathcal{C}_c(O)$ and $\nu_s^n\in\mathcal{M}(O;\mathbb{R})$ is a signed measure. Reasoning component by component, we can assume that $\nu^n$ is a signed measure for all $n\in\mathbb{N}$ and $f\in\mathcal{C}_c(O)$. We have for every $n\in\mathbb{N}$

$$\int_O f d\nu^n = \int_O f^+ d\nu^n - \int_O f^- d\nu^n$$

where $f^+ = \max\{f,0\}$ and $f^- = \max\{-f,0\}$. Set $M := \max\left(\sup_O f^+, \sup_O f^-\right) < \infty$. We can express this by

$$\int_O f d\nu^n = \int_0^M \nu^n\left([f^+ > t]\right) dt - \int_0^M \nu^n\left([f^- > t]\right) dt. \quad (26)$$

Set $U_t := [g > t] = \{x \in O : g(x) > t\}$ for all $t \geq 0$ and all $g \in \{f^+, f^-\}$. Our task is to prove that

$$\lim_{n\to\infty} \int_0^M \nu^n(U_t)\, dt = \int_0^M \nu(U_t)\, dt. \tag{27}$$

To establish this, we need the following lemma:

**Lemma 5.3.** *Let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset ]0,1[$ be a sequence satisfying $\lim_{n\to\infty} \varepsilon_n = 0$. Then for every $n \in \mathbb{N}$ and every open subset $U \subset O$*

$$\nu^n(U) - \nu(U) = \nu(U^n \setminus U) - \nu^n(U^n \setminus U). \tag{28}$$

*where $U^n := \bigcup_{j\in J_n} B_j^n$ with $J_n := \left\{i \in I_n : B_i^{\delta(\varepsilon_n)} \cap U \neq \emptyset\right\}$, $B_j^n := B_j^{\delta(\varepsilon_n)}$, $I_n := I_{\delta(\varepsilon_n)}$ and where*

$$\nu^n = \nu_{\varepsilon_n}^{\delta(\varepsilon_n)} = \sum_{i\in I_{\delta(\varepsilon_n)}} \nu_i^n \mathbb{1}_{B_i^n}$$

*with $\nu_i^n := \nu_{i,\varepsilon_n}^{\delta(\varepsilon_n)}$.*

**Proof of Lemma 5.3.** Fix $n \in \mathbb{N}$. Let $U \subset O$ be an open set. We have $\nu^n(U \setminus U^n) = 0$ since $\nu^n(U \cap U^n) = \nu^n(U)$. Indeed, we have

$$\nu^n(U^n \cap U) = \sum_{i\in I_n} \nu_i^n \left(B_i^n \cap \left(\bigcup_{j\in J_n} B_j^n\right) \cap U\right) = \sum_{j\in J_n} \nu_j^n \left(B_j^n \cap U\right) = \nu^n(U).$$

Moreover, we have

$$\nu^n(U^n) = \sum_{j\in J_n} \nu_j^n \left(B_j^n\right) = \sum_{j\in J_n} \nu\left(B_j^n\right) = \nu(U^n).$$

Thus, we can write, since $U^n \cup (U \setminus U^n) = U \cup (U^n \setminus U)$, that

$$\nu^n(U) = \nu(U) - \nu(U \setminus U^n) + \nu(U^n \setminus U) - \nu^n(U^n \setminus U). \tag{29}$$

But $\nu(U \setminus U^n) = 0$. Indeed, we have $U \setminus U^n = U \setminus \bigcup_{j\in I_n} B_j^n$ since $B_j^n \cap U = \emptyset$ when $j \notin J_n$, and by using the fact that $|\nu| \ll \mu_0 + |\nu_0^s|$, we obtain

$$|\nu(U \setminus U^n)| \leq |\nu| \left(U \setminus \bigcup_{j\in I_n} B_j^n\right) \leq |\nu| \left(O \setminus \bigcup_{j\in I_n} B_j^n\right) = 0.$$

Thus, (29) becomes (28). ■

By (28) we see that

$$\left|\int_0^M \nu^n(U_t) - \nu(U_t)\, dt\right| = \left|\int_0^M \nu(U_t^n \setminus U_t)\, dt - \int_0^M \nu^n(U_t^n \setminus U_t)\, dt\right|,$$

thus to prove (27), it is sufficient to show that

$$\overline{\lim_{n\to\infty}} \left(\left|\int_0^M \nu(U_t^n \setminus U_t)\, dt\right| + \left|\int_0^M \nu^n(U_t^n \setminus U_t)\, dt\right|\right) = 0.$$

But, by (25) we have for every $n \in \mathbb{N}$

$$\left|\int_0^M \nu^n(U_t^n \setminus U_t)\, dt\right| \leq \int_0^M |\nu^n|(U_t^n \setminus U_{t,n})\, dt \leq C' \int_0^M \alpha_n(U_t^n \setminus U_{t,n})\, dt,$$

with $\alpha_n := \widehat{\mu}_{\varepsilon_n} + |\nu| + \pi_0$,

$$U_t^n = [U_t]^{\delta(\varepsilon_n)} = \bigcup_{j\in\left\{i\in I_n : B_i^n \cap U_t \neq \emptyset\right\}} B_j^n \text{ and } U_{t,n} := [U_t]_{\delta(\varepsilon_n)} = \bigcup_{j\in\left\{i\in I_n : B_i^n \subset U_t\right\}} B_j^n.$$

Therefore, we are reduced to prove that $\overline{\lim}_{n\to\infty} \int_0^M \alpha_n(U_t^n \setminus U_{t,n})\, dt = 0$, which is the goal of the following lemma.

**Lemma 5.4.** *Let $g \in \mathcal{C}_c\left(O;\mathbb{R}^l\right)$ and set $U_t := [g > t]$ for all $t \in \mathbb{R}_+$. Set $M := \sup_O g$. For every sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset ]0,1[$ satisfying $\lim_{n\to\infty} \varepsilon_n = 0$ we have*

$$\overline{\lim_{n\to\infty}} \int_0^M \alpha_n\left(U_t^n \setminus U_{t,n}\right) dt = 0.$$

**Proof of Lemma 5.4.** Fix $n \in \mathbb{N}$. Since $\operatorname{diam}\left(B_i^n\right) < \delta\left(n\right)$, we have

$$U_t^n \subset \mathcal{N}_{\delta(n)}[U_t] := \left\{x \in O : \operatorname{dist}\left(x, U_t\right) < \delta\left(\varepsilon_n\right)\right\},$$

thus $\alpha_n\left(U_t^n \setminus U_{t,n}\right) \leq \alpha_n\left(\mathcal{N}_{\delta(n)}[U_t] \setminus U_{t,n}\right)$. Now, we give an estimate of $\alpha_n\left(\mathcal{N}_{\delta(n)}[U_t] \setminus U_{t,n}\right)$ from above as $n \to \infty$. Since (23) we can write

$$\begin{aligned} 0 \geq \overline{\lim_{n\to\infty}} \alpha_n\left(U_t \setminus \left(\bigcup_{i\in J_n} B_i^n \cup \bigcup_{i\in I_n\setminus J_n} B_i^n\right)\right) &= \overline{\lim_{n\to\infty}} \alpha_n\left(U_t \setminus \left(\left(\bigcup_{i\in J_n\setminus K_n} B_i^n\right) \cup U_{t,n}\right)\right) \\ &= \overline{\lim_{n\to\infty}} \alpha_n\left(\left(U_t \setminus U_{t,n}\right) \setminus F_n\right) \end{aligned}$$

where $F_n := \bigcup_{i\in J_n\setminus K_n} B_i^n$ and $K_n := K_{\delta(\varepsilon_n)}$. Observe that

$$F_n \subset \left\{x \in O : \operatorname{dist}\left(x, \partial U_t\right) \leq 2\delta\left(\varepsilon_n\right)\right\} =: \overline{\mathcal{N}}_{2\delta(n)}[\partial U_t],$$

it follows that

$$\overline{\lim_{n\to\infty}} \alpha_n\left(U_t \setminus U_{t,n}\right) \leq \overline{\lim_{n\to\infty}} \alpha_n\left(\left(U_t \setminus U_{t,n}\right) \setminus F_n\right) + \overline{\lim_{n\to\infty}} \alpha_n\left(\left(U_t \setminus U_{t,n}\right) \cap F_n\right) \leq \overline{\lim_{n\to\infty}} \alpha_n\left(F_n\right).$$

Now, for every $k \in \mathbb{N}$ there exists $N_k \in \mathbb{N}$ such that for every $n \geq N_k$ we have $\delta\left(\varepsilon_n\right) \leq \delta\left(\varepsilon_k\right)$, which implies that $\overline{\mathcal{N}}_{2\delta(n)}[\partial U_t] \subset \overline{\mathcal{N}}_{2\delta(k)}[\partial U_t]$. Therefore for every $k \in \mathbb{N}$

$$\overline{\lim_{n\to\infty}} \alpha_n\left(U_t \setminus U_{t,n}\right) \leq \overline{\lim_{n\to\infty}} \alpha_n\left(\overline{\mathcal{N}}_{2\delta(k)}[\partial U_t]\right) \leq \alpha_0\left(\overline{\mathcal{N}}_{2\delta(k)}[\partial U_t]\right),$$

where $\alpha_0 := \mu_0 + |\nu| + \pi_0$. Letting $k \to \infty$ we obtain

$$\overline{\lim_{n\to\infty}} \alpha_n\left(U_t \setminus U_{t,n}\right) \leq \alpha_0\left(\partial U_t\right).$$

In the same way, for every $k \in \mathbb{N}$

$$\overline{\lim_{n\to\infty}} \alpha_n\left(\mathcal{N}_{\delta(n)}[U_t] \setminus U_t\right) \leq \overline{\lim_{n\to\infty}} \alpha_n\left(\mathcal{N}_{\delta(k)}[U_t] \setminus U_t\right)$$

and letting $k \to \infty$

$$\overline{\lim_{n\to\infty}} \alpha_n\left(\mathcal{N}_{\delta(n)}[U_t] \setminus U_t\right) \leq \alpha_0\left(\partial U_t\right).$$

Now, we can write

$$\begin{aligned} \overline{\lim_{n\to\infty}} \alpha_n\left(U_t^n \setminus U_{t,n}\right) &\leq \overline{\lim_{n\to\infty}} \alpha_n\left(\mathcal{N}_{\delta(n)}[U_t] \setminus U_{t,n}\right) \\ &= \overline{\lim_{n\to\infty}} \left(\alpha_n\left(\left(\mathcal{N}_{\delta(n)}[U_t] \setminus U_t\right) \setminus U_{t,n}\right) + \alpha_n\left(U_t \setminus U_{t,n}\right)\right) \\ &\leq \overline{\lim_{n\to\infty}} \alpha_n\left(\mathcal{N}_{\delta(n)}[U_t] \setminus U_t\right) + \overline{\lim_{n\to\infty}} \alpha_n\left(U_t \setminus U_{t,n}\right) \leq 2\alpha_0\left(\partial U_t\right) \end{aligned}$$

Now, by noticing that $\partial U_t \subset [g = t]$ and $\int_0^M \alpha_0\left([g = t]\right) dt = 0$ (by Fubini-Tonelli theorem), we apply the Lebesgue dominated convergence theorem to find

$$\lim_{n\to\infty} \int_0^M \alpha_n\left(U_t^n \setminus U_{t,n}\right) dt = 0. \blacksquare$$

## 6. Auxiliary results

6.1. **Reduction to integrands with linear growth.** For each $j \in \mathbb{N}^*$ and each $\varepsilon > 0$, we set

$$I_\varepsilon^j(\nu, O) := \begin{cases} \displaystyle\int_O f_\varepsilon^j\left(x, \frac{d\nu}{d\mu_\varepsilon}(x)\right) d\mu_\varepsilon(x) & \text{if } \nu \ll \mu_\varepsilon \\ \infty & \text{otherwise,} \end{cases}$$

where $f_\varepsilon^j(x,v) := \inf\left\{f_\varepsilon(x,w) + j|v-w| : w \in \mathbb{R}^l\right\}$. Observe that $f_\varepsilon^j \le f_\varepsilon$, and then $I_\varepsilon^j \le I_\varepsilon$ for all $j \in \mathbb{N}^*$ and all $\varepsilon > 0$.

**Lemma 6.1.** *Assume that* $(C_1)$ *and* $(C_2)$ *hold. For every* $\nu \in \mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ *and every* $O \in \mathcal{O}(\Omega)$

$$\left(\Gamma(w^*)\text{-}\overline{\lim_{\varepsilon\to 0}} I_\varepsilon\right)(\nu, O) = \sup_{j\in\mathbb{N}^*}\left(\Gamma(w^*)\text{-}\overline{\lim_{\varepsilon\to 0}} I_\varepsilon^j\right)(\nu, O).$$

*Proof.* Let $\nu \in \mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ and let $O \in \mathcal{O}(\Omega)$. It is sufficient to show that

$$\infty > A_\nu := \sup_{j\in\mathbb{N}^*}\left(\Gamma(w^*)\text{-}\overline{\lim_{\varepsilon\to 0}} I_\varepsilon^j\right)(\nu, O) \ge \left(\Gamma(w^*)\text{-}\overline{\lim_{\varepsilon\to 0}} I_\varepsilon\right)(\nu, O).$$

Let $j \in \mathbb{N}$. There exists $\{\nu_\varepsilon^j\}_{\varepsilon>0} \subset \mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ such that $\nu_\varepsilon^j \ll \mu_\varepsilon$ for all $\varepsilon > 0$, $\nu_\varepsilon^j \overset{*}{\rightharpoonup} \nu$ as $\varepsilon \to 0$, and $A_\nu = \sup_{j\in\mathbb{N}} A_\nu^j$ where $A_\nu^j := \lim_{\varepsilon\to 0} I_\varepsilon^j\left(\nu_\varepsilon^j, O\right)$. For each $\varepsilon > 0$ and $j \in \mathbb{N}^*$, set $\Phi_\varepsilon^j(w, O) := \int_O f_\varepsilon\left(x, \frac{d\nu_\varepsilon^j}{d\mu_\varepsilon}(x) + w(x)\right) + j|w(x)|\, d\mu_\varepsilon(x)$ for all $w \in L^1_{\mu_\varepsilon}\left(O;\mathbb{R}^l\right)$, which satisfies

$$\Phi_\varepsilon^j\left(-\frac{d\nu_\varepsilon^j}{d\mu_\varepsilon}, O\right) = j|\nu_\varepsilon^j|(O) < \infty.$$

Applying Rockafellar interchange of infimum and integral theorem [Roc76, Theorem 3A], we obtain, for every $\varepsilon > 0$ and every $j \in \mathbb{N}^*$,

$$I_\varepsilon^j\left(\nu_\varepsilon^j, O\right) = \inf_{w\in L^1_{\mu_\varepsilon}\left(O;\mathbb{R}^l\right)} \Phi_\varepsilon^j(w, O).$$

There exists $C_\nu > 0$ (depending on $A_\nu$) such that for every $\varepsilon \in ]0,1[$ and every $j \in \mathbb{N}^*$ there exists $w_\varepsilon^j \in L^1_{\mu_\varepsilon}\left(O;\mathbb{R}^l\right)$ satisfying

$$1 + C_\nu \ge \varepsilon + I_\varepsilon^j\left(\nu_\varepsilon^j, O\right) \ge I_\varepsilon\left(\nu_\varepsilon^j + w_\varepsilon^j\mu_\varepsilon, O\right) + j\left|w_\varepsilon^j\right|_{L^1_{\mu_\varepsilon}\left(O;\mathbb{R}^l\right)}.$$

It follows, on one hand, that $\sup_{\varepsilon>0}\left|w_\varepsilon^j\right|_{L^1_{\mu_\varepsilon}\left(O;\mathbb{R}^l\right)} \le \frac{1+C_\nu}{j}$, and, on the other hand by the coercivity condition $(C_1)$, that

$$|\nu_\varepsilon^j|(O) \le r_\nu$$

for all $\varepsilon > 0$ and all $j \in \mathbb{N}^*$, where $r_\nu := \left(1 + c^{-1}\right)(1 + C_\nu)$. By the lower semicontinuity of the total variation, we infer $|\nu|(O) \le r_\nu$. Since $\mathcal{C}_0\left(O;\mathbb{R}^l\right)$ is a separable Banach space when endowed with the sup norm, the weak$^*$ topology on the ball $B_{r_\nu}(0) \subset \mathcal{M}\left(O;\mathbb{R}^l\right)$ of radius $r_\nu$ and center 0 is metrizable, thus there exists a metric $d_\nu$ on $B_{r_\nu}(0)$ such that $\lim_{\varepsilon\to 0} d_\nu\left(\nu_\varepsilon^j, \nu\right) = 0$ for all $j \in \mathbb{N}^*$. By a simultaneous diagonalization, there exists an increasing sequence $\{j(\varepsilon)\}_\varepsilon \subset \mathbb{N}^*$ such that

$$\overline{\lim_{\varepsilon\to 0}} I_\varepsilon\left(\nu_\varepsilon^{j(\varepsilon)} + w_\varepsilon^{j(\varepsilon)}\mu_\varepsilon, O\right) \le \overline{\lim_{j\to\infty}}\,\overline{\lim_{\varepsilon\to 0}} I_\varepsilon\left(\nu_\varepsilon^j + w_\varepsilon^j\mu_\varepsilon, O\right) \le A_\nu \quad\text{and}\quad \lim_{\varepsilon\to 0} d_\nu\left(\nu_\varepsilon^{j(\varepsilon)}, \nu\right) = 0.$$

Now, we see that for every $\varphi \in \mathcal{C}_0\left(O;\mathbb{R}^l\right)$ and every $\varepsilon > 0$

$$\begin{aligned}\left|\left\langle \nu_\varepsilon^{j(\varepsilon)} + w_\varepsilon^{j(\varepsilon)}\mu_\varepsilon - \nu, \varphi\right\rangle\right| &\le \left|\left\langle \nu_\varepsilon^{j(\varepsilon)} - \nu, \varphi\right\rangle\right| + \left|\left\langle w_\varepsilon^{j(\varepsilon)}\mu_\varepsilon, \varphi\right\rangle\right| \\ &\le \left|\left\langle \nu_\varepsilon^{j(\varepsilon)} - \nu, \varphi\right\rangle\right| + |\varphi|_\infty \frac{1 + C_\nu}{j(\varepsilon)},\end{aligned}$$

which shows that $\nu_\varepsilon^{j(\varepsilon)} + w_\varepsilon^{j(\varepsilon)}\mu_\varepsilon \overset{*}{\rightharpoonup} \nu$ in $\mathcal{M}\left(O;\mathbb{R}^l\right)$ as $\varepsilon \to 0$, and consequently

$$A_\nu \geq \left(\Gamma\left(w^*\right) \text{-} \overline{\lim_{\varepsilon\to 0}} I_\varepsilon\right)(\nu, O)\,. \blacksquare$$

6.2. **Properties of $m_\varepsilon$.** In this part, we assume that $(\mathrm{C}_1)$ holds. We begin with the following lemma.

**Lemma 6.2.** *Let $B \in \mathcal{O}(\Omega)$. For every $\varepsilon > 0$ and every $(\lambda, \nu) \in \mathcal{M}\left(\Omega;\mathbb{R}^l\right) \times \mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ we have*

$$|m_\varepsilon(\lambda, B) - m_\varepsilon(\nu, B)| \leq C\,|\lambda(B) - \nu(B)| \tag{30}$$

*with $C > 0$ is given by $(\mathrm{C}_1)$. In particular, we also have*

$$m_\varepsilon\left(\frac{\nu(B)}{\mu_\varepsilon(B)}\mu_\varepsilon, B\right) = m_\varepsilon(\nu, B)\,.$$

*Proof.* Fix $B \in \mathcal{O}(\Omega)$, $\varepsilon > 0$ and $\lambda, \nu \in \mathcal{M}\left(\Omega;\mathbb{R}^l\right)$. Let $\delta > 0$. There exists $\nu_\delta \in \mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ with $\nu_\delta \ll \mu$ and $\nu_\delta(B) = \nu(B)$ such that $m_\varepsilon(\nu, B) + \delta \geq I_\varepsilon(\nu_\delta, B)$. Let $k \in \mathbb{N}^*$ and $B_k := \left\{x \in B : \operatorname{dist}(x, \partial B) > \frac{1}{k}\right\}$. We set

$$\lambda_\delta := \nu_\delta \mathbb{1}_{B_k} + \left(\frac{1}{\mu_\varepsilon(B \setminus B_k)}\left(\lambda(B) - \nu_\delta(B_k)\right)\right)\mu_\varepsilon \mathbb{1}_{B\setminus B_k}\,.$$

We observe that $\lambda_\delta \ll \mu$ and $\lambda_\delta(B) = \lambda(B)$, so we can write

$$m_\varepsilon(\lambda, B) \leq I_\varepsilon(\lambda_\delta, B) = I_\varepsilon(\nu_\delta, B_k) + I_\varepsilon(\lambda_\delta, B \setminus B_k) \leq m_\varepsilon(\nu, B) + \delta + I_\varepsilon(\lambda_\delta, B \setminus B_k)\,. \tag{31}$$

Using the growth condition $(\mathrm{C}_1)$, we have

$$I_\varepsilon(\lambda_\delta, B \setminus B_k) \leq C \int_{B\setminus B_k} 1 + \left|\frac{1}{\mu_\varepsilon(B \setminus B_k)}\left(\lambda(B) - \nu_\delta(B_k)\right)\right| d\mu_\varepsilon.$$

Since $\lim_{k\to\infty} \nu_\delta(B_k) = \nu_\delta(B) = \nu(B)$ and $\lim_{k\to\infty} \mu_\varepsilon(B \setminus B_k) = 0$, we have

$$\overline{\lim_{k\to\infty}}\, I_\varepsilon(\lambda_\delta, B \setminus B_k) \leq C\,|\lambda(B) - \nu(B)|\,.$$

So, inequality (31) becomes

$$m_\varepsilon(\lambda, B) \leq m_\varepsilon(\nu, B) + \delta + C\,|\lambda(B) - \nu(B)|$$

and the proof of (30) is complete by letting $\delta \to 0$. $\blacksquare$

Using Lemma 6.2 we obtain the following consequence which is used in the proof of Proposition 2.2.

**Corollary 6.1.** *Let $\sigma \in \mathcal{M}_+(\Omega)$ and $\nu \in \mathcal{M}\left(\Omega;\mathbb{R}^l\right)$. Let $x \in \Omega$ be such that*

$$\lim_{\rho\to 0}\left|\frac{d\nu}{d\sigma}(x) - \frac{\nu(B_\rho(x))}{\sigma(B_\rho(x))}\right| = 0 \quad \text{and} \quad \lim_{\rho\to 0}\left|\frac{d|\nu|}{d\sigma}(x) - \frac{|\nu|(B_\rho(x))}{\sigma(B_\rho(x))}\right| = 0. \tag{32}$$

*Then for every $\phi \in \mathcal{C}\left(\overline{\Omega}\right)$*

$$\lim_{\rho\to 0}\sup_{\varepsilon>0}\left|\frac{m_\varepsilon\left(\phi\frac{d\nu}{d\sigma}(x)\,\sigma, B_\rho(x)\right)}{\sigma(B_\rho(x))} - \frac{m_\varepsilon(\phi\nu, B_\rho(x))}{\sigma(B_\rho(x))}\right| = 0. \tag{33}$$

*Proof.* Fix $x \in \Omega$ satisfying (32). Let $\phi \in \mathcal{C}\left(\overline{\Omega}\right)$, and let $\omega : [0,\infty[ \to [0,\infty]$ its modulus of continuity, i.e.

$$\omega(\rho) := \sup\left(\left\{\left|\phi(z) - \phi\left(z'\right)\right| : \overline{\Omega} \times \overline{\Omega} \ni \left(z, z'\right), \quad |z - z'| \leq \rho\right\}\right)$$

satisfies $\lim_{\rho\to 0}\omega(\rho)=0$. Using Lemma 6.2 we have for every $\rho>0$

$$\begin{aligned}
&\sup_{\varepsilon>0}\left|\frac{m_\varepsilon\left(\phi\frac{d\nu}{d\sigma}(x)\,\sigma,B_\rho(x)\right)}{\sigma(B_\rho(x))}-\frac{m_\varepsilon(\phi\nu,B_\rho(x))}{\sigma(B_\rho(x))}\right|\\
\leq& C\left|\fint_{B_\rho(x)}\frac{d\nu}{d\sigma}(x)\,\phi(z)\,d\sigma(z)-\frac{1}{\sigma(B_\rho(x))}\int_{B_\rho(x)}\phi(z)\,d\nu(z)\right|\\
\leq& C\left|\fint_{B_\rho(x)}\frac{d\nu}{d\sigma}(x)\,(\phi(z)-\phi(x))\,d\sigma(z)+\frac{d\nu}{d\sigma}(x)\,\phi(x)\right.\\
&\left.-\frac{1}{\sigma(B_\rho(x))}\int_{B_\rho(x)}\phi(z)-\phi(x)\,d\nu(z)-\frac{\nu(B_\rho(x))}{\sigma(B_\rho(x))}\phi(x)\right|\\
\leq& C\left|\frac{d\nu}{d\sigma}(x)\,\phi(x)-\frac{\nu(B_\rho(x))}{\sigma(B_\rho(x))}\phi(x)\right|+C\omega(\rho)\left(\left|\frac{d\nu}{d\sigma}(x)\right|+\frac{|\nu|(B_\rho(x))}{\sigma(B_\rho(x))}\right).
\end{aligned}$$

Passing to the limit $\rho\to 0$, the proof of (33) is complete. ■

6.3. **Integral representation of the Vitali envelope of a set function.** This section is dedicated to establishing the integral representation of the Vitali envelope of a set function defined on open subsets of $\Omega$. The inspiration for this approach partly stems from [BB00, DMM86, BFM98]. We then apply this representation to the set function $\overline{m}(\nu,\cdot):=\overline{\lim}_{\varepsilon\to 0}m_\varepsilon(\nu,\cdot)$.

Let $\mu\in\mathcal{M}_+(\Omega)$. For each open set $O\subset\Omega$, let $\mathscr{B}_0(O)\subset\mathcal{O}(O)$ be the set of all open balls $B$ in $O$ whose boundaries have zero measure, i.e. $\mu(\partial B)=0$.

Let $G:\mathscr{B}_0(\Omega)\to\mathbb{R}$ be a set function. We define the lower Vitali envelope of $G$ with respect to $\mu$

$$\mathcal{O}(\Omega)\ni O\longmapsto G^*_-(O):=\sup_{\varepsilon>0}\inf\left\{\sum_{i\in I}G(B_i):\{B_i\}_{i\in I}\in\mathcal{V}^\varepsilon(O)\right\}$$

and the upper Vitali envelope with respect to $\mu$

$$\mathcal{O}(\Omega)\ni O\longmapsto G^*_+(O):=\inf_{\varepsilon>0}\sup\left\{\sum_{i\in I}G(B_i):\{B_i\}_{i\in I}\in\mathcal{V}^\varepsilon(O)\right\},$$

where for every $\varepsilon>0$

$$\begin{aligned}
\mathcal{V}^\varepsilon(O):=\Bigg\{\{B_i\}_{i\in I}\subset\mathscr{B}_0(\Omega):I\text{ is countable, }\mu\left(O\setminus\bigcup_{i\in I}B_i\right)=0,\ \overline{B}_i\subset O,\\
\operatorname{diam}(B_i)\in]0,\varepsilon[\ \text{ and }\ \overline{B}_i\cap\overline{B}_j=\emptyset\ \text{ for all }\ i\neq j\Bigg\}.
\end{aligned}$$

We consider the following two conditions on a set function $G:\mathcal{O}(\Omega)\to\mathbb{R}$:

- $(\mathcal{V}_1)$ the set function $G$ *is dominated* by a positive measure absolutely continuous with respect to $\mu$, i.e. there exists $\alpha\in\mathcal{M}_+(\Omega)$ with $\alpha\ll\mu$ satisfying
$$|G(O)|\leq\alpha(O)\qquad\text{for all }O\in\mathcal{O}(\Omega).$$
- $(\mathcal{V}_2)$ the set function $G$ is *subadditive*, i.e. for every $U,V,O\in\mathcal{O}(\Omega)$ with $U\cap V=\emptyset$, $U\subset O$, $V\subset O$ and $\mu(O\setminus(U\cup V))=0$ it holds
$$G(O)\leq G(U)+G(V).$$

In [AHM18, Theorem 3.17, p. 65], we established the integral representation result for the Vitali envelopes of $G$ under $(\mathcal{V}_1)$ and $(\mathcal{V}_2)$. It is worth noting that the theorem is proved in the particular case where the measure of the boundaries of balls is zero. However, this result naturally extends to the more general case where the measure of the boundaries of balls

is non-zero, as shown in [AHMZ19]. Consequently, we introduce the set $R_\mu^x := \{\rho \in ]0,\infty[ : \mu(\partial B_\rho(x)) > 0\}$.

**Theorem 6.1.** *Let $\mu \in \mathcal{M}_+(\Omega)$. If $G : \mathcal{O}(\Omega) \to \mathbb{R}$ satisfies $(\mathcal{V}_1)$ and $(\mathcal{V}_2)$ then*

$$\Omega \ni x \longmapsto \lim_{R_\mu^x \not\ni \rho \to 0} \frac{G(B_\rho(x))}{\mu(B_\rho(x))} \in L^1_\mu(\Omega)$$

*and*

$$G^*_+(O) = G^*_-(O) = \int_O \lim_{R_\mu^x \not\ni \rho \to 0} \frac{G(B_\rho(x))}{\mu(B_\rho(x))} d\mu(x) \quad \text{for all } O \in \mathcal{O}(\Omega). \tag{34}$$

For each $\nu \in \mathcal{M}(\Omega;\mathbb{R}^l)$ we consider $\overline{m}^*_-(\nu,\cdot)$ the lower Vitali envelope of

$$\overline{m}(\nu,\cdot) := \overline{\lim_{\varepsilon\to 0}}\, m_\varepsilon(\nu,\cdot)$$

with respect to the measure $\pi_0 = \mu_0 + |\nu_0^s|$. The following result is deduced from Theorem (6.1):

**Lemma 6.3.** *Assume that $(C_1)$ holds. Then, for every $\nu \in \mathcal{M}(\Omega;\mathbb{R}^l)$, and every $\sigma \in \{\mu_0, |\nu_0^s|\}$ the function*

$$\Omega \ni x \longmapsto \lim_{\rho\to 0} \frac{\overline{m}(\nu, B_\rho(x))}{\sigma(B_\rho(x))} = \lim_{\rho\to 0}\overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon(\nu, B_\rho(x))}{\sigma(B_\rho(x))} \in L^1_\mu(\Omega)$$

*and for every $O \in \mathcal{O}_0(\Omega)$*

$$\begin{aligned}\overline{m}^*_-(\nu,O) &= \sum_{\sigma\in\{\mu_0,|\nu_0^s|\}} \int_O \lim_{\rho\to 0} \frac{\overline{m}(\nu,B_\rho(x))}{\sigma(B_\rho(x))} d\sigma(x) \\ &= \int_O \lim_{\rho\to 0}\overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon(\nu,B_\rho(x))}{\mu_0(B_\rho(x))} d\mu_0(x) + \int_O \lim_{\rho\to 0}\overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon(\nu,B_\rho(x))}{|\nu_0^s|(B_\rho(x))} d|\nu_0^s|(x).\end{aligned}$$

*Proof.* The proof is divided into two steps.

*Step 1:* We begin by verifying the assumptions $(\mathcal{V}_1)$ and $(\mathcal{V}_2)$ to apply Theorem 6.1 with $G(\cdot) = \overline{m}(\nu,\cdot)$ and $\pi_0 = \mu_0 + |\nu_0^s|$.

Let $\nu \in \mathcal{M}(\Omega;\mathbb{R}^l)$. We start by showing that the set function $\overline{m}(\nu,\cdot) : \mathcal{O}(\Omega) \to \mathbb{R}_+$ is subadditive. Let $U_1, U_2, W \in \mathcal{O}(\Omega)$ be such that $U_1 \subset W$, $U_2 \subset W$ with $\pi_0(W \setminus (U_1 \cup U_2)) = 0$ and $U_1 \cap U_2 = \emptyset$. We can assume that $\overline{m}(\nu,U_1) < \infty$ and $\overline{m}(\nu,U_2) < \infty$. Thus, there exists $\varepsilon_0 > 0$ such that $m_\varepsilon(\nu,U_i) < \infty$ for all $\varepsilon \in ]0,\varepsilon_0[$ and all $i \in \{1,2\}$. Let $\varepsilon \in ]0,\varepsilon_0[$. There exists $\lambda_i \in \mathcal{M}(\Omega;\mathbb{R}^l)$ such that $\lambda_i \ll \mu_\varepsilon$, $\lambda_i(U_i) = \nu(U_i)$ and

$$\varepsilon + \sum_{i=1}^2 m_\varepsilon(\nu,U_i) \geq \sum_{i=1}^2 I_\varepsilon(\lambda_i,U_i).$$

Set $\lambda_0 := \sum_{i=1}^2 \lambda_i \mathbb{1}_{U_i}$. We see that $\lambda_0 \ll \mu$ and $\lambda_0(W) = \lambda_0(U_1\cup U_2) = \lambda_1(U_1) + \lambda_2(U_2) = \nu(U_1\cup U_2) = \nu(W)$ since $\nu \ll \pi_0$. Therefore, we obtain

$$\varepsilon + \sum_{i=1}^2 m_\varepsilon(\nu,U_i) \geq I_\varepsilon(\lambda_0,W) \geq m_\varepsilon(\nu,W).$$

Passing to the limit $\varepsilon \to 0$, we obtain $(\mathcal{V}_2)$.

By $(C_1)$, we have for every $O \in \mathcal{O}_0(\Omega)$

$$\overline{m}(\nu,O) \leq \overline{\lim_{\varepsilon>0}} I_\varepsilon(\nu,O) \leq C\left(\overline{\lim_{\varepsilon\to 0}} \mu_\varepsilon(O) + |\nu|(O)\right) \leq \alpha_\nu(O)$$

where $\alpha_\nu := C\mu_0 + C|\nu| \ll \pi_0$ and, with $C > 0$ given by $(C_1)$. Thus $(\mathcal{V}_1)$ is satisfied. It follows that $\overline{m}(\nu,\cdot)$ is $\pi_0$-differentiable, and for every $O \in \mathcal{O}(\Omega)$

$$\overline{m}^*_-(\nu,O) = \int_O \lim_{R_{\pi_0}^x \not\ni \rho\to 0} \frac{\overline{m}(\nu,B_\rho(x))}{\pi_0(B_\rho(x))} d\pi_0(x).$$

*Step 2:* In this second step, we show that the limit in $\rho$ appearing in the density of the integral representation does not depend on $R^x_{\pi_0}$. So, let us show that for every $x \in \Omega$

$$\lim_{R^x_{\pi_0} \not\ni \rho \to 0} \frac{\overline{m}(\nu, B_\rho(x))}{\pi_0(B_\rho(x))} = \lim_{\rho \to 0} \frac{\overline{m}(\nu, B_\rho(x))}{\pi_0(B_\rho(x))}. \tag{35}$$

Fix $x \in \Omega$. The proof is divided into two substeps.

*Substep 2.1:* Let $\{\rho_n\}_{n\in\mathbb{N}} \subset ]0,1[$ satisfies $\lim_{n\to\infty} \rho_n = 0$ and

$$\sup_{n\in\mathbb{N}} \frac{\overline{m}(\nu, B_{\rho_n}(x))}{\pi_0(B_{\rho_n}(x))} < \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\overline{m}(\nu, B_{\rho_n}(x))}{\pi_0(B_{\rho_n}(x))} = \underline{\lim_{\rho\to 0}} \frac{\overline{m}(\nu, B_\rho(x))}{\pi_0(B_\rho(x))}.$$

Fix $n \in \mathbb{N}$ and $\delta > 0$. There exists $\varepsilon_n^\delta > 0$ such that for every $\varepsilon \in ]0, \varepsilon_n^\delta[$

$$\frac{m_\varepsilon(\nu, B_{\rho_n}(x))}{\pi_0(B_{\rho_n}(x))} < \infty.$$

Fix $\varepsilon \in ]0, \varepsilon_n^\delta[$. There exists $\lambda \in \mathcal{M}(\Omega; \mathbb{R}^l)$ such that $\lambda \ll \mu_\varepsilon$, $\lambda(B_{\rho_n}(x)) = \nu(B_{\rho_n}(x))$ and

$$\frac{\delta}{2}\pi_0(B_{\rho_n}(x)) + m_\varepsilon(\nu, B_{\rho_n}(x)) \geq I_\varepsilon(\lambda, B_{\rho_n}(x)).$$

We claim that there exists $\tau_{n,\delta} \in ]1,2]$ such that

$$]1, \tau_{n,\delta}] \subset \left\{ t \in ]1,2[ : \alpha_\nu(B_{t\rho_n}(x) \setminus B_{\rho_n}(x)) \leq \frac{\delta}{2}\pi_0(B_{\rho_n}(x)) \right\}.$$

Indeed, otherwise there exists a sequence $\{t_k\}_{k\in\mathbb{N}} \subset ]1,2[$ satisfying $t_k \in ]1, 1+\frac{1}{k+1}]$ and

$$\alpha_\nu(B_{t_k\rho_n}(x) \setminus B_{\rho_n}(x)) > \frac{\delta}{2}\pi_0(B_{\rho_n}(x)),$$

which implies

$$\alpha_\nu\left(B_{(1+\frac{1}{k+1})\rho_n}(x) \setminus B_{\rho_n}(x)\right) > \frac{\delta}{2}\pi_0(B_{\rho_n}(x))$$

for all $k \in \mathbb{N}$. By letting $k \to \infty$ we obtain $\frac{\delta}{2}\pi_0(B_{\rho_n}(x)) = 0$ a contradiction. Now, since $\pi_0(\Omega) < \infty$, the set $\{t \in ]1,2[ : \pi_0(\partial B_{t\rho_n}(x)) > 0\}$ is countable, thus

$$]1, \tau_{n,\delta}] \setminus \{t \in ]1,2[ : \pi_0(\partial B_{t\rho_n}(x)) > 0\} \neq \emptyset.$$

So, there is a sequence $\{t_n^\delta\}_{n\in\mathbb{N}} \subset ]1,2[$ which satisfies for every $n \in \mathbb{N}$

$$\alpha_\nu\left(B_{t_n^\delta\rho_n}(x) \setminus B_{\rho_n}(x)\right) \leq \frac{\delta}{2}\pi_0(B_{\rho_n}(x)) \text{ and } \pi_0\left(\partial B_{t_n^\delta\rho_n}(x)\right) = 0.$$

Set

$$\lambda_n^\delta := \lambda \mathbb{1}_{B_{\rho_n}(x)} + \frac{\nu\left(B_{t_n^\delta\rho_n}(x) \setminus B_{\rho_n}(x)\right)}{\mu_\varepsilon\left(B_{t_n^\delta\rho_n}(x) \setminus B_{\rho_n}(x)\right)} \mu_\varepsilon \mathbb{1}_{B_{t_n^\delta\rho_n}(x) \setminus B_{\rho_n}(x)}.$$

It is direct to see that $\lambda_n^\delta \ll \mu_\varepsilon$ and $\lambda_n^\delta\left(B_{t_n^\delta\rho_n}(x)\right) = \nu\left(B_{t_n^\delta\rho_n}(x)\right)$. We can write using growth conditions

$$\begin{aligned} m_\varepsilon\left(\nu, B_{t_n^\delta\rho_n}(x)\right) &\leq I_\varepsilon\left(\lambda_n^\delta, B_{\rho_n}(x)\right) + I_\varepsilon\left(\lambda_n^\delta, B_{t_n^\delta\rho_n}(x) \setminus B_{\rho_n}(x)\right) \\ &= I_\varepsilon(\lambda, B_{\rho_n}(x)) + I_\varepsilon\left(\lambda_n^\delta, B_{t_n^\delta\rho_n}(x) \setminus B_{\rho_n}(x)\right) \\ &\leq \frac{\delta}{2}\pi_0(B_{\rho_n}(x)) + m_\varepsilon(\nu, B_{\rho_n}(x)) + \alpha_\nu^\varepsilon\left(B_{t_n^\delta\rho_n}(x) \setminus B_{\rho_n}(x)\right) \end{aligned}$$

where $\alpha_\nu^\varepsilon := C\mu_\varepsilon + C|\nu|$. Dividing by $\pi_0\left(B_{t_n^\delta\rho_n}(x)\right)$

$$\frac{m_\varepsilon\left(\nu, B_{t_n^\delta\rho_n}(x)\right)}{\pi_0\left(B_{t_n^\delta\rho_n}(x)\right)} \leq \frac{\delta}{2} + \frac{m_\varepsilon(\nu, B_{\rho_n}(x))}{\pi_0(B_{\rho_n}(x))} + \frac{\alpha_\nu^\varepsilon\left(B_{t_n^\delta\rho_n}(x) \setminus B_{\rho_n}(x)\right)}{\pi_0(B_{\rho_n}(x))}$$

letting $\varepsilon \to 0$, since $\alpha_\nu\left(\partial B_{t_n^\delta \rho_n}(x)\right) = 0$ because $\alpha_\nu \ll \pi_0$, we obtain

$$\frac{\overline{m}\left(\nu, B_{t_n^\delta \rho_n}(x)\right)}{\pi_0\left(B_{t_n^\delta \rho_n}(x)\right)} \leq \frac{\delta}{2} + \frac{\overline{m}\left(\nu, B_{\rho_n}(x)\right)}{\pi_0\left(B_{\rho_n}(x)\right)} + \frac{\alpha_\nu\left(B_{t_n^\delta \rho_n}(x) \setminus B_{\rho_n}(x)\right)}{\pi_0\left(B_{\rho_n}(x)\right)} \leq \delta + \frac{\overline{m}\left(\nu, B_{\rho_n}(x)\right)}{\pi_0\left(B_{\rho_n}(x)\right)}.$$

Now, letting $n \to \infty$, one has $\lim_{n\to\infty} t_n^\delta \rho_n = 0$ and

$$\begin{aligned}\lim_{R_{\pi_0}^x \ni \rho \to 0} \frac{\overline{m}\left(\nu, B_\rho(x)\right)}{\pi_0\left(B_\rho(x)\right)} &= \lim_{n\to\infty} \frac{\overline{m}\left(\nu, B_{t_n^\delta \rho_n}(x)\right)}{\pi_0\left(B_{t_n^\delta \rho_n}(x)\right)} \\ &\leq \delta + \lim_{n\to\infty} \frac{\overline{m}\left(\nu, B_{\rho_n}(x)\right)}{\pi_0\left(B_{\rho_n}(x)\right)} = \delta + \underline{\lim}_{\rho\to 0} \frac{\overline{m}\left(\nu, B_\rho(x)\right)}{\pi_0\left(B_\rho(x)\right)}. \end{aligned} \tag{36}$$

*Substep 2.2:* Let $\{\rho_n\}_{n\in\mathbb{N}} \subset ]0,1[$ satisfies $\lim_{n\to\infty} \rho_n = 0$ and

$$\sup_{n\in\mathbb{N}} \frac{\overline{m}\left(\nu, B_{\rho_n}(x)\right)}{\pi_0\left(B_{\rho_n}(x)\right)} < \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\overline{m}\left(\nu, B_{\rho_n}(x)\right)}{\pi_0\left(B_{\rho_n}(x)\right)} = \overline{\lim_{\rho\to 0}} \frac{\overline{m}\left(\nu, B_\rho(x)\right)}{\pi_0\left(B_\rho(x)\right)}.$$

Fix $n \in \mathbb{N}$ and $\delta > 0$. There exists $\varepsilon_n^\delta > 0$ such that for every $\varepsilon \in ]0, \varepsilon_n^\delta[$

$$\frac{m_\varepsilon\left(\nu, B_{\rho_n}(x)\right)}{\pi_0\left(B_{\rho_n}(x)\right)} < \infty.$$

Fix $\varepsilon \in ]0, \varepsilon_n^\delta[$. There exists $\tau_{n,\delta} \in ]0,1[$ such that

$$[\tau_{n,\delta}, 1[ \subset \left\{ t \in ]0,1[ : \alpha_\nu\left(\overline{B}_{\rho_n}(x) \setminus \overline{B}_{t\rho_n}(x)\right) \leq \frac{\delta}{2} \pi_0\left(B_{\rho_n}(x)\right) \right\},$$

and

$$[\tau_{n,\delta}, 1[ \setminus \{ t \in ]0,1[ : \pi_0\left(\partial B_{t\rho_n}(x)\right) > 0 \} \neq \emptyset.$$

So, there exists a sequence $\{t_n^\delta\}_{n\in\mathbb{N}} \subset ]0,1[$ which satisfies for every $n \in \mathbb{N}$

$$\alpha_\nu\left(\overline{B}_{\rho_n}(x) \setminus \overline{B}_{t_n^\delta \rho_n}(x)\right) \leq \frac{\delta}{2} \pi_0\left(B_{\rho_n}(x)\right) \text{ and } \pi_0\left(\partial B_{t_n^\delta \rho_n}(x)\right) = 0.$$

There exists $\lambda \in \mathcal{M}\left(\Omega; \mathbb{R}^l\right)$ such that $\lambda \ll \mu_\varepsilon$, $\lambda\left(B_{t_n^\delta \rho_n}(x)\right) = \nu\left(B_{t_n^\delta \rho_n}(x)\right)$ and

$$\frac{\delta}{2} \pi_0\left(B_{t_n^\delta \rho_n}(x)\right) + m_\varepsilon\left(\nu, B_{t_n^\delta \rho_n}(x)\right) \geq I_\varepsilon\left(\lambda, B_{t_n^\delta \rho_n}(x)\right).$$

Set

$$\lambda_n^\delta := \lambda \mathbb{1}_{B_{t_n^\delta \rho_n}(x)} + \frac{\nu\left(B_{\rho_n}(x) \setminus B_{t_n^\delta \rho_n}(x)\right)}{\mu_\varepsilon\left(B_{\rho_n}(x) \setminus B_{t_n^\delta \rho_n}(x)\right)} \mu_\varepsilon \mathbb{1}_{B_{\rho_n}(x) \setminus B_{t_n^\delta \rho_n}(x)}.$$

It is direct to see that $\lambda_n^\delta \ll \mu_\varepsilon$ and $\lambda_n^\delta\left(B_{\rho_n}(x)\right) = \nu\left(B_{\rho_n}(x)\right)$. We can write using growth conditions

$$\begin{aligned} m_\varepsilon\left(\nu, B_{\rho_n}(x)\right) &\leq I_\varepsilon\left(\lambda_n^\delta, B_{t_n^\delta \rho_n}(x)\right) + I_\varepsilon\left(\lambda_n^\delta, B_{\rho_n}(x) \setminus B_{t_n^\delta \rho_n}(x)\right) \\ &= I_\varepsilon\left(\lambda, B_{\rho_n}(x)\right) + I_\varepsilon\left(\lambda_n^\delta, B_{\rho_n}(x) \setminus B_{t_n^\delta \rho_n}(x)\right) \\ &\leq \frac{\delta}{2} \pi_0\left(B_{\rho_n}(x)\right) + m_\varepsilon\left(\nu, B_{t_n^\delta \rho_n}(x)\right) + \alpha_\nu^\varepsilon\left(B_{\rho_n}(x) \setminus B_{t_n^\delta \rho_n}(x)\right). \end{aligned}$$

Dividing by $\pi_0\left(B_{\rho_n}(x)\right)$ we have

$$\frac{m_\varepsilon\left(\nu, B_{\rho_n}(x)\right)}{\pi_0\left(B_{\rho_n}(x)\right)} \leq \frac{\delta}{2} + \frac{m_\varepsilon\left(\nu, B_{t_n^\delta \rho_n}(x)\right)}{\pi_0\left(B_{t_n^\delta \rho_n}(x)\right)} + \frac{\alpha_\nu^\varepsilon\left(B_{\rho_n}(x) \setminus B_{t_n^\delta \rho_n}(x)\right)}{\pi_0\left(B_{\rho_n}(x)\right)}.$$

Letting $\varepsilon \to 0$ we obtain

$$\frac{\overline{m}\left(\nu, B_{\rho_n}(x)\right)}{\pi_0\left(B_{\rho_n}(x)\right)} \leq \frac{\delta}{2} + \frac{\overline{m}\left(\nu, B_{t_n^\delta \rho_n}(x)\right)}{\pi_0\left(B_{t_n^\delta \rho_n}(x)\right)} + \frac{\alpha_\nu\left(\overline{B}_{\rho_n}(x) \setminus \overline{B}_{t_n^\delta \rho_n}(x)\right)}{\pi_0\left(B_{\rho_n}(x)\right)}$$

$$\leq \frac{\overline{m}\left(\nu, B_{t_n^\delta \rho_n}(x)\right)}{\pi_0\left(B_{t_n^\delta \rho_n}(x)\right)} + \delta$$

since $\alpha_\nu\left(\partial B_{t_n^\delta \rho_n}(x)\right) = 0$ because $\alpha_\nu \ll \pi_0$. Letting $n \to \infty$

$$\overline{\lim_{\rho \to 0}} \frac{\overline{m}\left(\nu, B_\rho(x)\right)}{\pi_0\left(B_\rho(x)\right)} = \lim_{n \to \infty} \frac{\overline{m}\left(\nu, B_{\rho_n}(x)\right)}{\pi_0\left(B_{\rho_n}(x)\right)}$$

$$\leq \delta + \overline{\lim_{n \to \infty}} \frac{\overline{m}\left(\nu, B_{t_n^\delta \rho_n}(x)\right)}{\pi_0\left(B_{t_n^\delta \rho_n}(x)\right)} = \delta + \lim_{R_{\pi_0}^x \not\ni \rho \to 0} \frac{\overline{m}\left(\nu, B_\rho(x)\right)}{\pi_0\left(B_\rho(x)\right)}. \tag{37}$$

Combining (36) and (37) and letting $\delta \to 0$ we obtain (35). ■

6.4. **Proof of Theorem 3.1.** We see, by setting $f_\varepsilon(x,v) := \Phi(v)$ for all $(x,v) \in \Omega \times \mathbb{R}^l$, that $(C_1)$ and $(C_2)$ are fulfilled since (9). So, it remains to verify $(H^j_{\text{reg}})$ and $(H^j_{\text{sing}})$.

Let $\rho > 0$, $\varepsilon > 0$ and $x \in \Omega$. Let $\nu \in \mathcal{M}\left(\Omega; \mathbb{R}^l\right)$. For each $j \in \mathbb{N}$ we denote by $\Phi^j$ the Moreau-Yosida envelope of $\Phi$ defined by

$$\Phi^j(w) := \inf_{v \in \mathbb{R}^l} \Phi(w+v) + j|v|.$$

Note that $\Phi^j(0) = 0$ for all $j \in \mathbb{N}$ since (9), and

$$\left|\Phi^j(w) - \Phi^j\left(w'\right)\right| \leq j\left|w - w'\right| \quad \text{for all } \left(w, w'\right) \in \mathbb{R}^l \times \mathbb{R}^l.$$

Fix $j \in \mathbb{N}$. Let $t \in ]0,1[$. Consider $\varphi_t \in \mathcal{C}_c\left(B_\rho(x); [0,1]\right)$ be such that $\varphi_t \equiv 1$ on $\overline{B}_{t\rho}(x)$ and set $\widetilde{\mu}_\varepsilon := \varphi_t \mu_\varepsilon$. Using Lemma 6.2, one has for every $u \in \mathbb{R}^l$ and every $\sigma \in \{\mu_0, |\nu_0^s|\}$

$$\frac{m_\varepsilon^j\left(u\sigma, B_\rho(x)\right)}{\sigma\left(B_\rho(x)\right)} = \frac{m_\varepsilon^j\left(u\frac{\sigma\left(B_\rho(x)\right)}{\widetilde{\mu}_\varepsilon\left(B_\rho(x)\right)}\widetilde{\mu}_\varepsilon, B_\rho(x)\right)}{\sigma\left(B_\rho(x)\right)} \leq \frac{\widetilde{\mu}_\varepsilon\left(B_\rho(x)\right)}{\sigma\left(B_\rho(x)\right)} \Phi^j\left(u\frac{\sigma\left(B_\rho(x)\right)}{\widetilde{\mu}_\varepsilon\left(B_\rho(x)\right)}\right). \tag{38}$$

Therefore when $\sigma = \mu_0$

$$\frac{m_\varepsilon^j\left(u\sigma, B_\rho(x)\right)}{\mu_0\left(B_\rho(x)\right)} \leq \frac{\widetilde{\mu}_\varepsilon\left(B_\rho(x)\right)}{\mu_0\left(B_\rho(x)\right)} \Phi^j\left(u\frac{\mu_0\left(B_\rho(x)\right)}{\widetilde{\mu}_\varepsilon\left(B_\rho(x)\right)}\right).$$

Since $\widetilde{\mu}_\varepsilon\left(\partial B_\rho(x)\right) = 0$, we have $\lim_{\varepsilon \to 0} \widetilde{\mu}_\varepsilon\left(B_\rho(x)\right) = \int_{B_\rho(x)} \varphi_t d\mu_0$. We set $\varphi_\rho^t(x) := \fint_{B_\rho(x)} \varphi_t d\mu_0$. Passing to the limit $\varepsilon \to 0$, one has

$$\overline{\lim_{\varepsilon \to 0}} \frac{m_\varepsilon^j\left(u\sigma, B_\rho(x)\right)}{\mu_0\left(B_\rho(x)\right)} \leq \varphi_\rho^t(x)\, \Phi^j\left(u\frac{1}{\varphi_\rho^t(x)}\right)$$

$$\leq \varphi_\rho^t(x) \left|\Phi^j\left(u\frac{1}{\varphi_\rho^t(x)}\right) - \Phi^j(u)\right| + \varphi_\rho^t(x)\, \Phi^j(u)$$

$$\leq j\varphi_\rho^t(x)\,|u|\left(\frac{1}{\varphi_\rho^t(x)} - 1\right) + \varphi_\rho^t(x)\, \Phi^j(u)$$

$$= j|u|\left(1 - \varphi_\rho^t(x)\right) + \varphi_\rho^t(x)\, \Phi^j(u).$$

Since $1 \geq \varphi_t \geq \mathbb{1}_{B_{t\rho}(x)}$, by using $(M_0)$, we see that

$$\lim_{t \to 1} \overline{\lim_{\rho \to 0}} \varphi_\rho^t(x) = \lim_{t \to 1} \overline{\lim_{\rho \to 0}} \varphi_\rho^t(x) = 1. \tag{39}$$

Consequently, letting $\rho \to 0$

$$\begin{aligned}\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\frac{m_\varepsilon^j\left(u\sigma, B_\rho(x)\right)}{\mu_0\left(B_\rho(x)\right)} &\le \overline{\lim_{\rho\to 0}}\, j|u|\left(1-\varphi_\rho^t(x)\right)+\overline{\lim_{\rho\to 0}}\,\varphi_\rho^t(x)\,\Phi^j(u)\\ &= j|u|\left(1-\underline{\lim_{\rho\to 0}}\,\varphi_\rho^t(x)\right)+\overline{\lim_{\rho\to 0}}\,\varphi_\rho^t(x)\,\Phi^j(u)\end{aligned}$$

and passing to the limit $t \to 1$ one obtains

$$\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\frac{m_\varepsilon^j\left(u\sigma, B_\rho(x)\right)}{\mu_0\left(B_\rho(x)\right)} \le j|u|\left(1-\lim_{t\to 1}\underline{\lim_{\rho\to 0}}\,\varphi_\rho^t(x)\right)+\lim_{t\to 1}\overline{\lim_{\rho\to 0}}\,\varphi_\rho^t(x)\,\Phi^j(u)=\Phi^j(u).$$

Thus

$$\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\frac{m_\varepsilon^j\left(u\sigma, B_\rho(x)\right)}{\mu_0\left(B_\rho(x)\right)} \le \Phi^j(u). \tag{40}$$

Assume now that $\sigma = |\nu_0^s|$. We suppose

$$\lim_{r\to 0}\frac{|\nu_0^s|\left(B_r(x)\right)}{\mu_0\left(B_r(x)\right)}=\infty$$

and we set $t_\rho^x := \frac{|\nu_0^s|(B_\rho(x))}{\mu_0(B_\rho(x))}$. Using (38) one has

$$\begin{aligned}\overline{\lim_{\varepsilon\to 0}}\,\frac{m_\varepsilon^j\left(u|\nu_0^s|, B_\rho(x)\right)}{|\nu_0^s|\left(B_\rho(x)\right)} &\le \frac{\mu_0\left(B_\rho(x)\right)}{|\nu_0^s|\left(B_\rho(x)\right)}\varphi_\rho^t(x)\,\Phi^j\left(u\frac{|\nu_0^s|\left(B_\rho(x)\right)}{\displaystyle\int_{B_\rho(x)}\varphi_t d\mu_0}\right)\\ &\le \frac{1}{t_\rho^x}\varphi_\rho^t(x)\,\Phi^j\left(u\frac{t_\rho^x}{\varphi_\rho^t(x)}\right)\\ &\le \frac{1}{t_\rho^x}\varphi_\rho^t(x)\left|\Phi^j\left(u\frac{t_\rho^x}{\varphi_\rho^t(x)}\right)-\Phi^j\left(t_\rho^x u\right)\right|+\frac{1}{t_\rho^x}\varphi_\rho^t(x)\,\Phi^j\left(t_\rho^x u\right)\\ &\le \frac{j}{t_\rho^x}\varphi_\rho^t(x)\left|u\frac{t_\rho^x}{\varphi_\rho^t(x)}-t_\rho^x u\right|+\frac{1}{t_\rho^x}\varphi_\rho^t(x)\,\Phi^j\left(t_\rho^x u\right)\\ &\le j|u|\left(1-\varphi_\rho^t(x)\right)+\frac{1}{t_\rho^x}\varphi_\rho^t(x)\,\Phi^j\left(t_\rho^x u\right),\end{aligned}$$

letting $\rho \to 0$

$$\begin{aligned}\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\frac{m_\varepsilon^j\left(u|\nu_0^s|, B_\rho(x)\right)}{|\nu_0^s|\left(B_\rho(x)\right)} &\le j|u|\left(1-\underline{\lim_{\rho\to 0}}\,\varphi_\rho^t(x)\right)+\overline{\lim_{\rho\to 0}}\,\varphi_\rho^t(x)\,\frac{\Phi^j\left(t_\rho^x u\right)}{t_\rho^x}\\ &= j|u|\left(1-\underline{\lim_{\rho\to 0}}\,\varphi_\rho^t(x)\right)+\overline{\lim_{\rho\to 0}}\,\varphi_\rho^t(x)\left(\Phi^j\right)^\infty(u).\end{aligned}$$

and passing to the limit $t \to 1$ by using (39), one obtains

$$\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\frac{m_\varepsilon^j\left(u|\nu_0^s|, B_\rho(x)\right)}{|\nu_0^s|\left(B_\rho(x)\right)} \le \left(\Phi^j\right)^\infty(u).$$

Using a change of variable, the Jensen inequality and the Lipschitz property of $\Phi^j$ we have

$$\begin{aligned}
\frac{m_\varepsilon^j\left(u\sigma, B_\rho(x)\right)}{\sigma\left(B_\rho(x)\right)} =& \frac{m_\varepsilon^j\left(u\frac{\sigma\left(B_\rho(x)\right)}{\mu_\varepsilon\left(B_\rho(x)\right)}\mu_\varepsilon, B_\rho(x)\right)}{\sigma\left(B_\rho(x)\right)} \\
\geq & \inf_{\substack{w\in L^1_{\mu_\varepsilon}\left(B_\rho(x);\mathbb{R}^l\right) \\ \int_{B_\rho(x)} w d\mu_\varepsilon = 0}} \frac{\mu_\varepsilon\left(B_\rho(x)\right)}{\sigma\left(B_\rho(x)\right)}\Phi^j\left(\fint_{B_\rho(x)} w + u\frac{\sigma\left(B_\rho(x)\right)}{\mu_\varepsilon\left(B_\rho(x)\right)}d\mu_\varepsilon\right) \\
=& \frac{\mu_\varepsilon\left(B_\rho(x)\right)}{\sigma\left(B_\rho(x)\right)}\Phi^j\left(u\frac{\sigma\left(B_\rho(x)\right)}{\mu_\varepsilon\left(B_\rho(x)\right)}\right) \\
\geq & \frac{\mu_\varepsilon\left(B_\rho(x)\right)}{\sigma\left(B_\rho(x)\right)}\left(\Phi^j\left(u\frac{\sigma\left(B_\rho(x)\right)}{\mu_\varepsilon\left(B_\rho(x)\right)}\right) - \Phi^j\left(u\frac{\sigma\left(B_\rho(x)\right)}{\widetilde{\mu}_\varepsilon\left(B_\rho(x)\right)}\right)\right) \\
& + \frac{\widetilde{\mu}_\varepsilon\left(B_\rho(x)\right)}{\sigma\left(B_\rho(x)\right)}\Phi^j\left(u\frac{\sigma\left(B_\rho(x)\right)}{\widetilde{\mu}_\varepsilon\left(B_\rho(x)\right)}\right) \\
\geq & -\frac{\mu_\varepsilon\left(B_\rho(x)\right)}{\sigma\left(B_\rho(x)\right)} j\left|u\sigma\left(B_\rho(x)\right)\right|\left|\frac{1}{\mu_\varepsilon\left(B_\rho(x)\right)} - \frac{1}{\widetilde{\mu}_\varepsilon\left(B_\rho(x)\right)}\right| \\
& + \frac{\widetilde{\mu}_\varepsilon\left(B_\rho(x)\right)}{\sigma\left(B_\rho(x)\right)}\Phi^j\left(u\frac{\sigma\left(B_\rho(x)\right)}{\widetilde{\mu}_\varepsilon\left(B_\rho(x)\right)}\right) \\
=& j|u|\left(1 - \frac{\mu_\varepsilon\left(B_\rho(x)\right)}{\widetilde{\mu}_\varepsilon\left(B_\rho(x)\right)}\right) + \frac{\widetilde{\mu}_\varepsilon\left(B_\rho(x)\right)}{\sigma\left(B_\rho(x)\right)}\Phi^j\left(u\frac{\sigma\left(B_\rho(x)\right)}{\widetilde{\mu}_\varepsilon\left(B_\rho(x)\right)}\right).
\end{aligned}$$

But

$$\overline{\lim_{\varepsilon\to 0}}\frac{\mu_\varepsilon\left(B_\rho(x)\right)}{\widetilde{\mu}_\varepsilon\left(B_\rho(x)\right)} \leq \frac{\mu_0\left(\overline{B}_\rho(x)\right)}{\displaystyle\int_{B_\rho(x)}\varphi_t d\mu_0} \leq \frac{\mu_0\left(B_{\frac{1}{t}\rho}(x)\right)}{\mu_0\left(B_{t\rho}(x)\right)},$$

hence, by convexity, one has

$$\begin{aligned}
& \underline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon^j\left(u\sigma, B_\rho(x)\right)}{\sigma\left(B_\rho(x)\right)} \\
\geq & j|u|\left(1 - \frac{\mu_0\left(B_{\frac{1}{t}\rho}(x)\right)}{\mu_0\left(B_{t\rho}(x)\right)}\right) + \frac{\displaystyle\int_{B_\rho(x)}\varphi_t d\mu_0}{\sigma\left(B_\rho(x)\right)}\Phi^j\left(u\frac{\sigma\left(B_\rho(x)\right)}{\displaystyle\int_{B_\rho(x)}\varphi_t d\mu_0}\right) \\
= & j|u|\left(1 - \frac{\mu_0\left(B_{\frac{1}{t}\rho}(x)\right)}{\mu_0\left(B_{t\rho}(x)\right)}\right) + \frac{\mu_0\left(B_\rho(x)\right)}{\sigma\left(B_\rho(x)\right)}\varphi_\rho^t(x)\,\Phi^j\left(u\frac{\sigma\left(B_\rho(x)\right)}{\mu_0\left(B_\rho(x)\right)\varphi_\rho^t(x)}\right) \\
\geq & j|u|\left(1 - \frac{\mu_0\left(B_{\frac{1}{t}\rho}(x)\right)}{\mu_0\left(B_{t\rho}(x)\right)}\right) + \frac{\mu_0\left(B_\rho(x)\right)}{\sigma\left(B_\rho(x)\right)}\Phi^j\left(u\frac{\sigma\left(B_\rho(x)\right)}{\mu_0\left(B_\rho(x)\right)}\right) \\
= & j|u| + \frac{\mu_0\left(B_\rho(x)\right)}{\sigma\left(B_\rho(x)\right)}\Phi^j\left(u\frac{\sigma\left(B_\rho(x)\right)}{\mu_0\left(B_\rho(x)\right)}\right) - j|u|\frac{\mu_0\left(B_{\frac{1}{t}\rho}(x)\right)}{\mu_0\left(B_{t\rho}(x)\right)}.
\end{aligned}$$

Suppose $\sigma = \mu_0$. We have

$$\underline{\lim_{\varepsilon\to 0}}\frac{m_\varepsilon^j\left(u\mu_0, B_\rho(x)\right)}{\mu_0\left(B_\rho(x)\right)} \geq j|u| + \Phi^j(u) - j|u|\frac{\mu_0\left(B_{\frac{1}{t}\rho}(x)\right)}{\mu_0\left(B_{t\rho}(x)\right)}.$$

Letting $\rho \to 0$ one has

$$\begin{aligned}\overline{\lim_{\rho\to 0}} \underline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon^j\left(u\mu_0, B_\rho(x)\right)}{\mu_0\left(B_\rho(x)\right)} &\geq j|u| + \Phi^j(u) - j|u| \overline{\lim_{\rho\to 0}} \frac{\mu_0\left(B_{\frac{1}{t}\rho}(x)\right)}{\mu_0\left(B_{t\rho}(x)\right)} \\ &= j|u| + \Phi^j(u) - j|u| \frac{1}{\underline{\lim_{r\to 0}} \dfrac{\mu_0\left(B_{t^2 r}(x)\right)}{\mu_0\left(B_r(x)\right)}},\end{aligned}$$

and passing to the limit $t \to 1$ one obtains

$$\overline{\lim_{\rho\to 0}} \underline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon^j\left(u\mu_0, B_\rho(x)\right)}{\mu_0\left(B_\rho(x)\right)} \geq \Phi^j(u). \tag{41}$$

Suppose $\sigma = |\nu_0^s|$. Let $O \in \mathcal{O}(\Omega)$. Since Lemma 5.2, consider $x \in O$ such that for every $\alpha \in ]0,1[$

$$\lim_{r\to 0} \frac{|\nu_0^s|\left(B_r(x)\right)}{\mu_0\left(B_r(x)\right)} = \infty \quad \text{and} \quad \overline{\lim_{\rho\to 0}} \frac{|\nu_0^s|\left(B_{\alpha\rho}(x)\right)}{|\nu_0^s|\left(B_\rho(x)\right)} \geq \alpha^N. \tag{42}$$

Let $\rho > 0$. Set $t_\rho^x := \frac{|\nu_0^s|(B_\rho(x))}{\mu_0(B_\rho(x))}$. We have

$$\underline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon^j\left(u\sigma, B_\rho(x)\right)}{|\nu_0^s|\left(B_\rho(x)\right)} \geq j|u| + \frac{\Phi^j\left(ut_\rho^x\right)}{t_\rho^x} - j|u| \frac{\mu_0\left(B_{\frac{1}{t}\rho}(x)\right)}{\mu_0\left(B_{t\rho}(x)\right)}.$$

Letting $\rho \to 0$

$$\overline{\lim_{\rho\to 0}} \underline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon^j\left(u\sigma, B_\rho(x)\right)}{|\nu_0^s|\left(B_\rho(x)\right)} \geq j|u| + \left(\Phi^j\right)^\infty(u) - j|u| \frac{1}{\underline{\lim_{r\to 0}} \dfrac{\mu_0\left(B_{t^2 r}(x)\right)}{\mu_0\left(B_r(x)\right)}},$$

and letting $t \to 1$ one obtains

$$\overline{\lim_{\rho\to 0}} \underline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon^j\left(u\sigma, B_\rho(x)\right)}{|\nu_0^s|\left(B_\rho(x)\right)} \geq \left(\Phi^j\right)^\infty(u).$$

Now, we see that the formulas for the limit integrands are given by:

$$\overline{\lim_{\rho\to 0}} \overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon^j\left(u\mu_0, B_\rho(x)\right)}{\mu_0\left(B_\rho(x)\right)} = \sup_{j\in\mathbb{N}} \Phi^j(u) = \Phi(u) \quad \mu_0\text{-a.e. in } O,$$

and

$$\overline{\lim_{\rho\to 0}} \overline{\lim_{\varepsilon\to 0}} \frac{m_\varepsilon^j\left(v|\nu_0^s|, B_\rho(x)\right)}{|\nu_0^s|\left(B_\rho(x)\right)} = \sup_{j\in\mathbb{N}} \sup_{t>0} \frac{1}{t}\Phi^j(tv) = \sup_{t>0} \sup_{j\in\mathbb{N}} \frac{1}{t}\Phi^j(tv) = \Phi^\infty(v) \quad |\nu_0^s|\text{-a.e. in } O. \blacksquare$$

6.5. **Proof of Lemma 4.1** ($i$)**.** Fix $A \in \mathcal{B}\left(\mathbb{R}^N\right)$, $\omega \in \Sigma$ and $u \in \mathbb{R}^l$. First, observe that we can write

$$\begin{aligned}\mathcal{S}_u(A)(\omega) &:= \inf_{\substack{\psi\in L_\mu^1\left(\Omega;\mathbb{R}^l\right) \\ \int_A \psi d\mu = 0}} \int_A f\left(x, \psi(x) + u\frac{\mu_0(A)}{\mu(A)}, \omega\right) d\mu \\ &= \inf_{\psi\in L_\mu^1\left(\Omega;\mathbb{R}^l\right)} \int_A g\left(x, \psi(x) - \fint_A \psi d\mu, \omega\right) d\mu,\end{aligned}$$

where $g(x,v,\omega) := f\left(x, v + u\frac{\mu_0(A)}{\mu(A)}, \omega\right)$ satisfies

$$g(x,v,\omega) \leq C'(1+|v|) \tag{43}$$

for all $x \in \mathbb{R}^N$ and all $v \in \mathbb{R}^l$, with $C' := C\left(1+|u|\,\frac{\mu_0(A)}{\mu(A)}\right)$ since $(\mathrm{C}_1^\omega)$. As $L^1_\mu\left(\Omega;\mathbb{R}^l\right)$ is separable, there exists a countable set $D$ such that $\mathrm{cl}_{L^1_\mu(\Omega;\mathbb{R}^l)}(D) = L^1_\mu\left(\Omega;\mathbb{R}^N\right)$, and by using Vitali convergence theorem and the growth condition (43) we have

$$\inf_{\psi\in L^1_\mu(\Omega;\mathbb{R}^l)}\int_A g\left(x,\psi(x)-\fint_A \psi d\mu,\omega\right)d\mu = \inf_{\psi\in D}\int_A g\left(x,\psi(x)-\fint_A \psi d\mu,\omega\right)d\mu.$$

Now, for each $\psi \in D$ the map $\Sigma \ni \omega \longmapsto \int_A g\left(x,\psi(x)-\fint_A \psi d\mu,\omega\right)d\mu$ is measurable since for every $x \in \Omega$

$$f\left(x,\psi(x)-\fint_A \psi d\mu + u\frac{\mu_0(A)}{\mu(A)},\cdot\right) \text{ is } (\mathcal{T},\mathcal{B}(\mathbb{R}_+))\text{-measurable.}$$

It follows that $\Sigma \ni \omega \longmapsto \inf_{\psi\in L^1_\mu(\Omega;\mathbb{R}^l)}\int_A g\left(x,\psi(x)-\fint_A \psi d\mu,\omega\right)d\mu$ is $(\mathcal{T},\mathcal{B}(\mathbb{R}_+))$-measurable, which means that $\mathcal{S}_u(A)$ is a $(\mathcal{T},\mathcal{B}(\mathbb{R}_+))$-measurable function. Now, the condition $(\mathrm{C}_1^\omega)$ and (43) insures that $\mathcal{S}_u(A) \in L^1(\Sigma,\mathcal{T},\mathbb{P})$.

Let $A \subset [0,1[^N$ be a Borel set. We see that for $\mathbb{P}$-a.e. $\omega \in \Sigma$

$$\mathcal{S}_u(A)(\omega) \le \int_A f\left(x,\frac{u\mu_0(A)}{\mu(A)},\omega\right)d\mu \le C_0(1+|u|)$$

where $C_0 := C(\mu_0+\mu)\left([0,1[^N\right)(1+|u|)$, and with $C>0$ given by $(\mathrm{C}_1^\omega)$.

Next, we verify that $\mathcal{S}_u$ is a stationary process by using $(\mathrm{C}_3^\omega)$. For $\mathbb{P}$-a.e. $\omega \in \Sigma$ and for every $z \in \mathbb{Z}^N$, one has

$$\begin{aligned}
&\mathcal{S}_u(A+z)(\omega)\\
&= \inf\left\{\int_{A+z} f\left(x,\frac{d\lambda}{d\mu}(x),\omega\right)d\mu : \mathcal{M}\left(\Omega;\mathbb{R}^l\right)\ni\lambda\ll\mu \text{ and } \lambda(A+z)=u\mu_0(A+z)\right\}\\
&= \inf\left\{\int_A f\left(y+z,\frac{d\lambda}{d\mu}(y+z),\omega\right)d\mu : \mathcal{M}\left(\Omega;\mathbb{R}^l\right)\ni\lambda\ll\mu \text{ and } \lambda(A+z)=u\mu_0(A)\right\}\\
&= \inf\left\{\int_A f\left(y,\frac{d\lambda}{d\mu}(y),\tau_z\omega\right)d\mu : \mathcal{M}\left(\Omega;\mathbb{R}^l\right)\ni\lambda\ll\mu \text{ and } \lambda(A)=u\mu_0(A)\right\}\\
&= \mathcal{S}_u(A)(\tau_z\omega)
\end{aligned}$$

since $\mu_0 = p_0\mathcal{L}_N$ is invariant by translation.

It remains to show that $\mathcal{S}$ is subadditive, i.e. satisfies $(\mathscr{S}_1)$. Let $\omega \in \Sigma$ and let $(A,B) \in \mathcal{B}\left(\mathbb{R}^N\right)\times\mathcal{B}\left(\mathbb{R}^N\right)$ be such that $A\cap B=\emptyset$. Let $\varepsilon>0$. There exist $\lambda_A \in \mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ and $\lambda_B \in \mathcal{M}\left(\Omega;\mathbb{R}^l\right)$ such that $\lambda_A \ll \mu$, $\lambda_B \ll \mu$, $\lambda_A(A)=u\mu_0(A)$, $\lambda_B(B)=u\mu_0(B)$ and

$$\varepsilon + \mathcal{S}_u(A)(\omega)+\mathcal{S}_u(B)(\omega) \ge \int_A f\left(y,\frac{d\lambda_A}{d\mu}(y),\omega\right)d\mu + \int_B f\left(y,\frac{d\lambda_B}{d\mu}(y),\omega\right)d\mu.$$

Set $\lambda = \lambda_A\mathbb{1}_A+\lambda_B\mathbb{1}_B \in \mathcal{M}\left(\Omega;\mathbb{R}^l\right)$. It is direct to see that $\lambda\ll\mu$ and $\lambda(A\cup B)=u\mu_0(A\cup B)$. Thus

$$\begin{aligned}
\mathcal{S}_u(A\cup B)(\omega) &\le \int_{A\cup B} f\left(y,\frac{d\lambda}{d\mu}(y),\omega\right)d\mu\\
&\le \int_A f\left(y,\frac{d\lambda_A}{d\mu}(y),\omega\right)d\mu + \int_B f\left(y,\frac{d\lambda_B}{d\mu}(y),\omega\right)d\mu\\
&\le \varepsilon + \mathcal{S}_u(A)(\omega)+\mathcal{S}_u(B)(\omega). \blacksquare
\end{aligned}$$

6.6. **Proof of Lemma 4.1 (*ii*).** First, note that there exists $\Sigma' \in \mathcal{T}$ with $\mathbb{P}(\Sigma')=1$ such that for every $\omega \in \Sigma'$, every $u \in \mathbb{Q}^l$ and every cube $B \subset \mathbb{R}^N$,

$$\lim_{\varepsilon\to 0}\frac{\mathcal{S}_u\left(\frac{1}{\varepsilon}B\right)(\omega)}{\mu_0\left(\frac{1}{\varepsilon}B\right)} = \inf_{k\in\mathbb{N}^*}\mathbb{E}^{\mathcal{I}}\left[\frac{\mathcal{S}_u\left([0,k[^N\right)}{p_0k^N}\right](\omega).$$

Fix $\omega \in \Sigma'$. Let $\delta > 0$, $\overline{u} \in \mathbb{R}^l$ and $u \in \mathbb{Q}^l$ be such that $|u - \overline{u}| < \delta/2$. By Lemma 6.2, for every $\varepsilon > 0$ and every $k \in \mathbb{N}^*$

$$\begin{aligned}
&\left| \frac{\mathcal{S}_{\overline{u}}\left(\frac{1}{\varepsilon}B_\rho(x)\right)(\omega)}{\mu_0\left(\frac{1}{\varepsilon}B_\rho(x)\right)} - \mathbb{E}^{\mathcal{I}}\left[\frac{\mathcal{S}_{\overline{u}}\left([0,k[^N\right)}{p_0 k^N}\right](\omega) \right| \\
\leq &\left| \frac{\mathcal{S}_{\overline{u}}\left(\frac{1}{\varepsilon}B_\rho(x)\right)(\omega)}{\mu_0\left(\frac{1}{\varepsilon}B_\rho(x)\right)} - \frac{\mathcal{S}_{u}\left(\frac{1}{\varepsilon}B_\rho(x)\right)(\omega)}{\mu_0\left(\frac{1}{\varepsilon}B_\rho(x)\right)} \right| + \left| \frac{\mathcal{S}_{u}\left(\frac{1}{\varepsilon}B_\rho(x)\right)(\omega)}{\mu_0\left(\frac{1}{\varepsilon}B_\rho(x)\right)} - \mathbb{E}^{\mathcal{I}}\left[\frac{\mathcal{S}_{u}\left([0,k[^N\right)}{p_0 k^N}\right](\omega) \right| \\
&+ \left| \mathbb{E}^{\mathcal{I}}\left[\frac{\mathcal{S}_{u}\left([0,k[^N\right)}{p_0 k^N}\right](\omega) - \mathbb{E}^{\mathcal{I}}\left[\frac{\mathcal{S}_{\overline{u}}\left([0,k[^N\right)}{p_0 k^N}\right](\omega) \right| \\
\leq &\delta + \left| \frac{\mathcal{S}_{u}\left(\frac{1}{\varepsilon}B_\rho(x)\right)(\omega)}{\mu_0\left(\frac{1}{\varepsilon}B_\rho(x)\right)} - \mathbb{E}^{\mathcal{I}}\left[\frac{\mathcal{S}_{u}\left([0,k[^N\right)}{p_0 k^N}\right](\omega) \right|.
\end{aligned}$$

Passing to the limits $\varepsilon \to 0$, $k \to \infty$ and $\delta \to 0$, we conclude that

$$\lim_{\varepsilon \to 0} \frac{\mathcal{S}_{\overline{u}}\left(\frac{1}{\varepsilon}B\right)(\omega)}{\mu_0\left(\frac{1}{\varepsilon}B\right)} = \inf_{k \in \mathbb{N}^*} \mathbb{E}^{\mathcal{I}}\left[\frac{\mathcal{S}_{\overline{u}}\left([0,k[^N\right)}{p_0 k^N}\right](\omega). \blacksquare$$

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